Learning from Shared News: When Abundant Information Leads to Belief Polarization*

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August 31, 2020

Abstract

We study social learning in networks. Each period agents receive the same quantity and quality of first-hand information and have the opportunity to share it with friends. Some agents (possibly a few) share information selectively. Selective sharing generates heterogeneous news diets across agents, who, however, are aware of it and update beliefs via Bayes' rule. We show that, contrary to standard learning results, agents' beliefs can diverge in this environment. This occurs if and only if agents hold misperceptions (even minor) about friends' access to first-hand information and if its *quality* is low. We show that abundant information can exacerbate belief polarization. That is, when the *quantity* of first-hand information grows indefinitely agents can hold opposite degenerate beliefs. Polarization can worsen when the information quality rises or when the network expands, despite providing agents with more information. Information aggregation can mitigate, and even eliminate, polarization.

*We thank our discussant, Myles Ellis, Joel Sobel, participants in the UCSD Theory and Behavioral and the PhDEI workshops for helpful comments and suggestions

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1 Introduction

Social divisions have been linked to negative economic and political outcomes, including inequality, political gridlock, poor legislation, weak property rights, low trust, investment, and growth.[1] Recent decades have witnessed rising polarization in politics, media, and public opinions, particularly in the United States.[2] As Alesina et al. (2020) notes, "Americans are polarized not only in their views on policy issues and attitudes towards government and society, but also about their perceptions of the same, factual reality." Economists have, thus, been investigating the determinants of belief polarization and we contribute to this line of research.

Several authors have made a connection between rising polarization of the American public and growing use of the Internet as a source of information (Periser, 2011; Sunstein, 2017; Azzimonti and Fernandes, 2018; Tucker et al., 2019). Some have highlighted the effects of systematic *misinformation*—fake news, bots, and bad actors—leading to discussions about regulating social media to minimize these elements.[3] But, even if successful, will such regulation solve the problem of polarization? Can polarization simply result from how people consume and share information in social networks, even without misinformation? Also, does the information abundance brought by such networks lead to more or less polarization? This paper provides a theoretical framework to answer these questions.

Building on recent evidence on how people use information on social networks, we analyze how they individually learn from first-hand and shared news and whether this can cause their beliefs to become polarized. We highlight that polarization depends on people's information diet, which is driven by the composition of friends in their network, but also that information quality plays a key role. Our analysis suggests mechanisms whereby changes in people's information ecosystem brought by the Internet and the expansion of social networks may contribute to polarization. In addition, *low* quality of external information is crucial in generating belief polarization. Despite selective sharing relying on the same quality, it is capable to overcome correct learning only when that information quality falls below a certain threshold. We also discuss implications for policies aimed at curbing polarization.

We base our theory on evidence highlighting specific ways in which information flows on social networks. First, people tend to share information *selectively*.[4] For example, they share only information that favors their preferred political candidate or views on the importance of vaccinating children. Second, although most people have friends who share different kinds of information, the resulting information diet is likely to feature some *imbalance*. This is a distinctive aspect of so-called echo chambers or media bubbles, which appear in a wealth of evidence (see Levy and Razin, 2019a, for a review).

[1] See Zak and Knack (2001); Keefer and Knack (2002); Bartels (2008); Bishop (2009); McCarty et al. (2009); Gilens (2012); Barber and McCarty (2015).

[2] See Pew Research Center (2014, 2020); Desmet and Wacziarg (2018).

[3] See, for example, "Should the Government Regulate Social Media?", Wall Street Journal, June 25, 2019 and "Facebook Throws More Money at Wiping Out Hate Speech and Bad Actors", Wall Street Journal, May 15, 2018.

[4] See Shin and Thorson (2017); Weeks et al. (2017); Shin et al. (2018); Pogorelskiy and Shum (2019).

It seems intuitive that unbalanced selective sharing of information can lead to polarization: If a person takes at face value what her friends say and they support one view, her opinion can be swayed accordingly. On brief reflection, however, it is clear this reasoning as some flaws. First, people often receive first-hand information in addition to information received from friends, so this dampens any effect of selective sharing. Second, only a few of their friends may share information selectively, so exposure to selective sharing may be limited. Finally, even in the absence of these two mitigating factors, if a person fully understands how her friends select what to share, she will adjust for it (or ignore the received information altogether) and her beliefs will not be distorted.

What we assume about people's understanding of the selectivity of shared news turns out to be crucial. It is unrealistic to expect people to be completely naive about selective sharing, but experimental evidence suggests they do not fully take it into account either (Pogorelskiy and Shum, 2019). Consistent with this evidence, our theory allows for some *misperception* of selective sharing.

We incorporate selective sharing, echo chambers, and misperception into a simple model of learning from shared information. A binary state of the world, A or B, realizes in the first period. In every subsequent period, each agent directly observes an unbiased signal about the state. The signals have the same independent distribution across agents and periods. We refer to their informativeness as the information quality. After observing her signal, each agent can share it with her friends (i.e., neighbors in the network) or remain silent. This rules out fake news in the form of fabricated signals. Selective sharing arises as follows: Some agents—called *normal*—share every signal; other agents—called *dogmatic*—share only signals supporting a specific state. To fix ideas, some people (possibly a tiny minority) may hold a dogmatic view on whether to vaccinate children and share only information in its favor; others simply share any information. We refer to the composition of normal and dogmatic friends of an agent as her echo chamber. We model the agents' misperception of selective sharing in a way that renders them partially unresponsive to it (as found in Pogorelskiy and Shum (2019)), prevents it from unraveling, and is inspired by the psychology literature.[5] In a nutshell, each agent correctly interprets all received signals, but misinterprets her friends' silence by allowing for the possibility that they simply have received no information. This is akin to assuming that her friends do not read the newspaper as often as they do. Thus, our agents have a common misspecified model of selective sharing, based on which they update beliefs according to Bayes' rule.

We analyze learning and belief polarization by studying how beliefs responds to one round of signals (short-run learning) as well as infinite sequences of signals (long-run learning). Consider any agent and suppose that, besides possibly many normal friends, she has more dogmatic friends who favor state A than B. For the short run, we find that her *expected* posterior belief differs from her prior in the direction of A—at least when the signal quality is sufficiently low. For the long run, we identify a precise information quality threshold below which almost surely the agent's asymptotic belief assigns probability one to A, *irrespective* of the true state. For higher quality, beliefs converge to the truth despite the echo-chamber

[5]See Cross (1977); Svenson (1981); Odean (1998); Zuckerman and Jost (2001).

effect. Both results depart from standard Bayesian learning. Perhaps counterintuitively, by thinking that her friends might be uninformed, conditional on receiving a shared signal the agent views it as more informative than it is. This mechanism affects normal and dogmatic friends asymmetrically, giving the latter the power to distort beliefs. Moreover, instead of curbing this power, abundant information can boost it and exaggerate incorrect learning. Note that the echo-chamber imbalance can be arbitrarily small, yet offset many unbiased signals. This happens when information has low quality—for *all*—because starting from this level, the misperception boost to the perceived informativeness of shared signals is more consequential. Finally, we show that both selective sharing and misperception are necessary for our conclusions.

It is easy to see how these forces can cause beliefs to polarize. If some agents have echo chambers unbalanced towards different states *and* information quality is sufficiently low, their beliefs can move apart on average in the short-run and almost surely in the long run. One of our main contributions is to highlight this role of information quality, in conjunction with echo chambers. It is obvious that if the signals perfectly revealed the state, it would be impossible for echo chambers to distort beliefs. But in reality information is noisy. By quantifying the minimal quality that ensures correct learning, we open the door to feasible interventions aimed at curbing polarization. Our analysis goes to the heart of why new technologies and formats of communication enabled by the Internet can increase polarization. They raised the frequency of information arrival and possibly lowered its quality as a result—for instance, tweets and social-media posts tend to be short. Moreover, overwhelmed by the huge abundance of information, people may spread their limited attention across more sources, thereby absorbing less accurate information from each of them. All of this can be a driver of polarization, even without deliberate misinformation. Given this, one may expect that raising information quality would undoubtedly help curb polarization in a network. However, we find that raising the information quality may *increase* polarization. We identify conditions on the network that are necessary and sufficient for this effect.

We find that the expansion of connections as networks grow can also be a driver of polarization. Larger networks provide better access to information but also greater scope for echo chambers to distort beliefs. We provide sharp conditions on the growth rates of the different types of friends under which network expansion curbs or exacerbates polarization. In short, curbing polarization requires that normal friends grow sufficiently *faster* than dogmatic friends. Thus, fixing information quality, it is possible that society is not polarized when people have small echo chambers, but becomes polarized when they have similarly divided, but larger, echo chambers.

Equipped with these results, we return to the motivating question of what policies may reduce the effects of the Internet and social media on polarization. We highlighted several drivers of polarization: selective news sharing in echo chambers, misperceptions, and low information quality. The first seems hard to regulate without infringing personal freedom. The second may be addressed by improving digital literacy so that people can better take into account selective sharing. The last allows for perhaps easier and less intrusive interventions aimed at providing higher-quality information. Incentivizing independent news sources to

do so in a decentralized way may be hard. However, we show that an alternative route is to create institutional intermediaries that aggregate news for people, namely, that provide information in larger "digested" batches rather than many "raw" bits. Even if these less frequent news reports *lose* some of the information in the summarized facts, they can mitigate polarization by reducing the scope for echo chambers to cause people's beliefs to go awry. This also provides a rationale for authorities to commit to releasing information only rarely but of high quality.

Related Literature

The economics literature discusses at least three possible causes for belief polarization. One strand of papers that is most closely related (e.g., Levy and Razin, 2019b; Hoffmann et al., 2019; Enke et al., 2019) has studied polarization arising from behavioral biases. Our main contribution to this literature is to highlight the importance of the network structure coupled with selective sharing and misperception in generating polarization. Another strand studies polarization arising as a result of heterogeneity in preferences (see Dixit and Weibull (2007) and Pogorelskiy and Shum (2019)). Such heterogeneity of preferences would exacerbate the polarization we find, but is not required for generating our results about polarization. Finally, another reason for polarization that has been studied is biased or multidimensional information sources (e.g. Mullainathan and Shleifer, 2005; Andreoni and Mylovanov, 2012; Levendusky, 2013; Conroy-Krutz and Moehler, 2015; Reeves et al., 2016; Perego and Yuksel, 2018), where bias usually comes from media competition over viewers. In our model, the external information sources are assumed to be unbiased when reporting information and bias is generated in the network. Andreoni and Mylovanov (2012) show that one-dimensional opinions can diverge with two-dimensional information. We provide conditions under which one-dimensional opinions diverge with one-dimensional information, based on selective sharing and misperception.

This project also fits into the growing literature on the effects of model misspecification on social learning. Bohren (2016), Bohren and Hauser (2018), and Frick et al. (2019) analyze how model misspecification impacts long-term learning in environments where agents learn from private signals and the actions of other agents. In particular, Bohren and Hauser (2018) study when agents with different, yet reasonable, models have no limit beliefs (i.e. beliefs cycle) or have different limit beliefs (disagreement). The result about disagreement is close in spirit to what we find, but the driving mechanism is fundamentally different: In our model, all agents have the same misspecified model of the world and polarization results from exposure to selectively shared information through the social network. We also emphasize the important role played by the quality of information, which may suggest a simple way to alleviate polarization by aggregating signals. Molavi et al. (2018) study long-run learning on social networks when non-Bayesian agents exhibit imperfect recall. They show that such agents may overweigh evidence encountered early on relative to later information, which can lead to mislearning in the limit. Again, while this conclusion is related to ours, unlike these authors, we assume that agents update beliefs via Bayes' rule, albeit with a misspecified

model of the world.

A key element of our model is the network structure. There is a large literature on both Bayesian (Acemoglu et al., 2010; Pogorelskiy and Shum, 2019; Spiegler, 2019) and non-Bayesian learning in networks (Golub and Jackson, 2010; DeMarzo et al., 2003; Azzimonti and Fernandes, 2018; Eyster and Rabin, 2010). One closely related paper in this literature is Levy and Razin (2019a), which considers the effect of what they call a "Bayesian Peer Influence" updating heuristic on the limit beliefs in the network. One of the main results shows how beliefs in society can become polarized as a result of agents using this updating rule. However, the meaning of polarization in that paper is different from ours: Instead of groups in society becoming polarized, it is the entire society's consensus that shifts towards extreme beliefs, whereas we provide conditions under which beliefs of agents in society diverge upon arrival of new information.

Recent empirical studies show that social media is an important source of news for people and can lead to divergence of beliefs and attitudes (e.g. Allcott and Gentzkow, 2017; Bursztyn et al., 2019; Mosquera et al., 2019; Levy, 2020). Still other empirical evidence by Boxell et al. (2018) suggests that the Internet does not drive polarization. Our model can predict in which environments we expect to see the Internet drive polarization. We, thus, contribute to this literature by providing a theoretical framework to better understand how social media can contribute to polarization and to guide future empirical investigations of this phenomenon.

The remainder of the paper is organized as follows. Section 2 presents our stylized model of information sharing in a network. Section 3 presents the main results on learning in the short and long run. Section 4 considers what happens when the network expands. Section 5 considers the polarization in the entire network. Section 6 demonstrates how aggregating signals can mitigate polarization. We conclude with a discussion of the results in Section 7.

2 Model

We provide a stylized model of information sharing in a network. Time is discrete and denoted by $t = 0, \ldots, T$, where T may be infinite. There are two possible states of the world, $\omega \in \{A, B\}$, and the state is determined by nature in period 0. For example, the state could represent whether the preservation of the environment requires lower or higher national spending than the current level, or whether vaccinations can be harmful. There is an exogenously fixed network of agents who seek to match the state. Each period $t \geq 1$, each agent privately receives a signal $s_{it} \in \{a, b\}$ about the state, which is i.i.d. across agents and time.[6] The information quality of s_{it} is denoted by q, namely, the probability that the

[6]In reality, people receive news that are correlated. However, strong evidence suggests that people often neglect correlation, especially in network environments (Enke and Zimmermann (2017); Eyster et al. (2018); Pogorelskiy and Shum (2019)). Under correlation neglect, we can allow for arbitrary correlation between the agents' signals within each period without qualitatively changing the results.

signal matches the true state. Formally,

$$\begin{aligned}\mathbb{P}(s_{it} = a|\omega = A) &= \mathbb{P}(s_{it} = b|\omega = B) = q,\\ \mathbb{P}(s_{it} = b|\omega = A) &= \mathbb{P}(s_{it} = a|\omega = B) = 1 - q.\end{aligned}$$

We assume that $q > \frac{1}{2}$, so that the signals are at least partially informative. As q increases the signal becomes more precise, so we refer to q as the information *quality*.

Once an agent receives a signal, she can either stay silent or share the signal with her neighbors in the network, which we will call her friends. She cannot lie when sharing a signal, which rules out fake news. For example, an agent can choose to share a news article or not, but cannot edit the article's contents. In addition, she cannot choose with whom to share her information, i.e. no targeting: She either shares it with all her friends, or with none. By ruling out fake news and targeted sharing, we highlight the role of selective sharing in a baseline model to which these other effects can be added.

Our model aims to capture the evidence of selective sharing that Pogorelskiy and Shum (2019) and others find. To this end, we introduce three types of agents. We will call an agent *A-dogmatic* if she shares only signals $s_{it} = a$ and *B-dogmatic* if she only shares signals $s_{it} = b$. One interpretation is that such agents choose to share only information that supports the state they dogmatically believe in.[7] We will call an agent *normal* if she shares any signal she receives. Each agent's type is fixed and exogenous. Formally, each period the left and right dogmatic types will transmit signals as follows:

$$a_A(s_{it}) = \begin{cases}\text{transmit, if } s_{it} = a\\ \text{stay silent, if } s_{it} = b\end{cases}$$

$$a_B(s_{it}) = \begin{cases}\text{stay silent, if } s_{it} = a\\ \text{transmit, if } s_{it} = b\end{cases}.$$

The timing of period $t \geq 1$ is as follows: (1) signals s_{it} are independently realized; (2) agent i observes s_{it} privately; (3) agent i shares s_{it} with her friends according to her type and transmission strategy; (4) agents update beliefs based on all signals they have received, including their own signal.

