A Framework for Economic Growth with Capital-Embodied Technical Change

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Abstract

Technological advance is often embodied in capital inputs. This paper develops a model where capital innovations occur on two margins: (1) vertically, where a capital input becomes more productive at a given task; and (2) horizontally, where a capital input replaces labor at a given task. These two forms of technological advance engage in a macroeconomic "tug of war" when capital and labor have less than a unitary elasticity of substitution, and the resulting framework can meet numerous macroeconomic regularities. First, it can produce a balanced growth path and satisfy the Uzawa Growth Theorem – even though all technological progress occurs in capital inputs. Second, it can produce intuitive macroeconomic dynamics, adding perspectives on the apparent productivity slowdown and declining labor share of income. Third, it can produce rich industry dynamics and inform structural change, including declining GDP shares of agriculture and manufacturing, sectoral bottlenecks, the role of general purpose technologies, and the limited macroeconomic impacts of computing. Overall, this tractable framework can help resolve puzzling tensions between micro-level observations of technological advance and macroeconomic features of economic growth.

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1 Introduction

This paper provides a conceptual framework to address key tensions between microeconomic descriptions of technological progress and the macroeconomic features of economic growth. Specifically, the model connects three features that usually have trouble sitting together: (1) capital-embodied technical change; (2) balanced growth; and (3) a non-unitary elasticity of substitution between capital and labor. The model provides a tractable and intuitive approach for thinking about growth rates, income shares, industry dynamics and structural change, all based in forms of capital-embodied technical change. At the heart of the model is a surprise: a framework that can feature purely capital-embodied innovations at a micro level yet, at a macro level, purely labor-augmenting technological change.

To motivate this paper, consider first that major forms of technological progress appear to be embodied in physical capital. For example, advances in transportation have followed innovations in capital equipment (e.g., engines; also trains, automobiles, and airplanes). Advances in agriculture appear to follow capital-embodied innovations in machines (e.g., tractors, threshers, combine harvesters) and other non-labor inputs (e.g., seeds, fertilizers, and pesticides). Advances in manufacturing also appear in many capital-embodied forms (e.g., lathes, robots, photolithographic machines, 3D printers). Mokyr's classic *Lever of Riches* more generally studies productivity-enhancing innovations through human history, and it would be hard from this telling not to conclude that capital-embodied innovations are at the heart of technological progress (Mokyr 1990). Indeed, the major "general purpose technologies" (Bresnahan and Trajtenberg 1995, Helpman 1998) since the Industrial Revolution – engines, electrification, computing – are all embodied in capital equipment. See Figure 1.

While the importance of capital-embodied technical change seems self-evident, it creates surprising tensions for models of economic growth. To match the key stylized facts of a balanced growth path (Kaldor 1961), the Uzawa Growth Theorem shows that technological progress in the aggregate production function must be purely labor-augmenting (Uzawa 1961). Namely, write aggregate production as

$$Y_t = F(B_t K_t, A_t L_t) \tag{1}$$



Figure 1: Top: Agricultural laborers working with limited tools (left). An early tractor with a steam engine (middle). A modern combine harvester (right). A single Lexion 760 Terra Trac combine harvester in 2018 harvested 2.6 million kilograms of corn in 12 hours. Bottom: "Computers" at NASA in the 1950s (left). The IBM 7090 computer in 1958 (middle). The 10^{-12} decline in the cost of computer operations per second from Nordhaus (2001). (right).

where Y_t, K_t, and L_t are aggregate output, capital, and labor, respectively. The Uzawa Growth Theorem requires the capital-augmenting technology term to be constant, $B_t = B$, with per-capita income growth following only from an increasing labor-augmenting technology term, A_t. This result is surprising, even paradoxical, given that technological advance at a micro level appears substantially embodied in capital and yet a balanced growth path seems a reasonable description of the economy as a whole.

The traditional "fix" in growth models has been to utilize a Cobb-Douglas aggregate production function, a special case where capital-augmenting technological advance can be recast as labor-augmenting and balanced growth is indifferent to the source of technological advance. However, this special case raises additional tensions. Namely, there is substantial evidence for a non-unitary elasticity of substitution between capital and labor, and especially for an elasticity that is less than 1 (e.g., Antras 2004, Chirinko 2008, Oberfield and Raval 2021, Grossman et al. 2021). Further,

a Cobb-Douglas approach locks the labor share of income to an exogenous parameter, limiting its capacity to inform pertinent contemporary issues, including the apparent decline in the labor share (e.g., Elsby et al. 2013, Karabarbounis and Neiman 2014, Dao et al. 2017).

This paper works to resolve these tensions, developing a conceptual framework where technological advance can be embodied in capital inputs at a micro level yet match the macroeconomic regularities. The key idea is that there are two types of capital-embodied technological advance. Both margins operate in a task-based model of the economy. The first margin lies in automating activities or tasks that were formerly performed by labor (Zeira 1998, Acemoglu and Restrepo 2018, 2020). The second margin lies in improving the productivity of capital at the already-automated tasks. Ultimately the economy depends on two technology state variables. The first, β_t, is the share of tasks in the economy that are automated (i.e., performed by capital inputs). The second, Z_t, is an index of productivity across these capital inputs. So, for example, a new application of computers, replacing labor at a task, causes β_t to go up, while faster computers cause Z_t to go up. Ultimately the capital share proves to be a remarkably simple outcome based on these two technology indices:

$$Capital\ Share = \beta_t Z_t^{-1} \tag{2}$$

which situates the key intuition. Automation causes the capital share to rise (Acemoglu and Restrepo 2018, 2020). However, advances in the productivity of the capital inputs cause the capital share to fall (Aghion et al. 2019). This latter effect follows from the final key to the model: that the elasticity of substitution between capital and labor is less than 1.[1] In this context, advances in productivity of a given input cause its price to fall relatively quickly so that its share in GDP declines. Thus the two technology indices end up pushing in opposite directions with regard to the capital share and can lead to balanced growth.[2]

The approach builds on recent advances in the growth literature. The first advances are models of automation, which have extended how we conceptualize technological progress (e.g., Zeira 1998,

[1]The elasticity of substitution between capital and labor is debated (e.g., Chirinko 2008, Karabarbounis and Neimann 2014), but the balance of the empirical literature appears to favor an elasticity of substitution less than 1 (e.g., reviews by Hamermesh 1993 and Chirinko 2008; Antras 2004, Oberfield and Raval 2021, among others). See also Grossman et al. (2021).

[2]The possibility that a balanced growth path might emerge along these lines was first suggested, to our knowledge, in Aghion, Jones, and Jones (2019).

Acemoglu and Restrepo 2018). In contrast to the classic vertical approaches where innovations are modeled as directly productivity enhancing (i.e., a technology term weighting the quantity of the input, or the cost of producing the input), automation allows us to consider a horizontal form of innovation where one input (capital) replaces another (labor). It is the combinination of the classic view and the automation view that makes the model in this paper work. A second new strand of literature, which focuses more precisely on the Uzawa Theorem, introduces human capital into the production function in a way that holds the capital-augmenting term constant along the balanced growth path (Grossman et al. 2017, 2021). In the Grossman et al. approach, there is capital-augmenting technological progress, but human capital in a sense eats this physical capital with the result that the capital-augmenting term B_t can remain constant as both capital technology advances and human capital deepens.

This paper relates to both ideas, but with distinct foundations, forces, and intuitions, and a wide set of applications and results. First, at the aggregate level, the two forms of capital advancement acts as neutralizing forces in the aggregate production function. In particular, when aggregating from the task level up to total output, the technology indexes enter the aggregate production function in the form

$$Y_t = F((\beta_t Z_t^{-1})^{\theta})K_t, (1-\beta_t)^{\theta} L_t) \tag{3}$$

where $\theta < 0$, which further demonstrates key intuition. Essentially the two margins of technical progress are in a "tug of war" with each other in the aggregate. Balanced growth emerges when advances in automation (e.g., computers take over more tasks) and advances in the productivity of capital inputs (e.g., computers get faster) proceed at the same rate. This holds the capital share constant in (2), and, as we see in (3), meets the key Uzawa condition that the capital-augmenting term be constant in the production function to achieve balanced growth. Further, while these two capital-technology indices neutralize each other in the overall capital technology index, the balanced growth path is still driven by the capital-embodied improvements, as labor focuses on and increases production in the remaining non-automated tasks and capital deepens overall.

Second, in addition to being able to meet the requirements of the Uzawa Growth Theorem, this tractable framework provides novel and intuitive perspectives for how technology dynamics and economic dynamics can relate. For example, a decline in the labor share follows naturally if

automation accelerates or vertical improvements slow. Further, a slowdown in vertical technological progress leads both to a labor share decline and a growth slowdown, providing a potentially straightforward conceptual linkage between the observation that innovation may be "getting harder" (e.g., Jones 1995, Jones 2009, Bloom et al. 2020) and recent U.S. growth and income share observations.

Third, viewing different intervals of the economy's tasks as representing different sectors, capital automation and capital quality advances (and the tug-of-war between them) can occur differently within different sectors, allowing for distinct industry dynamics and structural change along a steady growth path. This process occurs in line with Baumol's cost disease, where high-productivity sectors tend to shrink as a share of GDP and laggard sectors grow as a share of GDP (Baumol 1967, Aghion et al. 2019). For example, if agriculture and manufacturing see relatively rapid technological advance compared to services, then the GDP share of agricultural outputs and manufacturing outputs will go down and the GDP share of services will go up. Yet the capital share in agriculture or manufacturing need not go up or down compared to services, and the capital share in the economy can remain steady. Thus this model can speak to structural change as well as balanced growth.

As another application, take computers. Moore's Law, which has increased floating-point operations per second by 10^{11} since World War II, is often seen as an essential technological force of our age, and yet overall productivity growth has been modest, and perhaps puzzlingly so (Solow 1987). A Baumol perspective can naturally help: Capital-embodied tasks (e.g., floating point operations) that advance rapidly become a smaller share of GDP, and growth becomes more determined by productivity at other, bottleneck tasks. Thus, the very success of Moore's Law engenders its ultimate impotence in the aggregate production function, if it only applies to a limited set of tasks (see also Aghion et al. 2019). However, at the same time, and unlike agriculture, computing equipment has not become a smaller share of GDP. Rather ICT capital equipment investment tends to sustain at high levels in advanced economies (OECD 2021). The model suggests that this sustained GDP share occurs through increasing automation: if computers are simultaneously taking over more tasks (e.g., search, machine control, artificial intelligence, etc.). This increasing breadth of tasks undertaken by computers can balance computer quality advances

and sustain computers' GDP share. The model may then also address both the power and the limits of computing, among other general purpose technologies.

This paper proceeds as follows. Section 2 develops the baseline model with exogenous technological advances on both the automation and productivity margins and considers conditions for a balanced growth path. Section 3 uses this model to examine broader technology dynamics and economic outcomes, with applications to dynamics in the overall growth rate and income shares, structural change, and general purpose technologies. Section 4 then develops an endogenous growth model, where R&D investment and rates of progress on both technology margins are choices by firms and a balanced growth path emerges based on fundamental economic parameters. Additional applications are then discussed, including further interpretations for the apparent productivity slowdown and declining labor share of income. Section 5 concludes.

2 Baseline Growth Model

Here we present the growth model with exogenous technological progress. We first lay out the assumptions regarding production and preferences, and consider equilibrium based on consumer and firm optimization. We then consider a balanced growth path. Namely, we will show the existence of a BGP – even though all technological progress is embodied in capital goods and the elasticity of substitution is less than 1.

2.1 Production Technology

The production technology features a unit interval of tasks, $i \in [0,1]$. Final output Y_t is given as

$$Y_t = \upsilon \left[\int_0^1 y_t(i)^\rho di \right]^{1/\rho}, \ \rho < 0 \tag{4}$$

where $y_t(i)$ is an intermediate good. By assumption, $\rho < 0$, so that intermediates are gross complements. The price of final output is the numeraire.

Intermediate good production is as follows. Each intermediate good can be produced with labor. However, intermediate goods on the interval $i \in [0, \beta_t]$ can also be produced by capital.

The production possibilities for the intermediates are

$$y_t(i) = \begin{cases} Al_t(i) & \text{for all } i \in [0,1] \\ z_t(i)^{\frac{\rho-1}{\rho}} x_t(i) & \text{for all } i \in [0,\beta_t] \end{cases} \tag{5}$$

where $l_t(i)$ is a labor input, $x_t(i)$ is a capital input, and $z_t(i)$ is a capital-input specific productivity term. The capital input can be made at a marginal cost of ψ units of the final good. Following standard vertical growth models (e.g., Aghion and Howitt 1992), we assume capital inputs depreciate fully with their use, and we will normalize the cost of machine production such that $\psi = \upsilon$. The exponent on the capital-productivity term is for notational convenience (and is positive, since $\rho < 0$).

Capital-embodied technology in this model is thus described on two key margins. First, there is the share of tasks, β_t, that have been automated. Second, there is the productivity, $z_t(i)$, of each automated task. Note as well that we are fixing the labor productivity via the constant A. This feature can easily be relaxed, but we fix labor productivity here to emphasize that all technological progress in this model can be embodied in capital – and yet we will have a balanced growth path. Figure 2 depicts the two capital-embodied technology frontiers, in productivity and automation, in this model.

Finally, it will be useful to summarize the capital-productivity terms using the index

$$Z_t = \left[\frac{1}{\beta_t}\int_0^{\beta_t} \frac{1}{z_t(i)} di\right]^{-1} \tag{6}$$

which is the harmonic average of the capital-embodied technology terms.

