Gazing at r^*: A Hysteresis Perspective *

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March 23, 2022

Abstract: Despite current high inflation and a monetary tightening cycle, the market's evaluation of long-term real interest rates remains very low in most advanced countries. This is consistent with the view that neither monetary policy nor inflation shocks — which are both nominal phenomena — are likely to effect long-run real interest rates. This paper presents both theory and evidence that put into question this simple dichotomy between real and nominal phenomena due to asset accumulation behavior that favours the emergence of more than one steady state value of real interest rates (r^*) and thereby creates hysteresis. Our main building block is household saving decisions that incorporate both inter-temporal substitution and retirement forces. When households trade off these two saving motives, we show how this can give rise to C-shaped asset demands and the possibility of more than one steady state equilibrium real interest rate. Since many macroeconomic models predict that long-run asset demands are increasing in interest rates, as opposed to C-shaped, we provide evidence from household balance sheets that runs counter to the former and favours the latter. A central contribution of the paper is to show that when r^* is not unique due to C-shaped asset demands, monetary policy can greatly influence long-run real interest rate outcomes. In particular, we show that an aggressive inflation targeting regime can make a high-real-rate outcome fragile to small negative inflation shocks and favour the convergence to a low (possibly negative) real-rate environment. However, we also show that either a large positive inflation shock or a large increase in public debt can bring back an equilibrium with high real rates, which could surprise the market in the current environment.

*This draft is incomplete. We thank Oleksiy Kryvtsov, Guillaume Rocheteau, Ludwig Straub, and Tim Willems for helpful comments and discussions. Matthew Tang provided outstanding research assistance. We also thank seminar participants at the Bank of Canada and the University of California, at Irvine for comments on a very early version of this paper. Some of the initial ideas in this paper circulated earlier under the title "Monetary Policy, Trends in Real Interest Rates and Depressed Demand ". All remaining errors are those of the authors. The views expressed in this paper are those of the authors and not necessarily of the Bank of Canada.

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1 Introduction

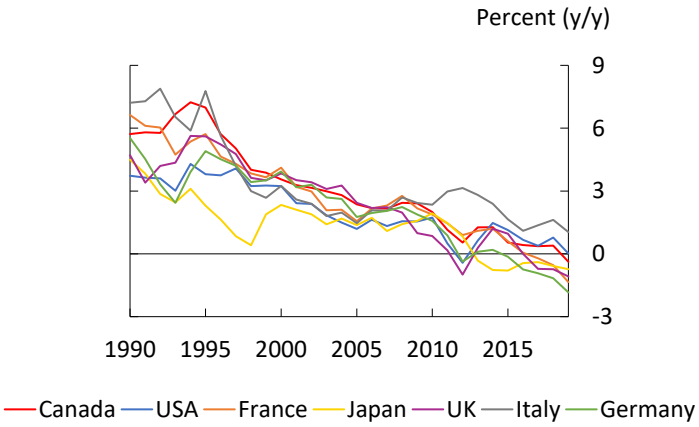
In most advanced economies, real interest rates have trended down over the last few decades (see Figure 1). The most common explanation for this trend is that advanced economies have experienced a secular fall in demand and that policy decisions to decrease interest rates and to increase government debt have been important mitigating factors that have helped offset this fall in demand. The forces cited for inducing such a fall in demand include reduced productivity growth, the aging of the population, and increased inequality.[1] While all these factors may be relevant, this one-way narrative from exogenous reductions in demand to policy response has nonetheless been put into question by many. In particular, several market commentators argue that monetary policy over the period possibly contributed to the long-term downward trend in real interest rates by decreasing interest rates aggressively in every downturn and being hesitant to increase them in upturns. This has also been highlighted by policymakers such as Borio, Disyatat, Juselius, and Rungcharoenkitkul (2017), who provide evidence that over a long history persistent changes in real interest rates coincide with changes in monetary regimes. Recently, Bianchi, Martin, and Ludvigson (2022) estimate that two-thirds of the fall in the real interest rate since the early 1980s may be due to shifts in the parameters of the monetary policy rule. All this points to a possibly underrated role of nominal factor associated with monetary policy in determining real interest rates over long horizons.[2]

The goal of this paper is to help advance the discussion around the potential forces that have weighed down real interest rates over the last few decades and provide new insights on how this may play out going forward. In particular, we aim to highlight how a multiple steady state real interest rate environment can provide an explanation to a set of outcomes,

[1] A vast literature examines the sources of the decreasing trend in real interest rates. Borio, Disyatat, Juselius, and Rungcharoenkitkul (2017) provide an excellent survey of the literature on these issues. Several hypotheses about these sources have been proposed: demographics (Summers (2014), Eggertsson and Mehrotra (2014), and Eichengreen (2015)); a productivity slowdown (Gordon (2017)); a global saving glut and/or lack of safe assets (Bernanke (2005), Caballero, Farhi, and Gourinchas (2008), Gourinchas and Rey (2016), Gourinchas, Rey, and Sauzet (2020), and Acharya and Dogra (2021)); a decline in desired investment (Rachel and Smith (2017)); a rise in inequality (Mian, Straub, and Sufi (2020), Auclert and Rognlie (2020), Fagereng, Blomhoff Holm, Moll, and Natvik (2019), and Rachel and Smith (2017)).

[2] Gourinchas and Rey (2016)'s and Gourinchas, Rey, and Sauzet (2020)'s focus on financial cycles, especially the leveraging cycle that accompanied the boom and bust in the 1930s and 2000s, for explaining the short-term real interest rate movements is consistent with the role of monetary policy. Their explanation centres on the relative demand for safe assets in the aftermath of a deleveraging shock. However, the association between the consumption-to-wealth ratio and subsequent short-term real risk-free interest rates could also be seen as reflecting the central bank's reaction function in boom and bust periods. That is, the abnormally low consumption-to-wealth ratio following financial busts tends to coincide with periods of aggressive monetary policy easing to support the economy, which in turn gives rise to the association with low short-term risk-free rates in the subsequent period.

Figure 1: Long-term interest rates for G7 countries from 1990 to 2019



10-year government bond yields minus 3-year moving average inflation (total CPI)
Sources: Jordà-Schularick-Taylor Macrohistory Database until 2016 and OECD Main Economic Indicators from 2017-2019
Last observation: 2019

including explaining why real interest rates have trended downward since the expansion of inflation targeting regimes. For example, we will show how an aggressive inflation target can create an asymmetry whereby a high-real-rate environment becomes fragile to small negative shocks to inflation and favour movements towards low real rates, while it takes a large positive shock to inflation to move the economy out of the resulting low-real-rate trap.

The dismissal of monetary policy and nominal factors in affecting the long-run real interest rates is most commonly discussed in the confines of models where the real side of the economy admits only one equilibrium long-run real interest rate (i.e., one r^*). In such environments, monetary policy is unlikely to affect long-run real interest rates. In contrast, when we are in an environment where the real side of the economy permits two (or more) steady state real interest rates, we show that nominal factors — and especially monetary policy — have the potential to affect long-run outcomes. This does not work through changing beliefs but reflects the power of monetary policy to influence the stability, emergence and basin of attraction of the different steady state real interest rates.

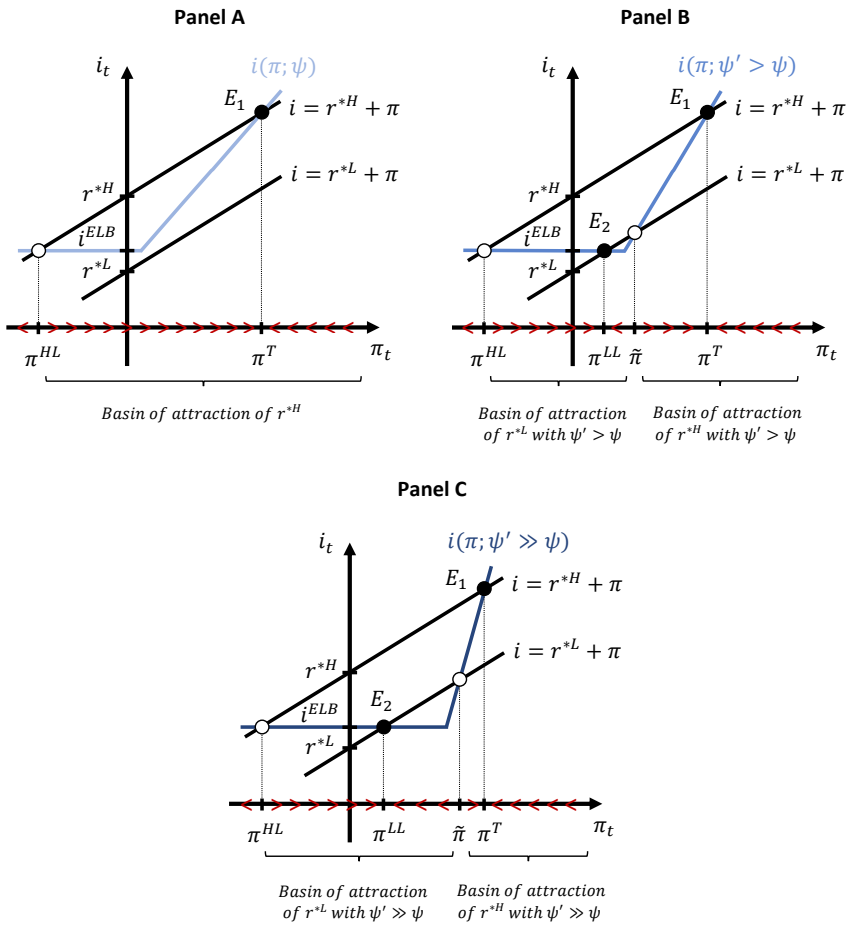
To provide an immediate sense of the argument, we use Figure 2 to depict some of the elements at play. We are condensing substantial information into the three panels of Figure 2 to summarize the mechanisms we will flesh out in the paper. In this figure, we plot two possible combinations of inflation and nominal interest rates in environments

where the real side of the economy allows for two steady state equilibrium real interest rates. These two potential real interest rates (r^{*H} and r^{*L}) are represented in the two steady state Fisher equation curves that have slopes of one. The reasons for these two potential steady state real interest rates will be shown to emerge quite naturally when households' asset accumulation decisions are influenced by both inter-temporal substitution and retirement motives. We then superimpose on this figure Taylor rule specifications for monetary policy, where they all satisfy the Taylor principle with slopes (ψ) greater than one when not constrained by an effective lower bound (ELB). In Panel A, we have the more standard configuration where there are only two possible equilibrium outcomes, both with the same real rate (r^{*H}): one outcome where the nominal interest rate is at the ELB and one where the nominal rate is above the ELB. If we disregard the second real rate locus, this figure is akin to that in Benhabib, Schmitt-Grohé, and Uribe (2002). This particular monetary policy stance implies that the low real interest rate (r^{*L}) is not possible as part of a nominal equilibrium. In contrast, in Panel B we have a more aggressive Taylor rule — as can be seen by it having a steeper slope (captured by a higher ψ) when not constrained by the ELB — and this causes the Taylor rule to support both high and low real interest rate outcomes. In this case, as we will discuss in detail, there will be two *stable* equilibrium outcomes and these are highlighted on the figure: one with a high nominal and real interest rate (point E_1) and one with the nominal rate at the ELB corresponding to a low real rate (point E_2). The two other equilibrium crossings on the figure will be shown to be unstable. On the horizontal axis, we represent the transitional dynamics for inflation in the presence of nominal frictions. In panel B, we see that whether the high-real-rate or the low-real-rate equilibrium emerges depends on where inflation starts. This is expected given the system exhibits *hysteresis* in the presence of two stable steady states.

In Panel C, monetary policy becomes even more aggressive as reflected by an even steeper Taylor rule above the ELB. In this case, the basins of attraction corresponding to the two different real rate equilibrium outcomes react asymmetrically. The basin of attraction of the low-real-rate equilibrium expands, while that of the high-real-rate equilibrium contracts. In addition, the basin of attraction of the high-real-rate equilibrium becomes itself very asymmetric. To the left of the high-inflation, high-real-rate equilibrium, the basin of attraction becomes small, implying that the high-real-rate equilibrium will not be robust to downward inflation shocks. In contrast, the low-real-rate equilibrium remains robust to both upward and downward shocks to inflation, and consequently it would take a large exogenous positive shock to inflation to move the economy from a low-real-rate equilibrium to a high-real-rate equilibrium.

The above discussion was aimed at hinting how the presence of more than one steady

Figure 2: The interaction of a Taylor rule with more than one equilibrium r^*



Note: The blue lines in the figure represent a Taylor rule which is constrained by a lower bound, aims for a target level of inflation π^T, and reacts to inflation as governed by the parameter ψ. The black parallel lines reflect different possible real interest rates.

state equilibrium real interest rate r^* can imply that nominal factors like monetary policy have the scope to affect long-run real outcomes, and especially that aggressive monetary policy could favour the emergence of a low-real-rate trap where a large shock is required to escape it. However, before being able to discuss such a possibility more rigorously, we need to step back and build up the elements. First, we need to ask which features of the environment — in the absence of nominal considerations — may give rise to multiple r^*s and whether there is empirical evidence which provides support for such a possibility. After this first step, we can proceed to explore how such real features interact with monetary policy, fiscal policy and inflation shocks in a New Keynesian set-up with sticky prices or wages.

To understand the type of features that can give rise to multiple steady state real interest rates, properties of savings and asset demand are key. In a standard infinitely lived representative agent model, the long-run asset demand by households is stark. If the real interest rate is below the households' subjective discount rate (adjusted for growth), then households will not want to hold any assets in the long run. If it is slightly above the subjective discount rate, then households want to hold an infinite amount. Such a set-up exemplifies an asset demand function that is increasing in interest rates, albeit in an extreme form. While there are several modifications that can make such a long-run asset demand function less extreme, many commonly used models maintain the property that the long-run asset demand, especially relative to consumption, is monotonically increasing in interest rates. However, there are many reasons to question our reliance on models where asset demand is monotonically increasing in real interest rates. As is well known, there exist income effects associated with interest rates that could reverse this property. For example, when interest rates fall, households may want to hold more assets — not fewer — if at least part of their asset holdings are for retirement motives, bequest motives or precautionary motives. A very simple example of such non-monotonicity is when the demand for assets has a C-shaped relationship with respect to real interest rates. If long-run asset demands are non-monotonic in interest rates, this opens the door to multiple steady states.

In this paper, we present a simple continuous time OLG environment that gives rise to long-run asset demands which are C-shaped in real interest rates. The model builds on the work of Blanchard (1985) and Yaari (1965), and is closest to Gertler (1999), in having stochastic transitions between stages of life. We use this environment to study the potential macroeconomic implications of C-shaped asset demands. In particular, we examine how monetary policy can affect long-run outcomes in an environment in which money is essentially neutral in the long run.[3]

An important element of our analysis is to show how a C-shaped asset demand can interact with a Taylor rule specification of monetary policy when the latter is subject to an ELB constraint. It is the interaction between these two forces that drives several of our results regarding the role of monetary policy. From Benhabib, Schmitt-Grohé, and Uribe (2001), Benhabib, Schmitt-Grohé, and Uribe (2002), and related literature, we know that an ELB constraint can give rise to multiple equilibria.[4] However, most of this literature is

[3]A large literature supports the notion that money is neutral in the long run: see King and Watson (1997) for a survey. However, an emerging literature is questioning such a view. For example, Jordà, Singh, and Taylor (2020) and Willems (2020) provide evidence of the non-neutrality of money over long periods; that is, they show that monetary policy has real effects that last more than a decade.

[4]Expectations-driven liquidity traps have also been applied to fiscal policy, optimal monetary policy and open economy issues. See for example, Mertens and Ravn (2014), Bilbiie (2018), Nakata and Schmidt

not aimed at explaining changes in real interest rates, as the long-run real interest rate in the ELB regime is the same as the one in the non-ELB regime. In contrast, in our set-up, we show that the real interest rate that emerges when the ELB constraint is binding is lower than when it is not binding. Therefore, a shift from a non-ELB-constrained equilibrium toward an ELB-constrained equilibrium is associated with a fall in real interest rates.[5]

Before delving into the above conceptual possibilities, we first present observations which run counter to the notion — inherent to many macroeconomic models — that long-run asset demands are monotonically increasing in real interest rates. To do this, we look into the decomposition of aggregate asset demand and saving rates in the US over the period of falling real rates. Given this focus, we examine changes in households' asset holdings over the period 1989-2019. For multiple real interest rates to be potentially supported as an equilibrium outcome in the long run, it must be the case that — at least over a certain range — households want to increase their demand for assets as the long-run return on assets falls. As is well known, when interest rates decrease, this likely leads to valuation effects that effectively act as a boost to asset supply. Given this, asset demands would need to rise sufficiently in response to a fall in interest rates if there is to be the possibility of an alternative equilibrium at a lower interest rate. Hence, a multiple steady state narrative behind the fall in real interest rates observed over 1989-2019 period requires that people of a given age and given real incomes in 2019 hold much more financial wealth than similar people thirty years ago. Such a *"within" group* narrative around changes in asset demands contrasts with the more common *"between" group (compositional effects)* narrative whereby higher asset demands over this period mainly reflect increases due to aging of the population and/or changes in the income distribution. Hence, the first step of our empirical analysis will be to examine the within and between group decomposition of the rise in asset holdings over the period 1989-2019. This will then be complemented by examining the within group changes in savings rates to see if increases in observed wealth holdings reflect increases in desired wealth holdings or, if alternatively, they reflect over-accumulation due to unexpected valuation effects.

Our main finding from this empirical analysis is that over the 30 year-period where interest rates were falling, within group desired wealth holdings increased substantially. In effect, we find that the increase in the wealth-to-income ratio observed over this period

(2021), Aruoba, Cuba-Borda, and Schorfheide (2018), and Kollmann (2018).

[5] Fernández-Villaverde, Marbet, Nuño, and Rachedi (2021) also consider how monetary policy can affect the long-run level of real interest rates. Specifically, they show, in a quantitative HANK model with an ELB constraint, that the interaction between the inflation target and wealth inequality is an important determinant for the level of real interest rates. However, their approach does not explore how monetary policy can affect the set of steady states and their basins of attraction as we do here.

is predominantly a within group phenomenon as opposed to resulting from changes in demographics or income distribution, and furthermore we show that the saving behavior supports the interpretation of the observed higher wealth holdings as reflecting desired increases as opposed to temporarily above target levels.

The remainder of the paper is organized as follows. Section 2 exploits household level data to provide evidence against the property – as implied by many macroeconomic models– that long-run asset demands are monotonically increasing in interest rates. Instead, we show that these data point toward either C-shaped asset demands or asset demands that are decreasing in interest rates. Section 3 presents an OLG model — similar in spirit to that of Gertler (1999) — that integrates both inter-temporal substitution forces and retirement preoccupations in wealth accumulation. These two forces are shown to interact in a manner that gives rise to C-shaped asset demands when the inter-temporal elasticity of substitution is not too high. Section 4 embeds such C-shaped asset demand in a general-equilibrium setting. The section begins with an environment without nominal rigidities to show how and when the real side of this economy generates more than one steady state real interest rate. Then the section introduces sticky wages/prices, with monetary policy following a Taylor rule subject to an ELB constraint. Here, we show how monetary policy can affect the emergence, stability and basin of attraction of different long-run real interest rate outcomes. Section 5 enriches the environment by including a claim on a productive asset — where the price of the asset increases when interest rates decrease — in order to examine the implications of asset valuation effects. The main additional feature that arises with the presence of a productive asset is that a low-real-rate trap does not necessarily happen only at the ELB. It can also arise with nominal interest rates above the ELB and with inflation close to target. Hence, this set-up offers an explanation for why economies can get stuck with low-real-interest rates at either the ELB or above the ELB, with a high valuation of productive assets. Section 6 returns to a more general formulation of the model for robustness analysis. Section 7 concludes.

2 The Between versus Within Household Decomposition of Aggregate Asset Holdings over 30 years: 1989-2019

While real interest rates were declining over the last several decades (as seen in Figure 1), Figure 3 indicates that the aggregate wealth-to-income ratio in the US increased significantly and the aggregate saving rate mildly decreased. The question we want to address is how best to interpret such observations; should they mainly be interpreted as reflecting between group (composition) effects or do they instead largely reflect within group choices. In

particular, we want to ask if measured within group changes in asset holdings over the period appear consistent with asset demands that are monotonically increasing in real interest rates or if they place in doubt such monotonicity and open the door to multiple steady state equilibrium interest rates.[6]

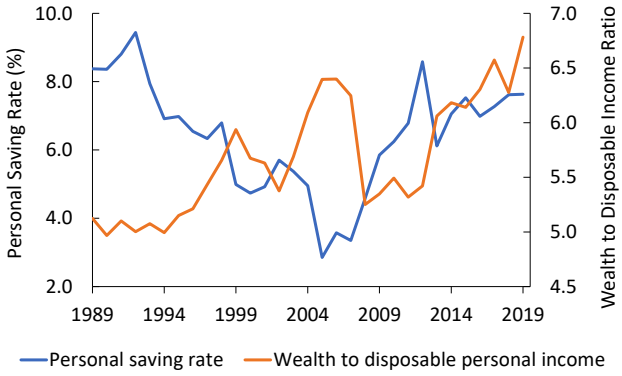
Between group (compositional effect) explanation. A common explanation to the increase in aggregate wealth-to-income ratio is that it reflects increase in demand for assets induced by changes in demographics and income distribution. As the population aged, and more income was concentrated in higher income groups, the demand for wealth increased. This put downward pressure on interest rates, which through valuation effects among others, raised the effective supply of wealth. The higher savings of the older and richer population was compensated by a decreased incentive to save by the population at large due to lower interest rates, leaving the overall savings rate relatively flat. Such narrative is essentially a "between" group narrative, that is, it is a narrative that relies on compositional changes in types of individuals to explain the increased demand for wealth. In particular, it suggests that, for similar age and income levels, as interest rates fell, households may have saved less and accumulated less wealth, but due to the changes in the age and income distributions of the population, the aggregates behaved very differently from individual level outcomes.[7]

Within group explanation. At the other end of the spectrum, a multiple steady state equilibrium story suggests that the joint pattern of increased aggregate wealth, lower interest rates and decreasing saving rates potentially reflects a "within" group phenomenon. In this alternative view, we still have that as interest rates decrease, the effective supply of wealth increases through valuation effects. However, now the endogenous increase in supply does not need to be primarily driven by an exogenous increase in demand. Instead, it can be accompanied by a simultaneous endogenous response of demand whereby households choose to hold more wealth at lower interest rates due to income effects. Note that such a multiple steady state equilibrium story would not negate the possibility of between group effects arising from demographics and inequality, but it does not rely on them. In fact, a change in demographics or income inequality could complement this type of multiple steady state equilibrium story by helping to explain why the economy may have switched from a

[6]The analysis of this section will not in itself provide any direct evidence of multiple steady state equilibria. Instead it is aimed at shedding light on whether long-run asset demands are likely monotonically increasing in real interest rates — which would make multiple steady states very unlikely — or whether they point to an alternative configuration where asset demands are at least decreasing in interest rates over a range — which opens the door to multiple steady state equilibria. The later case is a necessary condition for the source of multiple steady state equilibria we will analyze in theory section.