We are interested in studying the beliefs of normal agents. We will assume that all normal agents have a common prior that assigns probability $\pi \in (0,1)$ to state A. Although our results can be adjusted to allow for different priors, this assumption helps emphasize the drivers of polarization that we discover. Let i be any normal agent. The composition of her friends will play a key role. Suppose i has $d_{Ai} \geq 0$ A-dogmatic friends, $d_{Bi} \geq 0$ B-dogmatic friends, and $n_i \geq 0$ normal friends. We will refer to i's composition of friends (d_{Ai}, d_{Bi}, n_i) as her *echo chamber*, since the behavior of dogmatic friends to reinforce a particular view is similar to the common understanding of how echo chambers work on social networks. Our results will show that the composition of i's echo chamber has a crucial effect on her posterior

[7] Dogmatic agents can be thought of as having degenerate beliefs that do not change with new information.They can be thought of as stubborn, narrow minded, individuals who blindly follow and promote their convictions.

beliefs.

We argued that, besides selective sharing, a key part of our analysis is to allow for misperceptions about it. Selective sharing involves suppressing some information that one receives and replacing it with silence. Thus, one plausible way to misperceive selective sharing in our setting is to misinterpret the silence of a friend as the possibility that she genuinely received no information (i.e., no signal). This form of misperception has another interpretation that is consistent with our analysis. A manifestation of the so-called "illusory superiority" or "better-than-average" heuristic is thinking that other people may be less informed than they actually are (see, for example, Cross, 1977; Svenson, 1981; Odean, 1998; Zuckerman and Jost, 2001). People often have unjustifiably favorable views of themselves relative to the population average or even in person-to-person comparisons on various characteristics, which may include how well informed they are or how good they are at getting and understanding information.

We model such misperceptions as follows. Each period, agent i observes her signal s_{it} and the signals shared by her friends. When updating, i knows the types of her friends and is aware of their transmission strategies. However, she thinks that they are less informed than they are: In her view, in every period each friend observes a signal only with probability $\gamma < 1$ and no signal with probability $1 - \gamma$. Thus, from i's perspective, each of her friends faces the following timing in period $t \geq 1$: (1) signals s_{it} are independently realized; (2) friend i privately observes s_{it} with probability γ; (3) if s_{it} is observed, it is either shared or not according to i's type and strategy; (4) the agents update beliefs. Note that γ is the same for all normal agents. This way of modeling misperceptions of selective news sharing may be viewed as a minimal departure from a standard Bayesian model, where the agents continue to be Bayesian but relative to a slightly misspecified model of the world. This way of modeling misperceception also ensures that selective sharing does not unravel, which would prevent it from having any effect on beliefs.

3 Learning in the short and long run

To examine the effects of selective sharing and misperceptions in the short run, our first result looks at the expected posterior belief of agent i that results from one round of updating (i.e., $T = 1$). Since the focus of this section is normal agent i, we drop all i subscripts. Let $\mu(\mathbf{s}^1)$ be normal agent i's Bayesian posterior probability assigned to state A given all the information she obtains after one period, which we denote by $\mathbf{s}^1$ (i.e., her signal, her friends' signals, and their silence). We show that whenever there is an imbalance in i's echo chamber, i's expected posterior conditional on *any* state will be distorted towards the state preferred by the majority of her dogmatic friends when the signal quality q is sufficiently low.

Proposition 1. *Fix an agent such that $d_A > d_B$. There exists $\bar{q}_1(d_A, d_B, n, \gamma) > \frac{1}{2}$ such that, if $q < \bar{q}_1(d_A, d_B, n, \gamma)$, then for $\omega \in \{A, B\}$*

$$\mathbb{E}\left[\mu(\mathbf{s}^1)|\omega\right] > \pi.$$

The proof of Proposition 1, and henceforth all omitted proofs, can be found in the Appendix.

An immediate implication of this result is that a defining property of Bayesian updating no longer holds in our setting. Since the expected posterior is distorted conditional on any true state of the world, the *unconditional* expected posterior will also be distorted in the same direction. This is in contrast to Bayesian updating with a correctly specified model of the world, where the expected posterior would be *equal* to the prior.

We provide some intuition for the result by first deriving agent i's posterior after one period. Consider that agent i has echo chamber (d_A, d_B, n). Let a_A denote the number of a-signals that A-dogmatic friends of i have received, and a_B denote the number of a-signals that B-dogmatic friends of i have received. Since there are d_A A-dogmatic and d_B B-dogmatic friends (and they all receive independent signals), a_A is distributed as a Binomial random variable with probability $1-q$ and sample size d_A, whereas a_B is distributed as a Binomial random variable with probability $1-q$ and sample size d_B. Note that i also receives a private signal and n independent signals from her normal friends. This is equivalent to receiving $n+1$ independent private signals. Letting a_N denote the number of signals $s=a$ that are contained within them, a_N is a Binomial random variable with probability $1-q$ and sample size $N=n+1$. Note that agent i's information $\mathbf{s}$ is summarized by (a_A, a_B, a_N).

Given a fixed realization (a_A, a_B, a_N), by Bayes's rule agent i's posterior belief is given by

$$\mu^1(\omega = A | a_A, a_B, a_N) \;=\; \frac{\pi}{\pi + (1-\pi) Q^M \Gamma^S}, \tag{1}$$

where

$$\begin{aligned} Q &\equiv \frac{1-q}{q} \\ M &\equiv a_A - (d_B - a_B) + 2a_N - N \\ \Gamma &\equiv \frac{\gamma(1-q) + (1-\gamma)}{\gamma q + (1-\gamma)} \\ S &\equiv a_B - (d_A - a_A). \end{aligned}$$

Note that Q^M captures the correct interpretation of signals received. Agent i receives $a_A + a_N$ a-signals from dogmatic left and normal friends. She receives $(d_B - a_B) + (N - a_N)$ b-signals from dogmatic right and normal friends which counteract the left signals (and hence enter negatively in the exponent).

In addition to signals received, agent i observes silence from dogmatic left and right friends, which is incorrectly interpreted. This is captured by Γ^S. Agent i observes silence from a_B B-dogmatic friends, each of which agent i (incorrectly) attributes either to an a-signal with probability γ or to no signal with probability $1-\gamma$. Similarly, agent i observes silence from $d_A - a_A$ A-dogmatic friends and (incorrectly) attributes these to either a b-signal or no signal. Counterintuitively, the incorrectly interpreted silence captured by Γ^S provides a "boost" to the correctly interpreted signals Q^M. Note also that if $\gamma = 1$ then

$\Gamma = (1-q)/q$ so the boost disappears and silence is correctly interpreted.

Although the term Γ^S distorts the posterior, it is still unclear that the *average* distortion goes in any specific direction. For instance, they could inflate or deflate updating but still be correct on average. Intuitively, if i's echo chamber has more A-dogmatic than B-dogmatic friends ($d_A > d_B$), she tends to receive more signals that support A than B as the true state due to their selective sharing. However, i is fully aware of this and would be able to fully unravel the effect of selective sharing, except that her misperception complicates this. When hearing silence from a dogmatic friend, i could either (correctly) assume her friend is censoring, or (incorrectly) assume that friend simply received no signal. It is this latter incorrect assumption that allows i's belief to be biased. Clearly, the quality of the signals cannot be perfect (i.e., $q = 1$) for i's expected posterior to deviate from her prior. Yet, the result shows that the imbalance between dogmatic friends—however small—can always overcome the unbiased information coming from normal friends and i's own signal when the agents get signals of sufficiently low quality. Put differently, the misperceived selective sharing dominates even though dogmatic friends' signals become almost uninformative.

Proposition 1 has immediate consequences for belief polarization in societies. If i has more A-dogmatic than B-dogmatic friends and i' has more B-dogmatic than A-dogmatic friends—where again the imbalances can be small—then in expectation their posteriors will move apart, at least if the quality of signals is small. Thus, our result suggests that the composition of one's echo chamber can be a driver of belief polarization in spite of all agents having the same access to first-hand information. This is consistent with the narrative that social media can give rise to "echo chambers" where people are exposed to an unbalanced diet of opinions and as a result develop polarized beliefs. However, our result qualifies this narrative in two ways: First, it does not require the presence of fake news; second, it stresses that echo chambers also require low quality information in order to distort beliefs from the truth.

Our first result showed that one round of information arrival is enough to give rise to polarization, at least in expectation. However, it leaves open the possibility that when information becomes abundant (i.e., in the long run after many signals) polarization disappears. In fact, the opposite can occur. As long as information quality is sufficiently low, abundant information can exacerbate the effect of misperceived selective sharing and cause i's beliefs to be incorrect *with probability 1*. To show this, we consider the limit in probability of the posterior belief $\mu(\mathbf{s}^T)$ as $T \to \infty$, which we denote by $\operatorname{plim}_{T\to\infty} \mu(\mathbf{s}^T)$. If q is sufficiently small, i's posterior belief is pushed towards the degenerate belief on the state that is favored by the majority of her dogmatic friends, with probability 1 and irrespective of the true state. Denote, $I_{\{\omega=A\}}$ as the indicator function that equals 1 if $\omega = A$ and 0 otherwise.

Proposition 2. *Fix an agent such that $d_A > d_B$.*

1. *There exists a unique $1 > \bar{q}_2(d_A, d_B, n, \gamma) > \frac{1}{2}$ such that*

$$\operatorname*{plim}_{T\to\infty} \mu(\mathbf{s}^T) = \begin{cases} 1 & \text{if } q < \bar{q}_2(d_A, d_B, n, \gamma) \\ I_{\{\omega=A\}} & \text{if } q > \bar{q}_2(d_A, d_B, n, \gamma) \end{cases}$$

2. $\bar{q}_2(d_A, d_B, n, \gamma)$ *is increasing in* $d_A - d_B$ *and decreasing in* n *and* γ*.*

Consider the first part of Proposition 2. Intuitively, with many rounds of information arrival agent i's own signals provide a more accurate estimate of the state, which would result in perfect learning in a standard setting. However, i combines her signals with the signals from her friends, which provide more information, but this information is biased in ways she does not correctly take into account. Once again, the outcome of this race between the two kinds of information is a priori not clear. Yet, with low quality signals, the distortion introduced in each step of updating unveiled in Proposition 2.1 accumulates over time leading the posterior astray with certainty.

To see this more precisely, consider updating at each period t. Dogmatic friends tend to provide more $s = a$ than $s = b$ signals. Normal friends provide unbiased signals. Who wins the race as $T \to \infty$? Recall the correct updating term Q^M and incorrect updating term Γ^S. When q is close to $\frac{1}{2}$ (low informativeness), the correct updating term Q^M is close to 1 and thus the misperception boost to informativeness Γ^S dominates. By contrast, when q is close to 1 (maximal informativeness), the correct updating term Q^M is close to zero and hence the misperception boost to informativeness Γ^S is curtailed.

The second part of Proposition 2 provides important comparative statics on the lower bound of informativeness that avoids polarization. This lower bound increases with the absolute difference between the number of A-dogmatic and B-dogmatic friends, or the magnitude of the echo chamber imbalance, and increases with the degree of misperception. The lower bound decreases with number of normal friends. Therefore, given q, it is possible that if an agent belongs to a social group with a large number of normal friends, she learns correctly as information becomes abundant. By contrast, if the agent belongs to a smaller social group with fewer normal friends, the effect of dogmatic friends may prevail for the same q, causing her belief to polarize relative to other agents.

With regard to polarization, Propositions 1 and 2 can shed further light on the role of "echo chambers." In sum, they highlight the importance of the composition of an echo chamber, rather than its size. To the extent that interacting online promotes the formation of more unbalanced social groups, it may contribute to belief polarization even when the quality of information is unchanged.

Propositions 1 and 2 show that imbalance in selective sharing and misperception are sufficient for polarization to arise. But are they necessary? The answer is yes.

Proposition 3. *If either* $d_A = d_B$ *or* $\gamma = 1$*, then*

$$\mathbb{E}[\mu(\mathbf{s}^1)] = \pi \qquad \textit{and} \qquad \operatorname*{plim}_{T\to\infty} \mu(\mathbf{s}^T) = I_{\{\omega=A\}}.$$

Intuitively, if i has an equal number of A-dogmatic and B-dogmatic friends, they offset each other when sharing information selectively, even in the presence of misperception. Alternatively, if i fully understands their selective sharing—always interpreting their silence as the arrival of unfavorable signals—she fully unravels its effects even if there are more dogmatic friends on one side. Note that even a minor degree of misperceptions (γ close to but strictly less than 1) will make polarization possible.

4 Network expansion

Consider agent i once again and suppose that due to advances in technology (e.g. development of social media), the agent is now connected with more agents of all types, while still receiving the same quantity of information on her own. Specifically, assume that the number of normal, A-dogmatic and B-dogmatic friends expands according to proportions $(\lambda_A, \lambda_B, \lambda_N)$, respectively. Proposition 4 characterizes the impact on the informativeness required to avoid polarization.

Proposition 4. *Fix an agent such that $d_A > d_B$, then $\bar{q}_2(\lambda_A d_A, \lambda_B d_B, \lambda_N n, \gamma) \geq \bar{q}_2(d_A, d_B, n, \gamma)$ if and only if*

$$(\lambda_N - 1) \leq \left(\frac{\lambda_A d_A - \lambda_B d_B}{d_A - d_B} - 1 \right) \cdot \left(1 + \frac{1}{n} \right)$$

Proposition 4 states that the range of information quality q leading to incorrect learning shrinks if and only if normal friends grow sufficiently fast. Intuitively, there is a trade-off between access to information and the scope for echo-chambers to bias beliefs. Note that if scaling is proportional (i.e. $\lambda_A = \lambda_B = \lambda_N$) then $1 < 1 + \frac{1}{n}$ and hence the range of information quality leading to incorrect beliefs *expands*. Thus a proportional scaling of the network leads to more opportunities for polarization.

This result suggests that when there are changes in the network environment that increase connectivity of people (keeping proportions of different types of friends the same), but do not increase the amount of information people get on their own, it can contribute to belief polarization. For a fixed information quality q, it is possible that a person whose beliefs are not biased in a smaller network becomes polarized when the network expands.

The previous proposition deals with how large λ_N needs to be in order to decrease an individual's $\bar{q}_2$. This reduces the range of q in which the individual gets biased, but may still not be enough if the true q is far lower. The next proposition explores how large λ_N needs to be in order to drop $\bar{q}_2$ below any given value of q, so as to un-bias a given agent. Put another way: Given some $\hat{q}$, what λ_N suffices to ensure correct learning?

Proposition 5. *Fix $d_A > d_B$ and $\hat{q} \in \left(\frac{1}{2}, \bar{q}_2(d_A, d_B, n, \gamma) \right)$. Then $\bar{q}_2(\lambda d_A, \lambda d_B, \lambda_N n, \gamma) < \hat{q}$ if*

$$\lambda_N > \frac{d_A - \hat{q}(d_A + d_B)}{(2\hat{q} - 1)n} \lambda - \frac{1}{n}.$$

Given echo chamber (d_A, d_B, n) and growth rates of friends (which may be determined by data or algorithms), this result predicts if and when $\bar{q}_2$ crosses $\hat{q}$. This points to how link-formation algorithms may possibly be designed to minimize polarization. i.e. the result suggests the growth rate of normal friends required. Note that the right side of the inequality is increasing in d_A and λ, decreasing in d_B and $\hat{q}$. Moreover, the right side of the inequality is decreasing in n when it is positive (when $\hat{q} < \bar{q}_2$).

