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Blood Allocation with Replacement Donors*

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Abstract

In 56 developing and developed countries, blood component donations by volunteer non-remunerated donors can only meet less than 50% of the demand. In these countries, blood banks heavily rely on donor replacement programs that provide blood to patients in return for donations made by their close relatives or friends. Such programs appear to be highly disorganized, non-transparent, and inefficient despite the scarcity of blood components. We introduce the design of donor replacement programs and blood allocation mechanisms as a new application of market design. We formulate a general blood allocation and donation model including the blood bank, patients, volunteer non-remunerated and replacement donors. Within this framework, we introduce a class of mechanisms that sequentially accommodates various policy objectives of a blood bank while ensuring efficiency. Another novelty we introduce is a rich class of feasible allocation possibilities beyond the classical one-to-one exchange. This class accommodates endogenous exchange rates between donated and received blood units together with various fairness and efficiency objectives. Furthermore, these mechanisms provide correct incentives for the patients to bring forward as many replacement donors as possible.

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1 Introduction

Transfusions are commonly used to treat various medical conditions to replace lost or add inadequate components of blood. Replacement red blood cells and other blood components such as platelets, plasma, and clotting factors are essential for medical procedures such as surgeries, chemotherapy, obstetrics, for people with various blood diseases and trauma patients.[1] In the US, according to Pfuntner, Wier, and Stocks (2013), blood transfusion was the most common procedure performed during hospitalizations in 2011. While blood transfusion saves lives and improves health outcomes, many patients requiring transfusion around the world do not have timely access to safe blood due to significant shortages of supply.

Around the world, the collection and distribution of blood is organized through blood banks where donated blood is processed and stored.[2] Unlike most solid human organs and tissues, blood replenishes and a healthy donor can donate red blood cells regularly once in every two-three months and other components, such as platelets and plasma, much more frequently at a blood bank or collection center. Moreover, different compatibility requirements apply for each blood component, which makes the medical feasibility requirements of transfusion substantially different from organ transplantation.[3]

The most adequate and reliable supply of blood is through volunteer non-remunerated donors, who mostly donate blood through blood drives or other campaigns.[4] These donors provide the safest supply of blood, since the prevalence of blood-borne infections is lowest among this group of donors.[5] According to the World Health Organization (WHO), 79 countries (38 high-income, 33 middle-income, and 10 low-income) collect more than 90% of their blood supply from volunteer non-remunerated donors as of August 2020 (WHO, 2020). The World Health Assembly resolution WHA63.12 (Sixty-third World Health Assembly, 2010) urges all member states to develop national blood systems based on volunteer non-remunerated donations and to work towards the goal of self-sufficiency. Despite these warnings, volunteer non-remunerated donation remains insufficient to meet the demand for blood in many regions of the world.

Although it seems relatively costless and is even considered healthy to donate blood, there are severe blood shortages in many developing countries and seasonal shortages in many developed countries for blood components (Gilcher and McCombs, 2005).[6] Moreover, cultural and religious factors

[1] Whole blood is not commonly used is transfusion medicine. In addition to the blood components mentioned, plasma derivatives manufactured from pooled plasma donations in plasma fractionation centers (such as albumin, coagulation factors, and immunoglobulins) are used in the treatment of various conditions.

[2] The average blood donation takes about one hour per session. Donated blood is then separated into its components and each component has different storage needs. The red blood cells can be stored for 42 days while platelets can be kept only for five days. On the other hand, plasma can be preserved for a full year.

[3] See Section 2 for institutional and medical details of blood component transfusion including various compatibility requirements.

[4] Blood components are forbidden to be exchanged using valuable enumeration in most countries. Nevertheless, it is reported that 19 countries collect blood through paid donations as of 2015 (WHO, 2020).

[5] Paid donors are considered to be inferior as they may be in poorer health condition than volunteer non-remunerated donors. Such donors may also have incentives to hide their health status causing an adverse selection problem. The US is one of the 19 countries in which paid donation is allowed, as blood is not covered by the National Organ Transplant Act (NOTA) of 1984, which forbids sale of solid human organs and tissues. In spite of this fact, most of the US blood component supply is volunteer non-remunerated donation based because of this reason. Plasma is an exception. It is also collected by companies paying donors and turned into drugs after thorough fractionation.

[6] Especially in early winter and midsummer months, blood-type O red blood cell shortages occur in the US. Besides

create frictions deterring volunteer non-remunerated donations especially in some developing countries. Furthermore, some blood components, such as platelets, have short shelf life and are more difficult to collect than the others. Thus, platelet shortages occur even in the developed world.

In 56 countries wordlwide (9 high-income, 37 middle-income, and 10 low-income), more than 50% of the blood supply is met by *replacement donors* and, in some rare cases, through paid donors as of August 2020 (WHO, 2020). As an effective method to boost blood component reserves, blood banks in many places, including highly populated countries such as India and China, employ official or unofficial *donor replacement programs*. A donor replacement program requires each patient to nominate a number of willing donors, who are typically close friends or family members of the patient, to donate at a blood bank before or after the patient receives transfusion.[7]

Notwithstanding the important role they play in addressing blood shortages, existing donor replacement programs suffer from two major shortcomings. The first shortcoming is the loss of welfare due to the lack of optimized inventory management based on donor screening and needs of the blood bank. Although inventory management is often considered among the most important goals for a blood bank (see, for example,Indian Department of Health (2016) for a detailed description of the blood type and volume requirements to be kept in stock of each blood centre as mandated in India), as far as we know, no explicit optimization is pursued by blood banks in current donor replacement programs to achieve certain policy objectives. In the face of chronic supply shortages, one such natural objective can be to maximize transfused blood volume using the correct replacement donors.

Second, replacement donor programs generally operate on fixed exchange rates between units received and supplied (regardless of blood type), which creates efficiency, fairness, and ethical problems. A fixed exchange rate does not reward suppliers of most needed blood types. Furthermore, certain patients may not be able to recruit the required number of donors that they are obliged to bring, making it difficult to receive blood. This gives rise to issues of coercion and black markets through which patients without donors pay others to be their replacement donors (see Section 2 for other institutional details of how real-life replacement donor programs function).

In this paper, we introduce blood allocation with volunteer non-remunerated and replacement donors as a novel market design problem and propose a general blood allocation model together with solutions to address these shortcomings (see Section 3). In our model, each patient is represented by her blood type and her blood need, which is a two dimensional integer vector.[8] The first part

seasonal shortages, during catastrophic events such as an earthquake or a pandemic, blood shortages frequently occur. For example, during the recent COVID-19 pandemic blood components have had shortages in the US. For example, see, Red Cross of America (2020a).

[7]Within the medical community, disagreements exist with the stance of WHO at the regional level regarding volunteer non-remunerated donation being the most ethical and safest blood supply. Opponents of the view of WHO point out considerable evidence suggesting that the blood collected through replacement donors is as safe as volunteer non-remunerated donors. They argue that the motives of both types of donors and how they are directed toward donation are similar in many aspects. See for example Allain and Sibinga (2016) for an excellent survey of these views, empirical evidence, and the references therein.

[8]A healthy donor – depending on gender, height, weight, and total blood volume – can give up to two units of blood (measured in pints or approximately half liters) through automated blood collection, whereas a patient can be in need of multiple units of blood. It is also possible to donate particular blood components in various volumes, e.g., one donor may be able to donate two units of red blood cells, while another may donate two units of platelets but not both. We normalize the amount of blood component that can be donated by a donor to 1 unit and the blood component volume that a patient receives is represented as a multiple of it.

of her need is a *minimum threshold* that shall be given to each patient by the blood bank, which can be set at the minimum necessary units for an emergency procedure the patient needs such as an operation after a traffic accident. (On the other hand, the minimum threshold can be zero for elective procedures or during severe shortages.) The second part of a patient's blood need is a *maximum threshold*. She prefers receiving more blood to less within the range from her minimum to maximum need thresholds (see Section 2.2).[9] Each patient brings forward a (possibly empty) set of replacement donors. We assume that each donor, who is represented by his blood type, can donate one unit of a blood component without loss of generality. The blood bank is represented by the inventory units of each type of blood component.[10]

The blood bank chooses a medically acceptable allocation based on the blood types and needs of the patients, the blood types of the replacement donors, and its inventory. The schedule of a patient simply specifies the units she receives and which of her donors donate. However, a patient's schedule may depend upon the donors she brings forward, her disease characteristics, and the blood component she needs among other things. To accommodate various blood transfusion and donor replacement protocols, we introduce the notion of a *feasible schedule set*. This idiosyncratic set of each patient specifies possible combinations of received and supplied units of blood the patient can be assigned under an allocation. We view the design of feasible schedule sets as an important policy design tool and discuss various examples depending on the objectives of a blood bank.