[7]In this type of scenario, the between group effect should actually explain more than 100 percent of the increase in asset demand as the within component should be negative.

Figure 3: Household saving rates and aggregate wealth-to-income ratios in the US from 1989 to 2019



Sources: U.S. BEA for Personal Saving Rate and Disposable Personal Income; Board of Governors of the Federal Reserve System for Wealth. Retrieved from FRED.
Last Observation: 2019

high real interest rate to a low real interest rate at this time in history.

The above discussion underlines the relevance of understanding the relative roles of within versus between group effects in explaining the increased wealth holdings in the US over the last three decades. To do so, we use the Survey of Consumer Finances (SCF) and focus on the difference in asset holdings across cohorts between 1989 and 2019. We choose this period for our analysis as it corresponds quite closely to the period of decreasing real interest rates presented in Figure 1.[8] Furthermore, by looking at this thirty-year difference, we hope to minimize higher frequency movements in wealth accumulation dynamics associated with business cycles forces and crises.

The SCF is the most comprehensive source of data on household-level wealth and its components in the United States. It also has a consistent sampling methodology, over-sampling the rich, in all the survey waves between 1989 and 2019, which is useful for our analysis. The survey has between 3 and 5.5 thousand households, and we use weighted estimates in the analysis. In addition, we supplement the SCF with the estimates on defined benefit pensions of households from Sabelhaus and Volz (2020). In this section, we use estimates of wealth-to-income ratio as computed in the SCF, which show similar upward movement in the dynamics of household wealth-to-income ratio, but slightly overstate its magnitude relative to the aggregate data. We will also show that our findings are robust

[8]The exact years chosen reflect the availability of consistent SCF data.

to rescaling SCF estimates of wealth and income to match aggregates reported in the Financial Flow Accounts and National Income and Product Accounts. Since this is the case, we choose to focus on the results using raw SCF data, and use adjusted data later in the paper.

The aggregate wealth-to-income ratios in 1989 and 2019 we use for our decompositions are calculated from the SCF as the ratio of the sum of the wealth of each household to the sum of incomes of each household, respectively denoted $\left(\frac{w}{y}\right)_{89}$ and $\left(\frac{w}{y}\right)_{19}$. In our baseline, we include all household wealth reported in the SCF in our measure of wealth. To explore robustness, we also provide calculations when we exclude wealth in a primary residence and/or include wealth in defined benefit pension schemes. The aggregate wealth-to-income ratio in the SCF increased from 4.61 in 1989 to 7.03 in 2019, which is an increase of about 2.42. This is the increase associated with an inclusive wealth measure from the SCF. When we exclude net housing wealth from this measure, the increase in the ratio is of similar magnitude at 2.24. When accounting for wealth in defined benefit pension plans, the wealth-to-income ratios increase by respectively, 2.62 and 2.45 depending on whether housing wealth is included. These increases are all substantial as they correspond to more than 50 percent of 1989 levels.

To examine the within versus between components of increased wealth holdings, we pursue two complementary approaches.

Approach 1: shift share analysis. The first approach is a simple shift share analysis. We place households in I bins, with N_i households in a bin $i = 1, ..., I$. The change in the aggregate wealth-to-income ratio can be decomposed as follows:

$$\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89} = \underbrace{\sum_i \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89} \left[\left(\frac{y_i}{y}\right)_{19} - \left(\frac{y_i}{y}\right)_{89}\right]}_{\text{between group or compositional effect}} + \underbrace{\sum_i \left(\frac{y_i}{y}\right)_{19} \left[\left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{19} - \left(\frac{\bar{w}_i}{\bar{y}_i}\right)_{89}\right]}_{\text{within group}}, \quad (1)$$

where the first summation term represents the between group component and the second represents the within group component.[9] In this expression, y_i is the total income in bin i, $\bar{y}_i$ is the average income in bin i, $\bar{w}_i$ is the average wealth in bin i and finally y is the total income across all bins. All nominal variables are converted into real variables indexed in 2019 dollars. As can be seen from Equation (1), the changes in the total wealth-to-

[9] In the appendix discussing the regression approach, we clarify the close relationship between this shift-share approach and a regression approach. This discussion helps to highlight under what conditions the first component represents the between component and the second component represents a within component.

income ratio can be divided into the between group component determined by the shift in the share of income going to each of the individual groups (y_i/y) and the within group component determined by changes in the (average) wealth-to-income ratio of each group ($\bar{w}_i/\bar{y}_i$). If the wealth-to-income ratio of individual groups were stable across time (e.g., $(\bar{w}_i/\bar{y}_i)_{19} = (\bar{w}_i/\bar{y}_i)_{89}$ for all groups i), the change in the aggregate wealth-to-income ratio would need to be fully explained by the between group component (i.e., by the change in income share alone). However, at the other extreme, if the income and age distributions remained stable across time (e.g., if $(y_i/y)_{19} = (y_i/y)_{89}$ for all groups i), then the within group components would need to account for all the change in the aggregate wealth-to-income ratio.

We start by dividing the population households into age groups, defined by the age of the household, to look narrowly at the effects of demographic changes in isolation. Then, we divide the population of households into income groups to examine only the effects of changes in the income distribution. Finally, in our preferred specification, we combine the two and place households into 30 age-income specific bins. The results of the shift share analysis for these groups are presented in Tables 1 and 2.

Table 1: Shift Share Decomposition of the Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019

Groups	Total Change	Between	Within	Fraction due to Within (%)
		Age Groups		
5 age groups	2.415	0.722	1.693	70.1
12 age groups	2.415	0.750	1.665	68.9
		Income Groups		
6 income groups	2.415	0.221	2.194	90.9
12 income groups	2.415	0.270	2.145	88.8

Note: The 5 age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the 12 age groups are: <25, 25-29, 30-34, 35-39, 40-44, 45-49, 50-54, 55-59, 60-64, 65-69, 70-74, and 75+. The 6 income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+ and the 12 income groups: 0-20, 20-40, 40-60, 60-80, 80-120, 120-160, 200-250, 250-500, 500-750, 750-1250, 1250+.

In Table 1, we report results for the more narrow focus on either only age or income groups. For this table, we use the inclusive measure of wealth from the SCF. With respect

to the results based on demographics the table presents two breakdowns: one based on 5 age groups and one based on 12 age groups. For these two breakdowns, we get very similar results: the within component explains about 70 percent of the change in the wealth-to-income ratio. Then, we look at only two income groupings: one based on 6 groups and one based on 12 groups. In this case, the between component only explains 9-11 percent of the change, leaving the within component explaining between 89 and 91 percent of the change.

In Tables 2 and 3, we present our preferred specification where we allow for 30 groups as the product of 5 age groups and 6 income groups. In these tables, we report results using four different measures for wealth: our baseline inclusive measure from SCF, the baseline measure less net housing wealth (primary residence), the baseline measure plus the implied value of defined benefit pension schemes and finally this later measure minus net housing wealth. In all four of these cases in Table 2, we find that the within component — that is, the component associated with changes in the wealth-to-income ratio of different groups — accounts for more than 60 percent of the change with the between component explaining around 40 percent.[10]

Table 2: Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Shift Share Decomposition

Definition	Total Change	Within (%)	Between (%)
Wealth (Baseline)	2.415	61.5	38.5
Wealth less housing	2.244	61.2	38.8
Wealth plus DB	2.618	61.5	38.5
Wealth plus DB less housing	2.447	61.3	38.7

Note: DB refers to the value of defined benefit pension schemes. The decomposition is done for 30 groups which are the product of 5 age groups and 6 income groups. The age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+.

Approach 2: regression. In the second approach, we use the 1989 cross section to estimate a wealth holding function, which we denote by $F_{89}(age, y)$, where *age* represents

[10] Using the SCF data scaled to the aggregates and 30 groups we find the share of the within component to be between 40 and 50%.

Table 3: Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Decomposition Based on Regression

Definition	Total Change	Within (%)	Between (%)
Wealth (Baseline)	2.415	64.0	36.0
Wealth less housing	2.244	59.0	41.0
Wealth plus DB	2.618	67.4	32.6
Wealth plus DB less housing	2.447	63.1	36.9

Note: DB refers to the value of defined benefit pension schemes. The decomposition is done for 30 groups which are the product of 5 age groups and 6 income groups. The age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the income groups (in thousands of real 2019 dollars) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+.

the age of the household head and y represents real income of a household. Function F can take different forms. In this section, we focus on the polynomial function F in income and age.[11] Then, for each household in the 2019 cross section, we use estimated function $F_{89}(age, y)$ to create a predicted wealth holding, which we denote by $\hat{w}_{19}$. These predicted wealth levels allow us to create a predicted wealth-to-income ratio in 2019 by adding up $\hat{w}_{19}$ across households, and by dividing it by the aggregate income in 2019 (denoted $\left(\frac{\hat{w}}{y}\right)_{19}$). By using the same prediction function for the wealth in 2019, as in 1989, the predicted ratio reflects only the changes in the proportions of different groups in the population. Accordingly, the fraction of the change in the wealth-to-income ratio explained by the within component can be expressed as

$$1 - \left[\frac{\left(\frac{\hat{w}}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89}}{\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89}}\right]. \tag{2}$$

In Table 3, we also report the results of this exercise using our four measures of wealth. Using a third order polynomial in income and age (including cross terms) to build predicted

[11] We have run our predictive regressions using polynomials of order 3, 4, and 5 with and without cross-terms. Polynomial function of order 3 with cross-terms delivers the best prediction. In Appendix B we show that these results are also similar to using a regression with a set of dummy variables for income and age groups, which we refer to as a step-function approach.

wealth, we find that the between component accounts for between 32 percent and 39 percent of the change in the aggregate wealth-to-income ratio, leaving the within component again accounting for over 60 percent of the rise. While these findings still support an important role of changes in demographics and income inequality in explaining movements in the wealth-to-income ratio, they indicate that an even greater share is due to changes in wealth holdings keeping income and age constant.

It must be immediately noted that these decomposition results — by themselves — do not imply that within group *desired* wealth holdings have necessarily gone up. Instead, if households' wealth holdings are sticky, it could be that these high levels of within-group's increases in wealth holdings simply reflect the fact that falls in real interest rates have led to increased valuation of wealth, and that households in 2019 are holding much more wealth than they desire relative to similar households in 1989. This could be the case if people have not had enough time to readjust their portfolios. This is especially likely for housing, which is why we reported results excluding housing. As we saw, results are not driven by housing wealth. Nonetheless, to explore this possibility more thoroughly, we need to examine the changes in saving rates by age-income groups.

2.1 Within-group saving behavior: Are households in 2019 trying to shed their increased wealth?

In the previous section we documented that a large share of the increase in the aggregate wealth-to-income ratio in the US over the period 1989-2019 is accounted for by increases in wealth for given levels of age and income, that is, it is predominantly a within group phenomenon. There are at least two potential interpretations of such an observation. On the one hand, increases in wealth-to-income ratio could reflect increases in desired wealth holdings due to low expected returns on assets. On the other hand, such increases in wealth could reflect unanticipated valuation effects, where the observed higher wealth holdings are not representing higher desired wealth holdings, but instead are reflecting wealth holdings that are above their desired levels. To help discriminate between these two possibilities, in this section we look at the changes in within group saving patterns over the same period. In particular, if the observed within group increases in wealth-to-income ratios reflect wealth levels in 2019 that are above desired levels, then we should see household groups with large increases in wealth wanting to spend more and save less to get their wealth back down to its target level. Accordingly, we should see them decreasing their savings rates. Hence, the absence of a negative relationship between increased wealth and savings rates would indicate that the extra wealth holdings are likely desired not excessive.

In line with the previous section, we focus on within group changes in saving rates for the 30 groups we used for our analysis of changes in wealth-to-income ratios. We measure saving in the SCF using the wealth-based synthetic saving approach, widely adopted in the literature, which approximates saving by each group by netting out valuation effects from their changes in wealth between two SCF waves.[12]

To construct changes in wealth, we, first, follow the literature by scaling the SCF aggregates[13] for different asset classes using the factors that allow them to be matched to corresponding aggregates from the Financial Flow Accounts for the US.[14] For the purposes of constructing saving out of income we also scale SCF income to match the aggregates from the National Income and Product Accounts. We then distribute these scaled wealth and income values between our 30 groups using their weights in the SCF. For valuation effects we apply asset inflation factors from Mian, Straub, and Sufi (2020), which are aggregate in nature, and use their methodology to extend them to 2019 (in particular, for the residual equity inflation). Appendix A provides further details of the saving rate construction.

Table 4 reports changes in saving rates[15], changes in wealth-to-income ratios and changes in log wealth[16] for our 30 groups. In the table we flag with a "*" the groups that had an increase in their wealth levels over the period. Looking first at averages, we can see that within group saving rates slightly declined over the period. However, this slight decrease is not driven by groups that experienced an increase in their wealth-to-income ratios. If we look only at the groups that increased their wealth, the average change in saving rates is slightly positive.

If we turn to correlations, we see from the table that there is a slight positive correlation

[12]Thus, our saving rates are calculated over a three-year window. Saving rates for 1989-92 and 2016-2019 periods, respectively, correspond to the start and the end of our 30-year period used to analyze changes in the aggregate wealth-to-income ratio in the US. In our robustness exercises using the SCF, we additionally exclude net inheritances from changes in wealth. We also consider an income minus consumption approach to saving, followed by Mian, Straub, and Sufi (2021b).

[13]We also used the raw SCF data for calculating savings rates and especially correlations between wealth changes and saving rates. The correlation patterns, which represent the main results of this section, are similar whether or not we scale the data. For example, using raw SCF data to compute the correlation between group changes in wealth-to-income ratios and saving rates, results in a coefficient that is similarly close to 0, at -0.06, as that reported later in this section, using the scaled data. However, given that the saving rates constructed using raw SCF data were low relative to aggregate measures of saving rates in FFA/NIPA, in the main text we are using results based on the scaled SCF measures of saving rates.

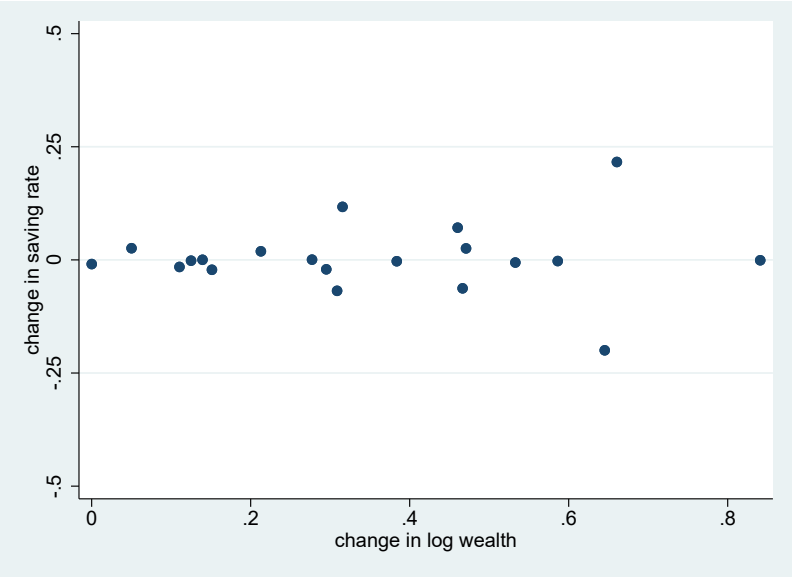
[14]This literature, for example, includes Feiveson and Sabelhaus (2019), Mian, Straub, and Sufi (2020), and Bauluz and Meyer (2019).

[15]For our baseline results we measure saving rates as a fraction of income, but using saving rates as a fraction of wealth gives very similar results.

[16]The change in log wealth is only calculated for the groups that had an increase in wealth.

between changes in saving rates and within group changes in wealth-to-income ratios. This correlation suggests that groups that experienced greater increases in wealth-to-income ratios do not appear to be systematically trying to reverse such accumulation by decreasing their rates of saving relative to groups that did not experience large increases. In Figure 4, we complement this correlation by plotting the changes in saving rates against the changes in log wealth for all the groups that experienced increases in wealth. As noted previously, the average change in savings rates for this subset is slightly positive. Moreover, as can be seen in the figure (and is confirmed by the correlation), higher increases in wealth are not on average associated with larger decreases in saving rates. It must be recognized that our measure of saving rates, which is common to the literature, is quite noisy. Accordingly, we witness substantial variation in saving rates. Nonetheless, we view these patterns as providing support to the notion that increases in within group wealth-to-income ratios documented in the previous section are more likely reflecting changes in desired wealth holdings as opposed to reflecting wealth holdings that exceed desired levels.[17]

Figure 4: Change in saving rates vs. change in log wealth for age-income groups with wealth increases between 1989 and 2019



Sources: Survey of Consumer Finances values scaled using aggregates from the Financial Flow Accounts and National Income and Product Accounts.

[17] In Appendix C we also repeat this analysis using scaled SCF aggregates for the construction of changes in both saving rates and wealth-to-income ratios. The results remain unchanged.

Table 4: Dynamics of Saving and Wealth-to-income ratios in 30 Age-income Groups

Group	Change in saving rate, pp	Change in wealth-to-income ratio	Change in log wealth
		Income 0-20K	
age 18-34	0.53	-1.27	
age 35-44*	-0.08	9.72	0.84
age 45-54*	0.03	0.31	0.14
age 55-64*	-0.03	11.30	0.59
age 65+*	-0.06	12.36	0.53
		Income 20-40K	
age 18-34	1.3	-1.12	
age 35-44	-1.57	-1.22	
age 45-54*	1.90	-1.18	0.21
age 55-64*	-2.08	-1.78	0.30
age 65+*	-1.56	0.08	0.11
		Income 40-60K	
age 18-34	-3.5	-0.59	
age 35-44*	-0.92	-0.12	0.00
age 45-54	-1.72	-1.74	
age 55-64*	2.58	-1.42	0.05
age 65+*	-6.83	-0.69	0.31
		Income 60-80K	
age 18-34	2.63	-1.17	
age 35-44	1.13	-0.88	
age 45-54*	-2.18	-0.23	0.15
age 55-64*	0.06	0.83	0.28
age 65+*	-6.28	0.67	0.47
		Income 80-120K	
age 18-34	-1.33	-0.50	
age 35-44	2.86	-0.10	
age 45-54	-8.55	-0.40	
age 55-64*	-0.3	0.92	0.38
age 65+*	2.54	-2.03	0.47
		Income 120K+	
age 18-34*	-0.17	-0.20	0.13
age 35-44*	11.73	1.39	0.32
age 45-54*	7.13	1.35	0.46
age 55-64*	-19.98	3.11	0.65
age 65+*	21.64	4.84	0.66
	1989-1992	2016-2019	
Aggregate s/y, annualized %	10.4	9.9	
	1989-2019		
Corr(s/y, w/y)	0.07		

Note: For each income-age group, the first column reports the changes in the three-year saving (%) over 1989-1992 and 2016-2019 periods normalized by aggregate incomes in 1989 and 2016, respectively. The second column reports the change in the wealth-to-income ratio between 1989 and 2019 (reported in Section 2). The third column reports the change in the log level of wealth for the groups, which have seen an increase in wealth over the 1989-2019 period. "*" refers to groups that experience increases in the level of wealth.

3 Model

The asset holding pattern presented in the previous section suggests that the quantity of asset holdings desired by households may be increasing with low frequency decreases in interest rates, at least over certain ranges. As indicated previously, such a property is not theoretically problematic as interest rate changes can have both income and substitution effects on desired wealth holdings. The question for us is how best to explain such an observation and what does it imply about the potential role of monetary policy in affecting long-run outcomes. The workhorse infinitely lived agent model is not a good starting point for asking these questions as it is not consistent with desired wealth holdings decreasing with higher interest rates. In contrast, the positive effect of lower interest rates on asset demands can in principle be easily captured in an OLG type framework. However, the perpetual youth OLG model of Blanchard (1985) and Yaari (1965), by omitting retirement savings needs, downplays precisely the potential income effects of interest rates which could help explain the pattern of interest to us. For these reasons, in this section we build on a model similar to that of Gertler (1999) that integrates both inter-temporal substitution forces and retirement preoccupations in wealth accumulation.[18] In particular, these two forces will be shown to interact in a manner that gives rise to C-shaped wealth demands where desired wealth holdings increase when long term interest rate decrease at low levels. In the subsequent section, we will embed this household decision model into a general equilibrium set-up to show how it can lead to multiple steady state real interest rates (r^*) and how it creates a role for monetary policy to influence long run real interest rates. In particular, the model will highlight why aggressive inflation targeting monetary policy may have contributed to the fall in real interest rates over the last thirty years. It is worth noting that the mechanisms we will highlight are not driven by monetary policy simply affecting beliefs in a multiple equilibrium setting but are instead associated with monetary policy affecting the inherent dynamics of a system with multiple steady states.