2.2 Preferences

A representative household has (CRRA) preferences,

$$U(t) = \int_t^{\infty} \frac{c(\tau)^{1-\theta} - 1}{1-\theta} e^{-\omega(\tau - t)} d\tau \tag{7}$$

where ω is the discount rate and $\theta \geq 0$. A household supplies one unit of labor inelastically to production, earning a wage rate w_t. Households earn wages as well as earn any capital income, holding a balanced portfolio of the firms in the economy. We assume the usual transversality condition. Consumption will proceed according to the Euler equation

$$g_c = \frac{1}{\theta}(r_t - \omega) \tag{8}$$

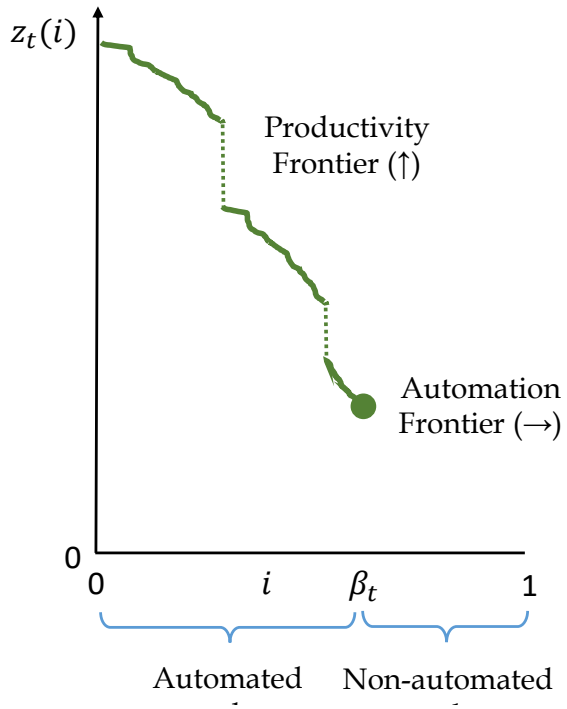


Figure 2: The two technology frontiers in the model.

2.3 Resource Constraints

The total supply of labor is L_t, where

$$L_t = \int_0^1 l_t(i)di \tag{9}$$

and L_t grows at rate n. The total investment cost for capital is

$$I_t = \int_0^1 \psi x_t(i)di \tag{10}$$

Output is either consumed or used to make capital inputs, giving the economy-wide constraint

$$I_t + C_t = Y_t$$

where $C_t = L_t c_t$ is total consumption.

2.4 Firm Optimization

Competitive final goods firms maximize profits using the final goods production function (4) and taking prices as given. These producers' demand for intermediate goods will thus take the

form

$$p_t(i) = \upsilon^{\rho} \left(\frac{Y_t}{y_t(i)} \right)^{1-\rho} \tag{11}$$

Intermediate firms maximize profits using the production possibilities (5), taking intermediate output prices, $p_t(i)$, and input prices for labor and capital, as given.

2.4.1 Non-Automated Tasks

Given the intermediate production technology (5) and competitive markets, the intermediate price for any non-automated task is

$$p_t(i) = w_t / A \tag{12}$$

Intermediate prices are thus the same across all labor-performed tasks. From (11), it then follows that $y_t(i)$ is the same for these labor-performed tasks. We will be interested in the equilibrium where automated tasks (on the interval $i \in [0, \beta_t]$) will indeed use capital inputs, so that labor input are used only on tasks $i \in (\beta_t, 1]$. Anticipating this feature, and using the labor resource constraint (9) we then have the equilibrium labor allocation

$$l_t(i) = \frac{L_t}{1-\beta_t},\ i \in (\beta_t, 1] \tag{13}$$

and thus the intermediate outputs

$$y_t(i) = \frac{AL_t}{1-\beta_t},\ i \in (\beta_t, 1] \tag{14}$$

for labor-produced tasks.

2.4.2 Automated Tasks

For tasks $i \in [0, \beta_t]$, firms can use capital inputs or labor inputs. For the exogenous growth model, these intermediate producers will be competitive and earn zero profits. If firms use capital inputs, the prices are

$$p_t(i) = \psi z_t(i)^{\frac{1-\rho}{\rho}},\ i \in [0, \beta_t] \tag{15}$$

To focus on the equilibrium of interest, where these firms do indeed use capital inputs, we require as a technical condition that automated-production is lower cost than labor-based production for automated tasks; i.e.,

$$w_t \geq \psi A z_t(i)^{\frac{1-\rho}{\rho}} \tag{16}$$

for all automated technologies $i \in [0, \beta_t]$. We will validate this technical condition later.

For automated tasks, the equilibrium capital allocation to each task is, using (11), (15), and the production technology (5),

$$x_t(i) = \upsilon^{-1} \frac{Y_t}{z_t(i)},\ i \in [0, \beta_t], \tag{17}$$

the intermediate outputs are

$$y_t(i) = \upsilon^{-1} z_t(i)^{-\frac{1}{\rho}} Y_t,\ i \in [0, \beta_t], \tag{18}$$

and the GDP share for a given automated task is

$$\frac{p_t(i)\, y_t(i)}{Y_t} = \frac{1}{z_t(i)},\ i \in [0, \beta_t]. \tag{19}$$

Thus productivity advances in a given automated task cause its output to go up but its GDP share to go down. This effect is part of Baumol's cost disease, where the GDP share falls for outputs the economy becomes especially good at producing.

2.5 Aggregation

A key feature of the model is how automated intermediates aggregate. Using the capital resource constraint, (10), and the equilibrium capital allocation to each task, (17), we have

$$I_t = Y_t \beta_t Z_t^{-1}$$

where we recall that Z_t is the harmonic average of the tasks-specific productivity terms (see (6)).

The capital share is then

$$s_{K_t} = \frac{\psi X_t}{Y_t} = \beta_t Z_t^{-1} \tag{20}$$

and the labor share is

$$s_{L_t} = \frac{w_t L_t}{Y_t} = 1 - \beta_t Z_t^{-1} \tag{21}$$

in terms of the exogenous technology variables.

Meanwhile, aggregate output, (4), follows from summing across the equilibrium intermediate outputs. Equilibrium aggregate output is

$$Y_t = \frac{\upsilon A}{\left(1 - \beta_t Z_t^{-1}\right)^{1/\rho}} \left(1 - \beta_t\right)^{\frac{1-\rho}{\rho}} L_t \tag{22}$$

in terms of the exogenous variables.

Equilibrium wages are, from (21) and (22)

$$w_t = \upsilon A \left(1 - \beta_t Z_t^{-1}\right)^{\frac{\rho-1}{\rho}} \left(1 - \beta_t\right)^{\frac{1-\rho}{\rho}} \tag{23}$$

and consumption per capita is the same as wages, $c_t = w_t$.[3]

2.6 Technology Adoption

Finally, recall that we require automated technologies to be sufficiently productive to be adopted. This condition is (16). Given the equilibrium wage, the technology assumption takes the form

$$z_t(i) \geq z_t^{\min} = \frac{1 - \beta_t}{1 - \beta_t Z_t^{-1}} \tag{24}$$

for all $i \in [0, \beta_t]$, where $z_t^{\min}$ is the minimum productivity level at which an automated task produces the intermediate output at lower cost than labor. Note that, since the index Z_t is the harmonic average and therefore must be greater than its minimum possible value, $Z_t \geq z_t^{\min}$, it then follows directly from (24) that

$$Z_t \geq 1 \tag{25}$$

along the economy's equilibrium path. Note that the adoption condition (24) thus also guarantees that $\beta_t Z_t^{-1} \leq 1$, so that the capital share in (20) is well defined.

2.7 The Balanced Growth Path

We now focus on a balanced growth path (BGP). The BGP is defined as the equilibrium that can match the Kaldor facts. Namely, aggregates Y_t, K_t, and C_t grow at a constant rate, while the interest rate r_t and the capital share of income are constant.

Looking at the aggregate results above, it is apparent that there are two technology conditions for the BGP. First, the BGP requires a constant capital (or labor) share. From (20), we therefore require

$$\beta_t Z_t^{-1} = s_{K_t} = s \tag{BGP1}$$

[3] In this exogenous growth model there are no profits and capital depreciates fully in use. Thus gross investment and capital income are equivalent, and consumption thus tracks wage income.

Here we see directly the role of the two forms of technological advance. The capital share is rising in the fraction of sectors that are automated (β_t) and declining in an index of these sectors' productivity (Z_t). These offsetting pressures are exactly why this model can maintain a constant capital share – even though all technological progress is embodied in capital. Note that the capital-share-reducing role of capital-embodied productivity gains (Z_t) follows because $\rho < 0$. That is, because intermediate outputs are gross complements, advancing productivity in a sector causes the GDP share of that sector to decline. Thus the BGP features declining GDP shares of each automated sector. Yet because the share of automated tasks (β_t) is rising, the total share of capital in GDP can remain constant. We will discuss this intuition further below, in light of the Uzawa Theorem.

Second, looking at (22), we see that growth in GDP per capita, g, will also take a simple form. With a constant capital share, $g = [(1-\rho)/\rho]\, g_{1-\beta_t}$, where $g_{1-\beta_t}$ is the rate of growth in $1-\beta_t$. The BGP thus requires a second technology condition. Steady-state growth occurs when the fraction of non-automated sectors is declining at a constant rate. We write this condition as

$$g_{1-\beta_t} = -q^h \tag{BGP2}$$

where the h superscript denotes the "horizontal" nature of this innovation, a process of automating tasks. We use a negative sign to emphasize that the fraction of tasks that are not automated, $1-\beta_t$, is decreasing as the automation advances. While we will use "horizontal" as an evocative shorthand, note that this mechanism corresponds closely to models of automation and much less to the features and intuition of love-of-variety growth models.

A natural way to think about the condition (BGP2) is that automation is getting harder as it proceeds. The fewer tasks that remain to be automated, the smaller the measure that are successfully automated each period. Anticipating the endogenous growth model, steady-state growth rates will follow from a simple process where innovators seek to automate non-automated tasks and succeed with probability q^h.

Under the two BGP technology conditions, (BGP1) and (BGP2), the steady-state growth rate of the economy is therefore

$$g = \frac{\rho - 1}{\rho} q^h \tag{26}$$

Putting the pieces together, on a balanced growth path the automation rate determines the steady-state growth rate, while the advance of productivity on automated tasks acts to neutralize the effect of automation on the capital share.[4]

We can encapsulate the technology conditions for balanced growth and the balanced growth path as follows.

Condition 1 *Let β_t and Z_t grow at the same rate (BGP1). Let $1-\beta_t$ decline at the rate q^h (BGP2). Further, let the $z_t(i)$ be sufficiently large to satisfy the technology adoption condition (24), and let $\omega - n > (1-\theta)\frac{\rho-1}{\rho}q^h$ to satisfy the transversality condition.*

Proposition 1 *(Balanced growth path). Under condition 1, a balanced growth path exists where the growth rate in per-capita output, per-capita consumption, the per-capita capital stock, and the wage are $g = \frac{\rho-1}{\rho}q^h$. The capital share is $s_{Kt} = s$. The discount rate is $r = \omega + \theta\frac{\rho-1}{\rho}q^h$.*

Proof. See Appendix. ■

2.7.1 Balanced Growth and the Uzawa Theorem

To gain further insight into the balanced growth path and how the model satisfies the Uzawa steady-state growth theorem, it is helpful to write aggregate output in the form $Y_t = F(B_tK_t, A_tL_t)$. The Uzawa Theorem tells us that, for a balanced growth path with a non-unitary elasticity of substitution between labor and capital, all technological progress must be labor augmenting. That is, we require the labor-augmenting technology term, A_t, to grow at a steady-state rate. Yet the capital-augmenting technology term, B_t, must be constant.

In the model, we can rewrite aggregate output as

$$Y_t = \upsilon\left[\left(\left(\beta_t/Z_t\right)^{\frac{1-\rho}{\rho}}\upsilon^{-1}K_t\right)^{\rho} + \left(\left(1-\beta_t\right)^{\frac{1-\rho}{\rho}}AL_t\right)^{\rho}\right]^{1/\rho} \tag{27}$$

This formulation shows us directly how the two technology conditions above produce a balanced growth path. First, we see that the condition $\beta_t Z_t^{-1} = s$ has the effect of making the capital-augmenting technology term constant. Second, the condition $g_{1-\beta} = -q^h$ has the effect of making

[4]The technology index Z_t is an aggregation of the $z_t(i)$ (see (6)), and we will consider specific innovation processes for $z_t(i)$ that produce a balanced growth path in Z_t further below.

the labor-augmenting technology term grow at a constant rate. In this way, although all technological advance is embodied in capital, aggregate production appears to have purely labor-augmenting technological progress and we can satisfy the Uzawa Theorem (and match the Kaldor facts). A key distinction between this approach to a BGP and others in the literature is that, in equilibrium, the production function above does not become Cobb-Douglas. Rather, we continue to have an elasticity of substitution between capital and labor that is less than 1.

2.7.2 Balanced Growth and Technology Limits

The central feature driving the balanced growth path is the behavior of the two forms of technological progress. These technology pathways are also interesting compared to standard theories and intuitions. Namely, the technology indices, Z_t and β_t, are not growing at exponential rates on the balanced growth path. Instead, they are approaching finite limits. Yet, despite these limits, steady-state growth in per-capita income can continue forever.

Specifically, the automation limit occurs as $\beta_t \to 1$, at which point all tasks would be automated. Balanced growth occurs, via (BGP2), not when β_t is growing at a constant rate but rather when $1 - \beta_t$ is shrinking at a constant rate: when there is proportional decline in the share of non-automated tasks each period. The process recalls Zeno's Paradox, where one takes an infinite number of steps yet never completes a journey of finite length. The remaining measure, $1 - \beta_t$, of non-automated tasks shrinks at rate q^h each period. But in finite time the economy never reaches full automation.

Further, along a BGP, the automation technology limit is inherited by the capital-quality index Z_t. Namely, if $\beta_t \to 1$, then the first BGP condition, (BGP1), implies that $Z_t \to 1/s$. Thus the capital-quality index also approaches a limit – even as steady-state growth in per-capita income continues forever and the capital share does not change. Further, the Z_t index approaches its limit through proportional decline in its remaining distance to that limit on the BGP.[5] Such vertical limits may be appealing if one believes that there is some kind of fishing out process at work in innovation, or if there are natural laws, like Carnot's efficiency maximum, that put finite limits on efficiency. Yet, again, steady-state growth continues forever despite these limits.

[5]The first BGP condition, $\beta_t/Z_t = s$, implies that β_t and Z_t have the same growth rates, which in turn implies that they approach their limits at the same proportional rate.