5 Network polarization

In this section, we move from the analysis of individual agents to the analysis of the network as a whole. Our focus is the set of normal agents in the network (i.e., excluding dogmatic ones). Interpreting dogmatic agents as those whose beliefs are degenerate implies that only the beliefs of normal agents respond to arriving information. Let the set of all normal agents be denoted by $\mathcal{N}$.

Before proceeding, we define a measure of polarization for our network. Without loss of generality, we will define it for beliefs in state A. Define polarization in period t as the average sum of differences of beliefs across all agents in $\mathcal{N}$:

$$\Pi^t = \frac{2}{|\mathcal{N}|^2} \sum_{i,j \in \mathcal{N}} \left| \mu_i(\mathbf{s}_i^t) - \mu_j(\mathbf{s}_j^t) \right|.$$

This expression captures the extent to which disagreement is present in the network. The scaling factor of 2 is introduced in order to ensure $\Pi^t \in [0,1]$.

We focus on asymptotic polarization of beliefs (as $T \to \infty$), when all normal agents' beliefs have already converged. Let $\mathcal{N}_B$ and $\mathcal{N}_A$ denote the sets of normal agents whose asymptotic beliefs put probability 1 on states B and A, respectively. We will also refer to these sets as "eventually incorrect" and "eventually correct" populations, depending on which state is true.[8] Define the limit polarization Π as

$$\Pi \equiv \operatorname*{plim}_{t \to \infty} \Pi^t = \frac{2}{|\mathcal{N}|^2} \cdot 2|\mathcal{N}_B||\mathcal{N}_A| \cdot |1-0| = \frac{4|\mathcal{N}_B||\mathcal{N}_A|}{|\mathcal{N}|^2}$$

The value of Π varies from 0 to 1, attaining its maximum when $\mathcal{N}_B$ and $\mathcal{N}_A$ have the same cardinality.

The first result in this section concerns the dynamics of $\mathcal{N}_B$ and $\mathcal{N}_A$ as q increases from $1/2$ to 1. Assume the true state is B and fix some $q \in (1/2, 1)$. Then, $\mathcal{N}_B$ consists of two sets of agents: (1) all agents for whom $\bar{q}_{2i}$ is below q (these agents learn correctly), *and* (2) all agents for whom B-dogmatic friends dominate A-dogmatic friends and $\bar{q}_{2i}$ is above q (these agents place probability 0 on $\omega = A$ in the long run irrespective of true state). Meanwhile, $\mathcal{N}_A$ consists of all agents for whom A-dogmatic friends dominate B-dogmatic friends and whose $\bar{q}_{2i}$ is above q.

As q increases, all agents that are already in $\mathcal{N}_B$ will remain there. Each agent for whom $\bar{q}_{2i}$ is below q will continue to be below q, and other agents for whom B-dogmatic friends dominate A-dogmatic friends may remain in that category or pass into the first category and still remain in $\mathcal{N}_B$. However, agents in $\mathcal{N}_A$ may switch to the other set. When q becomes larger than $\bar{q}_{2i}$ for an agent in $\mathcal{N}_A$, he or she is no longer biased by A-dogmatic friends, and thus will place probability 0 on $\omega = A$ in the limit instead of 1, passing into $\mathcal{N}_B$. This argument is formalized in the following lemma.

[8] If B is the true state, then $\mathcal{N}_B$ is the "eventually correct" population, whereas $\mathcal{N}_A$ is the "eventually incorrect" population—vice versa if the true state is A.

Lemma 1. *As q increases, the set of "eventually correct" agents is weakly expanding and the set of "eventually incorrect" agents is weakly contracting.*

What implications does this have for network polarization? A common intuition would suggest that as people receive better information, disagreements should decline. Lemma 1 places some limits on this intuition: if for low q the set of "eventually incorrect" agents is sufficiently large, polarization in society will be low. However, as q increases, it will cause a gradual shift into the set of "eventually correct" agents, and polarization in society will actually *increase* initially. Once the set of "eventually correct" agents outnumbers the set of "eventually incorrect" ones, polarization will start to decline as q increases. This implies that polarization in a social network may temporarily increase as quality of information is slowly improving.

The following proposition summarizes the intuition above and provides a necessary and sufficient condition for such non-monotonicity of network polarization to take place. Let $\mathcal{N}_B(q)$ and $\mathcal{N}_A(q)$ denote the sets of agents whose beliefs converge to state B and A, respectively, for a given value of q. Additionally, let $\mathcal{D}_B$ and $\mathcal{D}_A$ denote sets of agents who have a B- and A-slanted echo chambers, respectively. The remaining agents do not have any imbalance in their echo chamber.

Proposition 6. *Fix network with $\bar{q}_{2i} \neq \bar{q}_{2j}$ for all i, j and ω. Π is decreasing in q over $\left(\frac{1}{2}, 1\right)$ iff $|\mathcal{D}_{-\omega}| \leq \frac{1}{2}(|\mathcal{N}| + 1)$. Otherwise, Π is single peaked.*

6 Mitigating polarization

We have shown that selective sharing and misperceptions are enough to generate polarization of beliefs in a network of agents. How could a social planner combat this polarization, if he or she could affect only the information the agents receive (taking selective sharing and misperception as given)? That is, we want to see whether it is possible to eliminate polarization by changing only the external information structure of the agents. One obvious way to do this is to *directly* increase the quality of information q at the source. However, this may not be feasible due to technological or economic reasons. For example, this may involve forcing newspapers to spend more on reporters, data gathering, and fact checking. However, it is possible to increase the quality of information that the agents receive without changing the quality of the primitive signals s_{it}. This involves *signal aggregation*.

Signal aggregation consists in summarizing a set of signals into a single message. This summary can, of course, be done in many ways, emphasizing some aspects of the original signals and omitting others. It is important to note, however, that signal aggregation involves some loss of information relative to the totality of the aggregated signals. Nonetheless, the resulting message can have higher quality than each aggregated signal individually. It is this distinction that renders signal aggregation useful for our goal of reducing polarization, despite the loss of information. As such, signal aggregation involves a trade-off between

slowing short-run learning and debiasing long-run learning, which is a novel aspect of our learning environment.

Consider the following form of signal aggregation. Let M be an odd number and divide time into blocks of M periods. Define $\hat{s}^i_{Mt}$ as the aggregated signal that is released to agent i at the end of each time block and reports whether more $s = a$ or more $s = b$ signals realized over the previous M periods:

$$\hat{s}^i_{Mt} = \begin{cases} 0, \text{ if } \sum_{k=(t-1)M+1}^{tM} 1_{\{s_{ik}=a\}} < \frac{M}{2} \\ 1, \text{ if } \sum_{k=(t-1)M+1}^{tM} 1_{\{s_{ik}=a\}} > \frac{M}{2}. \end{cases}$$

A natural question is why a social planner would prefer to coarsen each agent's information structure in such a way. Clearly, $\hat{s}^i_{Mt}$ conveys less information than do the aggregated M signals together. However, on the one hand, if all agents observe only $\hat{s}^i_{Mt}$ every other M periods, they have *fewer* opportunities to selectively share information with their friends. But this is irrelevant in the long run. On the other hand—and more importantly—$\hat{s}^i_{Mt}$ has higher quality than each individual s_{it}. As a result, by substituting s_{it} information structure with $\hat{s}^i_{Mt}$ for each agent, we are worsening information quality for each agent individually, but we are also reducing the influence of selective sharing on agents' beliefs. Hence, this might restore the standard result of convergence to the truth in asymptotic learning. To see this, suppose $M = 3$, consider the first block of three periods, and assume $\omega = A$. We have

$$\begin{aligned} \mathbb{P}(\hat{s}^i_3 = 0|\omega = A) &= \mathbb{P}\left(\sum_{t=1}^{3} s_{it} \leq 1 \middle| \omega = A\right) \\ &= q^3 + 3q^2(1-q) \\ &= q \cdot \Big(q^2 + 2q(1-q) + q(1-q)\Big) \\ &> q \cdot \Big(q^2 + 2q(1-q) + (1-q)^2\Big) \\ &= q = \mathbb{P}(s_{it} = a|\omega = A). \end{aligned}$$

Thus, $\hat{s}^i_3$ is more informative than each s_{it} of quality q. In contrast to standard models where the quality of information does not matter in the long run, we saw that in our model a sufficiently high quality of information can allow all agents to learn correctly, thereby removing polarization. The remaining question is then whether aggregating signals according to $\hat{s}^i_{Mt}$ can achieve that quality level. The next proposition shows that this is always possible. Denote $\hat{\Pi}$ as the resulting limit polarization when signal s_{it} is replaced with aggregate signal $\hat{s}^i_{Mt}$ every M periods.

Proposition 7. *Fix any network and q such that $\Pi > 0$. There exists M such that, if each agent i observes signals $\hat{s}^i_{Mt}$, then the resulting limit polarization $\hat{\Pi}$ equals zero.*

7 Discussion

We explored conditions under which learning from shared news may lead to belief polarization. We develop a model reflecting two important facts about news sharing in networks: unbalanced selective transmission of information and misperception. We include misperception to account for the evidence that people are neither completely naive nor fully aware of selective sharing. We found that with these features polarization occurs if (and only if) information quality is sufficiently low. Thus, our results emphasize the importance of distinguishing between quality and quantity of information, a distinction that is irrelevant in standard models.

With this understanding, we also investigated ways to mitigate polarization. On the one hand, our analysis of the effects of network expansion suggests ways to influence network dynamics so as to limit the scope for polarization to occur. On the other hand, we show that aggregating the signals that agents receive may mitigate polarization. A benevolent authority may use this insight to stop the effects of misperceived selective sharing and debias beliefs. In practice, one way to achieve this is to create institutional intermediaries or platforms that aggregate news, namely, that provide people with information in larger digested batches rather than many raw bits. Even if these less frequent news reports *lose* some of the information in the summarized news, they can mitigate polarization by reducing the scope for echo chambers to cause people's beliefs to go awry.

Our analysis goes to the heart of why new technologies and formats of communication enabled by the Internet may increase polarization. Tweets and social-media posts tend to be short and hence of low quality. Moreover, overwhelmed by the abundance of information, people may spread their limited attention across more sources, thereby absorbing less accurate information from each of them. Finally, we also shed new light on two ways in which malevolent actors who thrive on social polarization benefit from how news is shared in social networks. One obvious way is to use fake news to directly lower the information quality or bias echo chambers in a particular direction so as to exacerbate their effects. A more subtle way is to release bits of news with high frequency so as to leverage the biasing power of misperceived selective sharing that we uncover and to draw attention away from high-quality information sources. Even though removing fake news is important for addressing polarization, our analysis suggests that it is not sufficient.

Another class of policies may be to intervene in network dynamics. Indeed, in our results expanding networks can help curb polarization if they also shrink the imbalance in echo chambers. We provide tools to precisely quantify how much the imbalance has to shrink and how long it will take, given the evolution of networks. These conditions can be easily tested using data about one's composition of friends, their news-sharing habits, and their growth rates. To the extent that social-network platforms influence this evolution with their algorithms suggesting new friends, our results may guide how to design them so as to render echo chambers more balanced.

Finally, our theory sheds new light on how malevolent actors who thrive on social polarization can benefit from how news is shared in social networks. One obvious way is to use

fake news to directly lower the information quality or render echo chambers even more unbalanced. A more subtle way is to release bits of news with high frequency so as to leverage the power of misperceived selective sharing and to draw attention away from high-quality information sources. These insights may be useful to guide policies aimed at dealing with malevolent actors. Even though removing fake news is important for addressing polarization, our analysis suggests that it is not sufficient.

Our analysis opens several directions for future research. So far, we have exogenously assumed selective sharing by having dogmatic agents who do not update their beliefs and so do not change how they selectively transmit their signals. However, in reality people often choose what information to share so as to promote their views or beliefs, which are not set in stone. One can interpret the current model as describing situations where dogmatic agents can change their beliefs, yet they do so much more slowly than normal agents. As a result, how they selectively share information is very persistent, which should leave the insights of our results unchanged.

Our results about the beliefs of a single agent can be used to further examine the distribution of beliefs in the network as a whole. We conjecture that there is a relationship between how the composition of echo chambers is distributed in the network and how beliefs are distributed across agents. For instance, if half the population has a majority of dogmatic-right friends, half a majority of dogmatic-left friends, and information quality is sufficiently low, then the beliefs of each half should move apart as information arrives. This is consistent with evidence showing that people on the left and on the right of the political spectrum tend to have more like-minded friends than not, and with the view that this may be a cause of the ongoing polarization (e.g., see Pew Research Center (2014)). Note that, according to our analysis, such polarization does not require that people look at the world in fundamentally incompatible ways, but only that they have different news diets—which may be easier to address.

Finally, the role of the network structure in our analysis begs the question of what would change if we allowed for endogenous network formation. On the one hand, people may tend to form more links with like-minded friends, which may enlarge the imbalance in their echo chambers. This would reinforce and magnify the effects of selective sharing and misperceptions we highlight. On the other hand, people may be more likely to link with normal friends than dogmatic friends, which would have opposite implications. Either way, this is ultimately an empirical question: Once we measure the rates of link formation with different types of friends, we can use our model to predict their consequences. Nonetheless, understanding the incentives to form links so as to obtain information is also important to answer these questions and guide further empirical investigations of polarization.

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Appendix

A Proof of Proposition 1

We consider agent i who has d_A A-dogmatic friends, d_B B-dogmatic friends and n normal friends. We drop all i-subscripts since we are focusing on a single agent.

Proof of Proposition 1. First, assume that the true state is $\omega = B$. As we noted in Equation 2, given fixed realizations of a_A, a_B and a_N, agent i's posterior belief is given by the following expression:

$$\mu_i(\omega = A|a_A, a_B, a_N) = \frac{\pi}{\pi + (1-\pi)Q^M\Gamma^S}, \tag{2}$$

where

$$\begin{aligned} Q &\equiv \frac{1-q}{q} \\ M &\equiv a_A - (d_B - a_B) + 2a_N - N = a_A + a_B - d_B + 2a_N - N \\ \Gamma &\equiv \frac{\gamma(1-q) + (1-\gamma)}{\gamma q + (1-\gamma)} \\ S &\equiv a_B - (d_A - a_A) = a_A + a_B - d_A. \end{aligned}$$

In terms of random variables, $\mu_i(\omega = A|a_A, a_B, a_N)$ depends only on the sum of a_A and a_B and on a_N. Let $a_D = a_A + a_B$ and note that due to independence of individual signals, a_D is a Binomial r.v. with probability $(1-q)$ and sample size $(d_A + d_B)$.

Agent i's posterior belief, given fixed realizations of a_D and a_N, is then given by:

$$\mu_i(\omega = A|a_D, a_N) = \frac{\pi}{\pi + (1-\pi)Q^{a_D - d_B + 2a_N - N}\Gamma^{a_D - d_A}}$$

Recall that a_D has Binomial distribution with parameters $(1-q)$ and $(d_A + d_B)$. Therefore, i's expected posterior belief, conditional on a_N and $\omega = B$, is given by the following expression:

$$\begin{aligned} \mathbb{E}\left[\mu_i(\omega = A) \mid a_N, \omega = B\right] = \sum_{k=0}^{d_A+d_B} & \left[\frac{(d_A+d_B)!}{k!(d_A+d_B-k)!} \cdot (1-q)^k q^{d_A+d_B-k} \right. \\ & \left. \cdot \frac{\pi}{\pi + (1-\pi)Q^{k - d_B + 2a_N - N}\Gamma^{k-d_A}}\right]. \end{aligned}$$

Recall that a_N is a Binomial r.v. with parameters $(1-q)$ and N. Hence, the expected posterior belief conditional on $\omega = B$ is given by

$$\mathbb{E}\left[\mu_i(\omega = A) \mid \omega = B\right] = \sum_{k=0}^{N} \frac{N!}{k!(N-k)!}(1-q)^k q^{N-k} \mathbb{E}\left(\mu_i(\omega = A)|k, \omega = B\right).$$

We will prove that there exists $\bar{q}_1 > \frac{1}{2}$ such that if $q \in \left(\frac{1}{2}, \bar{q}_1\right)$, $\mathbb{E}\left[\mu_i(\omega = A)|\omega = B\right] > \pi$. For that, we will first find the derivative of $\mathbb{E}\left[\mu_i(\omega = A)|a_N, \omega = B\right]$ with respect to q at $q = \frac{1}{2}$ for any $a_N \geq 0$, and then show that the derivative of $\mathbb{E}\left[\mu_i(\omega = A)|\omega = B\right]$ w.r.t. q at $q = \frac{1}{2}$ is positive. Then, using continuity of the expected posterior in q and the fact that at $q = \frac{1}{2}$ we have $\mathbb{E}\left[\mu_i(\omega = A)|\omega = B\right] = \pi$, this will imply the desired statement.

