Linear Pricing Mechanisms without Convexity

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Abstract

We introduce two new linear pricing mechanisms that lead to feasible, budget-balanced and approximately efficient outcomes even when preferences or production sets are not convex and Walrasian equilibrium does not exist. One mechanism permits different prices for buyers and sellers; the other uses a single price vector but permits some agents to be rationed. In these mechanisms, both the inefficiency-to-value ratio and the maximum benefit any single agent can gain from false reporting tend quickly to zero as the numbers of producers and consumers increase.

Keywords: Approximate efficiency, Approximate incentive-compatibility, Market design, Non-convexity, Prices, Rationing

JEL Codes: C620, D400, D440, D450, D470, D500, D510, D610.

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1 Introduction

Auction markets are often designed to promote approximate competitive equilibrium outcomes because of their several desirable properties. First, in competitive equilibrium, the allocation is always feasible, no unwanted goods are produced, and no demand is left unfulfilled. Second, in the "large market" limit, no agent has much influence over the prices, which leads to the further conclusion that the mechanism is approximately incentive-compatible.[1] Third, suppliers' payments are equal to what consumers pay; this budget-balance is a common goal of mechanism design because it means that no subsidy is required from a market operator or government. Fourth, in important special cases, the prices can be approximated efficiently as the solution of a convex program or as the rest point of a natural price adjustment process. Fifth, linear and anonymous pricing rules are familiar, simple and often seen as more fair than personalized payment rules.[2]

All of these properties, however, depend on the existence of a competitive equilibrium, which is proved in traditional general equilibrium theory using the assumption that preferences and production sets are convex. The convexity assumptions are problematic, especially on the production side, for many industries that operate auction-like markets. For example, in electricity markets, generating plants often incur fixed costs to operate or adjust production on any given day, and there are additional fixed costs of maintaining unused capacity. Similarly, in fisheries, there are fixed costs of operating a boat on any given day. There are many more such examples. With these non-convexities, there is no guarantee that linear prices exist to support efficient allocations or to equate supply and demand in these industries, so there is no mechanism that accepts truthful reports and always outputs an associated competitive equilibrium.

Our goal in this paper is to demonstrate mechanisms that achieve or approximate the desirable properties of price-guided allocations despite the failures of convexity. In our main mechanism, the key innovation is to permit prices to differ between buyers and sellers, with each side unrationed at its own prices. We also study mechanisms in which agents' supplies or demands may be rationed directly. We show that these linear pricing mechanisms sacrifice little in terms of the desirable properties of prices listed above, provided that the markets are sufficiently large, and we explore how large the markets must be for the losses to be small.

There are three kinds of challenges to designing mechanisms for markets without convexity, which we illustrate by example in Section 1.1. One is that in some efficient allocations, production

[1] Certain assumptions are needed to reach conclusions about large markets, which we discuss further in Section 4.3.

[2] With fixed linear prices, the allocation is unaffected if firms merge or break up, providing a further good incentive property.

strictly exceeds consumption, which makes *budget balance* with positive prices impossible.[3] Second, even when efficient allocations equalize production and consumption, there may be no prices such that all agents prefer the recommended allocation to the zero trade: *individual rationality* is violated. Third, even if there exist prices that make participation individually rational, some agents may need to be *rationed* at those prices, that is, some agents may strictly prefer a different bundle over the prescribed one. This may create an incentive for even a price-taking agent to report falsely in order to affect its allocation. These complications are avoidable when preferences and production sets are convex, but they provide challenges that need to overcome when designing pricing mechanisms for markets in which convexity is not guaranteed.

We address these challenges in two ways. Firstly, to achieve budget balance, we study *double-price* mechanisms in which buyers may be charged slightly higher prices than are paid to producers. The price premium can be used to cover the cost of any units which must be discarded by the market operator. Secondly, we also study mechanisms in which the mechanism designer may ration some agents.

Our analysis applies to economies in which all agents have quasilinear preferences. Our first key result provides an upper bound on the welfare loss from any feasible allocation. Given *any* feasible allocation and *any* price vector, the welfare loss from the allocation is at most equal to the sum of the budget deficit plus the agents' *rationed surplus* at the specified prices. We define an agent's rationed surplus to be the difference between the payoff (utility or profits) it would obtain from its most preferred bundle at the specified prices and its payoff at the given allocation. If the price and allocation pair is a competitive equilibrium, the allocation involves no rationing or budget deficit, so our Theorem 1 implies that the welfare loss is zero. With that special case in mind, we call this result the *Bound-Form First Welfare Theorem*.

We use this Bound-Form First Welfare Theorem to construct pricing mechanisms in which losses are small, despite non-convexities and failures of market clearing. In Section 4, we introduce a mechanism that produces an allocation and different prices for producers and consumers such that the allocation is approximately efficient and budget-balanced, and participation is individually-rational. We call this mechanism the *α-price mechanism*, because buyers pay prices that are $(1+\alpha)$ times the prices paid to sellers. We can choose α close to zero when the economy is large. Theorem 2 shows that the welfare loss of this mechanism is bounded by a constant independently of the market size, so the *fraction* of the efficient surplus lost, the *relative inefficiency* of the mechanism, is at most inversely proportional to the market size. Theorem 3 provides conditions

[3] We assume that buyers and sellers have free disposal, so there is always *some* efficient allocation with exact market-clearing. However, this efficient allocation may include bundles that would never be chosen by price-taking agents.

under which the benefits of misreporting in this mechanism are arbitrarily small for sufficiently large markets, so that the mechanism has good incentive properties similar to those of the Walrasian mechanism demonstrated by Roberts and Postlewaite (1976). The mechanism may also offer computational advantages in non-convex economies, because it computes optima only for convex problems, while computing the exactly efficient allocation is generally NP-hard.

In Section 4, we consider the case when non-convexities can occur only on one side of the market, that is, for producers but not consumers. In this case, we introduce a *single* price mechanism in which some agents are rationed. Assuming that consumer preferences are strongly convex (Watt, 2021), we find that the mechanism's *total* losses are inversely proportional to the market size. Consequently, their *relative inefficiency* is inversely proportional to the *square* of market size. This mechanism is budget-balanced, has the same incentive properties as the double-price mechanism and may be efficiently computed as the solution of a convex optimization problem. It is also approximately individually rational, and exactly so if buyers' preferences are separable.

Finally, we discuss some applications of our results to some real-world markets, including the fisheries auctions designed by Bichler, Fux, and Goeree (2018, 2019).

1.1 An example

Here we offer a simple example of a setting with indivisibilities to illustrate the three complications of non-convex markets discussed above: the need to dispose of excess production, the nonexistence of supporting prices, and the need for rationing at any prices. The example also suggests our two mechanisms – the double-price mechanism and a single-price mechanism with careful rationing – can result in approximately-efficient outcomes.

Consider a market with one good and a single producer. The producer can only make only non-negative integer numbers of units a marginal cost $c = \$0.50$. These indivisibilities make this a non-convex production technology.

There are N buyers indexed $n = 1, ..., N$. If the price is low enough, buyer n would like to purchase *exactly* q_n units of the good: buyers derive no value from receiving fewer than q_n units and no additional value for additional units beyond q_n. We refer to q_n as buyer n's *order size*. Buyers have quasilinear preferences, where buyer n's per-unit value for the good is v_n, so its utility $U_n(x_n, t)$ from paying $\$t$ for x_n units is

$$U_n(x_n, t) = \begin{cases} -t & \text{if } x_n < q_n \\ q_n v_n - t & \text{if } x_n \geq q_n. \end{cases}$$

This utility function is not quasiconcave, so that buyers' preferences do not satisfy the standard convexity assumption of general equilibrium theory.

First, consider two small versions of this market, each with three potential buyers, with values and order sizes as tabulated in Table 1 and Table 2 (differing only in the unit value for buyer 3).

Buyer n	1	2	3
Unit value v_n	0.9	0.8	0.6
Order size q_n	0.3	0.8	0.1

Table 1: Buyers in small market 1

Buyer n	1	2	3
Unit value v_n	0.9	0.8	0.4
Order size q_n	0.3	0.8	0.1

Table 2: Buyers in small market 2

In small market 1, the efficient allocation has the seller producing 1 unit of the good and buyers 2 and 3 being allocated 0.8 and 0.1 units of the good, respectively. The buyers have no value for the additional 0.1 units of the good, so if an efficient allocation were to be supported by prices, these 0.1 units would need to be discarded.[4] If this allocation is to be implemented with a price of \$0.50 (the seller's marginal cost), then buyer 1 must be rationed at this price, and at any other price, the seller would need to be rationed. The efficient allocation in small market 2 is identical, but note that this allocation *cannot* be implemented by any price in a way that makes participation individually rational: the seller is unwilling to accept any price below $p = \$0.50$, but buyer 3 refuses to purchase at price p or higher.[5]

Similar problems remain as the market size increases. For suppose that N is large, and that each buyer n's preference parameters q_n and v_n are drawn independently and uniformly from the interval $[0, 1]$. For any non-trivial feasible allocation (including an efficient one) to be supported by prices, the mechanism designer will almost always be required to dispose of some unallocated production. A realization of this market with ten buyers is plotted in Figure 1. In this realization, the efficient allocation involves no sale to the highest-value buyer but selling units to the next five highest-value buyers, so that the supporting price $p = \$0.50$ requires rationing. It is also straightforward to construct examples in which there are no supporting prices.

The efficient allocations can be obtained in these examples as the solutions to simple integer programming problems but, in general, such optimization problems are NP-hard and become computationally intractable with more agents and goods. To find a mechanism that remains easily computable as the problem scales, we need to look beyond exact solutions of integer program-

[4] If we add an assumption of free disposal, the 0.1 units of the good could be allocated to either of the buyers, but, with any linear pricing rule, both would prefer *not* to purchase the additional units.

[5] In addition, if the price were fixed at $p = \$0.50$ and the firm's quantity were fixed at one unit, buyer 1 could benefit by increasing its reported value to, say, \$2.70 to change the mechanism designer's rationing decision. Incentives to misreport in a mechanism with both prices and rationing must account for the effect of the report on both the price and the rationing.

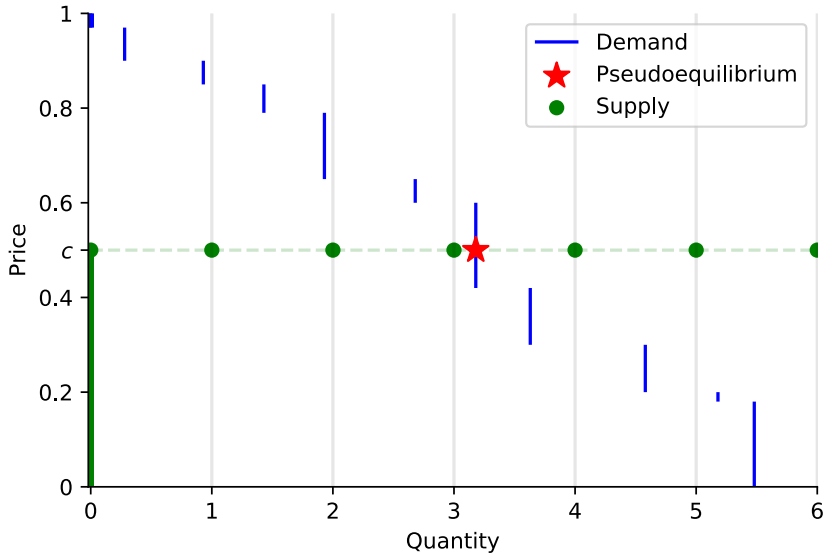


Figure 1: A larger market example

ming problems.