The model thus has a "limited innovation, limitless growth" feature. In a sense, one can be an extreme techno-pessimist and still see steady-state growth. Indeed, the growth rate in Z_t and β_t fall along the BGP and limit to zero. Nonetheless, the capacity for growth is preserved, intuitively, because automation relieves labor of having to do so many tasks. Labor can focus on a proportionally smaller share of tasks, and thus produce proportionally more of each, while the capital stock can grow with GDP and deploy across an ever widening fraction (β_t) of tasks. Effectively, even if technological progress in the overall quality index of capital inputs slows, we can increasingly leverage labor and allow ever-increasing capital deepening with regard to labor, driving continual and steady growth in output per capita through the use of machines.

We encapsulate these technology pathways on the balanced growth path in the following corollary.

Corollary 1 *(Limited innovation, limitless growth). On the balanced growth path, the technology indices shrink at constant rates,* $g_{1-\beta_t} = g_{1/s-Z_t} = -q^h$*, but grow at shrinking rates,* $g_{\beta_t} = g_{Z_t} = \frac{1-\beta_t}{\beta_t} q^h$*, and approach finite limits,* $\beta_t \to 1$ *and* $Z_t \to 1/s$*.*

Proof. See appendix. ■

This slowdown in the progress of the aggregate index Z_t on a BGP may seem restrictive or difficult to manage at a micro level. However, this BGP condition does not imply that any particular capital-embodied technologies face a slowdown and, as we will see next, one can deploy a more standard micro-level quality ladder approach within this framework.

2.7.3 Balanced Growth and Task-Level Technological Advance

The description of the balanced growth path has so far focused on aggregates and the BGP trajectories of β_t and Z_t. Here we focus on the task-level productivities, $z_t(i)$, that constitute the aggregate technology index Z_t. Specifically, we examine task-level technology pathways that can produce BGP behavior in Z_t.

Generally, the evolution of Z_t can be understood as being driven by two forces: productivity advances on already-automated lines and the initial productivity of newly-automated lines. In

practice, there are many ways one can proceed at the task-level so that a balanced growth path emerges. Using the definition (6), we can rewrite (BGP1) as

$$\int_0^{\beta_t} z_t(i)^{-1} di = s \tag{28}$$

Thus one can consider any innovation processes for which this is true. Natural processes of technological advance will see $z_t(i)$ increase with time on existing, automated lines. Meanwhile, newly automated technologies will add new initial productivity levels. The balanced growth path is possible when the productivities aggregate as in (28). To the extent that technology proceeds differently from (28), the economy will experience capital share dynamics and deviate from the BGP (which we will discuss in Section 3).

As a specific innovation process, consider a standard vertical innovation model as seen in the endogenous growth literature. In particular, let an innovation on an existing automated line, $i \in [0, \beta_t]$, increase that line's productivity proportionally to $(1+\phi) z_t(i)$. Let such an innovation occur with hazard rate q^v, where the superscript v denotes the "vertical" nature of such innovation. To satisfy (28), one then needs the new automation to enter as follows.

Case 1 *(A micro-innovation process for balanced growth). Let existing tasks increase their productivity proportionally by an amount ϕ with hazard rate q^v. Then a BGP will occur so long as newly automated tasks have productivity*

$$z_t(\beta_t) = h(1-\beta_t) \tag{29}$$

where $h \geq \frac{1}{1-s}$ and $s = \frac{q^h}{q^v \phi h}$.

Proof. See appendix. ■

This corollary provides one set of sufficient conditions for innovation at the task level, allowing the Z_t index to evolve to produce the BGP. In particular, the corollary tells us how newly automated technologies draw their productivities, when existing automated tasks follow a standard quality ladder model.

In this micro specification, the basic idea is that one automates the easy things first. Tasks that are harder to automate (they are automated later) enter initially with lower productivity. Specifically, in (29), the initial quality of the newly automated technologies is falling along the

growth path, tracking $1-\beta_t$. To further understand this case, note that wages are rising on the growth path (also tracking along $1-\beta_t$). Thus labor becomes an increasingly expensive production approach, which creates room for these new, lower-productivity automation technologies to still be adopted. The parametric condition $h \geq \frac{1}{1-s}$ ensures that newly-automated technologies are productive enough to be adopted given the equilibrium wage. The capital share in this technology process is $s = \frac{q^h}{q^v \phi h}$.

While the vertical innovation approach above is one way to satisfy (28) and generate a BGP in the exogenous growth model, it also sets up a tractable approach for an endogenous growth model. We will return to endogenous growth in Section 4 to further analyze balanced growth.

2.8 Unbalanced Growth

Stepping back from balanced growth, the model more generally develops analytic solutions for all the endogenous quantities and prices in the economy. Firm and consumer optimization produce these solutions, so that the results can speak to the relationship between technological change and economic dynamics, both on and off a balanced growth path. For example, the path of GDP is determined analytically in (22) in terms of exogenous variables and evolves according to the development of β_t and Z_t. We can encapsulate the equilibrium for more general sets of technology pathways (i.e., without assuming balanced growth) with a weak set of technology conditions as follows.

Condition 2 *Let the $z_t(i)$ be sufficiently large to satisfy the technology adoption condition (24), and let $\omega - n > (1-\theta)\overline{g}_\infty$ to satisfy the transversality condition.*

Proposition 2 *(Unbalanced growth path). Under condition 2, an equilibrium exists with a capital share $s_{K_t} = \beta_t Z_t^{-1}$ and explicitly determined paths of output, consumption, investment, the wage, the interest rate, and all other prices and quantities in the economy.*

Proof. See Appendix. ■

Studying more general relationships between technology dynamics and economic outcomes will be the focus of Section 3.

2.9 Summary

The key features of the model follow from having two margins of capital-embodied technological advance, creating a balance of forces that drive the trajectory of the economy. We have an automation process, with state variable β_t, featuring horizontal innovation as new tasks are automated. We separately have a capital-quality process, with state variable Z_t, encapsulating vertical innovation and the evolving productivity of the automated tasks. Thus, the model can have new capital goods (such as farming equipment, robots, or computers) taking over human tasks, while also having improvement in the capital goods over time (better farming equipment, better robots, or better computers). In the tandem of these forces, balanced growth can emerge. In effect, having two margins of capital-embodied technological progress allows the model to overcome the tension between capital-embodied technological progress, a non-unitary elasticity of substitution, and balanced growth. In addition to satisfying the Uzawa theorem and providing a balanced growth path, the model also provides closed-form solutions to study the economy off of a balanced growth path and thus engage a broad range of phenomena, which we turn to next.

3 Technology Dynamics and Economic Dynamics

The model allows for rich economic dynamics based in technological progress. We consider several such dynamic phenomena. Specifically, we develop further intuitions and applications, where technological progress proceeds in a potentially unbalanced manner. We first examine income shares and growth rates, showing how the two technological forces can provide a novel technological view of recent U.S. economic growth and income share dynamics. We then consider structural change, showing how the model may inform dynamics in sectoral GDP shares, such as the economic transformation from manufacturing and agriculture to services. Finally, we consider the role of a "general purpose technology" from the perspective of the model and the implications of extreme technological advance through computing.

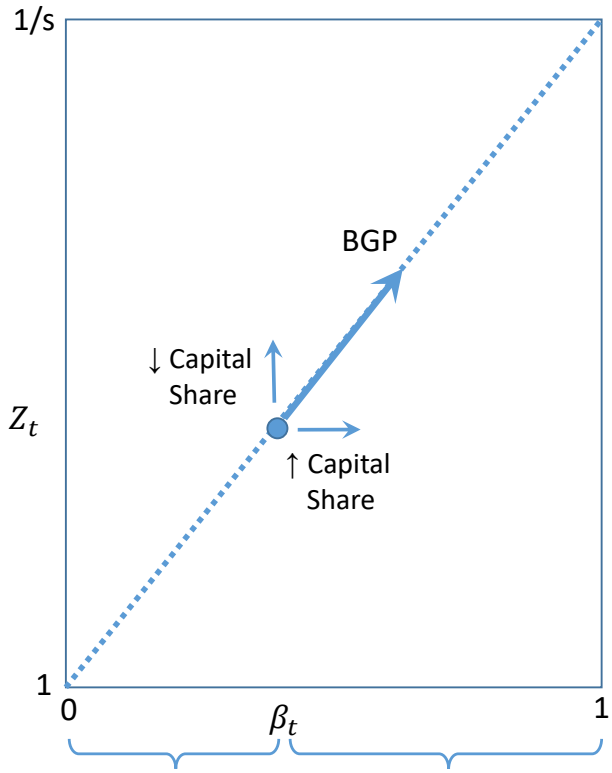


Figure 3: Technology dynamics on and off the balanced growth path.

3.1 Income Shares and Growth Dynamics

Consider first dynamics in the labor share of income and the growth rate. The model provides closed-form solutions for income shares and growth outside a balanced growth path, with the equilibrium results for these outcomes given in (20) and (22)). The income share dynamics are encapsulated in the following corollary.

Corollary 2 *(Labor share dynamics). The labor share is decreasing with time if* $g_{\beta_t} > g_{Z_t}$*, constant if* $g_{\beta_t} = g_{Z_t}$ *and increasing with time if* $g_{\beta_t} < g_{Z_t}$*.*

Proof. The result follows by inspection of (21). ■

The model thus presents two possible technological stories for a declining labor share in recent decades. First, the labor share will decline if technological progress in automation accelerates. Second, the labor share will decline if technological progress in capital productivity decelerates. See Figure 3. Put simply, one form of capital-embodied technological progress (vertical productivity

gains) is good for the labor share, but the other form of capital-embodied technological progress (increasing automation) is not.

To the extent that there has been a concomitant growth slowdown, this can sharpen the technological picture.

Corollary 3 *(Growth dynamics). The growth rate in income per-capita increases in g_{β_t} and g_{Z_t}.*

Proof. See appendix. ■

While still open to measurement debates, taking the common description of the U.S. economy as experiencing declines in both (i) the labor share and (ii) the growth rate, the model points toward a decline in g_{Z_t} as a force that can deliver both phenomena at once. Slowing growth in Z_t can in turn follow from declining innovation rates on newly automated lines or, alternatively, from new forms of automation that are of relatively low productivity.

Somewhat more subtly, while Z_t is growing on a balanced growth path, outside a balanced growth path the index Z_t may actually decline in equilibrium as automation advances. This can lead to a sharp decline in the labor share. In particular, newly automated technologies will be adopted so long as they are marginally lower cost than using labor, and the introduction of low productivity automation technologies can drag down Z_t (the harmonic average of the capital productivities). Further, if innovation is becoming harder in the sense that vertical improvements in existing automated technologies are small, or rare, then there is little pushing up on the productivity index and the newly-automated technologies becomes the central force in driving the evolution of Z_t. In this sense, weak vertical innovation rates and weakly successful automation technologies (meaning just productive enough to be adopted) can result in especially anemic gains from labor's perspective.[6]

We can formalize a "worst case" (for labor) automation-led style of growth as follows, where output per capita grows at a constant rate through automation but wages stagnate.

Corollary 4 *(Automation-led growth with constant wages) Let automation proceed at some rate $q^h > 0$ where newly automated technologies have productivity level z_t^{min}, the lowest level of pro-*

[6] See also Acemoglu and Restrepo (2019) for related discussion.

ductivity where they will still be adopted. Let prior automated technologies see no productivity improvement. Then wages remain constant. Income per-capita grows at rate q^h, and the labor share falls at rate q^h. The technology index Z_t declines at rate $q^h \frac{1-Z_t}{\beta_t} < 0$.

Proof. See appendix. ■

In this corollary, we have not only turned off vertical advances on existing lines but we have made newly-automated technologies barely good enough to be adopted. The result is a kind of weak but sustained growth in income per capita, with none of the gains going to labor. This result may be especially interesting as a technological context to consider recent dynamics in the U.S. economy.

3.1.1 Application to U.S. Growth and Income Share Series

Having established the theoretical results and their intuitions above, we can further consider an application of the model to U.S. macroeconomic data. In particular, one may use the output and labor share results, (22) and (21), to pin down β_t and Z_t each period to match the observed output and labor share series.[7] For this application we will use standard data series for U.S. labor productivity (output her hour) and the U.S. labor share of income from 1947-2019 (Bureau of Labor Statistics 2021a, 2021b). While the recent productivity growth slowdown and the recent decline in the labor share, which appear in these series, are subjects of ongoing measurement debates, here we take these data series as given and see how the model would explain these patterns.

Figure 4a presents the macro data series. Figure 4b presents the estimated technology paths.[8] These paths are presented as $1-\hat{\beta}_t$ and $c-\hat{Z}_t$, which is useful for seeing balanced growth behavior and deviations from this behavior. The estimation results suggest striking shifts in the nature of technological progress. First is an era of relatively stability. Horizontal and vertical innovation proceeded at broadly similar rates from the late 1940s into the early 1990s, consistent with broadly balanced growth.[9] Second, starting around the mid-1990s, we see episodes of decoupled technologi-

[7] As this model features full employment, among other simplifications, matching business cycles goes beyond what the model is meant to do. Longer-trends at full employment are more naturally within the conceptual grasp.

[8] The appendix provides further details on this estimation.

[9] The 1970s-era slowdown in labor productivity growth is reflected in slowdowns on both the vertical and horizontal technology dimensions so that, while the growth rate declined, the labor share of income remained relatively steady.

