Exchange Design and Efficiency*

Marzena Rostek† and Ji Hee Yoon‡

February, 2020

Abstract

Most assets clear independently rather than jointly. This paper presents a model based on the uniform-price double auction which accommodates arbitrary restrictions on market clearing, including asset by asset market clearing (allowed when demand for each asset is contingent only on the price of that asset) and joint market clearing for all assets (required when demand for each asset is contingent on prices of all assets). Introducing additional trading protocols for traded assets — neutral when the market clears jointly — or linking existent trading protocols are generally not redundant innovations, even if all traders participate in all protocols. Multiple protocols that clear independently can always be designed to be at least as efficient as joint market clearing for all assets. Separation in market clearing can enhance diversification and risk sharing. When traders have price impact, regulation of innovation in trading technology should be guided by market characteristics.

JEL Classification: D47, D53, G11, G12

Keywords: Imperfectly competitive market, Trading technology, Market design, Innovation, Decentralized market, Liquidity, Price impact, Uniform-price auction, Efficiency

* We thank numerous colleagues and participants at the Conference on "What We Know About Information in Financial Markets" (UC Santa Barbara, LEAF), the Cowles Conference on Economic Theory, the Philadelphia Search and Matching Conference (Federal Reserve Bank of Philadelphia), Conference on Finance-Related Theory and Experiments (NYU), Conference on "The Changing Regulatory and Technological Landscape of Market Structure" (Vanderbilt, The Financial Markets Research Center), the Conference on Advances in Mechanism and Market Design (NYU Abu Dhabi), the 5th European Meeting on Networks (UCL), the Spring Workshop in Economic Theory (Queen Mary University of London), and seminar participants at Arizona State University, University of Calgary (Finance), Carnegie Mellon University/Pittsburgh University, University of Chicago, Columbia University, ECARES, The Federal Reserve Board, HEC Paris (Finance), Higher School of Economics & ICEF (Moscow), IAQF &Thalesians (New York), INSEAD (Finance), Ohio State University, New Economic School (Moscow), Paris School of Economics, Princeton University, Stockholm School of Economics (Economics), Stockholm School of Economics (Finance), UCL, UCSD, University of Cambridge, Washington University (St. Louis) for comments and suggestions. Rostek gratefully acknowledges the financial support from the NSF Grant SES-1851857. Yoon acknowledges the financial support of the University of Wisconsin-Madison and UCL.

† University of Wisconsin-Madison, Department of Economics; E-mail: mrostek@ssc.wisc.edu.

‡ University College London, Department of Economics; E-mail: jihee.yoon@ucl.ac.uk.

Electronic copy available at: https://ssrn.com/abstract=3604976

1 Introduction

Today's financial markets are comprised of coexistent trading protocols for the same or distinct assets. Typically, venues for financial securities clear independently: orders submitted to one cannot be made contingent on prices of the assets traded in others. In certain markets, such as those for spectrum or electricity, traders are allowed to express their demands for one asset contingent on prices of other assets. If available, such contingent orders allow cross-asset conditioning among only a limited number of assets.[1] Feasibility might provide one rationale — with contingent schedules, the market-clearing prices must be determined *jointly* for all assets, thus requiring coordination in market clearing among trading venues that are private entities and market makers. Advances in technology have increased interest in cross-asset conditioning.[2]

This paper investigates how the separation in market clearing affects equilibrium, welfare, and design. We show that regulation that promotes joint clearing for some assets, if applied in disregard of market characteristics, can lower welfare in the Pareto sense. Nevertheless, multiple exchanges that clear independently can be designed to be at least as efficient as joint clearing for all assets *irrespective of* the characteristics of assets and traders. Thus, joint market clearing of all assets is either inessential or inefficient.

We study a double auction model for $I < \infty$ strategic traders and $K < \infty$ assets based on the canonical uniform-price mechanism (e.g., Wilson (1979), Klemperer and Meyer (1989), Kyle (1989), Vives (2011)). Our analysis is cast in the quadratic-Gaussian setting. Traders have private information about their endowments, which are independent across assets and possibly correlated across traders. We first examine markets with *uncontingent schedules* $q_k^i(\cdot) : \mathbb{R} \to \mathbb{R}$, each specifying the quantities demanded for any price realizations of a given asset (i.e., asset-by-asset market clearing), which we contrast with the standard in theory but less so in practice *contingent schedules* $q_k^{i,c}(\cdot) : \mathbb{R}^K \to \mathbb{R}$ specifying the quantities demanded of each asset for any realization of the price vector (i.e., a joint market clearing for all assets). We then extend the model in two ways to accommodate more general market structures and innovation in trading technology. More specifically, we permit arbitrary restrictions on cross-asset demand conditioning (i.e., a demand for an asset can condition on any subset of prices) and allow an asset to be traded in multiple venues. Namely, a *market structure* consists of *exchanges*, each defined by the subset of the K assets traded there; all traders participate in all exchanges (see also ft. 11). It is convenient to identify market clearing — i.e., a uniform-price trading protocol — with an exchange. Demand schedules in any exchange condition on the prices of the assets traded in that exchange and not on those in other exchanges; the market clears independently

[1] E.g., RegNMS and UTP in US stock exchanges de facto induce contingent demand schedules. Analogous rules do not apply in markets for other asset classes or stock markets abroad. See Budish, Lee, and Shim (2019). Variants of cross-asset conditioning is available in futures and options markets (e.g., multi-leg orders).

[2] Indeed, modern trading platforms allow and innovate on such orders (e.g., Active Trader Pro, Etrade, Street Smart, Tradehawk).

Electronic copy available at: https://ssrn.com/abstract=3604976

across exchanges.

Dispensing with the assumption that schedules are contingent — on which the standard competitive (e.g., general equilibrium) and imperfectly competitive model of equilibrium and asset pricing is based — requires new techniques to characterize equilibrium. First, in contrast to contingent trading, we cannot rely on the method of characterizing *ex post* optimization. Since a trader's demands no longer condition on the prices of all assets, they depend on the expected (rather than realized) trades of the assets in other exchanges. Due to cross-asset inference, unlike with contingent trading, a trader's own *best-response* demand coefficients must be characterized as a fixed point across assets.[3] Additionally, in the *equilibrium* problem, traders' expectations across trading venues create a complex fixed point with price impacts across traders and assets.

The methodological contribution of the paper characterizes the Bayesian Nash Equilibrium in markets with limited cross-asset conditioning. We show that the equilibrium fixed point in demand schedules is equivalent to a fixed point in price impact matrices, i.e., we endogenize all equilibrium demand coefficients — including expected trades — as functions of price impacts (Theorem 1). We provide the comparative statics of price impacts (Theorem 3, Proposition 7) with respect to the asset covariance and the number of traders. We prove the existence of a symmetric linear Bayesian Nash Equilibrium in the uniform-price double auction for $K \geq 2$ assets (Theorem 2) and equilibrium uniqueness for $K = 2$ assets. Furthermore, we show that comparative analysis of equilibrium, welfare, and innovation in trading technology across market structures with arbitrary exchanges can be reduced to the analysis of the endogenous *per-unit price impact* matrix alone (Theorem 4).

The paper's second contribution is its implications of separation in market clearing for welfare and design. Why might one expect market structures with multiple trading venues that clear independently to *strictly* increase welfare relative to the market structures in which all assets clear jointly? If the market were competitive ($I \to \infty$), joint market clearing would be weakly more efficient than any other market structure, as contingent schedules eliminate the information loss across exchanges. In imperfectly competitive markets ($I < \infty$), independent market clearing across venues can lower the trading costs associated with per-unit price impact for a given asset and/or across assets. Thus, multi-venue trading changes the traders' ability to share risk and diversify risk across assets, respectively, and increase welfare despite the information loss due to limited demand conditioning.

Central to the effects present with multi-venue trading which have no analogues with joint

[3]In both the contingent and uncontingent market, the first-order conditions in the multivariate optimization *best-response* problem define a fixed point for a trader's demand schedules across assets. With demands for all assets contingent on the price vector, the first-order conditions can be written as a single matrix equation and solved for the quantity vector pointwise with respect to the price vector. Thus, there is no need to characterize the demand coefficients as a fixed point.

Electronic copy available at: https://ssrn.com/abstract=3604976

clearing is that it severs the proportionality of the equilibrium price impact in the fundamental asset's payoff covariance that holds with contingent trading: the cross-venue price impact becomes zero[4] and the price impact for the same assets can differ in different venues due to differences in the variables demands can condition on. Consequently, demand substitutability (i.e., demand Jacobian) is endogenous and assets that would be demand substitutes with joint clearing can become demand complements.[5]

We present three main results. First, innovations such as introducing additional exchanges (trading protocols) for traded assets without changing the traders' endowments need not be redundant with exchange by exchange market clearing, even if all traders participate in all exchanges. Such innovation has no counterparts when the market clears jointly. Theorem 4 characterizes the nonneutrality of innovation by a condition on the market structure — equivalently, a condition on traders' per-unit equilibrium price impact matrix or (when $I < \infty$) cross-asset expectations. As a necessary condition for a new exchange to be nonredundant, some assets ought to become relatively more prevalent across venues — replicating all exchanges would be neutral. The key observation is that when equilibrium is not *ex post*, the creation of a new exchange or merging the assets between existent trading protocols that alters traders' cross-exchange inference, affects the price distribution, and hence price impacts.

The fact that traders have price impact motivates various forms of innovation in trading technology: the introduction of new trading protocols for traded assets, linking existing trading protocols (i.e., allowing joint clearing for some assets), and the inclusion of an asset traded in some trading protocol in another. In fact, Theorem 4 shows that any nonredundant innovation can be mapped in one of these three types.

Second, if suitably designed, multiple trading protocols which clear independently can be at least as efficient as the market in which assets clear jointly. We show that, for any primitive characteristics of assets and traders, one can design a market comprised of multiple protocols, none of which clears all assets, which can function like the market in which all assets clear jointly, i.e., equilibrium trades, prices, and traders' payoffs are the same. In designs that lower-bound welfare by its corresponding contingent-trading level, there are sufficiently many trading protocols for different assets so that the conditioning variables eliminate inference errors in traders' expectations. Equilibrium is *ex post* even if demands in no exchange condition on the prices of all assets and so no expectation about trades in other exchanges is perfect.

This equivalence result also characterizes the scope of designing new trading protocols that would not be redundant. For a market structure with demands simpler than the contingent

[4]Nevertheless, equilibrium behavior and outcome (i.e., prices and trades) are not independent across trading venues — unless the asset payoffs are independent. In the uncontingent market, the within-exchange price impact strictly increases except when *all* asset payoffs are independent, for any asset covariance.

[5]We also show that, in contrast to contingent trading, the covariance of equilibrium prices does not correspond to a scaled covariance of asset payoffs. The sign of the cross-asset price elasticity can differ from that of the asset covariance — notably, not a prerequisite for the demand elasticity to change the sign.

Electronic copy available at: https://ssrn.com/abstract=3604976

ones to implement equilibrium with joint clearing, one venue per a pair of assets suffice — the maximal number of nonredundant protocols is $\frac{K(K-1)}{2}$. Notably, not all new trading protocols affect welfare even in market structures that are not payoff-equivalent to joint clearing. For example, the introduction of a new trading protocol whose assets are not traded in an existing venue can be neutral.

Third, we ask which designs can strictly improve welfare relative to the lower bound ensured by designs equivalent to joint clearing. Hinting at the diversity of the trading protocols in practice, the market structure in which all assets clear jointly (i.e., contingent demands or a payoff-equivalent design) is not generally efficient; nor is the market structure in which every asset is traded in a single exchange (i.e., uncontingent demands) efficient irrespective of the market characteristics.

A key result (Proposition 4) shows that in markets with two assets, joint clearing minimizes the price impact per unit for each asset among all market structures — hence, the cost of risk sharing is minimal among all market structures. Consequenty, to increase welfare, multi-venue trading must lower the trading cost of diversification (i.e., cross-asset price impact). More generally, however, with multiple assets, innovation that increases demand conditioning can increase or lower *both* the price impact costs of diversification and risk sharing. Underlying these effects is the endogenous demand substitutability.

Our results recognize that the welfare-enhancing exchange design should respond to the number of traders, and the joint substitutability of the asset payoffs and the trading needs of market participants *across assets*.[6] Even if assets' payoffs are all either substitutes or complements, the efficient design depends on whether the market is "one-sided," i.e., traders want to buy or sell all assets (e.g., the primary market in Treasury auctions) or some assets are demanded while others supplied (e.g., intra-dealer markets). For any number of traders, the efficient market structure is not contingent for some distribution of endowments. This stands in sharp contrast with competitive markets: any changes in market structures that increase demand conditioning would increase welfare *irrespective of* the characteristics of assets and traders.

One might wonder — given that in practice traders can condition their demands in each trading venue on past outcomes from other venues when trading is dynamic — whether the separation in market clearing across trading venues has any effects. Conditioning on past outcomes allows information from past shocks to be (at least partially) incorporated in traders' demands — contingent or not. Our paper investigates how separation in market clearing affects the way current-round shocks impact behavior and outcomes.[7] Our results thus indicate a

[6] All innovations are neutral with multi-venue trading if and only if either (by design) equilibrium is *ex post* or the payoffs of *all* assets are either perfectly correlated (i.e., cross-asset inference is perfect) or independent (i.e., cross-asset inference is absent).

[7] In a dynamic model, traders have price impact in every trading round. In response, they reduce their

Electronic copy available at: https://ssrn.com/abstract=3604976

role for opaqueness in the form of restrictions on cross-asset conditioning, which have distinct implications from transparency requirements (i.e., conditioning on past outcomes).

Other related literature. Our paper contributes to the literature on imperfectly competitive trading (Kyle (1989), Vayanos (1999), Vives (2011), Garleanu and Pedersen (2013), Rostek and Weretka (2015), Bergemann, Heumann, and Morris (2015), Sannikov and Skrzypacz (2016), Du and Zhu (2017a,b), Antill and Duffie (2017), Kyle, Obizhaeva, and Wang (2017), Kyle and Lee (2017), Duffie (2018), Zhu (2018a,b)). To our knowledge, we are the first to examine equilibrium and welfare with arbitrary restrictions on cross-asset conditioning and to characterize the (non)redundant exchange design. In fact, little is known about markets with multiple heterogeneous assets outside of settings with *ex post* equilibria. Contemporaneously, in a model with two assets and random supply, Wittwer (2019) shows that traders trade the same amounts with contingent and uncontingent demands if and only if traders' private signals are perfectly correlated and supplies are either zero or perfectly correlated across assets. See also Chen and Duffie (2020).

Apart from the financial market applications, the techniques we introduce will be useful to researchers studying games in which agents interact through contracts over multiple goods, actions, or characteristics. One application is to package auctions with large traders who have price impact. Our results suggests an implementation of package bids via simpler than contingent schedules. The problem in which players submit uncontingent demand schedules in different trading venues is also related to those studied by the literature on "island" models (in competitive markets) and, more generally, the approach based on Nash-in-Nash.[8] A typical context where Nash-in-Nash has been applied is surplus division in bargaining with externalities — across contracts and agents — when negotiations are simultaneous. Likewise, in this paper, the demands a player submits simultaneously in different trading venues are essentially contracts that are contingent on subsets of prices. Like in this paper's model, the applications of the Nash-in-Nash solution have typically considered negotiated contracts, given the set of agreements. There are two differences. In Nash-in-Nash, a player agrees to the price in one con-

demands relative to their price-taking demands, thus realizing the gains from trade from any endowment shock over multiple trading rounds. With the gains from trade renewed by shocks (to endowment or information), the inefficiency and all of the effects that we identify are present in all rounds. Based on the results in the literature on dynamic trading, which is based on *contingent* demands, if demands are contingent *and* traders were price-takers ($I \to \infty$), the outcome would be efficient in every round. Whether the inefficiency of trade due to *limited* demand conditioning eventually vanishes depends on the relative frequencies of the shocks (which renew the gains from trade), market-clearing, and payoff realization (consumption). Moreover, relative to contingent demands, limited conditioning has new contemporaneous (this paper) and temporal effects on price impact. (See, e.g., Du and Zhu (2017b), Rostek and Yoon (2019).)

[8]This solution concept was introduced by Horn and Wolinsky (1988) and has become popular in the structural literature on decentralized markets. See, e.g., Collard-Wexler, Gowrisankaran, and Lee (2019) and references there. We are grateful to an anonymous referee for suggesting we explore the connection to the literature on "island" models.

Electronic copy available at: https://ssrn.com/abstract=3604976

tract holding fixed (i) the prices in his other contracts[9] and (ii) the prices other players agree to in these contracts. By its virtue of treating prices as contingent variables in traders' demands, the (noncooperative) game in demand functions allows accounting for the cross-contract externalities (in *and* off equilibrium) in a Bayesian Nash Equilibrium — without employing the Nash-in-Nash counterfactual (which holds prices fixed in other contracts) or restricting how beliefs can change off equilibrium (e.g., passive beliefs, weary beliefs.[10] Our model complements the Nash-in-Nash approach in applications where there is private information, inefficiencies in surplus sharing due to limited inference and imperfect competition, and contracts over multiple assets with cross-asset externalities. Accounting for strategic behavior sheds light on how the design of contracts over which agents bargain can enhance the equilibrium surplus when there are cross-contract externalities.

Our paper also contributes to the literature on decentralized trading. Markets with contingent schedules are *centralized* because a single market clearing applies to all assets (as in the standard, e.g., competitive, model). In the *decentralized market*, assets are traded in separate venues that clear independently. The assumption that schedules are contingent is the only assumption of the centralized market model that we relax. In particular, assuming that all traders trade all assets allows us to focus on those effects of a decentralized market that are due to incomplete conditioning as opposed to incomplete participation.[11] There are now several arguments as to why decentralized trading might be more efficient: it may improve traders' learning about the asset value (Babus and Kondor (2017), Glode and Opp (2017)) or asset price (Zhu (2014)); it may redistribute risk towards less risk averse traders (Malamud and Rostek (2017)); and it may be more stable than the centralized market (Peivandi and Vohra (2017)). This paper contributes another argument: even if all traders are equally risk averse, decentralized trading may improve risk sharing and/or diversification by lowering the trading

[9]This is typically justified using the "delegated agent" interpretation: a player involved in multiple bilateral bargains relies on separate agents for each negotiation, and these agents cannot communicate with one another during the course of bargaining.

[10]With price-elastic demands, all price realizations can occur in equilibrium for some realizations of endowments.

[11]In the centralized market assumption, two assumptions are implicit. First, demand conditioning is complete, i.e., (net) demand schedules are contingent on prices of all assets; then, a single aggregation applies to all assets. Second, trader participation in the market is *complete* in the sense that each trader trades all assets with all other traders. A growing literature on decentralized trading has explored the implications of incomplete participation modeled as fixed or random (hyper)graphs (e.g., Gale (1986a, b), Kranton and Minehart (2001), Duffie, Garleanu, and Pedersen (2005), Vayanos and Weill (2008), Afonso and Lagos (2012), Gofman (2014), Atkeson, Eisfeldt, and Weill (2015), Elliott (2015), Choi, Galeotti, and Goyal (2017), Condorelli, Galeotti, and Renou (2017), Hugonnier, Lester, and Weill (2017), Malamud and Rostek (2017), Chang and Zhang (2018)). Babus and Kondor (2017), Babus and Parlatore (2017), and Malamud and Rostek (2017) study markets with limited participation and contingent contracts. It is interesting that endogenous demand substitutability arises with trading decentralized in the sense of limited demand conditioning (this paper) and limited participation (Malamud and Rostek (2017)). The structure of demand substitutability induced by limited conditioning and limited participation and the underlying mechanisms are distinct.

Electronic copy available at: https://ssrn.com/abstract=3604976

costs due to price impact.[12,13]

2 Model

Notation. We use the following notation: $(x_i)_i$ is a vector in which the i^{th} element is x_i, and $(y_{ij})_{i,j}$ is a matrix such that the $(i,j)^{\text{th}}$ element is y_{ij}; sets of the respective elements are denoted by $\{x_i\}_i$ and $\{y_{ij}\}_{i,j}$. In addition, $diag(x_i)_i = diag(x_1, \cdots, x_K)$ is a diagonal matrix in $\mathbb{R}^{K\times K}$ in which the i^{th} diagonal element is x_i. The $(k,\ell)^{\text{th}}$ element of matrix $\mathbf{M}$ is denoted by $m_{k\ell}$ and the k^{th} row is denoted by M_k.

Market: traders, assets, and exchanges. Consider a market with $I \geq 3$ traders who trade K risky assets in N exchanges. An *exchange* is defined by the assets traded there; all traders participate in all exchanges. In Section 3, to ease exposition, we focus on markets with one asset per exchange, i.e., $N = K$; in Sections 4 and 5, we consider exchanges with multiple assets (Definition 4). We index traders by i and assets by k.

The payoffs of the K risky assets are jointly normally distributed $\mathbf{r} = (r_k)_k \sim \mathcal{N}(\boldsymbol{\delta}, \boldsymbol{\Sigma})$ with a vector of expected payoffs $\boldsymbol{\delta} = (\delta_k)_k \in \mathbb{R}^K$ and a positive semi-definite covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{K\times K}$. There is also a riskless asset with a zero interest rate (a numéraire).

Each trader i has a quadratic in the quantity of risky assets (mean-variance) utility:

$$u^i(\mathbf{q}^i) = \boldsymbol{\delta} \cdot (\mathbf{q}^i + \mathbf{q}_0^i) - \frac{\alpha^i}{2}(\mathbf{q}^i + \mathbf{q}_0^i) \cdot \boldsymbol{\Sigma}(\mathbf{q}^i + \mathbf{q}_0^i), \tag{1}$$

where $\mathbf{q}^i = (q_k^i)_k \in \mathbb{R}^K$ is trade, $\mathbf{q}_0^i = (q_{0,k}^i)_k \in \mathbb{R}^K$ represents the units of risky assets trader i is initially endowed with, and $\alpha^i \in \mathbb{R}_+$ is trader i's risk aversion. Endowments $\{\mathbf{q}_0^i\}_i$ are traders' private information and are independent of asset payoffs $\mathbf{r}$. Gains from trade come from risk sharing and diversification: endowments are heterogeneous. All traders are strategic.

In keeping with the literature, to ensure that the per capita aggregate endowment (equivalently, price) is random in the limit large market ($I \to \infty$), we allow for the common value component $\mathbf{q}_0^{cv} = (q_{0,k}^{cv})_k \in \mathbb{R}^K$ in traders' endowments. For each asset k, endowments $\{q_{0,k}^i\}_i$ are correlated among traders through $q_{0,k}^{cv} \sim \mathcal{N}(E[q_{0,k}^{cv}], \sigma_{cv}^2)$: for each i,

$$q_{0,k}^i = q_{0,k}^{cv} + q_{0,k}^{i,pv}; \qquad q_{0,k}^{i,pv} \overset{iid}{\sim} \mathcal{N}(E[q_{0,k}^{i,pv}], \sigma_{pv}^2).$$

[12] Lester, Shourideh, Venkateswaran, and Zetlin-Jones (2018a,b) examine the joint effects of an information friction and market power (induced by a search friction) in over-the-counter markets.

[13] In 2016, the Securities and Exchange Commission granted a request by the alternative trading group IEX Group to launch a new public stock exchange. Prior to 2007, equity markets in Europe were characterized by dominant exchanges in each domestic market. The Markets in Financial Instruments Directive (MiFID) in 2007 created more than 200 new venues for trading equities, bonds, and derivatives. The European Securities and Markets Authority evaluated the impact of these trading venues — the results were mixed.