Then we propose and study the class of *sequential targeting mechanisms* that operate in conjunction with a profile of feasible schedule sets as intuitive and natural blood allocation schemes (Section 4). This class of mechanisms allow for the use of various inventory management objectives of the bank based on sequential maximization or minimization of blood tranfusion volume to a sets of patients. We consider a flexible objective function for the bank that works in a lexicographic manner following a precedence order of objectives: we first satisfy target goal A, e.g., maximize the total transfused blood volume; subject to A being satisfied next target goal B, e.g., minimize the use of bank's own blood component resources; then goal C, e.g., maximize the blood received by the urgent-care patients; and so on. It is possible to imagine several lexicographic objectives involving particular blood types (e.g., rare vs. common types), patient types (e.g., urgent-care vs. non-essential elective surgery patients), and donor types (e.g., high-risk blood volunteers vs. safe donors) that may require specific attention.

These mechanisms and feasible schedule sets overcome the two shortcomings of current replacement donor programs. First, they address the lack of optimization based on donor screening in current programs. In particular, these mechanisms are both efficient (Theorem 1) and *donor monotonic* (Theorem 2 and Proposition 1), i.e., bringing forward a larger set of donors does not decrease the units of blood the patient receives. We also provide comparative static analysis for changes in target sets and feasible schedules of patients (Propositions 2 and 3).

[9]There are several reasons for considering a range of units of blood need for the same patient. It is reported that there is a wide range of blood units that can be needed for the same medical procedure. Alternatively, a patient's demand in excess of her minimum threshold can denote a safety buffer that is available to her in case she needs it during an operation. Moreover, for frequently repeated transfusions of blood components such as platelets, patients almost always prefer more units to less. For example, a five-day stock of platelets, where five days is their maximum shelf life, is preferred to a single dose.

[10]Each of these units can be thought as coming from volunteer non-remunerated donors.

The innovation of feasible schedule sets allows for various feasible exchange rates between units received and supplied, and the sequential targeting mechanisms determine endogenously these exchange rates based on the objectives, patients, and the available replacement donors. This property helps rectify the shortcomings caused by a fixed exchange rate in current programs.

Places with replacement donor programs appear to be highly non-transparent in their blood allocation operations. It is elusive to find existing guidelines that govern how these processes are overseen. As such, our approach provides a framework to assess and improve the effectiveness of the existing replacement donor programs. Our approach also makes it possible to offer rigor and transparency to the organization of these programs. We provide concrete policy designs and implementation details as to how these programs can be managed (Section 5).

Differently from living-donor organ exchanges that have attracted much attention in the last two decades in the market design literature and practice, blood allocation involves multi-unit demand and supply.[11] Many factors make this market design problem both practically and theoretically different from the analysis and functioning of solid organ exchanges. These include differences in the compatibility requirements for different blood components, the possibility of non-unit exchange rates between blood demanded and blood supplied, the non-simultaneity between receiving and donating blood, and the possibility to store blood components.

Our model and theoretical results are independent of the particular background of blood transfusion and can readily be applied to other applications and contexts with a subset of similar features; notably, the exchange of indivisible goods with compatibility-based monotonic preferences in units consumed. Some examples studied in the literature are shift exchanges of staff in a company (Manjunath and Westkamp, 2019), time bank and favor exchanges (Andersson et al., 2020), or international student exchanges (Delacrétaz, 2019). As far as we are aware, all previous exchange mechanisms in the literature use exogenous one-to-one exchange rates such that for each unit received at most one unit is supplied. As an important theoretical contribution, we overcome this limitation and introduce endogenous pricing of units while maintaining the good incentive properties of our mechanisms under certain assumptions. Moreover, our class of Pareto efficient and incentive compatible mechanisms generalize all known such mechanisms in similar exchange models in the literature (see Section 6 more on this and other related literature).

2 Background

2.1 Main Blood Components and Compatibility Requirements

Blood-type compatibility plays an important role for the feasibility of a blood component transfusion. Different medical compatibility protocols are used for red blood cell, platelet, and plasma transfusions. Other factors, such as the frequency of transfusion the patient needs, may also affect the compatibility requirements. Moreover, the medical practices in different parts of the world may vary.

[11] See Sönmez and Ünver (2017) for a recent survey of this literature and practical developments. Notable exceptions of unit-demand organ exchange are living dual-donor lobar lung transplantation, dual-graft living-donor liver transplantation, and simultaneous liver-kidney transplantation. However, no organized exchange program exists for these practices as of writing of this paper.

We focus on these three crucial and most-transfused blood components and provide a brief account of these components and how their compatibility requirements are determined.

Whole Blood and Red Blood Cells

Red blood cell transfusion – the de-facto modern day replacement for the older whole blood transfusion therapy – is mostly used for people with cancer and other blood diseases, followed by surgical patients (including open heart operations and burns), people with heart, stomach, or kidney diseases, and orthopedic patients among other uses. Whole blood transfusion is still used occasionally when a patient loses significant amount of blood due to trauma, surgeries, etc.

There are several human antigens two of which play a crucial role for compatibility in red blood cell transfusion, although there are more than 300 human blood groups.

The first one, the ABO blood-type system antigens is the most commonly known. There are four human ABO blood types: O, A, B, and AB named for the existence or non-existence of A or B red blood cell antigens. If a person does not have an antigen, then beginning from childhood she develops antibodies against the non-existent antigens. Antibodies attack red blood cells carrying the foreign antigen. As a result, in theory a patient can receive transfusion from a donor who does not have an antigen that she possesses: type O donors can donate to all patients, type A donors can donate to type A or AB patients, type B donors can donate to type B or AB patients, and type AB donors can only donate to type AB patients. This is known as the *ABO-cellular compatibility*. For red blood cells and platelets this is the main compatibility requirement. However, as the plasma of the blood of a donor also carries the antibodies against the antigens she does not possess, it is risky to receive non-identical blood-type whole blood for a patient as these antibodies attack the patient's own red blood cells. Thus, unless it is not an emergency, *ABO-identical transfusion* is generally practiced for whole blood transfusion. Similar practices are followed for transfusion of any non-plasma blood component that is bundled with significant amount of plasma (Harm and Dunbar, 2019).[12] Otherwise, in theory, red blood cells can be transfused using *ABO-cellular compatible transfusion*.

The second human antigen that plays a crucial role in blood compatibility is Rh D. Its non-existence causes (denoted with a minus sign "−") antibodies forming after a transfusion in the patient's blood stream. Since the donor blood has the same feature, although in theory an Rh D− blood can be given to an Rh D+ ABO-identical patient, it is avoided in practice whenever possible.[13]

Therefore, eight blood types are relevant for regular red blood cell transfusion. Moreover, in some populations such as those in Asia,[14] Rh D− blood is so rare that effectively there are only four blood types.

Platelets

Platelets are tiny cells in the blood that form clots and stop bleeding. Platelet transfusions are required when patients have a low platelet count. They are needed by the body to prevent or stop bleeding. These patients include people who are receiving chemotherapy, who have had a bone marrow

[12]For example, red blood cell packs can also carry varying amount of plasma.

[13]On the other hand, type O Rh D− blood is frequently transfused in emergency situations to ABO non-identical patients, such as during paramedic interventions. For this reason it is also dubbed as the *global-donor* blood type.

[14]For example, in China the Rh D antigen exists in more than 99% of the whole population.

transplant, who take medicines that interfere with platelet function, who are bleeding due to traumatic injuries, and people with chronic diseases.However, owing to concerns over bacterial contamination due to their storage at room temperature, platelets have a short shelf life compared to most other blood components. In many countries platelets can be stored between four and seven days.

The use of platelets has increased more than other blood components in the last 15 years (McCullough, 2010). According to the American Red Cross, every 30 seconds someone needs platelets (Red Cross of America, 2020b). Platelets are most effectively collected using a technique called *aphaeresis* once in every one-to-two weeks up to 24 times a year. This technique only takes platelets out of the donor's blood leaving the rest of the components in the blood stream. By this method, a donor can donate significantly more platelets than she would by donating her whole blood. Thus, typically a donor can donate either red blood cells (and whole blood) or platelets, but not both at the same time.

For platelets, compatibility practices vary tremendously among countries. As platelet transfusion is a repeated therapy for many patients, ABO-identical transfusion is preferred for such patients. In some countries ABO-identical platelet transfusion is required even for acute needs, e.g., in China, the policy is to adhere to ABO-identical transfusion unless there is a life threatening emergency (China, 2014). However, ABO-cellur compatible transfusions of platelets are common in other countries such as the U.K. (Norfolk, 2013).

For platelets, research has identified that Rh D compatibility is not required and as a result not practiced (for example, see Cid and Yazer, 2013).

Plasma

Plasma is the non-cellular, protein- and anti-body-rich liquid component of blood. Its transfusions are needed by patients with liver failure, heart surgery, severe infections, and serious burns. In particular, it is used to replace clotting factors and also in patients who need to quickly reverse the effects of blood thinning medications. Convalescent plasma, the antibody-rich plasma of a recovering patient from an infectious disease with no other known cure such as ebola and most recently COVID-19, is commonly used to directly treat or to produce drugs against the disease.