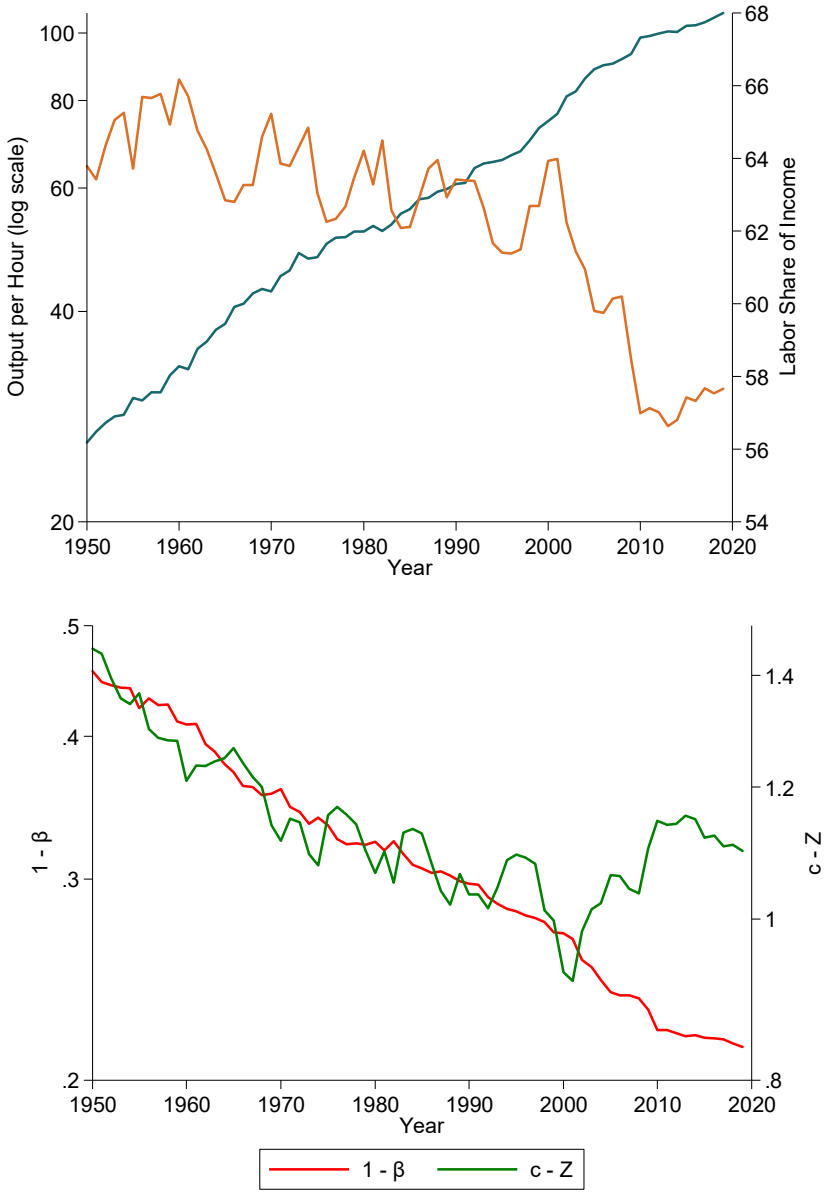


Figure 4: U.S. Macro Dynamics and the Two Technologies. Top: Output per hour and the labor share of income. Bottom: Estimated paths of automation and the capital productivity index.

cal progress and unbalanced growth, which continue until around 2010. Third, after 2010, we again see parallel but slower progress in both forms of innovation. This most recent period is consistent with innovation getting harder in a general sense while also maintaining a relatively stable labor share. Of course, the labor share is structurally lower after 2010 than in the latter half of the 20th century, which follows from highly unusual behavior from 2000-2010. This period follows the broad logic of Corollary 4: automation (in fact, a burst of automation) but a reversal in the capital productivity index, consistent with the entrance of new automation technologies that are not much lower cost than labor. The result is anemic growth, and especially anemic gains for labor.

3.2 Structural Change

The growth model can also be applied to consider structural change. Here we will consider how industry GDP shares and capital shares may evolve. For example, we can divide the economy into agriculture, manufacturing, and services.[10] The dynamics in these sectors can then follow from differential rates of automation and capital productivity advances within these sectors and provide a novel perspective on structural change.[11]

To formalize such ideas, one can label subsets of tasks as representing different sectors. Specifically, let there be J sectors indexed $j \in [1, J]$. Let each sector have a measure $u_t^j \leq 1$ of tasks, which divide up all the tasks in the economy so that $\sum_j u_t^j = 1$. Further, let the measure of automated tasks in sector j be $\beta_t^j \leq u_t^j$, which sum to the overall automated task share, $\sum_j \beta_t^j = \beta_t$. Finally, let each sector have a technology index Z_t^j, which is the harmonic average of the $z_t(i)$ in that sector.

It is now straightforward, by summing up output and capital investment within a sector, to write down the GDP share of the sector, as well as the labor share of income within that sector.

[10] We organize the analysis into these three broad sectors following long-standing orientations in studies of structural change (e.g., Fisher 1939, Kuznets 1966).

[11] The "two technologies" approach we develop here is distinguished from other perspectives in the vast literature on structural change, including those that emphasize non-homethetic preferences and income effects (e.g., Matsuyama 1992, Buera and Kaboski 2012, Matsuyama 2019); models that emphasize productivity shifts in the context of price-taking behavior (e.g., Hansen and Prescott 2002); and more general orientations in development economics that link development with structural change and engage frictions that inhibit development (e.g., Lewis 1954, Harris and Todaro 1970, Banerjee and Newman 1998).

In particular,

$$\frac{Y_t^j}{Y_t} = \beta_t^j \left(Z_t^j\right)^{-1} + s_{L_t} \frac{u_t^j - \beta_t^j}{1 - \beta_t} \tag{30}$$

$$s_{L_t}^j = 1 - \beta_t^j \left(Z_t^j\right)^{-1} \frac{Y_t}{Y_t^j} \tag{31}$$

where Y_t^j / Y_t is the GDP share of the sector and $s_{L_t}^j$ is the labor share of income within the sector.

With these expressions, we can consider the influence of technological advance on the sectoral GDP shares and income shares.[12]

Corollary 5 *(Structural change). Holding other sector technology levels fixed, the GDP share of a sector will decline with its automation level, β_t^j, or capital productivity level, Z_t^j. The labor share of income within the sector will decrease in the automation level, β_t^j, but increase in the capital productivity level, Z_t^j.*

Proof. See appendix. ■

These findings regarding GDP shares are intuitive and follow the basic logic of Baumol's cost disease. To the extent that sectors like agriculture and manufacturing deploy increasingly productive capital equipment relative to services, then the GDP shares of agriculture and manufacturing will decline. Similarly, sectors that are relatively highly automated will also see their GDP shares decline. The capital shares within the sectors, by contrast, follow opposing dynamics. Higher capital productivity lowers the sector's capital share while higher automation increases it.

3.2.1 Application to U.S. Structural Transformation

Having established these sectoral-level results and intuitions, we can once again consider an application of the model to U.S. macroeconomic data. In particular, we may describe each sector by the technology vector $\{u_t^j, \beta_t^j, Z_t^j\}$, representing the sector's share of tasks, automation rate, and

[12] On the growth path of the economy, technological advance may naturally be occurring in all sectors simultaneously. We may therefore also consider a variant of this corollary that emphasizes the relative evolution of sectors along a balanced growth path. That formulation produces similar results and intuition as Corollary 5, with technological advance now expressed in relative terms across sectors along a balanced growth path. That formulation and result are provided in the appendix as Corollary 5a.

capital-productivity index, where all sectors aggregate to the economy-wide measures $\{1, \beta_t, Z_t\}$. Using (30) and (31), we can then use data series for the sectoral GDP share and within-sector labor share of income to estimate $\hat{\beta}_t^j$ and $\hat{Z}_t^j$.

Figure 5 provides an illustrative application, focusing on three sectors: manufacturing, agriculture, and "other private sector" (which is mostly comprised of services). For data series, we use Mendieta-Munoz et al. (2020).[13] Figure 5 (top row) presents data for sectoral GDP shares and the labor shares of income within each sector. Figures 5 (bottom row) presents, for each sector, the estimated paths of automation and capital productivity. For these estimates, we determine the tasks shares using SIC and NAICS industrial coding schemes, as discussed in the appendix.

Looking at the raw data, Figure 5 shows a familiar picture of structural change. Agriculture (already at a low level) and especially manufacturing show declining GDP shares with time while the GDP share elsewhere rises. Further, we see that manufacturing exhibits a sharp decline in the labor share of income over the 1980-2010 period. By contrast, the labor share in other sectors is steadier (though much lower in agriculture and rising there to some degree).

Turning to the estimated technology paths, consider automation first. We see a durable rank ordering – agriculture is the most automated, followed by manufacturing, followed by services - even as each sector is becoming more automated with time. Notably, manufacturing is catching up to the high automation share in agriculture and automation proceeds especially quickly in manufacturing over the 1980-2010 period. This burst of automation is consistent with the declining labor share of income in manufacturing over this period.

Examining the capital productivity indices, we see that Z_t^{agr} exhibits much greater advance than the other sectors. This acts toward reducing the GDP share of agriculture and increasing the labor share in agriculture. By contrast, the advancing automation in manufacturing comes with weak and even retrograde movement in Z_t^{man} after 1980, which can further inform the declining labor share of income in this sector. Meanwhile, we see anemic growth in Z_t^{other}. Expanding service

[13] Mendieta-Munoz et al. (2020) provide value-added output and labor shares of income for 14 sectors. Their approach drops the public sector and housing sector. Here we study (1) manufacturing, (2) agriculture, forestry, and fishing; and (3) other, which includes arts, entertainment and recreation; construction; education, health and social services; finance and insurance; information; mining; retail trade and wholesale trade; transportation and warehousing, professional and business services; utilities; and other services.

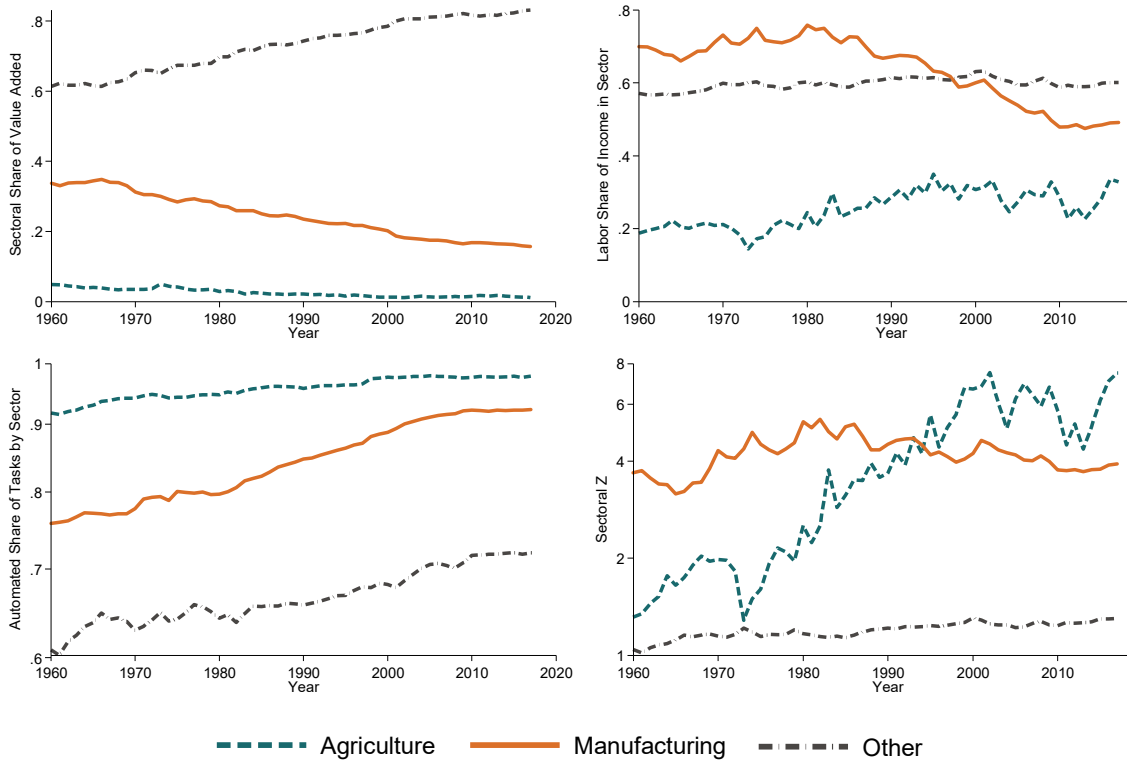


Figure 5: U.S. Structural Change and Technological Change. Top: Raw data for sectoral GDP shares (left) and labor shares of income in sector (right). Bottom: Estimated share of automated tasks within sector (left) and capital productivity index for sector (right). Sectoral data come from Mendieta-Munoz et al. (2020), where "other" is largely the service sector.

automation has not been coupled with new technologies that are substantially more productive than labor. The rising GDP share of services follows in part by this relatively anemic progress. Of course, as services become the increasingly dominant share of the economy, technological progress in this sector increasingly dominates the path of the economy overall. The relatively weak progress in this sector historically and recently thus suggests potentially enduring growth challenges.

3.3 General Purpose Technologies and Computing

As a related application to structural change, consider the GDP share of a general purpose technology (GPT) – for example, computers. Here we think of a type of capital input that takes on a widening variety of tasks, making it "general purpose." This can be new automation taking over from labor. For example, the word "computer" once referred to a worker type but now refers to a machine (see Figure 1). The general purpose capital input might also take over from other capital inputs; for example, computer storage can take over from printed books.

In the model, we can define a general purpose technology as a type of capital input that automates a measure β_t^{GPT} of tasks. This measure will increase as the GPT takes over more tasks. But then there are also advances in productivity, $z_t(i)$, for these general purpose capital inputs. These productivity gains will generally occur in heterogeneous ways across tasks. For example, the productivity at some computer-performed tasks (such as floating point operations per second) has risen at remarkable rates, while computer automation at other tasks (such as voice transcription) has shown more modest productivity improvements. The productivity index across the GPT tasks, Z_t^{GPT}, will then naturally grow (though determined relatively strongly by its lower-productivity tasks) as the GPT simultaneously extends its automation footprint. The GPT's share of GDP is then simply,[14]

$$\frac{Y_t^{GPT}}{Y_t} = \frac{\beta_t^{GPT}}{Z_t^{GPT}} \tag{32}$$

reflecting again the broad idea of a "tug of war" between automation and productivity. For example, here computers grow as a share of GDP as they take over more tasks, but the rising productivity of computers reduces their share of GDP.

[14]This follows directly by aggregating the capital inputs in (17) over a measure β_t^{GPT} of tasks performed by the GPT. Alternatively, using the lens of the structural change analysis above, we can define a tuple $\{u_t^{GPT}, \beta_t^{GPT}, Z_t^{GPT}\}$ where $u_t^{GPT} = \beta_t^{GPT}$ is the evolving measure of tasks performed by the GPT, and use (30).