Electronic copy available at: https://ssrn.com/abstract=3604976

Trader i knows his endowment $\mathbf{q}_0^i$ but not its components $\mathbf{q}_0^{cv}$ or $\mathbf{q}_0^{i,pv} = (q_{0,k}^{i,pv})_k \in \mathbb{R}^K$. The endowments $\{q_{0,k}^i\}_i$ and the common value $q_{0,k}^{cv}$ are independent across assets k.[14] The subsequent analysis accommodates a joint Gaussian distribution for correlated privately known endowments, including the independent private value model ($\sigma_{cv}^2 = 0$).[15]

Double auction. Each exchange is organized as the uniform-price double auction in which traders submit strictly downward-sloping[16] (net) demand schedules. For $q_k^i > 0$, trader i is a buyer of asset k; for $q_k^i < 0$, he is a seller. We consider two types of schedules: contingent and uncontingent. In Section 4, we analyze arbitrary cross-asset conditioning.

Definition 1 (Contingent and Uncontingent Schedules) *In a double auction with* contingent schedules*, each trader i submits K demand functions $\mathbf{q}^{i,c}(\cdot) \equiv \big(q_1^{i,c}(\mathbf{p}), \ldots, q_K^{i,c}(\mathbf{p})\big)$, each $q_k^{i,c}(\cdot) : \mathbb{R}^K \to \mathbb{R}$ specifying the quantity of asset k demanded for any price vector $\mathbf{p} = (p_1, \ldots, p_K)$.*

In a double auction with uncontingent schedules*, each trader i submits K demand functions $\mathbf{q}^i(\cdot) \equiv (q_1^i(p_1), \ldots, q_K^i(p_K))$, each $q_k^i(\cdot) : \mathbb{R} \to \mathbb{R}$ specifying the quantity of asset k demanded for any price p_k.*

Demand conditioning determines how the market clears. With uncontingent schedules, the market clears *exchange by exchange*: setting the aggregate net demand in each exchange k equal to zero, $\sum_i q_k^i(p_k) = 0$, determines the equilibrium price p_k. With contingent schedules, the K assets clear *jointly*: equilibrium price vector is determined by $\sum_i \mathbf{q}^{i,c}(p_1, \cdots, p_K) = 0 \in \mathbb{R}^K$. With either type of schedule, trader i trades $\{q_k^i\}_k$, pays $\sum_k p_k q_k^i$, and receives a payoff of $u^i(\mathbf{q}^i) - \mathbf{p} \cdot \mathbf{q}^i$.

Equilibrium. We study the Bayesian Nash Equilibrium in linear demand schedules (hereafter, *equilibrium*).

Definition 2 (Equilibrium) *A profile of (net) demand schedules $\{\{q_k^i(\cdot)\}_k\}_i$ is a Bayesian Nash equilibrium if, for each i, $\{q_k^i(\cdot)\}_k$ maximizes the expected payoff:*

$$\max_{\{q_k^i(\cdot)\}_k} E[\boldsymbol{\delta} \cdot (\mathbf{q}^i + \mathbf{q}_0^i) - \frac{\alpha^i}{2}(\mathbf{q}^i + \mathbf{q}_0^i) \cdot \boldsymbol{\Sigma}(\mathbf{q}^i + \mathbf{q}_0^i) - \mathbf{p} \cdot \mathbf{q}^i | \mathbf{q}_0^i], \tag{2}$$

given the schedules of other traders $\{\{q_k^j(\cdot)\}_k\}_{j \neq i}$ and market clearing $\sum_j q_k^j(\cdot) = 0$ for all k.

As is well known, in markets with contingent schedules, equilibrium is invariant to the distribution of private endowments; i.e., the linear Bayesian Nash Equilibrium with (possibly

[14] We assume the symmetry of variance across traders and the independent endowments across assets for simplicity; the results hold qualitatively without these assumptions.

[15] The common value component in $\{\mathbf{q}_0^i\}_i$ affects the magnitude of inference coefficients, but it does not affect any results qualitatively.

[16] This rules out trivial equilibria with no trade.

Electronic copy available at: https://ssrn.com/abstract=3604976

correlated) private endowments has an *ex post* property.[17] The contingent schedule allows a trader to choose his demand for each asset as a function of prices to-be-realized, which map one-to-one to realizations of quantities traded of other assets. With uncontingent schedules, equilibrium is not generally *ex post*.

Competitive market. The competitive market will often serve as a benchmark when evaluating the effects of incomplete conditioning with imperfectly competitive traders.[18]

Definition 3 (Competitive Market, Competitive Equilibrium) *Consider a market with $I < \infty$ traders. The* competitive market *is the limit game as $I \to \infty$, holding fixed all other primitives. Letting $\{\mathbf{q}^{i,I}(\cdot)\}_i$ be the equilibrium in the market with $I < \infty$ traders, the* competitive equilibrium *$\{\mathbf{q}^i(\cdot)\}_i$ is the limit of equilibria $\{\mathbf{q}^{i,I}(\cdot)\}_i$ as $I \to \infty$:*

$$\mathbf{q}^i(\cdot) = \lim_{I\to\infty} \mathbf{q}^{i,I}(\cdot) \qquad \forall i.$$

3 Equilibrium: Contingent vs. Uncontingent Demands

In this section, we characterize equilibrium in markets with uncontingent demands. We contrast it with equilibrium with contingent demands, which we review to facilitate the comparison.

Propositions 1 and 2 show the equivalence between optimization in demand functions (2) and *pointwise* optimization with respect to the realizations of the relevant contingent variables — $\mathbf{p}$ in the contingent and p_k in the uncontingent markets. This is motivated by the following observation. When traders submit demand schedules contingent on price realizations (of any subset of assets), it is useful to adopt the perspective of an individual trader who optimizes against a residual market $\{\{q_k^j(\cdot)\}_k\}_{j\neq i}$, for which the sufficient statistic is the profile of his residual supply functions $S_k^{-i}(\cdot) \equiv -\sum_{j\neq i} q_k^j(\cdot)$ for all k, defined by aggregation through market clearing of the other traders' submitted schedules. $S_k^{-i}(\cdot)$ is a function of $\mathbf{p} \in \mathbb{R}^K$ if demands

[17] Equilibrium is *linear* if schedules have the functional form of $q^i(\cdot) = \alpha_0 + \alpha_q^i q_0^i + \alpha_p^i p$. Equilibrium is *ex post* if equilibrium demands $\{q_k^i(\cdot;\mathbf{q}_0^i)\}_k$ are optimal for all i, given endowment realizations for *all* traders $\{\mathbf{q}_0^j\}_j$:

$$\{q_k^i(\cdot;\mathbf{q}_0^i)\}_k = \arg\max_{\{q_k^i(\cdot)\}_k} E[\boldsymbol{\delta}\cdot(\mathbf{q}^i+\mathbf{q}_0^i) - \frac{\alpha^i}{2}(\mathbf{q}^i+\mathbf{q}_0^i)\cdot\boldsymbol{\Sigma}(\mathbf{q}^i+\mathbf{q}_0^i) - \mathbf{p}\cdot\mathbf{q}^i|\{\mathbf{q}_0^j\}_j].$$

The *ex post* property holds because permitting pointwise optimization for each realization of the price vector produces contingent best-response schedules that are independent of traders' expectations about others' endowments and are thus optimal for any distribution of private information.

[18] The common value component $\mathbf{q}_0^{cv}$ in traders' endowments $\{\mathbf{q}_0^i\}_i$ ensures that the price (equivalently, the per capita aggregate endowment) is random in the limit large market ($I \to \infty$). To make the price variance $Var(\mathbf{p}|\mathbf{q}_0^i) = \boldsymbol{\Sigma} Var((\sum_j \frac{1}{\alpha^j})^{-1}\sum_j \alpha^j \mathbf{q}_0^j|\mathbf{q}_0^i)\boldsymbol{\Sigma}'$ independent of the number of traders I (so that the risk premium due to price uncertainty is independent of the number of participants in the market), the risk aversion $\{\alpha^i\}_i$ in utility (1) can be scaled according to $\alpha^{i,I} \equiv \alpha^i\sqrt{(\sigma_{cv}^2 + \frac{1}{I}\sigma_{pv}^2)^{-1}\sigma_{cv}^2}$. As $I \to \infty$, $\alpha^{i,I} \to \alpha^i > 0$ for all i. See Ft. 22.

Electronic copy available at: https://ssrn.com/abstract=3604976

are contingent ($\{q_k^{i,c}(\cdot) : \mathbb{R}^K \to \mathbb{R}\}_i$) and a function of $p_k \in \mathbb{R}$ if demands are uncontingent ($\{q_k^i(\cdot) : \mathbb{R} \to \mathbb{R}\}_i$).[19]

3.1 Equilibrium with Contingent Demands

Traders submit demand schedules $\mathbf{q}^{i,c}(\cdot) : \mathbb{R}^K \to \mathbb{R}^K$ contingent on price realizations for all assets $\mathbf{p} \in \mathbb{R}^K$. Since the demand for each asset is measurable with respect to $\{\mathbf{p}, \mathbf{q}_0^i\}$ (i.e., contingent variables $\mathbf{p}$ and privately known endowments $\mathbf{q}_0^i$) and, as we will show, price distribution has full support (see Remark 1), the expected payoff of trader i in the pointwise optimization is the same as the *ex post* payoff: for each asset k,

$$\max_{\mathbf{q}_k^{i,c} \in \mathbb{R}} \{\boldsymbol{\delta} \cdot (\mathbf{q}^{i,c} + \mathbf{q}_0^i) - \frac{\alpha^i}{2}(\mathbf{q}^{i,c} + \mathbf{q}_0^i) \cdot \boldsymbol{\Sigma}(\mathbf{q}^{i,c} + \mathbf{q}_0^i) - \mathbf{p} \cdot \mathbf{q}^{i,c}\} \qquad \forall \mathbf{p} \in \mathbb{R}^K, \tag{3}$$

given his residual supply function $\mathbf{S}^{-i,c}(\cdot) \equiv -\sum_{j \neq i} \mathbf{q}^{j,c}(\cdot) : \mathbb{R}^K \to \mathbb{R}^K$ for all assets and his demands for other assets $\{q_\ell^{i,c}(\cdot)\}_{\ell \neq k}$.

Pointwise optimization leads to an equilibrium characterization in terms of two simple conditions (Proposition 1), which can be derived in the following two steps.

Step 1 (Optimization, given price impact) The first-order condition with respect to the demand for each asset $q_k^{i,c}$ is: for each k,[20]

$$\underbrace{\delta_k - \alpha^i(\sigma_{kk}(q_k^{i,c} + q_{0,k}^i) + \sum_{\ell \neq k} \sigma_{k\ell}(q_\ell^{i,c} + q_{0,\ell}^i))}_{\text{Marginal utility for asset } k} = \underbrace{p_k + \frac{dp_k}{dq_k^i} q_k^{i,c} + \sum_{\ell \neq k} \frac{dp_\ell}{dq_k^i} q_\ell^{i,c}}_{\text{Marginal payment for asset } k} \quad \forall \mathbf{p} \in \mathbb{R}^K. \tag{4}$$

In a linear equilibrium,[21] $\frac{dp_\ell}{dq_k^{i,c}} \equiv \lambda_{k\ell}^{i,c}$ is constant for each k, ℓ and i. Written in matrix form, the first-order conditions become:

$$\boldsymbol{\delta} - \alpha^i \boldsymbol{\Sigma}\big(\mathbf{q}^{i,c} + \mathbf{q}_0^i\big) = \mathbf{p} + \boldsymbol{\Lambda}^{i,c} \mathbf{q}^{i,c} \qquad \forall \mathbf{p} \in \mathbb{R}^K, \tag{5}$$

where matrix $\boldsymbol{\Lambda}^{i,c} \equiv \frac{d\mathbf{p}}{d\mathbf{q}^{i,c}} \in \mathbb{R}_+^{K \times K}$ is the *price impact* of trader i (i.e., 'Kyle's lambda'). Its $(k,\ell)^{\text{th}}$ element $\lambda_{k\ell}^{i,c}$ represents the price change in asset ℓ following a demand change in asset k by trader i. The inverse of price impact is a common measure of *liquidity*: the lower the

[19] The idea of considering the optimization of a single trader, given the residual market goes back to Kyle (1989) and Klemperer and Meyer (1989). Rostek and Weretka (2015) introduced the equilibrium characterization in terms of the fixed point in price impacts (Proposition 1 below), showing the equivalence between the equilibrium conditions in Definition 2 and Proposition 1. The equilibrium characterization with contingent demands for heterogeneous risk aversions (Proposition 1) is from Malamud and Rostek (2017).

[20] A unilateral demand change for asset k is understood as an arbitrary twice differentiable function $\Delta q_k^i(\cdot) : \mathbb{R}^K \to \mathbb{R}$ in the contingent market, holding fixed demands $q_\ell^i(\cdot)$ for other assets $\ell \neq k$.

[21] More precisely, assuming that the best-response demands of traders $j \neq i$ are linear.

Electronic copy available at: https://ssrn.com/abstract=3604976

price impact, the smaller the price concession a trader must accept, the more liquid the market. From the first-order condition (5), the best response demand of trader i is:

$$\mathbf{q}^{i,c}(\mathbf{p}) = (\alpha^i \boldsymbol{\Sigma} + \boldsymbol{\Lambda}^{i,c})^{-1}(\boldsymbol{\delta} - \mathbf{p} - \alpha^i \boldsymbol{\Sigma} \mathbf{q}_0^i) \qquad \forall \mathbf{p} \in \mathbb{R}^K, \tag{6}$$

given his price impact $\boldsymbol{\Lambda}^{i,c}$, which is a sufficient statistic for trader i's residual supply function (see Remark 2) and is endogenized in Step 2.

Step 2 (Correct price impacts) In equilibrium, the price impact in the pointwise optimization (5) of trader i must be correct, i.e., it must equal the $K \times K$ Jacobian matrix of the inverse residual supply function of trader i. Applying market clearing to the best response demands (6) for traders $j \neq i$ gives the residual supply function of trader i $\mathbf{S}^{-i,c}(\cdot)$:

$$\mathbf{S}^{-i,c}(\mathbf{p}) = -\sum_{j \neq i} (\alpha^j \boldsymbol{\Sigma} + \boldsymbol{\Lambda}^{j,c})^{-1}(\boldsymbol{\delta} - \alpha^j \boldsymbol{\Sigma} \mathbf{q}_0^j) + \sum_{j \neq i} (\alpha^j \boldsymbol{\Sigma} + \boldsymbol{\Lambda}^{j,c})^{-1} \mathbf{p} \qquad \forall \mathbf{p} \in \mathbb{R}^K. \tag{7}$$

The price impact of trader i is the Jacobian of $(\mathbf{S}^{-i,c}(\cdot))^{-1}$, $\boldsymbol{\Lambda}^{i,c} \equiv \frac{d\mathbf{p}}{d\mathbf{q}^{i,c}} = -\big(\frac{\partial \mathbf{S}^{-i,c}(\cdot)}{\partial \mathbf{p}}\big)^{-1}$.

Proposition 1 gives an *equivalent* characterization of the (Bayesian Nash) equilibrium in demand schedules by two conditions: (i) traders optimize, given their assumed price impacts, (ii) which are correct.

Proposition 1 (Equilibrium: Contingent Trading) A profile of (net) demand schedules $\{\mathbf{q}^{i,c}(\cdot)\}_i$ is a linear Bayesian Nash equilibrium if and only if, for each i,

(i) (*Optimization, given price impact*) Demand schedules $\mathbf{q}^{i,c}(\cdot) : \mathbb{R}^K \to \mathbb{R}^K$ are uniquely determined by pointwise equalization of marginal utility and marginal payment in Eq. (6), given his price impact $\boldsymbol{\Lambda}^{i,c}$, such that:

(ii) (*Correct price impacts*) The price impact of trader i equals the slope of his inverse residual supply function:

$$\boldsymbol{\Lambda}^{i,c} = \big(\sum\nolimits_{j \neq i} (\alpha^j \boldsymbol{\Sigma} + \boldsymbol{\Lambda}^{j,c})^{-1}\big)^{-1}. \tag{8}$$

The fixed point in price impacts (8) can be solved in closed form with contingent demands:

$$\boldsymbol{\Lambda}^{i,c} = \beta^{i,c} \alpha^i \boldsymbol{\Sigma} \tag{9}$$

for each i, where $\beta^{i,c} = \frac{2 - \alpha^i b + \sqrt{(\alpha^i b)^2 + 4}}{2\alpha^i b} \in \mathbb{R}_+$ and $b \in \mathbb{R}_+$ is the unique solution to $\sum_j (\alpha^j b + 2 + \sqrt{(\alpha^j b)^2 + 4})^{-1} = 1/2$. With contingent trading, the price impact of every trader is *proportional* to the fundamental covariance matrix $\boldsymbol{\Sigma}$ (Eq. (9)). This linearity of incentives in risk has important implications for the contingent market (cf. Theorem 3, Proposition 7). If risk aversions are symmetric, i.e., $\alpha^i = \alpha$ for all i, then price impact is $\boldsymbol{\Lambda}^{i,c} = \frac{\alpha}{I-2} \boldsymbol{\Sigma}$.

Electronic copy available at: https://ssrn.com/abstract=3604976

Analyzing price impact directly offers insights into the role of imperfectly competitive behavior. As $I \to \infty$, then $\mathbf{\Lambda}^{i,c} \to 0$ for all i,[22] and the competitive limit demand coincides with the inverse marginal utility, given the quasilinearity of the payoff function. When price impact is positive, $\mathbf{\Lambda}^{i,c} > 0$, trader i demands (or sells) less than if he had submitted his competitive schedule.

Remarks. We note three properties, the last two of which do not hold in the uncontingent markets.

1. All price realizations $\mathbf{p} \in \mathbb{R}^K$ can occur in equilibrium for some realizations of endowments, given the downward-sloping demands of traders $j \neq i$ (i.e., the Jacobian $\frac{\partial \mathbf{q}^{j,c}(\cdot)}{\partial \mathbf{p}} = -(\alpha^j \mathbf{\Sigma} + \mathbf{\Lambda}^j)^{-1} < 0$)[23] and trader i (i.e., $\frac{\partial \mathbf{q}^{i,c}(\cdot)}{\partial \mathbf{p}} = -(\alpha^i \mathbf{\Sigma} + \mathbf{\Lambda}^i)^{-1} < 0$). Hence, the first-order conditions must hold for all prices and the price impact of each trader is determined by the requirement that optimization, Bayesian inference, and market clearing hold in equilibrium *and* following a unilateral demand change.[24]

2. A trader's own price impact $\mathbf{\Lambda}^{i,c}$ alone is a sufficient statistic for the residual supply function in the best-response problem. This holds due to the one-to-one map between the contingent variable (i.e., price vector $\mathbf{p}$) and the residual supply's intercept (i.e., the vector $\mathbf{s}^{-i} \equiv -\sum_{j \neq i} (\alpha^j \mathbf{\Sigma} + \mathbf{\Lambda}^{j,c})^{-1} (\boldsymbol{\delta} - \alpha^j \mathbf{\Sigma} \mathbf{q}_0^j) \in \mathbb{R}^K$ in Eq. (7)) for all assets.

3. Equilibrium is *ex post* given the one-to-one map.

3.2 Equilibrium with Uncontingent Demands

Consider the optimization problem of trader i who submits demand schedules $\{q_k^i(\cdot) : \mathbb{R} \to \mathbb{R}\}_k$ simultaneously in K exchanges, each for one asset, to maximize his expected payoff: for each k,

$$\max_{q_k^i \in \mathbb{R}} E[\boldsymbol{\delta} \cdot (\mathbf{q}^i + \mathbf{q}_0^i) - \frac{\alpha^i}{2} (\mathbf{q}^i + \mathbf{q}_0^i) \cdot \mathbf{\Sigma} (\mathbf{q}^i + \mathbf{q}_0^i) - \mathbf{p} \cdot \mathbf{q}^i | p_k, \mathbf{q}_0^i] \qquad \forall p_k \in \mathbb{R}, \tag{10}$$

given his residual supply functions $\{S_\ell^{-i}(\cdot) \equiv -\sum_{j \neq i} q_\ell^j(\cdot) : \mathbb{R} \to \mathbb{R}\}_\ell$ for all assets and his demands for other assets $\{q_\ell^i(\cdot)\}_{\ell \neq k}$.

The trader's objective function is the same as with contingent trading (Eq. (2)); in particular, his information set (i.e., $\mathbf{q}_0^i$) is. However, the choice variable differs: demand in the exchange for asset k is contingent on, and hence measurable with respect to, price p_k only, and

[22] Price impact converges to zero as $I \to \infty$ so long as the risk aversion $\alpha^{i,I}$ increases slower than linearly i.e., $\alpha^{i,I} \sim o(I^{1-\varepsilon})$ for some $\varepsilon > 0$; see Ft. 18. Aggregate endowment is random in the limit provided traders' endowments are correlated via $\mathbf{q}_0^{cv}$ ($\sigma_{cv}^2 > 0$).

[23] The Moore-Penrose pseudoinverse of matrix $\alpha^j \mathbf{\Sigma} + \mathbf{\Lambda}^j$ if the matrix is not invertible.

[24] The equilibrium market-clearing condition (Definition 2) is accounted for by condition (ii) for price impacts. Namely, by Eq. (8), the price impact of trader i is characterized as the price change at which other traders are willing to sell the extra units demanded by i (given that traders $j \neq i$ optimize according to (6)) so that the market clears.

Electronic copy available at: https://ssrn.com/abstract=3604976

the expected payoff (10) is not the same as the *ex post* payoff due to uncertainty about trade of other assets.

A trader maximizes his expected payoff pointwise for each asset k — with respect to $p_k \in \mathbb{R}$:

$$\underbrace{\delta_k - \alpha^i(\sigma_{kk}(q_k^i + q_{0,k}^i) + \sum_{\ell \neq k} \sigma_{k\ell}(\underbrace{E[q_\ell^i | p_k, \mathbf{q}_0^i]}_{\substack{\text{Expected trade} \\ \text{of asset } \ell}} + q_{0,\ell}^i))}_{\textit{Expected} \text{ marginal utility for asset } k} = \underbrace{p_k + \underbrace{\lambda_k^i}_{\substack{\text{Zero cross-exchange} \\ \text{price impact}}} q_k^i}_{\text{Marginal payment for asset } k} \quad \forall p_k \in \mathbb{R}, \quad (11)$$

given his demand for the other assets $\{q_\ell^i(\cdot)\}_{\ell \neq k}$[25], where $\lambda_k^i \equiv \frac{dp_k}{dq_k^i} \in \mathbb{R}_+$ is the price impact of trader i in the exchange for asset k. In a linear equilibrium, $\lambda_k^i \equiv \frac{dp_k}{dq_k^i}$ is constant.

Compared to contingent trading (Eq. (4)), the first-order condition differs in two ways. First, a trader's demand for asset k depends on *expected rather than realized trades* of other assets $\ell \neq k$, $E[q_\ell^i | p_k, \mathbf{q}_0^i]$. Second, *the cross-exchange price impact is zero*: $\lambda_{k\ell}^i \equiv \frac{dp_\ell}{dq_k^i} = 0$ for all $\ell \neq k$, since the residual supply function $S_k^{-i}(\cdot; \{\mathbf{q}_0^j\}_{j \neq i}) : \mathbb{R} \to \mathbb{R}$ is contingent on p_k but not $\{p_\ell\}_{\ell \neq k}$. It follows that, in contrast to the contingent market, where the price impacts of all traders are proportional to the fundamental covariance matrix $\mathbf{\Sigma}$ (Eq. (9)), the price impact matrices of all traders are diagonal:

$$\mathbf{\Lambda}^i \equiv (\frac{dp_\ell}{dq_k^i})_{k,\ell} = diag(\lambda_k^i)_k \in \mathbb{R}^{K \times K}. \quad (12)$$

Although the cross-exchange price impact is zero, equilibrium behavior and outcome (i.e., prices and allocations) are not independent across exchanges — unless all assets' payoffs are independent (i.e., $\sigma_{k\ell} = 0$ for all $\ell \neq k$) in which case traders' utility Hessian is separable.

3.2.1 Preview

Because of cross-exchange externalities (discussed in Section 3.2.3), equilibrium and welfare are not neutral to separation in market clearing allowed by limited demand conditioning. Moreover, innovation in trading technology is generally not neutral either. Before introducing a technique to characteirze equilibrium in uncontingent markets, Examples 1 and 2 give a preview of the results that follow.