Plasma has the longest shelf life among the three main blood components. For plasma transfusion, ABO compatibility is reversed as plasma does not have any red blood cells. The antibodies in plasma can attack the patient's red blood cell antigens. Thus, the patient should not possess the antigens that the donor does not have, either. As a result, type AB plasma can be transfused to all patients, while type O plasma can be transfused to only type O patients, and type A (B) plasma can be transfused to type O or A (B). This is known as the *ABO-plasma compatibility*. For plasma, like platelets, research has identified that Rh D compatibility is not required and as a result not practiced.

The preferred plasma collection method is called *plasmapheresis*, which keeps other blood components in the donor blood stream while only extracting plasma. A healthy donor can donate twice in every seven days up to 24 times a year using this method (Norfolk, 2013). A notable exception is convalescent plasma used to treat infectious diseases, as the donor's plasma cannot sustain forever the required antibodies to fight off the disease. Only a few donations can be made by a convalescent plasma donor. Although plasma can be obtained from donated whole blood together with red blood cells in almost equal amounts, for frequent and effective donation, plasmapheresis is used.

2.2 Blood Demand of a Patient

The amount of blood component needed to treat each medical condition is idiosyncratic. For example, for surgeries, although the actual consumption of red blood cells per patient is close to 3.5 units (Collins et al., 2015), this amount displays a large variance due to the patient's condition.

Besides the idiosyncratic demand, the patient needs are usually in a range of units such that each amount in the range can be transfused; however, more transfused units can be better under different outcome or preference metrics. We give three common examples of patient demand that have this common thread.

First, a range of units may indeed be feasible to be transfused to a patient with a particular medical condition such that more units lead to better expected patient outcomes. For example, platelets have prophylactic use to prevent bleeding in patients with bleeding disorders. In such cases either higher doses in lower frequency or lower doses in higher frequency are both practiced and used (Stroncek and Rebulla, 2007). Norol et al. (1998) show that high and very high dose lead to significantly better platelet outcomes in patients with respect to medium and lower doses, although both high and medium doses are used in practice.

Second, the need of a patient may be ex-ante uncertain. For example, surgery patients may have a range of red blood cell needs given by some minimum and maximum units to be transfused during the operation. For cautionary reasons, surgeons often order significantly more blood than the patient ends up using during many surgical procedures. Collins et al. (2015) report that 72% of the red blood cells ordered for surgeries go unused. In some operations the ratio of ordered to used red blood cells go as high as 11 to 1 on average in elective liver resection surgical procedures (Cockbain et al., 2010). These rates indicate that surgeons are quite risk averse and yet usually most surgeries ex-post end up using much less blood than ordered. Indeed, Collins et al. (2015) note that surgical blood loss can be unpredictable, so some leeway for ordering red blood cells that ultimately go unused is necessary for safe patient care.

Third, blood components such as platelets and red blood cells are transfused to patients with chronic conditions and are administered in small doses over time. For example, Marwaha and Sharma (2009) states that patients undergoing chemotherapy require platelet transfusion once in at least every three days, and when their bone marrow is adversely affected, every day. In such cases, more units are preferred to fewer in an interval time, although various units of transfusions can be conducted in this interval.

2.3 Blood-Bank Policies for Replacement Donors

Populous countries such as Pakistan, Brazil, and Mexico almost entirely collect their blood components through replacement donor programs. On the other hand, countries such as India and China, rely heavily on these programs to meet the demand not met through volunteer non-remunerated donations. In many cases, patients' replacement donors can either donate before or after the patient receives blood. Since direct donation from a donor to the patient (even if they are compatible) is not practiced in modern medicine due to health concerns (i.e., the donor blood may need to be tested for

certain diseases first), the blood bank is used as an intermediary.[15]

Blood banks work with hospitals and blood centers. Hospitals relay the needs of patients to the blood bank while blood centers collect donations form volunteer non-remunerated donors and replacement donors. Hospitals are often required to maintain a small inventory of their own (for example see Indian Department of Health, 2016).

The most common practice in current blood banks worldwide is to announce, either officially or unofficially, a preset *exchange rate* between the units of blood received and supplied irrespective of the blood type sought or donated. Blood banks provide the blood needs of patients exclusively based on these rates. We next give some examples of the policies practiced around different parts of the world by donor replacement programs.

In China, where donor replacement programs (known as *mutual help*) function semi-officially throughout the country, different policies are in place in different localities: In Beijing, the exchange rate has been one unit replacement per unit requested and there is no restriction on the patient blood type. In Xi'an, during periods of shortages, a patient has the priority of receiving three units of blood for every unit she has donated before, and she has priority of receiving one unit for every unit her replacement donors donate now. These priorities can only be used once and the patient may need to make up the difference. For instance, if she has donated one unit before and needs five units now, then his family has to donate two units now in order for him to get five units (She, 2020). In Guangzhou, there is not necessarily a fixed relation between the amounts received and the amounts donated (Chen, 2012).

India has the largest official donor replacement programs in the world after Pakistan. In Delhi, regardless of the blood volume she needs, the patient is required to bring forward one replacement donor, unless the intervention needed is an emergency surgery (Indian Department of Health, 2016).

In Cameroon and Congo, the exchange rate has been two replacement units per one unit of blood the patient requests, as almost 25% of the donations are not suitable for transfusion due to infections (Tagny, 2012). The same exchange rate is also used in Puerto Vallarta, Mexico (Thompson, 2020).

In Tucuman, Argentina the exchange rate is fixed at one unit replacement per unit requested, however it is not as strictly enforced.[16]

2.4 Logistical Constraints and Comparisons with Solid Organ Exchanges

The feasibility of blood transfusion primarily depends on the compatibility of the blood component types of the patient and the donor. Therefore, donor replacement programs operate on similar principles with the practice of organ exchanges. However, there are a number of important differences. To begin with, the logistical constraints of blood donation are negligible comparable to those in organ transplantation. The blood donation process takes only a few hours and its effects wear off in less than 24 hours. On the other hand, organ transplantations carry risks and require careful planning weeks before and after the operation. Once extracted, blood and blood components can be stored for a certain period of time, which can facilitate the designer's choice of optimal timing of assignments.

[15]Moreover, if the replacement donor is the husband of a female patient, a direct donation is not recommended, as it can lead to antibodies forming in the patient that may complicate future pregnancies (Memorial Sloan, 2020).

[16]Based on personal communication with the director of Tucuman Blood Bank, Dr. Felicitas Agote, on July 7, 2020.

Moreover, many blood banks and hospitals often operate in coordination making it possible to obtain the necessary blood units from neighboring facilities. These lead to the observation that in blood allocation with replacement donors, the possibility of reneging by a donor is not as much of a concern as in organ exchanges.

Unlike the case with organ transplantation, when a donor replacement problem is viewed as an exchange problem, the absence of logistical constraint together with the ability to store blood components make it possible to incorporate cycles and chains of arbitrary length into an allocation. Moreover, it does not really matter which donor donates to which patient as long as compatibility requirements are met. Thus, the matching feature of the problem is not as important as in organ exchanges and what is needed is keeping track of the overall allocations and feasibility requirements.

Another important distinguishing feature of the current problem is that while in organ allocation problems, the exchange rate is typically one-for-one, whereas in the current context, one can conceive of different exchange rates. Exchange rates can be imposed exogenously by the planner or may be determined endogenously.

The logistical ease and flexibility in blood allocation have led to different and innovative incentivization schemes to promote blood donation. The assignment of voucher credits have been a popular approach used in practice. For example, blood assurance programs in the US guarantee each volunteer non-remunerated donor or her tax-code dependents exactly the same amount of blood donated in the event of a need in the future.[17] Similar programs have been traditionally implemented in China. Recently, Kominers, Pathak, Sönmez, and Ünver (2020) proposed a similar incentive scheme for COVID-19 convalascent plasma donation.[18] Donor replacement programs differ from these proposals, as we are considering the improvement of the existing programs that do not have any voucher or memory features, nor the pay-it-backward or pay-it-forward features discussed in the literature. Thus, blood allocation is more in line with the analogy of organized organ exchanges without simultaneity and other severe constraints.

3 The Model

In this section, we consider the market for a single blood component, which we refer to simply as blood. Later in Section x, we discuss how to integrate different blood component markets together.

3.1 The Blood Allocation Problem

Let I be a set of **patients** and $\mathcal{B}$ be the set of **blood types**. Each $X \in \mathcal{B}$ denotes blood's specific type used in compatibility requirements.[19] Each patient $i \in I$ has type $\beta_i \in \mathcal{B}$ blood, and needs a **maximum** of $\overline{n}_i \in \mathbb{Z}_{++}$ units of blood. For each $X \in \mathcal{B}$, $\mathcal{C}(X) \subseteq \mathcal{B}$, $\mathcal{C}(X) \neq \emptyset$, is the set of blood types compatible with a type X patient. Each patient i also has a set of willing **replacement donors** D_i, which can possibly be empty, such that each donor $d \in D_i$ can provide one unit of type β_d blood.

