Additive Growth*

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Abstract

Growth theory is based on the assumption of multiplicative productivity growth. Across all countries and time periods I show that TFP growth is in fact additive. The additive growth model offers explanations for the TFP slowdown puzzle, the pre-war volatility puzzle, the Great moderation, and falling interest rates. In the long run, the model implies ever increasing increments to standards of living but growth rates that converge to zero.

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Following Solow (1956) the textbook model assumes that TFP growth follows a geometric process where the size of the next increment is proportional to the level of TFP (e.g., Barro and Sala-i-Martin, 2004). I examine data across many countries and time periods and I find that, in nearly all cases, productivity growth is in fact arithmetic. For instance, using data from Bergeaud et al. (2016), US TFP growth after World War 2 is well described by the following statement: Hicks-neutral TFP, normalized to 1 in 1947, increases by about 2.45 percentage points each year without any sign of increase or decrease over more than 80 years. Using data from Fernald (2012) for the *private* sector, the same statement holds with an annual increment of 2.76, as show in Table 3 column (iii).

I start my investigation with the US using data from Fernald (2012) – Fernald for short – and Bergeaud et al. (2016)–BCL. I show that TFP growth is linear during the post war period. The inverse marginal product of capital is also linear, as predicted by the first order condition for capital demand. And the linear TFP model predicts the correct amount of convexity in labor productivity growth. For long-term US TFP growth (1890-2019) the additive growth model's 10-year forecast errors are 10% to 25% lower than those of the multiplicative model.

I then consider the 23 countries in the BCL sample. I find that TFP and capital accumulation are better described as additive rather than multiplicative for all of the 23 countries both in the post war sample and in the long sample (1890-2019). I also consider Korea and Taiwan, that are not in the BCL sample, and I show that their growth since the 1960s has been additive.

The multiplicative growth model fails in two related dimensions. The first is that it predicts periods of geometric productivity growth that simply do not exist in the data. The second is that the trend growth rate is unstable. Tests of structural breaks in the multiplicative model find a large number of breaks. By contrast, the additive model displays few breaks and, at least in some cases, these breaks have a plausible economic interpretation in terms of General Purpose Technologies (GPTs). For example, the process of US TFP increments has only one break over the past 130 years, immediately after the Electrification revolution (Gordon, 2016; Jovanovic and Rousseau, 2005). In the BCL data, from 1890 to 1933, US TFP increases each year by 1.7 percentage points of its inital level in 1890, until the early 1930s. From 1933 to 2019 TFP increases each year by 4.3 pp of its 1933 level.

I then study the theoretical implications of the additive model for long term growth. The model has a balanced growth path with a constant capital/output ratio. The

capital labor ratio, labor productivity, and GDP per capita grow indefinitely, and with increasing increments. The model therefore does not predict stagnation: growth is ever increasing, but growth rates tend to zero. Finally, I propose a simple endogenous growth model that rationalizes this process.

Literature The literature on growth is enormous and cannot be summarized here. I will just highlight the areas where my result shed new light on existing puzzles. The first area is growth accounting, from Solow (1956) to Gordon (2016). I make two contributions here: I argue that the perceived TFP slowdown is the result of a misspecified model, since growth was never actually geometric, and I show that a well speficied model provides useful benchmarks and forecasts across a wide variety of countries and time periods.

The second area concerns the sources of growth. The additive model sheds new light on the role of human capital accumulation, as in Mankiw et al. (1992), and on the distinction between Hicks-neutral and Harrod-neutral progress.

Finally my results speak to the correct specification of models of endogenous growth, such as Romer (1986b), Lucas (1988) and Aghion and Howitt (1992). These models assume a knowledge production function that delivers a constant growth rate. If z is TFP, these models make assumptions that predict a balance growth. My results suggest that it should deliver constant productivity increments instead, with changes in the size of the increment only around the discovery of new GPTs. These insights relate to the recent work of Bloom et al. (2020) on the declining productivity of research and development activities. I argue that the benchmark growth model displays constant increments, not a constant growth rate. This implies that the research productivity puzzle is two or three times smaller than previously thought.

1 Evidence from US Growth

My main sources of data – Fernald (2012) and Bergeaud et al. (2016) – assume a Cobb-Douglas production function, so I will do the same in this section. Aggregate value value added (GDP, Y_t) is given by

$$Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}, \tag{1}$$

where z_t is TFP, K_t is the flow of capital services and L_t is the flow of labor services. I discuss more general production functions $F(K, L, Z)$ and the comparison of Hicks and Harrod's neutrality, which turns out to be important, in Section 3. My goal is to

understand the long-term dynamics of TFP. Since at least Solow (1956) economists have assumed that z follows a geometric process such that[1]

$$\mathbb{E}\left[z_{t+\tau} \mid z_t\right] = (1+g)^{\tau} z_t. \tag{2}$$

I am instead arguing that growth is additive and the process is better described by

$$\mathbb{E}\left[z_{t+\tau} \mid z_t\right] = z_t + \Delta\tau. \tag{3}$$

I start my investigation with post-war US data, as it is the most reliable and the most widely used.

1.1 Postwar U.S. TFP

My primary sources for TFP are Fernald (2012) (Fernald) and Bergeaud et al. (2016) (BCL). Let z_t^{bcl} and z_t^f denote the BCL and Fernald measures of TFP. There are several differences between these two datasets. BCL covers 23 countries from 1890 to 2019, so I will use these data also for a long sample analysis and for international comparisons. Fernald's series cover only the business sector, while BCL include households and the government. Fernald includes an adjustment for capacity utilization to make the series comparable to the theoretical benchmark. Finally, Fernald also includes an adjustment for human capital, following Mankiw et al. (1992). Formally, BCL assume that $L_t = H_t$, total hours worked, while Fernald assume $L_t = Q_t^L H_t$ where Q_t^L is an index of labor quality based on education. Using (1), we see that the Fernald's measure comparable to the BLC measure is