Our solution is to use the well known method of solving a related convex problem and rounding the solution. To do that, we replace each buyer's payoff function by the smallest concave function that everywhere exceeds the original function. We replace each seller's cost function by the largest convex function that is everywhere weakly less. This leads to a *pseudoequilibrium* allocation and supporting prices which may yield a higher surplus than an efficient allocation in the original problem. Such a pseudoequilibrium is illustrated in Figure 1. From this pseudoequilibrium, we can round the supplier's allocation to the next highest integer and reduce to zero the assignment of any buyer who is allocated less than q_n units in the pseudoequilibrium. The Shapley-Folkman Lemma ensures that if the optimum is unique, then there is at most one such buyer. The resulting allocation will not, in general, be market-clearing, so with any positive price there must be a budget deficit. To cover the budget deficit, we raise the price facing the buyers by a small amount. (We show later how to modify this construction, by perturbing the objective of the surplus maximization problem, in order to guarantee that the resulting price and allocation also satisfy individual rationality.) The required price increase becomes vanishingly small as the market gets large. Consequently, the resulting allocation will have total welfare close to that of the pseudoequilibrium, which is itself an upper bound on the efficient surplus, ensuring that the resulting allocation is approximately efficient. This is an intuitive description of our construction of approximately-efficient double-price mechanisms in Section 4.

1.2 Related literature

The problem of non-convexity for the existence of competitive equilibrium was discussed in a series of papers by Farrell (1959), Rothenberg (1960), Koopmans (1961) and Bator (1961). Much of the subsequent classical literature on non-convexity in general equilibrium theory focused on concepts of *approximate* equilibria which replace aggregate feasibility requirements with approximate feasibility, measured in terms of distance in the commodity space between the aggregate supply and demand, while maintaining the requirement that individual agents act optimally given the prices. Starr (1969) showed the existence of such an approximate equilibrium in non-convex production economies, in which the maximum imbalance is proportional to the number of goods and a measure of non-convexity. Heller (1972) proved a similar result with an alternative measure of non-convexity. More recently, Nguyen and Vohra (2020) proved a bound for markets with indivisible goods that depends only on a measure of preference complementarity of agents. We build on some of these results (summarizing the key results we employ in Appendix A), but depart from this literature by requiring that any feasible mechanism must always specify a feasible outcome. Influenced by computer scientists' approaches to approximations in mechanism design, we will be interested in approximations in surplus space, rather than in commodity space.[6]

A substantial literature has focused on identifying various *substitutes* conditions on valuation functions in quasilinear markets with indivisibilities, under which competitive equilibria exist despite non-convexities. Contributors include Bikhchandani and Mamer (1997), Gul and Stacchetti (1999), Danilov, Koshevoy, and Murota (2001), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), Baldwin and Klemperer (2019) and Nguyen and Vohra (2020). None of these results apply to markets with fixed costs such as those described above, in which case competitive equilibria do not generally exist. Our analysis seeks to develop practical mechanisms for those settings.

An alternative approach to establish equilibrium existence in markets with non-convexities is to study the large market limit with a continuum of agents. Aumann (1966) showed that the convexity assumption is not necessary for equilibrium existence in an economy with a continuum of traders and divisible goods, while Azevedo, Weyl, and White (2013) demonstrated a similar result for an economy with indivisibilities. In this paper, we study economies with a finite number of agents, which allows us to investigate properties of not only the limiting economy, but also rates of convergence, which are of practical significance to market designers who wish to apply mechanisms to finite economies. Another approach is to allow nonlinear pricing rules, as explored

[6]This is also the approach to approximation taken by Scarf (1967).

by Wilson (1993), Chavas and Briec (2012), Azizan, Su, Dvijotham, and Wierman (2020) and others, but we focus on what can be achieved by linear pricing rules, which seem to be desired for many applications.

Our study is motivated by other studies of market design with linear prices and non-convex production. In particular, we have taken inspiration from the novel market design for fisheries rights in New South Wales, Australia, introduced by Bichler et al. (2018, 2019). Other markets with non-convexities that have been studied in the market design literature include electricity markets, where large start-up and ramping costs are the source of the non-convexity, and spectrum auctions, where geographical complementarities cause exposure problems.[7]

1.3 Notation

In this paper, we will view consumption and production bundles as vectors in Euclidean space $\mathbb{R}^L$, and let $x \cdot y$ denote the standard inner product of x and y and $\|x\| = \sqrt{x \cdot x}$ denote the Euclidean norm of x. We use $\geq$ to denote the partial order on $\mathbb{R}^L$ so that $x \geq y$ if and only if $x_\ell \geq y_\ell$ for $\ell = 1, ..., L$. The set $\mathbb{R}^L_+$ is $\{x \in \mathbb{R}^L : x \geq 0\}$. The radius of a set $S \subseteq \mathbb{R}^L$ is defined by $\text{rad}(S) = \inf_{x \in \mathbb{R}^L} \sup_{y \in S} \|x - y\|$. The notation $|\cdot|$ represents either the absolute value (if its argument is a number) or the cardinality (if its argument is a set). The Hausdorff distance between sets $S, S' \subseteq \mathbb{R}^L$ is denoted d_H and is defined by letting $\text{dist}(x, S) = \inf_{y \in S} \|x - y\|$ and $d_H(S, S') = \sup_{x \in S'} \text{dist}(x, S)$.

The convex hull of a set $S \subseteq \mathbb{R}^L$ is denoted by $\text{co}(S) = \{x \in \mathbb{R}^L : x = \lambda y + (1 - \lambda)z \text{ for } y, z \in S \text{ and } \lambda \in [0, 1]\}$. The upper concave envelope of $f : S \to \mathbb{R}$ is the function $\text{cav}(f) : \text{co}(S) \to \mathbb{R}$ which is the (pointwise) smallest concave function g on $\text{co}(S)$ that satisfies $g(x) \geq f(x)$ for all $x \in S$. The lower convex envelope of $f : S \to \mathbb{R}$ is the function $\text{vex}(f) : \text{co}(S) \to \mathbb{R}$ which is the largest convex function g on $\text{co}(S)$ that satisfies $g(x) \leq f(x)$ for all $x \in S$.

We make regular use of the asymptotic notation due to Knuth (1976). For functions f and g both mapping $\mathbb{R}$ to $\mathbb{R}$, we write $f(x) = O(g(x))$ if $\limsup_{x \to \infty} \frac{|f(x)|}{g(x)} < \infty$. We write $f(x) = \Omega(g(x))$ if $\liminf_{x \to \infty} \frac{|f(x)|}{g(x)} > 0$. Finally, we write $f(x) = \Theta(g(x))$ if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

[7] See Liberopoulos and Andrianesis (2016) for a summary of pricing mechanisms used in electricity markets with non-convexities, most of which include "uplift" (or side-payments) in addition to linear pricing, and Ausubel and Milgrom (2002) for a discussion of complementarities in spectrum auctions.

2 Model and preliminaries

2.1 Model

We employ a Walrasian model with a set of **buyers** N and a set of **firms** or **sellers** F, both finite. Together we refer to $A = N \cup F$ as the set of **agents**. There are L varieties of consumable **goods** and a numeraire good, money.

Each buyer $n \in N$ can choose consumption bundles in X, a compact subset of $\mathbb{R}^L_+$ containing 0, called the **consumption possibility set**. Buyer n has quasilinear preferences[8] over bundles in X with a continuous **valuation function** $u_n : X \to \mathbb{R}$, so that the buyer's **utility** associated with allocation x_n after payment t is $U_n(x_n, t) = u_n(x_n) - t$. We suppose that the valuation functions are bounded, nondecreasing[9] with respect to the partial order $\geq$ on $\mathbb{R}^L_+$ and normalized so that $u_n(0) = 0$. We let $\mathscr{U}$ be the space of possible valuation functions for the buyers, which we assume to be an admissible function space in the sense of Aumann (1963) (that is, it is possible to define a measure on $\mathscr{U}$).[10]

Each seller $f \in F$ can choose a production bundle in the **production possibility set** Y, a compact subset of $\mathbb{R}^L_+$ containing 0. Seller f has a **cost function**[11] $c_f : Y \to \mathbb{R}_+$ which allows us to write f's **profit** from producing $y_f \in Y$ and receiving payment t as $\pi_f(y_f, t) = t - c_f(y_f)$. The cost functions are nondecreasing with respect to the partial order $\geq$ on $\mathbb{R}^L_+$ and normalized so that $c_f(0) = 0$. Let $\mathscr{C}$ be the space of sellers' cost functions, which we also assume to be admissible.

An **economy** $\mathscr{E}$ consists of buyers with their valuation functions and sellers with their cost functions, so that we may write $\mathscr{E} = \langle N, (u_n)_{n\in N}, F, (c_f)_{f\in F}\rangle$. When it is clear, we use the shorthand $\mathscr{E} = \langle N, F\rangle$. At times, it is also convenient to associate $\mathscr{E}$ with the normalized counting measures μ on $\mathscr{U}$ and ν on $\mathscr{C}$ defined by

$$\mu(u_n) = \frac{\#\text{ of buyers in } \mathscr{E} \text{ with valuation function } u_n}{|N|},$$
$$\chi(c_f) = \frac{\#\text{ of sellers in } \mathscr{E} \text{ with cost function } c_f}{|F|},$$

and to let $\phi = \frac{|F|}{|N|}$, so that $\langle N, \mu, \phi, \chi\rangle$ is an alternative specification of economy $\mathscr{E}$.

[8] The quasilinearity assumption allows our analysis to abstract from income effects, as is usual in mechanism design analyses. For more discussion of the role of income effects see Morimoto and Serizawa (2015).

[9] This form of monotonicity captures our assumption of free disposal.

[10] For example the set of bounded, continuous functions on a compact subset of $\mathbb{R}^L$ is admissible, as is the set of bounded functions with discontinuities of the first kind, or more generally, any subset of a Baire class (Aumann, 1963).

[11] Note that sellers in this economy could equivalently be thought of as buyers with valuations $-c_f(y_f)$ and payments $-t$. However, we will be interested in mechanisms that may charge buyers and sellers different prices, and so it is convenient to distinguish the two groups in our notation.

Allocations and efficiency An **allocation** $\omega = ((x_n)_{n\in N}, (y_f)_{f\in F})$ is an assignment of consumption bundles $x_n \in X$ to each buyer $n \in N$ and production bundles $y_f \in Y$ to each seller $f \in F$. An allocation is **feasible** if $\sum_{n\in N} x_n \leq \sum_{f\in F} y_f$. We denote by Ω the set of all feasible allocations.

We define the **surplus** $\mathcal{S}(\omega)$ associated with allocation $\omega \in \Omega$ by

$$\mathcal{S}(\omega) = \sum_{n\in N} u_n(x_n) - \sum_{f\in F} c_f(y_f).$$

The **efficient allocation** problem is to solve

$$\max_{\omega\in\Omega} \mathcal{S}(\omega), \tag{P}$$

with the resulting surplus denoted by $\mathcal{S}^*$.

For any allocation $\omega \in \Omega$, we will refer to $\mathcal{S}(\omega) - \mathcal{S}^*$ as the **deadweight loss** of ω and the ratio $\frac{\mathcal{S}(\omega)-\mathcal{S}^*}{\mathcal{S}^*}$ as the **relative inefficiency** of ω.[12]

Pricing rules We are interested in settings where buyers and sellers face a payment rule in the form of **linear prices**.

If a buyer n faces price $p \in \mathbb{R}^L_+$ and purchases bundle x, the resulting payment is $t = p \cdot x$, leading to utility $U_n(x, p \cdot x) = u_n(x) - p \cdot x$. The **demand correspondence** $D_n : \mathbb{R}^L_+ \rightrightarrows X$ maps each price vector p to the set of utility-maximizing bundles $D_n(p)$ which result in the **indirect utility**, $\hat{u}_n(p) = \max_{x\in X} u_n(x) - p \cdot x$.