Example 1 (Price Impact and Welfare) Consider a market with two imperfectly correlated assets, $0 < |\rho_{12}| < 1$. Suppose that two assets are payoff substitutes, i.e., $\rho_{12} > 0$. In the contingent market, by the proportionality of price impact in the covariance (Eq. (9)), the

[25]Lemma 5 in Appendix B shows that a trader's optimization of a demand $q_k^i(\cdot)$ for each asset k, given his demands $\{q_\ell^i(\cdot)\}_{\ell \neq k}$ for the other assets $\ell \neq k$ is without loss, by the differentiability of expected payoff (2) with respect to the profile of demands $\{q_k^i(\cdot)\}_k$.

Electronic copy available at: https://ssrn.com/abstract=3604976

cross-asset price impact inherits the covariance's sign. For the traders who buy both assets, the cross-asset price impact $\lambda_{12}^c > 0$ and $\lambda_{21}^c > 0$ increases the cost of trading:

$$p_1 + \lambda_1^c q_1^{i,c} + \lambda_{12}^c q_2^{i,c}, \qquad p_2 + \lambda_2^c q_2^{i,c} + \lambda_{21}^c q_1^{i,c}; \tag{13}$$

the cross-asset price impact λ_{12} and λ_{21} is zero with uncontingent demands. When the assets are payoff complements, i.e., $\rho_{12} < 0$, the cross-asset price impact $\lambda_{12}^c < 0$ and $\lambda_{21}^c < 0$ lowers the trading costs. We will show that when they enhance welfare, efficient multi-venue designs lower the trading cost associated with diversification across assets (i.e., the cross-asset price impact) when $K = 2$; more generally, in markets with $K > 2$ assets, multi-venue trading can lower the cost of trading associated with risk sharing (i.e., the price impact of a given asset) and diversification. □

Two-asset markets in Example 1 illustrate more general implications for design. When trading is imperfectly competitive, neither the market structure in which all assets clear jointly nor the market structure in which each asset is traded in a single exchange (i.e., uncontingent demands) is always efficient. More specifically, unlike the competitive markets, characteristics of assets and traders matter. In particular, the joint substitutability of asset payoffs (i.e., $\mathbf{\Sigma}$) and trading needs (i.e., $\{E[\bar{\mathbf{q}}_0] - E[\mathbf{q}_0^i]\}_i$) do. Section 4 shows that *for any characteristics of traders and assets*, one can design exchanges so that a market with multiple venues that clear independently can be as efficient as a single exchange which clears all assets jointly. Section 5 shows that market characteristics guide which multi-venue design can be strictly more efficient than joint clearing.

Example 2 (Innovation in Trading Technology) Suppose that a new exchange for some of the K assets is created to operate along with the existing exchanges *without altering traders' endowments of any asset*. In the contingent market, the introduction of $L \geq 1$ assets each of which payoff is perfectly correlated with a traded asset would be neutral for traders' equilibrium payoffs. This can be seen from the contingent demands $\mathbf{q}^{i,c}(p_1, ..., p_K, p_{K+1}, \cdots, p_{K+L}) : \mathbb{R}^{K+L} \to \mathbb{R}^{K+L}$ in Eq. (6):

$$(\alpha^i \mathbf{\Sigma}^+ + \mathbf{\Lambda}^{i,c})\mathbf{q}^{i,c} = \boldsymbol{\delta}^+ - \mathbf{p} - \alpha^i \mathbf{\Sigma}^+ \mathbf{q}_0^{i,+} \qquad \forall \mathbf{p} \in \mathbb{R}^{K+L}; \tag{14}$$

the split of endowments for the replicated assets is arbitrary. Using that price impact $\mathbf{\Lambda}^{i,c} = \beta^{i,c}\alpha^i \mathbf{\Sigma}^+ \in \mathbb{R}^{(K+L)\times(K+L)}$ is proportional to the covariance matrix for all i in the contingent market (Eq. (9)), and that the covariance matrix $\mathbf{\Sigma}^+ \in \mathbb{R}^{(K+L)\times(K+L)}$ is singular with the new assets, we conclude that condition (14) has a continuum of solutions $\mathbf{q}^{i,c} \in \mathbb{R}^{K+L}$ pointwise with respect to the price vector $\mathbf{p} \in \mathbb{R}^{K+L}$, including zero trades of the new assets $q_{K+\ell}^{i,c}(\cdot) = 0$

Electronic copy available at: https://ssrn.com/abstract=3604976

for $\ell \in L$. Even if the new assets are traded, traders' equilibrium payoffs are the same as in the market with K assets. □

Once one departs from the assumption that markets clear jointly for all assets, innovation in trading technology that would be neutral for traders' payoffs with joint clearing (if well defined at all) is no longer redundant, i.e., traders' equilibrium payoffs change. Section 4 identifies the types of innovation in trading technology that can be non-neutral — such as introducing new trading protocols (Example 2) or linking trading protocols by merging their assets — given any market structure.

3.2.2 Equilibrium as a fixed point in price impacts

Our main results in this section, Proposition 2 and Theorem 1, show that equilibrium in uncontingent markets can be characterized with conditions analogous to those with contingent trading (in Proposition 1): for each i,

(i) his demands are a best response, given i's residual supply (Step 1);

(ii) his residual supply function is correct (Step 2).

Uncontingent trading changes the structure of both the fixed point problem for a trader's best response (Step 1) and equilibrium (Step 2), compared to contingent demands — and makes the equilibrium characterization more challenging, as we explain next.

Equilibrium in contingent vs. uncontingent demands. In a trader's multvariate optimization problem (2) — contingent or uncontingent — the system of first-order conditions characterizes the fixed point problem for his best-response schedules. With contingent demands, the system can be written as a single matrix equation (5) for a quantity vector $\mathbf{q}^i = (q^i_k)_k$ and can be solved for $\mathbf{q}^i$ pointwise with respect to the price vector $\mathbf{p}$. The closed-form solution is given by Eq. (6).

In the uncontingent market, however, we cannot rely on the method of *ex post* optimization. That is, the best-response quantities demanded cannot be solved for pointwise with respect to the price vector $\mathbf{p}$: in the first-order condition (11) for asset k, expected trade $E[q^i_\ell | p_k, \mathbf{q}^i_0]$ depends on the *functional form* of $q^i_\ell(\cdot)$. Hence, even in a linear equilibrium, characterizing the best-response demands of a trader requires solving the fixed point for the trader's demand coefficients (on the contingent variables and private information) across assets. Additionally, unlike contingent trading, the price impact $\mathbf{\Lambda}^i$ is not a sufficient statistic for the residual supply of trader i (cf. Remark 2), given that the best response demands are not *ex post* and depend on the distribution of the conditioning variable $\mathbf{p}$.

Nevertheless, we show that *a fixed point in demand schedules* $\{\{q^i_k(\cdot)\}_k\}_i$ *is equivalent to a fixed point in price impact matrices*, i.e., we endogenize all demand coefficients (Step 1) —

Electronic copy available at: https://ssrn.com/abstract=3604976

including conditional expectations $E[q_\ell^i|p_k, \mathbf{q}_0^i]$ for all $\ell \neq k$ and k — and the distribution of the residual supply (Step 2) as functions of price impacts $\{\mathbf{\Lambda}^i\}_i$ (Theorem 1).

We begin by taking the intercept of trader i's residual supply s_k^{-i} rather than price as a contingent variable.[26] As we will show, price p_k is one-to-one with residual supply's intercept s_k^{-i} which has full support; however, unlike the distribution of p_k, that of s_k^{-i} is determined only by the schedules of traders $j \neq i$ and is thus exogenous in the best-response problem of trader i. This allows us to separate the best response and equilibrium problems analogously to Proposition 1: best response demands $\{\{q_k^i(\cdot)\}_k\}_i$ are equilibrium demands if and only if the residual supply functions are correct.

For each trader i, let $F(\mathbf{q}_0^{-i}|\mathbf{q}_0^i) \equiv F((\mathbf{q}_0^j)_{j\neq i}|\mathbf{q}_0^i)$ be the joint distribution of other traders' endowments and let $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ be the joint distribution of the intercepts $\mathbf{s}^{-i} \equiv (s_k^{-i})_k$ of the residual supplies of trader i — both conditional on trader i's privately known endowment. The former distribution is primitive, the latter is not but it is taken as given in trader i's best response problem; given the linear demands $\{\mathbf{q}^j(\cdot)\}_{j\neq i}$, $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ is jointly Normal.

Proposition 2 (Equilibrium: Uncontingent Trading) A profile of (net) demand schedules $\{\{q_k^i(\cdot)\}_k\}_i$ is a linear Bayesian Nash equilibrium if and only if, for each i,

(i) (*Optimization, given residual supply*) Demand schedules $q_k^i(\cdot) : \mathbb{R} \to \mathbb{R}$ are determined by equalization of expected marginal utility and marginal payment for each asset k:

$$\delta_k - \alpha^i \mathbf{\Sigma}_k E[\mathbf{q}^i + \mathbf{q}_0^i | s_k^{-i}, \mathbf{q}_0^i] \;=\; p_k + \lambda_k^i q_k^i \qquad \forall p_k \in \mathbb{R}, \tag{15}$$

given the trader's own demands for other assets $\{q_\ell^i(\cdot)\}_{\ell\neq k}$. Expected trades $E[q_\ell^i|s_k^{-i}, \mathbf{q}_0^i]$ for $\ell \neq k$ are characterized by the Projection Theorem, given the distribution $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ and price impact $\mathbf{\Lambda}^i = diag(\lambda_k^i)_k$.[27]

(ii) (*Correct residual supply*) The residual supply function $S_k^{-i}(\cdot) : \mathbb{R} \to \mathbb{R}$ is determined by applying market clearing to the best responses of traders $j \neq i$ $\{q_k^j(\cdot)\}_{j\neq i}$ that satisfy condition (i): for each k,

$$S_k^{-i}(\cdot) = -\sum_{j\neq i} q_k^j(\cdot).$$

[26] In the competitive market, price distribution can be taken as given in the best response problem. Then, the price distribution (i.e., the map between price realizations and random variables (or s_k^{-i})) does not differ in and off equilibrium and the expected trades and the distribution of their conditioning variable do not define a fixed point. When $I < \infty$, the distribution of the conditioning variable p_k creates a fixed point with the best-response demand $q_k^i(p_k)$ through the expected trade $E[q_\ell^i(p_\ell)|p_k, \mathbf{q}_0^i]$.

[27] Applying the Projection Theorem to the conditional distribution of the residual supply's intercept and prices endogenizes inference coefficients $\frac{\partial E[p_\ell|s_k^{-i}, \mathbf{q}_0^i]}{\partial s_k^{-i}} = \frac{Cov(p_\ell, s_k^{-i}|\mathbf{q}_0^i)}{Var(s_k^{-i}|\mathbf{q}_0^i)}$ in $E[p_\ell|s_k^{-i}, \mathbf{q}_0^i]$ as functions of price impact.

Electronic copy available at: https://ssrn.com/abstract=3604976

The price impact λ_k^i of trader i is characterized by the Jacobian of $\left(S_k^{-i}(\cdot)\right)^{-1}$ and the distribution $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ is characterized by the distribution of the residual supplies' intercepts, given $F(\mathbf{q}_0^{-i}|\mathbf{q}_0^i)$.

Best response fixed point. To tackle the characterization of the fixed point problem for a trader's best-response schedules $\{q_k^i(\cdot)\}_k$, we transform it into a fixed point among the trader's demand coefficients (Theorem 1). We treat a trader's best-responses for assets $\ell \neq k$ as linear functions of p_ℓ and $\mathbf{q}_0^i$:[28]

$$q_\ell^i(p_\ell) \equiv a_\ell^i - \mathbf{b}_\ell^i \mathbf{q}_0^i - c_\ell^i p_\ell \qquad \forall p_\ell \in \mathbb{R} \tag{16}$$

with the demand intercept $a_\ell^i \in \mathbb{R}$, the demand coefficients $\mathbf{b}_\ell^i \in \mathbb{R}^{1\times K}$ on $\mathbf{q}_0^i$, and the demand slope $c_\ell^i \in \mathbb{R}_+$ on p_ℓ.[29]

The parameterization of the best-response demands for assets $\ell \neq k$ and the change in the contingent variable (from p_k to s_k^{-i}) allow us to endogenize expected trades in the demand for asset k in terms of variables that are exogenous in the trader's best-response problem. The fixed point problem for best-response schedules $\{q_k^i(\cdot)\}_k$ thus becomes one for demand coefficients, given the residual supplies, i.e., $\mathbf{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$. When traders' risk aversions are the same, the best response fixed point has a unique solution.

Equilibrium as a fixed point in price impacts. The equilibrium fixed point problem has a large dimensionality: when schedules are not contingent, the price impact $\mathbf{\Lambda}^i$ is not by itself a sufficient statistic for the residual supply in the best response problem.[30] Theorem 1 shows that the joint distributions of the residual supply $\{F(\mathbf{s}^{-i}|\mathbf{q}_0^i)\}_i$ can be characterized as functions of only price impacts given the primitive distribution of endowments.

Applying market clearing to the best response schedules $\{q_k^j(\cdot)\}_{j\neq i}$ gives the residual supply functions of trader i (i.e., condition (ii) in Proposition 2): for each k,

$$S_k^{-i}(p_k) = \underbrace{-\sum_{j\neq i}(a_k^j - \mathbf{b}_k^j \mathbf{q}_0^j)}_{=s_k^{-i}} + \underbrace{\sum_{j\neq i} c_k^j}_{=(\lambda_k^i)^{-1}} p_k \qquad \forall p_k \in \mathbb{R}. \tag{17}$$

[28] A unilateral demand change for asset k is understood as an arbitrary twice differentiable function $\Delta q_k^i(\cdot) : \mathbb{R} \to \mathbb{R}$ in the uncontingent market, holding fixed demands $q_\ell^i(\cdot)$ for other assets $\ell \neq k$.

[29] Here, we used that the expected trades in exchange ℓ condition on price p_ℓ (the contingent variable in exchange ℓ) and endowment vector $\mathbf{q}_0^i$ (a trader's private information) by the one-to-one map between p_ℓ and s_ℓ^{-i} (to be established).

[30] Demand $q_k^i(p_k; \mathbf{q}_0^i)$ involves three demand coefficients (Eq. (16)) for each i, k, and residual supply functions $S_k^{-i}(p_k; \{\mathbf{q}_0^j\}_{j\neq i})$ involve $(I+1)$ coefficients (i.e., the intercept and the coefficients on p_k and $\mathbf{q}_0^j$ for each $j \neq i$) for each i, k — in total, the fixed point problem involves $(I+4)IK$ variables. Theorem 1 shows that equilibrium is equivalent to a fixed point for IK price impacts $\{\{\lambda_k^i\}_k\}_i$, thus reducing the dimension.

Electronic copy available at: https://ssrn.com/abstract=3604976

The distribution $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ and price impact $\{\lambda_k^i\}_k$ — taken as given in the trader's best response problem — are characterized by the distribution of the intercepts and the inverse slope of the residual supplies (17).

The joint distribution of intercepts $F(\mathbf{s}^{-i}|\mathbf{q}_0^i)$ is parameterized by demand coefficients $\{\{a_k^j, \mathbf{b}_k^j\}_k\}_{j\neq i}$, which — by Step 1 — are functions of price impacts.

$$F(\mathbf{s}^{-i}|\mathbf{q}_0^i) = \mathcal{N}\Big(\big(-\sum_{j\neq i}(a_k^j - \mathbf{b}_k^j E[\mathbf{q}_0^j|\mathbf{q}_0^i])\big)_k, \big(\sum_{j,h\neq i}\mathbf{b}_k^j Cov(\mathbf{q}_0^j, \mathbf{q}_0^h|\mathbf{q}_0^i)(\mathbf{b}_\ell^h)'\big)_{k,\ell}\Big). \tag{18}$$

The fixed point among the (moments of) traders' distributions $\{F(\mathbf{s}^{-i}|\mathbf{q}_0^i)\}_i$ becomes one for demand coefficients $\{\{a_k^j, \mathbf{b}_k^j\}_k\}_i$ and can be solved as a function of price impacts $\{\mathbf{\Lambda}^i\}_i$.

In each exchange, the equilibrium price impact $\lambda_k^i \equiv \frac{dp_k}{dq_k^i} \in \mathbb{R}_+$ must equal the slope of the inverse residual supply function: $\lambda_k^i = -\big(\sum_{j\neq i}\frac{\partial q_k^j(\cdot)}{\partial p_k}\big)^{-1} = (\sum_{j\neq i} c_k^j)^{-1}$ for all i and k.

Theorem 1 characterizes the equilibrium demand coefficients $\boldsymbol{a}^i \equiv (a_k^i)_k \in \mathbb{R}^K$, $\mathbf{B}^i \equiv (\mathbf{b}_k^i)_k \in \mathbb{R}^{K\times K}$, and $\mathbf{C}^i \equiv diag(c_k^i)_k \in \mathbb{R}^{K\times K}$ as functions of price impact — in matrix closed form — and characterizes equilibrium price impact in terms of primitives. Appendix C.1 derives demand coefficients for $K = 2$. In the main text, we present the characterization of the symmetric equilibrium[31] for simplicity of notation. In Appendix A, we state and prove the result for an asymmetric equilibrium.

Assumption (Symmetric Risk Preferences) Let $\alpha^i = \alpha$ for all i.

Let $[\cdot]_d : \mathbb{R}^{K\times K} \to \mathbb{R}^{K\times K}$ be an operator such that, for any matrix $\mathbf{M}$, $[\mathbf{M}]_d$ is a diagonal matrix with the $(k,\ell)^{\text{th}}$ element equal to zero for $k \neq \ell$ and the $(k,k)^{\text{th}}$ element equal to m_{kk} for any k.

Theorem 1 (Equilibrium: Fixed Point in Demand Schedules) In a symmetric equilibrium, the (net) demand schedules, defined by matrix coefficients $\{\boldsymbol{a}^i\}_i, \mathbf{B}$, and $\mathbf{C}$, and price impacts $\mathbf{\Lambda}$ are characterized by the following conditions: for each i,

(i) (*Optimization, given price impact* $\mathbf{\Lambda}$) Given price impact matrix $\mathbf{\Lambda}$, best-response coefficients $\boldsymbol{a}^i, \mathbf{B}$, and $\mathbf{C}$ are characterized by:

$$\boldsymbol{a}^i = \mathbf{C}\underbrace{\big(\boldsymbol{\delta} - (\alpha\mathbf{\Sigma} - \mathbf{C}^{-1}\mathbf{B})E[\bar{\mathbf{q}}_0]\big)}_{=\mathbf{p}-\mathbf{C}^{-1}\mathbf{B}\bar{\mathbf{q}}_0\equiv\mathbf{p}-\mathbf{Q}} + \underbrace{\big((\alpha\mathbf{\Sigma}+\mathbf{\Lambda})^{-1}\alpha\mathbf{\Sigma} - \mathbf{B}\big)(E[\bar{\mathbf{q}}_0] - E[\mathbf{q}_0^i])}_{\text{Adjustment due to cross-asset inference}}, \tag{19}$$

[31] Equilibrium is symmetric if for all k, price impacts satisfy $\lambda_k^i \equiv \lambda_k$ for all i and demand coefficients satisfy $c_k^i \equiv c_k$ and $\mathbf{b}_k^i \equiv \mathbf{b}_k$ for all i, and a_k^i is a symmetric function of $\{\{E[q_0^j]\}_{j\neq i}, E[q_0^i]\}$ across traders. We will suppress the superscript i except where it is helpful.

Electronic copy available at: https://ssrn.com/abstract=3604976

$$\mathbf{B} = \big((\alpha\mathbf{\Sigma} + \mathbf{\Lambda}) - \underbrace{\sigma_0^2(\alpha\mathbf{\Sigma} - (I-2)\mathbf{\Lambda})}_{\substack{\text{Adjustment due to} \\ \text{cross-asset inference}}}\big)^{-1}\alpha\mathbf{\Sigma}, \tag{20}$$

$$\mathbf{C} = \Big[(\alpha\mathbf{\Sigma} + \mathbf{\Lambda})\ \underbrace{(\mathbf{BB}')[\mathbf{BB}']_d^{-1}}_{\substack{\text{Inference coefficient} \\ \frac{Cov(s_\ell^{-i}, s_k^{-i}|\mathbf{q}_0^i)}{Var(s_k^{-i}|\mathbf{q}_0^i)^{-1}}}}\ \Big]_d^{-1}, \tag{21}$$

where $\bar{\mathbf{q}}_0 \equiv \frac{1}{I}\sum_j \mathbf{q}_0^j \in \mathbb{R}^K$ is the aggregate endowment and $\sigma_0^2 \equiv \frac{\sigma_{cv}^2 + \frac{1}{I}\sigma_{pv}^2}{\sigma_{cv}^2 + \sigma_{pv}^2} \in \mathbb{R}$.

(ii) (*Correct price impacts*) Price impact $\mathbf{\Lambda}$ equals the slope of his inverse residual supply function:

$$\mathbf{\Lambda} = \frac{1}{I-1}\mathbf{C}^{-1} = \frac{\alpha}{I-2}\Big[\mathbf{\Sigma}(\mathbf{BB}')[\mathbf{BB}']_d^{-1}\Big]_d. \tag{22}$$

Note that the price slope $\mathbf{C}$ is a diagonal matrix in the uncontingent market.[32]

Equilibrium outcome. Apart from characterizing equilibrium demands and distributions in terms of the profile of price impacts alone, Theorem 1 shows how separation in market clearing affects equilibrium outcome. To ease the comparison, if the market clears jointly, then $\mathbf{B}^c = (\alpha\mathbf{\Sigma} + \mathbf{\Lambda}^c)^{-1}\alpha\mathbf{\Sigma} = \frac{I-2}{I-1}\mathrm{Id}$, $\mathbf{C}^c = (\alpha\mathbf{\Sigma} + \mathbf{\Lambda}^c)^{-1}$, and $\mathbf{\Lambda}^c = \frac{1}{I-2}\alpha\mathbf{\Sigma}$ (Eq. (6) and (9)), where $\mathrm{Id} \in \mathbb{R}^{K\times K}$ is the identity matrix. In contrast, in the uncontingent market, traders' demands depend on the *distribution* of traders' endowments, as do price impacts, which are also not proportional to the fundamental risk $\mathbf{\Sigma}$.

Corollary 1 (Equilibrium Price and Allocations) Given the equilibrium demand coefficients $\{\boldsymbol{a}^i\}_i, \mathbf{B}, \mathbf{C}$, and price impact $\mathbf{\Lambda}$ in Theorem 1, equilibrium prices and allocations are:

$$\mathbf{p} = \boldsymbol{\delta} - (\alpha\mathbf{\Sigma} - \mathbf{C}^{-1}\mathbf{B})E[\bar{\mathbf{q}}_0] - \mathbf{C}^{-1}\mathbf{B}\bar{\mathbf{q}}_0, \tag{23}$$

$$\mathbf{q}^i + \mathbf{q}_0^i = \big((\alpha\mathbf{\Sigma} + \mathbf{\Lambda})^{-1}\alpha\mathbf{\Sigma} - \mathbf{B}\big)E[\bar{\mathbf{q}}_0 - \mathbf{q}_0^i] + \mathbf{B}\bar{\mathbf{q}}_0 + (\mathrm{Id} - \mathbf{B})\mathbf{q}_0^i. \tag{24}$$

In contrast to the contingent market, where $\mathbf{p}^c = \boldsymbol{\delta} - \alpha\mathbf{\Sigma}\bar{\mathbf{q}}_0$, the second moment $Var(\mathbf{p})$ of the distribution of equilibrium prices is not exogenous, but depends on price impact. In particular, due to the lack of proportionality between $\mathbf{\Lambda}$ and $\mathbf{\Sigma}$, $Var(\mathbf{p})$ depends on the endogenous demand coefficient $\mathbf{C}^{-1}\mathbf{B}$ rather than the exogenous asset covariance $\mathbf{\Sigma}$, and the price covariance of any assets depends on the second moment of the joint distribution of all assets.[33]

[32] The inference coefficient $(\mathbf{BB}')[\mathbf{BB}']_d^{-1}$ in Eq. (21) is derived from the distribution of the residual supply intercepts $\mathbf{s}^{-i}$ in Eq. (18), given the distribution of endowments $F(\mathbf{q}_0^{-i}|\mathbf{q}_0^i)$.