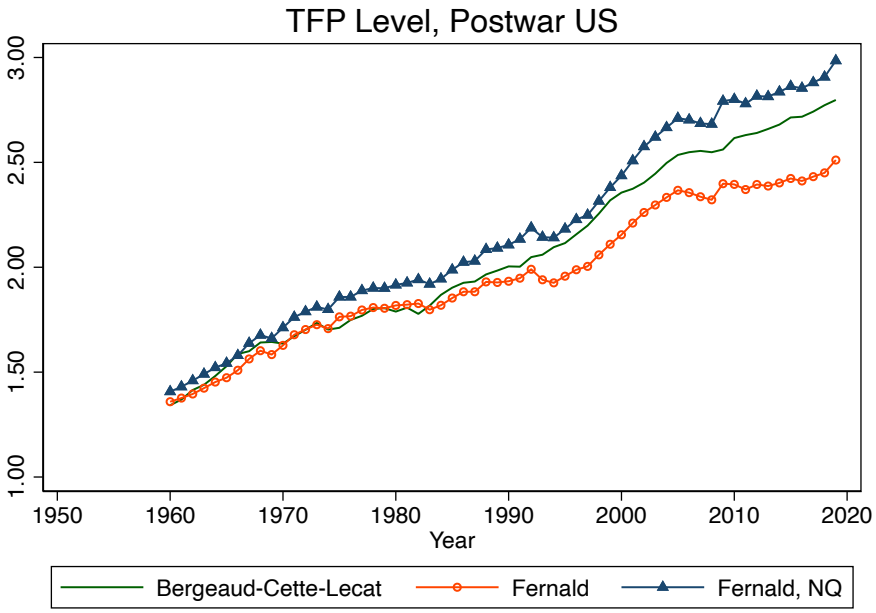
$$z_t^{nq} = z_t^f \left(Q_t^L\right)^{1-\alpha}$$

where z_t^f is Fernald's labor-quality-adjusted TFP measure. Figure 1 shows the three TFP series, where z_t^{bcl} is normalized to 1 in 1947 to be comparable with Fernald's measures. Fernald's unadjusted measure z_t^n grows somewhat faster than z_t^{bcl} because productivity growth is faster in the business sector than in the rest of the economy. On the other hand, z_t^f grows more slowly because some of the measure productivity is attributed to the labor quality factor.

The simplest way to start comparing model D and model G is to consider the following experiment. Suppose that two agents, *George* and *Daniela*, are asked in the middle

[1] Jensen's inequality terms are trivial in the calculations below.

Figure 1: US TFP Levels



Notes: TFP levels, z^{bcl}, z_t^f, and z_t^n. Data from Fernald (2012) and Bergeaud et al. (2016).

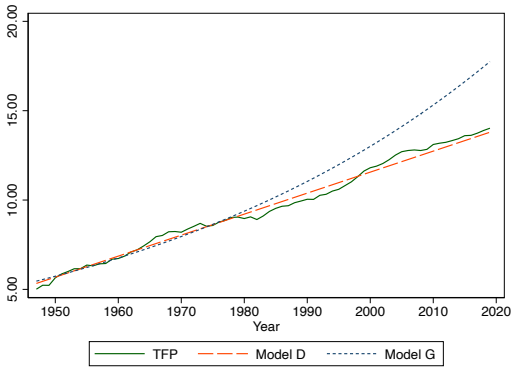
of the sample (1983) to predict the level of TFP for the second half of the sample. The agents have access to data from the end of World War 2 until 1983. The two agents have dogmatic beliefs regarding the correct model of economic growth. George believes in model G from equation (2) while Daniela believes in model D from equation (3). George fits a log linear model $\log(z_t) = \hat{\alpha}_g + \hat{g}t$ over the years $1947:1980$ and predicts future (log) TFP as

$$\log\left(\hat{z}_t^{(G)}\right) = \hat{\alpha} + \hat{g}_z t$$

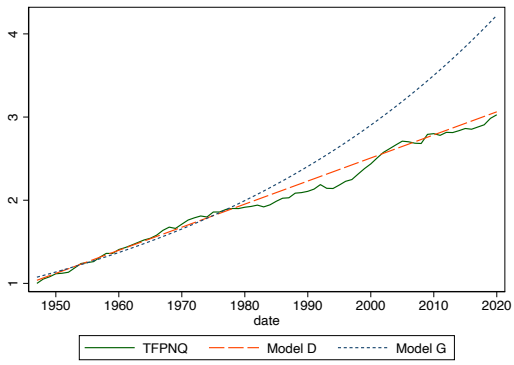
for $t = 1981:2019$. Daniela instead fits a linear model and predicts future TFP as $\hat{z}_t^{(D)} = \hat{\alpha}_z + \hat{\Delta}_z t$. Figure 2 shows that Danelia would have made a much better forecast than George. George is puzzled by the TFP slowdown while Daniela does not perceive an obvious long term break in her model (although there are some important medium term deviations that we analyze below). The results obtained from z_t^n and z_t^{bcl} are virtually identical so I focus on only one measure (BCL) for brevity in this section.

Figure 2: Out-of-Sample TFP Forecasts

(a) BCL Data



(b) Fernald Data



Notes: BCL TFP is in based on $US 2010. Fernald unadjusted TFP, z_t^n, is normalized to 1 in 1947. Models are estimated over 1947-1983. The forecast 1984-2019 is out-of-sample. Data source: Fernald (2012) and Bergeaud et al. (2016).

1.2 Capital Accumulation and Labor Productivity

Let us know study the accumulation of capital. Define the capital labor ratio as

$$k_t \equiv K_t / L_t,$$

where, in the BCL data, K_t is the real capital stock and L_t measures hours worked. The first order condition for capital demand in the neoclassical growth model equates the marginal product of capital (MPK) to the user cost (χ). BCL do not consider changes in the user cost. I therefore assume a constant user cost and the FOC is simply

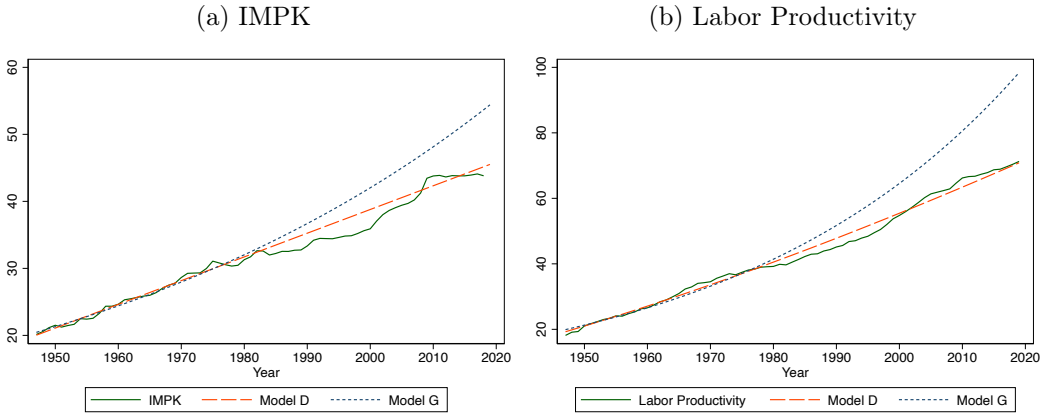
$$k_t^{1-\alpha} = \frac{\alpha}{\chi} z_t. \tag{4}$$

Equation (4) says that the inverse MPK (IMPK) is proportional to z.[2] Model G therefore predicts that $k_t^{1-\alpha}$ grows exponentially, while model D says that it grows linearly. Figure 3 presents the forecasts for $k_t^{1-\alpha}$ based on model D and G with $\alpha = 0.3$, the value used by BCL. For model D we have

$$\mathbb{E}\left[k_t^{1-\alpha}\right] = \hat{\alpha}_{impk} + \hat{\Delta}_{impk} t$$

[2]Users of model G typically interpret equation (4) as saying that capital grows grows exponentially, just like z, as a rate $(1+g)^{1/(1-\alpha)}$. Equivalently, if the model is written with Harrod-neutral technological progress, $Y_t = K_t^{\alpha} \left(\mathcal{Z}_t H_t\right)^{1-\alpha}$ then capital is proportional to $\mathcal{Z}_t$. I return to these issues in Section 3.