3.1 The household's decision problem with both inter-temporal substitution and retirement motives

When thinking about consumption and wealth accumulation decisions, it is common to think about people in different states. As is standard in simple OLG models, we can think of a household in one of three states: an active work state, a retirement state and a death

[18] We depart from Gertler (1999) by maintaining the more common CRRA utility specification instead of adopting RINSE preferences.

state. Following Blanchard (1985), Yaari (1965) and Gertler (1999) we want to think of these states as evolving stochastically. To be more precise, let us assume that a person starts life in a work state and transits out with instantaneous probability δ_1. In the absence of fixed retirement dates, this shock can be thought as a health shock. At this transition, with probability q, the person retires and with probability $(1-q)$, the health shock is severe, and the person dies. If the person retires, the person will die with instantaneous probability $\delta_2 \geq \delta_1$. If we denote the expected discounted utility of entering the retirement state at time t by V_t, we can express the utility of an active household, that is a household in the work state, as:

$$\int_0^\infty e^{-(\delta_1+\rho)t}\left[\frac{c_t^{1-\sigma_1}}{1-\sigma_1}+\delta_1 q V_t\right]dt, \qquad \sigma_1 > 0$$

where c_t is consumption, ρ is the subjective discount rate and $\sigma_1 > 0$ is the inverse of the elasticity of substitution $(1/\sigma_1)$, or alternatively the risk aversion parameter.

A retiree's decision problem. For the household in the retirement state, the preferences are given by:

$$\int_0^\infty e^{-(\delta_2+\rho)\tau}\frac{c_\tau^{1-\sigma_2}}{1-\sigma_2}d\tau, \qquad \sigma_2 \geq \sigma_1$$

We are allowing the parameters governing inter-temporal substitution, σ_1 and σ_2, to differ between the two states of life to illustrate important forces at play. Later, we will restrict attention to the more standard case where $\sigma_1 = \sigma_2$.

The budget constraint facing the retired household is given by:

$$\dot{a}_t = a_t r_t - c_t,$$

where a_t is the asset holding of a retired person at time t and r_t is the return on the asset a. As can be seen from the budget constraint of the retirees, moving into the retirement state is associated with the absence of labor income implying that households must rely only on asset income for consumption. The need to rely on asset income in retirement will play an important role in our results. Given this structure, the discounted expected utility of a household who retires at time t_1, V_{t_1}, can be solved explicitly and expressed as[19]

[19]The expected utility associated with the retirement state is found by first solving for the optimal consumption path, which is governed by the Euler equation $\frac{\dot{c}_t}{c_t} = \frac{r_t-\rho-\delta_2}{\sigma_2}$ and then integrating the implied utility flow over the expected duration of retirement.

$$V_{t_1} = \frac{a_{t_1}^{1-\sigma_2}}{1-\sigma_2}\left[\int_{t_1}^{\infty} e^{-\int_{t_1}^{t}\frac{1}{\sigma_2}[(\rho+\delta_2)-(1-\sigma_2)r(\tau)]d\tau}dt\right]^{\sigma_2},$$

where a_{t_1} is the level of assets held by the household at time of retirement. For convenience, we will also express V_{t_1} as

$$V_{t_1} = V(a_{t_1}, \Gamma_{t_1}) = \frac{a_{t_1}^{1-\sigma_2}}{1-\sigma_2}\left[\Gamma_{t_1}\right]^{\sigma_2},$$

where

$$\Gamma_{t_1} = \int_{t_1}^{\infty} e^{-\int_{t_1}^{t}\frac{1}{\sigma_2}[(\rho+\delta_2)-(1-\sigma_2)r(\tau)]d\tau}dt,$$

with Γ_{t_1} being a function of the whole future path of returns $\{r_t\}_{t_1}^{\infty}$. Expressing utility as $V(a_{t_1}, \Gamma_{t_1}) = \frac{a_{t_1}^{1-\sigma_2}}{1-\sigma_2}\left[\Gamma_{t_1}\right]^{\sigma_2}$ makes clear that the utility of someone who retires at time t_1 depends on both the asset at the time of retiring and the entire path of asset returns over the retirement period. As we shall see, the degree of inter-temporal substitution $\frac{1}{\sigma_2}$ will play an important role in controlling how asset returns affect marginal value of assets.

For future reference, it is useful to note that Γ_{t_1} obeys the following differential equation

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r_t\right]. \tag{3}$$

To see most easily how asset returns affect retirement utility, note that if the return on asset a is constant, $r_t = r$, then V_{t_1} can be expressed as

$$V_{t_1} = \frac{a_{t_1}^{1-\sigma_2}}{1-\sigma_2}\left[\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r\right]^{-\sigma_2}$$

Here we see that higher r increases utility in both the case where $\sigma_2 < 1$ or when $\sigma_2 > 1$, that is, retired individuals always like higher interest rates as this gives them a superior income stream . However, what will play an important role in our analysis is how higher r affects the marginal value of a_{t_1} to a retiree. This is given by the following key lemma

Lemma 1. *At fixed r, the marginal value of assets to a retiree, as given by $\frac{\partial^2 V_{t_1}}{\partial a_{t_1}\partial r} = a_{t_1}^{-\sigma_2}(1-\sigma_2)\left[\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r\right]^{-\sigma_2-1}$, is decreasing in interest rates when $\sigma_2 > 1$ and is increasing in interest rates when $\sigma_2 < 1$.*

In general, the effect of asset returns on the marginal value of assets for retirees depends

on σ_2. As noted in Lemma 1,[20] this marginal value is decreasing in r when $\sigma_2 > 1$. In other words, when a retiree has limited opportunities to inter-temporally substitute consumption across time, the retiree will view assets at time of retirement to have a greater marginal value when interest rates are low than when they are high. This property will influence the wealth accumulation behaviour of non-retirees as will be the focus below.[21] It is worth noting that, although we have not allowed for an annuity market for the effect of uncertainty about time of death, Lemma 1 is not dependent on the presence or not of such a market. The content of Lemma 1 would remain identical if we were to allow for an annuity market similar to that in Blanchard (1985).[22]

An active household's decision problem. Let us now turn to the decision problem of an active household. Its decision problem will incorporate the continuation value of assets in retirement and can be written as:

$$\int_0^\infty e^{-(\delta_1+\rho)t}\left[\frac{c_t^{1-\sigma_1}}{1-\sigma_1}+\delta_1 qV(a_t,\Gamma_t)\right]dt,$$

subject to

$$\dot{a}_t = y_t - c_t \tag{4}$$

with $y_t = w_t + r_t a_t - T_t$, where y_t is disposable income, w_t is labor income and T_t are taxes.

The consumption Euler equation for the active household becomes

[20] Lemma 1 can be trivially extended to include the case of log preferences. In this case, the marginal value of assets is independent of interest rates, i.e., $\frac{\partial^2 V_{t_1}}{\partial a_{t_1}\partial r}=0$.

[21] When $\sigma_2 > 1$, an increase in interest rates will cause the optimal path of post-retirement consumption to be higher at all dates and therefore the marginal value of assets is lower. This is easily understood and intuitive. In contrast, when $\sigma_2 < 1$ different interest rates will cause optimal paths of post-retirement consumption to cross; with the retiree consuming initially less in a higher interest rates environment but having his/her consumption decline more slowly over time. Because of this crossing property, the effect of interest rates on the marginal value of assets is not straightforward when $\sigma < 1$. Lemma 1 indicates that the net effect is that higher interest rates increase the marginal value of assets when $\sigma_2 < 1$ due to this crossing feature.

[22] Like Gertler (1999), a key assumption is the absence of a pension system which acts as a perfect insurance market against loss of labor income. The absence of such market implies consumption in retirement depends on the accumulated savings when active.

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma_1} + \frac{c_t^{\sigma_1}}{\sigma_1}\delta_1 q V_a(a_t, \Gamma_t) \tag{5}$$

Relative to a standard infinitely lived agent Euler equation, this Euler equation incorporates forces associated with both inter-temporal substitution and retirement preoccupations as in Gertler (1999). The first term in this Euler equation maintains the standard substitution effect of interest rates on consumption. However, this effect now relates to short-term interest rates movements holding the future path of interest rates constant. When both short-term and long-run interest rates move together the net effect is more involved. The additional term in the Euler equation — $\frac{c_t^{\sigma_1}}{\sigma_1}\delta_1 q V_a(a_t, \Gamma_t)$ — represents the incentive to save due to retirement motives and this is affected by future interest rates. Given this term is always positive, it implies that retirement adds a force towards postponing consumption and favouring asset accumulation.[23] The key element for us, and which will allow us to capture features of the data, is that the retirement incentive to save is affected by long run returns to savings. In particular, when interest rates are constant, $r_t = r$, we have seen that $V_{a,r}(a_t, r) < 0$ when $\sigma_2 > 1$. Hence, interest rates have two opposing effects in our set-up when $\sigma_2 > 1$. Low interest rates will favour higher consumption today due to inter-temporal substitution forces, while at the same time, low interest rates are an incentive for greater retirement savings if the low interest rates are viewed as persistent.

To help further highlight implications of this Euler equation, it is helpful to examine the implied long-run asset holdings of the active household when the return of asset a is constant and therefore $\Gamma_t = [\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r]^{-1}$. We will denote an active household's steady state asset holding function by $a^{a,ss}(y, r)$. Proposition 1 indicates that $a^{a,ss}(y, r)$ is attractive and describes the key properties of the function $a^{a,ss}(y, r)$.

Proposition 1. *The asset holdings of active households will converge to* $a^{a,ss}(y, r)$[24] *given by*

$$a^{a,ss}(y, r) = (\delta_1 q)^{\frac{1}{\sigma_2}} \left[\frac{\rho + \delta_2}{\sigma_2} - \frac{1 - \sigma_2}{\sigma_2} r\right]^{-1} [\rho + \delta_1 - r]^{\frac{-1}{\sigma_2}} y^{\frac{\sigma_1}{\sigma_2}}, \tag{6}$$

with r *in the interval defined by* $[\frac{\rho+\delta_1-r}{\delta_1 q}(\frac{\rho+\delta_2}{\sigma_2} - \frac{(1-\sigma_2)r}{\sigma_2})^{\sigma_2}]^{\frac{1}{\sigma_1}} > \max[0, r]$.

[23] This force is also present in models with warm-glow bequest motives, but in that case it does not depend on interest rates, which is the key feature for our purposes.

[24] In the case when $\sigma_1 = \sigma_2$, this equation implicitly defines the asset-to-income ratio as a function of interest rates.

The long-run asset holdings of active households $a^{a,ss}(y,r)$ are increasing in income. Moreover, if $\sigma_2 \leq 1$, then $a^{a,ss}(y,r)$ are monotonically increasing in asset return r, while if $\sigma_2 > 1$, they are C-shaped in r.

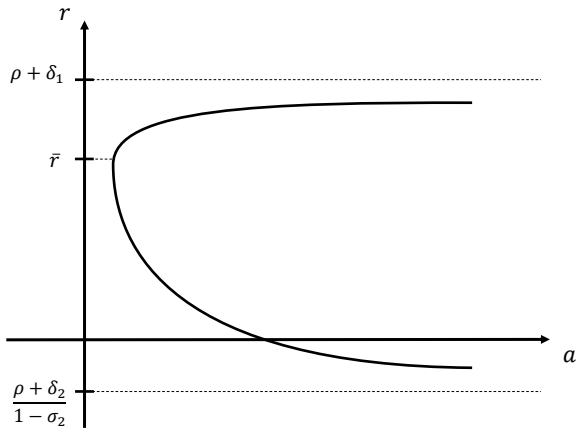
See Appendix D.1 for the proof.

The first property noted in Proposition 1 is straightforward. If an active household has greater income, its target level of asset holding will be higher. This remains true regardless of the degree of inter-temporal substitution. The most important element in Proposition 1 relates to the effects of steady state returns on desired long-run asset holdings. In particular, we see that if $\sigma_2 \leq 1$, then desired long run asset holdings would be monotonically increasing in r because the substitution effect always dominates retirement savings effect. However, recall from Section 2 that this property runs counter the within households asset accumulation over the few decades. For this reason, the case that interests us is when $\sigma_2 > 1$. In this case, the effects of returns on long-run asset holdings are non-monotonic. For high levels of returns, desired holdings are increasing in r, while for low returns they are decreasing in r. To understand this effect, recall that interest rates have two effects in this model. At low interest rates, households are encouraged to consume more, and accumulate less, through the standard inter-temporal substitution channel. However, retirement preoccupations play a counterbalancing role. When long-term interest rates are low and $\sigma_2 > 1$, active households have an increased marginal incentive to accumulate assets for retirement needs. What Proposition 1 indicates is that there will be a point of reversal of the effect of steady state r on accumulation incentives. When r is sufficiently high, a marginal increase in steady state r would lead to more accumulation as the positive substitution effect dominates the decreased retirement need effect even if $\sigma_2 > 1$. When interest rates are low then the increased need for retirement income will dominate the inter-temporal substitution effect and favour a greater accumulation of asset when $\sigma_2 > 1$.[25]

The shape of the active household's long-run asset demand $a^{a,ss}(y,r)$ is illustrated in Figure 5 when $\sigma_2 > 1$. Here we see the C-shape of the long-run asset demand keeping income, y, constant. Moreover, we can see that the long-run asset demand (when $\sigma_2 > 1$) is delimited by two levels of r. As r tends to $\rho + \delta_1$, desired asset demands relative to consumption tend to infinity. As r tends to $\frac{\rho+\delta_2}{1-\sigma_2} < 0$, desired asset demand relative to

[25] Our paper has some similarities with the work of Brunnermeier and Koby (2019) on the reversal interest rate. In their work, there is a reversal rate of interest whereby interest rates below the reversal rate become contractionary. Their reversal rate result comes from banking frictions. Our set-up can also be thought as having a reversal rate, which we denote $\bar{r}$. Our reversal rate arises from expected income effects in retirement that drive up households' desired savings while working and therefore depress consumption.

Figure 5: Active households' long-run asset demand



consumption will tend again toward infinity. When $\sigma_2 > 1$, there exists also a threshold or point of inflexion

$$\bar{r} = \left[\frac{\sigma_2(\sigma_2 - 1)(\rho + \delta_1) - (\rho + \delta_2)}{(\sigma_2 - 1)(\sigma_2 + 1)}\right],$$

such that the asset demand of active households is increasing in interest rates when r is above $\bar{r}$ and is decreasing in interest rates when r is below $\bar{r}$.

Even before we specify the general-equilibrium setting, one can see why this C-shaped property of asset demands by active households may create a situation with multiple steady states. An economy populated with such households will face a residual asset supply coming from the total asset supply in the economy minus that held by retired households. Even if this residual asset supply is monotonic and well-behaved, it is likely to cross the steady state asset demand of active households more than once. The object of the next section is to set up a general equilibrium structure where this possibility can be examined explicitly.

To simplify the presentation, the remaining sections of the paper will restrict attention to the case of interest where $\sigma_1 = \sigma_2 \equiv \sigma > 1$.

4 General equilibrium

We now want to look at the general equilibrium properties of an OLG economy populated with active and retired households with preferences as defined in the previous section. In

particular, we want to look at the implications of having active households whose long-run asset demands are non-monotonic in asset returns when $\sigma > 1$.[26] To begin, we will examine a setting without any nominal constraints. This will allow us to show how the real side of this economy is likely to generate more than one steady state real interest rate. In our baseline set-up, we will allow for only one asset and this will be a government bond. We will later include trade in a productive asset to allow for asset valuation effects.

In our model economy, we normalize the population to have a measure 1 of households, with the implied fraction $\phi \equiv \frac{\delta_2}{\delta_1 q + \delta_2}$ who are active and the fraction $1 - \phi$ who are retired. When a household dies it is replaced by the birth of a new active household.

The government in this one-good economy spends an amount G, has an outstanding debt in the amount B and levies taxes T_{1t} on active households. The taxes adjust to satisfy the government budget constraint

$$\phi T_{1t} = G + r_t B,$$

where r_t is the interest rate on government debt. Each active household is endowed with one unit of labor that produces w goods. Total production in this economy is given by output produced with the labor of active workers and therefore is equal ϕw. We will limit attention to cases where B is not so large that it could not be financed by active households. Since $\rho + \delta_1$ is the highest possible interest rate in this economy, we restrict attention to cases where $B(\rho + \delta_1) < \phi w$. Since we have not introduced annuity markets, private agents will generally have positive asset holdings when they die and therefore there will be unintended bequests. We assume that the unintended bequest of a household goes to the newborn household replacing that household. To keep the structure more tractable, we assume that the government ensures — through a tax T_{2t} on active households — that newborn households receiving bequest from retired parents have the same average starting wealth as the newborn households inheriting from active households. Under this assumption, if asset holdings are equal across active households at a point in time, then the system inherits a representative agent structure for active households.[27] The second tax on households, T_{2t}, is defined by the following budget condition.

$$\delta_1(1-q)\phi a_t + \delta_2(B - \phi a_t) = \left[\delta_1(1-q)\phi + \delta_2(1-\phi)\right] a_t + \phi T_{2t}.$$

[26] Note that we take the empirical observations we presented previously as placing in serious doubt the relevance of the case with $\sigma < 1$ as the observed pattern is not easily reconcilable with asset demands which would be monotonically increasing in real interest rates.

[27] Assuming that active households act like a large family as in Gertler, Kiyotaki, and Prestipino (2020) would lead also to maintain the tractability of the representative agent structure.

The first term on the left hand side of this equation is the total funds received from accidental bequests.[28] On the right hand side, the first term is the funds needed to give to newborn active households while the second term is the tax levied on all active households to equalize wealth between newborn that inherited from retired and active households. Rearranging the equation, we obtain that $T_{2t} = \delta_2(B - a_t)/\phi$.

Definition 1. *An equilibrium for this economy will be composed of a consumption profile and asset allocation profile for the different types of households, a time path of interest rates, and taxes such that (1) given interest rates, taxes, government expenditures and public debt, household consumption and asset allocation profiles maximize households' utility, (2) both the markets for goods and assets clear at each point in time, and (3) the government budget is balanced.*

Let us begin by examining the behavior of total asset demands in this economy in a steady state with constant interest rates and taxes. This demand is comprised of both the long-run asset demand function of active households, $a^{a,ss}(y, r)$, and that of retired households, denoted $a^{r,ss}$.[29] The steady state asset demand function of active households when interest rates are constant is given explicitly in Proposition 1 where it is shown to be C-shaped in r. Since long-run asset demands relative to consumption of active households go to ∞ when either r goes to $\rho + \delta_1$ or $-\frac{\rho+\delta_2}{\sigma-1}$, we can restrict attention to situations where $r \in \left(-\frac{\rho+\delta_2}{\sigma-1}, \rho + \delta_1\right)$ as this is the only feasible range for a steady state equilibrium.

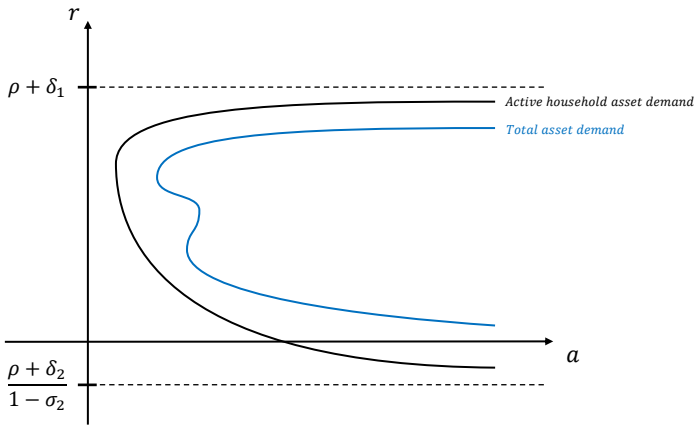
To get the steady state asset demand for retired households, we need to aggregate the asset holdings across the different retirement cohorts. With $r < \rho + \delta_1 \leq \rho + \delta_2$, retired households will be depleting their asset holdings as they age. In particular, this will cause the asset holdings of a retired household who retired τ periods ago with a assets to be given by $ae^{-(\frac{\rho+\delta_2-r}{\sigma})\tau}$.[30] Since in steady state, each retiree starts retirement with the same amount of assets, which is equal to the steady state asset holdings of active households ($a^{a,ss}$), the aggregate asset demand of retirees ($a^{r,ss}$) is given by

[28] This equation includes the asset market clearing condition $\phi a_t + a_t^r = B$ implying that the total asset demand of retirees is $a_t^r = B - \phi a_t$.

[29] We are focusing on potential steady states where all active households have the same wealth level in the steady state. There may be other types of steady states. However, if wealth levels of active households start from a position of equality, then they always stay equal because of the government tax-transfer scheme.

[30] Note that the consumption of retirees satisfies the relationship $c_t^r = a_t^r \Gamma^{-1}$, where a_t^r is the asset at time t. Hence, asset accumulation dynamics for constant interest rates are given by $\dot{a}_t^r = -\frac{\rho+\delta_2-r}{\sigma} a_r^r$.