Similarly, a seller f facing price p and supplying bundle y receives profit $\pi_f(y, p \cdot y)$. Seller f's **supply correspondence** $S_f : \mathbb{R}^L_+ \rightrightarrows Y$ maps each price vector p to the set of profit-maximizing bundles $S_f(p)$ which result in **producer surplus** $\hat{\pi}_f(p) = \max_{y\in Y} p \cdot y - c_f(y)$.

A pricing rule is **anonymous** if for all $n, n' \in N$, $p_n = p'_n$ and for all $f \in F$, $p_f = p'_f$. A pricing rule which is not anonymous is called **personalized**. A **single-pricing** rule imposes the same price vector p on all agents. Single-pricing has been the traditional emphasis of general equilibrium theory.

[12] Later we will make assumptions to rule out cases where $\mathcal{S}^* = 0$ so that this ratio is well-defined.

2.2 Convex quasilinear economies

Convexity is defined with respect to the set of payoff-improving allocations for an agent in the economy. The $\bar{u}$-**upper contour set** of buyer $n \in N$ is defined by

$$UC_n^{\bar{u}} = \{(x,t) \in X \times \mathbb{R} : U_n(x,t) \geq \bar{u}\},$$

while the $\bar{\pi}$-**upper contour set** of seller $f \in F$ is given by

$$UC_f^{\bar{\pi}} = \{(y,t) \in Y \times \mathbb{R} : \pi_f(y,t) \geq \bar{\pi}\}.$$

We say that buyer n has **convex preferences** if X is convex and $UC_n^{\bar{u}}$ is convex for all $\bar{u} \in \mathbb{R}$, while we say that seller f has **convex technology** if Y is convex and $UC_f^{\bar{\pi}}$ is convex for all $\bar{\pi} \in \mathbb{R}$. Equivalently, buyer n has convex preferences if and only if the utility function U_n is quasiconcave, and seller f has convex technology if and only if the profit function π_f is quasiconcave. In our quasilinear environment, convexity of preferences and technologies has the following well known characterization:

Proposition 1. *The utility function $U_n(x,t) = u_n(x) - t$ is quasiconcave if and only if the valuation function $u_n(x)$ is concave. Similarly, the profit function $\pi_f(y,t) = t - c_f(y)$ is quasiconcave if and only if the cost function $c_f(y)$ is convex.*

Under the assumption of quasilinearity and the convexity of agents' preferences and technologies, we have the following statement of the fundamental welfare theorems of Arrow (1951) and Debreu (1951). [13]

Proposition 2. *Suppose in (quasilinear) economy $\mathscr{E}$ that all buyers $n \in N$ have convex preferences and all sellers $f \in F$ have convex technology. Then a feasible allocation $\omega \in \Omega$ is efficient if and only if there exists $p \in \mathbb{R}^L_+, p > 0$ such that for all $n \in N$, $x_n \in D_n(p)$; for all $f \in F$, $y_f \in S_f(p)$; and $\sum_{n\in N} p \cdot x_n = \sum_{f\in F} p \cdot y_f$. The pair (p,ω) is a **competitive** or **Walrasian equilibrium**.*

[13] The statement of Proposition 2 is stronger than the classic statements of the welfare theorems in the 'only if' direction, which is possible due to the quasilinear form of the utility and profit functions. Without quasilinearity or an alternative assumption, it may only be possible to find prices so that agents are expenditure-minimizing for a given level of utility or profit, that is, a price quasiequilibrium with transfers. With quasilinearity and convexity, Proposition 1 implies that the efficient allocation program is convex. Since there exists a feasible allocation, Slater's Theorem (see, for example, Boyd and Vandenberghe (2004)) implies strong duality. A solution p^* to the dual program, $\inf_{p\in\mathbb{R}^L_+} \sum_{n\in N} \hat{u}_n(p) + \sum_{f\in F} \hat{\pi}_f(p)$, and an efficient allocation ω^* comprise a saddle point for the Lagrangian $\mathscr{L}(\omega,p) = \sum_{n\in N} u_n(x_n) - \sum_{f\in F} c_f(y_f) - p \cdot \left(\sum_{n\in N} x_n - \sum_{f\in F} y_f\right)$, so that for any $\omega \in \Omega$, $\mathscr{L}(\omega,p^*) \leq \mathscr{L}(\omega^*,p^*)$. Because the Lagrangian is separable across agents, the saddle point condition implies $x_n^* \in D_n(p^*)$ and $y_f^* \in S_f(p^*)$.

2.3 Measures of non-convexity and approximate equilibria

The non-convexity of a set S can be measured in several ways by comparing S and $\text{co}(S)$. We will work with the following measures of non-convexity of a set:

- The **inner radius** of S is $r(S) = \sup_{x \in \text{co}(S)} \inf_{T \subseteq S : x \in \text{co}(T)} \text{rad}(T)$.
- The **inner distance** of S is $\rho(S) = \sup_{x \in \text{co}(S)} \inf_{y \in S} \|x - y\|$.

For a convex set S, it is clear that $r(S) = 0 = \rho(S)$, so that these are appropriate measures of the non-convexity of S. These measures are illustrated in Figure 2.

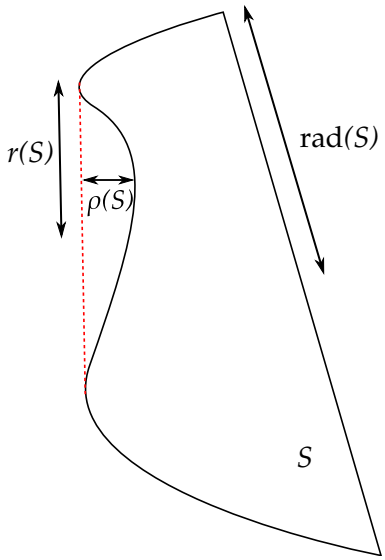


Figure 2: Measures of non-convexity of a set

The non-convexity of the preferences of buyer $n \in N$ may be measured by the largest inner radius or inner distance of their upper contour sets, that is $r_n = \sup_{\bar{u} \in \mathbb{R}} r(UC_n^{\bar{u}})$ or $\rho_n = \sup_{\bar{u} \in \mathbb{R}} \rho(UC_n^{\bar{u}})$. Similarly, the non-convexity of the technology of seller $f \in F$ may be measured by $r_f = \sup_{\bar{\pi} \in \mathbb{R}} r(UC_f^{\bar{\pi}})$ or $\rho_f = \sup_{\bar{\pi} \in \mathbb{R}} \rho(UC_f^{\bar{\pi}})$. Let $r_{\mathscr{E}}$ and $\rho_{\mathscr{E}}$ denote the largest of such measures among all the buyers and sellers in economy $\mathscr{E}$.

When agents' upper contour sets are not convex, the second welfare theorem may not hold and there may be no competitive equilibrium. Proposition 3, the Shapley-Folkman Lemma, assists in identifying allocations which are nearly competitive equilibria.

Proposition 3 (Shapley-Folkman Lemma[14]). *Let $S_i \subseteq \mathbb{R}^L$ for $i = 1, ..., M$, $S = \bigoplus_{i=1}^M S_i$ and $L' =$*

[14]It is perhaps most accurate to refer only to the result in the second sentence of Proposition 3 as the Shapley-Folkman Lemma, althought it was first reported by Starr (1969) as a result of private communication with Lloyd Shapley and Jon Folkman. Starr (1969) then proved the first half of sentence three of Proposition 3, while Heller (1972) proved the second half. For simplicity, we will refer to the whole of Proposition 3 as the Shapley-Folkman Lemma.

$\min(L, M)$. *Then for any* $x \in \text{co}(S)$, $x = \sum_{i=1}^{N} x_i$ *where* $x_i \in \text{co}(S_i)$ *and* $|i : x_i \in \text{co}(S_i) \setminus S_i| \leq L'$. *Moreover, there exists* $y, y' \in S$ *such that* $\|x - y\| \leq r_{\mathscr{E}}\sqrt{L'}$ *and* $\|x - y'\| \leq \rho_{\mathscr{E}} L'$.

Proposition 3 has been used to establish results about what are called *approximate equilibria*, which are constructed as follows. First, consider a convexified version of the non-convex economy in which the upper contour sets of all agents are replaced by their convex hulls. This is equivalent to replacing the buyers' valuation functions by their upper concave envelopes and sellers' cost functions by their lower convex envelopes. The **convexified economy** $\hat{\mathscr{E}}$ is then

$$\hat{\mathscr{E}} = \langle N, (\text{cav}(u_n))_{n \in N}, F, (\text{vex}(c_f))_{f \in F} \rangle.$$

The convexified economy has a competitive equilibrium which is efficient by Proposition 2. We call the resulting price-allocation pair (p, ω) a **pseudoequilibrium** of the actual economy $\mathscr{E}$. Proposition 3 implies that ω can be chosen so at most L' agents in $\mathscr{E}$ are not utility- or profit-maximizing at ω given prices p. Proposition 3 implies that there is a nearby allocation ω' such that *all* agents are maximizing given prices p,[15] but markets may not exactly clear at ω'. The price-allocation pair (p, ω') is called an **approximate equilibrium**.

Practical market design requires allocations to be feasible and voluntary, so that pseudoequilibria and approximate equilibria are often insufficient. We modify the approach to approximation of equilibria in Section 4 to obtain mechanisms whose allocations satisfy these requirements.

3 Rationing, inefficiency and incentives

When competitive equilibrium does not exist, allocations supported by a price vector must, by definition, entail rationing or non-exact market-clearing (and therefore budget imbalance), or both. Rationing occurs at a given price and feasible allocation if the demand of some buyers is unfulfilled or the supply of certain sellers is unneeded. For our purposes, it will be helpful to measure the extent of rationing for buyers and sellers by what we call the *rationed utility* and *rationed profit*, which are both defined by the difference between the payoff an agent would obtain at their preferred bundle given the prices and the payoff they instead receive in the prescribed allocation at those prices.

[15] To see this, note that if a buyer is assigned a bundle x_n in ω that is not utility-maximizing at p, then x_n must be the convex combination of bundles (x'_n) in X which are *exposed points* in u_n (i.e. where $\text{cav}(u_n) = u_n$), and that the agents in the convexified economy must be indifferent between x_n and these bundles. That is, the concavified portions of buyers' utility functions consist of (patches of) hyperplanes, and if an agent is assigned a bundle on such a patch, then the price vector must be normal to that hyperplane. This implies that the original buyer must be maximizing at bundles in (x'_n), which are on the relative boundaries of the patch of hyperplane.

Definition 3.1. The **rationed utility** $\mathscr{R}_n(p,x)$ **of buyer** n at price p and allocation x is

$$\mathscr{R}_n(p,x) = \hat{u}_n(p) - U_n(x, p\cdot x).$$

The **rationed profit** $\mathscr{R}_f(p,y)$ **of seller** f is

$$\mathscr{R}_f(p,y) = \hat{\pi}_f(p) - \pi_f(y, p\cdot y).$$

The **rationed surplus** of allocation $\omega = ((x_n)_{n\in N}, (y_f)_{f\in F})$ at price p is defined by

$$\mathscr{R}(p,\omega) = \sum_{n\in N} \mathscr{R}_n(p,x_n) + \sum_{f\in F} \mathscr{R}_f(p,y_f).$$

With this definition, we can state our first main result.