Figure 3: Out-of-Sample $IMPK$ and LP Forecasts



Notes: K/L is in \$US 2010 ppp per hour. In 2010, K/L was \$221 per hour and $IMPK = (K/L)^{0.7} = 43.8$. Models are estimated over 1947-1983. The forecast 1984-2019 is out-of-sample. Labor productivity is GDP per hour. Data source: Bergeaud et al. (2016).

For model G we have the formula in logs. Once again we find that the log-linear model with constant growth widely missed the mark, while the additive model gives a useful forecast.

Once we have a forecast for the capital labor ratio we can use our forecast for TFP to create a forecast for labor productivity λ_t, defined as output per hour:

$$\lambda_t \equiv \frac{Y_t}{L_t} = z_t k_t^{\alpha}. \tag{5}$$

Model D offers a forecast for labor productivity as

$$\hat{\lambda}_t = \left(\hat{\alpha}_z + \hat{\Delta}_z t\right)\left(\hat{\alpha}_{impk} + \hat{\Delta}_{impk} t\right)^{\frac{\alpha}{1-\alpha}}$$

Note that labor productivity is convex in time even under additive growth since it depends on the *product* of both TFP and capital intensity. I could use the forecasts for IMPK and TFP to similarly create a forecast for model G but that would be a straw man since we have seen that model G fails to predict either z or IMPK. To give model G a chance, I create directly a forecast of labor productivity by fitting the series for $\log(\lambda_t)$ in the first half of the sample. Model G therefore gains two degrees of freedom. Panel (b) in Figure 3 shows that the convex-linear forecast of model D predicts correctly the evolution of labor productivity in the long term. Model G does not.

US growth is better described as additive rather than multiplicative. Instead of stat-

ing that the average growth rate of TFP is 1.45%, which is correct but not particularly useful, it is more relevant to say that TFP increases by 0.0245 points each year starting from a normalized value of 1 in 1947.[3] For labor productivity, both the additive growth model and the multiplicative growth model predict an increasing size of productivity increments, but at different speeds. The additive model D predicts that labor productivity increments increase with the square of the time horizon, while the geometric model G predicts exponentially increasing increments. Model G does not describe the data with a constant growth rate. Model D describes the data relatively well with year-on-year increments of about $1560 per full time worker ($0.87 per hour, assuming 1800 hours worked in a year) around 2010.

Remark 1. Postwar US TFP growth is relatively well described as an additive process with increments of 0.0245 points each year starting from a normalized value of 1 in 1947.

1.3 U.S. TFP, 1890-2019

Let us know extend the methodology and the sample, taking into account that trend growth changes over time. Our agents now forecast time varying growth according to a standard recursive model

$$\mathbb{E}_t\left[\Delta_{t+1}\right] = (1-\zeta)\,\mathbb{E}_{t-1}\left[\Delta_t\right] + \zeta\Delta_t \tag{6}$$

for the additive growth model, and similarly for the growth model with $g_t = z_t/z_{t-1} - 1$ instead of Δ_t.[4] For the smoothing parameter I study values between 0.05 and 0.1 which correspond to an a priori plausible range. At 0.05, the sensitivity of the trend estimate to the most recent observation is the same as that of a 20-year moving average. Below 0.05 the model would take too long to adjust to changing growth. At 0.1, the sensitivity is the same as that of a 10-year moving average which is a standard way to estimate trend growth rates. The optimal smoothing parameter estimated on *pre-war* data for US TFP is $\zeta = 0.077$, in the middle of the a priori range.

Figure 4 shows the raw and smoothed series for $\zeta = 0.05$ and $\zeta = 0.1$. The data is from Bergeaud et al. (2016) and winsorized in the first and last percentiles to remove

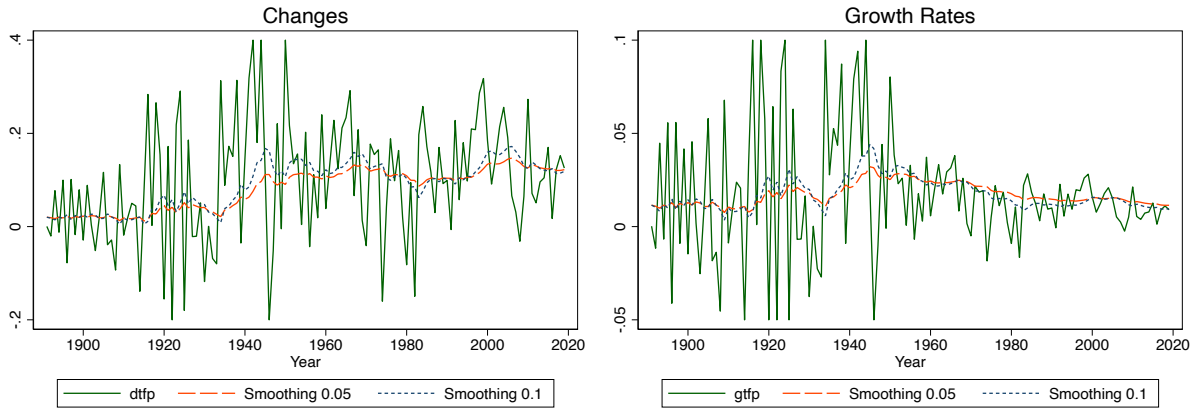
[3]The points of TFP need to be expressed in some units. Bergeaud et al. (2016) express TFP at $US 2010 ppp and the annual increase is 0.125 points each year. Alternatively, we can normalized initial TFP to 1 in 1947, as in Fernald (2012). Since TFP in 1947 in Bergeaud et al. (2016) is 5, we obtain $0.125/5 = 0.025$.

[4]Equivalently, in logs, $g_t = \log\left(z_t/z_{t-1}\right)$ and $\mathbb{E}^m_{t-10}\left[z_t\right] = z_{t-10}\left(1+\hat{g}_{t-10}\right)^{10}$. The quantitative results are virtually identical.

Figure 4: US TFP, 1890-2019

(a) TFP Changes. Model D.

(b) TFP Growth Rates. Model G.