Figure 6: Active households' asset demand and total asset demand



$$a^{r,ss} = a^{a,ss}(y,r)(1-\phi)\frac{\delta_2}{\frac{\rho+\delta_2-r}{\sigma}+\delta_2} = a^{a,ss}(y,r)(1-\phi)g(r) \qquad g'(r) > 0,$$

where

$$g(r) \equiv \frac{\delta_2}{\frac{\rho+\delta_2-r}{\sigma}+\delta_2}.$$

As a result, total asset demand in the steady state of this economy can be expressed as

$$a^{t,ss}(y,r) = \phi a^{a,ss}(y,r)\left(1+\frac{g(r)(1-\phi)}{\phi}\right).$$

Total asset demand in a steady state is therefore equal to the total asset demand — *i.e.*, $\phi a^{a,ss}(y,r)$ — of active households multiplied by the factor $1+\frac{g(r)(1-\phi)}{\phi}$. Accordingly, total asset demand will reflect several of the properties of the asset holdings of active households. In particular, this total asset demand will be non-monotonic in r with the additional property that as r goes to either $\rho+\delta_1$ or $-\frac{\rho+\delta_2}{\sigma-1}$, demand will go to infinity. However, even if total asset demand takes this form, it may not always inherit the simple C-shape of the active household's long-run asset demands. The possibility of a more complex non-monotonic shape for total asset demand is illustrated in Figure 6.

From Figure 6, we can see why such an economy is likely to have more than one steady state values for r. For a given level of total bonds B in the economy, there are likely to be more than one interest rate that clears the asset market. However, this simple argument is not complete as the income of active households, y, is being held fixed in this figure while in this set-up it is endogenous. Proposition 2 nonetheless confirms this line of reasoning.

Proposition 2. *When $\sigma > 1$ and bonds B are in fixed supply, then a unique steady state equilibrium interest rate – a unique r^*– is generically impossible.*[31] *There will either be more than one steady state value for r^* or, if the supply of bonds is sufficiently small, there will be no equilibrium.*

See Appendix D.2 for the proof.

In the next section, we will introduce nominal rigidities and show how an aggressive inflation targeting regime can effect which real interest rate arises in equilibrium, their stability properties and the size of their basin of attraction. In particular, we will show that an aggressive anti-inflation monetary policy can favour the emergence, stability and attractiveness of a low real interest rate outcome. In the subsequent section, we will also add productive assets to the analysis to explore the effects of valuation effects.

Prudent "perpetual" youth assumption. Much of our analysis could be conducted in the current set-up. However, to allow for an easier presentation of results we will now adopt a very useful simplifying assumption.[32] In the last section we will drop this assumption and come back to the general case to show that this assumption is not driving any of our main

[31] The generic property relates to the amounts of bonds. There can be one value for B where a unique equilibrium can exist if the bond supply happens to satisfy a precise tangency condition. However, such an equilibrium configuration would not be robust to any minor change in the amount of bonds.

[32] In the version of the model without nominal rigidities, the equilibrium dynamic system can still be analytically tractable if we are in a situation where the wealth of active households have converged to the same level. Once the active households have the same wealth, it will remain that way with active households acting like a representative household. In this case, the equilibrium behaviour is described by the following system of three dynamic equations:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q V_a(B, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r_t\right]$$

$$\dot{a}_t = w + r_t a_t + \frac{(B - a_t)\delta_2}{\phi} - \frac{G + B r_t}{\phi} - c_t$$

plus the goods market clearing condition

insights. To immediately get a sense of why we want to add a simplifying assumption, it is helpful to focus on the long run demand for asset by active households as presented in Figure 5 . As we have stressed, this demand is C-shaped when $\sigma > 1$. The equilibrium determination of r^* can then be viewed as depending on the interaction of this C-shaped demand curve for assets with a residual supply curve for assets. The relevant residual supply curve corresponds to the total supply of assets in this economy minus that held by retired agents. Depending on the properties of this residual supply curve, it is obvious from Figure 5 that many different equilibrium configurations could arise. In particular, as expressed in Proposition 2, there is likely more than one equilibrium value for r^*. In fact, depending on the shape of this residual supply curve, even if it is monotonic, there could be two, three or more equilibrium values for r^*. Our model in its full generality does not rule out any such possibilities. However, analyzing all these possibilities at once can be confusing. The following simplifying assumption will allow us to approach the problem in steps, where we first focus on a case which produces exactly two potential equilibrium values for r^*, then we introduce productive assets and discuss the case of three values. The general case embeds the features emphasized in these special cases but potentially allows for even more equilibria. At this point, we do not see any added insights from the cases with more than two or three potential equilibrium values for r^*. It is for this reason we find the adoption of following simplifying assumption useful.

In particular, consider a modification of the above setting where the probability q of surviving the health shock that moves one to retire has an objective component and a subjective component with the subjective value being denoted q^s while the objective value

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$$\phi c_t = \phi w - G - (B - \phi a_t)\Gamma_t^{-1}$$

where c_t is the consumption of the representative active household and a_t is its asset holdings. However, when we extend the model to include nominal rigidities this dynamic system expands to a 4^{th} and 5^{th} degree system making analytical results very difficult. It is for this reason, we choose to make the additional simplifying assumption of having q go to zero with $\epsilon > 0$. Under this addition assumption, the dynamic system is reduced in the absence of nominal rigidities to the pair of dynamic equations

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q^s V_a(B, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r_t\right]$$

plus the goods market clearing condition

$$c_t = w - G.$$

This lower dimensional system can be more easily extended to allow for nominal rigidities and still be analytically tractable.

is still given by q with $q^s = q + \epsilon$ ($\epsilon > 0$). In this setting, ϵ is governing the extent to which people are over-estimating the probability of needing their retirement savings. Now consider this model as q goes to zero. In this limit, we will have active agents that are saving for retirement but no actual retirees.[33] This simplifies the analysis by removing the need to track the wealth holdings of the retirees. In effect, under this assumption, the steady state demand for assets is now given entirely by the desired wealth holdings of the active population and has a simple C-shape.[34] Hence, steady state equilibrium real interest rates are given by the intersection of the C-shaped asset demand of active households and the exogenous supply of bonds as shown in Figure 7. As can be easily seen on this figure, under this simplifying assumption, the steady state will never be unique. This was also true in the more general case, but was harder to visualize. Moreover, if the supply of asset is sufficiently large, there will always be exactly two steady state values for r, which we will denote r^{*H} and r^{*L} for the high and low real steady state rate respectively. In the continuation, we will assume that B is sufficiently large such that an equilibrium exists as stated below. Lemma 2 indicates some key properties of r^{*H} and r^{*L}.

Going forward will will assume that the quantity of outstanding government bonds (B) is sufficiently large to guarantees the existence of an equilibrium, that is, $B > \bar{B} \equiv (\delta_1 q^s)^{1/\sigma}(\rho + \delta_1 - \bar{r})^{-1/\sigma}\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}\bar{r}\right]$.

Lemma 2. *The low and high natural interest rates r^{*H} and r^{*L} have the following properties: (i) $r^{*H} > \bar{r}$ and $r^{*L} < \bar{r}$; (ii) the high real interest rate (r^{*H}) increases with government bonds B while the low real rate r^{*L} decreases with B; and (iii) r^{*H} increases with the probability of death of retirees δ_2 while r^{*L} falls with δ_2.*

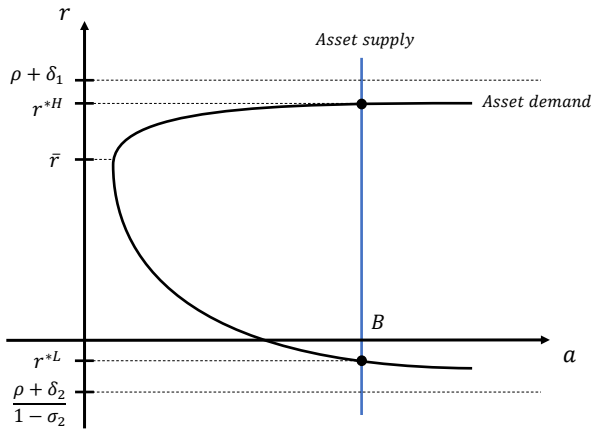
See Appendix D.10 for the proof.

The two steady state real interest rates depicted in Figure 7, r^{*H} and r^{*L}, will continue to play an important role in the presence of nominal rigidities. To foreshadow future results, monetary policy will be shown to potentially affect which of these real interest rates are stable and which are more likely to arise in the long run. However, monetary policy will simultaneously maintain a neutrality property, in that it will not affect the values of the potential long-run real interest rate that can arise. These will always be either r^{*H} or r^{*L}.[35]

[33] In similarity to the perpetual youth model of Blanchard (1985) and Yaari (1965), this version of our model can be considered as a *"prudent" perpetual youth* model where agents are prudently preparing for retirement even if they never retire. They are young, working and saving for retirement until they die.

[34] In the case without nominal rigidities, the income and consumption levels of the active household

Figure 7: Long-run asset demand and asset supply: multiple steady state real interest rates (r^*)



4.1 Introducing nominal rigidities and a vertical long run Phillips curve

In the environment considered up to now, we have not included any nominal rigidities. In this section, we extend the model to allow demand considerations to affect economic activity in the short run while maintaining that in the long run economic activity is entirely determined by the economy's productive capacity. In other words, we extend the model in a way that allows for a Philips curve which reflects a short-run tradeoff between inflation activity but not a long-run tradeoff. To this end, we slightly modify the environment and assume that output is produced using labour by a set of competitive firms. The production function is given by $y_t = Al_t$, where productivity $A > 0$ is constant. Goods prices p_t are perfectly flexible and therefore competition between firms will ensure that the price of the output good is equal $\frac{W_t}{A}$, where W_t is now the nominal wage. This implies that real wages are always equal to A. We denote $\bar{l}$ and $\bar{y} = A\bar{l}$ the natural rate of employment and output respectively.[36]

become exogenous with $c = y = w - G$.

[35] At this juncture, the low steady state real rate r^{*L} may appear unstable. However, as we shall show, in the presence of nominal rigidities, the stability properties around both r^{*L} and r^{*H} will depend on the nature of monetary policy.

[36] The household's budget constraint in real terms is now given by the following, where l_t will be endogenously determined: $c_t + \dot{a}_t = Al_t + r_t a_t - T_t$.

The key nominal rigidity is related to the wage determination process. Nominal wages are assumed to adjust slowly to excess labor supply according to:[37]

$$\frac{\dot{W}_t}{W_t} = \pi_t^e + \kappa'(l_t - \bar{l}), \qquad \kappa' > 0,$$

where π_t^e is the expected inflation rate and $(l_t - \bar{l})$ represents the deviation of employment from full employment $\bar{l}$. If we further assume that expected inflation:

$$\pi_t^e = \pi_t + \mu\dot{\pi}_t,$$

then we get the Phillips curve of the form

$$\dot{\pi_t} = \kappa(y_t - \bar{y}), \tag{7}$$

where $\kappa = \frac{\kappa'}{-A\mu}$. The parameter $\mu > 0$ governs the relationship between inflation and the output gap. If we assume perfect foresight in this simple setup, with $\mu = 0$, both the short term and long-run Phillips curve would be vertical. Hence, we allow $\mu \neq 0$.

With this formulation, when $\kappa > 0$ ($\mu < 0$), we have a backward-looking Phillips curve and π_t needs to be treated as a state variable. Alternatively, if $\kappa < 0$ ($\mu > 0$), we have a forward-looking Phillips curve and π_t needs to be treated as a jump variable. We will focus on the case with $\kappa > 0$ in the main text.[38] The characterization of stable steady states is similar in both cases. However, the narrative around inflation is potentially more compelling in the $\kappa > 0$ case as inflation is sluggish and without jumps.

Since we now allow for variable inflation, we now need to distinguish between real and nominal rates of interest. We will denote the nominal rate by i_t with the real rate given by $r_t = i_t - \pi_t$.

The equilibrium dynamics for this economy with nominal wage rigidities (and $q = 0$) is now governed by the following dynamic system

$$\dot{\pi} = \kappa(c_t + G - \bar{y})$$

$$\frac{\dot{c}_t}{c_t} = \frac{i_t - \pi_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q^s V_a(B, \Gamma_t)$$

[37] Note that in this model, wage inflation is equal to price inflation π_t.

[38] Details for the case with $\kappa < 0$ are available from the authors upon request.

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}(i_t - \pi_t) \right]$$

To complete this model, we need to specify how monetary policy sets the nominal interest rate i_t. Our main focus will be monetary policy that is governed by a simple Taylor rule of the form

$$i_t = \max\left\{0, i^T + \psi(\pi_t - \pi^T)\right\} \qquad \psi > 1, \tag{8}$$

where i^T is a nominal interest rate target, π^T is the central bank's inflation target, $\psi > 1$ satisfies the Taylor principle and the effective lower bound on interest rates is set to 0. However, instead of looking immediately at the implications of this constrained Taylor rule, it is easiest to start with the two embedded sub-cases: (1) $i_t = i^T + \psi(\pi_t - \pi^T)$, that is, disregard the ELB constraint and (2) $i_t = 0$, setting the interest rate at the ELB.

Propositions 3 and 4 highlight how monetary policy affects the stability of the system.[39] In particular, Proposition 3 indicates that if the nominal interest rate setting is unconstrained by an ELB and satisfies the Taylor principle, then the only equilibrium configuration that can be stable is one where the steady state equilibrium real interest rate is equal to r^{*H}. Moreover, if the central banks targets that real natural rate, it will achieve its target for π. Proposition 4 covers the converse case. It shows that if nominal interest rates are set at the ELB, then the only possible configuration for a stable steady state equilibrium is one where the real interest rate equals r^{*L}.

Proposition 3. *If $i_t = i^T + \psi(\pi_t - \pi^T)$ and $\psi > 1$, the economy admits only one stable steady state equilibrium.*[40] *In this equilibrium, the real interest rate equals r^{*H}. If $i^T = r^{*H} + \pi^T$, the central bank attains its inflation target.*

See Appendix D.3 for the proof.

Proposition 4. *If $i_t = 0$, the economy admits only one stable steady state equilibrium. In this equilibrium, the real interest rate equals r^{*L} and $\pi = -r^{*L}$.*

[39] The setup allows for a more general result: The high real interest rate steady state will be stable if nominal policy interest rates setting locally satisfies the Taylor principle, while the low real interest rate steady state will be stable if nominal policy interest rates setting locally does not satisfy the Taylor principle.

[40] When referring to a stable equilibrium here we are referring to a saddle path stable equilibrium where there are two roots of the system that are positive and one that is negative.

See Appendix D.4 for the proof.

To understand why the stability properties around two different equilibrium real interest rates can depend on the monetary regime, it is helpful to recognize that when inflation is high because consumption is high, endogenous dynamics must favour a reduction in consumption to induce stability. This is actually the case behind both Propositions 3 and 4. However, the underlying mechanisms are quite different. When the system is near the steady state real rate r^{*H} and nominal interest rates increase more than one-to-one with inflation, higher inflation pushes real rates above r^{*H}. With such higher real rates, consumption decreases because the inter-temporal substitution force dominates the retirement motive near r^{*H}. This makes this combination — being near r^{*H} and with real rates rising with inflation — locally stable. In contrast, when the system is near the steady state real rate r^{*L} and i_t is at the ELB, higher inflation pushes down real rates. These lower real rates then depress consumption near r^{*L} because the retirement motive of savings dominates the inter-temporal substitution motive. This makes the alternative combination —being near r^{*L} with real rates falling with inflation — locally stable.

Now, we turn to looking at possible equilibrium configurations when the Taylor rule is constrained by the ELB. To give more structure, let us assume that $i^T = r^{*H} + \pi^T$; that is, the central bank targets a real rate (i.e., a natural interest rate) equal to r^{*H}. Given Propositions 3 and 4, one may think that two stable equilibrium configurations would now always be possible with such a rule. However, that is not the case as implied by Proposition 5. Proposition 5 indicates that such a Taylor rule not only has the power to affect the stability properties of different steady state equilibrium real interest rates, it also has the power to affect which actually arise in equilibrium. In particular, if the policy is not very aggressive, that is if $\psi > 1$ is sufficiently close to 1, then only one type of stable equilibrium configuration will arise and that configuration has the real interest rate equal to r^{*H} and inflation on target. In contrast, if the policy is sufficiently aggressive (and π^T is not too small), then two stable equilibrium configurations are possible with two different real interest rates. The low real rate equilibrium is accompanied by a nominal rate at the ELB, while the high real rate is accompanied by with the nominal rate being at target.

Proposition 5. *When i_t is set according to $i_t = \max\{0, r^{*H} + \pi^T + \psi(\pi_t - \pi^T)\}$ with $\psi > 1$ and $\pi^T > -r^{*L}$, then there will be a cutoff level of monetary tightness $\bar{\psi} > 1$, such that the following holds*

- *if $\psi > \bar{\psi}$ (i.e., if monetary policy is sufficiently aggressive), the economy admits two*

stable steady state equilibrium outcomes; one with the real interest rate equal to r^{*H} and one with the real rate equal to r^{*L}. In the equilibrium with the real interest rate equal to r^{*H}, inflation is on target. In the equilibrium with the real interest rate equal to r^{*L}, inflation is below target and the policy rate i_t is at the ELB.

- *If $1 < \psi < \bar{\psi}$, then the economy admits only one stable equilibrium. In this equilibrium the real interest rate is equal to r^{*H} and inflation is on target.*

See Appendix D.5 for the proof.

To understand why monetary policy has the power to affect long-run real interest rate outcomes as indicated by Proposition 5, it is useful to recall how a constrained Taylor rule translates inflation into real rates. This is illustrated in Figure 8. As is indicated on the figure, the ELB constraint becomes binding at the inflation level $\pi^{ELB} \equiv \frac{(\psi-1)\pi^T - r^{*H}}{\psi}$. To the right of this binding level of inflation, real rates are increasing with inflation, while to the left, real rates are decreasing with inflation. A higher value of ψ implies that the ELB constraint will become binding at higher levels of inflation. Therefore, a smaller $\psi > 1$ allows for a smaller range of real interest rates. Accordingly, with a $\psi > 1$ sufficiently close to 1, r^{*L} will not be feasible, while r^{*H} would be feasible. As ψ increases, this allows for a larger range of real rates and generally makes an equilibrium with the real rate at r^{*L} feasible (as long as $\pi^T > -r^{*L}$).[41]

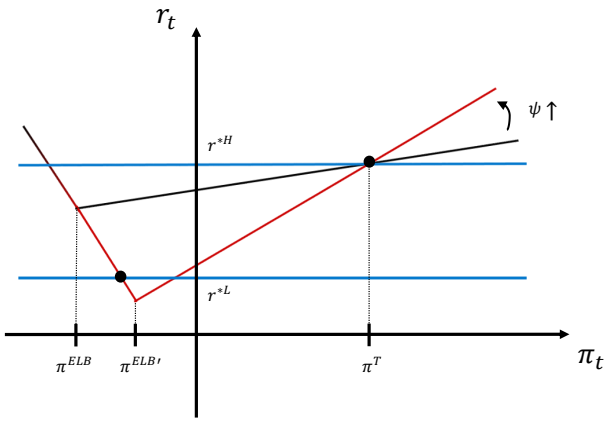
4.2 Illustrating transitional dynamics and how monetary policy can affect basins of attraction

We saw from Propositions 3, 4 and 5 that in the presence of multiple steady state real interest rates r^*, monetary policy can affect which real interest rate may arise in equilibrium and what stability properties it may have. In this section, we illustrate the transitional dynamics associated with the different possible outcomes. In particular, we want to show how aggressive monetary policy can go beyond allowing a low real interest rate equilibrium to emerge, but can also affect its basin of attraction and therefore make it more likely to arise the more aggressive policy is (higher ψ).

An illustration of transitional dynamics associated with the case of one stable steady state is represented in Figure 9, while the case with two stable steady states is represented

[41] From such a diagram one can also see that for a given feasible real rate, the system would allow for two associated outcomes. One at the ELB and one above the ELB. Proposition 5 indicates that the stable one will be the one above the ELB when the real rate is r^{*H} and the one at the ELB when the real rate is r^{*L}.

Figure 8: Link between real interest rates and inflation under a Taylor rule constrained by the ELB



in Figure 10. On this figure, we represent the steady state condition between c and π implied by the $\dot{c}_t = 0$ condition when $\dot{\Gamma}_t = 0$. This is best represented by pieces:

If $\pi < \pi^{ELB}$, the $\dot{c}_t = 0$ curve is given by

$$c = B\left(\delta_1 q^s\right)^{-1/\sigma}\left[\rho + \delta_1 + \pi\right]^{1/\sigma}\left[\frac{\rho + \delta_2}{\sigma} + \frac{1-\sigma}{\sigma}\pi\right]$$

and if $\pi \geq \pi^{ELB}$, the $\dot{c}_t = 0$ curve is given by

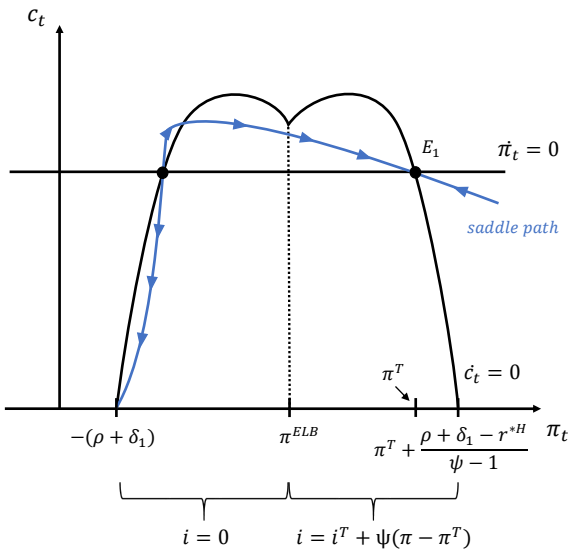
$$c = B\left(\delta_1 q^s\right)^{-1/\sigma}\left[\rho + \delta_1 - (i - \pi)\right]^{1/\sigma}\left[\frac{\rho + \delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right)(i - \pi)\right]$$

where $i - \pi = r^{*H} + (\psi - 1)(\pi - \pi^T)$.