Theorem 1 (Bound-Form First Welfare Theorem)**.** *Let $p \in \mathbb{R}^L_+$ be a price vector and $\omega = ((x_n)_{n\in N}, (y_f)_{f\in F})$ be any feasible allocation. Then, the deadweight loss of allocation ω satisfies*

$$\underbrace{\mathcal{S}^* - \mathcal{S}(\omega)}_{\textit{deadweight loss}} \leq \underbrace{\mathscr{R}(p,\omega)}_{\textit{rationed surplus}} + \underbrace{p\cdot\left(\sum_{f\in F} y_f - \sum_{n\in N} x_n\right)}_{\textit{budget deficit}}.$$

Proof. Fix any efficient allocation ω^*. By definition of the indirect utility and consumer surplus functions, the following must hold for any prices:

$$\hat{u}_n(p) \geq u_n(x_n^*) - p\cdot x_n^*$$
$$\hat{\pi}_f(p) \geq p\cdot y_f^* - c_f(y_f^*).$$

Summing these inequalities, we obtain

$$\sum_{n\in N} \hat{u}_n(p) + \sum_{f\in F} \hat{\pi}_f(p) + p\cdot\left(\sum_{n\in N} x_n^* - \sum_{f\in F} y_f^*\right) \geq \mathcal{S}^*.$$

Since ω^* is feasible, the third term is non-positive, which implies

$$\sum_{n\in N} \hat{u}_n(p) + \sum_{f\in F} \hat{\pi}_f(p) \geq \mathcal{S}^*$$

Subtracting $\mathcal{S}(\omega)$ and using our rationing definitions,

$$\mathcal{S}^* - \mathcal{S}(\omega) \leq \sum_{n \in N} \hat{u}_n(p) + \sum_{f \in F} \hat{\pi}_f(p) - \mathcal{S}(\omega) = \mathcal{R}(p, \omega) + p \cdot \left(\sum_{f \in F} y_f - \sum_{n \in N} x_n \right),$$

which is what we sought to prove. □

The Bound-Form First Welfare Theorem is a generalization of the First Welfare Theorem for quasilinear economies. Indeed, if p and ω are the price and allocation associated with a Walrasian equilibrium, then the right-hand side of the above inequality is zero and we recover the statement that Walrasian equilibria are efficient. One interpretation of the First Welfare Theorem is that prices act as a "certificate of optimality": if supporting prices exist for some allocation, then that allocation is efficient. As Scarf (1994) lamented, in the absence of convexity, there is, in general, no such optimality test. However, one interpretation of Theorem 1 is as an *approximate* optimality test: if we can show that the rationed surplus plus net subsidies to sellers is small, then the given allocation will be approximately efficient. Theorem 1 is also useful in the way that it links incentives to efficiency: since for any fixed price, agents would prefer not to be rationed, Theorem 1 suggests that a pricing mechanism with little rationing and in which individual agents have little influence over prices will have good incentive properties and small deadweight loss. We will exploit these properties of Theorem 1 to design pricing mechanisms in Section 4.

4 Two-sided non-convexities: the α^*-price mechanism

4.1 Pricing mechanisms

Taking a market design perspective, we study direct pricing mechanisms that map profiles reports about sellers' cost functions $(c_f)_{f \in F}$ and buyers' value functions $(u_n)_{n \in N}$ to a feasible allocation $\omega \in \Omega$ and prices for each agent $\boldsymbol{p} = ((p_n)_{n \in N}, (p_f)_{f \in F})$.[16] We do not delve into the complicated question of how agents communicate these potentially complicated costs and values to the mechanism.[17]

We use familiar definitions to describe certain properties of the mechanisms. A pricing mechanism is **incentive-compatible** if truthful reporting is a Nash equilibrium of the reporting game induced by the mechanism. The mechanism is **efficient** if the output allocation ω is an efficient

[16] For simplicity, we do not consider randomized mechanisms as we will be able to achieve our objectives using deterministic mechanisms.

[17] The design of reporting languages to report complex preferences for economic mechanisms has been studied by Milgrom (2009), Bichler, Goeree, Mayer, and Shabalin (2014), Bichler, Milgrom, and Schwarz (2020) and others.

allocation given the reported value functions and cost functions. It is **budget-balanced** if, for all report profiles, $\sum_{f\in F} p_f \cdot y_f = \sum_{n\in N} p_n \cdot x_n$, while it is **weakly budget-balanced** if $\sum_{f\in F} p_f \cdot y_f \leq \sum_{n\in N} p_n \cdot x_n$. Finally, the pricing mechanism is ε-**individually-rational** if, given reported value and cost functions, the allocation and price determined by the mechanism delivers each agent a payoff (utility or profit) of at least $-\varepsilon$. For $\varepsilon = 0$, we simply say the mechanism is **individually-rational**.

4.2 Approximately-efficient pricing mechanisms

In this subsection, we introduce a weakly budget-balanced double-pricing mechanism (one set of prices for buyers and another for sellers) which is *approximately* efficient, which means that the relative inefficency of the mechanism approaches zero as the economy becomes large, even if the preferences and/or production sets are not convex, provided that the radius of any non-convexity is bounded. We also provide a worst-case bound on the loss of surplus compared to the first-best allocation. Before providing a technical description of this pricing mechanism and its analysis, we sketch intuitively the steps of our approach.

For a weakly budget-balanced mechanism to allow free disposal, it must charge higher prices to buyers than are paid to the sellers. With that in mind, we focus on the class of *α-pricing mechanisms*, where payments to sellers are determined by a price vector p, while payments by buyers are determined by a price vector $(1+\alpha)p$ for some scalar $\alpha > 0$ to be chosen later. We select p to be the equilibrium price vector of a related economy with three changes from the actual economy: (1) every buyers' value function is replaced by the smaller function $u_n/(1+\alpha)$, (2) all values and costs are then replaced by their concave or convex hulls, respectively, and (3) we add demand for each good by the auctioneer in the amount of $R := \max\{r_{\mathscr{E}}\sqrt{L}, \rho_{\mathscr{E}} L\}$. We call the resulting price and allocation a *pseudoequilibrium* of our economy.

Our next step shows that this pseudoequilibrium allocation is nearly optimal for the adjusted economy without the $(1+\alpha)$ rescaling of buyers' values. Indeed, we show that the welfare loss from using the pseudoequilibrium allocation instead of the optimum with the unscaled values is of lower order than α. To prove that, we apply the Bound-Form First Welfare Theorem to the pseudoequilibrium prices and quantities and the Milgrom and Segal (2002) envelope theorem.

By construction, if the step (1) rescaling is omitted but buyers' prices are set to $(1+\alpha)p$ instead of p, the buyers' demands are unchanged, so with those prices the pseudoequilibrium allocation is still individually rational.

If the convexification of the economy matters for an agent, then its supply or demand without convexification is set-valued. Our next step is to apply the Shapley-Folkman lemma to round

the pseudoequilibrium allocation to some alternative demanded allocations for each agent, but in which the result changes the net demand for each good by at most R units – an amount that we propose to take from the units otherwise allocated to the auctioneer. The final allocation always has supply greater than or equal to demand and its excess supply is no more than $2R$ units of each good. This excess supply can result in a loss of efficiency, but as we have just seen, this loss is bounded by a constant, independently of the size of the market.

Since the excess supply of goods is bounded, the budget imbalance at price p is bounded as well. As trade increases with the size of the economy, the buyers' premium, α, needed to guarantee budget balance is inversely related to market size. Thus, the total welfare loss from this double-price mechanism is bounded by a constant plus a term that is inversely proportional to market size.

We are now ready to present the technical version of our main result. Let $\mathscr{E}_t = \langle N_t, \mu_t, \phi_t, \chi_t \rangle$ be a sequence of economies indexed by $t = 1, 2, ...$ For technical purposes, we assume the following

Assumption 1 (Existence of limit economy)**.** *As $t \to \infty$, $|N_t| \to \infty$ and $\phi_t \to \phi \in (0,1)$. Furthermore, μ_t converges weakly to probability measure μ_∞ on $\mathscr{U}$ and χ_t converges weakly to measure χ_∞ on $\mathscr{C}$.*

We make the following economic assumptions:

Assumption 2 (Individual non-convexities are bounded)**.** *There exists $R > 0$ with $R_{\mathscr{E}_t} < R$ for all t.*

Assumption 3 (Growing gains from trade)**.** *As $t \to \infty$, the efficient surplus $\mathcal{S}_t^*$ grows at least as quickly as $|N_t|$ asymptotically, that is, $\mathcal{S}_t^* = \Omega(|N_t|)$. Since utilities are bounded, equivalently, $\mathcal{S}_t^* = \Theta(|N_t|)$.*

Assumption 4 (Prices are bounded)**.** *There exists $M > 0$ such that $d_H\left(\sum_{n\in N_t} D_n(p), \sum_{f\in F_t} S_f(p)\right) \geq R$ for sufficiently large t and $p \in \mathbb{R}_+^L$ such that $\|p\| < \frac{1}{M}$ or $\|p\| > M$.*

Assumption 1 ensures that we may discuss meaningfully the mathematical properties of the mechanism as the number of agents becomes large. Assumption 2 prevents the economy from becoming more non-convex as the number of agents grow, while Assumption 3 reflects the natural economic assumption that the possible gains from trade grow with the size of the economy. Assumption 4 implies the compactness of the set of possible prices for which the aggregate demand and supply to be close to equal, so that pseudoequilibrium prices do not limit to zero or infinity as the economy grows. An alternative to Assumption 4 is to assume that all $u \in \mathscr{U}$ satisfy $\max_{x\in X\setminus\{0\}} \frac{u(x)}{\|x\|} < M$ and for all $c \in \mathscr{C}$, $\min_{y\in Y\setminus\{0\}} \frac{c(y)}{\|y\|} > \frac{1}{M}$.

The α^*-price mechanism is defined for an economy $\mathscr{E}$ as follows.

Definition 4.1 (α^*-price mechanism)**.** Let $\mathscr{E}$ be given. For $\alpha > 0$ to be chosen, define the convex program

$$\min_{p \in \mathbb{R}^L_+} \max_{x_n \in \text{co}(X), y_f \in \text{co}(Y)} \sum_{n \in N} \frac{\text{cav}(u_n)(x_n)}{1+\alpha} - \sum_{f \in F} \text{vex}(c_f)(y_f) - p \cdot \left(\sum_{n \in N} x_n + R1_L - \sum_{f \in F} y_f \right)$$

where 1_L is the vector of ones in $\mathbb{R}^L$. Let $(p^\alpha, \tilde{\omega}^\alpha)$ be a[18] solution to this program (it is a pseudoequilibrium of an economy with buyers' demand scaled down by $(1+\alpha)$ and R units of each good demanded by the mechanism designer).

From $\tilde{\omega}^\alpha$, obtain, via Proposition 3, an allocation ω^α with $\|\omega^\alpha - \tilde{\omega}^\alpha\| \leq R$ such that $x_n^\alpha \in \max_{x \in X} \frac{1}{1+\alpha} u_n(x) - p \cdot x$ for each $n \in N$ and $y_f^\alpha \in S_f(p)$. By construction, this ω^α will be feasible in $\mathscr{E}$.

Let α^* be the smallest α satisfying[19]

$$\alpha \sum_{n \in N} p^\alpha \cdot x_n^\alpha \geq p^\alpha \cdot \left(\sum_{f \in F} y_f^\alpha - \sum_{n \in N} x_n^\alpha \right). \tag{A}$$

Implement the allocation ω^{α^*} charging buyers the price $(1+\alpha^*)p^{\alpha^*}$ and paying sellers according to the price p^{α^*}.

A few properties of this mechanism follow directly from the definition. First, the mechanism is feasible and weakly budget-balanced.[20] Second, the allocation and payments are individually rational for each agent. For sellers, this follows since the price and production allocations obtain equal profits to the pseudoequilibrium[21] allocations of the adjusted convexified economy, in which the production bundles are individually optimal. For buyers, the pseudoequilibrium price p^α and consumption allocation in $\tilde{\omega}^\alpha$ satisfy

$$\frac{1}{1+\alpha} u_n(\tilde{x}_n^\alpha) - p^\alpha \cdot \tilde{x}_n^\alpha = \frac{1}{1+\alpha} u_n(x_n^\alpha) - p^\alpha \cdot x_n^\alpha \geq 0$$

so that $u_n(x_n^\alpha) - (1+\alpha) p^\alpha \cdot x_n^\alpha \geq 0$ as well.

Theorem 2, proved in Appendix B.2, concerns the approximate efficiency of this mechanism.