[33] One implication is that the effect of anticipated shocks (e.g., shocks to asset supply, assumed zero for simplicity) on prices and trades differs from those of unanticipated shocks. In contrast, in the contingent market, demand coefficients are the same whether shocks are anticipated or not.

Electronic copy available at: https://ssrn.com/abstract=3604976

Unlike the contingent market, the weights on the idiosyncratic and market risk (i.e., $\mathrm{Id} - \mathbf{B}$ and $\mathbf{B}$) depend on the asset covariance. Thus, asset payoff substitutability matters for which assets' allocation is more efficient.

In contrast to the contingent market, where $\mathbf{p}^c = \boldsymbol{\delta} - \alpha\boldsymbol{\Sigma}\bar{\mathbf{q}}_0$, the second moment $Var(\mathbf{p})$ of the distribution of equilibrium prices is not exogenous. In particular, due to the lack of proportionality between $\boldsymbol{\Lambda}$ and $\boldsymbol{\Sigma}$, $Var(\mathbf{p})$ depends on the endogenous demand coefficient $\mathbf{C}^{-1}\mathbf{B}$, rather than the exogenous asset covariance $\boldsymbol{\Sigma}$; the price covariance of any asset pair depends on the second moment of the joint distribution of *all* assets. [34] Unlike the contingent market, the weights on the idiosyncratic and market risk (i.e., $\mathrm{Id} - \mathbf{B}$ and $\mathbf{B}$) depend on the asset covariance. Thus, asset payoff substitutability matters for which assets' allocation is more efficient.

Having characterized the equilibrium fixed point, in what follows, we assume symmetric risk preferences.

Theorem 2 (Existence of the Symmetric Equilibrium) There exists a symmetric linear Bayesian Nash equilibrium. When $K = 2$, equilibrium is unique.

The argument differs in two ways from that in a double auction with contingent schedules. First, in the contingent market, the proportionality of price impact to asset covariance reduces the fixed point problem for $\{\boldsymbol{\Lambda}^{i,c}\}_i$ to one for scalars $\{\beta^{i,c} \in \mathbb{R}\}_i$ (Eq. (9)). In the uncontingent market, price impact matrices are not proportional to the covariance, and need to be found jointly for all assets and all traders due to cross-asset externalities. Second, due to the cross-asset inference (i.e., inference coefficient $(\mathbf{B}\mathbf{B}')[\mathbf{B}\mathbf{B}']_d^{-1}$), the fixed point for price impacts — Eq. (20) and (22) — is not a monotone map in price impact $\boldsymbol{\Lambda}$. Given Theorem 1's result that a fixed point in demand schedules can equivalently (for $I < \infty$) be represented as a fixed point in price impact matrices, equilibrium existence follows from the Brouwer fixed point theorem (Theorem 2) with the bounds of price impact being matrices (rather than scalars).[35]

[34] One implication is that the effect of anticipated shocks (e.g., shocks to asset supply, assumed zero for simplicity) on prices and trades differs from those of unanticipated shocks. In contrast, in the contingent market, demand coefficients are the same whether shocks are anticipated or not.

[35] We do not provide a uniqueness result for $K > 2$. In the symmetric market, for the many parameters $(I, \boldsymbol{\delta}, \boldsymbol{\Sigma}, \alpha)$ we analyzed, our numerical iteration algorithm for the fixed point problem in Theorem 1 gives a unique solution $(\{\boldsymbol{a}^i\}_i, \mathbf{B}, \mathbf{C}, \boldsymbol{\Lambda})$ such that $0 \leq \boldsymbol{\Lambda} \leq (\frac{\alpha}{I-2}\sum_k \sigma_{kk})\mathrm{Id}$ for an arbitrary starting point of the iteration. In the contingent model, the equilibrium uniqueness can be shown using the proportionality of price impact in the covariance matrix (Eq. (9)): it allows a diagonalization of traders' demand slopes with the same orthonormal basis for all i and applying the argument from a one-asset market (see Malamud and Rostek (2017)). Lambert, Ostrovsky, and Panov (2017) consider a game in which strategies are quantities (market orders) with one asset and one liquidity provider; the scalar price impact solves a quadratic equation that has a unique positive solution, which gives equilibrium uniqueness. We analyze games in demand and supply functions with multiple assets and price impacts that are matrices characterized by a system of nonlinear equations.

Electronic copy available at: https://ssrn.com/abstract=3604976

3.2.3 Price Impact

By Theorem 1, the implications of separation in market clearing for equilibrium, welfare, and design (Sections 4 and 5) can be understood through the structure of the endogenous price impact matrix. Thus far, we have noted that, in contrast to when markets clear jointly, first, the *cross-exchange* price impacts are zero, i.e., traders' price impact matrices are diagonal by the definition of uncontingent demands, and second, the *within-exchange* price impacts depend on cross-asset inference (Theorem 1). Theorem 3 shows that *the within-exchange price impacts* $\{\lambda_k\}_k$ is larger than its contingent counterpart irrespective of the substitutabilities and complementarities in the asset payoffs $\mathbf{\Sigma}$, except when asset payoffs are independent (Fig. 1). This result is due to cross-asset inference.

Price impact and cross-exchange inference. Consider the counterfactual that defines trader i's price impact in exchange k: what is the effect of increasing the demand for asset k at the margin by trader i? Price p_k increases so that other traders are willing to sell the extra units and the market clears. This is the *direct effect*, which is present in the contingent market as well. When the market is uncontingent (equilibrium is not *ex post*), the change in price p_k also has an *indirect inference effect* — through the conditioning variable p_k in expected trades $E[q^i_\ell|p_k, \mathbf{q}^i_0]$ (equivalently, $E[p_\ell|p_k, \mathbf{q}^i_0]$).[36] Applying the Chain Rule to the Jacobian $\frac{\partial q^j_k(\cdot)}{\partial p_k}$ gives:

$$\lambda^i_k = -\Big(\sum_{j\neq i} \underbrace{\frac{\partial q^j_k(\cdot)}{\partial p_k}}_{c^j_k}\Big)^{-1} = -\Big(\sum_{j\neq i}\Big(\underbrace{\frac{\partial q^j_k}{\partial p_k}}_{\text{Direct }(-)} + \underbrace{\sum_{\ell\neq k} \underbrace{\frac{\partial q^j_k}{\partial p_\ell}}_{sign(\sigma_{k\ell})} \underbrace{\frac{\partial E[p_\ell|p_k, \mathbf{q}^i_0]}{\partial p_k}}_{sign(\sigma_{k\ell})}}_{\text{Inference }(+)}\Big)\Big)^{-1}. \tag{25}$$

(For $K = 2$, see Eq. (135) in Appendix C.1.) To explain the inference effect in Eq. (25), in the counterfactual following the demand increase by i, consider what traders $j \neq i$ would infer from the higher price p_k. For assets that are payoff substitutes (i.e., $\sigma_{k\ell} > 0$), other traders, who assume that all others — including trader i — play equilibrium, would instead attribute the higher price p_k to a larger, on average, realization of endowments for all correlated assets, and expect higher prices and lower trades of those assets. This further increases the price at which they are willing to sell the units of the substitute asset k to trader i. Let $\rho_{k\ell} \equiv Corr(r_k, r_\ell) = \frac{\sigma_{k\ell}}{\sqrt{\sigma_{kk}\sigma_{\ell\ell}}}$.

Comparative statics of price impact.

Theorem 3 (Price Impact: Comparative Statics) Suppose that asset covariances $\sigma_{\ell m}$ are sufficiently symmetric for all ℓ and $m \neq \ell$. The within-exchange price impact λ_k satisfies the

[36] Price p_k affects the conditional expectation separately from endowments $\mathbf{q}^i_0$ provided asset payoffs are not independent: when asset prices are correlated, p_k contains information about endowments of other traders for *all* assets.

Electronic copy available at: https://ssrn.com/abstract=3604976

following properties for each k:

(1) (*Magnitude*) With K assets, price impact λ_k maximally increases K-fold relative to $\lambda_k^c = \frac{\alpha}{I-2}\sigma_{kk}$:

$$\frac{\alpha}{I-2}\sigma_{kk} \leq \lambda_k \leq \frac{\alpha}{I-2}\sum_{\ell}\sigma_{\ell\ell}.$$

The upper bound of the K-fold increase is attained if and only if $|\rho_{\ell m}| = 1$ for all ℓ and $m \neq \ell$.

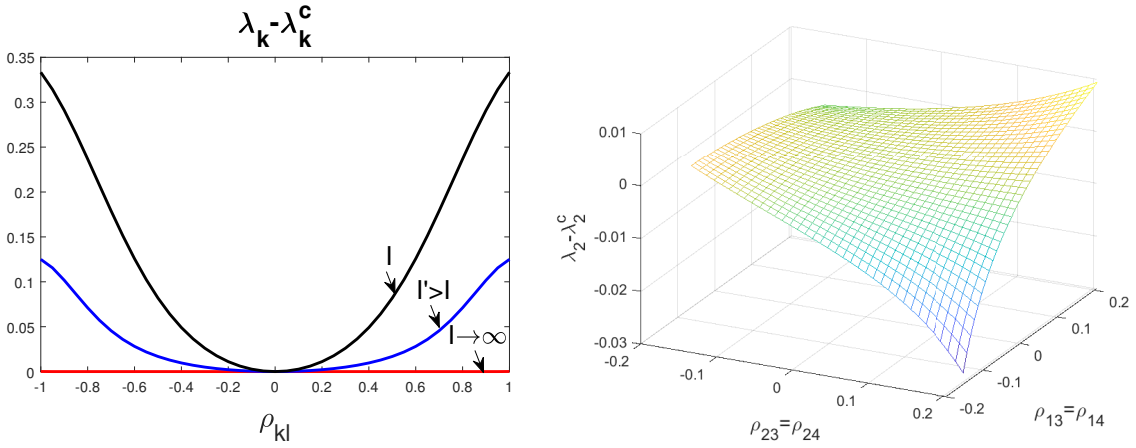
(2) (*Comparative Statics*) Relative to the contingent market:

(i) $\frac{\partial(\lambda_k - \lambda_k^c)}{\partial I} < 0$, i.e., the inference effect is decreasing in the number of traders I;

(ii) $\frac{\partial(\lambda_k - \lambda_k^c)}{\partial|\rho_{\ell m}|} > 0$, i.e., the inference effect is increasing in asset correlation $|\rho_{\ell m}|$ for all ℓ and $m \neq \ell$.

Price impact $\{\lambda_k\}_k$ increases less relative to $\lambda_k^c = \frac{\alpha}{I-2}\sigma_{kk}$ when the inference effect is weaker. This occurs with a larger number of traders I — then, the impact of a trader's demand change on other traders' conditioning statistics (prices) is smaller — and weaker correlations $|\rho|$ — price p_k is less informative about the average endowments of other traders. See Fig. 1. Price impact increases K-fold when the inference is perfect.

FIGURE 1: WITHIN-EXCHANGE PRICE IMPACT: INFERENCE EFFECT



(A) (B)

Notes: Panel A: The price impact difference $\lambda_k - \lambda_k^c$ is determined by the inference effect (Eq. (25)) — the direct effect is the same in the contingent and uncontingent market. The inference effect is larger (in absolute value) when prices are more strongly correlated (i.e., $|\rho|$ is larger) and the number of traders I is smaller. The black, blue and red curves assume, respectively, $I = 5$, $I' = 10$, and $I \to \infty$. Panel B: With heterogeneous correlations, price impact λ_k can be lower than λ_k^c.

Endogenous cross-price elasticity. When asset correlations are *heterogeneous* ($K > 2$), uncontingent trading can *lower* the price impact λ_k for *some* assets relative to contingent trading

Electronic copy available at: https://ssrn.com/abstract=3604976

(Panel B). This result is due to the nonproportionality of price impact to the fundamental covariance, which implies that the asset price covariance is endogenous and differs from the asset covariance $\mathbf{\Sigma}$. In fact, prices of assets that are payoff substitutes (i.e., $\sigma_{k\ell} > 0$) can act as complements (i.e., $Cov(p_k, p_\ell) < 0$): In the price equation (23), $Cov(p_k, p_\ell)$ is determined by

$$\left((\mathbf{C} + \kappa(\alpha\mathbf{\Sigma})^{-1})^{-1}\right)_{k\ell} = \alpha\sigma_{k\ell} - \alpha\mathbf{\Sigma}_k(\alpha\mathbf{\Sigma} + \kappa\mathbf{C}^{-1})^{-1}\alpha\mathbf{\Sigma}_\ell, \tag{26}$$

where $\kappa = \frac{1+(I-2)\sigma_0^2}{(I-1)(1-\sigma_0^2)} \in \mathbb{R}_+$. Because price distribution is determined by the endogenous demand coefficient $\mathbf{C}$, which is not proportional to $(\alpha\mathbf{\Sigma})^{-1}$, we can have $sign(\frac{\partial E[p_\ell|p_k,\mathbf{q}_0^i]}{\partial p_k}) = sign(Cov(p_k, p_\ell)) \neq sign(\sigma_{k\ell})$ for some $\ell \neq k$, and as a result, $\lambda_k < \lambda_k^c$ by Eq. (25). In the contingent market, price covariance matrix matches the asset covariance: substituting $\mathbf{C}^c = (\alpha\mathbf{\Sigma} + \mathbf{\Lambda}^c)^{-1} = \frac{I-2}{I-1}(\alpha\mathbf{\Sigma})^{-1}$ in Eq. (26),

$$\left((\mathbf{C}^c + \kappa(\alpha\mathbf{\Sigma})^{-1})^{-1}\right)_{k\ell} = \frac{(I-1)\kappa}{(I-1)\kappa + (I-2)}\alpha\sigma_{k\ell}.$$

Example 8 in Appendix C.2 shows that if assets k and ℓ are heterogeneously correlated with other assets (e.g., $\mathbf{\Sigma}_k > 0$ and $\mathbf{\Sigma}_\ell < 0$), then prices p_k and p_ℓ can be positively correlated $Cov(p_k, p_\ell) > 0$ even for complementary assets $\sigma_{k\ell} < 0$.[37]

The endogenous substitutability of price distribution creates incentives to for innovation in trading technology, which we explore next.

4 Changes in Market Structure

With multi-venue trading, innovation in trading technology that would be neutral for traders' payoffs with joint clearing are no longer redundant, i.e., traders' equilibrium payoffs change. We present three results about innovation. First, we characterize which innovation — and the changes in market structure it induces — is nonredundant (Theorem 4). Second, we show that markets with multiple exchanges that clear independently can be designed to function like a single exchange for all assets, if suitably designed (Corollary 2). Third, we then examine how market structure affects welfare (Section 5).

To analyze how changes in market structure and innovation in trading technology impact equilibrium and welfare (Sections 4 and 5), we extend the uncontingent model from Section 3.2 to a class of market structures that is more general in two ways: we permit arbitrary restrictions

[37]The proof of Theorem 3 shows that when asset correlations are symmetric, the substitutability of prices $sign(Cov(p_k, p_\ell))$ is the same as that of the asset payoffs $sign(\sigma_{k\ell})$. This ensures that the same counterfactual below Eq. (25) with $K = 2$ applies to markets with $K \geq 2$ assets with symmetric correlations. The inference effect is nonnegative, and thus, $\lambda_k \geq \lambda_k^c = \frac{\alpha}{I-2}\sigma_{kk}$. With heterogeneous correlations, the inference effects induced by a demand change for one asset may increase or lower prices of other assets and be negative.

Electronic copy available at: https://ssrn.com/abstract=3604976

on cross-asset demand conditioning "between" contingent and uncontingent and allow an asset to be traded in multiple venues. Given that all traders participate in all exchanges, we can identify an exchange with a subset of assets traded.

Notation. K denotes the number or the set of all assets; $K(n)$ denotes the number or the set of assets in exchange n.

Definition 4 (Exchanges, Market Structure) *Consider a market with I traders and K assets. An* exchange *n is defined by the subset of assets traded $K(n) \subseteq K$. The* market structure *is described by a set of N exchanges; i.e., $N = \{K(n)\}_n$.*

Exchanges clear independently: trader i submits a demand $q^i_{k,n}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}$ for each asset $k \in K(n)$ contingent on the prices of assets traded in exchange n, $\mathbf{p}_{K(n)} \equiv (p_{\ell,n})_{\ell \in K(n)} \in \mathbb{R}^{K(n)}$. For each exchange n, the market clearing price vector $\mathbf{p}_{K(n)}$ is determined by $\sum_j q^j_{k,n}(\mathbf{p}_{K(n)}) = 0$ jointly for all assets $k \in K(n)$ traded there.

Like in previous sections, the market clears exchange by exchange (but not necessarily asset by asset). The uncontingent market corresponds to K exchanges $N = \{\{k\}\}_k$, and the contingent market corresponds to a single exchange $N = \{K\}$.

We treat the same asset traded in different exchanges as distinct assets with perfectly correlated payoffs. For the fundamentals $\boldsymbol{\delta}, \boldsymbol{\Sigma}$ and $\{\mathbf{q}^i_0\}_i$, which are defined in $\mathbb{R}^K$, the superscript '+' indicates their counterparts in $\mathbb{R}^{\sum_n K(n)}$. Accordingly, the asset payoffs in N exchanges are jointly Normally distributed $\mathcal{N}(\boldsymbol{\delta}^+, \boldsymbol{\Sigma}^+)$, where $\boldsymbol{\delta}^+ \in \mathbb{R}^{\sum_n K(n)}$ and $\boldsymbol{\Sigma}^+ \in \mathbb{R}^{(\sum_n K(n)) \times (\sum_n K(n))}$. Given trader i's endowment $\mathbf{q}^i_0 = (q^i_{0,k})_k \in \mathbb{R}^K$, his endowment in $\mathbb{R}^{\sum_n K(n)}$ can be an arbitrary vector $\mathbf{q}^{i,+}_0 \equiv ((q^{i,+}_{0,k,n})_k)_n \in \mathbb{R}^{\sum_n K(n)}$ such that $q^i_{0,k} = \sum_{\{n|k \in K(n)\}} q^{i,+}_{0,k,n}$ for all k; i.e., the endowment for the same asset can be split in an arbitrary way across exchanges.[38] This is because the trader's demand for each asset depends on his *total* endowment of all assets (Eq. (11)), and hence do prices.

Generalizing from the first-order condition (15) for one asset per exchange, the best-response demand schedule for asset $k \in K(n)$ in exchange n is determined by:

$$\boldsymbol{\delta}^+_{K(n)} - \alpha^i \boldsymbol{\Sigma}^+_{K(n)} E[\mathbf{q}^i + \mathbf{q}^{i,+}_0 | \mathbf{p}_{K(n)}, \mathbf{q}^{i,+}_0] = \mathbf{p}_{K(n)} + \boldsymbol{\Lambda}^i_{K(n)} \mathbf{q}^i_{K(n)} \qquad \forall \mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}, \tag{27}$$

where $\boldsymbol{\Lambda}^i_{K(n)} \in \mathbb{R}^{K(n) \times K(n)}$ is trader i's price impact in exchange n.

To analyze equilibrium in markets with arbitrary demand conditioning (Definition 4) across assets, we need to extend Theorem 1. Theorem 5 in Appendix A characterizes equilibrium and Proposition 7 provides the comparative statics of equilibrium price impact. The proofs of Proposition 2 and Theorem 2 in Appendix A encompass general market structures as well.

[38] The endowments for the same asset in different exchanges are perfectly correlated: $Corr(q^{i,+}_{0,k,n}, q^{i,+}_{0,k,n'}) = 1$ for any n, n' such that $k \in K(n) \cap K(n')$.

Electronic copy available at: https://ssrn.com/abstract=3604976

As with the simpler market structures characterized in Theorem 1, the fixed point in demand schedules is equivalent to a fixed point in traders' price impacts.

We will invoke the following observation.

Lemma 1 (Prices for the Same Asset in Different Exchanges) Given a market structure N, the prices of asset k are the same in all exchanges in which k is traded:

$$p_{k,n} = p_{k,n'} \qquad \forall n,\ \forall n' \neq n \ \ s.t. \ \ k \in K(n) \cap K(n') \qquad \forall (\mathbf{q}_0^i)_i \in \mathbb{R}^{IK}.$$

Although the conditioning variables and price impacts differ across exchanges, prices of the same assets equalize: by market clearing applied to traders' demands exchange by exchange, price is characterized by the average marginal utility[39] and the sum of trades and expected trades for all traders is zero. Given Lemma 1, we dispense with the exchange index n for price $p_{k,n}$.

4.1 Multiple Exchanges: Equivalence with Joint Market Clearing

In this section, we ask: What is the scope for innovation in trading technology that would not be redundant in the market? The intuition for the general result can be seen in the following example.

Example 3 (Multiple Exchanges Can Be Equivalent to Market that Clears Jointly) Consider the market with exchanges $\{\{1,2\},\{2,3\},\{3,1\}\}$. Even though the market is comprised of multiple exchanges none of which contains all assets, traders' equilibrium payoffs are the same as in the market with a single exchange for all assets $\{\{1,2,3\}\}$. *Further linking trading protocols or introducing any additional exchange would be redundant.* □

To compare payoffs, price impacts are not useful, as they are defined for different exchanges and may have different dimensionality. As we show, *price impacts per unit* traded in all exchanges are. The following observation simplifies and illuminates the analysis of nonredundancy of innovation (Theorem 4) and welfare (Section 5). Each trader's payoff (2) is a function of *total trade* of each asset $\widehat{q}_k^i \equiv \sum_{\{n|k\in K(n)\}} q_{k,n}^i$, given price equalization of the same asset across exchanges (Lemma 1). One can thus recast a trader's best-response problem of choosing (net) demands in each exchange $\{\{q_{k,n}^i(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}\}_{k,n}\}_i$ as a problem of choosing total (net) demands $\{\widehat{q}_k^i(\cdot) \equiv \sum_{\{n|k\in K(n)\}} q_{k,n}^i(\cdot)\}_k$. Lemma 6 in Appendix B shows that these two different optimization problems yield the same equilibrium payoffs.[40] Total demand $\widehat{q}_k^i(\cdot)$ is a function

[39] A trader's marginal utility for each asset depends on the total trade of all assets (see Eq. (27)) and is, thus, the same for every exchange where this asset is traded.

[40] A trader's total demand for any asset is split across exchanges given the price impact in these exchanges. The proof of Lemma 6 characterizes how the total demand for each asset is split, given the market structure,

Electronic copy available at: https://ssrn.com/abstract=3604976

of prices of assets that are traded in the exchanges in which asset k is traded, denoted by $\widehat{K}(k) \equiv \cup_{\{n|k\in K(n)\}} K(n)$: for each k,

$$\mathbf{p}_{\widehat{K}(k)} \equiv \big(p_\ell | \ell = k, \text{ or } \exists n \in N \text{ s.t. } \ell \in K(n) \text{ and } k \in K(n)\big) \in \mathbb{R}^{\widehat{K}(k)}.$$

In line with Theorem 1, one can show that the profile of traders' *price impact per unit* of trade is a sufficient statistic for equilibrium in total demands:

$$\widehat{\mathbf{\Lambda}} \equiv \Big(\frac{\partial \widehat{\mathbf{S}}^{-i}(\cdot)}{\partial \mathbf{p}}\Big)^{-1} = -\Big(\sum_{j\neq i} \frac{\partial \widehat{q}^j_k(\cdot)}{\partial p_\ell}\Big)^{-1}_{k,\ell} \in \mathbb{R}^{K\times K}. \tag{28}$$

Theorem 4 then shows that one can compare equilibrium payoffs across market structures, and hence identify nonredundant innovation, through $\widehat{\mathbf{\Lambda}}$.