On this figure we also depict the steady state condition $\dot{\pi}_t = 0$ which corresponds to $c = A\bar{l} - G$. The crossing between these two curves gives us the set of steady states. Finally on the figure we plot transitional dynamics in blue which illustrate the stability properties of the steady state. These transitional dynamics should be viewed as a projection of the actual transitional dynamics which are in the three dimensional space $\{c_t, \pi_t, \Gamma_t\}$.[42] Note

[42] Alternatively, these transitional dynamics can be viewed as exact ones associated with having future expected values of r_t being set to the current rate r.

Figure 9: Equilibrium trajectories when monetary policy follows a not too aggressive Taylor rule: one stable steady state



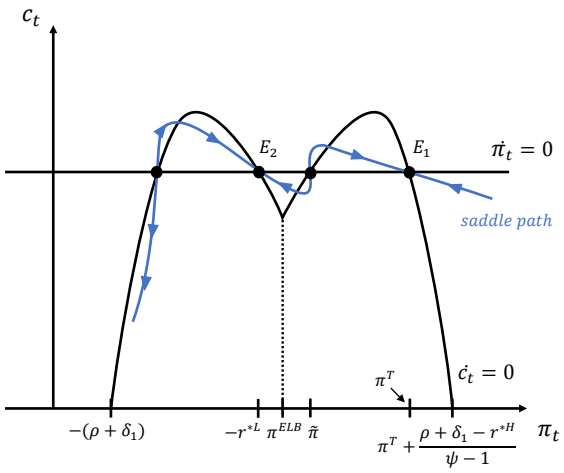
that the $\dot{c}_t = 0$ curve almost mirrors itself around the cutoff π^{ELB}.

In Figure 9, E_1 is the only stable steady state. E_1 is a high-real-interest-rate (r^{*H}) with inflation on target. There is also a low-inflation steady state in Figure 9, but it is not stable. The nominal interest rate at the unstable steady state in the figure is in the ELB region since the level of inflation arising from that equilibrium point is less than π^{ELB}. This type of configuration, where there is an unstable steady state at the ELB and a stable steady state with $i > 0$, echoes what arises in a standard infinitely lived representative agent environment without our bequest motives (see Benhabib, Schmitt-Grohé, and Uribe (2001)).[43] In contrast, in Figure 10 we now have two stable steady states. The high-real-interest stable steady state, denoted E_1, remains, but now we also have one low-real-rate (r^{*L}), low-inflation stable steady state denoted E_2. The E_2 steady state is in the ELB region, while the E_1 steady state remains in the region where $i > 0$ and where the Taylor principle is operative. Proposition 5 expresses this possibility. In this setting, given the two stable steady states, the system will exhibit *hysteresis*.[44] If inflation starts above the level $\tilde{\pi} = \pi^T + \frac{r^{*L} - r^{*H}}{\psi - 1}$ denoted on Figure 10, the system will converge to E_1, while if

[43] Recall that we are assuming a backward-looking Phillips curve in the main body of the text. When assuming a forward-looking Phillips curve, this equilibrium would exhibit indeterminacy.

[44] In the case where the parameter κ in the Phillips curve is negative, the same two steady states are determinate stable, and the system would jump to one of them instead of exhibiting hysteresis.

Figure 10: Equilibrium trajectories when monetary policy follows a sufficiently aggressive Taylor rule: two stable steady states



it starts below, it will tend to converge to E_2. In this set-up we can consider the effects of shocks, especially q^s shocks which increase the desire to accumulate more assets for retirement (precautionary) motives. For example, if the economy were to start at E_1, and there was a large temporary rise in q^s, the steady state equilibrium E_1 could temporarily disappear — the reason being that there would then be too much demand for assets relative to supply, which depresses demand. As a result, there would be a contractionary period with deflation. Once the shock reverses itself, the level of inflation would be starting from a lower level. If this new inflation level was below $\tilde{\pi}$, the economy would converge to the long-run equilibrium at E_2 even if it was at equilibrium point E_1 before the temporary shock to q^s.

In this setting, we can highlight the potential role of increasing the aggressiveness of monetary policy — as captured by high values of ψ — in making the low-inflation equilibrium outcome in Figure 10 more likely, that is, making it more likely that the economy converges to a low real interest rate.[45] Recall that we are always assuming that $\psi > 1$, so the Taylor

[45]We also examined the effect on equilibrium outcomes of changing the inflation target π^T. Among other results, we find that increasing π^T favours the status quo; that is, we find that the basin of attraction of neither the stable ELB equilibrium nor the non-ELB equilibrium decreases when π^T increases. Accordingly, if an economy were caught in a low-inflation, low-real-rate trap, increasing π^T would not help the economy exit this trap.

principle is active when not constrained by the ELB. If monetary policy is not too aggressive in the sense of ψ not being much greater than 1, then the equilibrium configuration will take the form we represented in Figure 9.[46] So in this case with monetary policy not too aggressive (but still satisfying the Taylor principle when above the ELB), the economy can only converge to the E_1 equilibrium. This has the desired outcome of supporting inflation close to target. However, as ψ increased the range of inflation that leads monetary authorities to set i at the ELB increases. A rise in ψ can therefore be seen as changing the equilibrium configuration from that depicted in Figure 9 to that depicted in Figure 10. In fact, as ψ gets very big, the equilibrium dynamics can make the high-real-rate equilibrium fragile. This can be seen in Figure 11. In this figure, we return to representing equilibria in the (i, π) space as this offers an alternative perspective to discuss dynamics. The two different real rates are represented in the panels of this figure as before by lines with slope of one and with a Taylor rule super-imposed. In this space, equilibrium dynamics can be summarized along the π axis, as π is the only state variable and the dynamics are driven by the stability of the different steady states for π. As can be seen in Figure 11, moving from Panel A to Panel B — that is, when monetary policy reacts more to below target inflation — leads the range of inflation rates above $\tilde{\pi}$ that support the higher-inflation equilibrium E_1 to become arbitrarily small. This implies that when such an economy in Panel B is subjected to shocks, even if it starts at the high-real-rate equilibrium with inflation on target, it is very likely to end up at the low-inflation ELB equilibrium. In this sense, a high ψ policy of reducing interest rates aggressively in response to deviation of inflation from target can contribute to the economy ending up at a low steady state real rate of interest. It is worth emphasizing that at this equilibrium, inflation is low (possibly negative), but it is nonetheless stable even if the Taylor principle does not hold.[47] Proposition 5 confirms that the existence of the E_2 equilibrium depicted in Figure 10 and Figure 11 actually depends on $\psi > 1$ being sufficiently large. If ψ is not sufficiently large, the configuration depicted in Figure 10 cannot arise.

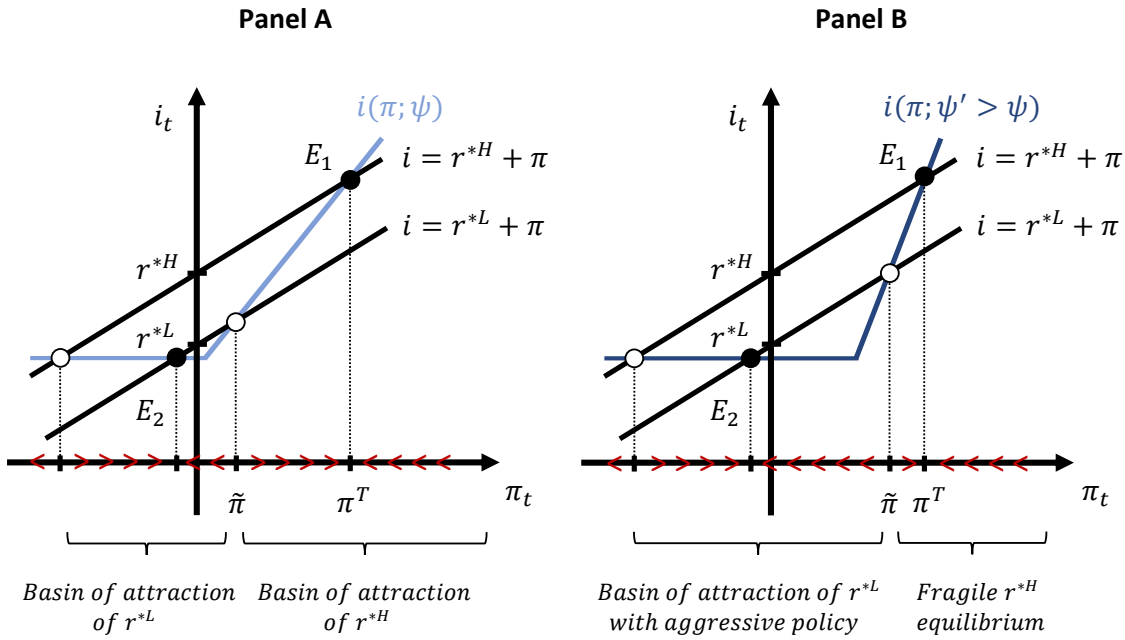
4.2.1 Real factors and the emergence of the low-real-rate, low-inflation trap

In the previous discussion, we emphasized how more aggressive monetary policy can simultaneously favour the emergence of a low-real-rate equilibrium at the ELB while also expanding its basin of attraction. We now want to briefly discuss the role of real factors

[46] For this precise equilibrium configuration, we are assuming that $\pi^T > -\bar{r}$ and $\psi > \frac{r^{*H}+\pi^T}{\bar{r}+\pi^T}$. See Appendix D.4.

[47] A downward spiral in inflation is nonetheless possible in this set-up if inflation gets sufficiently close to $-(\rho + \delta_1)$. See Cochrane (2017).

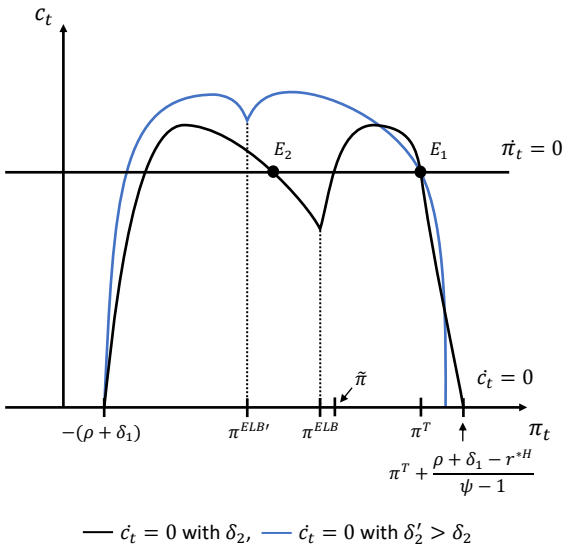
Figure 11: Equilibrium trajectories when monetary policy follows a very aggressive Taylor rule: two stable steady states



in allowing for such an equilibrium outcome. In particular, countries or regions that may have fallen into a low-inflation trap (such as Japan and Europe) do not appear to have a substantially more aggressive monetary policy than elsewhere. So monetary policy is unlikely to be the sole or main driver. Instead, these are countries that are generally viewed as having real factors that favour savings, and these are the factors that are commonly thought to contribute to demand being depressed and monetary policy being pushed to the ELB. In our set-up, real factors that favour savings play a very similar role to monetary policy in favouring the emergence of the low-real-rate, low-inflation trap. This is most easily seen by varying δ_2, which governs the expected duration of retirement. As indicated in Proposition 6, for a given monetary policy stance parameterized by ψ, δ_2 has to be sufficiently low for an equilibrium configuration such as in Figure 10 to arise. In Figure 12 we depict the effect of a change in δ_2 on the equilibrium configuration.[48] As illustrated in the figure, a higher δ_2 will make the E_2 equilibrium that arises with a low δ_2 disappear. So if an economy finds itself in a low-real-rate equilibrium like E_2 in the figure, it is both

[48] Under Lemma 2, a rise in δ_2 also increases r^{*H}.

Figure 12: Equilibrium trajectories when the probability of death δ_2 rises to $\delta_2' > \delta_2$



because δ_2 is sufficiently low and ψ is sufficiently high. In this sense, monetary policy can be viewed as contributing to the emergence of a low-real-rate trap, but it cannot be seen as the only driving factor. Real factors affecting savings are also key.

Proposition 6. *The existence of a stable low-real-rate, low-inflation trap at the ELB is only possible if δ_2 is sufficiently small (holding other parameters fixed).*

See Appendix D.6 for the proof.

4.2.2 Exiting the low-real-rate trap: the effects of Inflation shocks and expansionary fiscal policy

When the economy is in a low-real-rate trap, as represented by the equilibrium outcome E_2 shown in Figures 10 and 11, a sufficiently large exogenous shock to inflation could move inflation above the central bank's inflation target.[49] If such a high rate of inflation were to arise, the central bank would increase nominal interest rates aggressively causing real rates to rise also. This would place the economy temporarily in recession in order to reduce

[49]We are interpreting an inflation shock as an entirely unexpected shock to the production cost of firms, which is passed through to prices. In other words, it is an unexpected shock to the Phillips Curve equation which causes a discrete jump in inflation.

inflation. As inflation declines and the employment recovers, interest rates — both real and nominal — gradually decrease. However, the economy would not return to E_2. Instead, it would converge to the steady state E_1 with the high real rate. Hence, when the economy is at E_2 and there is a large inflation shock, this can cause the long-run real interest rate to increase from r^{*L} to r^{*H}.

Fiscal policy can also help create an exit from the low-real-rate steady state, but this exit will be non-monotonic. An increase in government debt B corresponds to an upward shift in the $\dot{c}_t = 0$ in Figure 10. This implies that the long-run equilibrium point E_2 will move to the right when B is larger, implying higher inflation and lower real interest rates (see Lemma 2). This is expressed in Proposition 7. However, the effect of changes in B on long-run interest rates and inflation will be discontinuous. As debt rises, there will come a point where the E_2 equilibrium will cease to exist. At that point, the only stable equilibrium will be E_1. Hence, both the long-run real interest rate and the rate of inflation in such an economy can change discretely in response to a large fiscal expansion. A sufficiently large increase in B can create a switch from the long-run equilibrium E_2 to the long-run equilibrium E_1. Fiscal policy in this case, is pushing the economy out of the low-real-rate, low-inflation steady state, but that is coming at the cost of a discrete jump in long-run inflation and r^*.[50]

Proposition 7. *The inflation rate at the ELB stable steady state is increasing in government debt B, while real interest rates are decreasing. However, when B becomes sufficiently large, the equilibrium at the ELB will cease to exist. At that point, long-run inflation and real interest rates will exhibit a discontinuous jump to higher levels.*

See Appendix D.7 for the proof.

5 Extending the Model to Include Productive Assets: Lucas Trees

Up to now we have been examining the equilibrium determination of long-run real interest rates — and the role of monetary policy — in the presence of only one asset: government bonds. In this section, we enrich the environment by introducing a claim on a productive asset, where the price of the asset increases when interest rates decrease, that is, we introduce valuation effects into the analysis. As we shall see, valuation effects render the analysis more complex but do not overturn our main results regarding both the

[50] Acharya and Dogra (2021), Eggertsson and Mehrotra (2014), and Mian, Straub, and Sufi (2021a) also find that rising public debt favours an escape from the ELB.

possibility of multiple r^* and the role of monetary policy in affecting which r^* is most likely to arise. It is for this reason that we left the introduction of valuation effects until now.

To introduce a second asset into our set-up, suppose there is a mass s of Lucas trees that produce a flow f of goods every period.[51] In order to introduce the possibility of something akin to a risk premium on these assets, we will assume that trees die at flow rate $\omega \geq 0$ and that dead trees are continuously replaced with new trees redistributed in a lump sum fashion to active households. In aggregate, trees are not risky, they simply decay at rate ω. A household can now hold a combination of bonds and trees. If we denote by z_t the price of a mass of one of trees at time t, then arbitrage between the two assets will cause z_t to satisfy the following asset pricing relationship

$$\frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - (r_t + \omega),$$

and households will be indifferent between holding bonds or trees. The advantage of allowing for trees to decay is that they permit situations where r can be zero and the price of trees is still finite. The household consumption Euler equation in this case can be re-written as

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \delta_1 q^s \frac{c_t^\sigma}{\sigma} V_a(\Omega_t, \Gamma_t),$$

where Ω denotes household wealth which includes both the holdings of bonds and trees. The main effect of introducing this second asset is that it causes the effective supply of assets that must be held by the market, $B + z_t s$, to contain valuation effects. To present this case more easily, it is helpful to recognize that the steady state of the consumption Euler equation can be presented as expressing a household's *desired* ratio of consumption-to-wealth as a function of interest rates. Note that this is simply a reinterpretation of the steady state condition for households' consumption decision in the previous sections. Accordingly, the desired consumption-to-wealth ratio maintains the hump shaped property because it balances inter-temporal substitution effects and retirement incentives. Furthermore, the desired consumption-to-wealth ratio maintains the property that it goes to zero as r goes to either $\rho + \delta_1$ or to $\frac{\rho+\delta_2}{1-\sigma}$. Relative to our analysis with only bonds, household desired wealth holding is unchanged with the introduction of trees. The feature that changes with the introduction of productive assets is the properties of the feasible aggregate long-run

[51] A Lucas tree set-up is one where we have productive assets but these assets are not themselves expandable. We have also explored the possibility of allowing for reproducible productive assets and have not found it to give novel insights relative to the case analyzed here. For this reason we chose to focus on the simpler case.

consumption-to-wealth ratio. Previously, the feasible long-run consumption-to-wealth ratio was $\frac{A\bar{l}-G}{B}$ and therefore independent of r. The economy's feasible long-run consumption-to-wealth ratios with trees is now given by

$$\frac{c}{\Omega} = \frac{A\bar{l} + sf - G}{B + \frac{sf}{r+\omega}},$$

where the numerator represents full employment output plus the flow of goods from trees less government consumption and the denominator represents the total value of assets in steady state. This is the aggregate consumption-to-wealth ratio that is consistent with full employment and $\dot{z}_t = 0$. This feasible consumption-to-wealth ratio is increasing in r for $r > \omega$, and starts from zero when $r = -\omega$. If $s = 0$, then this feasible consumption-to wealth ratio is independent of interest rates and we are back to our previous analysis where the only possible equilibrium configuration is one where there are two natural interest rates r^*. The introduction of trees increases the possible equilibrium configurations. This is due to it changing the shape of the feasible consumption-to-wealth ratios. In the absence of sticky prices, there now can be at least three equilibrium configurations. These are illustrated in Figure 13, and as can be seen in the figure, these depend on slope of the feasible consumption-to-wealth ratio curve compared to the slope of desired consumption-to-wealth ratio curve.

Figure 13 is aimed at representing the set of steady states for the following system of dynamic equations.

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \delta_1 q^s \frac{c_t^\sigma}{\sigma} V_a(\Omega_t, \Gamma_t),$$

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r_t \right]$$

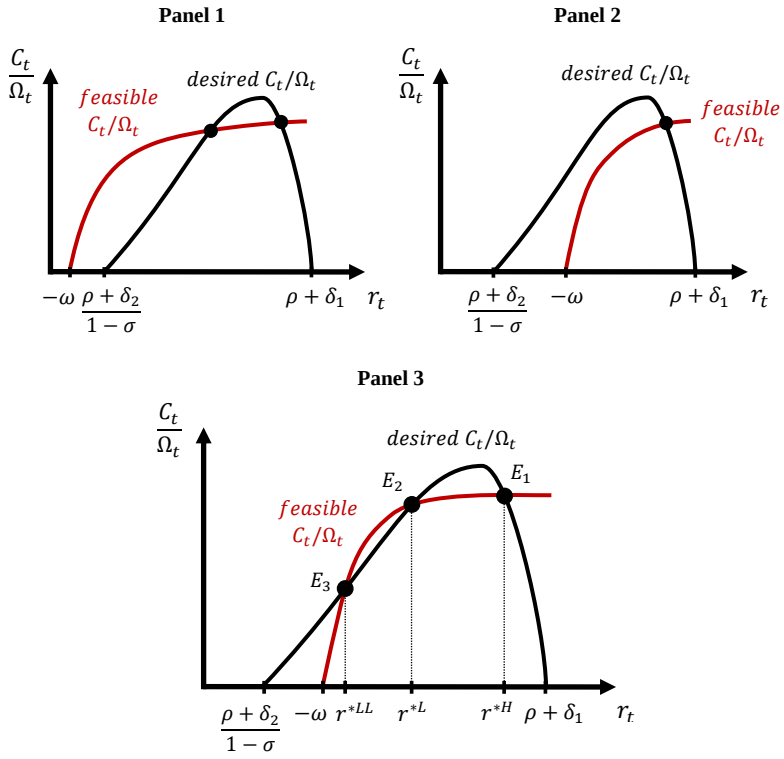
$$\frac{\dot{\Omega}_t}{\Omega_t - B} = \frac{fs}{\Omega_t - B} - (r_t + \omega),$$

where r needs to adjust to satisfy the market clearing condition

$$c_t = A\bar{l} + sf - G.$$

We refer to the desired consumption-to-wealth ratios as the consumption-to-wealth ratios c/Ω that satisfy $\dot{c}_t = \dot{\Omega}_t = \dot{\Gamma}_t = 0$ for different levels of r. The feasible consumption-to-wealth ratios are defined as the set of Ω_t that satisfy $\dot{\Gamma}_t = 0$, $\dot{\Omega}_t = 0$ and the market

Figure 13: Equilibrium trajectories in the presence of Lucas trees as function of the size of dividend flow f: three potential steady states



clearing condition.

The first panel of Figure 13 illustrates a case where only two equilibrium values for r^* arise. This requires ω to be sufficiently large. The analysis of the previous section extends directly to the current case, with the only difference being that the consumption-to-wealth ratio is now lower at r^{*L} than at r^{*H}. Given this minor difference, we will not dwell on this case in this section. The second panel illustrates a case where there is a unique equilibrium value of r^* despite the hump shape in desired consumption-to-wealth ratios. This was not possible in the absence of valuation effects. Such a configuration will arise if desired wealth holdings never outpace the valuation effects when r decreases. This case is important as it indicates that even in the presence of C-shaped asset demands, there is not necessarily more than one equilibrium. There can be a unique equilibrium if valuations effects play the

right role as illustrated in the second panel of the figure. Finally, the third panel illustrates the case where there are three possible values of r^*. Since this is the novel case, we will focus on it.