The computer application is perhaps especially worthy of consideration both because of its current relevance and because Moore's Law presents an especially remarkable and continuous improvement in certain productivity metrics. In fact, the growth slowdown often seems in tension with the extraordinary progress in computing and its many applications. As a final theoretical result, it is then useful to consider the limited power of seemingly amazing technologies in this model. In particular, recall from (6) that Z_t is the harmonic average of the $z_t(i)$. Recall also that harmonic averages are heavily weighted toward their smallest values. For example, the harmonic average of a finite number and infinity is twice the finite number. Thus, in this model, the productivity at any particular task or finite measure of automated tasks can advance enormously (e.g., via Moore's Law) and even go to infinity. However, Z_t (the harmonic average) will still remain below any finite positive limit so long as the productivity for some measure of automated tasks remains finite. This feature follows from the less-than-unitary elasticity of substitution (see also Aghion et al. 2019). Rather than having the highest-productivity sectors take over the economy, these sectors become smaller shares of the economy.

Formally, we can encapsulate the limited effect of extreme productivity at any particular measure of automated tasks, as follows.

Corollary 6 *(Extreme technological advance). Let a fraction α of the automated tasks have the same distribution of $z_t(i)$ as the other automated tasks. Holding other technologies constant, take $z_t(i) \to \infty$ for this fraction α of automated tasks. The capital share will decline by α percent. Income per capita will increase by $\Delta \ln(y_t) = -\frac{1}{\rho} \ln\left(1 + \alpha \frac{s_{K_t}}{s_{L_t}}\right)$.*

Proof. See appendix. ■

So, for example, if $\alpha = 10\%$ of automated tasks, and these suddenly became infinitely productive, then a capital share of, say, $s_K = .40$ would fall to .36. Taking an elasticity of substitution between capital and labor of 0.5 ($\rho = -1$), which is a typical value in the literature, then income per-capita would increase by 6.5%. Thus, even extreme (infinite!) advances in productivity for substantive shares of the economy's tasks, which would appear dramatic for the affected sectors, would have substantially muted macro effects. This suggests a possible view on the limited effects of computing advances and Moore's Law at an aggregate level.

4 Endogenous Growth Model

The baseline model of Section 2 can solve the challenge of the Uzawa Growth Theorem while also engaging rich industrial and macroeconomic dynamics. It is less clear, however, why a balanced growth path may tend to emerge amidst the tug of war between automation and capital quality improvements. And, more generally, the pathways of automation and capital productivity improvements are not determined inside the baseline model. In this section we extend the baseline model to allow for endogenous growth.

To model endogenous technological progress, we follow the standard set-up (Romer 1990, Aghion and Howitt 1992) of introducing profits into the intermediate goods sector and letting these profits pay for R&D. Thus we will edit the baseline growth model by (1) introducing market power on intermediate capital goods and (2) introducing knowledge production functions that relate innovation outcomes to input costs of R&D. A point of difference with standard endogenous growth models is that we now have both vertical innovations, increasing $z_t(i)$, and horizontal innovations, increasing β_t. For simplicity, we set $L_t = L$, a constant.

4.1 Capital Good Producers

As before, producers of final goods and non-automated intermediate goods remain competitive. However, producers of automated goods now have market power. We follow the vertical growth literature (Aghion and Howitt 1992), where the firm with the leading technology is a monopolist (supported, e.g., by an infinitely lived patent) but faces a competitive fringe that accesses a lower-productivity vintage of technology. While we will closely follow this standard vertical approach, some distinctions will occur here because $\rho < 0$.[15] Specifically, the limit price imposed by the competitive fringe always binds as the profit-maximizing price set by the leading firm. Thus, we replace the equilibrium price (15) in the baseline model with

$$p_t(i) = \psi z_t^e(i)^{\frac{1-\rho}{\rho}}, \; i \in [0, \beta_t] \tag{33}$$

where $z_t^e(i)$ is the technology accessed by the competitive fringe.

[15] This means that the final good producers' demand, given by (11), differs from the traditional Cobb-Douglas case.

Innovation steps are proportional increases of size ϕ in the prior technology level. We assume the competitive fringe accesses a level of technology

$$z_t^e(i) = \gamma z_t(i)$$

where $\gamma \in [\frac{1}{1+\phi}, 1]$.[16] This competitive fringe pins down the equilibrium price

$$p_t(i) = \psi \left[\gamma z_t(i)\right]^{\frac{1-\rho}{\rho}}, \; i \in [0, \beta_t] \tag{34}$$

which is the limit price used by the monopolist firm with the leading technology. Given this price, the demand for the good $y_t(i)$ is, using (11),

$$y_t(i) = \upsilon^{-1}\gamma^{-\frac{1}{\rho}} z_t(i)^{-\frac{1}{\rho}} Y_t, \; i \in [0, \beta_t]. \tag{35}$$

The innovator will fulfill this quantity demanded at the limit price, but using their new technology. This determines the scale of the new innovator's production, defining the $x_t(i)$. Setting demand equal to supply for $y_t(i)$ we have

$$x_t(i) = \upsilon^{-1}\gamma^{-\frac{1}{\rho}} \frac{Y_t}{z_t(i)}, \; i \in [0, \beta_t]. \tag{36}$$

And now there are profits. The flow profit for the new innovator is

$$\pi_t^v(i) = \mu \frac{Y_t}{z_t(i)} \tag{37}$$

where we define $\mu = \gamma^{-1} - \gamma^{-\frac{1}{\rho}}$, capturing the limit-price (markup) effect.

4.2 Innovation Process

Resources are now devoted endogenously to advance R&D. The resource constraint of the economy is

$$C_t + I_t + D_t = Y_t$$

where the new term, D_t, represents the total expenditure on R&D.

[16] Tying the competitive fringe's technology to exactly the prior technology vintage, $\gamma = \frac{1}{1+\phi}$ is a potential simplification. Here we decouple the markup from the technology step size, so that market power and technology step size may be distinct forces. We consider the competitive fringe technology to be in the interval $\gamma \in \left[\frac{1}{1+\phi}, 1\right]$, with the idea that the fringe imitates the leading technology to some extent.

There are two different types of innovations, corresponding to the margins of technology in the model. First, there are vertical innovations, where innovating firms follow a quality-ladder, improving technologies that have already been automated. Second, there are horizontal innovations, where innovating firms automate new tasks. We consider each in turn.

4.2.1 Vertical Innovation

An intermediate firm may invest in vertical research in a (stochastic) attempt to raise a given productivity, $z_t(i)$, by a proportional amount ϕ. The flow value of profits from such an innovation is given by (37). While the technology level, $z_t(i)$, is fixed for a given vintage and patent, this flow profit is growing as the scale of the market grows; i.e., at the growth rate of the economy, g_t. The firm will lose these profits through creative destruction when another firm innovates along this line, which occurs with arrival rate $q_t^v(i)$. Defining the present value of an innovation as $V_t(i)$, we have the Bellman equation,

$$r_t V_t(i) - \dot{V}_t(i) = \pi_t^v(i) - q_t^v(i) V_t(i) \tag{38}$$

We will examine the balanced growth path equilibrium where g_t, r_t, and $q_t^v(i)$ are all constants, and the value of an innovation along the BGP can thus be written

$$V_t(i) = \frac{\pi_t^v(i)}{r + q^v - g} = \frac{\mu}{r + q^v - g} \frac{Y_t}{z_t(i)} \tag{39}$$

The value of an innovation is a function of the task's market size. The market size for the task is increasing on the growth path as the overall economy expands. However, the intermediate good's market size is decreasing when its productivity rises.[17]

Entrant firms perform R&D with increased investment raising the arrival rate of innovation.[18] We specify a lab equipment model that features diminishing returns to effort but positive intertemporal spillovers on a given line. Namely, investing $d_t^v(i)$ in R&D at task i will generate an arrival rate of new ideas of

$$q_t^v(i) = \xi^v \left(\frac{z_t(i) d_t^v(i)}{Y_t} \right)^\alpha \tag{40}$$

[17] Note that the latter effect is only operable across rungs of the quality ladder - the $z_t(i)$ is fixed from the perspective of the leading firm.

[18] Following standard vertical models (Aghion and Howitt 1992), entrant firms do all the R&D and seek to displace the incumbent firm.

where ξ^v is a measure of how easy innovation is in the vertical direction and $\alpha < 1$.[19]

The zero profit entry condition in R&D implies that

$$q_t^v(i)V_t(i) = d^v(i)$$

so that the expected value of R&D effort is equated to its costs in equilibrium. Given the R&D technology, (40), and the value of an innovation on the BGP, (39), this equilibrium entry condition on the balanced growth path becomes

$$\frac{\xi^v \mu}{r + q^v - g} = \left(\frac{z_t(i)d_t^v(i)}{Y_t}\right)^{1-\alpha}, \ i \in [0, \beta_t) \tag{41}$$

On a BGP, it follows that q^v is a constant. In particular, from (41), the ratio $z_t(i)d_t^v(i)/Y_t$ is a constant on a BGP, so that more advanced lines (which have smaller shares of GDP and hence smaller market size) attract less effort in equilibrium. From (40), this leads to a constant hazard rate of vertical advance across automated lines.

4.2.2 Horizontal Innovation

The other innovation margin is horizontal, where innovators seek to automate currently non-automated sectors. We allow R&D to be conducted on any currently non-automated line. Success occurs with arrival rate q_t^h. Further, a new automation technology must come with some initial technology level, which we write $z_t(\beta_t)$. We assume that the initial automation quality follows the process

$$z_t(\beta_t) = h(1 - \beta_t) \tag{42}$$

for some constant h. This assumption (which is familiar from Section 2.7.3) says that the initial automation quality is decreasing over time, as β_t rises. One can interpret this feature as saying that production lines automated later are the harder ones: they take longer to successfully automate and their starting productivity will be lower.[20]

[19] We specify a lab-equipment model rather than a labor input model as it is slightly more tractable. In the lab equipment model, expenditure on R&D is normalized by GDP in (40). This will lead to a constant GDP share being spent on R&D. This specification is similar to relying instead on R&D labor, $l_t^v(i)$, and the knowledge production function $q_t^v(i) = \xi^v(z_t(i)l_t^v(i))^\alpha$. With labor paid the prevailing wage, this formulation also produces a constant R&D expenditure share of GDP and a balanced growth path. This variation is available from the authors upon request.

[20] We will consider below the condition on h that guarantees the newly automated technology will be adopted. In particular, although the initial productivity of a newly automated line is dropping along the growth path, and limits

Once a horizontal research investment succeeds, the newly-automated task becomes a vertical line, and further innovation proceeds in the same vertical manner as for previously automated tasks, described above. Specifically, we assume there is a markup based on a technology advantage γ, which defines the flow profits from this innovation as in (37).[21] The present value of this new innovation, $V_t(i)$, will then again be as in (38) and (39). Creative destruction occurs when further, vertical R&D improves on this newly-automated technology.

On the cost side, the horizontal knowledge production function will follow a similar structure of R&D costs as seen vertically, in (40), except that we allow the cost parameter to differ. Specifically, horizontal R&D investment, $d_t^h(i)$, which targets a non-automated line, $i \in (\beta_t, 1]$, will generate an arrival rate for a new automation of

$$q_t^h(i) = \xi^h \left(\frac{z_t(i) d_t^h(i)}{Y_t} \right)^{\alpha} \tag{43}$$

where ξ^h is a measure of how easy innovation is in the horizontal direction. The zero-profit entry condition is then,

$$q^h(i) V_t(i) = d_t^h(i).$$

On the balanced growth path, this entry condition becomes

$$\frac{\xi^h \mu}{r + q^v - g} = \left(\frac{z_t(i) d_t^h(i)}{Y_t} \right)^{1-\alpha}, \; i \in (\beta_t, 1]. \tag{44}$$

On the BGP, it follows that q^h is a constant. As with the vertical logic above, from (44), the ratio $z_t(i) d_t^h(i) / Y_t$ is a constant on a BGP. From (43), this leads to a constant hazard rate of automation among the non-automated lines.

We can now collect several results for R&D in the following lemma.

Lemma 1 *The vertical and horizontal hazard rates of innovation, q^v and q^h, are both constants on a BGP and have the ratio*

$$\frac{q^h}{q^v} = \left(\frac{\xi^h}{\xi^v} \right)^{\frac{1}{1-\alpha}}. \tag{45}$$

to zero as $\beta \to 1$, wage growth in the economy means that these increasingly low-quality initial technologies will still be cost-effective compared to labor. We show this below.

[21] That is, we assume that the newly automated production tasks engender a competitive fringe that can access a technology a proportion γ worse than the newly automated technology. This assumption enhances tractability by creating symmetry with the vertical case, but is not essential.

Aggregate R&D expenditure in the vertical direction and in the horizontal direction are both constant shares of GDP.

Proof. See appendix. ■

The relative innovation rate in the horizontal and vertical dimensions (q^h/q^v) is thus determined by the relative ease of innovating in these directions (ξ^h/ξ^v) and the degree of diminishing returns (α) when R&D effort crowds into one research avenue. With these innovation rates, together with the size of vertical steps, ϕ, and the corresponding "size" parameter for a horizontal step, h, we can determine the endogenous evolution of the technology indices, β_t and Z_t, and characterize the balanced growth path. To do so, we will first determine the economic aggregates and then return to the conditions that determine the BGP.

4.3 Aggregates

We calculate the capital stock, and the capital share of income, by summing up across the automated intermediate inputs using (36). The capital share of income is

$$s_{K_t} = \frac{\psi X_t}{Y_t} = \gamma^{-\frac{1}{\rho}} \beta_t Z_t^{-1} \tag{46}$$

Comparing this outcome to its value in the exogenous growth model, (20), we see that less capital is used for a given technological state. This is the usual result in endogenous growth, where market power in this sector and the consequent markup has reduced its output and thus its input demand.