Notes: Models are estimated over 1947-1980. The left panel show the prediction of a linear model. The right panel shows the prediction of a log-linear model. US TFP is from the updated work of Bergeaud et al. (2016).

limit extreme outliers during WW2. The model is initiated over the first 10 observations, 1891 to 1900.

As expected the trend growth of the economy changes over this long sample. The extreme heteroskedasticity of growth rates is apparent in Panel (b). TFP growth rates are much more volatile before than after WW2 – 4.5% vs 1.5% – and volatility declines further after 1980. Romer (1986a) discusses the first fact, McConnell and Perez-Quiros (2000) discuss the second fact, which became know as the great moderation puzzle. These puzzles are much less salient if we consider TFP changes rather than growth rates. Volatility of changes is 0.13 before WW2 and 0.11 since 1947.

Remark 2. The moderation puzzle is much less salient for the additive model.

Forecasts Let us know study the forecasting accuracy of the two models, but instead of performing once test pre/post 1980, we compute real-time rolling estimates and out-of-sample forecasts. Figure 5 shows the 10-year out-of-sample predictions for the level of TFP for the additive model (model D, as "differences")

$$\mathbb{E}_{t-10}^{D}\left[z_t\right] = z_{t-10} + 10\hat{\Delta}_{t-10} \tag{7}$$

Table 1: RMSE for US Growth Forecasts, 1901-2019

Smoothing Parameter	$\zeta = 0.05$		$\zeta = 0.1$	
Forecast Errors in Levels				
	TFP	IMPK	TFP	IMPK
Additive Model D	.57	3.00	.60	3.25
Multiplicative Model G	.72	3.45	.76	3.82
Forecast Errors in Percents				
Additive Model D	.108	.143	.113	.156
Multiplicative Model G	.117	.169	.131	.188

Notes: US TFP is from the updated work of Bergeaud et al. (2016)

and for the multiplicative model (model M)

$$\mathbb{E}_{t-10}^{G}\left[z_t\right] = z_{t-10}\left(1+\hat{g}_{t-10}\right)^{10}, \tag{8}$$

where $\hat{\Delta}_{t-10} \equiv \mathbb{E}_{t-10}\left[\Delta_{t-9}\right]$, $\hat{g}_{t-10} = \mathbb{E}_{t-10}\left[g_{t-9}\right]$ are the trends estimated 10 years before.[5] One could justify two ways to compute the forecast error: in points, or in percents. I use points as a baseline since the level of TFP is the variable of interest:

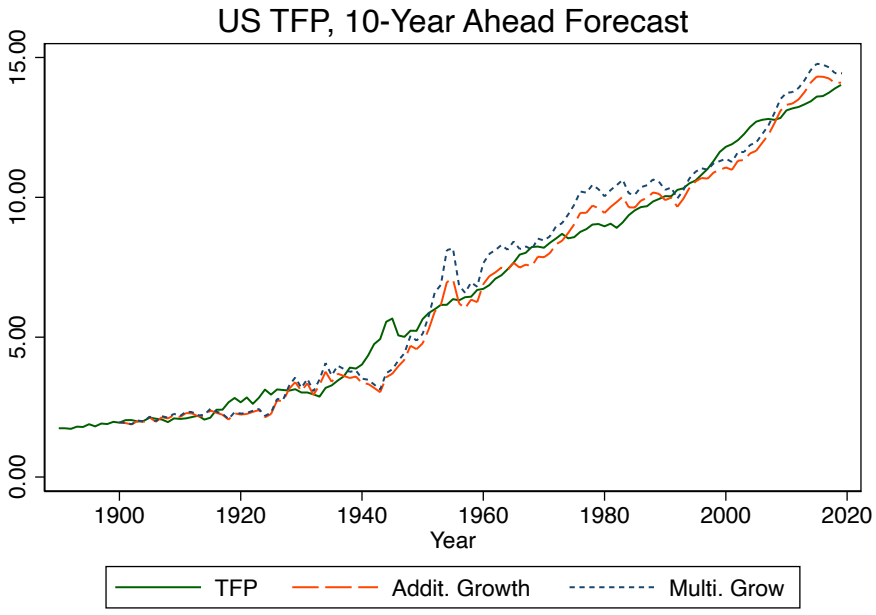
$$\epsilon_{i,t}^{D,G} = z_{i,t} - \mathbb{E}_{t-10}^{D,G}\left[z_{i,t}\right]$$

One could argue, however, that a 1 point error when TFP is 10 is more important than a one point error when TFP is 20, in which case the percent error $\frac{z_{i,t}-\mathbb{E}_{t-10}^{D,G}[z_{i,t}]}{z_{i,t}}$ is more appropriate. Table 1 reports the root mean square errors (RMSE) for both measures. Irrespective of the measure used the additive growth model outperforms the multiplicative growth model. With $\zeta = 0.05$ the RMSE is 0.57 points for the D model and 0.72 points for the G models, or 10.8% versus 11.7% for the percent errors. The gaps are larger for capital and for $\zeta = 0.1$. The main reason is that after a sequence of positive growth rates the multiplicative model extrapolates exponential growth for 10 years, which systematically fails to materialize. From an econometric perspective the level errors are more relevant because they are stationary. The percent level are much higher pre war when TFP is low.

Remark 3. For long-term US TFP growth (1890-2019) the multiplicative model's 10-year forecast errors are 10% to 25% higher than those of the additive model.

[5] Note that $\mathbb{E}_t\left[g_{t+k}\right] = \mathbb{E}_t\left[g_{t+1}\right]$ for all $k \geq 1$.

Figure 5: US TFP Forecasts, Long Sample



Notes: Forecast with smoothing parameter 0.075. US TFP is from the updated work of Bergeaud et al. (2016).

1.4 TFP and GPT

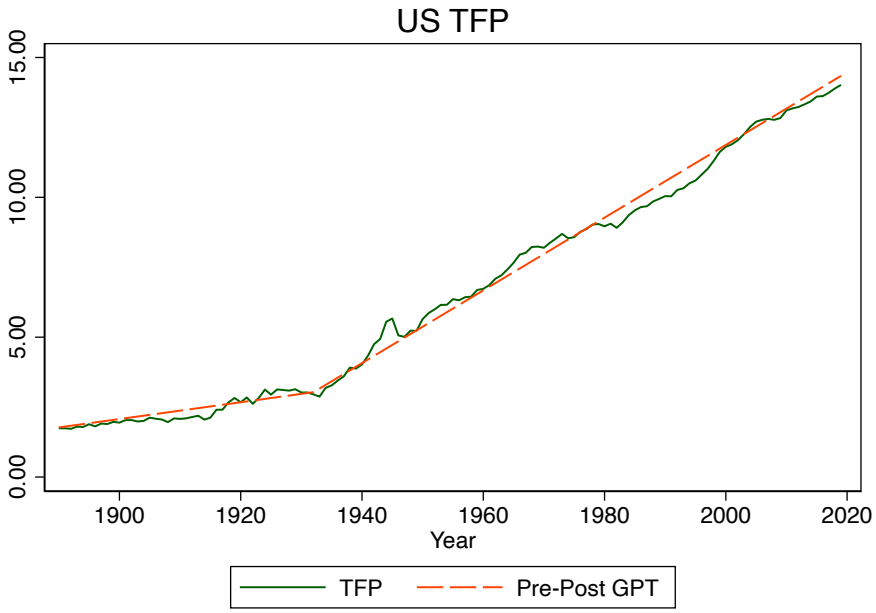
Figure 4 shows that model G is unstable. The estimated growth rates is constantly being revised. This is why model G is not useful as a long run growth model. Model D, on the other hand, appears to have only one break. We can formally test this idea following Bai and Perron (2003). The unconstrained test finds one break in the $\Delta[TFP]$ series, in 1933. We can test H0: no breaks versus H1: break in 1933. The W statistic is 21.72 and the p-value is 0.0.