[18] If there are multiple solutions to the convex program, we suppose there is some fixed way of choosing a solution (e.g., the lexicographically smallest solution), but our argument will not depend on this.

[19] Note that such an α^* always exists since as $\alpha \to \infty$, there is zero trade (which satisfies the inequality).

[20] Note that in general the mechanism *can* result in a budget surplus, because there may be no α that exactly equates the expressions in the inequality defining α^* (the expressions are not continuous, as p^α and ω^α depend on α in some complicated way).

[21] More accurately, because the constraint in the convex program defining $(p^\alpha, \tilde{\omega}^\alpha)$ is not exact market-clearing, this is not exactly a pseudoequilibrium, but we may imagine it as such under the interpretation that the mechanism designer acts as a consumer purchasing R units of each good.

Theorem 2. *Let $\mathscr{E}_t$ be a sequence of economies satisfying Assumptions 1–4. Then the deadweight loss of the allocation associated with the α^*-price mechanism is $O(1)$ as $|N_t| \to \infty$, so that the relative inefficiency is $O\left(\frac{1}{|N_t|}\right)$.*

Remark. Although the rate of convergence in Theorem 2 is stated in terms of $|N_t|$, by Assumption 1, the same asymptotic rate of convergence holds with respect to $|F_t|$ or $|A_t|$.

4.3 Incentives

Under the α^*-price mechanism, although sellers receive their optimal bundle given the price and buyers receive a nearly optimal bundle, agents may have incentives to misreport preferences to influence the prices. In this section, we show that these reporting incentives are effectively the same as those of the usual Walrasian equilibrium mechanism: under certain assumptions on the underlying preferences, the benefit from deviating to an alternative report tends quickly to zero.

The **benefit of misreporting** for an agent in some mechanism is defined as the payoff obtained by the agent under their optimal report, under the assumption that all other agents report truthfully. A mechanism is **limiting incentive-compatible** in a sequence of economies $(\mathscr{E}_t)_{t\in\mathbb{N}}$ if the maximum benefit of misreporting among agents in $\mathscr{E}_t$ tends to zero as $t \to \infty$.

Because the prices obtained in the α^*-price mechanism can be viewed as Walrasian equilibrium prices of a related economy (and in the limit the two mechanisms coincide), many of the key results on incentives in the Walrasian equilibrium can be adapted to the α^*-price mechanism. We now briefly discuss the key results on incentives in the Walrasian mechanism which we adapt to our setting.

For exchange economies, Roberts and Postlewaite (1976) showed that if the Walrasian price correspondence, mapping economies, viewed as measures μ_t on $\mathscr{U}$, to the set of Walrasian equilibrium prices, is continuous at the limiting measure $\lim_{t\to\infty} \mu_t$, then each agent's influence on the price must limit to zero as the number of agents approaches infinity. Jackson (1992) showed that an agent's optimal reported demand converges in the L^∞ norm to the demand associated with that agent's true preferences under the same continuity assumption. Continuity of the Walrasian price correspondence holds, for example, if the limiting economy has a unique clearing price. However, Roberts and Postlewaite (1976) provided an example of a sequence of economies where lower hemicontinuity fails at the limiting economy, and the Walrasian equilibrium mechanism is not limiting incentive-compatible.

Watt (2021) studied the rate of this convergence, showing that a condition on the demand and supply correspondences, called the **strong convexity property**, ensures the fast convergence of

incentives in the Walrasian mechanism. [22]

Definition 4.2. The demand correspondence $D : \mathbb{R}^L_+ \rightrightarrows X$ satisfies the **strong law of demand property** if for all p, p' and $d \in D(p)$, $d' \in D(p')$ we have

$$(p - p') \cdot (d' - d) \geq m\|p - p'\|^2 \text{ for some } m > 0.$$

The supply correspondence $S : \mathbb{R}^L_+ \rightrightarrows Y$ satisfies the **strong law of supply property** if for all p, p' and $s \in S(p)$, $s' \in S(p')$, we have

$$(p - p') \cdot (s - s') \geq m\|p - p'\|^2.$$

An economy has the **strong convexity property** if both the strong law of demand property and the strong law of supply property are satisfied.

If each buyer is drawn independently at random from some distribution μ over $\mathscr{U}$ and each seller is drawn independently at random from χ over $\mathscr{C}$ for which the expected demand and supply correspondences satisfy the strong convexity property, the maximum benefit of misreporting under the Walrasian mechanism is $O\left(\frac{1}{|N_t|}\right)$ in expectation.[23]

While the above results considered the incentives of agents under complete information, Azevedo and Budish (2019) studied Bayesian incentives and show that the Walrasian mechanism (with a finite set of buyer and seller types) is **strategy-proof in the large**. This implies that the benefit of misreporting for any agent against any full-support, independent and identically-distributed distribution of agent types tends to zero at $O(|N_t|^{-\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$.

Theorem 3, proved in Appendix B.3, adapts these results to the α^*-price mechanism.

Theorem 3. *Let $(\mathscr{E}_t)_{t\in\mathbb{N}} = (\langle N_t, \mu_t, \phi_t, \chi_t\rangle)_{t\in\mathbb{N}}$ be a sequence of economies satisfying Assumptions 1–4.*

(a) Let P be the set of Walrasian equilibrium prices for $\langle \mu_\infty, \phi_\infty, \chi_\infty\rangle$. Then the set P' of attainable prices for any agent in the α^*-price mechanism satisfies $d_H(P, P') \to 0$. In addition, if the α^*-price correspondence, mapping economies $\langle N, \mu, \phi, \chi\rangle$ into the set of prices under the α^*-price mechanism, is continuous[24] at the limit economy, then the mechanism is limiting incentive-compatible on $(\mathscr{E}_t)_{t\in\mathbb{N}}$.

[22] Furthermore, Watt (2021) showed that a slightly weakened version of the strong convexity property, called the local quadratic growth condition, is, in some sense, *necessary* for this rate of convergence. The incentive results in this paper can be easily adapted under this condition as well.

[23] Here the expectation is with respect to the measure over economies induced by draws of agents from μ and χ. We clarify the meaning of the "expected demand and supply correspondences" in Appendix B.3.

[24] We formalize both the α^*-price correspondence and the sense of continuity in Appendix B.3

(b) Suppose the type spaces $\mathcal{U}$ and $\mathcal{C}$ are finite sets and let full-support probability distributions μ and χ be defined (respectively) on them. Suppose that each buyer's type is drawn i.i.d. from $\mathcal{U}$ according to μ, the type of each seller is drawn i.i.d. from $\mathcal{C}$ according to χ, and the distributions μ and χ are common knowledge while realized types are private information. Then, for any agent, the expected benefit of misreporting under the α^*-price mechanism is $O(|N_t|^{-\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$.

(c) Suppose that buyers are drawn i.i.d. from $\mathcal{U}$ according to some probability measure μ and that the type of each seller in F_t is drawn i.i.d. from $\mathcal{C}$ according to a probability measure χ (here $\mathcal{U}$ and $\mathcal{C}$ need not be finite) for which the expected demand and supply correspondences satisfy the strong convexity property, then the maximum benefit for misreporting for any agent is $O\left(\frac{1}{|N_t|}\right)$ in expectation.

4.4 Computational properties

In general, the problem of identifying a Walrasian equilibrium is known to be computationally hard. Even in the case of an exchange economy with additively separable, piecewise-linear and concave utility functions, Chen, Dai, Du, and Teng (2009) show that the problem of calculating Walrasian equilibrium is PPAD-complete.[25] This complexity class is roughly characterized by three important properties: a solution (e.g. a Walrasian equilibrium in convex economies) is known to exist, solutions may be easily verified (in deterministic polynomial time) but can be computationally intensive to identify. For Walrasian equilibrium, a solution is easily verified by checking that all buyers and sellers are optimizing with respect to the identified prices. These prices act as a "certificate of optimality" for the efficient allocation problem in convex economies.

In contrast, the efficient allocation problem in non-convex economies has no such certificate of optimality. This implies immediately that the problem of identifying efficient allocations in non-convex economies is in a higher complexity class than PPAD. This problem is of practical significance for market designers: for example, the problem of identifying an optimal allocation in the fisheries market of Bichler et al. (2018) involved solving a large integer programming problem, which is known to be NP-complete. No efficient algorithms are known for such problems, although a variety of heuristic approaches may make specific instances of such problems computationally tractable.

The α^*-price mechanism involves solving, for *fixed* α, a convex optimization problem, which offers prices as a certificate of optimality. Therefore, for fixed α, known algorithms for the identification of Walrasian equilibria may be used, for example tâtonnement. In important special

[25] PPAD stands for 'polynomial parity arguments on directed graphs', and is a complexity class containing many fixed point problems (including the approximation of Nash equilibria) and is conjectured to be harder than P. We refer the reader to Daskalakis, Goldberg, and Papadimitriou (2009) for further information.

cases, for example, weak gross substitutes (Codenotti, McCune, & Varadarajan, 2005) and markets satisfying the strong convexity property (Watt, 2021), tâtonnement may be used to efficiently approximate Walrasian equilibrium. The identification of α^* may be more challenging, although a bisection search algorithm might be employed, in practice, to identify *some* α that satisfies equation (A) for sufficiently large economies (and therefore, *some* approximately-efficient allocation). Unfortunately, we are not able to identify theoretical guarantees related to the computation of α^*, the *smallest* such α, although the proof of Theorem 2 provides an upper bound on the search space for α^*.

4.5 Alternative mechanisms

The α^*-price mechanism is suggestive of several alternative mechanisms that could be employed in non-convex economies. The first is the **maximum surplus anonymous pricing** mechanism which was introduced by Bichler et al. (2018) and adapted for use in an exchange for fisheries licenses in New South Wales. The maximum surplus anonymous pricing mechanism solves the surplus optimization problem

$$\max_{\omega \in \Omega} \sum_{n \in N} u_n(x_n) - \sum_{f \in F} c_f(y_f),$$

subject to the constraint that there exist prices p_N and p_F satisfying

(a) Individual rationality: for all $n \in N, f \in F$, $u_n(x_n) - p_N \cdot x_n \geq 0$ and $p_F \cdot y_f - c_f(y_f) \geq 0$.

(b) Budget balance: $\sum_{n \in N} p_N \cdot x_n \geq \sum_{f \in F} p_F \cdot y_f$.

Another possibility is be the **maximum surplus $\alpha-$price mechanism**, which imposes the additional constraint that $p_N = (1 + \alpha) p_F$ for some $\alpha > 0$.

A corollary of Theorem 2 is that the deadweight loss of both of these mechanisms is $O(1)$ in $|N|$. However, we have focused our analysis on the α^*-price mechanism for several reasons. Firstly, the alternative mechanisms require the solution of non-convex optimization problems, while the α^*-mechanism involves an inner loop of a convex optimization problem (for fixed α). This offers a theoretical benefit of obtaining prices as the dual variables of a convex optimization problem (as in Walrasian equilibrium) and may have the added advantage of improved compuation, as discussed in Section 4.4. Secondly, optimizers of the alternative mechanisms may involve rationing at the prevailing prices, which may result in inferior incentive performance. With rationing, there may be an incentive to misreport to affect not only prices but also the quantity received. We show that it is possible to obtain many of the desirable properties of Walrasian equilibrium in non-convex economies without the need to ration agents at the prevailing prices. In the next section, we will

identify some settings in which rationing may be employed without substantial adverse effects on incentives.

5 One-sided non-convexities: the single-price mechanism with rationing

In this section, we consider the case where one side of the market—for simplicity, the buyers—has convex preferences, while the other side of the market has non-convex preferences. There are two reasons to conjecture that convexity on one side of the market might lead to more efficient pricing mechanisms. The first is that convexity on one side of the market makes it possible to obtain exact market-clearing outcomes via prices, since all feasible bundles are in the exposed set for the convex side of the market.[26] This eliminates one possible source of inefficiency in the market mechanism: unused supply.[27]The second reason is that in many familiar models, faster rates of convergence have been identified. For example, in the double auction model of Rustichini, Satterthwaite, and Williams (1994), the expected loss under the mechanism is $O\left(\frac{1}{|N|}\right)$, while simulation studies of the maximum surplus anonymous pricing mechanism due to Bichler et al. (2018) suggested a similar rate of convergence using simulated valuation functions.