Theorem 4 (Nonredundancy of Changes in Market Structure) Let $I < \infty$ and $K > 1$ and consider two market structures $N = \{K(n)\}_n$ and $N' = \{K(n')\}_{n'}$. Traders' equilibrium payoffs are the same if one of the following conditions holds:

(i) The price impact per unit is the same: $\widehat{\mathbf{\Lambda}} = \widehat{\mathbf{\Lambda}}'$.

(ii) The expected total trades of other assets are the same: $E[\widehat{q}^i_\ell | \mathbf{p}_{\widehat{K}(k)}, \mathbf{q}^i_0] = E[\widehat{q}^i_\ell | \mathbf{p}_{\widehat{K}'(k)}, \mathbf{q}^i_0]$ for all $\ell \neq k$ and k.

(iii) Imperfectly correlated assets k and $\ell \neq k$ (i.e., $0 < |\rho_{k\ell}| < 1$) are traded in an exchange in N if and only if they are both traded in some exchange in N': $k, \ell \in K(n)$ for some $n \in N$ if and only if $k, \ell \in K(n')$ for some $n' \in N'$.

The price impact per unit $\widehat{\mathbf{\Lambda}}$ corresponds to the unique positive semi-definite matrix such that if the price impact in a market structure that comprises a single exchange for K assets were $\widehat{\mathbf{\Lambda}}$, the expected trade of each asset $k \in K$ would equal the sum of the equilibrium expected trades for the assets that are perfectly correlated with k.[41] Market structures with different conditioning variables, and hence different price impact $\mathbf{\Lambda}$, can have the same price impact per unit $\widehat{\mathbf{\Lambda}}$. Theorem 4 compares market structures through this single-exchange counterfactual.

to characterize the map between the price impact per unit matrix $\widehat{\mathbf{\Lambda}}$ and price impact matrix $\mathbf{\Lambda}$. Example 5 in Section 4.2 shows the map between $\widehat{\mathbf{\Lambda}}$ and $\mathbf{\Lambda}$ when a single asset ($K = 1$) is traded in two exchanges ($N = \{\{1\}, \{1\}\}$).

[41] For all i, k,

$$\sum_{\{n|k\in K(n)\}} E[q^i_{k,n}] = (\alpha \mathbf{\Sigma} + \widehat{\mathbf{\Lambda}})^{-1}_k (\boldsymbol{\delta} - E[\mathbf{p}] - \alpha \mathbf{\Sigma} E[\mathbf{q}^i_0]), \tag{29}$$

where $\sum_{\{n|k\in K(n)\}} q^i_{k,n}$ is the total trade of asset k and $\mathbf{p} = (p_k)_k$ is the price vector in $\mathbb{R}^K$, given Lemma 1. It is clear that part (i) of Theorem 4 does not follow from Theorem 5 (which generalizes Theorem 1, see Appendix A). Price impact $\mathbf{\Lambda}$ and price impact per unit $\widehat{\mathbf{\Lambda}}$ are not one-to-one.

Electronic copy available at: https://ssrn.com/abstract=3604976

Example 1 Cont'd Consider trader i's total demand for asset 1, i.e., the sum of his trade of asset 1 in exchanges $\{1,2\}$ and $\{3,1\}$, i.e., $\mathbf{p}_{\widehat{K}(1)} = (p_1, p_2, p_3)$:

$$\widehat{q}_1^i(p_1,p_2,p_3) \equiv q^i_{1,\{1,2\}}(p_1,p_2) + q^i_{1,\{3,1\}}(p_1,p_3) \qquad \forall (p_1,p_2,p_3) \in \mathbb{R}^3.$$

The total demand $\widehat{q}_1^i(\cdot)$ is a linear function of prices of all assets p_1, p_2, p_3 (by Lemma 1 we can drop the exchange subscript). Each of the prices p_1, p_2 and p_3 is a different linear combination of the random variables in the market, i.e., the average of privately known endowments $\bar{\mathbf{q}}_0 \equiv (\bar{q}_{0,1}, \bar{q}_{0,2}, \bar{q}_{0,3}) \in \mathbb{R}^3$ (Eq. (23)).

Crucially, because the conditioning variables in the total demands for assets 2 and 3 also include prices of all assets, the trader's total demands for all assets are *as if they were contingent on the price vector* (p_1, p_2, p_3). Equilibrium total demands $\{\widehat{\mathbf{q}}^i(\cdot) : \mathbb{R}^3 \to \mathbb{R}^3\}_i$ are equivalent to the contingent demands (Lemma 6). Equilibrium is *ex post* even if in each exchange, the demand conditions only on a subset of prices of all assets (i.e., $K(n) \subsetneq K$ for all n) and so no expectation about trade is perfect: $E[q^i_{\ell,n'} | \mathbf{p}_{K(n)}, \mathbf{q}_0^{i,+}] \neq q^i_{\ell,n'},\ n' \neq n$.[42] □

Corollary 2 gives a necessary and sufficient condition on the market structure $\{K(n)\}_n$ itself for any changes in the market structure to be redundant.

Corollary 2 (Redundancy of Changes in Market Structure: A Condition on Exchanges) Suppose that $0 < |\rho_{k\ell}| < 1$ for some k and $\ell \neq k$. When $I < \infty$, the following statements are equivalent:

(i) Introducing any additional exchange n' is redundant;

(ii) Equilibrium is *ex post*;

(iii) For every pair of assets k' and $\ell' \neq k'$ such that $0 < |\rho_{k'\ell'}| < 1$, there is an exchange n in which these assets are traded, i.e., $k', \ell' \in K(n)$.

The equivalence between conditions (i) and (iii) puts a bound on the number of exchanges that can be introduced in a market or the ways trading protocols can be linked (by merging assets offered by two venues) and be still nonredundant. The maximal number of such innovations is $\frac{K(K-1)}{2}$; introducing additional exchanges or linking trading protocols will not change the number of the contingent variables in any asset's total demand.

The equivalence between conditions (ii) and (iii) answers the following question: Which market structures with multiple exchanges function like the market that clears jointly, given the set of assets, and when does equilibrium behavior differ? For all market structures characterized in condition (iii) of Corollary 2, even if traders' demands in a given exchange are not contingent

[42] With three random variables and three contingent variables, the *total* expected trades of assets 2 and 3 in trader i's total trade of asset 1 are the same as the total *realized* trades.

Electronic copy available at: https://ssrn.com/abstract=3604976

on the prices in other exchanges and exchanges clear independently, equilibrium is *ex post*. Thus, the introduction of sufficiently many exchanges for subsets of assets can implement the contingent-market equilibrium via simpler schedules. Two assets per exchange suffice.

When the market structure satisfies condition (iii), while the cross-exchange price impacts are zero, the price impact per unit matrix $\widehat{\mathbf{\Lambda}}$ is the same as in the contingent market — in particular, it is proportional to the fundamental asset covariance $\mathbf{\Sigma}$. In essence, price effects in a market with multiple exchanges mimic cross-asset price impact with joint market clearing.

4.2 Nonredundant Changes in Market Structure

We now consider market structures that are not payoff-equivalent to joint clearing. Example 4 shows that even then, not all innovations are nonredundant. If joint clearing of all assets is not desired (see Section 5), clearing some assets jointly might be. Theorem 4 implies that one can implement joint clearing for some assets with schedules simpler than contingent ones for those assets and, if linking trading protocols is desired, which links among trading protocols impact welfare. Example 4 illustrates these observations.

Example 4 ((Non)Redundant Changes in the Market Structure) The assets considered in each example are imperfectly correlated, for simplicity.

(a) (*Creation of a new exchange*) Consider the market with two exchanges $N = \{\{1,2,3\},\{4\}\}$. The introduction of exchange $\{1,2\}$ is redundant whereas $\{1,4\}$ is not. The assets in exchange $\{1,2\}$ are a (weak) subset of those in an existing exchange $\{1,2,3\}$. Hence, the prices in the additional exchange $\{1,2\}$ are the same linear combinations of the random variables in the market, i.e., $\bar{\mathbf{q}}_0 \equiv (\bar{q}_{0,k})_k$, as the prices in the existing exchange $\{1,2,3\}$; hence, $\{1,2\}$ does not change total demands (equivalently, per-unit price impact) and thus payoffs (Theorem 4 (ii)). In contrast, adding $\{1,4\}$ allows demands for asset 1 to condition on p_4 which is not a linear combination of $\{p_1,p_2,p_3\}$.

(b) (*Merging assets between exchanges*) In market $N = \{\{1,2\},\{2,3\},\{3,1\},\{4\}\}$, merging exchanges $\{1,2\}$ and $\{2,3\}$ is redundant, whereas $\{1,2\}$ and $\{4\}$ is not. Unlike in Example (a), the assets in the new exchange that is redundant are not a subset of those in an existing exchange. Yet, when $\{1,2\}$ and $\{2,3\}$ are merged, the imperfectly correlated conditioning variables do not change in the total demand for each asset (Theorem 4 (iii)).

Moreover, exchanges $\{1,2\}$ and $\{2,3\}$ implement joint clearing "locally" for assets 1, 2, and 3 — total demands for assets 1, 2, and 3 are equivalent to submitting $\mathbb{R}^3 \to \mathbb{R}^3$ and the outcome in N is the same as in $N' = \{\{1,2,3\},\{4\}\}$.[43] Equilibrium is, however, not *ex post* — demands for

[43] Likewise, in market $N = \{\{1,2,3\},\{2,4\},\{3,4\}\}$ in (c), the introduction of exchange $\{2,3,4\}$ is redundant. Notably, the price impact for the assets that *de facto* clear jointly depends on the exchanges of other assets. For example, the price impact for assets 2, 3, and 4 generally differs in $N = \{\{1,2,3\},\{2,4\},\{3,4\}\}$ $N = \{\{1\},\{2,3\},\{2,4\},\{3,4\}\}$.

Electronic copy available at: https://ssrn.com/abstract=3604976

assets $1, 2, 3$ are not $\mathbb{R}^4 \to \mathbb{R}^4$, thus, violating condition (iii) in Corollary 2 (iii). In contrast to Example 3, the inference error ($E[q^i_\ell | \mathbf{p}_{K(n)}, \mathbf{q}_0^{i,+}] \neq q^i_\ell$ for $\ell \neq k$, $k \in K(n)$) does not cancel out in the total demands for assets $1, 2, 3$; the equivalence with joint clearing implies that the error is the same in the *total* demand for each asset. Likewise, in market $N = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}\}$, merging exchanges $\{2, 4\}$ and $\{3, 4\}$ is redundant.

(c) (*Inclusion of an asset in an exchange where it was not previously traded*) In market $N = \{\{1, 2, 3\}, \{2, 4\}, \{3, 4\}\}$, the inclusion of asset 2 in exchange $\{3, 4\}$ is redundant; the inclusion of asset 1 is not. In the total demands for assets $2, 3, 4$, conditioning variables in N include prices p_2, p_3, p_4; hence, the inclusion of asset 2 in $\{3, 4\}$ does not change the total demand for each asset (Theorem 4 (iii)). The inclusion of asset 1 changes the conditioning variable of the total demand for asset 4 to p_1, p_2, p_3, p_4. In fact, with the inclusion of asset 1 in exchange $\{3, 4\}$ (i.e., $N' = \{\{1, 2, 3\}, \{2, 4\}, \{1, 3, 4\}\}$), equilibrium is *ex post* (Corollary 2 (iii)). □

Theorem 4 implies that any nonredundant innovation maps into the three types of changes in the market structure illustrated in Example 4. When, as in the example, they all increase the set of (imperfectly correlated) conditioning variables in total demands, they lead to a market structure with fewer, the same number, or more exchanges, respectively.[44]

Corollary 2 gives a condition on the market structure for any of these innovations to be redundant. Corollary 3 shows that as long as some assets in the market are *imperfectly* correlated, some innovations will not be redundant.

Corollary 3 (Nonredundancy of Changes in Market Structure: A Condition on Primitives) All market structures $\{K(n)\}_n$ give the same equilibrium payoff if and only if the payoffs of all assets are either perfectly correlated or independent.

Example 5 (Market with one asset) In a market with one asset ($K = 1$) and one exchange, suppose that the second exchange for the same asset (or a perfectly correlated asset, i.e., $\rho^+_{k\ell} = \pm 1$) opens. By Theorem 3, in the unique solution λ, the introduction of the new exchange doubles price impact (see Eq. (135) in Appendix C.1):

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \frac{2\alpha}{I-2} & 0 \\ 0 & \frac{2\alpha}{I-2} \end{bmatrix}.$$

The price impact *per unit* of the perfectly correlated asset is the same as in the contingent market (i.e., $\widehat{\lambda} = \frac{1}{2}\lambda = \frac{\alpha}{I-2}$). Traders' trade is split equally between the two exchanges. □

[44] In light of Theorem 4, the relevant metric for the notion of "more fragmented" or "more centralized" markets is not the number of exchanges but the number of conditioning variables in traders' total demand for each asset.

Electronic copy available at: https://ssrn.com/abstract=3604976

The role of strategic behavior is worth highlighting. For innovation to be nonredundant with price-taking traders, it must reduce the inference error. This contrasts with imperfectly competitive markets, where an innovation can be nonredundant *even when there is no information loss* (i.e., $(\sigma_{cv}^2, \sigma_{pv}^2) \to 0$ and $\sigma_0^2 < 1$ so that $E[\bar{\mathbf{q}}_0|\mathbf{q}_0^i] = \bar{\mathbf{q}}_0$ for all i). Moreover, whereas in a competitive market, the creation of a new exchange or the inclusion of an asset in an existing exchange always weakly increase *ex ante* welfare, nonredundant innovations can either reduce or improve welfare when traders have price impact (see Section 5).

5 Welfare and Separation in Market Clearing

In this section, we consider the welfare implications of separation in market clearing. An important implication of Corollary 2 is that *when combined with suitable exchange design, markets with multiple exchanges that clear independently can be as efficient as a single exchange that clears all assets jointly, for any distributions of asset payoffs and endowments*. The main observation in this section is that markets with multiple exchanges can *strictly* improve *ex ante* welfare relative to the joint market clearing for all assets.

We first ask: Why might a market with multiple trading venues give rise to higher welfare compared to a single exchange for all assets? If the market is competitive ($I \to \infty$), the joint clearing gives higher welfare compared to any other market structure — it eliminates the information loss across exchanges. In imperfectly competitive markets, lower trading costs associated with price impact can countervail the cost due to inference error.

We then ask in which trading environments, welfare is higher with multi-venue trading — unlike the competitive market, efficient design depends on market characteristics. Propositions 3 and 4 and Examples 1 and 6 give and illustrate the corresponding condition.

5.1 Welfare-Improving Designs

By Theorem 4, welfare effects can be understood in terms of the per-unit price impact matrix itself. To identify the welfare gains from multi-venue trading, we characterize the indirect utility as a function of price impact per unit $\widehat{\mathbf{\Lambda}}$ by substituting the equilibrium price $\mathbf{p}$ (Eq. (23)) and trade $\mathbf{q}^i$ (Eq. (24)) into $u^i(\cdot) - \mathbf{p} \cdot \mathbf{q}^i$ (Eq. (2)). Then, the *ex ante* welfare is:

$$\sum_i E[u^i - \mathbf{p} \cdot \mathbf{q}^i] = \underbrace{\sum_i E[\boldsymbol{\delta} \cdot \mathbf{q}_0^i - \frac{1}{2}\mathbf{q}_0^i \cdot \alpha\mathbf{\Sigma}\mathbf{q}_0^i]}_{\text{Welfare without trade}} + \underbrace{\sum_i (E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) \cdot \mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}})(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i])}_{\text{Equilibrium surplus from trade}}$$
$$+ \underbrace{(I-1)\sigma_{pv}^2 tr\Big(\alpha\mathbf{\Sigma}\widehat{\mathbf{B}} - \frac{1}{2}\widehat{\mathbf{B}}'\alpha\mathbf{\Sigma}\widehat{\mathbf{B}}\Big)}_{\text{Welfare term due to } Var(\bar{\mathbf{q}}_0|\mathbf{q}_0^i) > 0}, \tag{30}$$

Electronic copy available at: https://ssrn.com/abstract=3604976

where $\widehat{\mathbf{B}} \equiv \big((\alpha\mathbf{\Sigma} + \widehat{\mathbf{\Lambda}}) - \sigma_0^2(\alpha\mathbf{\Sigma} - (I-2)\widehat{\mathbf{\Lambda}})\big)^{-1}\alpha\mathbf{\Sigma}$ is the coefficient of the total demand of trader i on privately known endowment $\mathbf{q}_0^i$ (cf. Eq. (20)), $tr(\mathbf{M}) \equiv \sum_k m_{kk}$ is the trace of a matrix $\mathbf{M}$ (i.e., the sum of its diagonal elements), and $\mathbf{\Upsilon}(\widehat{\mathbf{\Lambda}}) \in \mathbb{R}^{K\times K}$ represents the marginal payoff per unit of *ex ante* trading needs:[45]

$$\Upsilon(\widehat{\mathbf{\Lambda}}) \equiv \alpha\mathbf{\Sigma}(\alpha\mathbf{\Sigma} + \widehat{\mathbf{\Lambda}})^{-1}(\frac{1}{2}\alpha\mathbf{\Sigma} + \widehat{\mathbf{\Lambda}})(\alpha\mathbf{\Sigma} + \widehat{\mathbf{\Lambda}})^{-1}\alpha\mathbf{\Sigma}.$$

When the market structure changes, the corresponding welfare change can be decomposed into three effects associated with: (1) inference error, (2) price impact for a given asset (i.e., diagonal elements of $\widehat{\mathbf{\Lambda}}$), and (3) cross-asset price impact (i.e., off-diagonal elements of $\widehat{\mathbf{\Lambda}}$). (2) represents the *trading cost of risk-sharing*, whereas (3) captures the *trading costs due to diversification*. Proposition 3 shows that it is important to distinguish between the trading cost components (2) and (3).

Proposition 3 (Price Impact and Market Structure) Fix two market structures N and N' such that if $\{k,\ell\} \subset K(n')$ for some $n' \in N$, $\ell \neq k$, then $\{k,\ell\} \subset K(n)$ for some $n \in N$ and there exists $\{k',\ell'\}$ such that $\{k',\ell'\} \subset K(n)$ for some $n \in N$ but $\{k',\ell'\} \not\subset K(n')$ for all $n' \in N'$.[46] Let $\widehat{\mathbf{\Lambda}}^N$ and $\widehat{\mathbf{\Lambda}}^{N'}$ be the corresponding per-unit price impact matrices.

(i) If $\widehat{\mathbf{\Lambda}}^N$ is proportional to $\mathbf{\Sigma}$ and $K = 2$, then $\widehat{\lambda}_{kk}^N \leq \widehat{\lambda}_{kk}^{N'}$ for all k.

(ii) In general, $\widehat{\lambda}_{kk}^N \not\leq \widehat{\lambda}_{kk}^{N'}$, i.e., the on-diagonal price impact per unit needs not be lower in the market structure N compared to N'.

For two-asset markets, Proposition 3 establishes a trade-off between risk sharing and diversification. Namely, when $K = 2$, joint clearing always minimizes the per-unit on-diagonal price

[45] In the contingent market, by the proportionality of price impact Λ^c to $\mathbf{\Sigma}$, the *ex ante* welfare (30) is:

$$\sum_i E[u^{i,c} - \mathbf{p}^c\cdot\mathbf{q}^{i,c}] = \underbrace{\sum_i E[\boldsymbol{\delta}\cdot\mathbf{q}_0^i - \frac{1}{2}\mathbf{q}_0^i\cdot\alpha\mathbf{\Sigma}\mathbf{q}_0^i]}_{\text{Welfare without trade}} + \underbrace{\sum_i (E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i])\cdot\mathbf{\Upsilon}(\mathbf{\Lambda}^c)(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}_0^i])}_{\text{Equilibrium surplus from trade}} + \underbrace{\frac{I(I-2)}{2(I-1)}\sigma_{pv}^2 tr(\alpha\mathbf{\Sigma})}_{\text{Welfare term due to } Var(\bar{\mathbf{q}}_0|\mathbf{q}_0^i) > 0},$$

where

$$\Upsilon(\mathbf{\Lambda}^c) \equiv \frac{I(I-2)}{2(I-1)^2}\alpha\mathbf{\Sigma}.$$

Relative to the contingent market (i.e., *ex post* equilibrium), the inference error with multiple venues in the last term of $\sum_i E[u^i - \mathbf{p}\cdot\mathbf{q}^i]$ in Eq. (30) is:

$$(I-1)\sigma_{pv}^2 tr\Big(\frac{1}{2}(\widehat{\mathbf{B}} - \frac{I-2}{I-1}\mathrm{Id})'\alpha\mathbf{\Sigma}(\widehat{\mathbf{B}} - \frac{I-2}{I-1}\mathrm{Id}) + \frac{1}{I-1}\alpha\mathbf{\Sigma}(\widehat{\mathbf{B}} - \frac{I-2}{I-1}\mathrm{Id})\Big).$$

In the contingent market, this term is zero since $\mathbf{B}^c = \frac{I-2}{I-1}\mathrm{Id}$.

[46] The condition on k' and ℓ' excludes markets that have the same price impact per unit.

Electronic copy available at: https://ssrn.com/abstract=3604976

impact for each asset — the cost of risk-sharing is lower (part (i)).[47] Nevertheless, multi-venue trading can give strictly higher welfare by lowering the trading costs of diversification due to cross-asset price impact; Example 1 illustrates this. Example 7 in Appendix C.2 provides a complete welfare analysis of two-asset markets.

More generally, in markets with $K > 2$, multi-venue trading might increase welfare by lowering the trading costs due to price impact for a given asset, across assets, or *both* (part (ii); Example 6). Likewise, so can further limiting demand conditioning in markets with multiple exchanges. The key property of equilibrium price impact per unit underlying Proposition 3 is that, in contrast to joint clearing, it need not be proportional to the fundamental covariance in markets with multiple exchanges. The welfare increase with multi-venue trading — relative to joint clearing or more generally — can be accomplished in the Pareto sense.

5.2 When Are Multi-Venue Markets More Efficient?

We further ask: in which trading environments is multi-venue trading more efficient than joint clearing? Proposition 4 translates Proposition 3's price-impact effects on risk sharing and diversification to market characteristics to provide a sufficient condition for multi-venue trading to increase welfare relative to joint clearing.

Proposition 4 (Welfare with Multiple Exchanges vs. Joint Market Clearing) Suppose that there is no information loss: i.e., $(\sigma^2_{cv}, \sigma^2_{pv}) \to 0$ and $\sigma^2_0 < 1$.[48] Given a market structure with multiple exchanges, the *ex ante* welfare is strictly larger than that in a single exchange for all assets if the *ex ante* trading needs $E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]$ are proportional for each trader to an eigenvector of the equilibrium surplus matrix difference $\boldsymbol{\Upsilon}(\boldsymbol{\Lambda}^c) - \boldsymbol{\Upsilon}(\widehat{\boldsymbol{\Lambda}})$ that corresponds to a negative eigenvalue.

Under the sufficient condition — on the *relative trading needs across assets* — one can design exchanges so that the *ex ante* total welfare is strictly larger than in the centralized market. By Theorem 4, this can be accomplished by linking trading protocols, the introduction of new exchanges, or the inclusion in a trading protocol of an asset previously available only in another protocol.[49]

[47] Part (i) of Proposition 3 holds in more general symmetric markets with $K \geq 2$ assets, symmetric covariances (i.e., $\rho_{k,\ell} = \rho$ for all $\ell \neq k$), and market structures defined by partitions of the set of assets into exchanges with the same number of assets. (See Proposition 7 in Appendix B.)

[48] One can then focus on the surplus from trade term in Eq. (30).