The important element to note in Panel 3 of Figure 13 is that there are potentially three real interest rates (r^*) compatible with full employment. There are two steady states which resemble E_1 and E_2 in terms of how the curves cross, but now a third equilibrium appears. This third equilibrium, which we will denote E_3, has an associated real interest rate which we denoted by r^{*LL}.[52] This equilibrium arises with both very low real interest rates and high asset holdings on the part of households. The households exhibit a very low consumption-to-wealth ratio in this equilibrium. In the absence of Lucas trees, this configuration was not possible as the feasible consumption-to-wealth ratio did not change with r. However, with the Lucas trees, the high demand for assets at r^{*LL} is satisfied by the large valuation of Lucas trees which acts as to endogenously increase the supply of assets.

Assuming that the real side of the economy takes the form as in Panel 3 of Figure 13, we can re-introduce sticky prices and a Taylor rule to look at the joint determination of c_t and π_t as we did before.[53]

There are now two sub-cases to consider. The easy case is when r^{*LL} is small relative to the inflation target π^T in the Taylor rule, that is, when $r^{*LL} < -\pi^T$. In such a case, monetary policy is ruling out the E_3 type equilibrium, and all our previous results again carry over. In particular, if monetary policy is not very aggressive (but still satisfying the Taylor principle), then there can be only one stable steady state equilibrium and that corresponds to the high-real-rate equilibrium E_1. As monetary policy gets more aggressive, the equivalent of equilibrium E_2 will appear as a stable steady state of the system with

[52] Note that r^{*LL} may well be negative.

[53] In the presence of Lucas trees, the set of dynamic equations representing equilibrium with sticky prices and a Taylor rule can be reduced to

$$\frac{\dot{c}_t}{c_t} = \frac{i_t - \pi_t - \rho - \delta_1}{\sigma} + \delta_1 q^s \frac{c_t^\sigma}{\sigma} V_a(\Omega_t, \Gamma_t),$$

$$\dot{\pi}_t = \kappa(c_t + G - A\bar{l} - fs), \qquad \kappa > 0$$

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{i_t - \pi_t + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}(i_t - \pi_t) \right]$$

$$\frac{\dot{\Omega}}{\Omega_t - B} = \frac{fs}{\Omega_t - B} - (i_t - \pi_t + \omega).$$

$$i_t = \max\left\{0, r^{*H} + \pi^T + \psi(\pi_t - \pi^T)\right\} \qquad \psi > 1$$

nominal frictions.[54] E_2 will again be associated with the nominal interest rate being at the ELB. And as monetary policy becomes gradually more aggressive, the basin of attraction of this E_2 equilibrium will expand while that of E_1 will become small. In this sense, our previous analysis extends directly to this case even when the configuration in the absence of sticky prices is of the form given by Panel 3 in Figure 13 as long as $r^{*LL} < -\pi^T$.

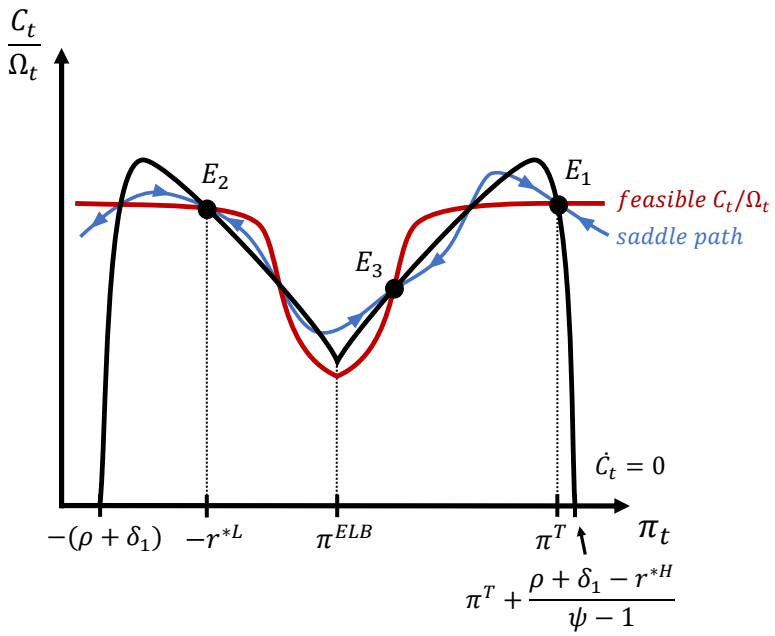
Now if $r^{*LL} > -\pi^T$, then the equilibrium dynamics can get more complex than that presented with only bonds. For example, it can take the form as given in Figure 14. In this case, it is possible to have three stable steady states with different levels of inflation and different real rates. The high-real-rate equilibrium corresponding to E_1 remains. As before, an ELB equilibrium with a low real rate at r^{*L} will also be present when monetary policy is sufficiently aggressive. This is point E_2. But, now we get the possibility of a third equilibrium; this one implements the real rate r^{*LL} and is not in the ELB region. This equilibrium has a low real rate — even lower than that of the E_2 equilibrium — even though the nominal interest rate is positive. The price of the Lucas trees at the E_3 equilibrium, which is given by $z = \frac{f}{\omega+i-\pi}$ in steady state, will be higher in the E_3 equilibrium than in both the E_2 and E_1 steady state equilibria. With such a configuration, if the economy were to start in the E_1 equilibrium and be subject to a set of inflation shocks[55], it would likely go from E_1 to E_3, with a drop in inflation and a rise in asset prices.

Let us now examine how the equilibrium configuration depicted in Figure 14 changes as monetary policy gets more aggressive. This is depicted in Figure 15. Like previously, we return to representing equilibria in the (i, π) space. The three different real rates are represented in the panels of Figure 15 as before by lines with slope of one and with a Taylor rule super-imposed. In this space, equilibrium dynamics can be summarized along the π axis, as π is the only state variable and the dynamics are driven by the stability of the different steady states for π. As can be seen in Figure 15, when policy becomes more aggressive ($\psi > 1$ becomes larger, ie., moving from Panel A to Panel B), the inflation level at the E_3 steady state equilibrium gets closer and closer to that at the E_1 equilibrium. Hence, with very aggressive monetary policy we can get a situation where the two steady state equilibria E_1 and E_3 are very close together in terms of inflation outcomes, but far apart in real interest rate outcomes. This arises because the nominal rate is much lower at the E_3 equilibrium than at the E_1 equilibrium since monetary policy is very aggressive in cutting rates when inflation is below target. Furthermore, in this case, both the E_3 and the E_1 inherit a fragility property. As seen previously, the E_1 becomes fragile with respect

[54] The proof of the stability of these steady states is similar to that given for Propositions 3, 4, and 5. It is available upon request.

[55] We are interpreting inflation shocks as shocks that change the initial level of inflation.

Figure 14: Equilibrium trajectories in the presence of Lucas trees when the inflation target is sufficiently high: three stable steady states

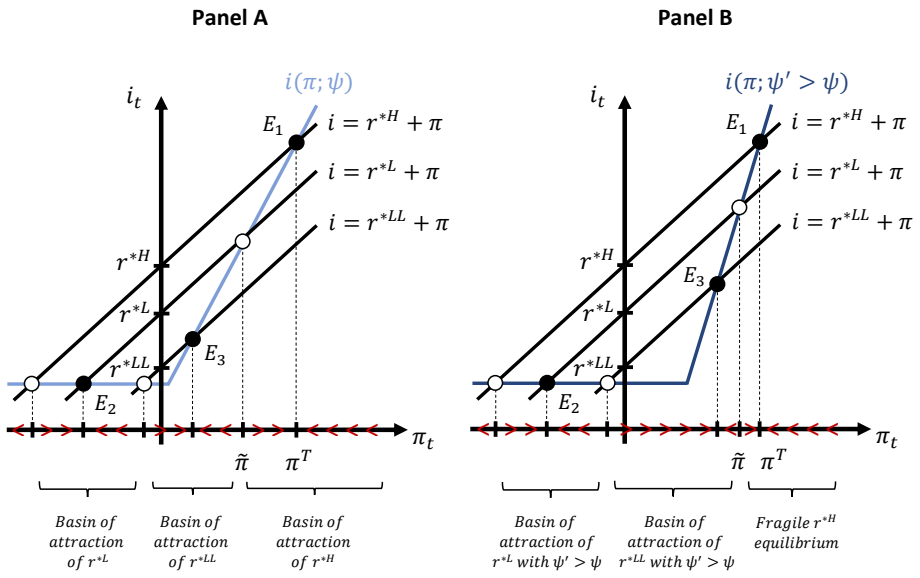


to downward shocks to inflation. In contrast, the E_3 equilibrium will be quite robust to downward shocks to inflation as its basin of attraction to its left actually expands as policy gets more aggressive. However, when the policy is very aggressive, the E_3 equilibrium will become fragile to positive shocks to inflation as the relevant basin of attraction to its right can become arbitrarily small.

When the economy is at E_3 or E_1, it could still be pushed to the ELB equilibrium at E_2, This would require a sufficiently large downward shock to inflation. A move from E_3 to E_2 would cause a drop in inflation, but it would be associated with a fall in asset prices. The mechanism could also work in reverse. If the economy is at either E_3 or E_2 and there were a sufficiently large exogenous positive shock to inflation, then the economy could find itself back to E_1.

In summary, the presence of a productive asset in the form of a Lucas tree enriches our previous analysis but it does not change the basic messages. Because there can be more than one real natural interest rate r^*, monetary policy becomes an important force

Figure 15: Equilibrium trajectories in the presence of Lucas trees when the inflation target is sufficiently high and monetary policy is very aggressive

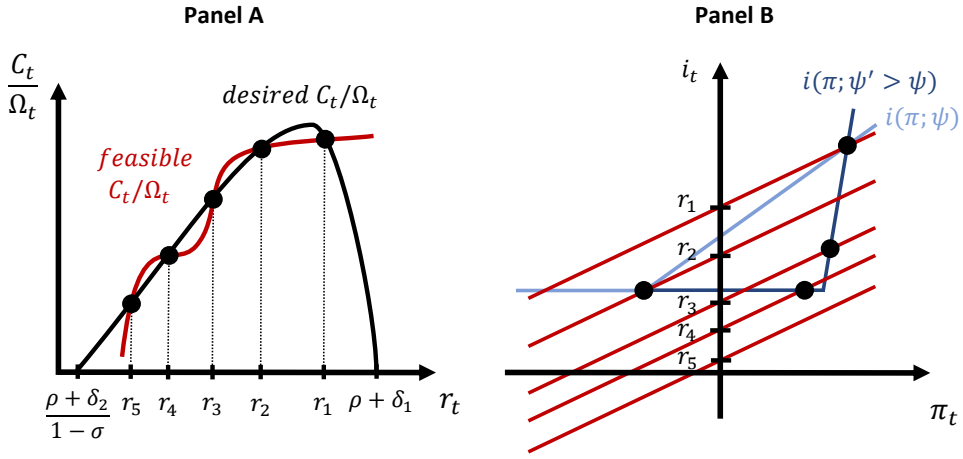


in determining long-run real rate outcomes. In particular, the more monetary policy aggressively targets inflation, the more likely it is to cause the high-real-rate equilibrium to be fragile to negative shocks to inflation. This makes the economy likely to converge to a low-real-rate equilibrium. The main additional property that arises with the presence of a Lucas tree is that a lower-real-rate equilibrium does not necessarily happen only at the ELB. It can also arise with nominal interest rates above the ELB and with inflation close to target. Hence this set-up offers an explanation for why economies can get stuck with low real interest rates at either the ELB or above the ELB, where in both cases we would have a high valuation of productive assets.

6 Back to full model

In the previous sections we have been analyzing the monetary policy implications of having C-shaped asset demands by active households, where the C-shaped arose due to the competing motives of inter-temporal substitution and retirement. However, we have been conducting most of our analysis under the simplifying assumption that active households were actually the only type of living households. Recall that we assumed toward the end of Section 4 that active households perceived a risk of needing assets to pay for retirement,

Figure 16: Multiple equilibrium real interest rates in the general model with both active households and retirees



Note: The red lines in Panel B represent the steady state Fisher equation $i = r_i + \pi$ for $i = 1, \ldots, 5$. The dark blue and light blue lines in the figure represent a Taylor rule which is constrained by a lower bound, aims for a target level of inflation π^T, and reacts to inflation as governed by the parameter ψ.

but that they actually died before needing these funds. This gave rise to a perpetual youth type setup where households always stayed young but nevertheless saved for a possible retirement that never actually happened. In introducing this assumption we claimed that it was not driving our main results. In this Section, we return to the more general case where we remove this assumption and allow active and retired households to co-exist. Our goal in this section is to illustrate why our results carry through to this more general case.

The implications of dropping this assumption (i.e., dropping $q^s \neq q$) can most easily be seen on Panel A of Figure 16. In this figure, we plot the desired long-run consumption-to-wealth ratio of active households against real interest rates. This locus, which is in black, is now familiar and it is not changed with the drop of the assumption. The red line represents the feasible long-run consumption-to-wealth ratio of active households, where now the feasible outcome includes the fact that retired individuals are both consuming resources and holding assets. This figure is abstracting from sticky prices and we are allowing for both bonds and productive assets to be present.[56] This figure is very similar to that we

[56]The equilibrium behaviour is now described by the following system of four dynamic equations:

presented in Section 5 where we introduced productive assets. However, there is one main difference which relates to the precise properties of the feasibility locus. Previously, this feasibility locus was monotonically increasing in r and concave. However, in the more general model this feasibility locus can be less well behaved, leading it to potentially cross the locus of desired consumption-to-wealth ratio of active households several times. To be clear, in the more general case, there may still only be two or three crossings as before, but we can't rule out more crossings. Hence, as shown in the figure, we could, for example, have five crossings.

In Panel B, we translate this five equilibrium example in Panel A into its implications in terms of feasible stable steady state equilibria in the presence of sticky prices and an ELB constrained monetary policy. The five parallel red linear lines in Panel B represent the five potential real interest rates from Panel A, while the two different blue lines represent two different monetary policy rules, one being more aggressive than the other (both reflecting our previous specification of a Taylor rule satisfying the Taylor principle when not constrained by the ELB). The black dots at the intersections of the lines represent stable outcomes. In this more general set-up, we can see the main properties we have previously emphasized. First, the stability around the different real interest rates depends on the local property of monetary policy. The highest rate interest rates will be stable if monetary policy locally satisfies the Taylor principle; the next highest real rate will be stable if monetary policy locally does not satisfy the Taylor principle as is the case at the ELB.[57] Any additional potential real rate equilibrium will reflect the same pattern, alternating between being

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho - \delta_1}{\sigma} + \frac{c_t^{\sigma}}{\sigma}\delta_1 q V_a(\Omega_t, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r_t\right]$$

$$\dot{\Omega}_t = w + r_t\Omega_t + \frac{(B + z_t s - \Omega_t)\delta_2}{\phi} - \frac{G + Br_t}{\phi} - c_t$$

$$\frac{\dot{z}_t}{z_t} = \frac{f}{z_t} - (r_t + \omega)$$

plus the good market clearing condition

$$\phi c_t = \phi w - G - (B + z_t s - \phi\Omega_t)\Gamma_t^{-1}$$

.

This system governs the holding of asset across active and retired households. However, it does not give a breakdown of holdings of trees versus bonds, as the two are perfect substitutes in equilibrium. An easy fix to this indeterminacy is to assume that both active households and retirees holds the same fraction of wealth in bonds and in trees.

[57] The proof of this statement is available from the authors.

stable under a monetary policy that satisfies the Taylor principle or the inverse. The second feature is how monetary policy affects the set of effective steady state equilibria. If monetary policy is not too aggressive, then the set of steady state equilibria will be more limited. In the figure, the light blue line represents a monetary policy with limited aggressiveness which results in only one equilibrium. The dark blue line reflects a more aggressive monetary policy and results in four possible steady states: two above the ELB and two at the ELB. In this later case, the lowest real rate from Panel A remains unattainable as part of a nominal equilibrium with this particular monetary policy. [58]

The main message to convey from Figure 16 is that much of our previous analysis– which relied on an assumption that eliminated retirees but not retirement savings– provided insights regarding monetary policy that are robust to eliminating the assumption.

7 Conclusion

The idea that monetary policy may have contributed to the secular decline in real interest rates is a popular theme among many financial market participants and economic commentators. However, evaluating this type of claim is difficult without first specifying the mechanisms that could in theory generate such an outcome. Motivated by observations regarding within-group changes in wealth-to-income ratios since the late 1980s, we showed how savings behavior which is influenced by both inter-temporal substitution and retirement motives could support/rationalize such claims. In particular, we showed how such savings behavior can give rise to long-run asset demands that are C-shaped with respect to real interest rates and favour multiple steady state equilibrium real rates. Moreover, we showed that in such an environment, an aggressive inflation targeting can render the higher-real-rate equilibrium fragile and favour the convergence to a low-real-rate trap. However, the resulting low-real-rate trap is not insurmountable. In particular, we showed that the economy may return to a high-real-rate equilibrium if it is subjected to either a large exogenous increase in inflation or to a large increase in public debt. Since both these forces are currently at play, it raises the possibility that the future could involve a much higher real rate.

[58] One aspect that is more complicated to present in this general case is the basin of attraction of the different steady states because the state space is larger.

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Appendix

A Data

For the main analysis we use four waves of the US Survey of Consumer Finances for 1989, 1992, 2016 and 2019. The 1989 and 2019 SCFs are used for the wealth-to-income analysis, while the 1989-1992 and 2016-2019 SCFs are used for the construction of saving rates corresponding to the beginning (1989) and the end (2019) of our period of interest. For this latter saving rate analysis, we combine the SCF with household-level aggregates reported in the US Flow of Funds Accounts and the National Income and Product Accounts.

Household wealth in the SCF is defined to include all assets of households (both real and financial) net of their liabilities. On the one hand, household non-financial (real) assets include primary and other residential real estate, non-residential equity, as well as equity holdings in privately held businesses (both corporate and non-corporate) and other assets. Financial assets, on the other hand, include fixed-income assets, e.g. bonds, deposits, as well as mutual fund holdings, and directly and indirectly held stocks, and other financial assets. On the liability side, we include both mortgage and non-mortgage household debt obligations.

For the total wealth-to-income ratio analysis, we adjust SCF income to match NIPA aggregates by using scaling factors of 1.21, 1.41, 1.33 and 1.36 for the year 1989, 1992, 2016 and 2019, respectively. For the same years, the adjustments for total net worth are 1.18, 1.34, 1.03 and 1.14.

For the construction of saving rates using the wealth-based approach, we use the same definition of wealth as in Mian, Straub, and Sufi (2020), which excludes the value of durables on household balance sheets. The pure price inflation factors are aggregate in nature (same factors apply to all households). We use the series available from Mian, Straub, and Sufi (2020) replication package until 2016, and extend them to 2019 using their methodology, in particular, for the residual equity price inflation.

B Results using a step-function regression approach and relationship with the shift-share analysis

The regression approach to estimating the between component using a function F with a set of age and income group dummies produces the first term in the decomposition below:

$$\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89} = \sum_i \left[\frac{\bar{w}_{i,89} N_{i,19}}{y_{19}} - \frac{\bar{w}_{i,89} N_{i,89}}{y_{89}}\right] + \sum_i \left(\frac{N_{i,19}}{y_{19}}\right) [\bar{w}_{i,19} - \bar{w}_{i,89}] \tag{B1}$$

Table B1 reports the shares of within and between components using this decomposition. These shares are very similar to those obtained using both a baseline shift-share decomposition and a regression approach. In fact, this decomposition can written in a manner that makes an easy comparison with our baseline shift-share decomposition

$$\left(\frac{w}{y}\right)_{19} - \left(\frac{w}{y}\right)_{89} = \sum_i \frac{\bar{w}_{i,89}}{\bar{y}_{i,89}} \left[\Theta \frac{y_{i,19}}{y_{19}} - \frac{y_{i,89}}{y_{89}}\right] + \sum_i \left(\frac{N_{i,19}\bar{y}_{i,89}}{y_{19}}\right) \left[\frac{\bar{y}_{i,19}}{\bar{y}_{i,89}} \frac{\bar{w}_{i,19}}{\bar{y}_{i,19}} - \frac{\bar{w}_{i,89}}{\bar{y}_{i,89}}\right] \tag{B2}$$

where $\Theta_i = \frac{N_{i,19}\bar{y}_{i,89}}{N_{i,19}\bar{y}_{i,19}}$. If the average within group earning $\bar{y}_i$ don't change much over time, then Θ_i will be close to 1, making the two decompositions very close,

Table B1: Total Change in the Aggregate Wealth-to-Income Ratio Between 1989 and 2019 and the Fraction of the Change due to Within and Between Effects: Decomposition Using Step-function

Definition	Total Change	Within (%)	Between (%)
Wealth (Baseline)	2.415	64.1	35.9
Wealth less housing	2.244	62.9	37.1
Wealth plus DB	2.618	64.5	35.5
Wealth plus DB less housing	2.447	63.3	36.7

Note: DB refers to the value of defined benefit pension schemes. The decomposition is done for 30 groups which are the product of 5 age groups and 6 income groups. The age groups are: 18-34, 34-35, 35-44, 45-54, 54-64, 65+ and the income groups (in thousands) are: 0-20, 20-40, 40-60, 60-80, 80-120, 120+.