[17] An example is North Carolina Cape Fear Valley Blood Bank's program (Cape Fear Valley, 2020).

[18] Also a similar voucher-based scheme is used for kidney exchange in the US (Veale et al., 2017), and it has been proposed for compatible pairs to participate in kidney exchange (Sönmez, Ünver, and Yenmez, 2020).

[19] For red blood cells or whole blood there are eight blood types ($O+, O-, A+, A-, B+, B-, AB+, AB-$), for platelets or plasma there are four blood types (O, A, B, AB). However, our model and results are independent of the particular structure of blood-type compatibility or number of blood types.

Let $\mathcal{D}_i$ be the collection of all possible donor sets that a patient $i \in I$ can bring forward. Assume that if $D_i \in \mathcal{D}_i$, then $D_i' \in \mathcal{D}_i$ for every $D_i' \subseteq D_i$. Let $\overline{n} = (\overline{n}_i)_{i\in I}$, $D = (D_i)_{i\in I}$, $\mathcal{D} = \Pi_{i\in I}\mathcal{D}_i$, $\beta_I = (\beta_i)_{i\in I}$, and $\beta_D = (\beta_d)_{d\in\cup_{i\in I}D_i}$.

The **blood bank**, denoted as b, has v_X units of type X blood in its **inventory** for each $X \in \mathcal{B}$. Let $v = (v_X)_{X\in\mathcal{B}}$. The blood bank guarantees a **minimum** $\underline{n}_i \in \{0, 1, \ldots, \overline{n}_i\}$ units of blood for each patient $i \in I$ if the patient participates in the program, i.e., if she provides a qualified set of donors. Let $\underline{n} = (\underline{n}_i)_{i\in I}$. Assume that for any non-empty subset of blood types $\mathcal{B}' \subseteq \mathcal{B}$,

$$\sum_{i\in I:\beta_i\in\mathcal{B}'} \underline{n}_i \leq \sum_{X\in\cup_{Y\in\mathcal{B}'}\mathcal{C}(Y)} v_X.$$

Therefore, the blood bank carries enough blood to meet the minimum guarantees regardless of the replacement donors that will be brought by the patients.[20]

Minimum guarantee is a long-run policy variable unlike the maximum threshold, which is determined by patient's medical condition. Minimum guarantee can refer to the minimum threshold needed by the patient to treat the patient's condition or depending on the inventory of the blood bank, this can be set lower to satisfy the above displayed equation.[21]

3.2 Feasible Schedules and Allocations

Since each patient demands and (possibly) supplies blood through her replacement donors, we want to impose restrictions on the relationship between the amount of blood received and the amount of blood supplied. A **schedule** is a pair of non-negative integers (r, s), where r denotes the amount of blood received and s denotes the amount of blood supplied. For every patient $i \in I$, her **feasible schedule function** $\mathcal{S}_i(\cdot)$ assigns a non-empty set of schedules $\mathcal{S}_i(D_i)$ to each donor set $D_i \in \mathcal{D}_i$ such that

- $\mathcal{S}_i(D_i) \subseteq \{0, \underline{n}_i, \ldots, \overline{n}_i\} \times \{0, \ldots, |D_i|\}$, and
- $\mathcal{S}_i(D_i) = \{(0, 0)\}$, or $\min\{r : (r, s) \in \mathcal{S}_i(D_i)\} = \underline{n}_i$.

The definition of a feasible schedule function captures possible complementarities between units received and supplied. If $\mathcal{S}_i(D_i) = \{(0, 0)\}$, the patient's donor set does not meet the minimum requirement by the blood bank for receiving her minimum guarantee of $\underline{n}_i$ units of blood. On the other hand, if $\min\{r : (r, s) \in \mathcal{S}_i(D_i)\} = \underline{n}_i$, then the donor set D_i satisfies the requirement, and thus, the bank is obliged to give her $\underline{n}_i$ units of blood. We assume that the feasible schedule functions are fixed throughout the paper except in the comparative static analysis in Section 4.4.

We pause the presentation of the model and emphasize the flexibility and generality embedded in our model through the following examples.

[20] Specifically, it follows from Hall's theorem that this assumption is necessary and sufficient for the blood bank to be able to provide $\underline{n}_i$ units of compatible blood to each patient i using its inventory.

[21] Minimum guarantees may also apply due to the good deeds she has done in the past. Some countries use blood assurance (such as the US) or voucher programs (such as China) so that a patient who has donated blood in the past can receive credits for transfusions. Thus, she may be covered up to a certain units of blood even if she does not bring forward any donors now (see Section 5.1.5).

Example 1. (Patient Examples).

- **Urgent care patients:** A patient i with a medical urgency and exactly determined need can be represented by her minimum guarantee being equal to her maximum need, i.e., $\underline{n}_i = \overline{n}_i$. Moreover, for any set of donors including empty-set she brings forward, she receives $\overline{n}_i$ units of blood, and the blood bank does not charge her any donors. Thus, for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \big\{(\overline{n}_i, 0)\big\}.$$

- **Elective surgery patients:** A patient with an elective surgery requires on average $\underline{n}_i$ units of blood and she may, with some chance, need excess blood. So $\overline{n}_i > \underline{n}_i$. Moreover, the exchange rate for her is to supply one unit of blood for every unit received. Thus, her feasible schedule function is given as, for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \begin{cases} \big\{(0,0)\big\} & \text{if } \big|D_i\big| < \underline{n}_i \\ \big\{(r,s) \in \mathbb{Z}_+^2 : s = r \text{ and } \underline{n}_i \leq r \leq \min\{\overline{n}_i, \big|D_i\big|\}\big\} & \text{otherwise} \end{cases}.$$

- **Patients waiting for platelets:** A chemotherapy patient requires platelet treatment and needs a transfusion of one unit of platelet every day. She can have the platelets stored for at most 5 days. The exchange rate is again one-to-one: for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \big\{(r,s) \in \mathbb{Z}_+^2 : s = r \text{ and } 0 \leq r \leq \min\{\overline{n}_i, \big|D_i\big|\}\big\},$$

  where $\overline{n}_i = 5$. ∎

A feasible schedule function is also a policy lever of the blood bank. We model some policies (for non-urgent care patients) used around the world as possible feasible schedule functions.

Example 2. (Real-Life Policy Examples).

- **The Standard One-to-One Policy:** Many blood allocation programs with replacement donors use one unit received per unit supplied exchange rate.[22] This leads to the following feasible schedule function: for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \begin{cases} \big\{(0,0)\big\} & \text{if } \big|D_i\big| < \underline{n}_i \\ \big\{(r,s) \in \mathbb{Z}_+^2 : s = r \text{ and } \underline{n}_i \leq r \leq \min\{\overline{n}_i, \big|D_i\big|\}\big\} & \text{otherwise} \end{cases}.$$

- **The Delhi Policy:** According to Delhi, India, guidelines (Indian Department of Health, 2016), each patient needs to provide one donor regardless of the amount of blood she consumes. This

[22] This is also consistent with the focus of real-life living-donor organ exchange programs, as well as the market design literature on them: $\overline{n}_i = 1$ for kidney and liver exchange. Moreover, for multi-unit exchange of indivisible goods, the recent literature on time banks and shift exchange (Andersson et al., 2020; Manjunath and Westkamp, 2019) has focused on the exchange of one unit of endowment in return of one unit of consumption.

leads to the following feasible schedule function: for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \begin{cases} \{(0,0)\} & \text{if } D_i = \emptyset \\ \{(r,s) \in \mathbb{Z}^2_+ : 0 \leq s \leq 1, \text{ and } \underline{n}_i \leq r \leq \overline{n}_i\} \setminus \{(0,1)\} & \text{otherwise} \end{cases}.$$

- **The Cameroon, Congo and Mexico Policy:** In Cameroon, Congo and Mexico, for each unit of blood received, two units of blood have to be supplied (Tagny, 2012; Thompson, 2020). Therefore, for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \begin{cases} \{(0,0)\} & \text{if } |D_i| < 2\underline{n}_i \\ \{(r,s) \in \mathbb{Z}^2_+ : s = 2r \text{ and } \underline{n}_i \leq r \leq \min\{\overline{n}_i, \lfloor |D_i|/2 \rfloor\}\} & \text{otherwise} \end{cases}.$$

- **The Xi'an Policy:** In Xi'an, China, a patient is guaranteed three units for each unit she has donated before, and the exchange rate is one-to-one beyond this guarantee (She, 2020). Let x_i be the amount of previous donation from the patient, which is assumed to be exogenous to the problem. This leads to the following feasible schedule function.