[49] The condition is given by the eigenvector of the matrix difference of equilibrium surplus for a given market structure N relative to that in a single exchange for all assets: using (30),

$$(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]) \cdot \big(\boldsymbol{\Upsilon}(\boldsymbol{\Lambda}^c) - \boldsymbol{\Upsilon}(\widehat{\boldsymbol{\Lambda}})\big)(E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]). \tag{31}$$

If $E[\overline{\mathbf{q}}_0] - E[\mathbf{q}^i_0]$ is the eigenvector that corresponds to a negative eigenvalue $\mu < 0$ of the surplus difference matrix

Electronic copy available at: https://ssrn.com/abstract=3604976

Intuitively, the sufficient condition captures that the trading needs across assets are such that the net welfare benefit from diversification and/or from risk sharing exceeds the welfare cost due to worsened risk sharing for other assets, if any. We highlight three implications of Proposition 4.

First, a market structure with multiple venues is more efficient than joint clearing for some distributions of endowments. In fact, given any market structure with multiple exchanges in which $\boldsymbol{\Upsilon}(\boldsymbol{\Lambda}^c) - \boldsymbol{\Upsilon}(\widehat{\boldsymbol{\Lambda}})$ has a negative eigenvalue, *ex ante* welfare is strictly larger than that in a single exchange for all assets for some distribution of endowments (i.e., $\{E[\mathbf{q}_0^i]\}_i$ and $Var((\mathbf{q}_0^i)_i)$) or a supply vector.

For some market structures, the equilibrium surplus matrix difference $\boldsymbol{\Upsilon}(\boldsymbol{\Lambda}^c) - \boldsymbol{\Upsilon}(\widehat{\boldsymbol{\Lambda}})$ may not have a negative eigenvalue, in which case the joint market clearing is more efficient for any distribution of endowments. Lemma 7 in Appendix B shows that $\boldsymbol{\Upsilon}(\boldsymbol{\Lambda}^c) - \boldsymbol{\Upsilon}(\widehat{\boldsymbol{\Lambda}})$ has a negative eigenvalue for any market structure whose exchanges partition the K assets traded (i.e., $K(n) \cap K(n') = \emptyset$ for all n and $n' \neq n$), except when N implements equilibrium with joint market clearing (i.e., $\boldsymbol{\Upsilon}(\boldsymbol{\Lambda}^c) - \boldsymbol{\Upsilon}(\widehat{\boldsymbol{\Lambda}}) = 0$). Thus, under the condition on the relative trading needs, simply breaking up a single exchange for all assets into multiple venues can increase welfare.

Second, which exchanges should be introduced and demands of which assets should be linked depends on the joint substitutability in asset payoffs (i.e., $\boldsymbol{\Sigma}$) — by Proposition 3, and the trading needs of market participants *across assets* (i.e., $\{|E[\bar{q}_{0,k}] - E[q_{0,k}^i]|\}_{i,k}$) — by Proposition 4. In particular, given the asset payoffs' substitutability, the efficient design depends on whether the market is "one-sided" (i.e., traders want to buy or sell all assets, e.g., the primary market in Treasury auctions) or traders buy some assets and sell others (e.g., intra-dealer markets).[50]

Third, by Proposition 4, the *heterogeneity* in the asset payoffs' substitution $\boldsymbol{\Sigma}$ as well as trading needs across assets $\{E[\bar{\mathbf{q}}_0] - E[\mathbf{q}_0^i]\}_i$ determines whether the net benefit from diversification and risk sharing with multi-venue trading dominates. For $K > 2$, the heterogeneity in trading needs and asset covariances can favor a market structure "intermediate" between contingent or uncontingent. In two-asset markets, Propositions 3 and 4 imply a necessary and sufficient condition on $\boldsymbol{\Sigma}$ and $\{E[\bar{\mathbf{q}}_0] - E[\mathbf{q}_0^i]\}_i$ for multi-venue trading dominate joint clearing in welfare terms.[51]

$\boldsymbol{\Upsilon}(\boldsymbol{\Lambda}^c) - \boldsymbol{\Upsilon}(\widehat{\boldsymbol{\Lambda}})$, then the difference $(E[\bar{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) \cdot \big(\boldsymbol{\Upsilon}(\boldsymbol{\Lambda}^c) - \boldsymbol{\Upsilon}(\widehat{\boldsymbol{\Lambda}})\big)(E[\bar{\mathbf{q}}_0] - E[\mathbf{q}_0^i]) = \mu \|E[\bar{\mathbf{q}}_0] - E[\mathbf{q}_0^i]\|^2 < 0$, and thus, the welfare is higher. The proportionality constant is positive for some traders and negative for others, and can be zero for some but not all traders.

[50] Example 1 above illustrates that in markets with $K = 2$, the zero cross-asset price impact is beneficial when traders take the same (buying or selling) position for assets with payoff substitutes or the opposite position for assets with payoff complements.

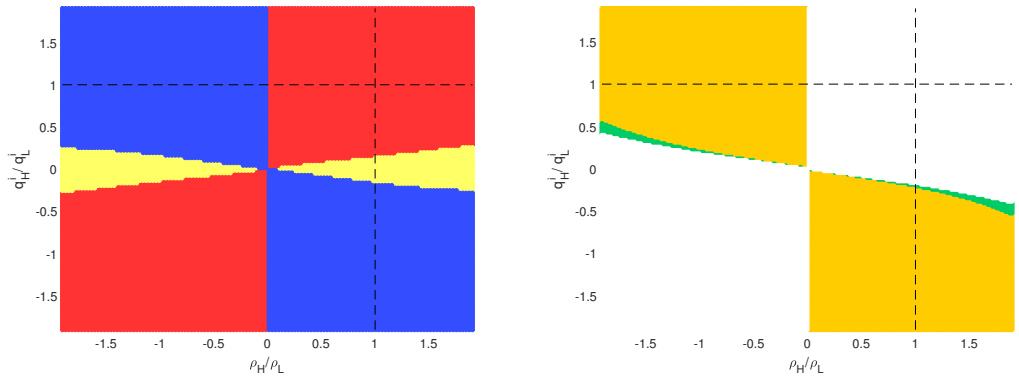
[51] When $K = 2$, the eigenvector condition in Proposition 4 captures that with sufficiently symmetric trading needs across assets, diversification benefit dominates the risk sharing cost. Example 7 in Appendix C.2 completely characterizes the condition for $K = 2$.

Electronic copy available at: https://ssrn.com/abstract=3604976

Example 6 illustrates these observations. Example 8 in Appendix C.2 illustrates how, when $K > 2$, the heterogeneity in asset payoffs matters for welfare benefit from risk sharing, thus providing a counterexample for Proposition 3.

Example 6 (Heterogeneity in Trading Needs And Efficient Market Structure) Consider a market with $K = 3$ assets two of which, 2 and 3, are symmetric with respect to correlations and *ex ante* trading needs. There are six payoff-relevant market structures, including the contingent and the uncontingent ones. Fig. 2 plots the welfare-maximizing design as a function of the heterogeneity in asset correlations $\rho_{12}/\rho_{23} = \rho_{13}/\rho_{23} \equiv \rho_H/\rho_L$ — on the horizontal axis — and the heterogeneity in trading needs $(E[\bar{q}_{0,1}] - E[q^i_{0,1}])/(E[\bar{q}_{0,2}] - E[q^i_{0,2}]) = (E[\bar{q}_{0,1}] - E[q^i_{0,1}])/(E[\bar{q}_{0,3}] - E[q^i_{0,3}]) \equiv q^i_H/q^i_L$ — on the vertical axis.

FIGURE 2: HETEROGENEOUS ASSET CORRELATIONS AND TRADING NEEDS



(A) $\rho_L = 0.25$ (B) $\rho_L = -0.25$

Notes: Each color indicates which market structure provides the highest *ex ante* welfare. Red $= \{\{1\},\{2\},\{3\}\}$ (i.e., the uncontingent market); Orange $= \{\{1\},\{2,3\}\}$; Yellow $= \{\{1,2\},\{3\}\}$; Green $= \{\{1,2\},\{2,3\}\}$; Blue $= \{\{1,2\},\{1,3\}\}$; and White $= \{\{1,2,3\}\}$ (i.e., the contingent market). Information loss is sufficiently small not to dominate the welfare benefit from diversification ($\sigma^2_{cv} = 0, \sigma^2_{pv} = 0.01$). The number of traders is $I = 12$. The trading needs for assets 2 and 3 are $|E[\bar{q}_{0,L}] - E[q^i_{0,L}]| = 1$ for all i. Panel (A) assumes the asset payoff correlation $\rho_L = 0.25$ (i.e., substitutes) and panel (B) assumes $\rho_L = -0.25$ (i.e., complements).

- If the asset correlations and trading needs are symmetric across assets (i.e., $\rho_H = \rho_L$ and $q^i_H = q^i_L$), either the contingent or the uncontingent market structure is efficient.

- In one-sided markets (i.e., when traders either buy or sell all assets), the efficient market structure depends on the asset payoff substitutability:

  If asset payoffs are complements (i.e., $\rho_H < 0$ and $\rho_L < 0$; Fig. 2 (B)) and traders buy both assets (i.e., $q^i_H > 0$ and $q^i_L > 0$), then the contingent market is efficient, irrespective of the heterogeneity in $\{\rho_{k\ell}\}_{k,\ell}$ and $\{E[\bar{q}_{0,k}] - E[q^i_{0,k}]\}_k$.

Electronic copy available at: https://ssrn.com/abstract=3604976

If asset payoffs are substitutes (i.e., $\rho_H > 0$ and $\rho_L > 0$; Fig. 2 (A)), then the heterogeneity in trading needs matters. If trading needs are symmetric, the uncontingent market is efficient. With sufficiently heterogeneous trading needs, a market structure other than the contingent or uncontingent maximizes welfare.

- The efficient market structures in which some but not all assets clear jointly either link assets to reduce the trading cost of diversification for those assets (orange and blue areas) or link assets with the most heterogeneous trading needs to balance the tradeoff between risk sharing and diversification — even when linking the assets increases the trading costs due to diversification (yellow and green areas).[52] □

6 Discussion

This paper introduces a model with limited demand conditioning, taking a step towards exploring the implications of separation in market clearing. Our framework and equilibrium characterization as a fixed point in price impacts can be adapted to allow the study of the effects of other innovations in market structure. In Rostek and Yoon (2018), we study one such class: derivatives, i.e., securities whose payoffs are defined as bundles (linear combinations) of the existing assets. We investigate the distinct effects that derivatives have on welfare compared to new exchanges or merging trading protocols. Thus, exchange design studied in this paper and security innovation provide separate instruments for impacting markets' performance.

The standard equilibrium model of asset pricing (competitive or imperfectly competitive) assumes contingent schedules. This paper identifies market characteristics which — unlike contingent trading — affect the equilibrium price distribution and diversification. It would be worthwhile to further develop the asset-pricing implications of separation in market clearing and the equilibrium properties it induces, such as the nonproportionality and asymmetry of price impact. Another important direction would be exploring how the non-neutrality of innovation in trading technology can be leveraged to enhance revenue or other objectives.

Separation of market clearing: implementation. If traders themselves could decide whether to submit uncontingent or contingent schedules, individual optimization entails that a contingent schedule would be a best response, taking as given the schedules submitted by others. Submission of contingent schedules by all traders would be the unique equilibrium, eliminating the welfare improving (possibly in the Pareto sense) effects of limited conditioning. Thus, implementation of uncontingent trading involves a restriction of the cross-exchange conditioning. This is the prevalent practice.

[52] Whether linking trading protocols of some assets is efficient depends on the trading needs and the payoff substitutability *of all assets* — a result of the non-proportionality in price impact to asset covariance in markets other than the contingent one.

Electronic copy available at: https://ssrn.com/abstract=3604976

Role of the uniform-price mechanism. One might wonder whether the uniform-price mechanism matters for the conclusions. The key to the welfare effects of new exchanges is the inefficiency of equilibrium allocation due to price impact, which new exchanges alter, given incomplete conditioning. With (two-sided) private information, one expects allocation to be inefficient and the effects of new exchanges to exist for other pricing mechanisms.

Heterogeneous participation.[53] Decentralizing a market by allowing some traders to participate in exchanges for only a subset of all assets or trade with only a subset of all traders (while submitting contingent schedules) can improve *allocation of risk across traders* provided traders' risk preferences differ; with symmetric risk preferences, the centralized market maximizes welfare (Malamud and Rostek (2017)). Our results suggest that restrictions on conditioning can provide an effective instrument to facilitate efficient allocation (*risk sharing across traders and risk diversification across assets*) in ways not feasible with heterogeneous participation or contingent trading.

Correlated endowments across assets. We have assumed that endowments are independent across assets. Our equilibrium characterization in Appendix A allows for correlated endowments across assets. The (non)neutrality of the changes in the market structure continues to hold, so long as endowments are not perfectly correlated across assets and are symmetrically correlated across traders.

References

Afonso, G. and R. Lagos (2015): "Trade Dynamics in the Market for Federal Funds," Econometrica, 83, 1: 263-313.

Antill, S. and D. Duffie (2017): "Augmenting Markets with Mechanisms," Working Paper.

Atkeson, A. G., A. L. Eisfeldt, and P.-O. Weill (2015): "Entry and Exit in OTC Derivatives Markets," Econometrica 83: 2231-2292.

Babus, A. and P. Kondor (2017): "Trading and Information Diffusion in Over-the-Counter Markets," Working Paper.

Babus, A. and C. Parlatore (2017): "Strategic Fragmented Markets," Working Paper.

Bergemann, D., T. Heumann, and S. Morris (2015): "Information and Market Power," Cowles Foundation Discussion Paper No. 2017.

Budish, E., R. Lee, and J. Shim (2019): "A Theory of Stock Exchange Competition and Innovation: Will the Market Fix the Market?," Working Paper.

[53] Babus and Kondor (2017), Babus and Parlatore (2017), and Malamud and Rostek (2017) study markets with limited participation and contingent contracts.

Electronic copy available at: https://ssrn.com/abstract=3604976

Chang, B. and S. Zhang (2018): "Endogenous Market Making and Network Formation," Working Paper.

Chen, D. and D. Duffie (2020): "Market Fragmentation," Working Paper.

Choi, M., A. Galeotti, and S. Goyal (2017): "Trading in Networks: Theory and Experiment," Journal of the European Economic Association, 15, 4: 784-817.

Collard-Wexler, A., G. Gowrisankaran, and R. S. Lee (2019): ""Nash-in-Nash" Bargaining: A Microfoundation for Applied Work," Journal of Political Economy, 127, 1: 163-195.

Condorelli , D., A. Galeotti, and L. Renou (2017): "Bilateral Trading in Networks," Review of Economic Studies, 84, 1: 82-105.

Duffie, D. (2012): Dark Markets, Princeton Lectures in Finance, Y. Aït-Sahalia (ed.), Princeton University Press.

Duffie, D. (2018): "Notes on LIBOR Conversion," Working Paper.

Duffie, D., N. Garleanu, and L. H. Pedersen (2005): "Over-the-Counter Markets," Econometrica, 73, 6: 1815-1847.

Du, S. and H. Zhu (2017a): "Bilateral Trading in Divisible Double Auctions," Journal of Economic Theory, 167: 285-311.

Du, S. and H. Zhu (2017b): "What is the Optimal Trading Frequency in Financial Markets?," Review of Economic Studies, 84, 4: 1606-1651.

Elliott, M. (2015): "Inefficiencies in Networked Markets," American Economic Journal: Microeconomics, 7, 4: 43-82.

The Financial Stability Oversight Council (2017): "The Annual Report, 2017," https://www.treasury.gov/initiatives/fsoc/studies-reports.

Gale, D. (1986a):"Bargaining and Competition Part I: Characterization," Econometrica, 54, 4: 785-806.

Gale, D. (1986b):"Bargaining and Competition Part II: Existence," Econometrica, 54, 4: 807-818.

Garleanu, N. and L. Pedersen (2013): "Dynamic Trading with Predictable Returns and Transaction Costs," Journal of Finance, 68, 6: 2309-2340.

Glode, V. and C. Opp (2016): "Asymmetric Information and Intermediation Chains," American Economic Review, 106, 9: 2699-2721.

Gofman, M. (2014): "A Network-Based Analysis of Over-the-Counter Markets," Working Paper.

Horn, H., and A. Wolinsky (1988): "Bilateral Monopolies and Incentives for Merger," The RAND Journal of Economics, 19, 3, 408-419.

Hugonnier, J., B. Lester and P.-O. Weill (2017): "Heterogeneity in Decentralized Asset Markets," Working Paper.

Klemperer, P. and M. Meyer (1989): "Supply Function Equilibria in Oligopoly under Uncertainty," Econometrica, 57, 6: 1243-1277.

Knight Capital Group (2010): Current Perspectives on Modern Equity Markets: A Collection of Essays by Financial Industry Experts: Knight Capital Group, Inc.

Electronic copy available at: https://ssrn.com/abstract=3604976

Kranton, R. E. and D. F. Minehart (2001): "A Theory of Buyer-Seller Networks," American Economic Review, 91, 3: 485-508.

Kyle, A. S. (1989): "Informed Speculation and Imperfect Competition," Review of Economic Studies, 56, 3: 517-556.

Kyle, A. S. and J. Lee (2017): "Information and Competition with Speculation and Hedging," Working Paper.

Kyle, A. S., A. A. Obizhaeva, and Y. Wang (2017): "Smooth Trading with Overconfidence and Market Power," Review of Economic Studies, 85, 1: 611-662.

Lambert, N., M. Ostrovsky, and M. Panov (2018): "Strategic Trading in Informationally Complex Environments," Econometrica, 86, 4: 1119-1157.

Lester, B., A. Shourideh, V. Venkateswaran, and A. Zetlin-Jones (2018a): "Screening and Adverse Selection in a Frictional Market" Journal of Political Economy, forthcoming.

Lester, B., A. Shourideh, V. Venkateswaran, and A. Zetlin-Jones (2018b): "Market-making with Search and Information Frictions," Working Paper.

Malamud, S. and M. Rostek (2017): "Decentralized Exchange," American Economic Review, 107, 11: 3320-3362.

Peivandi, A. and R. Vohra (2017): "On Fragmentation of Markets," Working Paper.

Rostek, M. and M. Weretka (2015): "Dynamic Thin Markets," Review of Financial Studies, 28, 10: 2946-2992.

Rostek, M. and J. H. Yoon (2018): "Financial Innovation in Decentralized Markets," Working Paper.

Rostek, M. and J. H. Yoon (2019): "Dynamic Imperfectly Competitive Markets with Private Information," Working Paper.

Sannikov, Y. and A. Skrzypacz (2016): "Dynamic Trading: Price Inertia and Front-Running," Working Paper.

Vayanos D. (1999): "Strategic Trading and Welfare in a Dynamic Market," Review of Economic Studies, 66, 2: 219-54.

Weill, P.-O. and D. Vayanos (2008): "A Search-based Theory of the On-the-run Phenomenon," Journal of Finance, 63, 3: 1351-1389.

Vives, X. (2008): Information and Learning in Markets: The Impact of Market Microstructure. Princeton: Princeton University Press.

Vives, X. (2011): "Strategic Supply Function Competition with Private Information," Econometrica, 79, 6: 1919-66.

Wilson, R. (1979): "Auctions of Shares," Quarterly Journal of Economics, 93, 4: 675-689.

Wittwer, M. (2019): "Connecting Disconnected Markets? An Irrelevance Result," Working Paper.

Zhu, H. (2014): "Do Dark Pools Harm Price Discovery?" Review of Financial Studies, 27, 3: 747-789.

Zhu, H. (2018a): "A multi-maturity clock auction design," Market Design Proposal.

Zhu, H. (2018b): "Design of CCP Default Management Auctions," Market Design Proposal.

Electronic copy available at: https://ssrn.com/abstract=3604976

Appendix

Appendix A: Equilibrium Characterization (Proofs of Proposition 2 and Theorems 1, 2, and 5).

Appendix B: All Other Proofs (Supplementary Material).

Appendix C: Additional Examples and Results (Supplementary Material).

A Equilibrium Characterization

Theorem 5 characterizes equilibrium for general market structures (Definition 4). In a market with exchanges $N = \{K(n)\}_n$, let us define an operator $[\cdot]_N : \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))} \to \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$ maps a matrix $\mathbf{M}$ to a block-diagonal matrix $[\mathbf{M}]_N$ with $\big([\mathbf{M}]_N\big)_{K(n),K(n')} \equiv 0$ for $n \neq n'$ and $\big([\mathbf{M}]_N\big)_{K(n),K(n)} \equiv \mathbf{M}_{K(n),K(n)}$ for any n. Furthermore, we allow correlated endowments across assets: $\mathbf{\Omega} = \big(Cov(q^i_{0,k}, q^i_{0,\ell})\big)_{k,\ell} \in \mathbb{R}^{K\times K}$ is a positive semidefinite matrix. The distribution of the asset returns in N exchanges is jointly normal, $\mathcal{N}(\boldsymbol{\delta}^+, \mathbf{\Sigma}^+)$, where $\boldsymbol{\delta}^+ \in \mathbb{R}^{\sum_n K(n)}$ and $\mathbf{\Sigma}^+ \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$. Given trader i's endowment vector for K assets $\mathbf{q}^i_0 \in \mathbb{R}^K$, his endowment in N exchanges is an arbitrary vector $\mathbf{q}^{i,+}_0 \in \mathbb{R}^{\sum_n K(n)}$ such that $q^i_{0,k} = \sum_{\{n|k\in K(n)\}} q^{i,+}_{0,k,n}$ for all k. The covariance matrix of $\mathbf{q}^{i,+}_0$ is denoted by $\mathbf{\Omega}^+ \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$ for each i.

Theorem 5 (Equilibrium: Fixed Point in Demand Schedules (General Design)) Consider a market with N exchanges; assets $K(n) \subseteq K$ are traded in exchange $n \in N$. In equilibrium, the (net) demand schedules, defined by matrix coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}_i$, and price impacts $\{\mathbf{\Lambda}^i\}_i$ are characterized by the following conditions: for each i,

(i) (*Optimization, given price impact* $\mathbf{\Lambda}^i$) Given price impact matrices $\mathbf{\Lambda}^i \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$, best response coefficients $\boldsymbol{a}^i \in \mathbb{R}^{\sum_n K(n)}$, $\mathbf{B}^i \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$, and $\mathbf{C}^i \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}$ are characterized by:

$$\begin{aligned}\boldsymbol{a}^i = \mathbf{C}^i\boldsymbol{\delta}^+ + \big(\mathbf{B}^i - (\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)^{-1}\alpha^i\mathbf{\Sigma}^+\big)E[\mathbf{q}^{i,+}_0] \\ -(\mathbf{C}^i - (\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)^{-1})\underbrace{(\sum_j(\alpha^j\mathbf{\Sigma}^+ + \mathbf{\Lambda}^j)^{-1})^{-1}\sum_j(\alpha^j\mathbf{\Sigma}^+ + \mathbf{\Lambda}^j)^{-1}\alpha^j\mathbf{\Sigma}^+E[\mathbf{q}^{j,+}_0]}_{=\boldsymbol{\delta}-E[\mathbf{p}]},\end{aligned} \tag{32}$$

$$\mathbf{B}^i = (\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)^{-1}\alpha^i\mathbf{\Sigma}^+ - \underbrace{((\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)^{-1} - \mathbf{C}^i)(\sum_j \mathbf{C}^j)^{-1}\Big(\mathbf{B}^i + \frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}(\sum_{j\neq i}\mathbf{B}^j)\Big)}_{\text{Adjustment due to cross-asset inference}}, \tag{33}$$

$$\Big[(\mathrm{Id} - (\alpha^i\mathbf{\Sigma}^+ + \mathbf{\Lambda}^i)\mathbf{C}^i)\underbrace{(\sum_j \mathbf{C}^j)^{-1}\Big(\sum_{j\neq i}\mathbf{B}^j\mathbf{\Omega}^+\big(\mathbf{B}^j + \frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}\sum_{h\neq i}\mathbf{B}^h\big)'\Big)}_{\text{Inference coefficient } \frac{Cov(p_\ell, s_k^{-i}|\mathbf{q}^i_0)}{Var(s_k^{-i}|\mathbf{q}^i_0)}}\Big]_N = 0. \tag{34}$$

Electronic copy available at: https://ssrn.com/abstract=3604976

(ii) (*Correct price impacts*) Price impact $\mathbf{\Lambda}^i$ equals the slope of trader i's inverse residual supply:

$$\mathbf{\Lambda}^i = (\sum_{j \neq i} \mathbf{C}^j)^{-1} \qquad \forall i. \tag{35}$$

Note. With one asset per exchange (i.e., $N = \{\{k\}\}_k$), the statement and proof of Theorem 5 specialize to those of Theorem 1.[54]

Proof of Proposition 2 (Equilibrium: Uncontingent Trading). Consider a market with exchanges $N = \{K(n)\}_n$.