The break in 1933 corresponds to the end of the electrification GPT discussed in Jovanovic and Rousseau (2005) and Gordon (2016). Along these lines, Figure 6 proposes one interpretation of US TFP from 1890 to 2019, using linear growth with one structural break in 1933 after the electrification revolution.

We can summarize this idea in the following remark, keeping in mind that in BCL data, US TFP in 1890 is 1.74.

Remark 4. From 1890 to 1933, TFP increases by 0.03 points (1.7 percentage points of its initial level) each year until it reaches a level around 3 in the early 1930s. From 1933 to 2019 TFP increases by 0.13 points each year (4.3 p.p. of its initial level in 1933) to

Figure 6: US TFP under Electrification GPT Interpretation



Notes: US TFP is from the updated work of Bergeaud et al. (2016).

reach a level around 14 in the late 2010s.

2 Country-Level International Evidence

Bergeaud et al. (2016) provide data for 23 countries.[6] I focus on the post war period because many countries (e.g. Japan, Germany) experience extreme shocks during the 1940s which render the forecasts from the multiplicative model unstable. The trend growths are estimated with the recursive learning model (6) with parameter $\zeta = 0.05$ initiated at US predicted values in 1950. As before, all the forecasts are out-of-sample. For each country $i = 1 : 23$ and each year $t = 1959 : 2019$ I compute the relative forecast error in percent for ease of comparison across countries

$$\epsilon_{i,t}^{D,G} = \frac{z_{i,t} - \mathbb{E}_{t-10}^{D,G}\left[z_{i,t}\right]}{z_{i,t}}$$

[6]Australia, Austria, Belgium, Canada, Switzerland, Chile, Germany, Denmark, Spain, Finland, France, United Kingdom, Greece, Ireland, Italy, Japan, Mexico, Netherlands, Norway, New Zealand, Portugal, Sweden and United States. The sample covers 1890–2019. The main variables are GDP, labor, and capital. Labor is constructed from data on total employment and working time. Capital is constructed by the perpetual inventory method applied equipment and buildings.

Table 2: RMSE for 23 Countries over 1959-2019

Average RMSE	TFP	IMPK
Additive Model D	.0887	.1037
Multiplicative Model G	.1319	.1369
N. Obs.	23	23

Notes: Data from Bergeaud et al. (2016). Sample 1950-2019.

where the expectation are taken under the additive D and multiplicative M models. For the multiplicative model, for instance, $\hat{g}_t = \mathbb{E}_t [g_{t+1}] = (1-\zeta)\mathbb{E}_{t-1}[g_t] + \zeta g_t$. Finally, I compute the root mean square error for each country

$$\texttt{RMSE}_i^{D,G} = \sqrt{\frac{1}{61}\sum_{t=1959}^{2019}\left(\epsilon_{i,t}^{a,m}\right)^2}$$

2.1 Post-War Sample

If we average across all countries, the TFP-forecast RSME is 4.3 percentage points higher for the G-model than for the D-model, as shown in Table 2.

Figure 7 shows the RMSE of the two models for TFP and for Capital Intensity. The additive model performs better than the multiplicative one in essentially all cases. The differences are larger than in Table 1 because many countries experience more volatile growth sequences than the US, which makes it easier to separate the two models.

I have studied 46 long run series (TFP and capital intensity for 23 countries) over the period 1950-2019. In two cases (TFP in Ireland, capital intensity in Chile), the two models have the same RMSE. In the 44 other cases the RMSE is lower for the additive model, and often much lower.

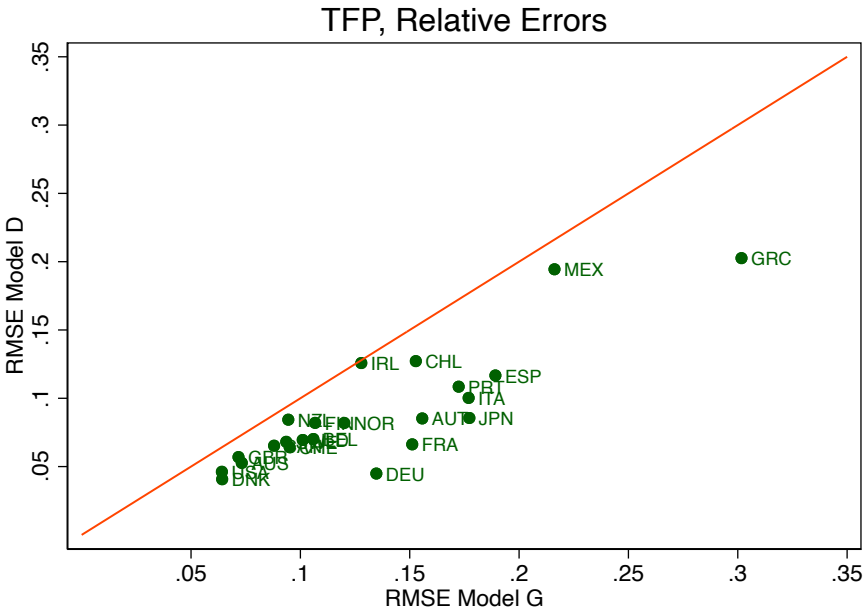
Remark 5. TFP and Capital Intensity are better described as additive rather than multiplicative for the 23 countries covered by Bergeaud et al. (2016) since 1950.