If $d_A + d_B$ is odd[9], we can rewrite one of the sums above as follows:

$$\mathbb{E}\left[\mu_i(\omega = A)|a_N, \omega = B\right] = \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\Big(f(k,q,a_N) + f(d_A+d_B-k,q,a_N)\Big),$$

where

$$f(k,q,a_N) = (1-q)^k q^{d_A+d_B-k} \frac{\pi}{\pi + (1-\pi)\left(\frac{1-q}{q}\right)^{k-d_B+2a_N-N}\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)^{k-d_A}}$$

For simplifying subsequent algebra, define a function $z(q,\gamma)$ as

$$z(q,\gamma) = \ln(\Gamma)\left[\ln(Q)\right]^{-1} = \ln\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)\left[\ln\left(\frac{1-q}{q}\right)\right]^{-1}.$$

Consider the derivative of $f(k,q,a_N)$ w.r.t. q at $q = \frac{1}{2}$:

$$\begin{aligned}
\frac{\partial}{\partial q} f(k,q,a_N) &= \Big((d_A+d_B-k)(1-q)^k q^{d_A+d_B-k-1} - k(1-q)^{k-1}q^{d_A+d_B-k}\Big) \cdot \\
&\frac{\pi}{\pi + (1-\pi)\left(\frac{1-q}{q}\right)^{k-d_B+2a_N-N+(k-d_A)z(q,\gamma)}} \\
&+(1-q)^k q^{d_A+d_B-k} \cdot \pi(1-\pi) \cdot \Bigg[\frac{(k-d_B+2a_N-N)Q^{k-d_B+2a_N-N-1+(k-d_A)z(q,\gamma)}\frac{1}{q^2}}{\left(\pi+(1-\pi)Q^{k-d_B+2a_N-N}\Gamma^{k-d_A}\right)^2} \\
&\qquad + \frac{(k-d_A)Q^{k-d_B+2a_N-N+(k-d_A-1)z(q,\gamma)}\frac{(2-\gamma)\gamma}{(\gamma q+(1-\gamma))^2}}{\left(\pi+(1-\pi)Q^{k-d_B+2a_N-N}\Gamma^{k-d_A}\right)^2}\Bigg] \\
&\overset{(q=\frac{1}{2})}{=} \left(\frac{1}{2}\right)^{d_A+d_B-1}(d_A+d_B-2k)\frac{\pi}{\pi+(1-\pi)} + \left(\frac{1}{2}\right)^{d_A+d_B} 4\pi(1-\pi) \cdot \\
&\qquad \cdot \frac{(k-d_B+2a_N-N)+(k-d_A)\frac{\gamma}{2-\gamma}}{(\pi+(1-\pi))^2} \\
&= \left(\frac{1}{2}\right)^{d_A+d_B-1} \cdot \Big((d_A+d_B-2k)\pi + 2\pi(1-\pi)\Big((k-d_B+2a_N-N)+(k-d_A)\frac{\gamma}{2-\gamma}\Big)\Big).
\end{aligned}$$

[9]We consider the case of $d_A + d_B$ being even at the end of the proof.

Therefore, at $q = \frac{1}{2}$ we have

$$\begin{aligned}
\frac{\partial}{\partial q}\mathbb{E}\left[\mu_i(\omega = A)|a_N, \omega = B\right] &= \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\left(\frac{\partial}{\partial q}f(k,q,a_N) + \frac{\partial}{\partial q}f(d_A+d_B-k,q,a_N)\right) \\
&= \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\left(\frac{1}{2}\right)^{d_A+d_B-1} 2\pi(1-\pi)\cdot \\
&\quad \cdot\left(2(2a_N - N) + (d_A - d_B) - (k - d_A)\frac{\gamma}{2-\gamma}\right) \\
&= 4\pi(1-\pi)(2a_N - N) + \\
&\quad + \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\left(\frac{1}{2}\right)^{d_A+d_B-1} 2\pi(1-\pi)\left(d_A - d_B\right)\frac{2-2\gamma}{2-\gamma}.
\end{aligned}$$

Note that the second term (a sum) in the expression above is strictly positive for any $a_N \geq 0$, since $d_A > d_B$ and $\gamma \in (0,1)$ hold. Thus, when we compute the derivative of $\mathbb{E}\left[\mu_i(\omega = A)|\omega = B\right]$, this part of the sum will remain strictly positive. The only term to be concerned about is the one with $(2a_N - N)$ in it.

Let

$$H = \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\left(\frac{1}{2}\right)^{d_A+d_B-1} 2\pi(1-\pi)\left(d_A - d_B\right)\frac{2-2\gamma}{2-\gamma}.$$

Then the derivative of the conditional expected posterior at $q = \frac{1}{2}$ is equal to

$$\begin{aligned}
\frac{\partial}{\partial q}\mathbb{E}\left[\mu_i(\omega = A)|\omega = B\right] = \sum_{k=0}^{N} \frac{N!}{k!(N-k)!}\left(\frac{1}{2}\right)^{N-1}(-k + (N-k))\,\mathbb{E}\left[\mu_i(\omega = A)|k, \omega = B, q = \tfrac{1}{2}\right] + \\
+ \sum_{k=0}^{N} \frac{N!}{k!(N-k)!}\left(\frac{1}{2}\right)^{N}\left[H + 4\pi(1-\pi)(2k - N)\right].
\end{aligned}$$

We know $\mathbb{E}\left[\mu_i(\omega = A)|k, \omega = B, q = \frac{1}{2}\right] = \pi > 0$ and $H > 0$. Thus,

$$\begin{aligned}
\frac{\partial}{\partial q}\mathbb{E}\left[\mu_i^1(\omega = A)|\omega = B\right] &= H + \sum_{k=0}^{N} \frac{N!}{k!(N-k)!}\left(\frac{1}{2}\right)^{N-1}(\pi - 2\pi(1-\pi))\,(N-2k) \\
&= H + (\pi - 2\pi(1-\pi))\left(\frac{1}{2}\right)^{N-1}\sum_{k=0}^{N} \frac{N!}{k!(N-k)!}\,(N-2k)
\end{aligned}$$

Note that the sum $\sum_{k=0}^{N} \frac{N!}{k!(N-k)!}\,(N-2k)$ is symmetric around $k = \frac{N}{2}$. Specifically, for $k \leq \frac{N}{2}$ there is exactly identical term with an opposite sign for $j = N - k$. This implies that the sum will annihilate itself into zero. This will mean that the derivative is simply equal to H, which is positive, i.e. the conditional expected posterior is *increasing* at $q = \frac{1}{2}$.

Now consider the case $\omega = A$. The updating rule of the agent will be the same as in the previous case. What changes is the distribution of a_A, a_B and a_N, as they now follow

Binomial distributions with probability parameter q (rather than $1-q$). Hence,

$$\mathbb{E}\left[\mu_i(\omega = A) \mid a_N, \omega = A\right] = \sum_{k=0}^{d_A+d_B} \left[\frac{(d_A+d_B)!}{k!(d_A+d_B-k)!} \cdot q^k(1-q)^{d_A+d_B-k} \cdot \frac{\pi}{\pi + (1-\pi)Q^{k-d_B+2a_N-N}\Gamma^{k-d_A}}\right].$$

Recall that a_N is a Binomial r.v. with parameters q and N. Hence, the expected posterior belief conditional on $\omega = B$ is given by

$$\mathbb{E}\left[\mu_i(\omega = A) \mid \omega = A\right] = \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} q^k(1-q)^{N-k} \mathbb{E}\left(\mu_i(\omega = A) | k, \omega = A\right).$$

Following a similar proof method, suppose $d_A + d_B$ is odd and rewrite one of the conditional expected posteriors above as follows:

$$\mathbb{E}\left[\mu_i(\omega = A) | a_N, \omega = L\right] = \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!} \Big(\hat{f}(k, q, a_N) + \hat{f}(d_A+d_B-k, q, a_N)\Big),$$

where

$$\hat{f}(k, q, a_N) = q^k(1-q)^{d_A+d_B-k} \frac{\pi}{\pi + (1-\pi)Q^{k-d_B+2a_N-n_i}\Gamma^{k-d_A}}$$

Consider the derivative of $\hat{f}(k, q, a_N)$ w.r.t. q at $q = \frac{1}{2}$:

$$\frac{\partial}{\partial q}\hat{f}(k, q, a_N) = \Big(-(d_A+d_B-k)q^k(1-q)^{d_A+d_B-k-1} + kq^{k-1}(1-q)^{d_A+d_B-k}\Big) \cdot \frac{\pi}{\pi+(1-\pi)Q^{k-d_B+2a_N-N+(k-d_A)z(q,\gamma)}}$$

$$+q^k(1-q)^{d_A+d_B-k} \cdot \pi(1-\pi) \cdot \left[\frac{(k-d_B+2a_N-N)Q^{k-d_B+2a_N-N-1+(k-d_A)z(q,\gamma)}\frac{1}{q^2}}{\left(\pi+(1-\pi)Q^{k-d_B+2a_N-N}\Gamma^{k-d_A}\right)^2}\right.$$

$$\left. + \frac{(k-d_A)Q^{k-d_B+2a_N-N+(k-d_A-1)z(q,\gamma)}\frac{(2-\gamma)\gamma}{(\gamma q+(1-\gamma))^2}}{\left(\pi+(1-\pi)Q^{k-d_B+2a_N-N}\Gamma^{k-d_A}\right)^2}\right]$$

$$\overset{(q=\frac{1}{2})}{=} \left(\frac{1}{2}\right)^{d_A+d_B-1}(2k-d_A-d_B)\frac{\pi}{\pi+(1-\pi)} + \left(\frac{1}{2}\right)^{d_A+d_B} 4\pi(1-\pi) \cdot \frac{(k-d_B+2a_N-N)+(k-d_A)\frac{\gamma}{2-\gamma}}{(\pi+(1-\pi))^2}$$

$$= \left(\frac{1}{2}\right)^{d_A+d_B-1} \cdot \Big((2k-d_A-d_B)\pi + 2\pi(1-\pi)\Big((k-d_B+2a_N-N) + (k-d_A)\frac{\gamma}{2-\gamma}\Big)\Big).$$

Therefore, at $q = \frac{1}{2}$ we have

$$\begin{aligned}
\frac{\partial}{\partial q}\mathbb{E}\left[\mu_i(\omega = A)|a_N, \omega = A\right] &= \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\left(\frac{\partial}{\partial q}\hat{f}(k,q,a_N) + \frac{\partial}{\partial q}\hat{f}(d_A+d_B-k,q,a_N)\right) \\
&= \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\left(\frac{1}{2}\right)^{d_A+d_B-1} 2\pi(1-\pi)\cdot \\
&\quad \cdot\left(2(2a_N - N) + (d_A - d_B) - (k - d_A)\frac{\gamma}{2-\gamma}\right) \\
&= 4\pi(1-\pi)(2a_N - N) + \\
&\quad + \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\left(\frac{1}{2}\right)^{d_A+d_B-1} 2\pi(1-\pi)\left(d_A - d_B\right)\frac{2-2\gamma}{2-\gamma}.
\end{aligned}$$

Note that the second term (a sum) in the expression above is strictly positive for any $a_N \geq 0$, since $d_A > d_B$ and $\gamma \in (0,1)$ hold. Thus, when we compute the derivative of $\mathbb{E}\left[\mu_i(\omega = A)|\omega = A\right]$, this part of the sum will remain strictly positive. The only term to be concerned about is the one with $(2a_N - N)$ in it.

Let

$$H = \sum_{k=0}^{\frac{d_A+d_B-1}{2}} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\left(\frac{1}{2}\right)^{d_A+d_B-1} 2\pi(1-\pi)\left(d_A - d_B\right)\frac{2-2\gamma}{2-\gamma}.$$

Then the derivative of the conditional expected posterior at $q = \frac{1}{2}$ is equal to

$$\begin{aligned}
\frac{\partial}{\partial q}\mathbb{E}\left[\mu_i(\omega = A)|\omega = A\right] &= \sum_{k=0}^{N} \frac{N!}{k!(N-k)!}\left(\frac{1}{2}\right)^{N-1}\left(k - (N-k)\right)\mathbb{E}\left[\mu_i(\omega = A)|k, \omega = A, q = \tfrac{1}{2}\right] + \\
&\quad + \sum_{k=0}^{N} \frac{N!}{k!(N-k)!}\left(\frac{1}{2}\right)^{N}\left[H + 4\pi(1-\pi)(2k-N)\right] \\
&= H + \sum_{k=0}^{N} \frac{N!}{k!(N-k)!}\left(\frac{1}{2}\right)^{N-1}\left[2\pi + 4\pi(1-\pi)\right](2k-N)
\end{aligned}$$

We know that $H > 0$, whereas the second sum is symmetric around $k = \frac{N}{2}$ and will therefore annihilate to 0. Thus, the derivative of the expected posterior is positive at $q = \frac{1}{2}$, which then implies that the expected posterior is bigger than prior for some range above $q = \frac{1}{2}$. In fact, given that $\omega = A$ is the true state, it is reasonable to expect that it is bigger than prior for all $q > \frac{1}{2}$, as both the received information and the dogmatic-friends bias work in the same direction.

We have proved that both conditional expected posteriors will be larger than the prior π in some range $\left(\frac{1}{2}, \bar{q}_1\right)$. This also implies that the unconditional expected posterior belief of i in state $\omega = A$ must be strictly higher than π for any $q \in \left(\frac{1}{2}, \bar{q}_1\right)$.

The final part of the proof will touch on the case where $d_A + d_B$ is even. We will do it for the case $\omega = B$, with $\omega = A$ being completely analogous. When $d_A + d_B$ is even, we

can rewrite the expression for $\mathbb{E}\left[\mu_i(\omega = A)|\omega = B\right]$ as follows:

$$\begin{aligned}\mathbb{E}\left[\mu_i(\omega = A)|a_N, \omega = B\right] &= \sum_{k=0}^{d_A+d_B} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!} \cdot (1-q)^k q^{d_A+d_B-k} \cdot \\ &\quad \cdot \frac{\pi}{\pi + (1-\pi)Q^{k-d_B+2a_N-N}\Gamma^{k-d_A}} \\ &= \sum_{k=0}^{\frac{d_A+d_B}{2}-1} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\Big(f(k,q,a_N) + f(d_A+d_B-k,q,a_N)\Big) + \\ &\quad + \frac{(d_A+d_B)!}{\left(\frac{d_A+d_B}{2}\right)!\left(\frac{d_A+d_B}{2}\right)!}\left(f\left(\tfrac{d_A+d_B}{2},q,a_N\right) + f\left(\tfrac{d_A+d_B}{2},q,a_N\right)\right)\end{aligned}$$

The derivative of the first sum at $q = \frac{1}{2}$ will be positive, as we showed above. As for the second term, note that at $q = \frac{1}{2}$ we have

$$\begin{aligned}\frac{\partial}{\partial q} f\left(\frac{d_A+d_B}{2}, q, a_N\right)\Big|_{q=\frac{1}{2}} &= \left(\tfrac{1}{2}\right)^{d_A+d_B-1}\left(0 + 2\pi(1-\pi)\left(\left(\tfrac{d_A+d_B}{2} - d_B + 2a_N - N\right) + \left(\tfrac{d_A+d_B}{2} - d_A\right)\tfrac{\gamma}{2-\gamma}\right)\right) \\ &= 2\pi(1-\pi)\left(\tfrac{1}{2}\right)^{d_A+d_B-1} \cdot \left(2a_N - N + \tfrac{d_A-d_B}{2}\left(1 - \tfrac{\gamma}{2-\gamma}\right)\right).\end{aligned}$$

Note that the term $(2a_N - N)$ is symmetric around $a_N = \frac{N}{2}$, and so will annihilate to zero when we compute expectation over a_N. The second term is positive for all $a_N \geq 0$. Hence, $\frac{\partial}{\partial q}\mathbb{E}\left[\mu_i(\omega = A)\right] > 0$ still holds at $q = \frac{1}{2}$, and the argument follows exactly as in the case of $d_A + d_B$ being odd. We will omit it here for brevity.