(Part If) Given the market structure N, we show that equilibrium is characterized by two conditions: (i) trader i chooses demand schedules $q^i_{k,n}(\cdot) : \mathbb{R}^{K(n)} \to \mathbb{R}$ for each asset $k \in K(n)$ in exchange n to maximize the expected payoff

$$E\big[\boldsymbol{\delta}^+ \cdot (\mathbf{q}^i + \mathbf{q}_0^{i,+}) - \frac{\alpha^i}{2}(\mathbf{q}^i + \mathbf{q}_0^{i,+}) \cdot \mathbf{\Sigma}^+ (\mathbf{q}^i + \mathbf{q}_0^{i,+}) - \mathbf{p} \cdot \mathbf{q}^i \big| \mathbf{q}_0^{i,+}\big],$$

given his residual supply functions $S^{-i}_{k,n}(\cdot; \{\mathbf{q}_0^{j,+}\}_j) \equiv -\sum_{j \neq i} q^j_{k,n}(\cdot; \mathbf{q}_0^{j,+}) : \mathbb{R}^{K(n)} \to \mathbb{R}$ for all k and n, (ii) which is correct. We denote the intercept of the residual supply by $s^{-i}_{k,n} \equiv S^{-i}_{k,n}(0; \{\mathbf{q}_0^{j,+}\}_j) \in \mathbb{R}$ for each k in exchange n. For each i, the vector of the residual supply intercepts is $\mathbf{s}^{-i} \equiv ((s^{-i}_{k,n})_k)_n \in \mathbb{R}^{\sum_n K(n)}$ and its conditional distribution is $F(\mathbf{s}^{-i} | \mathbf{q}_0^{i,+})$.

Step 1 (Part (i): Optimization, given residual supply $\mathbf{\Lambda}^i$ and $F(\mathbf{s}^{-i}|\mathbf{q}_0^{i,+})$) Assume that the price vector $\mathbf{p}_{K(n)} \equiv (p_{k,n})_{k \in K(n)} \in \mathbb{R}^{K(n)}$ in exchange n has full support, as we will show later. The first-order condition of trader i for asset $k \in K(n)$ is:

$$\delta_k - \alpha^i \mathbf{\Sigma}^+_k E[\mathbf{q}^i + \mathbf{q}_0^{i,+} | \mathbf{p}_{K(n)}, \mathbf{q}_0^{i,+}] = p_{k,n} + \boldsymbol{\lambda}^i_{k,K(n)} \mathbf{q}^i_{K(n)} \qquad \forall \mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}, \tag{36}$$

given his own demands in other exchanges $\mathbf{q}^i_{K(n')}(\cdot) = (q^i_{\ell,n'}(\cdot))_\ell : \mathbb{R}^{K(n')} \to \mathbb{R}^{K(n')}$ for all $n' \neq n$, the distribution of his residual supply intercepts $F(\mathbf{s}^{-i}|\mathbf{q}_0^{i,+})$, and his price impact $\mathbf{\Lambda}^i$.

Eq. (36) becomes Eq. (15) if the market is uncontingent, i.e. $N = \{\{k\}\}_k$. In matrix form, the first-order condition (36) of trader i in each exchange n can be written as a single matrix equation:

$$\boldsymbol{\delta}^+_{K(n)} - \alpha^i \mathbf{\Sigma}^+_{K(n)} E[\mathbf{q}^i + \mathbf{q}_0^{i,+} | \mathbf{p}_{K(n)}, \mathbf{q}_0^{i,+}] = \mathbf{p}_{K(n)} + \mathbf{\Lambda}^i_{K(n)} \mathbf{q}^i_{K(n)} \quad \forall \mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)} \quad \forall n \in N, \tag{37}$$

where $\mathbf{\Lambda}^i_{K(n)} \equiv \frac{d\mathbf{p}_{K(n)}}{d\mathbf{q}^i_{K(n)}} \in \mathbb{R}^{K(n) \times K(n)}$ is his price impact in exchange n. In the quadratic-Gaussian setting, $\mathbf{\Lambda}^i_{K(n)}$ is a constant matrix for all n, given that other traders $j \neq i$ submit linear schedules. The second-order condition is $-\alpha^i \mathbf{\Sigma}^+_{K(n)} - 2\mathbf{\Lambda}^i_{K(n)} < 0$ for each n.

Due to the presence of expected trades $E[\mathbf{q}^i | \mathbf{p}_{K(n)}, \mathbf{q}_0^{i,+}]$ in the first-order condition (37), characterizing the best-response demands of trader i, $\{\mathbf{q}^i_{K(n)}(\cdot)\}_n$, requires solving the fixed point for the trader's demand coefficients (on the contingent variables and private information) across exchanges.

[54] Equilibrium is characterized by the same fixed point equations in matrix form as in Theorem 5 except that the operator $[\cdot]_d$ replaces $[\cdot]_N$ in the demand coefficients.

Electronic copy available at: https://ssrn.com/abstract=3604976

Step 1.1 (Parameterization of Demands in Other Exchanges $n' \neq n$) To characterize the best response demand of trader i in exchange n, take as given his demands in other exchanges $\{\mathbf{q}^i_{K(n')}(\cdot)\}_{n'\neq n}$. We parameterize these demands as linear functions: for each $n' \neq n$,

$$\mathbf{q}^i_{K(n')}(\mathbf{p}_{K(n')}) = \boldsymbol{a}^i_{K(n')} - \mathbf{B}^i_{K(n')}\mathbf{q}^{i,+}_0 - \mathbf{C}^i_{K(n')}\mathbf{p}_{K(n')}, \qquad \forall \mathbf{p}_{K(n')} \in \mathbb{R}^{K(n')} \tag{38}$$

where $\boldsymbol{a}^i_{K(n')} \in \mathbb{R}^{K(n')}, \mathbf{B}^i_{K(n')} \in \mathbb{R}^{K(n')\times(\sum_n K(n))}$, and $\mathbf{C}^i_{K(n')} \in \mathbb{R}^{K(n')\times K(n')}$.

Step 1.2 (Expected Trades, given $F(\mathbf{s}^{-i}|\mathbf{q}^{i,+}_0)$) To endogenize the expected trades in trader i's demands in exchanges n, we characterize the distributions of prices $\mathbf{p}_{K(n')}$ and his trades $\{q^i_{\ell,n'}(\mathbf{p}_{K(n')})\}_{\ell\in K(n')}$ in other exchanges $n' \neq n$, using the parameterized demands (38).

Applying market clearing to trader i's demands (38) and his residual supplies $\mathbf{S}^{-i}_{K(n')}(\cdot)$ determines the price vector $\mathbf{p}_{K(n')}$: for each n',

$$\boldsymbol{a}^i_{K(n')} - \mathbf{B}^i_{K(n')}\mathbf{q}^{i,+}_0 - \mathbf{C}^i_{K(n')}\mathbf{p}_{K(n')} = \mathbf{s}^{-i}_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1}\mathbf{p}_{K(n')} \qquad \forall \mathbf{s}^{-i}_{K(n')} \in \mathbb{R}^{K(n')}. \tag{39}$$

Hence, in each exchange $n' \neq n$, price $\mathbf{p}_{K(n')}$ maps one-to-one to the residual supply intercept $\mathbf{s}^{-i}_{K(n')}$ as a function of $\boldsymbol{a}^i_{K(n')}, \mathbf{B}^i_{K(n')}, \mathbf{C}^i_{K(n')}$, and $\boldsymbol{\Lambda}^i_{K(n')}$. One can treat $\mathbf{s}^{-i}_{K(n')}$ as a contingent variable (rather than $\mathbf{p}_{K(n')}$) in trader i's demands in exchange n':

$$\mathbf{q}^i_{K(n')}(\mathbf{s}^i_{K(n')}) = (\boldsymbol{\Lambda}^i_{K(n')})^{-1}(\mathbf{C}^i_{K(n')}+(\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1}(\boldsymbol{a}^i_{K(n')}-\mathbf{B}^i_{K(n')}\mathbf{q}^{i,+}_0)+\mathbf{C}^i_{K(n')}(\mathbf{C}^i_{K(n')}+(\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1}\mathbf{s}^{-i}_{K(n')}. \tag{40}$$

The distribution of trades $\mathbf{q}^i_{K(n')}$ is characterized by Eq. (40), given $F(\mathbf{s}^{-i}|\mathbf{q}^{i,+}_0)$ and $\boldsymbol{a}^i_{K(n')}, \mathbf{B}^i_{K(n')}$, and $\mathbf{C}^i_{K(n')}$, and is denoted by $F(\mathbf{q}^i_{K(n')}|\mathbf{q}^{i,+}_0)$.

Given trader i's demands in other exchanges $n' \neq n$ (Eq. (40)), the expected trades $E[\mathbf{q}^i|\mathbf{p}_{K(n)}, \mathbf{q}^{i,+}_0]$ in the first-order condition (37) are characterized conditionally on the residual supply intercept $\mathbf{s}^{-i}_{K(n)}$:

$$E[\mathbf{q}^i_{K(n')}(\mathbf{p}_{K(n')})|\mathbf{p}_{K(n)}, \mathbf{q}^i_0] = E[\mathbf{q}^i_{K(n')}(\mathbf{s}^{-i}_{K(n')})|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^{i,+}_0], \tag{41}$$

where, from Eq. (40), the expected trades vector $E[\mathbf{q}^i|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^{i,+}_0]$ is a linear function of the expected intercepts $E[\mathbf{s}^{-i}_{K(n')}|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^{i,+}_0]$. Given the distribution of intercepts $F(\mathbf{s}^{-i}|\mathbf{q}^{i,+}_0)$, we apply the Projection Theorem to the expected intercepts:

$$E[\mathbf{s}^{-i}_{K(n')}|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^{i,+}_0] = \mathbf{x}^{-i}_{n',n} + \mathbf{Y}^{-i}_{n',n}\mathbf{s}^{-i}_{K(n)} + \mathbf{Z}^{-i}_{n',n}\mathbf{q}^{i,+}_0, \tag{42}$$

where $\mathbf{x}^{-i}_{n',n} \in \mathbb{R}^{K(n')}, \mathbf{Y}^{-i}_{n',n} \in \mathbb{R}^{K(n')\times K(n)}$, and $\mathbf{Z}^{-i}_{n',n} \in \mathbb{R}^{K(n')\times(\sum_n K(n))}$ (which we endogenize later in Eqs. (49)-(51), given correct distribution $F(\mathbf{s}^{-i}|\mathbf{q}^{i,+}_0)$). Substituting the expected intercepts (42) in Eq. (41) characterizes the expected trades $E[\mathbf{q}^i|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^{i,+}_0]$, given the demand coefficients in other exchanges $\{\boldsymbol{a}^i_{K(n')}, \mathbf{B}^i_{K(n')}, \mathbf{C}^i_{K(n')}\}_{n'\neq n}$ and the inference coefficients $\{\mathbf{x}^{-i}_{n',n}, \mathbf{Y}^{-i}_{n',n}, \mathbf{Z}^{-i}_{n',n}\}_{n'\neq n}$.

Step 1.3 (Best Response in Exchange n, Given Demands in Other Exchanges) Substituting

Electronic copy available at: https://ssrn.com/abstract=3604976

expectated trades (41) in the first-order condition (37) gives the best response $\mathbf{q}^i_{K(n)}(\cdot)$ in exchange n:

$$\begin{aligned}
&\big(\alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n)} + \boldsymbol{\Lambda}^i_{K(n)}\big)\mathbf{q}^i_{K(n)} \\
&= \boldsymbol{\delta}^+_{K(n)} - \mathbf{p}_{K(n)} - \alpha^i \boldsymbol{\Sigma}^+_{K(n)} \mathbf{q}_0^{i,+} - \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} (\boldsymbol{\Lambda}^i_{K(n')})^{-1} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} (\boldsymbol{a}^i_{K(n')} - \mathbf{B}^i_{K(n')} \mathbf{q}_0^{i,+}) \\
&- \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} \mathbf{C}^i_{K(n')} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \big(\mathbf{x}^{-i}_{n',n} + \mathbf{Y}^{-i}_{n',n} \mathbf{s}^{-i}_{K(n)} + \mathbf{Z}^{-i}_{n',n} \mathbf{q}_0^{i,+}\big). \qquad (43)
\end{aligned}$$

Hence, the best response $\mathbf{q}^i_{K(n)}(\cdot)$ is a linear function of $\mathbf{p}_{K(n)}, \mathbf{s}^{-i}_{K(n)}$, and $\mathbf{q}_0^{i,+}$. We can write $\mathbf{q}^i_{K(n)}(\cdot)$ as follows:

$$\mathbf{q}^i_{K(n)}(\mathbf{p}_{K(n)}) = \boldsymbol{a}^i_{K(n)} - \mathbf{B}^i_{K(n)} \mathbf{q}_0^{i,+} - \mathbf{C}^i_{K(n)} \mathbf{p}_{K(n)} \qquad \forall \mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}, \qquad (44)$$

using that $\mathbf{p}_{K(n)}$ maps one-to-one to $\mathbf{s}^{-i}_{K(n)}$, given the linear best response $\mathbf{q}^i_{K(n)}(\cdot)$ (Eq. (39) for exchange n). Substituting $\mathbf{s}^{-i}_{K(n)}$ for $\mathbf{p}_{K(n)}$ in Eq. (43) gives the best response $\mathbf{q}^i_{K(n)}(\cdot)$ as a function of his private information $\mathbf{q}_0^{i,+}$ and contingent variable $\mathbf{p}_{K(n)}$.

Step 1.4 (Fixed Point of Best Response Coefficients) The endogenous characterization of demand coefficients $\boldsymbol{a}^i_{K(n)}, \mathbf{B}^i_{K(n)}$, and $\mathbf{C}^i_{K(n)}$ in exchange n (Eqs. (39) and (44)) can be written as functions of those in other exchanges, $\{\boldsymbol{a}^i_{K(n')}, \mathbf{B}^i_{K(n')}, \mathbf{C}^i_{K(n')}\}_{n' \neq n}$, given the coefficients of the residual supplies $\{\mathbf{x}^{-i}_{n',n}, \mathbf{Y}^{-i}_{n',n}, \mathbf{Z}^{-i}_{n',n}\}_{n' \neq n}$ and price impact $\{\boldsymbol{\Lambda}^i_{K(n')}\}_{n'}$. Respectively, for $\boldsymbol{a}^i_{K(n)}, \mathbf{B}^i_{K(n)}$, and $\mathbf{C}^i_{K(n)}$, we have:

$$\begin{aligned}
&\big(\alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n)} + \boldsymbol{\Lambda}^i_{K(n)} + \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}_{K(n),K(n')} \mathbf{C}^i_{K(n')} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \mathbf{Y}^{-i}_{n',n}\big) \boldsymbol{a}^i_{K(n)} \\
&= \boldsymbol{\delta}^+_{K(n)} - \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} (\boldsymbol{\Lambda}^i_{K(n')})^{-1} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \boldsymbol{a}^i_{K(n')} \\
&\quad - \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} \mathbf{C}^i_{K(n')} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \mathbf{x}^{-i}_{n',n}; \qquad (45) \\
&\big(\alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n)} + \boldsymbol{\Lambda}^i_{K(n)} + \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} \mathbf{C}^i_{K(n')} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \mathbf{Y}^{-i}_{n',n}\big) \mathbf{B}^i_{K(n)} \\
&= \alpha^i \boldsymbol{\Sigma}^+_{K(n)} - \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} (\boldsymbol{\Lambda}^i_{K(n')})^{-1} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \mathbf{B}^i_{K(n')} \\
&\quad + \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} \mathbf{C}^i_{K(n')} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \mathbf{Z}^{-i}_{n',n}; \qquad (46) \\
&\big(\alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n)} + \boldsymbol{\Lambda}^i_{K(n)} + \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} \mathbf{C}^i_{K(n')} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \mathbf{Y}^{-i}_{n',n}\big) \mathbf{C}^i_{K(n)} \\
&= \mathrm{Id} - \sum_{n' \neq n} \alpha^i \boldsymbol{\Sigma}^+_{K(n),K(n')} \mathbf{C}^i_{K(n')} (\mathbf{C}^i_{K(n')} + (\boldsymbol{\Lambda}^i_{K(n')})^{-1})^{-1} \mathbf{Y}^{-i}_{n',n} (\boldsymbol{\Lambda}^i_{K(n)})^{-1}. \qquad (47)
\end{aligned}$$

The system of equations (45)-(47) for all exchange n determines trader i's demand coefficients $\{\boldsymbol{a}^i_{K(n)}, \mathbf{B}^i_{K(n)}, \mathbf{C}^i_{K(n)}\}_n$ as a fixed point across exchanges, given $\{\boldsymbol{\Lambda}^i_{K(n)}\}_n$ and $F(\mathbf{s}^{-i}|\mathbf{q}_0^{i,+})$ (equivalently, given $\{\boldsymbol{\Lambda}^i_{K(n)}\}_n$ and $\{\mathbf{x}^{-i}_{n',n}, \mathbf{Y}^{-i}_{n',n}, \mathbf{Z}^{-i}_{n',n}\}_{n,n' \neq n}$).

Step 2 (Part (ii): Correct residual supply) Applying market clearing to the best response demands

Electronic copy available at: https://ssrn.com/abstract=3604976

(44) of other traders $j \neq i$ gives the residual supply of trader i in exchange n:

$$\mathbf{S}^{-i}_{K(n)}(\mathbf{p}_{K(n)}) \equiv -\sum_{j\neq i} \mathbf{q}^{j}_{K(n)}(\mathbf{p}_{K(n)}) = -\sum_{j\neq i}(\boldsymbol{a}^{j}_{K(n)} - \mathbf{B}^{j}_{K(n)}\mathbf{q}^{j,+}_{0}) + \sum_{j\neq i}\mathbf{C}^{j}_{K(n)}\mathbf{p}_{K(n)} \quad \forall \mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}. \tag{48}$$

Step 2.1 (Correct Distribution of Residual Supply Intercepts) The vector of intercepts $\mathbf{s}^{-i}_{K(n)} = -\sum_{j\neq i}(\boldsymbol{a}^{j}_{K(n)} - \mathbf{B}^{j}_{K(n)}\mathbf{q}^{j,+}_{0})$ for each exchange n is jointly normally distributed, given the demand coefficients of other traders $\{\boldsymbol{a}^{j}_{K(n)}, \mathbf{B}^{j}_{K(n)}\}_{n,j\neq i}$ and the primitive joint distribution of their endowments $\{F(\mathbf{q}^{j,+}_{0}|\mathbf{q}^{i,+}_{0})\}_{j\neq i}$:

$$F(\mathbf{s}^{-i}|\mathbf{q}^{i,+}_{0}) = \mathcal{N}\Big(-\sum_{j\neq i}(\boldsymbol{a}^{j}_{K(n)} - \mathbf{B}^{j}_{K(n)}E[\mathbf{q}^{j,+}_{0}|\mathbf{q}^{i,+}_{0}]), \sum_{j,h\neq i}\mathbf{B}^{j}_{K(n)}Cov(\mathbf{q}^{j,+}_{0}, \mathbf{q}^{h,+}_{0}|\mathbf{q}^{i,+}_{0})(\mathbf{B}^{h}_{K(n)})'\Big).$$

Applying the Projection Theorem to $E[\mathbf{s}^{-i}_{K(n')}|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}^{i,+}_{0}]$ determines the inference coefficients $\mathbf{x}^{-i}_{n',n}$, $\mathbf{Y}^{-i}_{n',n}$, and $\mathbf{Z}^{-i}_{n',n}$ in expected intercepts (42) as functions of demand coefficients $\{\boldsymbol{a}^{j}_{K(n)}, \mathbf{B}^{j}_{K(n)}, \mathbf{C}^{j}_{K(n)}\}_{n,j\neq i}$, given $\{\boldsymbol{\Lambda}^{j}_{K(n)}\}_{n,j\neq i}$: for each n and $n' \neq n$,

$$\mathbf{x}^{-i}_{n',n} = -\sum_{j\neq i}\Big(\boldsymbol{a}^{j}_{K(n')} - \mathbf{Y}^{-i}_{n',n}\boldsymbol{a}^{j}_{K(n)} - (\mathbf{B}^{j}_{K(n')} - \mathbf{Y}^{-i}_{n',n}\mathbf{B}^{j}_{K(n)})E[\mathbf{q}^{j,+}_{0}]\Big) - \mathbf{Z}^{-i}_{n',n}E[\mathbf{q}^{i,+}_{0}]; \tag{49}$$

$$\mathbf{Y}^{-i}_{n',n} = \Big(\sum_{j\neq i}\mathbf{B}^{j}_{K(n')}\Big(\frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}\mathbf{B}^{j}_{K(n)} + \sum_{h\neq i}\mathbf{B}^{h}_{K(n)}\Big)'\Big)\Big(\sum_{j\neq i}\mathbf{B}^{j}_{K(n)}\Big(\frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}\mathbf{B}^{j}_{K(n)} + \sum_{h\neq i}\mathbf{B}^{h}_{K(n)}\Big)'\Big)^{-1}; \tag{50}$$

$$\mathbf{Z}^{-i}_{n',n} = \sum_{j\neq i}\frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}\mathbf{B}^{j}_{K(n')} - \mathbf{Y}^{-i}_{n',n}\sum_{j\neq i}\frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}\mathbf{B}^{j}_{K(n)}. \tag{51}$$

Step 2.2 (Correct Price Impacts) The Jacobian of the inverse residual supply $(\mathbf{S}^{-i}_{K(n)}(\cdot))^{-1}$: $\boldsymbol{\Lambda}^{i}_{K(n)} = (\sum_{j\neq i}\mathbf{C}^{j}_{K(n)})^{-1}$ characterizes the price impact matrix $\boldsymbol{\Lambda}^{i}_{K(n)}$ for each i and n. Given the previous steps, equilibrium is characterized as a fixed point for price impacts $\{\boldsymbol{\Lambda}^{i}_{K(n)}\}_{i,n}$.

(Part Only If) The argument mimics the proof of Lemma 1 in Rostek and Weretka (2015). ■

Proof of Theorem 5 (Equilibrium: Fixed Point in Demand Schedules (General Design)).
Step 1 (Part (i): Optimization of trader i, given price impact $\boldsymbol{\Lambda}^{i}$) By Proposition 2, the demand function of trader i in exchange n can be written as:

$$\mathbf{q}^{i}_{K(n)}(\mathbf{p}_{K(n)}) = \boldsymbol{a}^{i}_{K(n)} - \mathbf{B}^{i}_{K(n)}\mathbf{q}^{i,+}_{0} - \mathbf{C}^{i}_{K(n)}\mathbf{p}_{K(n)} \qquad \forall \mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}. \tag{52}$$

Step 1.1 (Fixed Point of Each Demand Coefficient in a Single Matrix Equation) We represent the demand schedules (52) in matrix form (Eq. (53) below). The matrix form allows us to write the fixed point problem (45)-(47) for trader i's best responses across exchanges as a single matrix equation for each demand coefficient $\boldsymbol{a}^{i}, \mathbf{B}^{i}$, and $\mathbf{C}^{i}$, given $\boldsymbol{\Lambda}^{i}$ and $F(\mathbf{s}^{-i}|\mathbf{q}^{i,+}_{0})$.