The aggregate profit is calculated by summing up profits across the automated lines using (37).[22]

$$\frac{\Pi_t}{Y_t} = 1 - \mu \beta_t Z_t^{-1} \tag{47}$$

The labor income is the remaining part of the output,

$$s_{L_t} = 1 - \gamma^{-1} \beta_t Z_t^{-1} \tag{48}$$

Aggregating the intermediate outputs, we find a similar result to the exogenous growth model. The difference is that monopoly power reduces GDP compared to the exogenous case with fully

[22] Recall that non-automated lines are produced competitively using labor.

competitive markets. Adding up intermediate outputs and prices, aggregate GDP is[23]

$$Y_t^* = \frac{vAL\left(1-\beta_t\right)^{\frac{1-\rho}{\rho}}}{\left(1-\gamma^{-1}\beta_t Z_t^{-1}\right)^{1/\rho}}. \tag{49}$$

4.4 The Endogenous Balanced Growth Path

As with the exogenous growth model, we need the same, two macro-level conditions for the economy to present a balanced growth path. First, we require $\beta_t Z_t^{-1} = s$, for some constant s, as before (BGP1). This condition produces a constant capital share, profit share, and labor share (see (46), (47), and (48)). Second, with a constant capital share, we see that income per capita in (49) will grow at $g = \frac{\rho-1}{\rho} g_{1-\beta_t}$. Balanced growth thus requires that $g_{1-\beta_t} = -q_t^h$, a constant (BGP2).

We can now confirm that the endogenous innovation processes will meet these conditions. First, consider the horizontal innovation sector. We know via Lemma 1 that q^h is constant with time in equilibrium. Thus the horizontal innovation sector provides $g_{1-\beta_t} = -q_t^h$, a constant. This directly satisfies (BGP2).

Second, consider the vertical innovation sector. We have a step size of ϕ on each line and, again per Lemma 1, a constant rate of innovation, q^v, on the BGP. Further, newly automated technologies are drawn according to (42). It therefore follows that the ratio of the technology indices on the BGP is pinned down as[24]

$$\beta_t Z_t^{-1} = \frac{1}{h\phi}\left(\frac{\xi^h}{\xi^v}\right)^{\frac{1}{1-\alpha}} \tag{50}$$

The technology indices β_t and Z_t thus evolve in a constant ratio, satisfying (BGP1).

This constant ratio in turn implies that the capital, profit, and labor shares of income will be constant on the BGP (see (46), (47), and (48)). The labor share specifically is

$$s^L = 1 - \frac{\gamma^{-1}}{h\phi}\left(\frac{\xi^h}{\xi^v}\right)^{\frac{1}{1-\alpha}}$$

[23]This form is familiar from the exogenous growth case. The difference is the markup parameter, γ, which acts to reduce GDP.

[24]This follows by combining the process of new productivity draws, (42), the condition (29) for newly automated technologies to maintain a BGP, and the relative rates of horizontal and vertical innovation (45).

Thus not only are the income shares constant with this innovation model, producing a balanced growth path, but they can be expressed in a simple form based on the parameters of the vertical and horizontal knowledge production functions and the markup parameter.

Finally, we can confirm that these newly automated technologies are in fact adopted - that is, that they are more cost effective than using labor. This requires that the newly automated technologies be sufficiently productive initially.[25] This parametric condition is readily verified as follows, which we assume holds.

Lemma 2 *Automated technologies are used on the BGP if* $h \geq \gamma^{-1} \frac{1}{\phi} \left(\frac{\xi^h}{\xi^v} \right)^{\frac{1}{1-\alpha}}$.

Proof. See appendix. ■

This condition is the endogenous growth analogue to (24) in the exogenous growth model. It further guarantees that the capital share in (50) is less than 1.

With these results, we have determined the income shares in terms of the exogenous variables, and now turn to the growth rate itself. For the general case, we have a system of four equations that determine four endogenous variables $\{q^v, q^h, r, g\}$. These four equations govern the vertical innovation rate (from (40) and (41)), the relationship between the vertical and horizontal innovation rates (45), the relationship between the growth rate and horizontal innovation rate (26), and the Euler equation (8), which we collect here.

$$q^v = \xi^{v\frac{1}{\alpha}} \mu q^{v\frac{\alpha-1}{\alpha}} - (r-g) \qquad \text{Vertical Innovation}$$

$$\frac{q^h}{q^v} = (\frac{\xi^h}{\xi^v})^{\frac{1}{1-\alpha}} \qquad \text{V\&H Relationship}$$

$$g = \frac{1}{\theta}(r-\omega) \qquad \text{Euler Equation}$$

$$g = \frac{\rho-1}{\rho} q^h \qquad \text{Growth Rate}$$

To study properties of the growth path, we focus on the case where $\theta \geq 1$.[26]

[25] Noting that wages are rising on the growth path, being productive enough to be adopted when it is initially automated guarantees that the technology remains productive enough to continue using over time, since labor becomes increasingly expensive.

[26] Empirical evidence commonly suggests the case where $\theta \geq 1$. See for example Campbell (2003) and Vissing-Jorgensen (2002). The model can also be considered where $\theta < 1$, with some further parametric restrictions, but this case is analytically more complex in addition to being less salient empirically.

Proposition 3 *(Existence and uniqueness). The balanced growth path exists and is unique if* $\theta \geq 1$.

Proof. See Appendix. ■

The model thus produces a balanced growth path, emerging endogenously from R&D effort on the vertical and horizontal dimensions. A primary intuition for the balanced result follows from the diminishing returns to R&D effort at a point in time on a given line, which we can think of as a crowding or duplication externality among competing R&D firms. This force acts to spread R&D effort out across lines, with the balance of vertical and horizontal progress depending on the knowledge production function parameters. This diminishing returns force also allows the equilibrium to move away from requiring knife-edge relationships among the knowledge production function parameters, and it creates conceptual degrees of freedom in the model, separating automation forces from capital productivity improvement forces in informing macroeconomic outcomes.

4.5 Growth and the Labor Share of Income

Tracking the two dimensions of technological progress, we can return to consider the apparent growth slowdown and decline in the labor share. As usual with an endogenous growth model, we think of dynamics as a shift between two different balanced growth paths, where we have moved from a relatively high growth and high labor share equilibrium to a relatively low growth and low labor share equilibrium. With the capital share (and hence labor share) given explicitly by (50) and the growth rate determined implicitly in the four-equation system above, we can now consider the implications of changing parameters of the model. Key comparative statics are collected here.

Proposition 4 *(Endogenous growth). The labor share on a balanced growth path increases with* ξ^v, h, ϕ, *and* γ *and decreases with* ξ^h. *The growth rate on a balanced growth path increases with* ξ^h *and* μ, *decreases with* ξ^v, *and is unchanging in* h.

Proof. See Appendix. ■

In light of these results, consider the apparent recent downshift in both the growth rate and the labor share (e.g., Gordon 2016, Elsby et al. 2013, Karabarbounis and Neiman 2014). First, consider an increasing difficulty in innovation and let's assume this happens on all dimensions.

Specifically, let the ease of innovation, ξ^h and ξ^v, decline proportionally in both horizontal and vertical dimensions and let the size of innovations, h and ϕ, also decline. If "innovation has become harder," both the growth rate and the labor share will decline. The decline in growth is intuitive when innovation gets harder. The decline in the labor share follows because advancing capital technology has become more difficult. With a less than unitary elasticity of substitution, a lower productivity capital stock makes it relatively costly, and it is cheap capital that supports the labor share.[27]

Second, one can look at more precise dimensions of knowledge production. First, consider the automation dimension. Here, we can think of automation as becoming harder, in two senses. We could imagine that the ease of discovering automation goes down (ξ^h decreases) and the productivity of the new automation becomes worse (h declines). First, the decline in h acts to pull down the labor share. Second, a decline in ξ^h will cause the growth rate to decline.[28] For example, if we think that recent automation has been barely good enough to replace labor (e.g., replacing customer service workers or secretarial workers with automated systems, but these automated systems perform poorly), this low automation quality will cause the labor share to fall. What is bad for labor share of income is being replaced by machines that aren't very good, so that there is little productivity gain to offset the displacement effect of increased automation. This "so-so" technology effect is featured in Acemoglu and Restrepo (2018, 2019). Here this force is seen as a balance between the rate and size parameters governing horizontal innovation, while also integrating with vertical technological progress.

Third, consider the vertical technology dimension. To focus purely on technology, consider for the moment the natural idea that the technological distance between the leader and the competitive fringe (γ^{-1}) is positively related to the size of the advance the leader made in the line's technology ($1+\phi$).[29] Then a decline in the vertical step size, ϕ, will weaken profitability of an incumbent monopolist and reduce incentives to innovate. We will again see lower growth and a lower labor

[27] This is the opposite of how one thinks about capital productivity and income shares with an elasticity of substitution greater than 1 (e.g., Karabarbounis and Neiman (2014)). With an elasticity of substitution less than 1, we can instead link a declining labor share and declining productivity growth in one frame.

[28] For the labor share to decline on net, we would need the percentage decrease in h to exceed $\frac{1}{1-\alpha}$ times the percentage decrease in ξ^h (see (50)).

[29] To be specific, let $\gamma^{-1} = \lambda(1+\phi)$ for some $\lambda > 0$.

share of income. A decline in the vertical technology step size can then provide an especially parsimonious force for both the macroeconomic effects.

Finally, consider a political economy dimension, focusing on incumbent market power. Here we let the markup parameter, γ, move separately from the vertical step size, acknowledging that markups also may depend on institutions. A rise in the markup (i.e., a decline in γ) will cause the labor share to decline. This follows because, again, we have a balanced growth path that still features a less than unitary elasticity of substitution between capital and labor. A larger markup reduces the size of the capital stock, and this reduce the labor share. Thus we have a simple link between increased market power and a declining labor share of income. However, from a growth point of view, a rise in the markup makes innovation more profitable and the growth rate should then increase. Thus, the markup story must be counterbalanced by other forces to further match the declining growth rate.[30]

5 Conclusion

This paper seeks to resolve key tensions between micro and macro descriptions of technological progress. Specifically, the model engages the microeconomic regularities of capital-embodied technological progress and the macroeconomic regularities of balanced growth. By engaging two salient technology dimensions – the automation frontier and the productivity frontier - balanced growth emerges in a surprisingly tractable framework. These two frontiers engage in a "tug of war," with their balance determining the income shares in a transparent manner while allowing steady growth.

In addition to providing a novel type of solution to the puzzle of the Uzawa steady-state growth theorem, the model can also inform economic dynamics. The interplay of the two technology frontiers can provide insights about shifts in growth rates and income shares as well as industry dynamics, with applications to structural change (on or off a balanced growth path) and general purpose technologies. With exogenous technological progress, the model engages such dynamics in a straightforward manner, and indeed is flexible enough to fit macroeconomic and sectoral dynamics exactly. The endogenous growth model, which is necessarily more constrained, delivers further insights about why a balanced growth path would emerge amidst the tug of war between automation

[30]Skeptics about the growth slowdown, who may see it a measurement artifact, may not see a tension here.

and capital-productivity improvements. The endogenous growth model may further inform long-run shifts in growth rates and income shares from the perspective of underlying economic parameters.

The results and intuitions of the model build in part from a less than unitary elasticity of substitution between capital and labor. This feature puts the two technology margins in opposition in the evolution of income shares, allowing for balanced growth, while also allowing for intuitive dynamics in growth and income shares, and further applications to structural change, where sectors with relatively slow technological advance take on a larger share of the economy, as with Baumol's cost disease. Further applications engage the macroeconomic implications of general purpose technologies and the ultimately limited aggregate effects of remarkable technologies, like computing.

Numerous extensions are possible. First, we have not emphasized policy implications, but for example the role of tax policy would be nuanced and distinct, with capital taxation having different implications for labor shares depending on whether it limits horizontal or vertical progress. Second, to emphasize the novel forces and intuitions at work in this model, we have focused purely on capital-embodied technological change, but future extensions can incorporate labor-embodied productivity gains, human capital, and capital-skill complementarity to engage additional forms of productivity growth as well as skill-biased technical change. Third, the tractability of the technology dynamics may also allow extensions to business cycles, where frictions may lead to less than full employment. The displacement effects of increased automation may then have additional labor market effects. Fourth, alternative forms of the endogenous growth theory may prove insightful. Overall, micro-foundations emphasizing the two dimensions of capital-embodied technology advance appear to provide a tractable and rich framework for engaging economic growth and related phenomena.

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6 Appendix

6.1 Proofs

Proposition 1: *(Balanced growth path). Under condition 1, a balanced growth path exists where the capital share is* $s = \beta_t Z_t^{-1}$ *and the growth rate in per-capita output, per-capita consumption, per-capita capital stock, and the wage are* $g = \frac{\rho-1}{\rho} q^h$*. The discount rate is* $r = \omega + \theta \frac{\rho-1}{\rho} q^h$*.*

Proof. On the production side, firm optimization presents explicit solutions at each point t for intermediate outputs and prices ($y_t(i)$ and $p_t(i)$), factor inputs ($l_t(i)$ and $x_t(i)$), aggregate GDP and investment (Y_t and I_t), and wages (w_t). Specifically, Y_t, w_t, and $l_t(i)$ are given explicitly in terms of exogenous variables by (22), (23), and (13) respectively. The explicit solution for $x_t(i)$ in terms of exogenous variables follows from (17) and replacing Y_t using (22). Similarly one finds explicit solutions in terms of endogenous variables for $y_t(i)$, $p_t(i)$, and I_t in a straightforward manner.

On the consumption side, household optimization gives the Euler equation (8). Market clearing further implies that consumption equates to wages in this baseline model. Specifically, household income is $Y_t = c_t L_t + S_t$ and factor payments total $Y_t = w_t L_t + \psi X_t$. Savings equals investment, $S_t = I_t$, and the full depreciation each period equates capital factor payments to gross investment, $I_t = \psi X_t$. Hence, equating income and expenditure, $c_t = w_t$.[31] Per condition 1, the growth rate in per-capita income is $g = \frac{\rho-1}{\rho} q^h$, and with a constant labor share we have $g_c = g_w = g$. The Euler equation thus implies $r = \omega + \theta g = \omega + \theta \frac{\rho-1}{\rho} q^h$. The transversality condition is $\omega - n > (1-\theta)g$, which is satisfied by the assumption in condition 1.