2.2 Long-Sample, 1890-2019

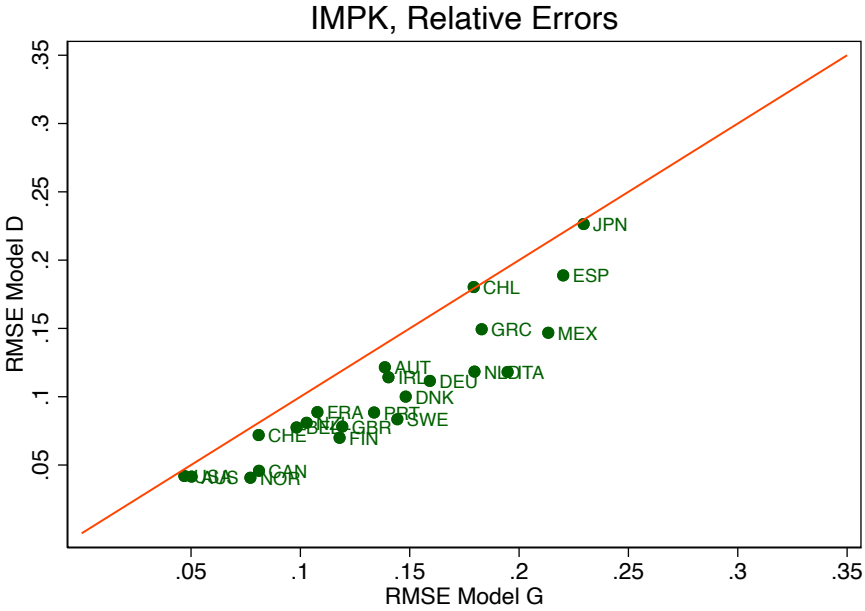
I now run the model over the whole sample, 1890-2019, initializing over the first 10 years. This adds a lot of volatility in many countries during and immediately after the wars, but figure 8 shows that the D-model continues to out-perform the G-model. 8 uses the absolute forecast errors to avoid giving excessive weight to early observation when TFP is low is many countries. In the long sample the D-model provides better forecasts than

Figure 7: Root Mean Square Errors (%), Postwar Sample

(a) TFP Forecasts

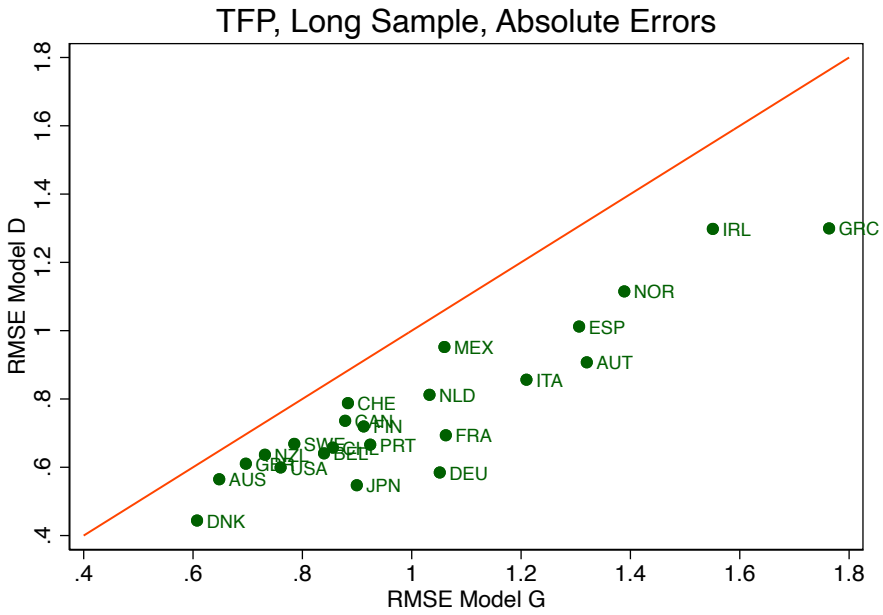


(b) IMPK Forecasts



Notes: Model G on the horizontal axis, model D on the vertical axis. Out-of-sample, 10-year forecasts with smoothing parameter 0.05. Data from Bergeaud et al. (2016). Sample 1950-2019.

Figure 8: Root Mean Square Errors (levels), Long Sample



Notes: US TFP is from the updated work of Bergeaud et al. (2016).

the G model for 23 of the 23 TFP series.

3 Implications for Long Term Growth

The goal of this section is to highlight the most important features of additive growth in the aggregate. I leave industry and firm dynamics for future research. To draw long-run implications from the theory we must first revisit the exact nature of the production function.

3.1 Finding Linearity

As explained in Barro and Sala-i-Martin (2004), balanced growth requires labour-augmenting (Harrod-neutral) technology

$$Y_t = F\left(K_t, \mathcal{Z}_t L_t\right) \tag{9}$$

where $\mathcal{Z}_t$ is labor-augmenting, or Harrod-neutral, technological progress. We need an exact mapping between this functional form and the evidence discussed so far. This is important even if F is Cobb-Douglas, as Fernald (2012) and Bergeaud et al. (2016)

assume. In that case we can of course renormalize our productivity measure as $z = \mathcal{Z}^{1-\alpha}$ so that (9) becomes $Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}$, but this does not answer the question of whether z or $\mathcal{Z}$ (or perhaps neither) is best described as a linear process. To be concrete, if $\mathcal{Z}$ is linear then z is concave in time. If instead z is linear then $\mathcal{Z}$ is convex in time. These distinctions matter for long-run growth.

Fernald (2012) provides a good framework to discuss these issues. He writes the following production function

$$Y_t = z_t^f K_t^{\alpha} \left(Q_t^L H_t\right)^{1-\alpha},$$

where z_t^f Fernald's headline TFP measure, capital services K adjusted for variable utilization are constructed from disaggregated series on structures, equipments and IPs, and Q_t^L is a labor-quality index constructed from rolling Mincer wage regressions in the Current Population Survey. I consider 5 hypotheses:

- (i) GzQ: the Hicks-neutral series net of labor quality improvements, z_t^f, features constant *geometric* growth.
- (ii) DzQ: z_t^f features constant *additive* growth.
- (iii) DzNQ: $z_t^n = z_t^f \left(Q_t^L\right)^{1-\alpha}$, the Hicks-neutral TFP *including* educational improvements is additive. Whether or not one should net out the effect of education when measuring TFP depends on the question at hand. Solow (1957) explains that he uses "the phrase "technical change" as a short-hand expression for any kind of shift in the production function. Thus [..] improvements in the education of the labor force, and all sorts of things will appear as "technical change"." If one follows this line of reasoning, then z_t^n is the more relevant concept.
- (iv) DZQ: $\left(z_t^f\right)^{\frac{1}{1-\alpha}}$, the Harrod-neutral series adjusted for labor quality, is additive. If the true model is $Y_t = F\left(K_t, \mathcal{Z}_t Q_t^L H_t\right)$ with $\mathcal{Z}_t$ additive, then we should find that $\left(z_t^f\right)^{\frac{1}{1-\alpha}}$ is additive.
- (v) DZNQ : $\left(z_t^f\right)^{\frac{1}{1-\alpha}} Q_t^L$, the Harrod-neutral series including educational improvements is additive. If the true model is $Y_t = F\left(K_t, \mathcal{Z}_t H_t\right)$ with $\mathcal{Z}_t$ additive, then we should find that $\left(z_t^f\right)^{\frac{1}{1-\alpha}} Q_t^L$ is additive.