We show that a **single-price mechanism with rationing** can lead to rapid convergence to efficiency under certain additional assumptions on buyers' preferences.

Definition 5.1 (Single-price mechanism with rationing)**.** Let $(p, \tilde{\omega})$ be a solution to

$$\min_{p \in \mathbb{R}^L_+} \max_{\omega \in X^{|N|} \times Y^{|F|}} \sum_{n \in N} u_n(x_n) - \sum_{f \in F} \text{vex}(c_f)(y_f) - p \cdot \left(\sum_{n \in N} x_n - \sum_{f \in F} y_f - R_{\mathscr{E}} \right). \tag{SPM}$$

Using Proposition 3, identify ω satisfying $\|\tilde{\omega} - \omega\| \leq R_{\mathscr{E}}$ with $y_f \in S_f(p)$ for each $f \in F$ and let $s = \sum_{f \in F} y_f$ be the resulting supply vector.

Let $\mathbf{x}' = (x'_n)_{n \in N}$ solve

$$\max_{(x_n)_{n \in N} \in X^N} \sum_{n \in N} u_n(x_n) \text{ subject to } \sum_{n \in N} x_n \leq s. \tag{SPM2}$$

The single-price mechanism with rationing purchases y_f from seller f at price p, and allocates x'_n to buyer n at price p.

[26]That is, given any feasible consumption bundle x, there is some price p such that agent n demands x at p.

[27]Note that in order to apply Theorem 1 with a single price and obtain an $O\left(\frac{1}{|N_t|}\right)$ bound, we effectively require that the identified allocation is exactly market-clearing, else the budget deficit term in the First Welfare Bound is $O(1)$.

As in the α^*-price mechanism, we use the pseudoequilibrium of a perturbed economy to identify a price vector. The key difference to the α^*-price mechanism is that in this case we solve for the pseudoequilibrium with an additional R units of *supply* rather than demand. This allows us to obtain, by applying Proposition 3, an allocation ω at which all agents optimize with respect to p, but with (at most) $2R$ units of excess demand. While we previously sought to avoid excess demand, here we use the existence of excess demand to our advantage. We then ration buyers in order to attain exact market-clearing. To do so in the least-cost way, we re-solve the surplus maximization problem with the supply vector fixed. Under certain assumptions on buyers' preferences, only buyers with values near the prices will be rationed so that the resulting allocation is, via Theorem 1, approximately-efficient. While rationing, we also need to account for the effect on individual rationality and incentives, but we show that this is not a significant impediment under certain assumptions.

Theorem 4. *Let $(\mathscr{E}_t)_{t\in\mathbb{N}}$ be a sequence of economies satisfying Assumptions 1–4 and that all buyers have convex preferences. Suppose in addition that the type of each buyer is drawn i.i.d. from $\mathscr{U}$ according to some probability measure μ and that the type of each seller in F_t is drawn i.i.d. from $\mathscr{C}$ according to a probability measure χ (here $\mathscr{U}$ and $\mathscr{C}$ need not be finite) for which the expected demand and supply correspondences satisfy the strong convexity property. Then the expected deadweight loss of the allocation associated with the single-price mechanism with rationing is $O\left(\frac{1}{|N_t|}\right)$, so that the relative inefficiency is $O\left(\frac{1}{|N_t|^2}\right)$. The expected maximum benefit of misreporting for any agent under the mechanism is $O\left(\frac{1}{|N_t|}\right)$ and the mechanism is individually rational for all sellers and $O\left(\frac{1}{|N_t|}\right)$-individually rational for buyers. If the buyers' preferences are, in addition, separable across goods, the mechanism is exactly individually rational for buyers.*

The strong convexity assumption in Theorem 4 ensures that the prices in the mechanism (determined by (SPM)) and the shadow prices associated with (SPM2) are increasingly close as the economy gets large. This implies that the buyers' final allocation (determined by (SPM2)) will be almost optimal for them given the mechanism's prices. The proof of Theorem 4 is in Appendix B.4.

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A Non-convexity and approximate equilibria

We begin with a slightly stronger statement of the Shapley-Folkman Lemma that is used in general equilibrium theorem with non-convexities.

Proposition 4. *Let $S_i \subseteq \mathbb{R}^L$ for $i = 1, ..., N$, $S = \bigoplus_{i=1}^N S_i$ and $L' = \min(L, N)$. Then for any $x \in \text{co}(S)$:*

(a) (Shapley-Folkman Lemma) $x = \sum_{i=1}^N x_i$ where $x_i \in \text{co}(S_i)$ and $|i : x_i \in \text{co}(S_i) \setminus S_i| \leq L'$.

(b) (Starr, 1969) If S_i is ordered so that $r(S_i)$ is nonincreasing in i, then there is $y \in S$ such that $|x - y| \leq \sqrt{\sum_{i=1}^{L'} r(S_i)^2}$.

(c) (Heller, 1972) If S_i is ordered so that $\rho(S_i)$ is nonincreasing in i, then there is $y \in S$ such that $|x - y| \leq \sqrt{\sum_{i=1}^{L'} \rho(S_i)^2}$.

These results have been used in the general equilibrium context to obtain **approximate equilibria**, which are price-allocation pairs (p, ω) such that $x_n \in D_n(p)$ for all n, $y_f \in S_f(p)$ for all f but $\left|\sum_{n \in N} x_n - \sum_{f \in F} y_f\right| \leq s$ for some $s > 0$. In particular, the allocation associated with an

approximate equilibrium may have excess demand and therefore be infeasible. The approximate equilibrium is obtained by identifying the competitive equilibrium associated with a convexified version of the economy (in which each agent's upper contour set is replaced by its convex hull) and applying the results of Proposition 4 to the resulting allocation. The approximate equilibrium analogues of Proposition 4 are contained in Proposition 5 below.

Proposition 5. *For economy $\mathscr{E} = (N, F)$:*

(a) (Starr, 1969) There is $\omega \in \text{co}(\Omega)$ and $p \in \mathbb{R}^L_+, p > 0$ such that $|n : x_n \in \text{co}(X)| + |f : y_f \in \text{co}(Y)| \leq L$ and for all other agents, $x_n \in D_n(p)$ and $y_f \in S_f(p)$.

(b) Let $r_n = \sup_{\bar{u} \in \mathbb{R}} r(UC_n^{\bar{u}})$ and $r_f = \sup_{\bar{\pi} \in \mathbb{R}} r(UC_f^{\bar{\pi}})$. Let $\delta \geq 0$ satisfy $r_n \leq \delta$ for all $n \in N$ and $r_f \leq \delta$ for all $f \in F$. Then there exists $p \in \mathbb{R}^L_+, p > 0$, $x_n \in X$ and $y_f \in Y$ such that $x_n \in D_n(p)$ for all $n \in N$, $y_f \in S_f(p)$ for all $f \in F$ and $\left|\sum_{n \in N} x_n - \sum_{f \in F} y_f\right| \leq \delta\sqrt{L}$.

(c) Let $\rho_n = \sup_{\bar{u} \in \mathbb{R}} \rho(UC_n^{\bar{u}})$ and $\rho_f = \sup_{\bar{\pi} \in \mathbb{R}} \rho(UC_f^{\bar{\pi}})$. Let $\delta' \geq 0$ satisfy $\rho_n \leq \delta'$ for all $n \in N$ and $\rho_f \leq \delta'$ for all $f \in F$. Then there exists $p \in \mathbb{R}^L_+, p > 0$, $x_n \in X$ and $y_f \in Y$ such that $x_n \in D_n(p)$ for all $n \in N$, $y_f \in S_f(p)$ for all $f \in F$ and $\left|\sum_{n \in N} x_n - \sum_{f \in F} y_f\right| \leq \delta' L$.

The statement of Proposition 5(a) is standard, but the statements of Proposition 5(b) and (c) are stronger than the classical statement due to Starr (1969) and Heller (1972). Again, the quasilinearity of agent preferences allows us to conclude that agents are utility- and profit-maximizing, rather than just expenditure-minimizing.

Finally, we introduce a more general class of quasilinear preferences to which many of our results also apply and which offer a meaningful interpretation in terms of perceived complementarity and substitutability of goods. Nguyen and Vohra (2020) introduced the concept of the **generalized Δ-single improvement property**, which is a generalization of the well-known single improvement property.

Definition A.1. The preferences of buyer $n \in N$ satisfies the generalized Δ-single improvement property for some $\Delta > 0$ if for any price vector $p > 0$, any two bundles $x, y \in D_n(p)$ and any price change δp such that $\delta p \cdot x > \delta p \cdot y$, there exist $a \leq (x - y)^+$ and $b \leq (y - x)^+$ such that:

(a) $|a| + |b| \leq \Delta$

(b) $\delta p \cdot a > \delta p \cdot b$, and

(c) $x - a + b \in D_n(p)$.

Here $(x - y)^+$ denotes the vector whose ℓ^{th} component is $\max(x_\ell - y_\ell, 0)$.

The Δ in this definition captures a measure of the substitutability and complementarity between goods. Preferences with the single improvement property of Gul and Stacchetti (1999) are contained in the class with $\Delta = 2$.

By our assumption on the compactness of X and Y, all preferences and technologies satisfy the general Δ-improvement property for some Δ. But the following stronger relationship between the inner radii of preferences and the Δ-single improvement property also holds.

Proposition 6. *Let $r_n = \sup_{\bar{u}\in\mathbb{R}} r(UC_n^{\bar{u}})$. Then the preferences of buyer $n \in N$ satisfy the generalized $\Delta-$single improvement property for all $\Delta > 2\sqrt{2}r_n$.*

Proof. Let the preferences of buyer n satisfy $r_n = \sup_{\bar{u}\in\mathbb{R}} r(UC_n^{\bar{u}})$. Let $x, y \in X$ and $p \in \mathbb{R}_+^L$ be given such that $x, y \in D_n(p)$. Suppose $|(x-y)^+| + |(y-x)^+| \geq 2r_n$ (else the preferences immediately satisfy the Δ improvement property for $\Delta = 2r_n$).

For any $\epsilon > 0$, let $z \in \mathbb{R}_+^L$ be the unique convex combination of x and y such that $|x - z| = r_n + \epsilon$ and write $z = \lambda x + (1-\lambda)y$. By construction $(z, p\cdot z) \in \text{co}(UC_n^{u_n(x)-p\cdot x})$. Then by the bound on the non-convexity of the preferences, there is a set $T \subseteq UC_n^{u(x)-p\cdot x}$ with $\text{rad}(T) \leq r_n$ such that $(z, p\cdot z) = \sum_{(x',t')\in T} \alpha_{(x',t')}(x',t')$ where $\sum_{(x',t')\in T} \alpha_{(x',t')} = 1$.

We now argue that for all $(x',t') \in T$, $x' \in D_n(p)$ and $t' = p\cdot x'$. To see this, note that $x \in D_n(p)$ implies $u_n(x') - p\cdot x' \leq u_n(x) - p\cdot x$. Summing, we have

$$\begin{aligned} u_n(x) - p\cdot x &\geq \sum_{(x',t')\in T} \alpha_{(x',t')}[u_n(x') - p\cdot x'] \\ &= \sum_{(x',t')\in T} \alpha_{(x',t')} u_n(x') - p\cdot z \\ &= \sum_{(x',t')\in T} \alpha_{(x',t')}[u_n(x') - t'] \end{aligned}$$

On the other hand, since $(x',t') \in UC_n^{u(x)-p\cdot x}$ we have $u_n(x') - t' \geq u(x) - p\cdot x$. The only way these can simultaneously hold is if $u_n(x') - t' = u(x) - p\cdot x$ for all $(x',t') \in T$.