This completes the proof. ■

B Proof of Proposition 2

Proof of Proposition 2. Suppose that the true state is $\omega = B$. Recall that i has d_A A-dogmatic friends, d_B B-dogmatic friends, and n "normal" friends. Suppose that T periods have passed; this means that S has sent T time-independent and individual-independent signals to all agents in the network. Denote the number of signals $s = a$ received by:

- agent i as a_i
- A-dogmatic friends of agent j as a_j^A, $j \in \{1, 2, \ldots, d_A\}$
- B-dogmatic friends of agent j as a_j^B, $j \in \{1, 2, \ldots, d_B\}$
- normal friends of agent i as a_j^n, $j \in \{1, 2, \ldots, n\}$

Then the number of signals $s = b$ received by the same agents will be given by:

- agent i – it is $(T - a_i)$
- dogmatic-left friends of agent i – it is $(T - a_j^A)$, $j \in \{1, 2, \ldots, d_A\}$
- dogmatic-right friends of agent i – it is $(T - a_j^B)$, $j \in \{1, 2, \ldots, d_B\}$
- normal friends of agent i – it is $(T - a_j^n)$, $j \in \{1, 2, \ldots, n\}$

Additionally, an A-dogmatic friend j has stayed silent exactly $(T - a_j^A)$ times, and a B-dogmatic friend k has stayed silent exactly a_k^B times. Thus, agent i's posterior beliefs at the end of period T is given by the following:

$$\mu_i(\omega = A|\mathbf{s}_i^T) = \pi \cdot \left[\pi + (1-\pi) \cdot Q^{(2a_i - T) + \sum_{j=1}^{n}(2a_j^n - T) + \sum_{j=1}^{d_A} a_j^A - \sum_{j=1}^{d_B}(T - a_j^B)} \cdot \Gamma^{\sum_{j=1}^{d_B} a_j^B - \sum_{j=1}^{d_A}(T - a_j^A)}\right]^{-1},$$

where $Q = \frac{1-q}{q}$ and $\Gamma = \frac{\gamma(1-q)+(1-\gamma)}{\gamma q + (1-\gamma)}$.

This posterior belief converges to 1 with probability 1 as $T \to \infty$ if and only if

$$\left(\frac{1-q}{q}\right)^{(2a_i - T) + \sum_{j=1}^{n}(2a_j^n - T) + \sum_{j=1}^{d_A} a_j^A - \sum_{j=1}^{d_B}(T - a_j^B)} \cdot \left(\frac{\gamma(1-q) + (1-\gamma)}{\gamma q + (1-\gamma)}\right)^{\sum_{j=1}^{d_B} a_j^B - \sum_{j=1}^{d_B}(T - a_j^A)}$$

converges to zero with probability 1 as $T \to \infty$. This is equivalent to requiring that the natural log of this expression converges to $-\infty$ with probability 1 as $T \to \infty$, where the natural log equals $\ln\left(\frac{1-q}{q}\right) K(\mathbf{x}, T; q, \gamma)$, where

$$\begin{aligned} K(\mathbf{x}, T; q, \gamma) &= (2a_i - T) + \sum_{j=1}^{n}(2a_j^n - T) + \sum_{i=1}^{d_A} a_j^A - \sum_{j=1}^{d_B}(T - a_j^B) \\ &\quad + \left(\sum_{j=1}^{d_B} a_j^B - \sum_{j=1}^{d_A}(T - a_j^A)\right) z(q, \gamma), \end{aligned}$$

after we define

$$\mathbf{x} = (a_i, (a_j^{n_i})_{j=1}^{n}, (a_j^A)_{j=1}^{d_A}, (a_j^B)_{j=1}^{d_B})$$

and

$$z(q, \gamma) = \ln\left(\frac{\gamma(1-q) + (1-\gamma)}{\gamma q + (1-\gamma)}\right)\left[\ln\left(\frac{1-q}{q}\right)\right]^{-1}.$$

Given $\ln\left(\frac{1-q}{q}\right) < 0$, we require that $K(\mathbf{x}, T; q, \gamma)$ converges to $+\infty$ with probability 1 as $T \to \infty$. Now note that

$$\lim_{T\to\infty} K(\mathbf{x}, T; q, \gamma) = \lim_{T\to\infty} T\left(\frac{K(\mathbf{x}, T; q, \gamma)}{T}\right).$$

By the Law of Large Numbers, we have that

$$\begin{aligned}\operatorname*{plim}_{T\to\infty}\frac{K(\mathbf{x},T;q,\gamma)}{T} &= (2(1-q)-1)+\sum_{j=1}^{n}(2(1-q)-1)\\ &\quad+\sum_{j=1}^{d_A}(1-q)-\sum_{j=1}^{d_B}(1-(1-q))+\\ &\quad+\left(\sum_{j=1}^{d_B}(1-q)-\sum_{j=1}^{d_A}(1-(1-q))\right)z(q,\gamma)\\ &= (d_A+2n+2)(1-q)-(n+1)-d_Bq\\ &\quad+(d_B(1-q)-d_Aq)\,z(q,\gamma).\end{aligned}$$

Given this, $\operatorname{plim}_{T\to\infty}K(\mathbf{x},T;q,\gamma)=+\infty$ if and only if

$$(d_A+2n+2)(1-q)-(n+1)-d_Bq+(d_B(1-q)-d_Aq)\,z(q,\gamma)>0,$$

which is equivalent to

$$q<\bar{q}_2(q)\equiv\frac{1+d_A+n+d_Bz(q,\gamma)}{2+2n+(d_A+d_B)(1+z(q,\gamma))}.$$

To check whether there are values of $q\in(\frac{1}{2};1)$ for which the last inequality holds, let us check whether it holds strictly for $q=\frac{1}{2}$. If it does, then due to continuity of $\bar{q}_2(q)$, there will be a range of values of q for which the inequality must hold.

Before we proceed, note the following:

$$\begin{aligned}\lim_{q\to\frac{1}{2}^+}z(q,\gamma) &= \lim_{q\to\frac{1}{2}^+}\frac{\ln(\gamma(1-q)+(1-\gamma))-\ln(\gamma q+(1-\gamma))}{\ln(1-q)-\ln(q)}\\ &= \lim_{q\to\frac{1}{2}^+}\frac{-\frac{\gamma}{\gamma(1-q)+(1-\gamma)}-\frac{\gamma}{\gamma q+(1-\gamma)}}{-\frac{1}{1-q}-\frac{1}{q}}\\ &= \frac{-\frac{\gamma}{\frac{1}{2}\gamma+(1-\gamma)}-\frac{\gamma}{\frac{1}{2}\gamma+(1-\gamma)}}{-\frac{1}{1-\frac{1}{2}}-\frac{1}{\frac{1}{2}}}\\ &= \frac{2\gamma}{4\cdot\left(\frac{1}{2}\gamma+(1-\gamma)\right)}=\frac{\gamma}{2-\gamma}\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{q\to\frac{1}{2}} \bar{q}_2(q) &= \frac{1+d_A+n+d_B\cdot\frac{\gamma}{2-\gamma}}{2+2n+(d_A+d_B)(1+\frac{\gamma}{2-\gamma})} \\ &= \frac{(1+d_A+n)(2-\gamma)+d_B\gamma}{2(2-\gamma)(1+n)+2(d_A+d_B)} \\ &= \frac{(2-\gamma)(1+n)+(2-\gamma)d_A+d_B\gamma}{2(2-\gamma)(1+n)+2(d_A+d_B)} \\ &= \frac{(2-\gamma)(1+n)+(d_A+d_B)+(d_A-d_B)(1-\gamma)}{2(2-\gamma)(1+n)+2(d_A+d_B)} \\ &= \frac{1}{2}+\frac{(d_A-d_B)(1-\gamma)}{2(2-\gamma)(1+n)+2(d_A+d_B)}\end{aligned}$$

This is strictly bigger than $\frac{1}{2}$ in any situation where $d_A > d_B$ and $\gamma < 1$. Thus, $q < \bar{q}_2(q)$ holds at $q = \frac{1}{2}$, and, due to continuity of $\bar{q}_2(q)$, it holds for an interval to the right of $q = \frac{1}{2}$. Defining $\bar{q}_2$ through the implicit equation $q = \bar{q}_2(q)$, we can conclude that the posterior belief of agent i converges to 1 with probability 1 as $T \to \infty$ for any q in the range $\left(\frac{1}{2}, \bar{q}_2\right)$.

Note that

$$\begin{aligned}\lim_{q\to 1^-} z(q,\gamma) &= \lim_{q\to 1^-} \frac{\ln(\gamma(1-q)+(1-\gamma)) - \ln(\gamma q+(1-\gamma))}{\ln(1-q)-\ln(q)} \\ &= \lim_{q\to 1^-} \frac{-\frac{\gamma}{\gamma(1-q)+(1-\gamma)} - \frac{\gamma}{\gamma q+(1-\gamma)}}{-\frac{1}{1-q}-\frac{1}{q}} = 0\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{q\to 1^-} \bar{q}_2(q) &= \frac{1+d_A+n}{2+2n+d_A+d_B} \\ &= \frac{1+d_A+n}{1+d_A+n+1+d_B+n} \in \left(\frac{1}{2}, 1\right),\end{aligned}$$

because $d_A > d_B$ by assumption. By continuity of $\bar{q}_2(q)$, this implies that there is an interval to the left of $q = 1$ such that $q > \bar{q}_2(q)$. Note that if $q > \bar{q}_2$, we have $\operatorname{plim}_{T\to\infty} K(\mathbf{x}, T; q, \gamma) = \infty$, which means that $\operatorname{plim}_{T\to\infty} \mu_i(\omega = A|\mathbf{s}_i^T) = 0$ and therefore agent i learns the state $\omega = B$ correctly.

When $q = \bar{q}_2$, the limit $\operatorname{plim}_{T\to\infty} K(\mathbf{x}, T; q, \gamma)$ is indeterminate, but we can ignore this knife-edge case.

In the last part of the proof, we will touch on the case $\omega = A$. This only affects distribution of a_A, a_B and a_N in their probability parameter, which changes from $1-q$ to q. Therefore, the proof above still applies up to the application of the Law of Large numbers.

The posterior belief of the agent will converge to 1 with probability 1 as $T \to \infty$ if and

only if

$$\begin{aligned} K(\mathbf{x},T;q,\gamma) &= (2a_i - T) + \sum_{j=1}^{n}(2a_j^n - T) + \sum_{j=1}^{d_A} a_j^A - \sum_{j=1}^{d_B}(T - a_j^B) \\ &\quad + \left(\sum_{j=1}^{d_B} a_j^B - \sum_{j=1}^{d_A}(T - a_j^A)\right) z(q,\gamma) \end{aligned}$$

converges to $+\infty$ with probability 1 as $T \to \infty$. Similarly to before, note

$$K(\mathbf{x},T;q,\gamma) = T \cdot \frac{K(\mathbf{x},T;q,\gamma)}{T}$$

and apply the Law of Large Numbers to the fraction:

$$\begin{aligned} \operatorname*{plim}_{T\to\infty} \frac{K(\mathbf{x},T;q,\gamma)}{T} &= (2q-1) + \sum_{j=1}^{n}(2q-1) + \sum_{j=1}^{d_A} q - \sum_{j=1}^{d_B}(1-q) + \left(\sum_{j=1}^{d_B} q - \sum_{j=1}^{d_A}(1-q)\right) z(q,\gamma) \\ &= (d_A + 2n + 2)q - (n+1) - d_B(1-q) + (d_B q - d_A(1-q))\, z(q,\gamma). \end{aligned}$$

Given this, $\operatorname{plim}_{T\to\infty} K(\mathbf{x},T;q,\gamma) = +\infty$ if and only if

$$(d_A + 2n + 2)q - (n+1) - d_B(1-q) + (d_B q - d_A(1-q))\, z(q,\gamma) > 0,$$

which is equivalent to

$$\frac{1 + n + d_B + d_A z(q,\gamma)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))} < q.$$

Note

$$\frac{1 + n + d_B + d_A z(q,\gamma)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))} = \frac{1}{2} - \frac{\left(1 - \frac{1+z(q,\gamma)}{2}\right)(d_A - d_B)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))} < \frac{1}{2},$$

which implies that the desired inequality on q holds for any $q > \frac{1}{2}$. Thus, given $\omega = A$, $d_A > d_B$ and $\gamma \in (0,1)$, the agent's posterior belief converges to probability 1 on state $\omega = L$ as $T \to \infty$ with probability 1 for any $q \in \left(\frac{1}{2}, 1\right)$.

Therefore, provided that $q \in \left(\frac{1}{2}, \bar{q}_2\right)$, the posterior belief will converge to $\mu_i^\infty(\omega = A) = 1$ with probability 1 as $T \to \infty$ irrespective of what the true state is.

■

Moreover, as Lemmas 2 and 3 show below, the function $\bar{q}_2(q)$ is increasing in q, which implies that we can put a lower bound on the value of $\bar{q}_2$. As it is defined by the equation $q = \bar{q}_2(q)$, we can conclude that $\bar{q}_2 \geq \bar{q}_2\left(\frac{1}{2}\right)$, i.e.

$$\bar{q}_2 \geq \frac{1}{2} + \frac{(d_A - d_B)(1-\gamma)}{2(2-\gamma)(1+n) + 2(d_A + d_B)}.$$

Lemma 2. *$\bar{q}_2(q)$ is increasing in q.*

Proof. To prove Lemma 2 we first establish the following intermediate result.