  If $\overline{n}_i \leq 3x_i$, then for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \{(\overline{n}_i, 0)\}.$$

  If $\overline{n}_i > 3x_i$, then for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \{(r,s) \in \mathbb{Z}^2_+ : s = r - \underline{n}_i \text{ and } \underline{n}_i \leq r \leq \min\{|D_i| + \underline{n}_i, \overline{n}_i\}\},$$

  where $\underline{n}_i = 3x_i$.

- **The Jiangsu Policy:** In Jiangsu, China, the standard one-to-one policy is used with the restriction that the type of the blood supplied should be identical to the type of the patient (Chen, 2012): for every $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \begin{cases} \{(0,0)\} & \text{if } |\{d \in D_i : \beta_d = \beta_i\}| < \underline{n}_i \\ \{(r,s) \in \mathbb{Z}^2_+ : s = r, \ \underline{n}_i \leq r \leq \min\{\overline{n}_i, |\{d \in D_i : \beta_d = \beta_i\}|\}\} & \text{otherwise} \end{cases}.$$

  This is akin to no exchange (autarky) treatment and all the blood used is given by the patients' donors. ∎

A **blood allocation problem with replacement donors** is denoted as $\mathcal{P} = \langle I, \beta_I, \overline{n}, D, \beta_D, v, \underline{n}, \mathcal{S} \rangle$. The profile of minimum guarantees $\underline{n}$, inventory vector v, and feasible schedule functions $\mathcal{S}$ are interrelated and can be policy levers of the blood bank although in many places they are mandated by bylaws (for example see Indian Department of Health, 2016) and they do not typically change as easily as the other parameters of the problem.[23] Thus, we fix every component of a problem except

[23] In such cases v can be interpreted as the minimum required inventory level to be kept in stock (see Indian Department of Health, 2016). This is mostly ensured through a blood exchange program between blood banks and this is commonly practiced in many parts of the world.

D.[24] Then a problem is simply denoted as a donor profile D.[25]

Given a problem $D \in \mathcal{D}$, an **allocation** α consists of non-negative integers $\alpha_X(i)$ for each $i \in I$ and $X \in \mathcal{C}(\beta_i)$, and $\alpha(d) \in \{0,1\}$ for each $d \in \cup_{i\in I} D_i$ such that

1. for every $X \in \mathcal{B}$, $\sum_{i\in I: X\in\mathcal{C}(\beta_i)} \alpha_X(i) \leq v_X + \sum_{d\in\cup_{i\in I} D_i:\beta_d=X} \alpha(d)$,
2. for every $i \in I$, $\big(\alpha(i), \sum_{d\in D_i} \alpha(d)\big) \in \mathcal{S}_i(D_i)$, where $\alpha(i) = \sum_{X\in\mathcal{C}(\beta_i)} \alpha_X(i)$.

In an allocation, the patients only receive blood that are deemed medically compatible with them. An allocation specifies the amount of blood of each compatible type that a patient receives, as well as which of her donors donate. The first condition in the definition of an allocation makes sure that, for each blood type, the allocated blood is not more than the sum of the existing blood in the blood bank and the collected blood from the patients' donors. Thus, it is a market clearing condition. The second condition requires that each patient's schedule induced by the allocation is in her feasible schedule set, which is determined by the set of donors that she brings forward. There always exists an allocation by definition. Denote the set of all the allocation for D as $\mathcal{A}(D)$.

3.3 Preferences, Mechanisms, and Incentives

We next introduce the patients' preferences over the allocations. Each patient first and foremost cares about the units of blood she receives. She has monotonic preferences over the units of blood in her range of needs $\{\underline{n}_i, \underline{n}_i + 1, \ldots, \overline{n}_i\}$.[26] Fixing the amount of blood she receives, she would like fewer of her donors to donate and thus, has monotonic preferences over less donation by her donors.

Formally, given a problem $D \in \mathcal{D}$, each patient $i \in I$ has a preference relation, denoted by $\succsim_i$, over the allocations $\mathcal{A}(D)$. Its asymmetric part, the strict preference relation $\succ_i$, is defined for every $\alpha, \alpha' \in \mathcal{A}(D)$ as:

$$\alpha \succ_i \alpha' \iff \alpha(i) > \alpha'(i) \text{ or } \Big[\, \alpha(i) = \alpha'(i) \text{ and } \sum_{d\in D_i} \alpha(d) < \sum_{d\in D_i} \alpha'(d) \,\Big],$$

and its symmetric part, the indifference relation $\sim_i$, is defined as:

$$\alpha \sim_i \alpha' \iff \alpha(i) = \alpha'(i) \text{ and } \sum_{d\in D_i} \alpha(d) = \sum_{d\in D_i} \alpha'(d).$$

Thus, a patient strictly prefers the allocation in which she receives more blood, and if she receives the same amounts of blood under the two allocations, then she prefers the allocation in which her donors donate less blood in total. Two allocations α and α' are **welfare equivalent** if $\alpha \sim_i \alpha'$ for every $i \in I$, i.e., for each patient the amounts of blood received and supplied are the same at the two allocations. An allocation α is **efficient** if it is not Pareto dominated by another allocation, i.e., there is no allocation $\alpha' \in \mathcal{A}(D)$ such that $\alpha' \succsim_i \alpha$ for every $i \in I$ and $\alpha' \succ_j \alpha$ for some $j \in I$. A **mechanism** is a function f that maps each problem $D \in \mathcal{D}$ to an allocation $f(D) \in \mathcal{A}(D)$. A mechanism f is **efficient** if for every $D \in \mathcal{D}$, $f(D)$ is efficient.

[24] Without loss of generality, we use this notation for brevity, assuming β_D is determined once D is given.
[25] Also later we will discuss the affects of changes in these policy levers.
[26] Monotonicity of a patient's preferences in the units of blood she receives was motivated in Section 3.1.

We consider the patients' incentives for bringing forward their donors. As alluded in Section 2.4, blood donation is not as costly as solid organ donation, leading to a much less invasive procedure and very fast replenishment of blood. Therefore, as long as the patient does not receive less blood, providing more donors to the system may not be as undesirable for the patient. Based on this motivation, we first introduce a weaker incentive compatibility concept. Specifically, we require that having a larger set of donors never causes the patient to receive less blood.

A mechanism f is **donor monotonic** if for any $D \in \mathcal{D}, i \in I$ and $D'_i \subseteq D_i$ we have

$$f(D)(i) \geq f(D'_i, D_{-i})(i).$$

Next, we introduce our stronger incentive compatibility concept. A mechanism f is **strongly donor monotonic** if for any $D \in \mathcal{D}, i \in I$ and $D'_i \subseteq D_i$ we have

$$f(D) \succsim_i f(D'_i, D_{-i}).$$

A mechanism is strongly donor monotonic if bringing forward any subset of her donors does not make any patient strictly better off.

4 Sequential Targeting

We seek mechanisms that guarantee efficiency together with donor monotonicity or strong donor monotonicity. The following class of mechanisms will be key in achieving our objectives.

Let $\{N_k\}_{k=1}^{\bar{k}}$, $\bar{k} \geq 2$, be a sequence of nonempty subsets of patients, which we refer to as **target sets**, and $\tau : \{2, \ldots, \bar{k}\} \to \{\max, \min\}$ be a **target function** that designates for each subset N_k with $k \geq 2$ whether a maximization or minimization target will be achieved for the patients in N_k, given that all the previous targets designated for the patients in $N_1, \ldots, N_{k-1}$ are already achieved.

Maximization, denoted by $\tau(k) = \max$, means that the total amount of blood received by the patients in N_k is maximized given that the previous objectives are already satisfied.

Minimization, denoted by $\tau(k) = \min$, means that the total amount of blood supplied by the patients in N_k is minimized given that the previous objectives are already satisfied.

Our first target is always to maximize the total amount of blood received by the patients in N_1.[27] To define our main class of mechanisms, we introduce an iterative procedure given the target sets $\{N_k\}_{k=1}^{\bar{k}}$ and the target function τ.

Sequential Targeting Procedure:

Given a problem $D \in \mathcal{D}$, we construct a sequence of subsets of allocations $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \ldots \supseteq \mathcal{A}_{\bar{k}}$ in $\bar{k}$ steps after initializing $\mathcal{A}_0 = \mathcal{A}(D)$.

Step 1. Let $\mathcal{A}_1 \subseteq \mathcal{A}_0$ be the subset of allocations that maximize the amount of blood received by the patients in N_1, that is

$$\mathcal{A}_1 = \arg\max_{\alpha \in \mathcal{A}_0} \sum_{i \in N_1} \alpha(i).$$

[27] See also the Condition (1) below the procedure definition.

$\vdots$

Given that $\mathcal{A}_{k-1}$ is constructed in Step $k-1$, Step k is defined as follows.