However, we then have $\sum_{(x',t')\in T} \alpha_{(x',t')}[u_n(x') - p\cdot x'] = u(x) - p\cdot x$. This implies that at least one of $u_n(x') - p\cdot x' \geq u_n(x) - p\cdot x$. But then $x \in D_n(p)$ implies that $u_n(x') - p\cdot x' = u_n(x) - p\cdot x$ for all x', so $x' \in D_n(p)$.

By construction, $|x - x'| \leq 2r_n + \epsilon$. But then $||x - x'||_1 \leq 2\sqrt{2}r_n + \epsilon$ as well. □

Clearly, the generalized Δ-single property can be readily extended to sellers, by replacing the expressions for utility with those for profits, and an analogue of Proposition 6 also holds.

Nguyen and Vohra (2020) demonstrate the following approximate equilibrium result in a setting with indivisibilities (so that $X \subseteq \mathbb{Z}_+^L$ and $Y \subseteq \mathbb{Z}_+^L$).

Proposition 7. *Suppose all buyers' preferences and sellers' technologies satisfy the generalized Δ-improvement property and that $X \subseteq \mathbb{Z}_+^L$ and $Y \subseteq \mathbb{Z}_+^L$. Then there exists $p \in \mathbb{R}_+^L$, $p > 0$, $x_n \in X$ and $y_f \in Y$ such that $x_n \in D_n(p)$ for all $n \in N$, $y_f \in S_n(p)$ and for each $\ell \in L$, $\left|\sum_{n\in N} x_{nl} - \sum_{f\in F} y_{fl}\right| \leq \Delta - 1$.*

Note that the concept of approximate equilibrium in this result is somewhat stronger than the previous results since the maximum imbalance in supply and demand is bounded good-by-good, rather than in terms of Euclidean distance in commodity space. However, depending on the relative size of Δ, the inner radii of non-convexity and the breadths of non-convexity of preferences, any of the approximate equilibrium bounds in Proposition 5(b), 5(c) or 7 may be strongest for our purposes.

B Proofs omitted from the main text

B.1 Proof of Proposition 1

Suppose $U_n(x,t) = u_n(x) - t$ is quasiconcave, so that its upper contour set UC_n^c is convex for any $c \in \mathbb{R}$. Recall that $u_n(\cdot)$ is concave iff $\operatorname{hyp} u = \{(x,\mu) : x \in \mathbb{R}_+^L, \mu \in \mathbb{R}, \mu \leq f(x)\}$ is convex. Let $(x,\mu),(x',\mu') \in \operatorname{hyp} u$ be given and suppose, without loss of generality, that $\mu \leq \mu'$. Then $u_n(x) \geq \mu$, $u_n(x') \geq \mu'$ implies $u_n(x') - (\mu' - \mu) \geq \mu$. Thus, $(x,0)$ and $(x',\mu'-\mu)$ are in the upper contour set UC_n^μ of $U_n(\cdot,\cdot)$. This implies for any $\lambda \in [0,1]$, $u_n(\lambda x + (1-\lambda)y) - (1-\lambda)(u'-\mu)$ is in UC_n^μ. This implies that $u_n(\lambda x + (1-\lambda)y) \geq \mu + (1-\lambda)(\mu'-\mu) \geq \mu$.

Now suppose that $u_n(x)$ is concave. Let $(x,t),(x',t') \in UC_n^c$, an upper contour set of $U_n(\cdot,\cdot)$, be given. Then $u_n(x) \geq c + t$, $u_n(x') \geq c + t'$ so that $(x, c+t) \in \operatorname{hyp} u$ and $(x', c+t') \in \operatorname{hyp} u$. By concavity of u_n, we have $(\lambda x + (1-\lambda)y, \lambda t + (1-\lambda)t') \in \operatorname{hyp} u$, which implies $u_n(\lambda x + (1-\lambda)y) \geq c + \lambda t + (1-\lambda)t'$, which on reorganization tells us that $\lambda(x,t) + (1-\lambda)(x',t') \in UC_n^c$, and so $U_n(\cdot,\cdot)$ is quasiconcave.

An analogous argument applies to the cost and profit functions.

B.2 Proof of Theorem 2

Theorem 2 is proved in two parts. First, in Lemma 1, we establish that the premium α_t^* is $O\left(\frac{1}{|N_t|}\right)$. Then we use Theorem 1 to show that this implies that the loss of the associated allocation is $O(1)$.

Lemma 1. *Under Assumptions 1–4, the α_t^* in Definition 4.1 is $O\left(\frac{1}{|N_t|}\right)$.*

Proof. For notational simplicity we drop the index for t in the prices, premiums and allocations.

The construction in Definition 4.1 ensures, via Proposition 3, that $\sum_{f\in F} y_f^\alpha - \sum_{n\in N} x_n^\alpha \leq (2R)1_L$. Thus, it suffices to show that for sufficiently large $|N_t|$, there is an α such that

$$\alpha \sum_{n\in N} p^\alpha \cdot x_n^\alpha \geq (2R)p^\alpha \cdot 1_L.$$

Moreover, if this $\alpha = O\left(\frac{1}{|N_t|}\right)$, then since $\alpha^* < \alpha$, Lemma 1 will follow. To arrive at this result, we will show that for fixed $\alpha > 0$ close enough to zero, $\sum_{n\in N} p^\alpha \cdot x_n^\alpha$ is $\Omega(|N_t|)$, while $p^\alpha \cdot 1_L$ is $O(1)$.

Let $\mathcal{S}^\alpha$ be the value of the saddle point problem

$$\min_{p\in\mathbb{R}^L_+} \max_{\omega\in\Omega} \sum_{n\in N} \frac{\text{cav}(u_n)(x_n)}{1+\alpha} - \sum_{f\in F} \text{vex}(c_f)(y_f) - p\cdot\left(\sum_{n\in N} x_n + R1_L - \sum_{f\in F} y_F\right).$$

First, note that $\mathcal{S}^\alpha$ is $\Theta(|N_t|)$ for sufficiently small α by Assumption 3. To see this, denote by $f(|N_t|) = \sum_{n\in N_t} u_n(x_n^*)$ and $g(|N_t|) = \sum_{f\in F_t} c_f(y_f^*)$. Then by Assumption 3 and the definition of $\Theta(\cdot)$, $\liminf_{N\to\infty} \frac{f(N)}{N} = u > 0$, say, and $\limsup_{N\to\infty} \frac{g(N)}{N} = c > 0$, with $u - c > 0$. Then $\mathcal{S}^\alpha \geq \liminf_{N\to\infty} \frac{f(N)}{(1+\alpha)N} - g(N) = \frac{u}{1+\alpha} - c$, which is positive for sufficiently small α.

Now we show that this implies $\sum_{n\in N} p^\alpha \cdot x_n^\alpha$ is $\Omega(|N_t|)$ for small, fixed α. To see this, note that since $\sum_{f\in F_t} c_f(y_f^\alpha)$ is $\Omega(|N_t|)$, individual rationality of the sellers (in the perturbed economy) implies that $\sum_{f\in F_t} p^\alpha \cdot y_f^\alpha$ is $\Omega(|N_t|)$. But then by complementary slackness $\sum_{n\in N_t} p^\alpha \cdot x_n^\alpha = \sum_{f\in F_t} p^\alpha \cdot y_f^\alpha - Rp^\alpha \cdot 1_L$, and since Assumption 4 implies $\|p\| \leq M$, we must have that $\sum_{n\in N} p^\alpha \cdot x_n^\alpha$ is $\Omega(|N_t|)$.

Since for α near zero, $\sum_{n\in N} p^\alpha \cdot x_n^\alpha$ is $\Omega(|N_t|)$ and $(2R)p^\alpha \cdot 1_L$ is $O(1)$, for sufficiently large $|N_t|$, there is a least α such that

$$\alpha \sum_{n\in N} p^\alpha \cdot x_n^\alpha \geq (2R)p^\alpha \cdot 1_L,$$

and furthermore, this α is $O\left(\frac{1}{|N_t|}\right)$. Since $\alpha^* < \alpha$, we have that α^* is $O\left(\frac{1}{|N_t|}\right)$, as required. □

We now apply the First Welfare Bound to show that the allocation ω^{α^*} is approximately efficient. In order to satisfy the assumptions on prices in Theorem 1, we imagine ω^{α^*} was implemented with a single price vector p^{α^*} and (therefore) a budget deficit. Theorem 1 tells us that

$$\mathcal{S}(\omega^*) - \mathcal{S}(\omega^{\alpha^*}) \leq \mathcal{R}(p^{\alpha^*}, \omega^{\alpha^*}) + p^{\alpha^*}\cdot\left(\sum_{f\in F_t} y_f^{\alpha^*} - \sum_{n\in N_t} x_n^{\alpha^*}\right). \tag{1}$$

By construction, in ω^{α^*} at prices p^{α^*}, no sellers are rationed, while at prices $(1+\alpha^*)p^{\alpha^*}$, no buyers are rationed. We now use Lemma 1 to show that since α^* is small for large $|N_t|$, that the total

rationing of buyers at (counterfactual) price p^{α^*} is $O(1)$. To do so, we prove in Lemma 2 that $\mathscr{R}_n(p^{\alpha^*}, x_n^{\alpha^*})$ is $O\left(\frac{1}{|N_t|}\right)$ so that $\sum_{n\in N_t} \mathscr{R}_n(p^{\alpha^*}, x_n^{\alpha^*})$ is $O(1)$.

Lemma 2. *Let (p, ω) be a competitive equilibrium and $p' \gg p$ be an alternative price. Then the rationing loss of buyer n at allocation ω given price p' is $O(\|p - p'\|)$*

Proof. Let x_n be buyer n's consumption bundle in ω. Since (p, ω) is a competitive equilibrium, we have that $x_n \in D_n(p)$, the agent's demand correspondence at price p. By the definition of rationing,

$$\begin{aligned}\mathscr{R}_n(p', x_n) &= \hat{u}_n(p') - (u_n(x_n) - p' \cdot x_n) \\ &= \hat{u}_n(p') - (u_n(x_n) - p \cdot x_n) - p \cdot x_n + p' \cdot x_n \\ &= \hat{u}_n(p') - \hat{u}_n(p) + (p' - p) \cdot x_n.\end{aligned}$$

Now let $p(t) = (1-t)p + tp'$ for $t \in [0, 1]$ and consider the parametrized utility maximization problem

$$\hat{u}_n(p(t)) = \max_{x\in X} u_n(x) - p(t) \cdot x.$$

The conditions for the Milgrom and Segal (2002) envelope theorem hold, namely the objective function is differentiable in t given any $x \in X$ and the demand correspondence is always non-empty. The derivative of the objective with respect to t is $(p - p') \cdot x$, which is bounded (since X is assumed compact)

Then for any selection of $d(t) \in D_n(p(t))$, we have that

$$\hat{u}_n(p') = \hat{u}_n(p) - \int_0^1 (p' - p) \cdot d(t)dt \leq \hat{u}_n(p) - (p' - p) \cdot d',$$

for any d' in $D_n(p')$ since D_n is nonincreasing along $p(t)$.

Substituting into the expression for $\mathscr{R}_n$ above, we obtain

$$\mathscr{R}_n(p', x_n) \leq (p' - p) \cdot (x_n - d').$$

With no assumptions on demand, we see that $\mathscr{R}_n(p', x_n)$ is $O(\|p - p'\|)$, since the consumption set is bounded. □

Since α^* is $O\left(\frac{1}{|N_t|}\right)$ and $\|p^{\alpha^*}\|$ is bounded, this implies that $\mathscr{R}_n(p^{\alpha^*}, x_n^{\alpha^*})$ is $O\left(\frac{1}{|N_t|}\right)$, and so $\mathscr{R}(p^{\alpha^*}, \omega^{\alpha^*})$ is $O(1)$.