Lemma 3. *$z(q,\gamma)$ is weakly decreasing for any $q \in \left(\frac{1}{2};1\right)$.*

Proof. Note:

$$\begin{aligned}
\frac{\partial z}{\partial q} &= \frac{\partial}{\partial q}\frac{\ln\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)}{\ln\left(\frac{1-q}{q}\right)} \\
&= \frac{\frac{\gamma q+(1-\gamma)}{\gamma(1-q)+(1-\gamma)}\cdot\frac{-\gamma(\gamma q+(1-\gamma))-\gamma(\gamma(1-q)+(1-\gamma))}{(\gamma q+(1-\gamma))^2}\cdot\ln\left(\frac{1-q}{q}\right)-\ln\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)\cdot\frac{q}{1-q}\cdot\left(-\frac{1}{q^2}\right)}{\ln^2\left(\frac{1-q}{q}\right)} \\
&= \frac{-\frac{\gamma^2 q+\gamma(1-\gamma)+\gamma^2(1-q)+\gamma(1-\gamma)}{(\gamma(1-q)+(1-\gamma))(\gamma q+(1-\gamma))}\cdot\ln\left(\frac{1-q}{q}\right)+\ln\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)\cdot\frac{1}{q(1-q)}}{\ln^2\left(\frac{1-q}{q}\right)} \\
&= \frac{-\frac{\gamma(2-\gamma)}{(\gamma(1-q)+(1-\gamma))(\gamma q+(1-\gamma))}\cdot\ln\left(\frac{1-q}{q}\right)+\ln\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)\cdot\frac{1}{q(1-q)}}{\ln^2\left(\frac{1-q}{q}\right)} \\
&= \frac{-\frac{\gamma(2-\gamma)}{\gamma^2 q(1-q)+(1-\gamma)}\cdot\ln\left(\frac{1-q}{q}\right)+\ln\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)\cdot\frac{1}{q(1-q)}}{\ln^2\left(\frac{1-q}{q}\right)} \\
&= \frac{-\frac{\gamma(2-\gamma)}{\gamma^2 q(1-q)+(1-\gamma)}\cdot\ln\left(\frac{1-q}{q}\right)+\ln\left(\frac{1-q}{q}\right)\cdot\frac{z(q,\gamma)}{q(1-q)}}{\ln^2\left(\frac{1-q}{q}\right)} \\
&= \frac{-\gamma(2-\gamma)q(1-q)+\gamma^2 q(1-q)z(q,\gamma)+(1-\gamma)z(q,\gamma)}{\ln\left(\frac{1-q}{q}\right)\cdot\left(\gamma^2 q(1-q)+(1-\gamma)\right)q(1-q)} \\
&= \frac{(1-\gamma+\gamma^2 q(1-q))z(q,\gamma)-(2-\gamma)\gamma q(1-q)}{\ln\left(\frac{1-q}{q}\right)\cdot\left(\gamma^2 q(1-q)+(1-\gamma)\right)q(1-q)} \\
&= \frac{\left(\frac{1-\gamma}{q(1-q)}+\gamma^2\right)z(q,\gamma)-(2-\gamma)\gamma}{\ln\left(\frac{1-q}{q}\right)\cdot\left(\gamma^2 q(1-q)+(1-\gamma)\right)}
\end{aligned}$$

Note that $z\left(\frac{1}{2},\gamma\right) = \frac{\gamma}{2-\gamma} > 0 = z(1,\gamma)$. As $z(q,\gamma)$ is continuously differentiable for $q \in \left(\frac{1}{2},1\right)$, it is enough to prove that there are no local maxima on that interval in order to show that $\frac{\partial z}{\partial q} \leq 0$ holds on that interval. At an intermediate local maximum, $\frac{\partial z}{\partial q} = 0$ must hold. Consider:

$$\frac{\partial z}{\partial q} = 0 \;\Rightarrow\; \left(\frac{1-\gamma}{q(1-q)} + \gamma^2\right) z(q,\gamma) - (2-\gamma)\gamma = 0$$

$$\begin{aligned} z(q,\gamma) &= \frac{\gamma(2-\gamma)}{\gamma^2 + \frac{1-\gamma}{q(1-q)}} \\ &\leq \frac{\gamma(2-\gamma)}{\gamma^2 + \frac{1-\gamma}{\frac{1}{4}}} \\ &= \frac{\gamma(2-\gamma)}{(2-\gamma)^2} \\ &= \frac{\gamma}{2-\gamma} \end{aligned}$$

Hence, $z(q,\gamma) \leq \frac{\gamma}{2-\gamma}$ must hold at any intermediate local maximum in $\left(\frac{1}{2},1\right)$. This immediately rules our the possibility that $z(q,\gamma)$ is increasing at $q = \frac{1}{2}$, since otherwise it would need to achieve a local maximum with value above $\frac{\gamma}{2-\gamma}$.

Consider again the implication of $\frac{\partial z}{\partial q} = 0$:

$$z(q,\gamma) = \frac{\gamma(2-\gamma)}{\gamma^2 + \frac{1-\gamma}{q(1-q)}}$$

Note that the right-hand side is strictly decreasing in q for $q \in \left(\frac{1}{2};1\right)$. If $z(q,\gamma)$ was to decrease at first (as q rises from $\frac{1}{2}$) and then increase before going down to 0, the value of $z(q,\gamma)$ at the corresponding local maximum would be necessarily above the right-hand side. This is a contradiction, and thus $z(q,\gamma)$ cannot be strictly increasing for any $q \in \left(\frac{1}{2};1\right)$.

One final case to rule out is that $z(q,\gamma)$ is decreasing at first, passing through a local minimum, and then is increasing all the way until $q = 1$. That, however, would mean that the value at the local minimum is less than $z(1,\gamma)$, which is equal to 0. Since $z(q,\gamma) > 0$ for any $q \in (\frac{1}{2};1)$ and $\gamma \in (0;1)$, this case is also impossible.

These considerations prove that $z(q,\gamma)$ is weakly decreasing for any $q \in \left(\frac{1}{2};1\right)$. ■

To complete the proof of Lemma 2, note that

$$\begin{aligned}
\frac{\partial \bar{q}_2(q)}{\partial q} &= \frac{d_B \frac{\partial z}{\partial q}(2+2n+(d_A+d_B)(1+z(q,\gamma))) - (1+d_A+n+d_B z(q,\gamma))(d_A+d_B)\frac{\partial z}{\partial q}}{(2+2n+(d_A+d_B)(1+z(q,\gamma)))^2} \\
&= \frac{\frac{\partial z}{\partial q}\left(2d_B+2d_Bk+(d_Ad_B+d_B^2)(1+z(q,\gamma))-((1+d_A+n)(d_A+d_B)+(d_A+d_B)d_Bz(q,\gamma))\right)}{(2+2n+(d_A+d_B)(1+z(q,\gamma)))^2} \\
&= \frac{\frac{\partial z}{\partial q}\left(2d_B(1+n)+d_B(d_A+d_B)-(1+d_A+n)(d_A+d_B)\right)}{(2+2n+(d_A+d_B)(1+z(q,\gamma)))^2} \\
&= \frac{\frac{\partial z}{\partial q}(d_B-d_A)(1+d_B+n_i)}{(2+2n+(d_A+d_B)(1+z(q,\gamma)))^2} \geq 0,
\end{aligned}$$

since $(d_B - d_A) < 0$ and, $\frac{\partial z}{\partial q} \leq 0$ by Lemma 3 we have the result.

■

Proof that $\bar{q}_2$ is unique.

We would like to prove that $\bar{q}_2(q)$ is convex in q. This, along with $\bar{q}_2(\frac{1}{2}) > \frac{1}{2}$ and $\bar{q}_2(1) < 1$, will imply that $\bar{q}_2(q) = q$ has a unique solution. For the purposes of this argument, we will still assume $d_A > d_B$ and a corresponding functional form of $\bar{q}_2(q)$.

Note that

$$\bar{q}_2(q) = \frac{1+d_A+n+d_Bz(q,\gamma)}{2+2n+(d_A+d_B)(1+z(q,\gamma))} = \frac{A+B\cdot z(q,\gamma)}{C+D\cdot z(q,\gamma)},$$

where

$$A = 1+d_A+n,\ B = d_B,\ C = 2+2n+d_A+d_B,\ D = d_A+d_B.$$

Therefore, we can represent

$$\bar{q}_2(q) = \frac{B}{D} + \frac{A-\frac{BC}{D}}{C+D\cdot z(q,\gamma)} = \frac{B}{D} + \frac{AD-BC}{D(C+Dz(q,\gamma))}.$$

Note that

$$\begin{aligned}
AD-BC &= (1+d_A+n)(d_A+d_B) - d_B(2+2n+d_A+d_B) \\
&= (d_A-d_B)(d_A+d_B) + (1+n)(d_A+d_B) - 2d_B(1+n) \\
&= (d_A-d_B)(d_A+d_B) + (1+n)(d_A-d_B) \\
&= (d_A-d_B)(1+n+d_A+d_B) > 0.
\end{aligned}$$

$\bar{q}_2(q)$ is convex in q if and only if $\frac{AD-BC}{D(C+Dz(q,\gamma))}$ is convex in q. Since $AD-BC > 0$ and $D > 0$, it follows that $\bar{q}_2(q)$ is convex in q if and only if

$$g(q) = \frac{1}{C+Dz(q,\gamma))} \text{ is convex in } q.$$

Note:

$$g'(q) = -\frac{D}{\left(C + Dz(q,\gamma)\right)^2} \cdot z_q(q,\gamma)$$

$$\begin{aligned} g''(q) &= \frac{2D^2}{\left(C + Dz(q,\gamma)\right)^3} \cdot \left(z_q(q,\gamma)\right)^2 - \frac{D}{\left(C + Dz(q,\gamma)\right)^2} \cdot z_{qq}(q,\gamma) \\ &= \frac{2D^2\left(z_q(q,\gamma)\right)^2 - D(C + Dz(q,\gamma))z_{qq}(q,\gamma)}{\left(C + Dz(q,\gamma)\right)^3} \end{aligned}$$

If we are able to prove that $z_{qq}(q,\gamma) < 0$ for all $q \in \left(\frac{1}{2}, 1\right)$, then we will have proven that $g''(q) > 0$, implying that $\bar{q}_2(q)$ is convex for all $q \in \left(\frac{1}{2}, 1\right)$.

Recall from Lemma 3:

$$z_q(q,\gamma) = \frac{\left(\frac{1-\gamma}{q(1-q)} + \gamma^2\right) z(q,\gamma) - \gamma(2-\gamma)}{\ln\left(\frac{1-q}{q}\right)\left(\gamma^2 q(1-q) + (1-\gamma)\right)}$$

Let $K(q) = \frac{1}{\ln^2\left(\frac{1-q}{q}\right)\left(\gamma^2 q(1-q)+(1-\gamma)\right)^2}$. Then:

$$\begin{aligned} z_{qq}(q,\gamma) &= K(q)\Bigg[\left(-\tfrac{(1-\gamma)(1-2q)}{q^2(1-q)^2} + \left(\tfrac{1-\gamma}{q(1-q)} + \gamma^2\right) z_q(q,\gamma)\right) \ln\left(\tfrac{1-q}{q}\right)\left(\gamma^2 q(1-q) + (1-\gamma)\right) \\ &- \left(\left(\tfrac{1-\gamma}{q(1-q)} + \gamma^2\right) z(q,\gamma) - \gamma(2-\gamma)\right)\left(\tfrac{-1}{q(1-q)}\left(\gamma^2 q(1-q) + (1-\gamma)\right) + \ln\left(\tfrac{1-q}{q}\right)\gamma^2(1-2q)\right)\Bigg] \\ &= K(q)\Bigg[\left(\tfrac{(1-\gamma)(2q-1)}{q^2(1-q)^2} + \left(\tfrac{1-\gamma}{q(1-q)} + \gamma^2\right) z_q(q,\gamma)\right) \ln\left(\tfrac{1-q}{q}\right)\left(\gamma^2 q(1-q) + (1-\gamma)\right) + \\ &+ \left(\left(\tfrac{1-\gamma}{q(1-q)} + \gamma^2\right) z(q,\gamma) - \gamma(2-\gamma)\right)\left(\tfrac{1}{q(1-q)}\left(\gamma^2 q(1-q) + (1-\gamma)\right) + \ln\left(\tfrac{1-q}{q}\right)\gamma^2(2q-1)\right)\Bigg]. \end{aligned}$$

Let

$$\begin{aligned} C_1(q) &= \frac{(1-\gamma)(2q-1)}{q^2(1-q)^2} + \left(\frac{1-\gamma}{q(1-q)} + \gamma^2\right) z_q(q,\gamma) \\ C_2(q) &= \left(\frac{1-\gamma}{q(1-q)} + \gamma^2\right) z(q,\gamma) - \gamma(2-\gamma) \\ C_3(q) &= \frac{1}{q(1-q)}\left(\gamma^2 q(1-q) + (1-\gamma)\right) + \ln\left(\frac{1-q}{q}\right)\gamma^2(2q-1) \end{aligned}$$

Then we can write

$$z_{qq}(q,\gamma) = K(q)\left[C_1(q)\ln\left(\frac{1-q}{q}\right)\left(\gamma^2 q(1-q) + (1-\gamma)\right) + C_2(q)C_3(q)\right]$$

Consider $C_1(q)$:

$$\begin{aligned}
C_1(q) &= \frac{(1-\gamma)(2q-1)}{q^2(1-q)^2} + \left(\frac{1-\gamma}{q(1-q)} + \gamma^2\right) z_q(q,\gamma) \\
&= \frac{(1-\gamma)(2q-1)}{q^2(1-q)^2} + \frac{\gamma^2 q(1-q) + (1-\gamma)}{q(1-q)} \cdot \frac{\left(\frac{1-\gamma}{q(1-q)} + \gamma^2\right) z(q,\gamma) - \gamma(2-\gamma)}{\ln\left(\frac{1-q}{q}\right)\left(\gamma^2 q(1-q) + (1-\gamma)\right)} \\
&= \frac{(1-\gamma)(2q-1)\ln\left(\frac{1-q}{q}\right) + \left(\gamma^2 q(1-q) + (1-\gamma)\right) z(q,\gamma) - \gamma(2-\gamma)q(1-q)}{q^2(1-q)^2 \ln\left(\frac{1-q}{q}\right)} \\
&= \frac{(1-\gamma)(2q-1)\ln\left(\frac{q}{1-q}\right) + \gamma(2-\gamma)q(1-q) - \left(\gamma^2 q(1-q) + (1-\gamma)\right) z(q,\gamma)}{q^2(1-q)^2 \ln\left(\frac{q}{1-q}\right)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
C_1(q)\ln\left(\frac{1-q}{q}\right)\left(\gamma^2 q(1-q) + (1-\gamma)\right) &= -\left(\gamma^2 q(1-q) + (1-\gamma)\right) \cdot \\
&\cdot \frac{(1-\gamma)(2q-1)\ln\left(\frac{q}{1-q}\right) + \gamma(2-\gamma)q(1-q) - \left(\gamma^2 q(1-q) + (1-\gamma)\right) z(q,\gamma)}{q^2(1-q)^2}
\end{aligned}$$

Similarly, consider:

$$\begin{aligned}
C_2(q)C_3(q) = &\frac{\left(\gamma^2 q(1-q)+(1-\gamma)\right) z(q,\gamma) \cdot \left[\left(\gamma^2 q(1-q)+(1-\gamma)\right) + \ln\left(\frac{1-q}{q}\right)\gamma^2(2q-1)q(1-q)\right]}{q^2(1-q)^2} \\
&-\gamma(2-\gamma)\frac{\left(\gamma^2 q(1-q)+(1-\gamma)\right) q(1-q) + \ln\left(\frac{1-q}{q}\right)\gamma^2(2q-1)q^2(1-q)^2}{q^2(1-q)^2}
\end{aligned}$$

Therefore, we can write:

$$\begin{aligned}
\frac{z_{qq}(q,\gamma)q^2(1-q)^2}{K(q)} = &\left(2\left(\gamma^2 q(1-q) + (1-\gamma)\right) + \ln\left(\frac{1-q}{q}\right)\gamma^2(2q-1)(1-q)\right) \cdot \\
&\cdot \left(\gamma^2 q(1-q) + (1-\gamma)\right) z(q,\gamma) \\
&+ \left(\gamma^2 q(1-q) + (1-\gamma)\right)\left[(1-\gamma)(2q-1)\ln\left(\frac{1-q}{q}\right) - 2\gamma(2-\gamma)q(1-q)\right] \\
&+ \ln\left(\frac{q}{1-q}\right)\gamma^3(2-\gamma)(2q-1)q^2(1-q)^2 \\
= &\, 2\left(\gamma^2 q(1-q) + (1-\gamma)\right)^2 z(q,\gamma) + \ln\left(\frac{q}{1-q}\right)\gamma^3(2-\gamma)(2q-1)q^2(1-q)^2 \\
&- \left(\gamma^2 q(1-q) + (1-\gamma)\right)\ln\left(\frac{q}{1-q}\right)(2q-1)\left[\gamma^2(1-q)z(q,\gamma) + (1-\gamma)\right] \\
&-2\left(\gamma^2 q(1-q) + (1-\gamma)\right)\gamma(2-\gamma)q(1-q)
\end{aligned}$$