Step k. Two cases are possible:

- If $\tau(k) = \max$, let $\mathcal{A}_k \subseteq \mathcal{A}_{k-1}$ be the subset of allocations in $\mathcal{A}_{k-1}$ that maximize the amount of blood received by the patients in N_k, that is:

$$\mathcal{A}_k = \arg\max_{\alpha \in \mathcal{A}_{k-1}} \sum_{i \in N_k} \alpha(i).$$

- If $\tau(k) = \min$, let $\mathcal{A}_k \subseteq \mathcal{A}_{k-1}$ be the subset of allocations in $\mathcal{A}_{k-1}$ that minimize the amount of blood supplied by the patients in N_k, that is:

$$\mathcal{A}_k = \arg\min_{\alpha \in \mathcal{A}_{k-1}} \sum_{d \in \cup_{i \in N_k} D_i} \alpha(d).$$

The set of allocations $\mathcal{A}_{\bar{k}}$ is the outcome of the procedure.

Observe that $\mathcal{A}_{\bar{k}}$ may involve allocations that are not welfare equivalent. To have a well-defined mechanism that achieves our desiderata, we make two assumptions jointly on the target sets $\{N_k\}_{k=1}^{\bar{k}}$ and the target function τ:

1. For every $k \in \{2, \ldots, \bar{k}\}$, if $\tau(k) = \min$, then for any $i \in N_k$ there exists $k' < k$ such that $i \in N_{k'}$ and $\tau(k') = \max$. That is, if we are going to minimize the blood supplied by a group of patients, then for each of those patients, we should have maximized the blood received at an earlier step as part of some group.

2. $\bar{k} \geq 2\big|I\big|$ and the last $2\big|I\big|$ sets, $\{N_k\}_{k=\bar{k}-2|I|+1}^{\bar{k}}$, are each a singleton such that each patient $i \in I$ appears exactly twice as $N_k = N_\ell = \{i\}$ for some distinct $k, \ell \geq \bar{k} - 2\big|I\big| + 1$, with one target as $\tau(k) = \max$ and the other target as $\tau(\ell) = \min$.

As the preferences of the patients are lexicographic in receiving more and then supplying less, the first condition will ensure the efficiency of sequential targeting. The second condition is assumed to guarantee that the outcome allocations of the procedure are welfare equivalent: We use the last $2\big|I\big|$ targets as tie breakers among the patients, in case the previous targets lead to multiplicity of allocations in terms of welfare levels. Some target sets may be redundant as the previous targets may have already achieved welfare uniqueness.

A **sequential targeting mechanism** is induced by the above procedure with respect to a sequence of target sets $\{N_k\}_{k=1}^{\bar{k}}$ and a target function τ that satisfy the above two conditions: it chooses an allocation from the outcome set of the procedure, $\mathcal{A}_{\bar{k}}$, executed for each problem $D \in \mathcal{D}$.

Different target set sequences and target functions induce different sequential targeting mechanisms. This class is rich with many possibilities, including two interesting and important examples, priority mechanisms and maximal mechanisms with priority tie-breakers, which were examined by Manjunath

and Westkamp (2019) and Andersson et al. (2020), respectively, in similar setups. In our problem, these two mechanisms are more broadly defined due to the more general nature of our setup.

In a **priority mechanism**, patients are processed one at a time using a priority order; let $i_1, i_2, \ldots, i_n$ be this order: the mechanism first maximizes the welfare of patient i_1; then, among all allocations that achieve this goal, it maximizes the welfare of patient i_2, and then patient i_3, so on. Formally, target sets are singletons such that $N_{2k-1} = N_{2k} = \{i_k\}$ for every $k \in \{1, 2, \ldots, n\}$. Target function τ is defined as $\tau(2) = \min$, $\tau(2k-1) = \max$ and $\tau(2k) = \min$ for every $k \in \{2, 3, \ldots, n\}$.[28]

In a **maximal mechanism with priority tie-breakers**, the total blood allocation to patients is maximized and the total blood donation by all replacement donors is minimized. Then using a tie-breaker priority order, such as $i_1, i_2, \ldots, i_n$, among all total welfare maximizing allocations, the welfare of patient i_1 is maximized. Subject to these goals being satisfied, the welfare of patient of i_2 is maximized, and so on. Formally, the sequential targeting mechanism implementation of these mechanisms require the first two target sets to be the whole set of patients: $N_1 = I$ and $N_2 = I$, the remaining target sets are singletons such that $N_{2k-1} = N_{2k} = \{i_{k-1}\}$ for every $k \in \{2, 3, \ldots, n+1\}$, and target function τ is defined as $\tau(2) = \min$, $\tau(2k-1) = \max$ and $\tau(2k) = \min$ for every $k \in \{2, 3, \ldots, n+1\}$.[29]

4.1 Efficiency

Our first main result shows that any sequential targeting mechanism achieves a Pareto efficient allocation:

Theorem 1. *Every sequential targeting mechanism is efficient.*

In a sequential targeting mechanism, by definition each patient $i \in I$ appears in at least one step's target set with maximization target, before she ever appears in a step with minimization target. This takes care of the lexicographic nature of patient preferences in received blood first. The key observation is that in a target set with a maximization target, making some patient better off by assigning her more blood would make some patient who is processed before or in the same step worse off or would not be feasible. Similarly, in a target set with a minimization target, making a patient better off by reducing the donation made by her replacement donors, would make some patient processed before or in the same step worse off or would not be feasible.

As mentioned at the end of previous section, welfare maximization could be one of the possible objectives in Steps 1 and 2 of a sequential targeting mechanisms. However, under different Step 1 targets, resulting sequential targeting mechanisms may not be maximal, i.e., may not maximize the total amount of blood allocated to patients. However, our first theorem shows that the resulting mechanism is necessarily Pareto efficient.

[28] Priority mechanisms are counterpart of serial dictatorships that are widely studied in the context of object allocation with multi-unit demand and strict preferences.

[29] Maximal mechanisms have found wide-spread application in the context of kidney exchange, which involves single-unit demand for each patient. For example, in the US the UNOS National Kidney Exchange Program and Alliance for Paired Donation have adopted maximal mechanisms, though they use different tie-breakers from our patient-based priority approach (Sönmez and Ünver, 2017).

4.2 Donor Monotonicity

In this subsection we explore the incentives faced by the patients in bringing forward their full set of donors to the blood bank.

For a general profile of feasible schedule functions $(\mathcal{S}_i)_{i\in I}$, although the sequential targeting mechanisms are efficient, they may not be incentive compatible even in the donor monotonicity sense. We will state regularity conditions on feasible schedule functions to which many real-life policies such as one-to-one exchange obey.

We make three assumptions which make sure that the sequential targeting mechanisms are donor monotonic. They all have natural explanations. The first assumption is about the convexity of a feasible schedule set for a given set of donors. Generally, a set $S \subseteq \mathbb{Z}^2_+$ is **L-convex** (where L stands for lattice) if for every $x, y \in S$, we have

$$\left\lfloor \frac{x+y}{2} \right\rfloor, \left\lceil \frac{x+y}{2} \right\rceil \in S.$$

L-convexity is one of the two most used generalizations of convex sets and functions to discrete domains (see Murota (2013) for the general use of discrete convexity notions and discrete convex analysis).

Assumption 1 (L-convexity of feasible schedule set)**.** *The feasible schedule set $\mathcal{S}_i(D_i)$ is L-convex for every $i \in I$ and $D_i \in \mathcal{D}_i$.*

A geometric illustration with three examples of L-convex feasible schedule sets is given in Figure 1. Assumption 1 also guarantees that an outcome allocation of a sequential targeting mechanism can be found in polynomial time, as we show in the appendix.

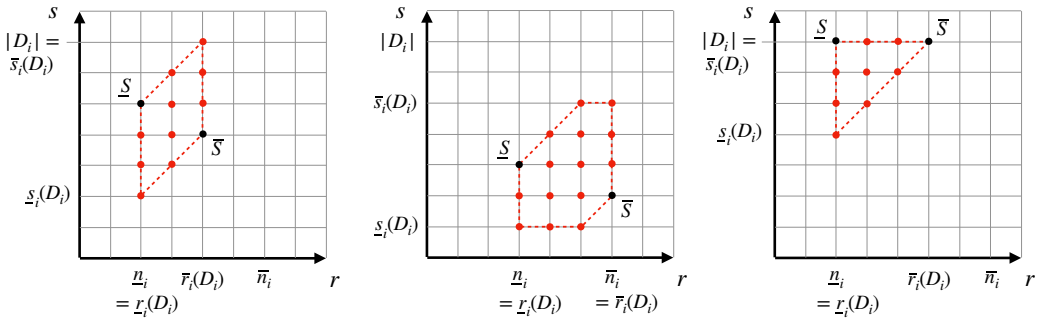


Figure 1: Illustration of Assumption 1, L-convexity: The feasible schedule set $\mathcal{S}_i(D_i)$ is the integral points of a convex polygon with integral corners and at most six edges of slopes 1, 0, or ∞. It is a lattice with the minimum schedule being marked as $(\underline{r}_i(D_i), \underline{s}_i(D_i))$ and the maximum schedule being marked as $(\overline{r}_i(D_i), \overline{s}_i(D_i))$. Observe that by the definition of feasible schedule functions $|D_i| \geq \overline{s}_i(D_i)$ and $\overline{n}_i \geq \overline{r}_i(D_i)$. The best schedule $\overline{S}$ and worst schedule $\underline{S}$ are also marked in each graph to show the lexicographic orientation of the patient's preferences in more blood received first and less blood supplied second.