■

Corollary 1: *(Limited innovation, limitless growth). On the balanced growth path, the technology indices shrink at constant rates,* $g_{1-\beta_t} = g_{1/s-Z_t} = -q^h$*, but grow at shrinking rates,* $g_{\beta_t} = g_{Z_t} = \frac{1-\beta_t}{\beta_t} q^h$*, and approach finite limits,* $\beta_t \to 1$ *and* $Z_t \to 1/s$*.*

[31] This will not be true in the endogneous technology model of Section 4, for the usual reason in endogneous growth where capital inputs still fully depreciate but there are profits from market power adding to household income and, on the expenditure side, there are additional investments in R&D that drive technological progress.

Proof. From (BGP2), we require $g_{1-\beta_t} = -q^h$. It follows that $g_{\beta_t} = \frac{\dot{\beta}_t}{\beta_t} = \frac{1-\beta_t}{\beta_t}\frac{\dot{\beta}_t}{1-\beta_t} = \frac{1-\beta_t}{\beta_t}q^h$. From (BGP1), we require $\frac{\beta_t}{Z_t} = s$. Therefore, $Z_t \to 1/s$ as $\beta_t \to 1$. Further, $g_{\beta_t} = g_{Z_t}$. The growth rate at which Z_t approaches its limit is $g_{1/s-Z_t} = -\frac{\dot{Z}_t}{1/s-Z_t} = -\frac{\dot{Z}_t}{Z_t}\frac{Z_t}{1/s-Z_t} = -\frac{1-\beta_t}{\beta_t}q^h\frac{1}{1/\beta_t-1} = q^h$. ■

Case 1: *(A micro-innovation process for balanced growth). Let existing tasks increase their productivity proportionally by an amount ϕ with hazard rate q^v. Then a BGP will occur with so long as newly automated tasks have productivity $z_t(\beta_t) = h(1-\beta_t)$ where $h \geq \frac{1}{1-s}$ and $s = \frac{q^h}{q^v\phi h}$.*

Proof. Differentiate the first BGP condition in the form (29) using Leibniz's rule. This gives

$$\int_0^{\beta_t} \frac{\dot{z}_t(i)}{z_t(i)^2}di = \frac{\dot{\beta}_t}{z_t(\beta_t)} \tag{51}$$

In the vertical innovation process, a given automated line increases its productivity proportionally by an amount ϕ with hazard rate q^v. Then we have $\dot{z}_t(i)/z_t(i) = \phi q^v$ in expectation, given $z_t(i)$. Therefore

$$\int_0^{\beta_t} \frac{\dot{z}_t(i)}{z_t(i)^2}di = \phi q^v \int_0^{\beta_t} \frac{1}{z_t(i)}di = s\phi q^v$$

Meanwhile, the horizontal process of automated new tasks occurs as the rate $\dot{\beta}_t = q^h(1-\beta_t)$ on the BGP. Thus we can write the BGP condition (51) as

$$\frac{q^h(1-\beta_t)}{z_t(\beta_t)} = s\phi q^v$$

Rearranging this produces the necessary condition of the Corollary 1, equation (29), governing the productivity draws for new technologies. Namely, with $z_t(\beta_t) = h(1-\beta_t)$ we have $s = \frac{q^h}{q^v\phi h}$.

Next consider the condition that all automated technologies be adopted (as opposed to using labor). First note that the automated technology with the lowest productivity will be the newly automated one. This follows because, from (29), the initial productivity is declining on the growth path and, once automated, the productivity on any line can only be increasing. Thus all automated technologies will be used so long as the newly automated technology is used. Second, from (24), the newly-automated technology will be adopted if $z_t(\beta_t) \geq \frac{1-\beta_t}{1-s}$. Using (29), this condition is $h \geq \frac{1}{1-s}$ with $s = \frac{q^h}{q^v\phi h}$ as above. ■

Proposition 2 *(Unbalanced growth path). Under condition 2, an equilibrium exists with a capital share* $s_{K_t} = \beta_t Z_t^{-1}$ *and explicitly determined paths of output, consumption, investment, the wage, the interest rate, and all other prices and quantities in the economy.*

Proof. On the production side firm optimization presents solutions at each point t for intermediate outputs and prices ($y_t(i)$ and $p_t(i)$), factor inputs ($l_t(i)$ and $x_t(i)$), aggregate GDP and investment (Y_t and I_t), and wages (w_t). On the consumption side, we have $c_t = w_t$. The Euler equation is therefore $r_t = \omega + \theta g_{w_t}$. The interest rate is then determined directly by taking the growth rate of w_t from (23). The interest rate is

$$r_t = \omega + \theta \frac{\rho - 1}{\rho} \left(\frac{\beta_t Z_t^{-1}}{1 - \beta_t Z_t^{-1}} (g_{\beta_t} - g_{Z_t}) - g_{1-\beta_t} \right) \tag{52}$$

The transversality condition requires $\lim_{t\to\infty} \psi X_t \exp - \int_0^t r_s ds = 0$. From the Euler equation, the integral can be solved from the path of wages, and the transversality condition can be written as

$$\lim_{t\to\infty} e^{-(\omega - n - (1-\theta)\overline{g}_t)t} \frac{\beta_t Z_t^{-1}}{\left(1 - \beta_t Z_t^{-1}\right)^{\theta}} = 0$$

where $\overline{g}_t$ is the mean growth rate in per-capita income from time 0 to time t. Under condition 2, we have $\omega - n > (1-\theta)\overline{g}_\infty$ and the transversality condition is therefore satisfied.[32]

The economy thus has a well-defined equilibrium, with quantities and prices following explicitly defined paths as given in the relevant equations of Section 2. The capital share is $s_{Kt} = \beta_t Z_t^{-1}$ from (20). The path of GDP is given in (22). The path of wages is given by (23), which also determines per-capita consumption. And the interest rate is (52). All other prices and quantities are found in Section 2, given either directly in terms of exogenous parameters there, or indirectly by replacing the GDP or wage term with its explicit solution.

■

Corollary 3: *(Growth dynamics). The growth rate in income per-capita increases in* g_{β_t} *and* g_{Z_t}*.*

[32]Note also that we require that the capital share ($\beta_t Z_t^{-1}$) be less than 1 as $t \to \infty$. This is a very weak condition because the technology adoption condition already implies $Z_t \geq 1$, as discussed in Section 2.6, and as long as capital productivities advance beyond this minimum floor, so that $Z_\infty > 1$, the capital share will be less than 1 in the limit.

Proof. Consider GDP as given by (22). Take logs and differentiate with respect to time. Then rearrange terms to write

$$g_{y_t} = -\frac{1}{\rho}\left(\frac{\beta_t Z_t^{-1}}{1-\beta_t Z_t^{-1}}\right) g_{Z_t} - \frac{1}{\rho}\left((1-\rho)\frac{\beta_t}{1-\beta_t} - \frac{\beta_t Z_t^{-1}}{1-\beta_t Z_t^{-1}}\right) g_{\beta_t} \tag{53}$$

Note that $\rho < 0$. Further note that $\beta_t Z_t^{-1} < 1$. Therefore we see that g_{y_t} is increasing in g_{Z_t}. Noting again that $\rho < 0$, we see that g_{y_t} is increasing in g_{β_t} if

$$\frac{\beta_t}{1-\beta_t} - \frac{\beta_t Z_t^{-1}}{1-\beta_t Z_t^{-1}} > 0 \tag{54}$$

The left hand side terms can be combined into the expression

$$\frac{\beta_t(1-Z_t^{-1})}{(1-\beta_t)(1-\beta_t Z_t^{-1})} > 0 \tag{55}$$

This is positive so long as $Z_t > 1$, which is guaranteed by (25), an implication of the adoption condition (24). ■

Corollary 4: *(Automation-led growth with constant wages) Let automation proceed at some rate $q^h > 0$ where newly automated technologies have productivity level z_t^{min}, the lowest level of productivity where they will still be adopted. Let prior automated technologies see no productivity improvement. Then wages remain constant. Income per-capita grows at rate q^h, and the labor share falls at rate q^h. The technology index Z_t declines at rate $q^h \frac{1-Z_t}{\beta_t} < 0$.*

Proof. Consider first the growth rate in Z_t. Take the definition of Z_t as the harmonic average of the $z_t(i)$, as in (6). Differentiating (6) with respect to time and using Leibniz's rule gives

$$g_{Z_t} = g_{\beta_t} - Z_t \frac{g_{\beta_t}}{z_t(\beta_t)} + \frac{Z_t}{\beta_t}\int_0^{\beta_t} \frac{1}{z_t\,(i)} g_{z_t(i)} di \tag{56}$$

Following the technological pathways defined in the Corollary, we have

1. $g_{1-\beta_t} = -q^h$, which is equivalently $g_{\beta_t} = ((1-\beta_t)/\beta_t)q^h$;
2. $z_t(\beta_t) = z_t^{min} = (1-\beta_t)/(1-\beta_t Z_t^{-1})$;
3. $g_{z_t(i)} = 0$ for all $i < \beta_t$ (no vertical progress).

Under these conditions, the growth rate of Z_t in (57) becomes

$$g_{Z_t} = \frac{1 - Z_t}{\beta_t} q^h \tag{57}$$

which is less than zero because $Z_t > 1$.

The labor share is $s_{L_t} = 1 - \beta_t Z_t^{-1}$. Taking logs and differentiating with respect to time, we have

$$g_{s_{L_t}} = \frac{\beta_t Z_t^{-1}}{1 - \beta_t Z_t^{-1}} \left(g_{\beta_t} - g_{Z_t} \right) \tag{58}$$

Using the above technology paths for β_t and Z_t this simplifies to $g_{s_{L_t}} = -q^h$.

The wage is given by (23). Taking logs and differentiating with respect to time gives

$$g_{w_t} = \frac{\rho - 1}{\rho} \left(g_{s_{L_t}} - g_{1-\beta_t} \right) \tag{59}$$

Using the results above, we have $g_{s_{L_t}} = -q^h = g_{1-\beta_t}$ and so $g_{w_t} = 0$.

Finally, given that wages are constant and that the labor share of income, $s_{L_t} = \frac{w_t L_t}{Y_t}$, falls at rate q^h, it follows directly that Y_t / L_t grows at rate q^h. ■

Corollary 5: *(Sectoral advance). Holding other sector technology levels fixed, the GDP share of a sector will decline with its automation level, β_t^j, or capital productivity level, Z_t^j. The labor share of income within the sector will decrease in the automation level, β_t^j, but increase in the capital productivity level, Z_t^j.*

Proof. From (30), write the GDP share of the sector as

$$\frac{Y_t^j}{Y_t} = \beta_t^j \left(Z_t^j \right)^{-1} + \left(1 - \sum_i \beta_t^i \left(Z_t^i \right)^{-1} \right) \frac{u_t^j - \beta_t^j}{1 - \sum_i \beta_t^i} \tag{60}$$

Differentiate with respect to Z_t^j. After simplification, this can be written as

$$\frac{\partial}{\partial Z_t^j} \left(\frac{Y_t^j}{Y_t} \right) = -\beta_t^j \left(Z_t^j \right)^{-2} \left(\frac{\sum_{i \neq j} u_t^i - \beta_t^i}{1 - \beta_t} \right) \tag{61}$$

which by inspection is less than zero.

Next differentiate (60) with respect to β_t^j. After simplification, this can be written as

$$\frac{\partial}{\partial \beta_t^j}\left(\frac{Y_t^j}{Y_t}\right) = (Z_t^j)^{-1}\left(\frac{s_{L_t} Z_t^j}{1-\beta_t} - 1\right)\left(\frac{u_t^j - \beta_t^j}{1-\beta_t} - 1\right) \tag{62}$$

This is also less than zero. To see this, note that $\frac{u_t^j - \beta_t^j}{1-\beta_t} \leq 1$ and $\frac{s_{L_t} Z_t^j}{1-\beta_t} \geq 1$. The former is true by inspection, as the measure of non-automated tasks in the sector, $u_t^j - \beta_t^j$, must be weakly less than the measure of non-automated tasks in the economy overall, $1 - \beta_t$. To show the latter, which requires that $s_{L_t} Z_t^j \geq 1 - \beta_t$, we can rewrite this expression using the definition of $s_{L_t} = 1 - \beta_t Z_t^{-1}$ to produce the equivalent condition

$$1 \geq \beta_t Z_t^{-1}\left(\frac{Z_t^j - Z_t}{Z_t^j - 1}\right) \tag{63}$$

which holds because $\beta_t Z_t^{-1} \leq 1$ and $Z_t \geq 1$. Thus the sector's GDP share is increasing in Z_t^j and β_t^j.

Turning to the income share within a sector, we find that

$$\frac{\partial s_{L_t}^j}{\partial \beta_t^j} \geq 0 \tag{64}$$

This follows by inspection of (31), since we have β_t^j increasing and the GDP share of the sector falling.

Differentiating the sectoral labor share of income in (31) by Z_t^j we find, after some simplification, that

$$\frac{\partial s_{L_t}^j}{\partial Z_t^j} = -\beta_t^j\left(\frac{u_t^j - \beta_t^j}{1-\beta_t}\right)^2 \leq 0 \tag{65}$$

■

As a related result, on the growth path of the economy, technological advance may proceed in all sectors simultaneously. We may therefore also consider a variant of this corollary that emphasizes the relative evolution of sectors along a balanced growth path. To do so, we can define the relative technology state of different sectors. Specifically, define the relative automation rate in sector j as $\eta_t^j = (\beta_t^j/u_t^j)/\beta_t$.[33] Similarly, define the relative capital productivity in sector j as $\varphi_t^j = Z_t^j/Z_t$.

[33] This is the relative automation rate in that β_t^j/u_t^j is the share of tasks in the sector that are automated, which is compared to β_t, the share of all tasks that are automated.

Thus a sector with $\eta_t^j > 1$ is relatively highly automated compared to the economy at large, and a sector with $\varphi_t^j > 1$ has relatively advanced capital productivity.

With these definitions, we can sum up sector-specific tasks and write the GDP share and capital share for a given sector in the following, relative technology form,

$$\frac{Y_t^j}{Y_t} = u_t^j \left(\frac{\eta_t^j}{\varphi_t^j} s_{K_t} + \frac{1 - \eta_t^j \beta_t}{1 - \beta_t} (1 - s_{K_t}) \right) \tag{66}$$

$$s_{K_t}^j = s_{K_t} \left(s_{K_t} + \varphi_t^j \frac{1/\eta_t^j - \beta_t}{1 - \beta_t} (1 - s_{K_t}) \right)^{-1} \tag{67}$$

We can then consider structural change in the economy along a balanced growth path, as follows.