C Saving Results: SCF Scaled to the Aggregates

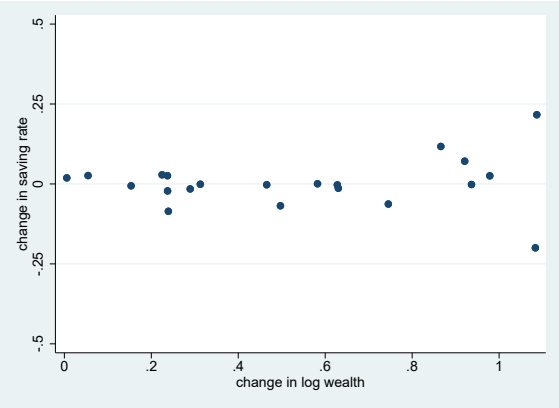
In this section, we repeat the analysis in Section 2.1 using data from the SCF scaled to the aggregates in the Financial Flow Accounts and National Income and Product Accounts for both saving rates and wealth-to-income ratios. The results are simiar. As shown in Table C2, the correlation between changes in saving rates and changes in wealth-to-income ratios is still positive and small. Figure C1 also supports the finding in the main text that there is no systematic relationship between increases in wealth and declines in saving rates and thus no desire by people to reduce their wealth holdings.

Table C2: Dynamics of Saving and Wealth-to-income ratios in 30 Age-income Groups: SCF scaled data

Group	Change in saving rate, pp	Change in wealth-to-income ratio	Change in log wealth
		Income 0-20K	
age 18-34	0.53	-6.87	
age 35-44*	-0.08	0.74	0.31
age 45-54	0.03	-15.50	
age 55-64*	-0.03	-0.88	0.47
age 65+*	-0.06	-8.81	0.15
		Income 20-40K	
age 18-34	1.3	-3.5	
age 35-44	-1.57	-14.02	
age 45-54*	1.90	-13.46	0.01
age 55-64	-2.08	-22.8	
age 65+*	-1.56	-2.9	0.29
		Income 40-60K	
age 18-34	-3.5	-1.67	
age 35-44	-0.92	-5.18	
age 45-54	-1.72	-8.31	
age 55-64*	2.58	-3.48	0.24
age 65+*	-6.83	-3.61	0.50
		Income 60-80K	
age 18-34*	2.63	-0.59	0.05
age 35-44	1.13	-1.53	
age 45-54*	-2.18	-2.51	0.24
age 55-64*	0.06	0.23	0.58
age 65+*	-6.28	-0.64	0.75
		Income 80-120K	
age 18-34*	-1.33	0.26	0.63
age 35-44*	2.86	-0.02	0.22
age 45-54*	-8.55	-0.76	0.24
age 55-64*	-0.3	-0.36	0.63
age 65+*	2.54	1.24	0.98
		Income 120K+	
age 18-34*	-0.17	0.95	0.94
age 35-44*	11.73	2.29	0.87
age 45-54*	7.13	2.14	0.92
age 55-64*	-19.98	3.72	1.08
age 65+*	21.64	5.30	1.09
	1989-1992	2016-2019	
Aggregate s/y, annualized %	10.4	9.9	
	1989-2019		
Corr(s/y, w/y)	0.14		

Note: For each income-age group, the first column reports the changes in the three-year saving (%) over 1989-1992 and 2016-2019 periods normalized by aggregate incomes in 1989 and 2016, respectively. The second column reports the change in the wealth-to-income ratio between 1989 and 2019 with SCF data scaled up to the Flow of Funds Accounts and National Income and Product Accounts. The third column reports the change in the log level of wealth for the groups, which have seen an increase in wealth over the 1989-2019 period. "*" refers to groups that experience increases in the level of wealth.

Figure C1: Change in saving rates vs. change in log wealth for age-income groups with wealth increases between 1989 and 2019 (scaled to FFA and NIPA aggregates)



D Proofs of Propositions and Lemmas

D.1 Proof of Proposition 1

Recall the steady state asset holdings $a^{a,ss}(y,r)$ defined in equation (6)

$$a^{a,ss}(y,r) = (\delta_1 q)^{\frac{1}{\sigma_2}} \left[\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r\right]^{-1} [\rho+\delta_1-r]^{\frac{-1}{\sigma_2}} y^{\frac{\sigma_1}{\sigma_2}}.$$

See also Section D.9 for the conditions of convergence to $a^{a,ss}(y,r)$.

Let us take the derivative of $a^{a,ss}$ with respect to income y

$$\frac{da^{a,ss}}{dy} = \frac{\sigma_1}{\sigma_2}(\delta_1 q)^{\frac{1}{\sigma_2}} \left[\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2} r\right]^{-1} [\rho+\delta_1-r]^{\frac{-1}{\sigma_2}} y^{\frac{\sigma_1}{\sigma_2}-1}.$$

$\frac{da^{a,ss}}{dy} > 0$ since $r \in \left(-\frac{\rho+\delta_2}{\sigma-1}, \rho+\delta_1\right)$. Hence, the long-run asset holdings of active households are increasing in income y.

Taking the derivative of $a^{a,ss}$ with respect to r, we have

$$\frac{da^{a,ss}}{dr} = (\delta_1 q)^{\frac{1}{\sigma_2}} (\rho+\delta_1-r)^{\frac{-1}{\sigma_2}-1} y^{\frac{\sigma_1}{\sigma_2}} \left(\frac{1}{\rho+\delta_2+(\sigma_2-1)r}\right) \left[1 - \frac{\sigma_2(\sigma_2-1)(\rho+\delta_1-r)}{\rho+\delta_2+(\sigma_2-1)r}\right].$$

If $\sigma_2 \leq 1$, $\frac{da^{a,ss}}{dr} \geq 0$ and hence the steady state asset holdings of active households are increasing in the interest rate.

Now let us assume that $\sigma_2 > 1$. When $r = \bar{r}$, we have $\frac{da^{a,ss}}{dr} = 0$ where

$$\bar{r} \equiv \frac{\sigma_2(\sigma_2-1)(\rho+\delta_1) - (\rho+\delta_2)}{(\sigma_2-1)(\sigma_2+1)}.$$

If $r > \bar{r}$, $\frac{da^{a,ss}}{dr} > 0$. And if $r < \bar{r}$, $\frac{da^{a,ss}}{dr} < 0$. As a result, $a^{a,ss}$ is increasing (decreasing) in the interest rate when r is above (below) $\bar{r}$. Hence, $a^{a,ss}$ is C-shaped in the space (r, a).

Q.E.D.

D.2 Proof of Proposition 2

In the steady state $\dot{c}_t = 0$, $\dot{\Gamma}_t = 0$, and $\dot{a}_t = 0$. Combining $\dot{c}_t = 0$ and $\dot{\Gamma}_t = 0$ we obtain the desired consumption-to-wealth ratio (c/a):

$$\frac{c}{a} = (\delta_1 q)^{-\frac{1}{\sigma}} (\rho + \delta_1 - r)^{\frac{1}{\sigma}} \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r \right] \equiv D(r), \tag{D3}$$

where we denote the desired consumption-to-wealth ratio $D(r)$.

Combing the asset $([\phi + (1-\phi)]a = B)$ and goods $(\phi c = \phi w - G - (B - \phi a)\Gamma^{-1})$ markets clearing conditions leads to the feasible consumption-to-wealth ratio (c/a)

$$\frac{c}{a} = \frac{y}{B} \left[1 + \frac{(1-\phi)}{\phi} g(r) \right] - \frac{(1-\phi)}{\phi} g(r) \Gamma^{-1} \equiv F(r), \tag{D4}$$

where $F(r)$ represents the feasible c/a ratio, $\Gamma = \left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma} r \right]^{-1}$, $g(r) = \frac{\delta_2 \sigma}{\rho + \delta_2 - r + \sigma \delta_2}$, and $y = \phi w - G$.

The function $D(r)$ has the following properties

- D is hump shape and continuous over the interval $[\frac{\rho+\delta_2}{1-\sigma}, \rho + \delta_1]$. $D(\bar{r}) = 0$, if $r < \bar{r}$, $D' < 0$ and if $r > \bar{r}$, $D' > 0$,
- $D(\rho + \delta_1) = 0$ and $D\left(\frac{\rho+\delta_2}{1-\sigma}\right) = 0$.

Similarly, the function $F(r)$ has these properties

- F is continuous over the interval $[\frac{\rho+\delta_2}{1-\sigma}, \rho + \delta_1]$,
- $F(\rho + \delta_1) > 0$ if $\frac{B}{y} < \frac{1 + \frac{(1-\phi)}{\phi} g(\rho+\delta_1)}{\frac{(1-\phi)}{\phi} \Gamma^{-1}}$ and $F\left(\frac{\rho+\delta_2}{1-\sigma}\right) > 0$.

The steady state equilibrium is obtained when the desired and feasible consumption-to-wealth ratios cross, that is, when $D(r) = F(r)$.

Given that $F\left(\frac{\rho+\delta_2}{1-\sigma}\right) > 0 = D\left(\frac{\rho+\delta_2}{1-\sigma}\right)$, if D and F cross once, they must cross at least one more time again since F and D are continuous over the interval $[\frac{\rho+\delta_2}{1-\sigma}, \rho + \delta_1]$, D is hump shape and $F\left(\frac{\rho+\delta_2}{1-\sigma}\right) > 0 = D\left(\frac{\rho+\delta_2}{1-\sigma}\right)$.

Q.E.D.

D.3 Proof of Proposition 3

Recall that $\pi^{ELB} = \frac{(\psi-1)\pi^T - r^{*H}}{\psi}$ where the ELB constraint is binding when $\pi \leq \pi^{ELB}$ and it is non binding when $\pi > \pi^{ELB}$. π^{ELB} is increasing in ψ. Assume that the ELB constraint is not binding and hence the Taylor rule is given by $i_t = r^{*H} + \pi^T + \psi(\pi_t - \pi^T)$ with $\psi > 1$. Let's also recall the equilibrium dynamics for the economy with nominal wage rigidities is now governed by the following dynamic system

$$\dot{\pi}_t = \kappa(c_t + G - \bar{y})$$

$$\frac{\dot{c}_t}{c_t} = \frac{i_t - \pi_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q^s V_a(B, \Gamma_t)$$

$$\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}(i_t - \pi_t)\right]$$

In the steady state, $\dot{\pi}_t = 0$, $\dot{c}_t = 0$, and $\dot{\Gamma}_t = 0$. The $\dot{\pi}_t = 0$ curve is given by $c = \bar{y} - G$. The $\dot{\Gamma}_t = 0$ curve is $\Gamma = \left[\frac{\rho+\delta_2}{\sigma} + \frac{1-\sigma}{\sigma}\pi\right]^{-1}$, while the $\dot{c}_t = 0$ curve (together with $\dot{\Gamma}_t = 0$) is given by

$$c = (\delta_1 q^s)^{-1/\sigma} B \left[\rho + \delta_1 - r^{*H} - (\psi-1)(\pi - \pi^T)\right]^{1/\sigma} \left[\frac{\rho+\delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right)\left(r^{*H} + (\psi-1)(\pi - \pi^T)\right)\right]. \tag{D5}$$

Some properties of the $\dot{c}_t = 0$ curve. We denote the $\dot{c}_t = 0$ curve $F^c(\pi)$. When $\pi > \pi^{ELB}$, $c = 0$ if $\pi = \pi^T + \frac{\rho+\delta_1-r^{*H}}{\psi-1} > \pi^T$. The derivative of $F^c(\pi)$ with respect to π is

$$F^{c'}(\pi) = -(\delta_1 q^s)^{-\frac{1}{\sigma}} B \left[\rho + \delta_1 - r^{*H} - (\psi-1)(\pi - \pi^T)\right]^{\frac{1}{\sigma}-1} \left(\frac{\psi-1}{\sigma^2}\right) \left[\rho + \delta_2 - (1-\sigma)(r^{*H} + (\psi-1)(\pi - \pi^T)) + \sigma(1-\sigma)\left(\rho + \delta_1 - r^{*H} - (\psi-1)(\pi - \pi^T)\right)\right],$$

where $F^{c'}(\pi) = 0$ when $\pi^{opt} = \pi^T + \frac{(\rho+\delta_2)+(\sigma-1)((1+\sigma)r^{*H}-\sigma(\rho+\delta_1))}{(1-\sigma)(\sigma+1)(\psi-1)}$, that is,

$$\pi^{opt} = \pi^T + \frac{\bar{r} - r^{*H}}{\psi - 1},$$

If $\pi < \pi^{opt}$, $F^{c'} > 0$ and if $\pi > \pi^{opt}$, $F^{c'} < 0$. Hence, the $\dot{c}_t = 0$ curve is hump shaped in π with the optimal consumption being equal to $F^c(\pi^{opt}) = (\delta_1 q)^{-1/\sigma} B(\rho + \delta_1 - \bar{r})^{1/\sigma}\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}\bar{r}\right]$. The $\dot{c}_t = 0$ curve (given by F^c) and the $\dot{\pi}_t = 0$ curve are displayed in Figures 9 and 10.

Existence of steady equilibria. The steady state equilibrium is determined by the intersection of the $\dot{\pi}_t = 0$ and $\dot{c}_t = 0$ curves, that is

$$F^c(\pi) = \bar{y} - G.$$

A necessary condition for an equilibrium to exist is $0 < \bar{y} - G < F^c(\pi^{opt})$ which is satisfied when

$$B > (\delta_1 q^s)^{1/\sigma}(\rho + \delta_1 - \bar{r})^{-1/\sigma}\left[\frac{\rho + \delta_2}{\sigma} - \frac{1-\sigma}{\sigma}\bar{r}\right] \equiv \bar{B}.$$

It is helpful to consider two cases for the discussion of the equilibrium. In the first case, we assume that $F^c(\pi^{ELB}) < \bar{y} - G$ (see the proof of Proposition 5 in Section D.5 for the condition). We start with the scenario where $\pi > \pi^{opt}$ and F^c is strictly decreasing in π. Since $0 < \bar{y} - G < F^c(\pi^{opt})$ and F^c is decreasing, there is an equilibrium inflation $\pi_1 = F^{c-1}(\bar{y} - G)$. We denote this equilibrium E_1. We also consider the scenario where $\pi^{ELB} < \pi < \pi^{opt}$ and F^c is strictly increasing in π. Similarly, since F^c is strictly increasing there is a second equilibrium $\tilde{\pi} = F^{c-1}(\bar{y} - G)$ (see Section D.4 for the value of $\tilde{\pi}$). We denote this equilibrium $\tilde{E}_1$. The first case shows that there are two equilibria. In the second case where $F^c(\pi^{ELB}) > \bar{y} - G$, only equilibrium E_1 exists.

Real rate and inflation at equilibrium E_1. Are r^{*H} and π^T the real interest rate and inflation rate at E_1 respectively? To answer this question, first recall that the r^{*H} (in the model without nominal rigidities) is determined by the following equations

$$\begin{aligned}
&F(r) \equiv (\delta_1 q^s)^{-1/\sigma} B\left[\rho + \delta_1 - r\right]^{1/\sigma}\left[\frac{\rho + \delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right) r\right] = \bar{y} - G,\\
&r^{*H} = F^{-1}(\bar{y} - G) > \bar{r},\\
&F'(r) < 0 \quad \text{if} \quad r > \bar{r}.
\end{aligned}$$

Now note that $F(i - \pi^T) = F^c(\pi^T)$. Hence $i - \pi^T = F^{-1}(\bar{y} - G) = r^{*H}$ and the inflation rate at the equilibrium E_1 is $\pi_1 = \pi^T$. We also need to check whether r^{*H} is higher $\bar{r}$ in the present of nominal rigidities. We know that at the equilibrium E_1, $\pi^T > \pi^{opt}$ and $i - \pi^T < i - \pi^{opt}$. Using the definition of π^{opt}, we obtain $i - \pi^T = r^{*H} > \bar{r}$. Therefore, at the equilibrium E_1, inflation is at target $\pi = \pi^T$ and as a result the real interest rate is $r^{*H} > \bar{r}$.

Stability. The stability analysis of these two steady states is given by the following dynamic system:

$$\begin{pmatrix} \dot{\hat{\pi}}_t \\ \dot{\hat{c}}_t \\ \dot{\hat{\Gamma}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \kappa & 0 \\ \frac{(\psi-1)c}{\sigma} & J_{22} & J_{23} \\ -\Gamma(\psi-1)\left(\frac{1-\sigma}{\sigma}\right) & 0 & J_{33} \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{\Gamma}_t \end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x,

$$J_{22} = (\rho + \delta_1) - (r^{*H} + (\psi - 1)(\pi - \pi^T)),$$

$$J_{23} = [(\rho + \delta_1) - (r^{*H} + (\psi - 1)(\pi - \pi^T))]\frac{c}{\Gamma},$$

and

$$J_{33} = \frac{\rho + \delta_2}{\sigma} - \frac{1 - \sigma}{\sigma}(r^{*H} + (\psi - 1)(\pi - \pi^T)).$$

The determinant of the 3x3 Jacobian J is given by

$$det(J) = -\frac{\kappa(\psi - 1)c}{\sigma^2} \left[\rho + \delta_2 - (1 - \sigma)(1 + \sigma)(\psi - 1)(\pi - \pi^T) - (1 - \sigma)(1 + \sigma)r^{*H} + \sigma(1 - \sigma)(\rho + \delta_1)\right].$$

If $\pi > \pi^{opt}$, then $det(J) < 0$, implying that the steady state equilibrium E_1 is saddle stable since $det(J) = \lambda_1\lambda_2\lambda_3$ and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ have opposite signs. If $\pi < \pi^{opt}$, then $det(J) > 0$, meaning that the steady state with $\tilde{\pi}_1 < \pi^{opt}$ is unstable. Hence only one stable steady state equilibrium exists.

Q.E.D.

D.4 Proof of Proposition 4

The proof of this proposition is similar to the one of Proposition 3 except that now $\pi < \pi^{ELB}$ and $i_t = 0$. Let's recall the equilibrium dynamics for the economy is governed by the following dynamic system

$$\dot{\pi}_t = \kappa(c_t + G - \bar{y}),$$

$$\frac{\dot{c}_t}{c_t} = \frac{-\pi_t - \rho - \delta_1}{\sigma} + \frac{c_t^\sigma}{\sigma}\delta_1 q^s V_a(B, \Gamma_t),$$

$$\dot{\Gamma}_t = -1 + \Gamma_t\left[\frac{\rho + \delta_2}{\sigma} + \frac{1 - \sigma}{\sigma}\pi_t\right].$$

In the steady state, $\dot{\pi}_t = 0$, $\dot{c}_t = 0$, and $\dot{\Gamma}_t = 0$. The $\dot{\pi}_t = 0$ curve is given by $c = \bar{y} - G$. The $\dot{\Gamma}_t = 0$ curve is $\Gamma = \left[\frac{\rho+\delta_2}{\sigma} + \frac{1-\sigma}{\sigma}\pi\right]^{-1}$, while the $\dot{c}_t = 0$ curve (together with $\dot{\Gamma}_t = 0$) is represented by

$$c = (\delta_1 q^s)^{-1/\sigma} B\left[\rho + \delta_1 + \pi\right]^{1/\sigma}\left[\frac{\rho + \delta_2}{\sigma} + \left(\frac{1 - \sigma}{\sigma}\right)\pi\right] \equiv H^c(\pi). \tag{D6}$$

The $\dot{c}_t = 0$ curve given by H^c and the $\dot{\pi}_t = 0$ are displayed in Figures 9 and 10.

Some properties of $\dot{c}_t = 0$ curve. When $\pi < \pi^{ELB}$, $c = 0$ if $\pi = -(\rho + \delta_1)$. The derivative of $H^c(\pi)$ with respect to π is

$$H^{c'}(\pi) = (\delta_1 q^s)^{-\frac{1}{\sigma}} B \left[\rho + \delta_1 + \pi\right]^{\frac{1}{\sigma}-1} \left(\frac{1}{\sigma^2}\right) \left[\rho + \delta_2 + (1-\sigma)\pi + \sigma(1-\sigma)\left(\rho + \delta_1 + \pi\right)\right].$$

$H^{c'}(\tilde{\pi}^{opt}) = 0$ where

$$\tilde{\pi}^{opt} = \frac{-(\rho + \delta_2) - \sigma(1-\sigma)(\rho + \delta_1)}{(1-\sigma)(\sigma+1)} = -\bar{r}.$$

If $\pi < \tilde{\pi}^{opt}$, $H^{c'} > 0$ and if $\pi > \tilde{\pi}^{opt}$, $H^{c'} < 0$. Hence the $\dot{c}_t = 0$ curve is hump shaped in π with maximal being given by $H^c(\tilde{\pi}^{opt}) = (\delta_1 q^s)^{-1/\sigma} B(\rho + \delta_1 - \bar{r})^{1/\sigma} \left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}\bar{r}\right]$.[59] In this configuration, the condition $\tilde{\pi}^{opt} < \pi^{ELB}$ must hold since the ELB binds. This is satisfied if $\pi^T > -\bar{r}$ and $\psi > \frac{r^{*H}+\pi^T}{\bar{r}+\pi^T}$.

Existence of equilibria. The steady state equilibrium is determined by the intersection of the $\dot{\pi}_t = 0$ and $\dot{c}_t = 0$ curves, that is

$$H^c(\pi) = \bar{y} - G.$$

A necessary condition for an equilibrium to exist is $0 < \bar{y} - G < H^c(\tilde{\pi}^{opt})$ which is satisfied when $B > \bar{B}$.

It is useful to consider two cases for the discussion of the equilibrium. In the first case, we assume that $H^c(\pi^{ELB}) < \bar{y} - G$ (see the proof of Proposition 5 in Section D.5 for the condition). We start with the scenario where $\pi > \tilde{\pi}^{opt}$ and H^c is strictly decreasing in π. Since $0 < \bar{y} - G < H^c(\tilde{\pi}^{opt})$ and H^c is decreasing, there is an equilibrium inflation $\pi_2 = H^{c-1}(\bar{y} - G)$. We denote this equilibrium E_2. We also consider the scenario where $\pi^{ELB} < \pi < \tilde{\pi}^{opt}$ and H^c is strictly increasing in π. Similarly, since H^c is strictly increasing there is a second equilibrium $\pi_2' = H^{c-1}(\bar{y} - G)$. We denote this equilibrium $\tilde{E}_2$. The first case shows that there are two equilibria. In the second case where $H^c(\pi^{ELB}) > \bar{y} - G$, only equilibrium $\tilde{E}_2$ exists.