A very special case is the feasible schedule set induced by the classical one-to-one exchange rate between the blood received and supplied, as depicted in Figure 2.

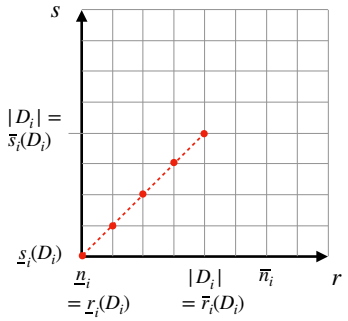


Figure 2: A special case that satisfies Assumption 1 is the classical one-to-one exchange rate policy. In this example we assume $\overline{n}_i > |D_i|$; if $|D_i| \geq \overline{n}_i$, then $\overline{r}_i(D_i) = \overline{n}_i$ is the maximum amount of blood that can be received.

The second assumption generalizes the idea that each unit of blood has a positive "price". It says that when a patient receives more (or less) blood, there is a feasible schedule in which her donors also donate more (or less) blood. Note that the patient does not have to supply more blood when she receives an additional unit, but this assumption says that such a schedule is feasible. It is stated as follows:

Assumption 2 (Feasibility of positive price)**.** *For every patient $i \in I$ and donor set $D_i \in \mathcal{D}_i$, the feasible schedule set $\mathcal{S}_i(D_i)$ satisfies the following:*

- *if $(r,s),(r',s') \in \mathcal{S}_i(D_i)$, $r' > r$ and $s < |D_i|$, then there exists $s'' > s$ such that $(r',s'') \in \mathcal{S}_i(D_i)$; and*
- *if $(r,s),(r',s') \in \mathcal{S}_i(D_i)$, $r' < r$ and $s > 0$, then there exists $s'' < s$ such that $(r',s'') \in \mathcal{S}_i(D_i)$.*

That is, given a feasible schedule, if the patient can potentially receive a larger (or smaller) amount of blood, then the patient can potentially receive this amount by supplying more (or less) as long as the supply does not exceed her number of donors (or is non-negative). The one-to-one exchange rate policy satisfies the feasibility of positive price assumption: each additional unit received costs exactly one unit supplied.

L-convexity and feasibility of positive price are independent. For example, the two-to-one exchange rate policy satisfies feasibility of positive price but not L-convexity; the second graph in Figure 1 violates feasibility of positive price as it has a "flat top" at $s = \overline{s}_i(D_i) < |D_i|$ and a "flat bottom" at $s = \underline{s}_i(D_i) > 0$, while it is L-convex. The other graphs in this figure satisfy feasibility of positive price, although the third one has a "flat top." This is because the "flat top" occurs at the maximum possible supply $s = |D_i|$.

Before stating our final assumption, we introduce a concept regarding the ranking of feasible schedule sets for the patients, which will also be useful in the comparative static analysis in Section 4.4. Given a patient $i \in I$, a donor set $D_i \in \mathcal{D}_i$ and two sets $S, S' \subseteq \mathbb{Z}^2_+$, we say S is **weakly more favorable than** S' **at** D_i if the following holds:

- if $(r,s) \in S'$ and $r \geq \underline{n}_i$, then there exists $s' \leq s$ such that $(r,s') \in S$; and

- if $(r, s) \in S$, $s \leq |D_i|$ and $(r, s') \in S'$, then there exists $s'' \geq s$ such that $(r, s'') \in S'$.

When S and S' are schedule sets for patient i, S is weakly more favorable than S' at D_i if (i) for any schedule in S', supplying weakly less blood is feasible for the patient at S while she receives the same positive amount of blood, and (ii) for any schedule in S, whenever receiving the same amount of blood is feasible for the patient at S', it is feasible at S' to receive this amount by supplying weakly more blood as long as the supply does not exceed the number of donors. Using this concept, we make the following assumption regarding the relation between feasible schedule sets when a patient reports different sets of donors.

Assumption 3 (Non-diminishing favorability in donors)**.** *For every patient $i \in I$ and donor sets $D_i, D_i' \in \mathcal{D}_i$ such that $D_i' \subseteq D_i$, $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}_i(D_i')$ at D_i'.*

Favorability manifests itself geometrically as $\mathcal{S}_i(D_i)$ being an expansion and/or a downward shift of $\mathcal{S}_i(D_i')$ in the direction of receiving more blood. We illustrate the implications of Assumption 3 in conjunction with Assumptions 1 and 2 with two examples in Figures 3 and 4.

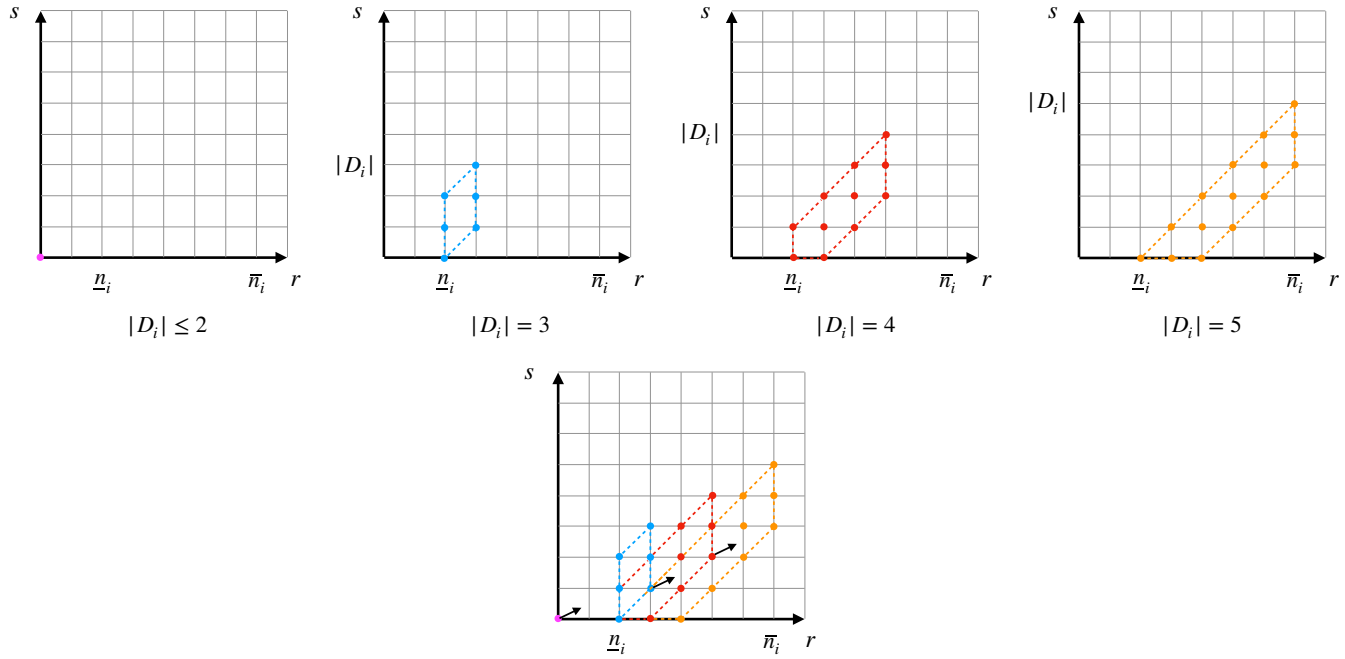


Figure 3: An illustration of a feasible schedule function $\mathcal{S}_i(\cdot)$ satisfying Assumptions 1, 2, and 3. This particular policy relies only on the number of donors brought forward $|D_i|$ but not other specifics of the donor set. The minimum guarantee of patient i is $\underline{n}_i = 2$ while her maximum need is $\overline{n}_i = 7$. The top graphs illustrate $\mathcal{S}_i(D_i)$ when $|D_i|$ changes from 0 to 5, while the bottom graph illustrates how the feasible schedule sets change as the number of donors increases.