Table 3 shows the estimates for the US from Fernald's data, together with one estimate from the BCL data for comparison. Column (i) documents the well-know TFP

Table 3: Distribution of TFP Growth, Top 500 US Employers

100∗	(i) $\frac{\Delta[z_t^f]}{z_t^f}$	(ii) $\Delta\left[z_t^f\right]$	(iii) $\Delta\left[z_t^f\left(Q_t^L\right)^{1-\alpha}\right]$	(iv) $\Delta\left[\left(z_t^f\right)^{\frac{1}{1-\alpha}}\right]$	(v) $\Delta\left[\left(z_t^f\right)^{\frac{1}{1-\alpha}} Q_t^L\right]$	(vi) $\Delta\left[z_t^{bcl}\right]$
(Year-1983)	-0.026	-0.015	-0.002	-0.005	0.038	-0.003
t	-3.3	-1.1	-0.1	-0.2	1.1	-0.3
Constant	1.311	2.107	2.758	4.131	5.740	2.498
t	7.9	7.2	8.5	6.7	7.9	9.9
N	72	72	72	72	72	72
R^2	0.133	0.017	0.000	0.000	0.017	0.001

Notes: US, 1947-2019. All dependent variables scaled by 100. In (i) the dependent variable is the growth rate of TFP. In (ii-vi) the dependent variable is the first difference of TFP. Data from Fernald (2012) in (i-v) and Bergeaud et al. (2016) in (vi). The independent variable is year minus sample mean (1983) so that the constant can be readily interpreted as the sample average growth.

"slowdown" which is a puzzle for the geometric model GzQ (the puzzle is the same for all G models, omitted for brevity). The sample average TFP growth is 1.3% per year, but loses 2.6 basis point each year, from around 2% in the early 1950s down to only 0.5% after 2010. Column (ii) shows that the puzzle is much reduced, but not entirely eliminated, by the DzQ model.

Column (iii) and (iv) show that DzNQ and DZQ are two equally plausible way to characterize additive growth. DZQ assume linear labor-augmenting productivity applied to quality adjusted labor $Q_t^L H_t$. DzNQ folds educational improvements into Hicks-TFP growth instead of netting them out. Column (v) shows that doing both adjustments simultaneously might be excessive in the US. Column (vi) shows that the BCL series is similar to the Hicks-neutral series based on raw labor DzNQ in (iii), which is consistent with our discussion in Section 1.

The data therefore suggests that models (iii) and (iv) provide reasonable descriptions of additive productivity growth. An important point is that both predict increasing improvements in labor productivity, but for slightly different reasons. In model DZQ, $\mathcal{Z}_t$ is linear, which by itself generates linear labor productivity, but it applies to an increasingly qualified quantity of labor. In model DzNQ the combination of educational and non educational improvements generates Hicks-neutral additive growth.

There is no clear statistical reason to prefer the Hicks-additive model (iii) to model (iv), but (iii) has three practical advantages. It is more directly comparable to the BCL TFP series; it requires only one factor (z_t) instead of two ($\mathcal{Z}_t$ and Q_t^L); and it is easier to describe. In fact we can describe post-war US growth in two sentences.

Remark 6. The following model correctly describes US private sector growth since 1947: Output is a Cobb-Douglass function of capital and hours worked $Y_t = z_t K_t^{\alpha} H_t^{1-\alpha}$, and Hicks-TFP z_t, normalized to 1 in 1947, increases by about 2.76 percentage points each year.

After twenty years, TFP is 1.55, after forty years it is 2.1, and so on. This model accounts well for the evolution of TFP in the US since 1947. There is no slowdown in TFP increments. The point estimates of -0.002 is rather precisely estimated at 0.

3.2 Additive Dynamics

Let us now turn to the theoretical dynamics of the Hicks-additive model. I use continuous time to simplify the notations. Output is $Y_t = z_t K_t^{\alpha} H_t^{1-\alpha}$, TFP grows according to $\frac{dz_t}{dt} = \Delta$, and capital accumulates as

$$\frac{dK_t}{dt} = I_t - \delta K_t.$$

Hours grow at the constant population growth rate n: $\frac{dH_t}{dt} = nH_t$.

Fixed Saving Rate Let us start with a textbook Solow model with a fixed saving rate s: $I_t = sY_t$. Define $\mathcal{Z}_t = z_t^{\frac{1}{1-\alpha}}$ and the scaled capital stock as

$$\kappa_t \equiv \frac{K_t}{\mathcal{Z}_t H_t}$$

to obtain the usual differential equation

$$\dot{\kappa}_t = s\kappa_t^{\alpha} - \left(\delta + n + \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t}\right)\kappa_t \tag{10}$$

with $\frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t} = \frac{1}{1-\alpha}\frac{\dot{z}_t}{z_t}$. Geometric predicts that $\frac{\dot{z}_t}{z_t}$ is constant. With additive growth $\frac{\dot{z}_t}{z_t} = \frac{\Delta}{z_t}$ is not constant. Since $\lim_{t\to\infty} z_t = \infty$ we have the following proposition.

Proposition 1. *The long-run balanced growth path is characterized by* $\kappa_\infty = \left(\frac{s}{\delta+n}\right)^{\frac{1}{1-\alpha}}$. *The capital labor ratio grows as* $k_t = \kappa_\infty z_t^{\frac{1}{1-\alpha}}$ *and labor productivity (or GDP per capita) as* $\lambda_t = \kappa_\infty^{\alpha} z_t^{\frac{1}{1-\alpha}}$. *The increments of* k_t *and* λ_t *increase to infinity*

$$\lim_{t\to\infty} \frac{d\lambda_t}{dt} = \infty,$$

but their growth rates converge to zero

$$\lim_{t\to\infty} \frac{d\log\lambda_t}{dt} = 0.$$

Let me now discuss a few implications and extensions of the model.

Long term growth Note that for large t we have $\frac{d\lambda_t}{dt} \approx \frac{1}{1-\alpha}\kappa_\infty^\alpha \left(z_0 + \Delta t\right)^{\frac{\alpha}{1-\alpha}} \Delta$. This shows that labor productivity, and thus living standards, grow as an increasing pace when we assume Hicks-linear growth. If we instead assume (counter-factually as discussed above) Harrod-linear growth, $\dot{\mathcal{Z}}_t = \Delta$, then improvements in living standards would not go to infinity but instead converge to a finite limit: $\frac{d\lambda_t}{dt} \to \kappa_\infty^\alpha \Delta$. Models DzNQ (iii) and DZNQ (v) therefore make very different predictions about long run growth. The good news is that the available data seems to support model DzNQ.