Let

$$D_1(q) = 2\left(\gamma^2 q(1-q) + (1-\gamma)\right) z(q,\gamma) - \ln\left(\frac{q}{1-q}\right)(2q-1)(1-\gamma) - 2\gamma(2-\gamma)q(1-q)$$

and

$$D_2(q) = \gamma^3(2-\gamma)q^2(1-q)^2 - \left(\gamma^2 q(1-q) + (1-\gamma)\right)\gamma^2(1-q)z(q,\gamma)$$

Then we have

$$\frac{z_{qq}(q,\gamma)q^2(1-q)^2}{K(q)} = \left(\gamma^2 q(1-q) + (1-\gamma)\right) D_1(q) + \ln\left(\frac{q}{1-q}\right)(2q-1)D_2(q).$$

Note:

$$\begin{aligned}
D_1(q) &\leq 2\left(\gamma^2 q(1-q) + (1-\gamma)\right)\tfrac{\gamma}{2-\gamma} - \ln\left(\tfrac{q}{1-q}\right)(2q-1)(1-\gamma) - 2\gamma(2-\gamma)q(1-q) \\
&= \tfrac{1}{2-\gamma}\left[2\gamma^3 q(1-q) + 2\gamma(1-\gamma) - \ln\left(\tfrac{q}{1-q}\right)(2q-1)(1-\gamma)(2-\gamma) - 2\gamma(2-\gamma)^2 q(1-q)\right] \\
&= \tfrac{1}{2-\gamma}\left[2\gamma q(1-q)\left(\gamma^2 - (2-\gamma)^2\right) + 2\gamma(1-\gamma) - \ln\left(\tfrac{q}{1-q}\right)(2q-1)(1-\gamma)(2-\gamma)\right] \\
&= \tfrac{1}{2-\gamma}\left[-8q(1-q)\gamma(1-\gamma) + 2\gamma(1-\gamma) - \ln\left(\tfrac{q}{1-q}\right)(2q-1)(1-\gamma)(2-\gamma)\right] \\
&= \tfrac{1-\gamma}{2-\gamma}\left[2\gamma(1-4q(1-q)) - \ln\left(\tfrac{q}{1-q}\right)(2q-1)(2-\gamma)\right] \\
&= \frac{1-\gamma}{2-\gamma}E(q)
\end{aligned}$$

for $E(q) = 2\gamma(1-4q(1-q)) - \ln\left(\frac{q}{1-q}\right)(2q-1)(2-\gamma)$. Differentiating this expression with respect to q, observe:

$$\begin{aligned}
E'(q) &= 2\gamma\cdot 4(2q-1) - \frac{1}{q(1-q)}(2q-1)(2-\gamma) - 2\ln\left(\frac{q}{1-q}\right)(2-\gamma) \\
&= (2q-1)\left(4\gamma - \frac{2-\gamma}{q(1-q)}\right) - 2\ln\left(\frac{q}{1-q}\right)(2-\gamma) \\
&< (2q-1)\left(4\gamma - 4(2-\gamma)\right) - 2\ln\left(\frac{q}{1-q}\right)(2-\gamma) \\
&< 0 \text{ for any } q \in \left(\frac{1}{2},1\right)
\end{aligned}$$

Therefore, $E(q) < E\left(\frac{1}{2}\right)$ for any $q \in \left(\frac{1}{2},1\right)$. Note that

$$E\left(\frac{1}{2}\right) = 2\gamma\left(1 - 4\cdot\frac{1}{4}\right) - \ln(1)\left(2\cdot\frac{1}{2} - 1\right)(2-\gamma) = 0.$$

Therefore, we can conclude

$$D_1(q) \leq \frac{1-\gamma}{2-\gamma}E(q) < 0 \text{ for any } q \in \left(\frac{1}{2},1\right).$$

Returning to $D_2(q)$, note:

$$\begin{aligned} D_2(q) &= \gamma^2(1-q)\left[\gamma(2-\gamma)q^2(1-q) - \left(\gamma^2 q(1-q) + (1-\gamma)\right) z(q,\gamma)\right] \\ &< \gamma^2(1-q)\left[\gamma(2-\gamma)q(1-q) - \left(\gamma^2 q(1-q) + (1-\gamma)\right) z(q,\gamma)\right] \end{aligned}$$

The expression in the brackets is the negative of the numerator in $z_q(q,\gamma)$. Given that $z_q(q,\gamma)$ is negative and that it includes $\ln\left(\frac{1-q}{q}\right)$, it follows that the numerator has to be positive. This implies that the expression above is negative, and therefore, $D_2(q)$ must be negative as well.

Combining $D_1(q) < 0$ and $D_2(q) < 0$ (for any $q \in \left(\frac{1}{2}, 1\right)$ and recalling

$$\frac{z_{qq}(q,\gamma)q^2(1-q)^2}{K(q)} = \left(\gamma^2 q(1-q) + (1-\gamma)\right) D_1(q) + \ln\left(\frac{q}{1-q}\right)(2q-1)D_2(q),$$

we can conclude that $z_{qq}(q,\gamma) < 0$ for any $q \in \left(\frac{1}{2}, 1\right)$.[10]

Returning to the function $\bar{q}_2(q)$, recall that we showed it is convex in q if $z(q,\gamma)$ is concave. Given that $z_{qq}(q,\gamma) < 0$, it follows that $\bar{q}_2(q)$ is convex. Combining this result with the inequalities

$$\bar{q}_2\left(\frac{1}{2}\right) > \frac{1}{2} \quad \text{and} \quad \bar{q}_2(1) < 1,$$

we can finally conclude that the equality $\bar{q}_2(q) = q$ is satisfied only once, which implies that $\bar{q}_2$ is unique.

Comparative statics of $\bar{q}_2$.

Here we will prove that $\bar{q}_2$ is increasing in d_A and decreasing in d_B, n and γ. The value of $\bar{q}_2$ is determined by the equation

$$q = \frac{1 + d_A + n + d_B z(q,\gamma)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))}.$$

We can rewrite it as

$$q - \frac{1}{2} = \frac{(d_A - d_B)\frac{1-z(q,\gamma)}{2}}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))}$$

The right-hand side is strictly decreasing in d_B and n (while the left-hand side is unaffected), which implies that the fixed point of this equation, $\bar{q}_2$, is decreasing in these two variables.

For γ, recall that we can rewrite the equation as

$$q - \frac{B}{D} = \frac{AD - BC}{D(C + Dz(q,\gamma))},$$

[10] This is due to the fact that $K(q) > 0$.

where

$$A = 1 + d_A + n,\ B = d_B,\ C = 2 + 2n + d_A + d_B,\ D = d_A + d_B.$$

We will show that $z(q,\gamma)$ is increasing in γ, which will imply[11] that the right-hand side is decreasing in γ. This, in turn, will imply that $\bar{q}_2$ is decreasing in γ. Consider:

$$\begin{aligned}
\frac{\partial}{\partial\gamma} z(q,\gamma) &= \left[\ln\left(\frac{1-q}{q}\right)\right]^{-1} \cdot \frac{-(1+q)(\gamma q + (1-\gamma)) + (1-q)(\gamma(1-q) + (1-\gamma))}{(\gamma(1-q) + (1-\gamma))(\gamma q + (1-\gamma))} \\
&= \left[\ln\left(\frac{1-q}{q}\right)\right]^{-1} \cdot \frac{-2q(1-\gamma) - (1+q)\gamma q + \gamma(1-q)^2}{(\gamma(1-q) + (1-\gamma))(\gamma q + (1-\gamma))} \\
&= \left[\ln\left(\frac{1-q}{q}\right)\right]^{-1} \cdot \frac{-2q + \gamma(1-q)}{(\gamma(1-q) + (1-\gamma))(\gamma q + (1-\gamma))} > 0,
\end{aligned}$$

where the last inequality holds because $(1-q) < q$, $\gamma < 2$ and $\ln\left(\frac{1-q}{q}\right) < 0$.

Finally, consider d_A. Return to the equation

$$q = \frac{1 + d_A + n + d_B z(q,\gamma)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))}.$$

Denote the right-hand side by h and take the derivative w.r.t. d_A:

$$\begin{aligned}
\frac{\partial h}{\partial d_A} &= \frac{2 + 2n + (d_A + d_B)(1 + z(q,\gamma)) - (1 + z(q,\gamma))(1 + d_A + n + d_B z(q,\gamma))}{(2 + 2n + (d_A + d_B)(1 + z(q,\gamma)))^2} \\
&= \frac{(1 - z(q,\gamma))\,(1 + n + d_B(1 + z(q,\gamma)))}{(2 + 2n + (d_A + d_B)(1 + z(q,\gamma)))^2} > 0,
\end{aligned}$$

since $z(q,\gamma) \leq \frac{\gamma}{2-\gamma} < 1$ for any $\gamma \in (0,1)$ and $q \in (\frac{1}{2}, 1)$.

C Proof of Proposition 3

Proof of Proposition 3. First, consider Proposition 2, since it is simpler. To prove that there is no polarization, it is sufficient to prove that $\bar{q}_2$ becomes $\frac{1}{2}$, because that will mean that the range $(\frac{1}{2}, \bar{q}_2)$ collapses to nothing. The value of $\bar{q}_2$ is implicitly determined by the equation $q = \bar{q}_2(q)$, where

$$\bar{q}_2(q) = \frac{1 + d_A + n + d_B z(q,\gamma)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))}$$

Note that

$$\bar{q}_2\left(\frac{1}{2}\right) = \frac{1}{2} + \frac{(d_A - d_B)(1-\gamma)}{2(2-\gamma)(1+n) + 2(d_A + d_B)},$$

[11] Recall that $AD - BC$ is positive.

which collapses to $\bar{q}_2\left(\frac{1}{2}\right) = \frac{1}{2}$ iff either $d_A = d_B$ or $\gamma = 1$. Thus, if either of these conditions holds, the fixed point of the equation $q = \bar{q}_2(q)$ is achieved at $q = \frac{1}{2}$, which means that $\bar{q}_2 = \frac{1}{2}$.

Now consider Proposition 1. Assume that i initially holds a correct prior: $\mathbb{P}(\omega = A) = \mu_i^0(\omega = A) = \pi$.

If $\gamma = 1$, then the situation is equivalent to agent i having $n + d_A + d_B$ normal neighbors, since she is able to completely unravel selective sharing. In that case, i's expected posterior belief $\mathbb{E}[\mu_i(\omega = A)]$ must be equal to her prior π due to a standard argument about a fully Bayesian agent who is correctly specified with respect to her information structure.

Now suppose $\gamma < 1$ and $d_A = d_B$. Using the conditional expected posterior beliefs from the proof of Proposition 1, we can show that i's unconditional expected posterior belief from ex-ante perspective (before ω is determined) is given by

$$\begin{aligned}
\mathbb{E}\left[\mu_i(\omega = A)\right] &= (1-\pi)\mathbb{E}\left[\mu_i^1(\omega = A) \mid \omega = B\right] + \pi\mathbb{E}[\mu_i^1(\omega = A) \mid \omega = A] \\
&= (1-\pi)\Bigg[\sum_{k=0}^{\frac{d_A+d_B}{2}-1} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\Big(f(k,q) + f(d_A+d_B-k,q)\Big) + \\
&\qquad + \frac{(d_A+d_B)!}{\left(\frac{d_A+d_B}{2}\right)!\left(\frac{d_A+d_B}{2}\right)!} f\left(\frac{d_A+d_B}{2}, q\right)\Bigg] + \\
&\quad + \pi\Bigg[\sum_{k=0}^{\frac{d_A+d_B}{2}-1} \frac{(d_A+d_B)!}{k!(d_A+d_B-k)!}\Big(\hat{f}(k,q) + \hat{f}(d_A+d_B-k,q)\Big) + \\
&\qquad + \frac{(d_A+d_B)!}{\left(\frac{d_A+d_B}{2}\right)!\left(\frac{d_A+d_B}{2}\right)!} \hat{f}\left(\frac{d_A+d_B}{2}, q\right)\Bigg],
\end{aligned}$$

where

$$f(k,q) = \frac{\pi(1-q)^k q^{d_A+d_B-k}}{\pi + (1-\pi)\left(\frac{1-q}{q}\right)^{k-d_B}\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)^{k-d_A}}$$

and

$$\hat{f}(k,q) = \frac{\pi q^k (1-q)^{d_A+d_B-k}}{\pi + (1-\pi)\left(\frac{1-q}{q}\right)^{k-d_B}\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)^{k-d_A}}.$$

Note that the expected belief in $\omega = A$ is completely symmetric to the expected belief in $\omega = B$ with respect to numbers of friends and priors. That is, if we were to let $\hat{\pi} = 1 - \pi$, $\hat{d}_A = d_B$ and $\hat{d}_B = d_A$, we would get the same expression for $\mathbb{E}[\mu_i^1(\omega = B)]$ as for

$\mathbb{E}[\mu_i^1(\omega = A)]$ above:

$$\begin{aligned}
\mathbb{E}\left[\mu_i(\omega = B)\right] &= (1-\hat{\pi})\mathbb{E}\left[\mu_i(\omega = B) \mid \omega = A\right] + \hat{\pi}\mathbb{E}[\mu_i(\omega = B) \mid \omega = B] \\
&= (1-\hat{\pi})\Bigg[\sum_{k=0}^{\frac{\hat{d}_B+\hat{d}_A}{2}-1} \frac{(\hat{d}_B+\hat{d}_A)!}{k!(\hat{d}_B+\hat{d}_A-k)!}\Big(f(k,q) + f(\hat{d}_B+\hat{d}_A-k,q)\Big) + \\
&\qquad + \frac{(\hat{d}_B+\hat{d}_A)!}{\left(\frac{\hat{d}_B+\hat{d}_A}{2}\right)!\left(\frac{\hat{d}_B+\hat{d}_A}{2}\right)!} f\left(\frac{\hat{d}_B+\hat{d}_A}{2}, q\right)\Bigg] + \\
&\quad + \hat{\pi}\Bigg[\sum_{k=0}^{\frac{\hat{d}_B+\hat{d}_A}{2}-1} \frac{(\hat{d}_B+\hat{d}_A)!}{k!(\hat{d}_B+\hat{d}_A-k)!}\Big(\hat{f}(k,q) + \hat{f}(\hat{d}_B+\hat{d}_A-k,q)\Big) + \\
&\qquad + \frac{(\hat{d}_B+\hat{d}_A)!}{\left(\frac{\hat{d}_B+\hat{d}_A}{2}\right)!\left(\frac{\hat{d}_B+\hat{d}_A}{2}\right)!} \hat{f}\left(\frac{\hat{d}_B+\hat{d}_A}{2}, q\right)\Bigg],
\end{aligned}$$

where

$$f(k,q) = \frac{\hat{\pi}(1-q)^k q^{\hat{d}_B+\hat{d}_A-k}}{\hat{\pi} + (1-\hat{\pi})\left(\frac{1-q}{q}\right)^{k-\hat{d}_A}\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)^{k-\hat{d}_B}}$$

and

$$\hat{f}(k,q) = \frac{\hat{\pi}q^k(1-q)^{\hat{d}_B+\hat{d}_A-k}}{\hat{\pi} + (1-\hat{\pi})\left(\frac{1-q}{q}\right)^{k-\hat{d}_A}\left(\frac{\gamma(1-q)+(1-\gamma)}{\gamma q+(1-\gamma)}\right)^{k-\hat{d}_B}}.$$

This symmetry implies the following conclusion. Suppose that for some d_A and d_B the expected belief $\mathbb{E}[\mu_i(\omega = A)]$ is biased in some direction; for the sake of argument, assume $\mathbb{E}[\mu_i(\omega = A)] > \pi$. Because of the symmetry above, it then follows that for $\hat{d}_A = d_B$ and $\hat{d}_B = d_A$ (swapping the number of friends), we will have $\mathbb{E}[\mu_i(\omega = B)] > \hat{\pi} = 1 - \pi$. In other words, if A-dogmatic friends are able to bias i's expected belief towards state $\omega = A$, B-dogmatic friends would be able to do the same with state $\omega = B$ under symmetric circumstances.