We begin by defining demand coefficients for all N exchanges by treating the same asset in different

Electronic copy available at: https://ssrn.com/abstract=3604976

exchanges as (perfectly correlated) different assets:

$$\boldsymbol{a}^i \equiv (\boldsymbol{a}^i_{K(n)})_n \in \mathbb{R}^{\sum_n K(n)};\ \mathbf{B}^i \equiv (\mathbf{B}^i_{K(n)})_n \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))};\ \mathbf{C}^i \equiv diag(\mathbf{C}^i_{K(n)})_n \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))}.$$

With the matrix coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}_i$, trader i's demand schedule $\mathbf{q}^i(\cdot) = ((q^i_{k,n}(\cdot) : \mathbb{R}^{\sum_n K(n)} \to \mathbb{R})_k)_n$ is represented by a function of *common* across exchanges contingent variables $\mathbf{p} \in \mathbb{R}^{\sum_n K(n)}$ for all n rather than the prices of assets in exchange n only $\mathbf{p}_{K(n)} \in \mathbb{R}^{K(n)}$:

$$\mathbf{q}^i(\mathbf{p}) = \boldsymbol{a}^i - \mathbf{B}^i\mathbf{q}_0^{i,+} - \mathbf{C}^i\mathbf{p} \qquad \forall \mathbf{p} \in \mathbb{R}^{\sum_n K(n)}. \tag{53}$$

The matrix slope $\mathbf{C}^i = diag(\mathbf{C}^i_{K(n)})_n$ is a block diagonal matrix; each block corresponds to an exchange in N.

Similarly, we write the inference coefficients (42) of trader i's expected intercepts $E[\mathbf{s}^{-i}_{K(n')}|\mathbf{s}^{-i}_{K(n)}, \mathbf{q}_0^{i,+}]$ in a matrix form:

$$(\mathbf{x}^{-i}_{n',n})_{n'} \in \mathbb{R}^{\sum_n K(n)};\quad \mathbf{Y}^{-i} \equiv (Y^{-i}_{n',n})_{n',n} \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))};\quad (\mathbf{Z}^{-i}_{n',n})_{n'} \in \mathbb{R}^{(\sum_n K(n))\times(\sum_n K(n))},$$

where $\mathbf{x}^{-i}_{n,n} \equiv 0, \mathbf{Y}^{-i}_{n,n} \equiv \mathrm{Id}$, and $\mathbf{Z}^{-i}_{n,n} \equiv 0$ for all n and i.

Using the matrix demand coefficients and the matrix inference coefficients, the fixed point (45)-(47) for trader i's best responses across exchanges simplifies to three matrix equations, one for each demand coefficient:

$$\big(\alpha^i\boldsymbol{\Sigma}^+(\boldsymbol{\Lambda}^i)^{-1} + 2\mathrm{Id}\big)(\mathbf{C}^i + (\boldsymbol{\Lambda}^i)^{-1})^{-1}\boldsymbol{a}^i = \boldsymbol{\delta}^+ - \big(\alpha^i\boldsymbol{\Sigma}^+_{K(n)}\mathbf{C}^i(\mathbf{C}^i + (\boldsymbol{\Lambda}^i)^{-1})^{-1}(\mathbf{x}^{-i}_{n',n})_{n'}\big)_n; \tag{54}$$

$$\big(\alpha^i\boldsymbol{\Sigma}^+(\boldsymbol{\Lambda}^i)^{-1} + 2\mathrm{Id}\big)(\mathbf{C}^i + (\boldsymbol{\Lambda}^i)^{-1})^{-1}\mathbf{B}^i = \alpha^i\boldsymbol{\Sigma}^+ + \big(\alpha^i\boldsymbol{\Sigma}^+_{K(n)}\mathbf{C}^i(\mathbf{C}^i + (\boldsymbol{\Lambda}^i)^{-1})^{-1}(\mathbf{Z}^{-i}_{n',n})_{n'}\big)_n; \tag{55}$$

$$\boldsymbol{\Lambda}^i\mathbf{C}^i + \big[\alpha^i\boldsymbol{\Sigma}^+\mathbf{C}^i(\mathbf{C}^i + (\boldsymbol{\Lambda}^i)^{-1})^{-1}\mathbf{Y}^{-i}\big]_N(\mathbf{C}^i + (\boldsymbol{\Lambda}^i)^{-1}) = \mathrm{Id}. \tag{56}$$

Step 1.2 (Distribution of Residual Supply Intercepts, given Price Impacts) The intercept distribution $F(\mathbf{s}^{-i}|\mathbf{q}_0^{i,+})$ determines the matrix inference coefficients $(\mathbf{x}^{-i}_{n',n})_{n'}$, $\mathbf{Y}^{-i}$, and $(\mathbf{Z}^{-i}_{n',n})_{n'}$ — equivalently, by Eqs. (54)-(56), the matrix demand coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}_i$ — as a fixed point across traders, given $\{\boldsymbol{\Lambda}^i\}_i$.

From Eqs. (49)-(51), matrices $\{(\mathbf{x}^{-i}_{n',n})_{n'}\}_n$,$\mathbf{Y}^{-i}$, and $\{(\mathbf{Z}^{-i}_{n',n})_{n'}\}_n$ of trader i are functions of $\{\boldsymbol{a}^j, \mathbf{B}^j\}_{j\neq i}$:

$$\begin{aligned}
(\mathbf{x}^{-i}_{n',n})_{n'} &= -\sum_{j\neq i}\big(\boldsymbol{a}^j - (\mathbf{Y}^{-i}_{n',n})_{n'}\boldsymbol{a}^j_{K(n)}\big) + \frac{\sigma^2_{pv}}{\sigma^2_{cv}+\sigma^2_{pv}}\sum_{j\neq i}(\mathbf{B}^j - (\mathbf{Y}^{-i}_{n',n})_{n'}\mathbf{B}^j_{K(n)})E[\mathbf{q}_0^j];\\
(\mathbf{Z}^{-i}_{n',n})_{n'} &= \frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}\sum_{j\neq i}(\mathbf{B}^j - (\mathbf{Y}^{-i}_{n',n})_{n'}\mathbf{B}^j_{K(n)}),\\
\mathbf{Y}^{-i} &= \Big(\sum_{j\neq i}\mathbf{B}^j\boldsymbol{\Omega}^+(\frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}\mathbf{B}^j + \sum_{h\neq i}\mathbf{B}^h)'\Big)\Big[\sum_{j\neq i}\mathbf{B}^j\boldsymbol{\Omega}^+(\frac{\sigma^2_{cv}}{\sigma^2_{cv}+\sigma^2_{pv}}\mathbf{B}^j + \sum_{h\neq i}\mathbf{B}^h)'\Big]^{-1}_N.
\end{aligned}$$

Substituting the inference coefficients into Eqs. (54)-(56) gives a system of equations (32)-(34) for

Electronic copy available at: https://ssrn.com/abstract=3604976

demand coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}_i$, given the price impact matrices $\{\boldsymbol{\Lambda}^i\}_i$.

Step 2 (Part (ii): Correct price impacts) Given the best response demands of other traders $j \neq i$, trader i's equilibrium price impact $\boldsymbol{\Lambda}^i \equiv \text{diag}(\boldsymbol{\Lambda}^i_{K(n)})_n$ is characterized by the Jacobian of his inverse residual supply $\left(\mathbf{S}^{-i}(\cdot)\right)^{-1}$ in a single matrix equation for all exchanges: for each i,

$$\boldsymbol{\Lambda}^i = (\sum_{j \neq i} \mathbf{C}^j)^{-1}. \tag{57}$$

The system of equations (57) for all traders characterizes the equilibrium price impact $\{\boldsymbol{\Lambda}^i\}_i$ as a fixed point across traders. ■

From now on, we assume symmetric risk preferences unless otherwise specified. Then, $\mathbf{B}^i = \mathbf{B}, \mathbf{C}^i = \mathbf{C}$, and $\boldsymbol{\Lambda}^i = \boldsymbol{\Lambda}$ for all i.

Assumption (Symmetric Risk Preferences) Let $\alpha^i = \alpha$ for all i.

In the symmetric equilibrium, Eqs. (33)-(35) in Theorem 5 simplify to:

Corollary 4 (Symmetric Equilibrium (General Design)) Fix a market structure N. In a symmetric equilibrium, the (net) demand schedules are determined by matrix coefficients $\boldsymbol{a}^i, \mathbf{B}$, and $\mathbf{C}$, and price impact $\boldsymbol{\Lambda}$ by the following conditions: for each i,

(i) (*Optimization, given price impact* $\boldsymbol{\Lambda}$) Given price impact matrix $\boldsymbol{\Lambda}$, best-response coefficients $\boldsymbol{a}^i, \mathbf{B}$, and $\mathbf{C}$ are characterized by:

$$\boldsymbol{a}^i = \mathbf{C}\big(\boldsymbol{\delta}^+ - (\alpha\boldsymbol{\Sigma}^+ - \mathbf{C}^{-1}\mathbf{B})E[\bar{\mathbf{q}}_0^+] + ((\alpha\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda})^{-1}\alpha\boldsymbol{\Sigma}^+ - \mathbf{B})(E[\bar{\mathbf{q}}_0^+] - E[\mathbf{q}_0^{i,+}]), \tag{58}$$

$$\mathbf{B} = \big((\alpha\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}) - \sigma_0^2(\alpha\boldsymbol{\Sigma}^+ - (I-2)\boldsymbol{\Lambda})\big)^{-1}\alpha\boldsymbol{\Sigma}^+, \tag{59}$$

$$\mathbf{C} = \big[(\alpha\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda})(\mathbf{B}\boldsymbol{\Omega}^+\mathbf{B}')\big[\mathbf{B}\boldsymbol{\Omega}^+\mathbf{B}'\big]_N^{-1}\big]_N^{-1}, \tag{60}$$

where $\sigma_0^2 \equiv \frac{\sigma_{cv}^2 + \frac{1}{I}\sigma_{pv}^2}{\sigma_{cv}^2 + \sigma_{pv}^2} = \frac{Cov(q^i_{0,k}, \bar{q}_{0,k})}{Var(q^i_{0,k})}$.

(ii) (*Correct price impacts*) Price impact $\boldsymbol{\Lambda}$ equals the slope of the trader's inverse residual supply function:

$$\boldsymbol{\Lambda} = \frac{1}{I-1}\mathbf{C}^{-1}. \tag{61}$$

Note that the price slope $\mathbf{C}^{-1} = diag(\mathbf{C}^{-1}_{K(n)})_n$ is a block diagonal matrix.

Proof of Theorem 2 (Existence of Symmetric Equilibrium). Consider a market with exchanges $N = \{K(n)\}_n$. Let $\mathcal{M}$ be the set of all $(\sum_n K(n))$-dimensional block-diagonal matrices, such that, for any $\mathbf{M} \in \mathcal{M}$, $\mathbf{M}_{K(n),K(n')} = 0$ for distinct exchanges $n \neq n'$. Given that price impacts are block-diagonal, we introduce a partial order on $\mathcal{M}$: $\boldsymbol{\Lambda}^1 \leq \boldsymbol{\Lambda}^2$ if $(\boldsymbol{\Lambda}^2 - \boldsymbol{\Lambda}^1)$ is positive semidefinite, which, by definition of the positive semidefiniteness, is equivalent to $\boldsymbol{\Lambda}^1_{K(n)} \leq \boldsymbol{\Lambda}^2_{K(n)}$ for all n, where $\boldsymbol{\Lambda}_{K(n)}$ is the $(K(n), K(n))$ sub-matrix of the block-diagonal matrix $\boldsymbol{\Lambda}$.

Electronic copy available at: https://ssrn.com/abstract=3604976

In the symmetric equilibrium, substituting $\mathbf{B}$ from Eq. (59) and $\mathbf{C} = \frac{1}{I-1}\mathbf{\Lambda}^{-1}$ from Eq. (61),

$$\mathbf{B} = \big(\sigma_0^2\mathbf{C}^{-1} + (1-\sigma_0^2)(\alpha\mathbf{\Sigma}^+ + \mathbf{\Lambda})\big)^{-1}\alpha\mathbf{\Sigma}^+ = \big((1-\sigma_0^2)\alpha\mathbf{\Sigma}^+ + (1+(I-2)\sigma_0^2)\mathbf{\Lambda}\big)^{-1}\alpha\mathbf{\Sigma}^+,$$

into Eq. (60), the fixed point equation (57) for $\mathbf{\Lambda}$ become:

$$L(\mathbf{\Lambda}) \equiv \big[(\alpha\mathbf{\Sigma}^+ - (I-2)\mathbf{\Lambda})\big(\alpha\mathbf{\Sigma}^+ + \frac{1+(I-2)\sigma_0^2}{1-\sigma_0^2}\mathbf{\Lambda}\big)^{-1}\alpha\mathbf{\Sigma}^+\mathbf{\Omega}^+\alpha\mathbf{\Sigma}^+\big(\alpha\mathbf{\Sigma}^+ + \frac{1+(I-2)\sigma_0^2}{1-\sigma_0^2}\mathbf{\Lambda}'\big)^{-1}\big]_N = 0. \tag{62}$$

Define a map $L : \mathcal{M} \to \mathcal{M}$ using the LHS of Eq. (62). To demonstrate that there exists $\mathbf{\Lambda}$ such that $L(\mathbf{\Lambda}) = 0$, we first show that there exist two bounds $(\underline{\mathbf{\Lambda}}, \overline{\mathbf{\Lambda}})$ such that $L(\underline{\mathbf{\Lambda}}) \geq 0$ and $L(\overline{\mathbf{\Lambda}}) \leq 0$. By the Brouwer fixed point theorem, given that the set of diagonal matrices defined by the bounds $(\underline{\mathbf{\Lambda}}, \overline{\mathbf{\Lambda}})$ is compact and the map $L(\mathbf{\Lambda})$ is continuous, there exists a solution $\mathbf{\Lambda}$ to the fixed point problem $L(\mathbf{\Lambda}) = 0$.

Let $\underline{\mathbf{\Lambda}} \equiv 0$ and $\overline{\mathbf{\Lambda}} \equiv \frac{\alpha}{I-2}N[\mathbf{\Sigma}^+]_N$. It is immediate that $\underline{\mathbf{\Lambda}}$ satisfies the desired condition: substituting $\underline{\mathbf{\Lambda}} = 0$ into the LHS of Eq. (62), we have $L(\underline{\mathbf{\Lambda}}) = [\alpha\mathbf{\Sigma}^+\mathbf{\Omega}^+]_N \gneq 0$, by the positive semi-definiteness of $\mathbf{\Omega}^+$ and the positive definiteness of $[\alpha\mathbf{\Sigma}^+]_N > 0$.

Substituting $\overline{\mathbf{\Lambda}}$ into map $L(\cdot)$, we have:

$$L(\overline{\mathbf{\Lambda}}) = \alpha\big[(\mathrm{Id} + \frac{2N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)\mathbf{\Omega}^+(\mathrm{Id} + \frac{2N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}\big]_N. \tag{63}$$

Given that $\mathbf{\Sigma}^+$ is positive semidefinite, $(\mathrm{Id} + \frac{2N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}$ is positive definite, where we used that the inverse of a positive definite matrix is positive definite. In turn, matrix $(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)\mathbf{\Omega}^+ = \mathbf{\Omega}^+(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)$ in Eq. (63) is negative definite (it is negative semidefinite if and only if all $\sum_n K(n)$ assets are perfectly correlated). To prove this, observe that for any vector $\mathbf{v} \equiv (\mathbf{v}_{K(n)})_n \in \mathbb{R}^{\sum_n K(n)}$, we have:

$$Cov(\mathbf{v}_{K(n)} \cdot \mathbf{r}_{K(n)}, \mathbf{v}_{K(n')} \cdot \mathbf{r}_{K(n')}) \;\leq\; \frac{1}{2}Var(\mathbf{v}_{K(n)} \cdot \mathbf{r}_{K(n)}) + \frac{1}{2}Var(\mathbf{v}_{K(n')} \cdot \mathbf{r}_{K(n')}) \quad \forall n, n' \in N. \tag{64}$$

Using that the covariance matrix of the distribution of asset returns $\mathbf{r} = (\mathbf{r}_{K(n)})_n$ is $\mathbf{\Sigma}^+$, inequality (64) is equivalent to:

$$\mathbf{v}'_{K(n)}\mathbf{\Sigma}^+_{K(n),K(n')}\mathbf{v}_{K(n')} \;\leq\; \frac{1}{2}\mathbf{v}'_{K(n)}\mathbf{\Sigma}^+_{K(n),K(n)}\mathbf{v}_{K(n)} + \frac{1}{2}\mathbf{v}'_{K(n')}\mathbf{\Sigma}^+_{K(n'),K(n')}\mathbf{v}_{K(n')} \quad \forall n, n' \in N. \tag{65}$$

Summing each side of Eq. (65) over n and n' gives:

$$\mathbf{v}'\mathbf{\Sigma}^+\mathbf{v} = \sum_{n,n'}\mathbf{v}'_{K(n)}\mathbf{\Sigma}^+_{K(n),K(n')}\mathbf{v}_{K(n')} \;\leq\; N\sum_n \mathbf{v}'_{K(n)}\mathbf{\Sigma}^+_{K(n),K(n)}\mathbf{v}_{K(n)} = \mathbf{v}'(N[\mathbf{\Sigma}^+]_N)\mathbf{v},$$

and hence, $\mathbf{v}'(\mathbf{\Sigma}^+ - N[\mathbf{\Sigma}^+]_N)\mathbf{v} \leq 0$ for any vector $\mathbf{v}$.

It follows from the positive semi-definiteness of $(\mathrm{Id} + \frac{2N}{I-2}[\mathbf{\Sigma}^+]_N(\mathbf{\Sigma}^+)^{-1})^{-1}$ and $\mathbf{\Omega}^+$ and the negative

Electronic copy available at: https://ssrn.com/abstract=3604976

semi-definiteness of $(\boldsymbol{\Sigma}^+ - N[\boldsymbol{\Sigma}^+]_N)$ that the RHS of Eq. (63) becomes

$$(\mathrm{Id} + \frac{2N}{I-2}[\boldsymbol{\Sigma}^+]_N(\boldsymbol{\Sigma}^+)^{-1})^{-1}(\boldsymbol{\Sigma}^+\boldsymbol{\Omega}^+ - N[\boldsymbol{\Sigma}^+]_N\boldsymbol{\Omega}^+)(\mathrm{Id} + \frac{2N}{I-2}[\boldsymbol{\Sigma}^+]_N(\boldsymbol{\Sigma}^+)^{-1})^{-1} \;\leq\; 0. \tag{66}$$

Consequently, $L(\overline{\boldsymbol{\Lambda}}) \leq 0$; the equality holds if and only if all $\sum_n K(n)$ assets are perfectly correlated.[55] This completes the argument.

Suppose $K = 2$. We now show the uniqueness of equilibrium in uncontingent market for $K = 2$. As Appendix C shows, the equilibrium fixed point (62) for $\boldsymbol{\Lambda} = diag(\lambda, \lambda)$ simplifies to:

$$\lambda = \frac{\alpha}{I-2} + \frac{\alpha\rho}{I-2}\frac{2xy}{x^2+y^2}, \tag{67}$$

where $x \equiv (1-\sigma_0^2)(1-\rho^2)\alpha + (1+(I-2)\sigma_0^2)\lambda$ and $y \equiv \rho(1+(I-2)\sigma_0^2)\lambda$ characterize each row of $\mathbf{B}$ in Eq. (59): $\mathbf{b}_1 = (x, y)$ and $\mathbf{b}_2 = (y, x)$. Rearranging Eq. (67) gives a third-order polynomial of λ:

$$\begin{aligned} 0 \;=\; & -(I-2)(1+\rho^2)(1+(I-2)\sigma_0^2)^2\lambda^3 + \big(4-(1-\rho^2)(2I-1)+(I-2)(3+\rho^2)\sigma_0^2\big)(1+(I-2)\sigma_0^2)\alpha\lambda^2 \\ & + \big(4-(1-\rho^2)I+(I-2)(3+\rho^2)\sigma_0^2\big)(1-\sigma_0^2)(1-\rho^2)\alpha^2\lambda + \alpha^3(1-\sigma_0^2)^2(1-\rho^2)^2. \end{aligned}$$

By the Descartes' sign rule, there exists a unique positive solution λ. ■

Proof of Corollary 1 (Equilibrium Price and Allocations). We characterize equilibrium prices and allocations given the equilibrium demand coefficients $\{\boldsymbol{a}^i, \mathbf{B}^i, \mathbf{C}^i\}_i$, and price impacts $\{\boldsymbol{\Lambda}^i\}_i$ characterized in Theorem 5. Applying the market clearing to demand schedules (53), the price vector is:

$$\mathbf{p} = (\sum_i \mathbf{C}^i)^{-1}(\sum_i \boldsymbol{a}^i - \sum_i \mathbf{B}^i\mathbf{q}_0^{i,+}). \tag{68}$$

Summing the intercepts $\{\boldsymbol{a}^i\}_i$ in Eq. (32) gives:

$$\sum_i \boldsymbol{a}^i = (\sum_i \mathbf{C}^i)\Big(\boldsymbol{\delta}^+ - (\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1})^{-1}\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1}\alpha^j\boldsymbol{\Sigma}^+E[\mathbf{q}_0^{j,+}]\Big) + \sum_i \mathbf{B}^iE[\mathbf{q}_0^{i,+}].$$

Substituting $\sum_i \boldsymbol{a}^i$ into the price equation (68), we have:

$$\mathbf{p} = \boldsymbol{\delta}^+ - (\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1})^{-1}\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1}\alpha^j\boldsymbol{\Sigma}^+E[\mathbf{q}_0^{j,+}] - (\sum_j \mathbf{C}^j)^{-1}\sum_j \mathbf{B}^j(\mathbf{q}_0^{j,+} - E[\mathbf{q}_0^{j,+}]). \tag{69}$$

Letting $\mathbf{Q} \equiv (\sum_j \mathbf{C}^j)^{-1}\sum_j \mathbf{B}^j\mathbf{q}_0^{j,+}$ be the aggregate risk given price impact $\{\boldsymbol{\Lambda}^i\}_i$, the price vector can be characterized as:

$$\mathbf{p} = \boldsymbol{\delta}^+ - E[\mathbf{Q}^c] - (\mathbf{Q} - E[\mathbf{Q}]),$$

where $\mathbf{Q}^c \equiv (\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1})^{-1}\sum_j(\alpha^j\boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^j)^{-1}\alpha^j\boldsymbol{\Sigma}^+\mathbf{q}_0^{j,+}$ is the aggregate risk in the contingent market.

[55] When all $\sum_n K(n)$ assets are perfectly correlated, the equality (66) implies that $\boldsymbol{\Lambda} = \frac{N}{I-2}[\boldsymbol{\Sigma}^+]_N$ is the solution to Eq. (62).

Electronic copy available at: https://ssrn.com/abstract=3604976

To characterize the equilibrium allocation of trader i, substitute equilibrium price $\mathbf{p}$ and demand coefficient $\boldsymbol{a}^i$ into the trader's demand (53): for each i,

$$\mathbf{q}^i + \mathbf{q}_0^{i,+} = (\alpha^i \boldsymbol{\Sigma}^+ + \boldsymbol{\Lambda}^i)^{-1}(E[\mathbf{Q}^c] - \alpha^i \boldsymbol{\Sigma}^+ E[\mathbf{q}_0^{i,+}]) + \mathbf{C}^i(\mathbf{Q} - E[\mathbf{Q}]) - \mathbf{B}^i(\mathbf{q}_0^{i,+} - E[\mathbf{q}_0^{i,+}]) + \mathbf{q}_0^{i,+}. \quad (70)$$

In the symmetric equilibrium (i.e., assuming $\alpha^i = \alpha$ for all i), equilibrium price (69) and allocations (70) become equations (23) and (24). ■

Electronic copy available at: https://ssrn.com/abstract=3604976