Finally, we note that the budget deficit (the second term on the right of Equation (1)) is $O(1)$ since the excess supply is bounded by construction and each component of p^{α^*} is $O(1)$ (as argued previously). Thus Theorem 2 follows.

B.3 Proof of Theorem 3

Most of the results in Theorem 3 follow by simple application of the corresponding theorems for Walrasian equilibrium, noting that at the limiting economy, the prices determined by the Walrasian mechanism and the α^*-price mechanism coincide (the R units purchased by the mechanism designer do not matter in the limit). Therefore, any conditions that ensure that agents have diminishing influence over prices in Walrasian equilibrium imply that agents have diminishing influence over prices in the α^*-price mechanism. We will only provide sufficient details to make clear the adaptations of the arguments required for the α^*-price mechanism and refer readers to Roberts and Postlewaite (1976), Azevedo and Budish (2019) and Watt (2021) for more details.

Part (a) Let Q be the Walrasian equilibrium correspondence mapping normalized economies $\langle \mu, \phi, \chi \rangle$ to Walrasian prices, which we assume is continuous at the limiting normalized economy of $(\mathscr{E}_t)_{t\in\mathbb{N}}$, $\langle \mu_\infty, \phi_\infty, \chi_\infty \rangle$. Let Q_α map economies $\langle N, \mu, \phi, \chi \rangle$ to the set of (seller) prices under the $\alpha-$price mechanism for fixed $\alpha > 0$. By dividing the objective in Definition 4.1 by $|N|$ and rewriting, $Q_\alpha(\langle N, \mu, \phi, \chi \rangle)$ are the solutions to the normalized dual problem

$$\min_{p\in\mathscr{Q}} \left\{ \int_{\mathscr{U}} \frac{\hat{u}_n\left(\frac{p}{1+\alpha}\right)}{1+\alpha} d\mu + \phi \int_{\mathscr{C}} \hat{\pi}_f(p) d\nu - \frac{p \cdot R1_L}{|N|} \right\},$$

where $\mathscr{Q} = \{p \in \mathbb{R}^L_+ : \frac{1}{M} \leq \|p\| \leq M\}$ (the optimizers in $\mathbb{R}^L_+$ lie in $\mathscr{Q}$ by Assumption 4). Note that in the limit as $t \to \infty$, since $|N| \to \infty$ and $\alpha \to 0$, the objective of Q_{α_t} converges pointwise and therefore uniformly on P (by compactness) to the dual objective of the Walrasian equilibrium problem (even if a single agent misreports). But this implies that $d_H(P, P') \to 0$ (by, for example, Sendov (1990) Corollary 2.1).

To discuss continuity of Q_α, we need to describe a topology on the domain of Q_α, which is the product space $\mathbb{R}_+ \times (\mathbb{N} \cup \{\infty\}) \times \mathscr{P}(\mathscr{U}) \times \mathbb{R}_+ \times \mathscr{P}(\mathscr{C})$ (corresponding to $\alpha, N, \mu, \phi, \chi$), where $\mathscr{P}(\mathscr{M})$ is the set of probability measures on a measurable space $\mathscr{M}$. We use the product metric, where we use the Lévy-Prokhorov metric[28] on $\mathscr{P}(\mathscr{U})$ and $\mathscr{P}(\mathscr{C})$, the usual metric on $\mathbb{R}_+$

[28] Recall the Lévy-Prokhorov metric on $\mathscr{P}(\mathscr{M})$ on measurable space $\mathscr{M}$ with Borel sigma algebra $\mathscr{B}(\mathscr{M})$ is defined by $\rho(\nu, \mu) = \inf\left\{\varepsilon > 0 \mid \nu(E) \leq \mu\left(B_\varepsilon(E)\right) + \varepsilon \text{ and } \mu(E) \leq \nu\left(B_\varepsilon(E)\right) + \varepsilon \text{ for every Borel set } E \subseteq \mathscr{B}(\mathscr{M})\right\}$, where $B_\varepsilon(E)$

and the metric on $\mathbb{N} \cup \infty$ given by $d(M,N) = \left|\frac{1}{M} - \frac{1}{N}\right|$. We now assume that Q_α is continuous at $(0, \infty, \mu_\infty, \phi_\infty, \chi_\infty)$ in this product topology. Since for the true economy $(\alpha_t, |N_t|, \mu_t, \phi_t, \chi_t) \to (0, \infty, \mu_\infty, \phi_\infty, \chi_\infty)$ in the product metric and for any misreport of a single agent, the resulting $(\alpha_t', |N_t|, \mu_t', \phi_t, \chi_t') \to (0, \infty, \mu_\infty, \phi_\infty, \chi_\infty)$, the triangle inequality will imply that the resulting prices will be arbitrarily close in the Hausdorff metric (see Roberts and Postlewaite (1976) for this argument). Finally, noting that for each agent $\hat{u}_n(p)$ and $\hat{\pi}_f(p)$ are (locally) Lipschitz continuous functions of p, this implies that the maximum benefit of misreporting must tend to zero as well.

Part (b) This follows by simply noting that the α^*-price mechanism is envy-free, and so Theorem 1 of Azevedo and Budish (2019) implies the result.

Part (c) We prove this result only for buyers - the proof for sellers is completely analogous. We first note the useful implication of strong convexity that

$$\|d - d'\| \geq m\|p - p'\|, \tag{*}$$

which is obtained by applying the Cauchy-Schwarz inequality to the definition of the strong law of demand property.

We now formally define the expected supply and demand correspondences, using the concept of a set-valued integral due originally to Aumann (1965). For any fixed p, the probability measure μ induces a probability measure over sets $D_n(p)$ (associated with the utility function $u_n \in \mathcal{U}$). A selection $\xi : \mathcal{U} \to X$ is a single-valued random vector such that $\xi(u_n)$ almost surely belongs to $D_n(p)$ for each $u_n \in \mathcal{U}$. Then $\mathbb{E}_\mu[D_n(p)]$ is defined as $\text{cl}(\{\mathbb{E}\xi\})$ for integrable selections ξ. We assume that the strong law of demand property applies to this correspondence (and analogously for the supply correspondence). Importantly, a central limit theorem result applies to $\mathbb{E}_\mu[D_n(p)]$ so that $d_H(\frac{1}{|N_t|}\sum_{n \in N_t} D_n(p), \mathbb{E}_\mu[D_n(p)]) \to 0$ (Weil, 1982).

Let us consider the incentives to misreport for a single buyer $n_0 \in N_t$. It suffices to consider misreports in which the buyer announces a price-quantity pair (p', x') which is the price and their assignment in the resulting α^*-price mechanism, or equivalently, it reports the valuation function $\max\{(p' - p) \cdot x', 0\}$ to the auctioneer (see Watt (2021) for more details). Then, let $\bar{R}_t(p) = \frac{1}{|N_t|}\sum_{n \in N_t \setminus \{n_0\}} D_n(p)$ be the average residual demand correspondence. Note that $\bar{R}_t(p)$ converges (in Hausdorff distance) to $\mathbb{E}_\mu[D_n(p)]$ as well.

For any realization of $\mathcal{E}_t$ obtained by draws from μ and χ, let Q_t be the set of α^*-price mech-

denotes the ε -neighborhood of E

anism prices for buyers under truthful reporting and Q'_t the set of α^*-price mechanism prices for buyers under any misreport by buyer n_0.

Suppose, for a contradiction, that $\limsup_{t\to\infty} |N_t|\mathbb{E}_\mu[d_H(Q'_t, Q_t)] = \infty$. Since for any $\mathscr{E}_t$, $p_t \in Q_t$ and $p'_t \in Q'_t$ with $\|p_t - p'_t\| = d_H(Q'_t, Q_t)$, we must have that there are selections $r_t \in \bar{R}_t(p_t)$ and $r'_t \in \bar{R}_t(p'_t)$ such that $\|r_t - r'_t\| \leq \frac{C}{|N_t|}$ (since the misreport quantity $\|x'_t\| \leq C$ by compactness of X and the approximate market-clearing property of the α^*-price mechanism), we have

$$\mathbb{E}_\mu\left[\frac{\|r_t - r'_t\|}{\|p_t - p'_t\|}\right] \leq \mathbb{E}_\mu\left[\frac{C}{|N_t|\|p_t - p'_t\|}\right] \leq \frac{C}{\mathbb{E}_\mu[|N_t| d_H(Q_t, Q'_t)]} \to 0,$$

by hypothesis.

On the other hand, by the strong convexity property and the (Hausdorff) convergence of $\bar{R}_t(p)$ to $\mathbb{E}_\mu[D_n(p)]$, $\mathbb{E}_\mu[\frac{\|r_t - r'_t\|}{\|p_t - p'_t\|}] \geq m$, which is a contradiction. Thus $\limsup_{t\to\infty} |N_t|\mathbb{E}[d_H(Q'_t, Q_t)] < \infty$, so that the expected influence of any buyer on prices, $\mathbb{E}_\mu[d_H(Q'_t, Q_t)] \leq O\left(\frac{1}{|N_t|}\right)$, as required. Since the buyer's indirect utility function is Lipschitz, this implies that the maximum benefit of misreporting is $O\left(\frac{1}{|N_t|}\right)$ as well.

B.4 Proof of Theorem 4

Under the strong convexity property, Theorem 2 in Watt (2021) implies that the price p from the saddle point of Equation (SPM) and the shadow price p' which is the dual variable associated with Equation (SPM2) satisfy $\|p - p'\| \leq O\left(\frac{1}{|N_t|}\right)$.

Then by the expression obtained for $\mathscr{R}_n(p, x'_n)$ in Lemma 2, we have for all $d_n \in D_n(p)$ and $d'_n \in D_n(p')$

$$\mathscr{R}_n(p, x'_n) \leq (p' - p) \cdot (d_n - d'_n).$$

Note that demand selections $d_n \in D_n(p)$ and $d'_n \in D_n(p')$ can be chosen to satisfy $\sum_n d_n = s + R1_L$ and $\sum_n d'_n = s$ by complementary slackness and the monotonicity of buyers' preferences (if these equations are not satisfied for some good for some demand selections, complementary slackness implies that the price of that good is zero and any agent would be willing to accept the excess supply of that good for free by monotonicity). Thus, by summing, we have

$$\mathscr{R}(p, x'_n) \leq (p' - p) \cdot R1_L \leq O\left(\frac{1}{|N_t|}\right).$$

Employing Theorem 1 with the prices and allocation from the single-price mechanism with rationing (noting that it is budget-balanced and no sellers are rationed) thus implies that the dead-

weight loss is $O\left(\frac{1}{|N_t|}\right)$.

The proof of incentive-compatibility is almost identical to the proof of Theorem 3(c), with the added complication that a buyer may have an incentive to misreport in order to avoid rationing. However, since each buyer's rationing is $O\left(\frac{1}{|N_t|}\right)$, the maximum expected benefit of misreporting is $O\left(\frac{1}{|N_t|}\right)$ as well (comprising the $O\left(\frac{1}{|N_t|}\right)$ capacity to influence prices and the $O\left(\frac{1}{|N_t|}\right)$ maximum benefit of avoiding rationing conditional on the change in prices). Finally, the approximate indivdual rationality follows from the observation that $x'_n \in \max_x u_n(x) - p' \cdot x$ so that $u_n(x'_n) - p' \cdot x \geq 0$. Then $u_n(x'_n) - p \cdot x = u_n(x'_n) - p' \cdot x + (p' - p) \cdot x \geq (p' - p) \cdot x$, which may be less than zero. However, since $p' - p$ is $O\left(\frac{1}{|N_t|}\right)$ and $\|x\| \leq O(1)$, at worst we have that the resulting allocation is $O\left(\frac{1}{|N_t|}\right)$ −individually rational.