This implies that, when $d_A = d_B$ holds, if $\mathbb{E}[\mu_i(\omega = A)] > \pi$, then $\mathbb{E}[\mu_i(\omega = B)] > 1 - \pi$ should also hold. However, this means that the expected posterior probabilities do not sum up to 1, which is a contradiction. Therefore, when $d_A = d_B$ holds, it must be the case that $\mathbb{E}[\mu_i(\omega = A)] = \pi$ and $\mathbb{E}[\mu_i(\omega = B)] = 1 - \pi$. This finishes the proof of the proposition. ■

D Proof of Proposition 4

Proof of Proposition 4. To prove the result, we need to consider the implicit equation that defines $\bar{q}_2(d_L, d_R, n, \gamma)$:

$$q = \bar{q}_2(q) \;\Leftrightarrow\; q = \frac{1 + d_A + n + d_B z(q,\gamma)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))} = \frac{1}{2} + \frac{d_A\left(1 - \frac{1+z(q,\gamma)}{2}\right) + d_B\left(z(q,\gamma) - \frac{1+z(q,\gamma)}{2}\right)}{2+2n+(d_A+d_B)(1+z(q,\gamma))}$$

Consider the right-hand side of this equation when d_A, d_B and n increase by a factor of $\lambda > 1$:

$$\begin{aligned}
\frac{1}{2} + \frac{\lambda d_A\left(1 - \frac{1+z(q,\gamma)}{2}\right) + \lambda d_B\left(z(q,\gamma) - \frac{1+z(q,\gamma)}{2}\right)}{2+\lambda(2n+(d_A+d_B)(1+z(q,\gamma)))} &> \frac{1}{2} + \frac{\lambda d_A\left(1 - \frac{1+z(q,\gamma)}{2}\right) + \lambda d_B\left(z(q,\gamma) - \frac{1+z(q,\gamma)}{2}\right)}{2\lambda+\lambda(2n+(d_A+d_B)(1+z(q,\gamma)))} \\
&= \frac{1}{2} + \frac{d_A\left(1 - \frac{1+z(q,\gamma)}{2}\right) + d_B\left(z(q,\gamma) - \frac{1+z(q,\gamma)}{2}\right)}{2+2n+(d_A+d_B)(1+z(q,\gamma))} \\
&= \frac{1 + d_A + n + d_B z(q,\gamma)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))} \\
&= \bar{q}_2(q)
\end{aligned}$$

Hence, the right-hand side of the equation $q = \bar{q}_2(q)$ increases when d_A, d_B and n all increase by a factor of $\lambda > 1$, which means that we need a higher q on the left-hand side to satisfy the equation. Therefore, if $\bar{q}_2(d_A, d_B, n, \gamma)$ solves $q = \bar{q}_2(q)$, then

$$\bar{q}_2(\lambda d_A, \lambda d_B, \lambda n, \gamma) > \bar{q}_2(d_A, d_B, n, \gamma).$$

Consider arbitrary $\lambda_A > 1$, $\lambda_B > 1$ and $\lambda_N > 1$ and look at the expression for $\bar{q}_2(\lambda d_A, \lambda d_B, \lambda_N n, \gamma)$:

$$\bar{q}_2(\lambda_A d_A, \lambda_B d_B, \lambda_N n, \gamma) = \frac{1}{2} + \frac{\lambda_A d_A\left(1 - \frac{1+z(q,\gamma)}{2}\right) + \lambda_B d_B\left(z(q,\gamma) - \frac{1+z(q,\gamma)}{2}\right)}{2 + 2\lambda_N n + (\lambda_A d_A + \lambda_B d_B)(1 + z(q,\gamma))}$$

The inequality $\bar{q}_2(d_A, d_B, n, \gamma) \leq \bar{q}_2(\lambda_A d_A, \lambda_B d_B, \lambda_N n, \gamma)$ will hold if and only if the

following holds:

$$\begin{aligned}
\frac{(\lambda_A d_A - \lambda_B d_B)\left(1 - \frac{1+z(q,\gamma)}{2}\right)}{2 + 2\lambda_N n + (\lambda_A d_A + \lambda_B d_B)(1 + z(q,\gamma))} &\geq \frac{(d_A - d_B)\left(1 - \frac{1+z(q,\gamma)}{2}\right)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))} \\
\frac{(\lambda_A d_A - \lambda_B d_B)}{2 + 2\lambda_N n + (\lambda_A d_A + \lambda_B d_B)(1 + z(q,\gamma))} &\geq \frac{(d_A - d_B)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))} \\
(\lambda_A d_A - \lambda_B d_B)\left(2 + 2n + (d_A + d_B)(1 + z(q,\gamma))\right) &\geq (d_A - d_B) \times \\
&\quad (2 + 2\lambda_N n + (\lambda_A d_A + \lambda_B d_B)(1 + z(q,\gamma))) \\
(\lambda_A d_A - \lambda_B d_B)(2 + 2n) &\geq (d_A - d_B)(2 + 2\lambda_N n) \\
(\lambda_A d_A - \lambda_B d_B)(1 + n) &\geq (d_A - d_B)(1 + \lambda_N n) \\
((\lambda_A - 1)d_A - (\lambda_B - 1)d_B) + n(\lambda_A d_A - \lambda_B d_B) &\geq (d_A - d_B)\lambda_N n \\
\frac{((\lambda_A - 1)d_A - (\lambda_B - 1)d_B) + n(\lambda_A d_A - \lambda_B d_B)}{(d_A - d_B)n} &\geq \lambda_N \\
\frac{((\lambda_A - 1)d_A - (\lambda_B - 1)d_B) + n((\lambda_A - 1)d_A - (\lambda_B - 1)d_B)}{(d_A - d_B)n} &\geq \lambda_N - 1 \\
\frac{((\lambda_A - 1)d_A - (\lambda_B - 1)d_B)(1 + n)}{(d_A - d_B)n} &\geq \lambda_N - 1 \\
\frac{(\lambda_A - 1)d_A - (\lambda_B - 1)d_B}{(d_A - d_B)} \cdot \left(1 + \frac{1}{n}\right) &\geq \lambda_N - 1
\end{aligned}$$

■

E Proof of Proposition 5

Proof of Proposition 5. Recall that $q = \bar{q}_2(d_A, d_B, n_i, \gamma)$ is defined by

$$q = \frac{1 + d_A + n + d_B z(q,\gamma)}{2 + 2n + (d_A + d_B)(1 + z(q,\gamma))}$$

Fix d_A, d_B, n_i, λ and $\hat{q}$. We need to find λ_N such that

$$\hat{q} \geq \frac{1 + \lambda d_A + \lambda_N n + \lambda d_B z(q,\gamma)}{2 + 2\lambda_N n + \lambda(d_A + d_B)(1 + z(q,\gamma))} = \frac{1}{2} + \frac{\lambda(d_A - d_B)\frac{1 - z(q,\gamma)}{2}}{2 + 2\lambda_N n + \lambda(d_A + d_B)(1 + z(q,\gamma))}$$

Note that the right-hand side is decreasing in $z(q,\gamma)$. So a sufficient condition would be to impose the inequality for the lowest value of $z(q,\gamma)$, which is 0. This gives us the following

inequality:

$$\begin{aligned}
\hat{q} &\geq \frac{1}{2} + \frac{\frac{1}{2}\lambda(d_A - d_B)}{2 + 2\lambda_N n + \lambda(d_A + d_B)} \\
2\hat{q} - 1 &\geq \frac{\lambda(d_A - d_B)}{2 + 2\lambda_N n + \lambda(d_A + d_B)} \\
\lambda(d_A - d_B) &\leq (2\hat{q} - 1)(2 + 2\lambda_N n + \lambda(d_A + d_B)) \\
\lambda(d_A - d_B) &\leq (2\hat{q} - 1)(2 + \lambda(d_A + d_B)) + 2(2\hat{q} - 1)\lambda_N n \\
\lambda_N &\geq \frac{2(1 - \hat{q})\lambda d_A - 2(2\hat{q} - 1) - 2\hat{q}\lambda d_B}{2(2\hat{q} - 1)n} \\
\lambda_N &\geq \frac{(1 - \hat{q})\lambda d_A - \hat{q}\lambda d_B - (2\hat{q} - 1)}{(2\hat{q} - 1)n} \\
\lambda_N &\geq \frac{(1 - \hat{q})d_A - \hat{q} d_B}{(2\hat{q} - 1)n}\lambda - \frac{1}{n} \\
\lambda_N &\geq \frac{d_A - \hat{q}(d_A + d_B)}{(2\hat{q} - 1)n}\lambda - \frac{1}{n}
\end{aligned}$$

■

F Proof of Proposition 6

Proof. First, we will quickly prove Lemma 1. Consider agent i who is "eventually correct" at a given $q = \hat{q}$. There are two possibilities: either $\bar{q}_{2i} > \hat{q}$ or $\bar{q}_{2i} < \hat{q}$ (we omit the knife-edge case). If $\bar{q}_{2i} > \hat{q}$, then the agent must have dogmatic imbalance towards the correct state. Increasing q beyond $\bar{q}_{2i}$ will lead to the agent learning correctly and having beliefs converge to the same state as before. Hence, the agent will not leave the set of "eventually correct" agents as q increases. If $\bar{q}_{2i} < \hat{q}$, then the agent is already learning correctly, and increasing q further will not alter her asymptotic beliefs. Thus, she will remain "eventually correct" as q goes up. All of this implies that the set of "eventually correct" agents is not contracting as q increases.

Now consider agent j who is "eventually incorrect" at a given $q = \hat{q}$. This can only occur if the agent is learning incorrectly, meaning she has a dogmatic imbalance towards the wrong state that is sufficiently large, i.e. $\bar{q}_{2i} > \hat{q}$. Increasing q beyond $\bar{q}_{2i}$ will make the agent begin to learn correctly, which means she will leave the "eventually incorrect" set of agents and will now be in the "eventually correct" set. This finishes the proof of Lemma 1.

Fix some $q = \hat{q}$ and considers sets $\mathcal{N}_A(q)$ and $\mathcal{N}_B(q)$. For definiteness, let $\omega = A$ be the true state. Let $\bar{q}_{min}(\hat{q}) = \min_{i \in \mathcal{N}}\{\bar{q}_{2i} \mid \bar{q}_{2i} > \hat{q}\}$ be the lowest $\bar{q}_{2i}$ among agents who are "eventually incorrect" at $q = \hat{q}$. As q increases and reaches $\bar{q}_{min}(\hat{q})$, that agent will flip from being "eventually incorrect" to being "eventually correct". Since $\omega = A$ is the true

state, this implies

$$|\mathcal{N}_A(\bar{q}_{min}(\hat{q}))| = |\mathcal{N}_A(\hat{q})| + 1 \text{ and } |\mathcal{N}_B(\bar{q}_{min}(\hat{q}))| = |\mathcal{N}_B(\hat{q})| - 1.$$

Consider the network polarization $\Pi(q)$ at $q = \hat{q}$ and $q = \bar{q}_{min}(\hat{q})$:

$$\Pi(\hat{q}) = \frac{4}{|\mathcal{N}|} \cdot |\mathcal{N}_A(\hat{q})||\mathcal{N}_B(\hat{q})|$$

$$\Pi(\bar{q}_{min}(\hat{q})) = \frac{4}{|\mathcal{N}|} \cdot (|\mathcal{N}_A(\hat{q})| + 1)(\mathcal{N}_B(\hat{q}) - 1)$$

Note that $\Pi(\hat{q}) \geq \Pi(\bar{q}_{min}(\hat{q}))$ if and only if

$$|\mathcal{N}_A(\hat{q})||\mathcal{N}_B(\hat{q})| \geq (|\mathcal{N}_A(\hat{q})| + 1)(\mathcal{N}_B(\hat{q}) - 1),$$

which is equivalent to

$$|\mathcal{N}_B(\hat{q})| \leq |\mathcal{N}_A(\hat{q})| + 1.$$

Hence, the network polarization weakly decreases with q if and only if the set of "eventually incorrect" agents initially (at $q = \hat{q}$) is smaller than the set of "eventually correct" agents plus one. Since $\mathcal{N}_B(q)$ is weakly contracting in q, a necessary and sufficient condition for $\Pi(q)$ to be always weakly decreasing in q is that $|\mathcal{N}_B(\frac{1}{2})| = |\mathcal{D}_R|$ is weakly smaller than $|\mathcal{N}| - |\mathcal{D}_B| + 1$, which is equivalent to $|\mathcal{D}_B| \leq \frac{1}{2}(|\mathcal{N}| + 1)$.

Therefore, the network polarization $\Pi(q)$ is weakly decreasing in q over $q \in \left(\frac{1}{2}, 1\right)$ if and only if $|\mathcal{D}_{-\omega}| \leq \frac{1}{2}(|\mathcal{N}| + 1)$. ■

G Proof of Proposition 7

Proof. The proof of this proposition is short, and requires us to prove that by aggregating enough signals, we can push quality of information above any given threshold. In other words, if $\hat{s}^i_{Mt}$ denotes an M-aggregated signal, we need to prove

$$\lim_{M\to\infty} \mathbb{P}(\hat{s}^i_{Mt} = a|\omega = A) = \lim_{M\to\infty} \mathbb{P}(\hat{s}^i_{Mt} = b|\omega = B) = 1.$$

Suppose that we aggregate M (odd number) signals together, and offer agents the following information structure:

$$\hat{s}^i_{Mt} = \begin{cases} 0, \text{ if } \sum_{t=1}^{M} s_{it} < \frac{M}{2} \\ 1, \text{ if } \sum_{t=1}^{M} s_{it} > \frac{M}{2} \end{cases}$$

For simplicity, assume that the true state is $\omega = A$, and consider the quality of signal $\hat{s}^i_{Mt}$:

$$\begin{aligned}
\mathbb{P}\left(\hat{s}^i_{Mt} = a \middle| \omega = L\right) &= \mathbb{P}\left(\sum_{t=1}^{M} s_{it} < \frac{M}{2} \middle| \omega = L\right) \\
&= \mathbb{P}\left(\frac{1}{M}\sum_{t=1}^{M} s_{it} < \frac{1}{2} \middle| \omega = L\right) \\
&= 1 - \mathbb{P}\left(\frac{1}{M}\sum_{t=1}^{M} s_{it} \geq \frac{1}{2} \middle| \omega = L\right) \\
&= 1 - \mathbb{P}\left(\frac{1}{M}\sum_{t=1}^{M} s_{it} - (1-q) \geq \frac{1}{2} - (1-q) \middle| \omega = L\right) \\
&= 1 - \mathbb{P}\left(a_M - (1-q) \geq \frac{1}{2} - (1-q) \middle| \omega = L\right) \\
&\overset{\text{as } M\to\infty}{\to} 1 - 0 = 1,
\end{aligned}$$

where the last line is due to the weak Law of Large Numbers on a Binomial random variable a_M with parameters M and $(1-q)$. Hence, the limit of signal's quality as $M \to \infty$ is equal to 1, meaning that there must exist a finite number of signals such that the quality surpasses the bound $\bar{q}_1$. For such aggregated signal, information quality is too high for Proposition 2 to apply, which implies that there is no divergence of beliefs in the network. ■