Real rate and inflation at equilibrium E_2. We now show that at the equilibrium E_2, inflation $\pi = -r^{*L}$ where the r^{*L} is the low real interest rate. To do so, first recall that the r^{*L} (in the model without nominal rigidities) is determined by the following equations

$$\begin{aligned}
&F(r) \equiv (\delta_1 q^s)^{-1/\sigma} B \left[\rho + \delta_1 - r\right]^{1/\sigma} \left[\frac{\rho + \delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right) r\right] = \bar{y} - G, \\
&r^{*L} = F^{-1}(\bar{y} - G) < \bar{r}, \\
&F'(r) > 0 \quad \text{if} \quad r < \bar{r}.
\end{aligned}$$

[59] Note that $H^c(\tilde{\pi}^{opt}) = F^c(\pi^{opt})$.

Now note that $F(-\pi) = H^c(\pi)$. Hence, in equilibrium $-\pi = F^{-1}(\bar{y} - G) = r^{*L}$. Since $\pi < \tilde{\pi}^{opt} = -\bar{r}$ we also have that $-\pi < \bar{r}$. As a result, in equilibrium E_2, inflation $\pi = -r^{*L}$.

Stability. The stability analysis of these two steady states is given by the following dynamic system:

$$\begin{pmatrix} \dot{\hat{\pi}}_t \\ \dot{\hat{c}}_t \\ \dot{\hat{\Gamma}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \kappa & 0 \\ -\frac{c}{\sigma} & J_{22} & J_{23} \\ \Gamma\left(\frac{1-\sigma}{\sigma}\right) & 0 & J_{33} \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{\pi}_t \\ \hat{c}_t \\ \hat{\Gamma}_t \end{pmatrix},$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x,

$$J_{22} = \rho + \delta_1 + \pi,$$

$$J_{23} = (\rho + \delta_1 + \pi)\frac{c}{\Gamma},$$

and

$$J_{33} = \frac{\rho + \delta_2}{\sigma} + \frac{1-\sigma}{\sigma}\pi.$$

The determinant of the 3x3 Jacobian J is given by

$$det(J) = \frac{\kappa c}{\sigma^2}\left[(1-\sigma)(1+\sigma)\pi + \rho + \delta_2 + \sigma(1-\sigma)(\rho + \delta_1)\right]$$

If $\pi > \tilde{\pi}^{opt}$, then $det(J) < 0$, implying that the steady state equilibrium E_2 is saddle stable since $det(J) = \lambda_1\lambda_2\lambda_3$ and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ have opposite signs. If $\pi < \tilde{\pi}^{opt}$, then $det(J) > 0$, meaning that the steady state with $\tilde{\pi}_2 < \tilde{\pi}^{opt}$ is unstable. Hence only the steady state equilibrium E_2 is stable.

Note to find $\tilde{\pi}$. At $\tilde{\pi}$, we must have $F^c(\tilde{\pi}) = H^c(-r^{*L})$. Rearranging this equation leads to

$$\tilde{\pi} = \pi^T + \frac{r^{*L} - r^{*H}}{\psi - 1},$$

where $\tilde{\pi} < \pi^{opt} = \pi^T + (\bar{r} - r^{*H})/(\psi - 1)$.

Q.E.D.

D.5 Proof of Proposition 5

This proof builds on the proofs of Propositions 3 and 4. We start by noting that at π^{ELB}, $F^c(\pi^{ELB}) = H^c(\pi^{ELB})$ where F^c and H^c are given by equations (D5) and (D6).[60]

[60] Note that at the π^{ELB}, the real interest rates are identical: $-\pi^{ELB} = r^{*H} + (\psi - 1)(\pi^{ELB} - \pi^T)$.

Recall that $\pi^{ELB} = \frac{(\psi-1)\pi^T - r^{*H}}{\psi}$ and is increasing in ψ.

From the proofs of Propositions 3 and 4, we know that the steady state equilibrium E_1 is always stable. For the second stable steady state equilibrium E_2 to exist, we must have $H^c(\pi^{ELB}) < \bar{y} - G$. Since $\pi^{ELB} > \tilde{\pi}^{opt} = -\bar{r}$, H^c is decreasing which implies that $\pi^{ELB} > H^{c-1}(\bar{y} - G)$. Using the definition of π^{ELB}, we obtain $\psi > \frac{r^{*H}+\pi^T}{\pi^T - H^{c-1}(\bar{y}-G)}$. Knowing from the proof of Proposition 4 that $H^{c-1}(\bar{y} - G) = -r^{*L}$ implies that

$$\psi > \frac{r^{*H} + \pi^T}{r^{*L} + \pi^T}.$$

Hence there exists a threshold[61] $\bar{\psi} \equiv \frac{r^{*H}+\pi^T}{r^{*L}+\pi^T} > 1$ such that if $\psi > \bar{\psi}$, there are two stable steady state equilibria given by E_1 and E_2. If $\psi < \bar{\psi}$ only equilibrium E_1 exists.

Q.E.D.

D.6 Proof of Proposition 6

This proof is also similar to the ones of Propositions 3, 4, and 5. The $\dot{c}_t = 0$ curve is

$$H^c(\pi; \delta_2) = (\delta_1 q)^{-1/\sigma} B \left[\rho + \delta_1 + \pi\right]^{1/\sigma} \left[\frac{\rho + \delta_2}{\sigma} + \left(\frac{1-\sigma}{\sigma}\right)\pi\right].$$

For given parameters, $H^c(\pi; \delta_2) \equiv \tilde{F}(\delta_2)$ is also increasing in δ_2 for any inflation rate π. At the low-real-rate, low-inflation equilibrium E_2, the function $H^c(\pi; \delta_2)$ is decreasing in π (since $\pi > \tilde{\pi}^{opt} = -\bar{r}$).

We start by assuming that π^{ELB} takes i^T as a given constant (that is, the definition of $i^T = r^{*H} + \pi^T$ is not taken into account), implying that $\pi^{ELB} = \frac{-i^T + \psi\pi^T}{\psi}$ is independent of δ_2.

For an equilibrium E_2 to exist in the decreasing part of H^c, the following relationship must hold:

$$H(\pi^{ELB}; \delta_2) \equiv \tilde{F}(\delta_2) \leq \bar{y} - G.$$

This implies that:

$$\delta_2 \leq \tilde{F}^{-1}(\bar{y} - G).$$

As a result, for sufficiently low δ_2, (i.e., δ_2 below a cutoff), the low-real-rate, low-inflation steady state (E_2) exists. The steady state equilibrium E_2 is also stable (see Section D.4).

When we take into account the definition of $i^T = r^{*H} + \pi^T$, we obtain that $\pi^{ELB} = \frac{-r^{*H} + (\psi-1)\pi^T}{\psi}$ and it is decreasing in δ_2 since $\frac{dr^{*H}}{d\delta_2} > 0$. That is, a decrease in δ_2 leads to

[61] Note that $\psi > \frac{r^{*H}+\pi^T}{r^{*L}+\pi^T} > \frac{r^{*H}+\pi^T}{\bar{r}+\pi^T} > 1$ since $r^{*L} < \bar{r} < r^{*H}$. This shows that when $\psi > \bar{\psi}$, both $H^c(\pi^{ELB}) > \bar{y} - G$ and $\pi < \pi^{ELB}$.

an increase in π^{ELB}, which in turn leads $H^c(\pi^{ELB})$ to fall.[62] This reinforces the result that a low-real-rate, low-inflation equilibrium exists only for sufficiently small δ_2.

Q.E.D.

D.7 Proof of Proposition 7

The proof is similar to the proof of Proposition 6. Recall that the inflation rate at the ELB stable steady state (E_2) is $-r^{*L}$. Since $r^{*L} < \bar{r}$ we have $\frac{dr^{*L}}{dB} < 0$. Therefore, the inflation rate $-r^{*L}$ at the equilibrium is increasing in government debt B.

First note that $H^c(\pi; B) \equiv \hat{F}(B)$ increases with B for any inflation rate π. It is also important to note that $\frac{\partial \hat{F}(B)}{\partial B} = \frac{\partial H^c(\pi^{ELB};B)}{\partial \delta_2} + \frac{\partial H^c(\pi^{ELB};B)}{\partial \pi^{ELB}} \frac{\partial \pi^{ELB}}{\partial B} > 0$ since $\pi^{ELB} > \tilde{\pi}^{opt}$, $\frac{\partial H^c}{\partial \pi^{ELB}} < 0$, $\frac{\partial \pi^{ELB}}{\partial B} < 0$, and $\frac{\partial r^{*H}}{\partial B} > 0$. Therefore, $\hat{F}(B)$ is strictly increasing in B.

For the ELB equilibrium (E_2) to cease to exist, the following relationship must hold:

$$H^c(\pi^{ELB}; B) \equiv \hat{F}(B) > \bar{y} - G.$$

This implies that:

$$B > \hat{F}^{-1}(\bar{y} - G) \equiv B^{cutoff}.$$

Consequently, when $B > B^{cutoff}$, the ELB equilibrium ceases to exist.

If $B < \bar{B}$, $\lim_{B \to B^{cutoff}}(\pi) = \pi^{ELB}$. At the cutoff B^{cutoff}, there is a discontinuity and the stable ELB equilibrium disappears.

Q.E.D.

D.8 Proof of Proposition 8

This proof is similar to the one of Proposition 3 in Section D.3. The equilibrium dynamics of the economy are the same except that now the Taylor rule does not satisfy the Taylor principle

$$i_t = \max\left\{0, r^{*H} + \pi^T + \psi(\pi_t - \pi^T)\right\} \qquad \psi < 1, \quad \pi^T > 0.$$

Consider the case where $i > 0$ (i.e., $\pi > \pi^{ELB}$). In the steady state, $\dot{\pi}_t = 0$, $\dot{c}_t = 0$, and $\dot{\Gamma}_t = 0$. The $\dot{c}_t = 0$ curve is given by

$$\begin{aligned} c = (\delta_1 q)^{-1/\sigma} B \left[\rho + \delta_1 - r^{*H} - (\psi - 1)(\pi - \pi^T)\right]^{1/\sigma} \\ \left[\frac{\rho + \delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right)(r^{*H} + (\psi - 1)(\pi - \pi^T))\right] \equiv F^{cc}(\pi). \end{aligned} \tag{D7}$$

[62] Note that $\frac{\partial \tilde{F}(\delta_2)}{\partial \delta_2} = \frac{\partial H^c(\pi^{ELB};\delta_2)}{\partial \delta_2} + \frac{\partial H^c(\pi^{ELB};\delta_2)}{\partial \pi^{ELB}} \frac{\partial \pi^{ELB}}{\partial \delta_2} > 0$ since $\pi^{ELB} > \tilde{\pi}^{opt}$, $\frac{\partial H^c}{\partial \pi^{ELB}} < 0$, $\frac{\partial \pi^{ELB}}{\partial \delta_2} < 0$, and $\frac{\partial H^c}{\partial \delta_2} > 0$.

Some properties of $\dot{c}_t = 0$ curve. We denote the $\dot{c}_t = 0$ curve $F^{cc}(\pi)$. When $\pi > \pi^{ELB}$, $c = 0$ if $\pi = \pi^T + \frac{\rho+\delta_1-r^{*H}}{\psi-1} < \pi^T$. For consumption to be positive, we must have $\pi > \pi^T + \frac{\rho+\delta_1-r^{*H}}{\psi-1}$. The derivative of $F^{cc}(\pi)$ with respect to π is

$$F^{cc'}(\pi) = -(\delta_1 q)^{-\frac{1}{\sigma}} B \left[\rho + \delta_1 - r^{*H} - (\psi-1)(\pi-\pi^T)\right]^{\frac{1}{\sigma}-1} \left(\frac{\psi-1}{\sigma^2}\right)$$
$$\left[\rho + \delta_2 - (1-\sigma)(r^{*H} + (\psi-1)(\pi-\pi^T)) + \sigma(1-\sigma)\left(\rho + \delta_1 - r^{*H} - (\psi-1)(\pi-\pi^T)\right)\right],$$

where $F^{cc'}(\pi) = 0$ when $\pi^{opt1} = \pi^T + \frac{\bar{r}-r^{*H}}{\psi-1} > \pi^T$. If $\pi < \pi^{opt1}$, $F^{cc'} > 0$ and if $\pi > \pi^{opt1}$, $F^{cc'} < 0$.

Existence of equilibria. The steady state equilibrium is determined by the intersection of the $\dot{\pi}_t = 0$ and $\dot{c}_t = 0$ curves, that is $F^{cc}(\pi) = \bar{y} - G$. A necessary condition for an equilibrium to exist is $0 < \bar{y} - G < F^{cc}(\pi^{opt1})$ which is satisfied when $B > \bar{B}$.

It is helpful to consider two cases for the discussion of the equilibrium. In the first case, we assume that $F^{cc}(\pi^{ELB}) < \bar{y} - G$ (see the proof of Proposition 5 in Section D.5 for the condition). We start with the scenario where $\pi > \pi^{opt1}$ and F^{cc} is strictly decreasing in π. Since $0 < \bar{y} - G < F^{c}(\pi^{opt1})$ and F^{cc} is decreasing, there is an equilibrium inflation $\pi_1 = F^{cc-1}(\bar{y} - G)$. We denote this equilibrium A_1. We also consider the scenario where $\pi^{ELB} < \pi < \pi^{opt1}$ and F^{cc} is strictly increasing in π. Similarly, since F^{cc} is strictly increasing there is a second equilibrium $\pi_1^{'} = F^{cc-1}(\bar{y} - G)$. We denote this equilibrium $\tilde{A}_1$. The first case shows that there are two equilibria. In the second case where $F^{cc}(\pi^{ELB}) > \bar{y} - G$, only equilibrium A_1 exists.

Stability. The stability analysis of these two steady states is identical to the Proposition 3 (see Section D.3). The determinant of the 3x3 Jacobian J is still represented by

$$det(J) = -\frac{\kappa(\psi-1)c}{\sigma^2}$$
$$\left[\rho + \delta_2 - (1-\sigma)(1+\sigma)(\psi-1)(\pi-\pi^T) - (1-\sigma)(1+\sigma)r^{*H} + \sigma(1-\sigma)(\rho+\delta_1)\right],$$

except that now $\psi < 1$. If $\pi > \pi^{opt} = \pi^T + \frac{r^{*H}-\bar{r}}{1-\psi}$, then $det(J) < 0$, implying that the steady state equilibrium A_1 is saddle stable since $det(J) = \lambda_1\lambda_2\lambda_3$ and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ have opposite signs. If $\pi < \pi^{opt}$, then $det(J) > 0$, meaning that the steady state with $\tilde{\pi}_1 < \pi^{opt}$ is unstable. Hence only one stable steady state equilibrium exists. Note that the unique (stable) steady state is such that $\pi > \pi^T + \frac{r^{*H}-\bar{r}}{1-\psi} > \pi^T$ since $\psi < 1$ and $r^{*H} > \bar{r}$. As ψ converges to 1, $\pi^T + \frac{r^{*H}-\bar{r}}{1-\psi}$ converges to infinity and consequently π goes to infinity.

Real interest rate at the stable steady state. We need show that r^{*L} is the real interest rate at the stable steady state equilibrium. To do so, first recall that r^{*L} (in the

model without nominal rigidities) is determined by the following equations

$$
\begin{aligned}
&F(r) \equiv (\delta_1 q)^{-1/\sigma} B \left[\rho + \delta_1 - r\right]^{1/\sigma} \left[\frac{\rho + \delta_2}{\sigma} - \left(\frac{1-\sigma}{\sigma}\right) r\right] = \bar{y} - G, \\
&r^{*L} = F^{-1}(\bar{y} - G) < \bar{r}, \\
&F'(r) > 0 \quad \text{if} \quad r < \bar{r}.
\end{aligned}
$$

Now note that $F(i - \pi) = F^{cc}(\pi)$. To show that the real interest rate $i - \pi$ at the stable steady state equilibrium is r^{*L}, we need to show that$i - \pi$ is less than $\bar{r}$. This is shown as follows:

$$
\begin{aligned}
&i - \pi < \bar{r}, \\
&r^{*H} + (\psi - 1)(\pi - \pi^T) < \bar{r}, \\
&\pi > \pi^T + \frac{r^{*H} - \bar{r}}{1 - \psi} = \pi^{opt1}.
\end{aligned}
$$

Consequently, at the stable equilibrium the real interest rate is $r^{*L} < \bar{r}$.

Q.E.D.

D.9 Proof of Lemma 2

Let's recall the dynamics of the optimization problem

$$
\dot{c}_t = \left(\frac{r_t - \rho - \delta_1}{\sigma_1}\right) c_t + \frac{c_t^{\sigma_1 + 1}}{\sigma_1} \delta_1 q a_t^{-\sigma_2} \Gamma_t^{\sigma_2},
$$

$$
\dot{a}_t = r_t a_t + w_t - T_t - c_t,
$$

$$
\dot{\Gamma}_t = -1 + \Gamma_t \left[\frac{\rho + \delta_2}{\sigma_2} - \frac{1 - \sigma_2}{\sigma_2} r_t\right].
$$

Linearizing this system around the steady state ($\dot{c}_t = 0$, $\dot{a}_t = 0$, and $\dot{\Gamma}_t = 0$) with $r_t = r$ leads to the dynamic system:

$$
\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{a}}_t \\ \dot{\hat{\Gamma}}_t \end{pmatrix} = \underbrace{\begin{bmatrix} \rho + \delta_1 - r & -\frac{\sigma_2}{\sigma_1}\frac{c}{a}(\rho + \delta_1 - r) & \frac{\sigma_2}{\sigma_1}\frac{c}{\Gamma}(\rho + \delta_1 - r) \\ -1 & r & 0 \\ 0 & 0 & \frac{\rho + \delta_2}{\sigma_2} - \frac{1 - \sigma_2}{\sigma_2} r \end{bmatrix}}_{J \text{ Jacobian evaluated at the steady state}} \begin{pmatrix} \hat{c}_t \\ \hat{a}_t \\ \hat{\Gamma}_t \end{pmatrix},
$$

where $\hat{x}_t \equiv x_t - x$ means the deviation of a variable x_t from its steady state x, and $\rho + \delta_1 - r = \delta_1 q c^{\sigma_1} a^{-\sigma_2} \Gamma^{\sigma_2}$.

The determinant of the 3x3 Jacobian J is given by

$$det(J) = (\rho + \delta_1 - r)\left(\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r\right)\left(r - \frac{\sigma_2}{\sigma_1}\frac{c}{a}\right)$$

If $r < \frac{\sigma_2}{\sigma_1}\frac{c}{a}$, then $det(J) < 0$, implying that the steady state is saddle stable since $det(J) = \lambda_1\lambda_2\lambda_3$ and the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ have opposite signs.

Combining $\rho + \delta_1 - r = \delta_1 q c^{\sigma_1} a^{-\sigma_2}\Gamma^{\sigma_2}$ and $\Gamma^{-1} = \frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r$ leads to

$$\frac{c}{a} = \left[\left(\frac{\rho+\delta_1-r}{\delta_1 q}\right)\left(\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r\right)^{\sigma_2}\right]^{\frac{1}{\sigma_1}} a^{1-\frac{\sigma_2}{\sigma_1}}.$$

Therefore, the stability condition is $r < \left[\left(\frac{\rho+\delta_1-r}{\delta_1 q}\right)\left(\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r\right)^{\sigma_2}\right]^{\frac{1}{\sigma_1}}\left(\frac{\sigma_2}{\sigma_1}\right)a^{1-\frac{\sigma_2}{\sigma_1}}$. This represents a necessary condition. A sufficient condition is

$$\max\{r, 0\} < \left[\left(\frac{\rho+\delta_1-r}{\delta_1 q}\right)\left(\frac{\rho+\delta_2}{\sigma_2} - \frac{1-\sigma_2}{\sigma_2}r\right)^{\sigma_2}\right]^{\frac{1}{\sigma_1}},$$

where $\max\{r, 0\}$ guarantees consumption to be non-negative.

Q.E.D.

D.10 Proof of Lemma 3

First, we examine the effects government debt B on r^{*L} and r^{*H}. Recall the Euler equation in the steady state which include the asset and goods market clearing conditions

$$\rho + \delta_1 - r = \delta_1 q c^{\sigma} B^{-\delta}\left[\frac{\rho+\delta_2}{\sigma} - \frac{1-\sigma}{\sigma}r\right]^{-\sigma}.$$

The derivative of this equation with respect to B is

$$\frac{dr}{dB} = \frac{-\sigma B^{-1}(\rho+\delta_1-r)(\rho+\delta_2-(1-\sigma)r)}{(1-\sigma)(1+\sigma)r + \sigma(\sigma-1)(\rho+\delta_1) - (\rho+\delta_2)}.$$

The numerator is negative, so the sign of $\frac{dr}{dB}$ depends on the denominator. Hence, we have:

$$\frac{dr^{*H}}{dB} > 0 \quad \text{if} \quad r^{*H} > \bar{r} \quad \text{and} \quad \frac{dr^{*L}}{dB} < 0 \quad \text{if} \quad r^{*L} < \bar{r}.$$

Second, we show how r^{*L} and r^{*H} change with δ_2: $\frac{dr}{d\delta_2} = \frac{-\sigma(\rho+\delta_1-r)}{(1-\sigma)(1+\sigma)r+\sigma(\sigma-1)(\rho+\delta_1)-(\rho+\delta_2)}$. Similarly we obtain that

$$\frac{dr^{*H}}{d\delta_2} > 0 \quad \text{if} \quad r^{*H} > \bar{r} \quad \text{and} \quad \frac{dr^{*L}}{d\delta_2} < 0 \quad \text{if} \quad r^{*L} < \bar{r}.$$

Q.E.D.