Convergence Transitional dynamics are essentially the same as in the standard model, so results on conditional convergence, discussed for instance in Barro and Sala-i-Martin (2004) are unchanged. Unconditional convergence depends on how Δ varies across countries and over time. Permanent differences in Δ predict infinitely increasing inequality even though growth rates converge to zero in all countries. Changes in Δ over time predict time-varying catch. Imagine a country that manages to permanently increase its Δ. The catch up would be initially quick but then slower.

Neoclassical Production Function The results in Proposition 1 generalize beyond the Cobb-Douglas case. Define $Z\left(z\right)$ as the Harrod growth in a neoclassical production function $Y_t = F\left(K_t, \mathcal{Z}_t H_t\right)$. The dynamic equation becomes

$$\dot{\kappa}_t = sf\left(\kappa_t\right) - \left(\delta + n + \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t}\right)\kappa_t$$

where $f\left(\kappa\right) \equiv F\left(\kappa, 1\right)$. As before the limit solves $sf\left(\kappa_\infty\right) = \left(\delta + n\right)\kappa_\infty$. Define

$$\alpha_\infty \equiv \lim_{\kappa\to\kappa_\infty} \alpha\left(\kappa\right)$$

where $\alpha\left(\kappa\right) \equiv \frac{KF_K}{F}$ estimated at the point $\kappa = \frac{K}{\mathcal{Z}H}$. Thus α_∞ is simply the capital elasticity (or capital share) estimated at κ_∞. This model is consistent with the evidence on additive Hicks productivity growth if and only if $\mathcal{Z}\left(z\right) = z^{\frac{1}{1-\alpha_\infty}}$.

Ramsey and Interest Rates Let us finally discuss the case where savings are endogenous. I again assume a textbook model where infinitely-lived households have CRRA preferences and fixed labor supply. The equilibrium is pinned down by capital accumulation and the households' Euler equation (and the usual transversality condition, omitted here):

$$\dot{\kappa}_t = f(\kappa_t) - \hat{c}_t - \left(\delta + n + \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t}\right)\kappa_t,$$
$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \sigma\left(f'(\kappa_t) - \delta - \rho\right) - \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t},$$

where $\hat{c}_t = \frac{C_t}{H_t \mathcal{Z}_t}$ is normalized consumption per capita, σ is the EIS and ρ the rate of time preference. As before, we have $\lim_{t\to\infty} \frac{\dot{\mathcal{Z}}_t}{\mathcal{Z}_t} = 0$ so the long-term balanced growth path is given by

$$f'(\kappa_\infty) = \delta + \rho$$

and

$$\hat{c}_\infty = f(\kappa_\infty) - (\delta + n)\kappa_\infty$$

All per capital variables grow with $\mathcal{Z}_t$. For instance, long run per capita consumption is $c_t = \hat{c}_\infty \mathcal{Z}_t$. What is interesting, however, is the behavior of interest rates. The model features decreasing growth rates, so if we assume CRRA preferences, the model predicts that interest rates fall over time and eventually converge to ρ.

4 A Simple Additive Endogenous Growth Model

TBC

5 Conclusion

I have shown that the data is well described by additive TFP growth. Not surprisingly the implications are important for topics that depend on long term growth. One is the valuation of long term assets, such as stocks or pensions. The model predicts falling growth rates and falling interest rates so valuation effect will depend on preferences. Another is convergence of income across countries, along the lines of Barro (1991), as briefly discussed above.

For models of endogenous growth additive growth speaks to the appropriate functional form of the innovation technology. Bloom et al. (2017) recast the TFP slowdown puzzle in terms of decreasing returns to R&D. If growth is additive it seems natural to think that ideas are also additive, which would solve the slowdown puzzle, but not necessarily the reseach effort puzzle. A more interesting question, however, may be to think about circumstances when knowledge creation is likely to be additive and when it is likely to be geometric.

Additive growth has implications for industry dynamics and structural transformation as in Baumol (1967), and for firms dynamics as in Luttmer (2007) and Gabaix (2011). Philippon (2022b) and Philippon (2022a) provide some early evidence on these issues.

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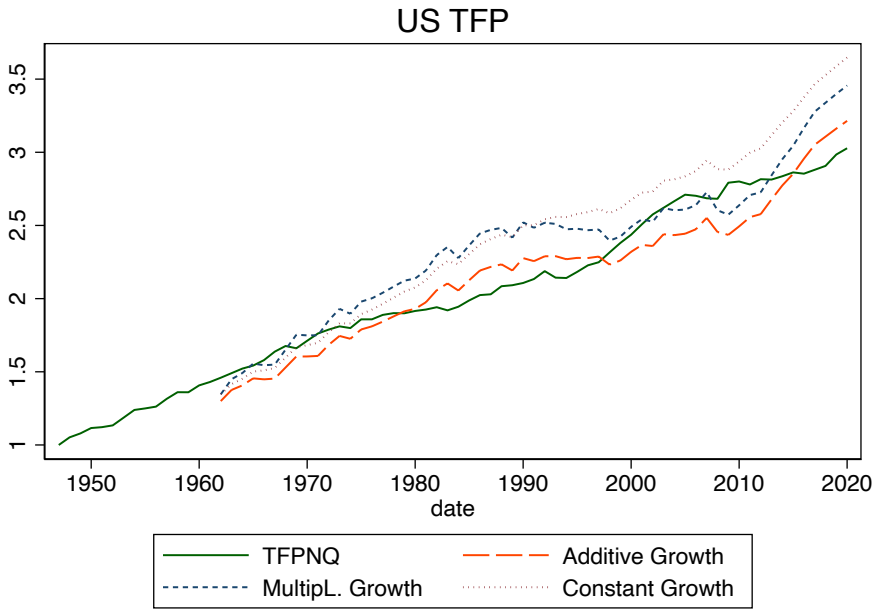
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Figure 9: US Business Sector TFP



Notes: TFP is an index normalized to 1 in 1947. The 10-year ahead forecasts use smoothing parameter 0.1. The constant growth model assumes a constant growth rate of 0.02. Data from Fernald (2012). Sample 1947-2020.

Appendix

A Fernald's Labor Quality Adjusted TFP

Figure9 shows Fernald's Labor Quality Adjusted TFP series together with the 10-year, out-of-sample forecasts of the additive model, the multiplicative model, and, for comparison, a model that assumes a constant growth rate of 2% per annum. The z_t^f series displays a stronger TFP slowdown that the z_t^f and z_t^n series because of the human capital adjustment. Because of this slowdown, the constant growth model makes much larger errors than the models with time varying growth. As with the other series, however, the additive growth model outperforms the multiplicative model, with a RMSE of 5.7% compared to 7.3%, in line with my other results.