Corollary 5a: *(Structural change). Along a balanced growth path, an increase in the relative productivity, φ_t^j, of the sector's capital inputs or relative automation level, η_t^j, will cause the sector's GDP share to decline. An increase in the sector's relative automation, η_t^j, will cause its labor share to decline while an increase in the relative productivity, φ_t^j, of the sector's capital inputs will cause its labor share to rise.*

Proof. Consider the within-sector capital share, as in (67). Consider dynamics where the overall capital share in the economy is fixed (i.e., the economy is on a balanced growth path). Focus on a particular sector j. By inspection of (67), the capital share in that sector will rise if its relative automation rate, η_t^j, increases, and the capital share in that sector will fall if its relative productivity level, φ_t^j, increases.

Next consider the sector's share of GDP, as in (66). By inspection, an increase in the sector's relative capital productivity level, φ_t^j, will cause the GDP share of that sector to decline. The effect of higher relative automation cannot be seen by inspection, however. Differentiate the sectoral GDP share by its relative automation rate, holding the economy wide technology indices fixed. We have

$$\frac{\partial (Y_t^j / Y_t)}{\partial \eta_t^j} = u^j \left(\frac{1}{\varphi_t^j} s_{K_t} - \frac{\beta_t}{1 - \beta_t} s_{L_t} \right) \tag{68}$$

Thus the GDP share of the sector is declining in its relative automation rate if the term on parentheses is negative. Recalling the definition $\varphi_t^j = Z_t^j / Z_t$, and that $s_{K_t} = 1 - s_{L_t} = \beta_t Z_t^{-1}$

we can write

$$\frac{\partial(Y_t^j/Y_t)}{\partial \eta_t^j} < 0 \text{ iff } Z_t^j > \frac{1-\beta_t}{1-\beta_t Z_t^{-1}} \tag{69}$$

Now recall from (24) that

$$z_t^{\min} \geq \frac{1-\beta_t}{1-\beta_t Z_t^{-1}} \tag{70}$$

Since the harmonic average Z_t^j must exceed $z_t^{\min}$, the above condition must hold. ■

Corollary 6: *(Extreme technological advance). Let a fraction α of the automated tasks have the same distribution of $z_t(i)$ as the other automated tasks. Holding other technologies constant, take $z_t(i) \to \infty$ for this fraction α of automated tasks. The capital share will decline by α percent. Income per capita will increase by $\Delta \ln(y_t) = -\frac{1}{\rho} \ln\left(1 + \alpha \frac{s_{K_t}}{s_{L_t}}\right)$.*

Proof. Consider a fraction α of the automated tasks. Define the harmonic average of the $z_t(i)$ for this fraction of tasks as $Z_{t,\alpha}$. Define the harmonic average of the $z_t(i)$ for the remaining $1-\alpha$ fraction of tasks as $Z_{t,1-\alpha}$. Therefore we can write

$$Z_t = \left(\alpha Z_{t,\alpha}^{-1} + (1-\alpha) Z_{t,1-\alpha}^{-1}\right)^{-1} \tag{71}$$

For simplicity, consider the initial state at time t where the harmonic average is the same for the fraction α of tasks as for the all automated tasks. Then $Z_{t,1-\alpha} = Z_{t,\alpha} = Z_t$. Now, for the fraction α of automated tasks, let the technology level $z_t(i) \to \infty$ at time $t = t'$, holding β_t and the other $z_t(i)$ fixed. The index $Z_{t'}$ becomes

$$Z_{t'} = \frac{1}{1-\alpha} Z_{t',1-\alpha} = \frac{1}{1-\alpha} Z_t \tag{72}$$

and the capital share becomes

$$s_{K_{t'}} = \beta_{t'} Z_{t'}^{-1} = (1-\alpha)\beta_t Z_t^{-1} = (1-\alpha) s_{K_t} \tag{73}$$

Hence the capital share falls by α percent.

For income per capita, consider GDP given by (22). With β_t fixed, the change in income per capita is

$$\frac{y_{t'}}{y_t} = \left(\frac{1-\beta_{t'} Z_{t'}^{-1}}{1-\beta_t Z_t^{-1}}\right)^{-1/\rho} = \left(\frac{1-(1-\alpha)s_{K_t}}{s_L}\right)^{-1/\rho} \tag{74}$$

Taking logs and using $1 - s_{K_t} = s_{L_t}$ produces the result in the corollary. ■

Lemma 1: *The vertical and horizontal hazard rates of innovation, q^v and q^h, are both constants on a BGP and have the ratio $\frac{q^h}{q^v} = \left(\frac{\xi^h}{\xi^v}\right)^{\frac{1}{1-\alpha}}$. Aggregate R&D expenditure in the vertical direction and in the horizontal direction are both constant shares of GDP.*

Proof. By (40) and (41), we can write the equilibrium vertical rate of innovation as

$$q^v = \xi^{v\frac{1}{\alpha}} \mu q^{v\frac{\alpha-1}{\alpha}} - (r-g) \tag{75}$$

Similarly, from the horizontal side, (43) and (44) imply

$$q^v = \xi^{h\frac{1}{\alpha}} \mu q^{h\frac{\alpha-1}{\alpha}} - (r-g). \tag{76}$$

Combining these produces the ratio $\frac{q^h}{q^v} = \left(\frac{\xi^h}{\xi^v}\right)^{\frac{1}{1-\alpha}}$, as was to be shown.

Further, recall that the ratio $z_t(i)d_t^v(i)/Y_t$ is a constant for vertical lines. Define this constant as χ^v. We can then write R&D expenditure on a given line as $d_t^v(i) = \chi^v Y_t / z_t(i)$. Total vertical research investment across the automated lines then adds up as

$$\frac{D_t^v}{Y_t} = \chi^v \beta_t Z_t^{-1}. \tag{77}$$

which will be a constant share of GDP on the BGP.

Similarly, for horizontal lines, the BGP features $z_t(i)d_t^h(i) = \chi^h Y_t$, where χ^h is a constant. Recalling that the initial quality of any newly automated line is (42), the R&D effort will then be the same across these horizontal lines. The aggregate investment in the horizontal research sector then adds up as

$$\frac{D_t^h}{Y_t} = \chi^h h^{-1}, \tag{78}$$

so that horizontal R&D expenditure is also a constant share of GDP, as was to be shown. ■

Lemma 2: *All automated technologies will be adopted on the BGP if $h \geq \gamma^{-1}\frac{1}{\phi}\left(\frac{\xi^h}{\xi^v}\right)^{\frac{1}{1-\alpha}}$.*

Proof. The adoption condition is that the price of using an automated technology, $p_t(i) = \psi\left[\gamma z_t(i)\right]^{\frac{1-\rho}{\rho}}$, is less than the price when using labor, $\hat{p}_t(i) = w_t/A$. This will be satisfied for

all automated sectors if it is satisfied for the automated sector with the lowest productivity. The lowest productivity level will be the one for the marginally automated technology, $z_t(\beta_t)$. Thus we require

$$w_t/A \geq \psi \left[\gamma z_t(\beta_t)\right]^{\frac{1-\rho}{\rho}}$$

Using the initial productivity for $z_t(\beta_t)$, (42), and the equilibrium wage via the labor share, (48), this becomes

$$(1-\gamma^{-1}\beta_t Z_t^{-1})\frac{Y_t}{L} \geq \psi A h^{\frac{1-\rho}{\rho}}(1-\beta_t)^{\frac{1-\rho}{\rho}}$$

Using GDP, (49), this simplifies as

$$1-\gamma^{-1}\beta_t Z_t^{-1} \geq h^{-1}$$

where $\gamma^{-1}\beta_t Z_t^{-1}$ is the capital share. Using the result for the capital share, (46), this produces the statement in the Corollary. ■

Proposition 3: *The balanced growth path exists and is unique if* $\theta \geq 1$.

Proof. We will first consider the existence and uniqueness of the horizontal innovation rate, q^h. Using the system of four equations ((8), (26), (75), and (45)) we substitute out the other endogenous variables and write an implicit function for q^h in terms of the exogenous parameters. This expression is

$$q^h\left(\left(\frac{\xi^v}{\xi^h}\right)^{\frac{1}{1-\alpha}} + (\theta-1)\frac{\rho-1}{\rho}\right) = \frac{\xi^{h\frac{1}{\alpha}}\mu}{q^{h\frac{1-\alpha}{\alpha}}} - \omega \tag{79}$$

The left hand side of this equation is a linear function of q^h. Note that $\theta \geq 1$ is a sufficient condition for the expression in parentheses to be positive. Therefore this function starts at the origin and rises monotonically in q^h and without bound as $q^h \to \infty$. Meanwhile, the right hand side of this equation is a function that declines in q^h. The function is unbounded at $q^h = 0$ and declines monotonically, crossing zero for some large q^h. Therefore there is a single crossing property in these two functions at some unique positive value of q^h.

Next, note that a unique positive value of q^h implies a unique, positive q^v (via (45)) and a unique, positive g (via (26). A unique r is then determined uniquely from the Euler equation (8)

(and we must also have $r > g$ with $\theta \geq 1$). Therefore there exists a set of values $\{q^v, q^h, r, g\}$ that are unique and create a balanced growth path equilibrium. ■

Proposition 4: *(Endogenous growth comparative statics). The labor share on a balanced growth path increases with ξ^v, h, ϕ, and γ and decreases with ξ^h. The growth rate on a balanced growth path increases with ξ^h and μ, decreases with ξ^v, and is unchanging in h.*

Proof. Write the labor share as

$$s^L = 1 - \frac{\gamma^{-1}}{h\phi}\left(\frac{\xi^h}{\xi^v}\right)^{\frac{1}{1-\alpha}}$$

The comparative statics for the labor share with respect to ξ^v, h, ϕ, and ξ^h follow by inspection.

For the growth rate, note that it is linear and monotonic in q^h, from (26). So we will consider the comparative statics in terms of the behavior of q^h. In particular, turn again to (79) and the single crossing property analyzed in the proof of Proposition 3. Consider the intersection point of the increasing function of q^h on the left hand side and the decreasing function of q^h on the right hand side of (3).

By inspection, an increase in ξ^h decreases the slope on the left hand side and shifts rightward the function on the right hand side. Both forces cause the equilibrium q^h to rise. By inspection, an an increase in ξ^v increases the slope on the left hand side (while the function on the right hand side does not change), causing the equilibrium q^h to fall. By inspection, a rise in the markup, μ, cause the right hand side function to shift rightward (while the function on the left hand side does not change), causing the equilibrium q^h to rise. By inspection, changes in h have no effect on q^h. (In fact, h does not appear in the four equation system and thus has no influence on the growth rate, innovation rates, or interest rate.) ■

6.2 Data Analysis

The paper provides two illustrative empirical applications of the model. We discuss the data sets and estimation strategies in further detail here.

6.2.1 Income and Growth Dynamics

In Figure 4, we use standard data series from the Bureau of Labor Statistics for U.S. labor productivity (output her hour) and the U.S. labor share of income. These data were drawn from FRED, with links to both data series provided in the references to this paper (BLS 2021a, 2021b). Both series run from 1947-2020.

The two technology paths are then estimated using (21) and (22). To pin down the technology paths we need one initial condition and we set $Z_0 = 1.5$ for the first year of the data series. Given this initial condition, the initial automation rate is pinned down by the initial capital share, $\beta_0 = Z_0 s_{K_0}$. One then pin down νA given the observed initial output per worker level, Y_0/L_0, thus normalizing the output per worker measure. Having normalized this measure, one can then proceed in each period to estimate the two unknowns, β_t and Z_t, from the two equations (21) and (22) and the observed output per hour and labor share data series.

Figure 4 presents the technology pathways in their limits form as $1 - \beta_t$ and $c - Z_t$. This is useful visually because (in logs) a common, constant slope then appears as a balanced growth path, allowing one to see a BGP and deviations from a BGP more easily. On a balanced growth path, the limit of Z_t is $c = 1/s$; i.e., the inverse of the capital share. For visualization purposes, we take $c = 1/\min[s_{Kt}]$.

6.2.2 Sectoral Dynamics

In Figure 5, we provide data on sectoral GDP share and labor shares of income. The raw data come from Mendieta-Munoz et al. (2020), who calculate output and labor compensation shares for 14 sectors (leaving out the public sector and housing). We consider agriculture, manufacturing, and the remaining sectors as one group (see main text).

To estimate the path of the sectoral technology parameters, β_t^j and Z_t^j, we can use (30) and (31). However, we also need information on the task shares, u_t^j, and these are not determined within the model. These task shares may also be evolving to some extent with time. For example, information services may replace certain manufactured goods as the leading technology for performing certain tasks (as when Internet search services replace dictionaries, phone books, etc.).

To provide some external grounding for the task shares, we use SIC and NAICS codes. The idea is to estimate the task share using the given industrial categorization scheme. Specifically, we count the number of six-digit subsectors in the 2012 NAICS, grouped according to their two-digit definitions (11 for agriculture, 31-33 for manufacturing). We drop the public sector, consistent with the Mendieta-Munoz et al. (2020) data, and group the remaining six-digit industries as "other". We similarly apportion industries in the 1987 SIC classification system. The NAICS results produce $\{u^{agr}_{2012}, u^{man}_{2012}, u^{oth}_{2012}\} = \{.0618, .3514, .5868\}$. The SIC results produce $\{u^{agr}_{1987}, u^{man}_{1987}, u^{oth}_{1987}\} = \{.0592, .4192, .5216\}$. We see that both schemes agree quite closely on the share of different subsectors that constitute agriculture. Over time, however, we see that the share of manufacturing subsectors appears by this measure to have declined compared to services. For illustration purposes, we take this shift as substantive (as information services do seem, e.g., to have led to the creative destruction in some manufactured goods used for some tasks), and we assign $\{u^{agr}_t, u^{man}_t, u^{oth}_t\}$ as linear trends that match the SIC and NAICS measures in the appropriate years.