The main result of this section is as follows:

Theorem 2. *Under Assumptions 1, 2, and 3 every sequential targeting mechanism is donor monotonic.*

We give a sketch of the proof idea. The proof involves defining an auxiliary matching market (which we refer to as an extended problem) that is a pure exchange economy and isomorphic to our

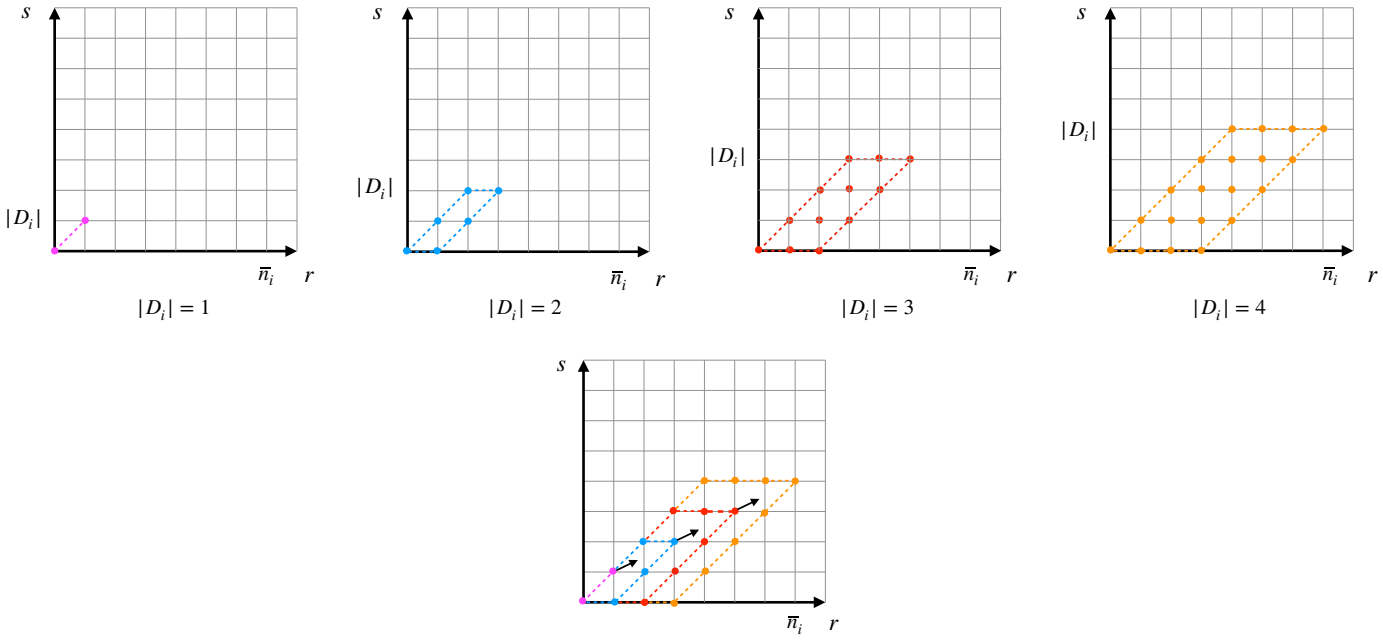


Figure 4: An illustration of a feasible schedule function $\mathcal{S}_i(\cdot)$ satisfying Assumptions 1, 2, and 3. This particular policy relies only on the number of donors brought forward $|D_i|$ but not other specifics of the donor set. The minimum guarantee of patient i is $\underline{n}_i = 0$ while her maximum need is $\overline{n}_i = 7$. The top graphs illustrate $\mathcal{S}_i(D_i)$ when $|D_i|$ changes from 1 to 4. The bottom graph illustrates how the feasible schedule sets change as the number of donors increases.

blood allocation problem with replacement donors. In this market, each allocation is represented by a matching that describes which donors are matched with each patient. Besides real patients and donors, the blood bank is represented as a pseudo-patient and its inventory is represented by pseudo-donors who are paired with it. For tractability purposes, if a patient is compatible with one of her donors, instead of directly receiving blood from this donor, we assume that she receives blood from a dummy donor paired with a dummy patient defined for each blood type. We define sequential targeting matching rules for the auxiliary market, which we show by Lemma 2 to be isomorphic to the sequential targeting mechanisms. Using this result, we prove the theorem for sequential targeting matching rules in the auxiliary matching market through two lemmata.

The first one, Lemma 3, is the most crucial result in the proof. This lemma gives a necessary condition for profitable manipulation under any mechanism. We consider two problems, the original problem and the one induced by patient i concealing exactly one of her donors. Let M be a matching for the original problem and M' be a matching for the induced problem such that i receives more blood under M'. Then Lemma 3 shows the existence of a particular graph theoretical structure, a cycle or a chain, relating these two matchings, M and M'. Cycles and chains do not necessarily represent Pareto improvements or losses. They are induced by the feasibility conditions in the definition of an allocation and show the swaps of donors that need to be done if we want to reach the matching M' from the matching M in one or multiple steps, such that each step results in a well-defined matching. Due to Assumptions 1, 2, and 3, the definition of a cycle and a chain has to be carefully tailored to respect feasibility.

Finally, Lemma 4 states that the sequential targeting matching rules are donor monotonic. We proceed by contradiction. Assume that there exists a patient i who can receive more blood by concealing a donor. Lemma 3 implies that there is a cycle or a chain between the outcomes of the particular sequential targeting matching rules used. We show that due to the optimization problems solved in each step of the sequential targeting procedure, whenever such a cycle or a chain is executed to make one matching closer to the other one, the welfare of the patients does not change. Thus, after a finite number of iterations, the two outcomes have to be welfare equivalent, and hence a contradiction is reached.

Moreover, each of the three assumptions is needed for the donor monotonicity of the sequential targeting mechanisms. In Section 5, Example 4 shows that Assumption 1 is necessary, even when Assumptions 2 and 3 are satisfied. It is straightforward to show that Assumption 3 is necessary. For example, when a patient i brings no donors, she receives her minimum guarantee of $\underline{n}_i = 1$ unit of blood: her feasible schdule set is $\{(1,0)\}$, and if she brings forward any donor, then her feasible schedule set shrinks to $\{(0,0)\}$. Such a feasible schedule function violates Assumption 3, but satisfies Assumptions 1 and 2. In this case, any mechanism is manipulable by patient i including a sequential targeting mechanism. Below, we provide an example to show that Assumption 2 is similarly necessary.

Example 3. Suppose that the set of patients is $I = \{1,2,3,4\}$. For every $i \in I$, $\underline{n}_i = 0$. For every $i \in I \setminus \{1\}$, $\overline{n}_i = 1$ and the exchange rate is one-to-one. That is, for every donor set $D_i \in \mathcal{D}_i$, where $i \in I \setminus \{1\}$,

$$\mathcal{S}_i(D_i) = \begin{cases} \{(0,0)\} & \text{if } D_i = \emptyset \\ \{(0,0),(1,1)\} & \text{otherwise} \end{cases}.$$

Moreover, $\overline{n}_1 = 2$ and Patient 1 can receive blood up to her maximal need by supplying at most one unit: for every $D_1 \in \mathcal{D}_1$,

$$\mathcal{S}_1(D_1) = \begin{cases} \{(0,0)\} & \text{if } D_1 = \emptyset \\ \{(0,0),(1,0),(1,1),(2,0),(2,1)\} & \text{otherwise} \end{cases}.$$

This is a special case of the Delhi policy in Example 2. Note that Assumptions 1 and 3 are satisfied for all the feasible schedule functions. However, when Patient 1 has two donors, Assumption 2 is not satisfied since $(2,2)$ is not a feasible schedule.

Each patient's blood type and donor set are given as follows.

- $\beta_1 = A$, and Patient 1 has one type B donor and one type O donor.
- $\beta_2 = B$, and Patient 2 has one type AB donor.
- $\beta_3 = AB$, and Patient 3 has one type A donor and one type O donor.
- $\beta_4 = O$, and Patient 4 has one type A donor.

In addition, the blood bank only has one unit of type AB blood in its inventory. Assume ABO-identical transfusion. Let f be a sequential targeting mechanism with respect to target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ such that $N_1 = I$, $N_2 = \{2\}$ and $\tau(2) = \max$. Then f selects the following allocation.

- Each $i \in I$ receives one unit of type β_i blood.[30]
- Patient 1's type B donor donates, Patient 3's type O donor donates, and the donor of $i \in \{2, 4\}$ donates.

If Patient 1 conceals her type B donor, then f selects the following allocation.

- Patient 1 receives two units of type A blood and her type O donor donates.
- Patient 2 receives nothing and her donor does not donate.
- Patient 3 receives one unit of type AB blood and her type A donor donates.
- Patient 4 receives one unit of type O blood and her type A donor donates.

Therefore, Patient 1 successfully manipulates.[31] ∎

4.3 Strong Donor Monotonicity

In order for the sequential targeting mechanisms to be strongly donor monotonic, we need a stronger restriction on the relation between feasible schedule sets when a patient reports different donor sets.

Assumption 4. *For every patient $i \in I$ and donor sets $D_i, D_i' \in \mathcal{D}_i$ such that $D_i' \subseteq D_i$, we have*

- *if $(r, s) \in \mathcal{S}_i(D_i')$ and $r \geq \underline{n}_i$, then there exists s' such that $(r, s') \in \mathcal{S}_i(D_i)$,*
- *if $(r, s) \in \mathcal{S}_i(D_i)$ and $(r, s') \in \mathcal{S}_i(D_i')$, then $s \leq s'$.*

It is straightforward to see that Assumption 4 implies Assumption 3. Moreover, if a patient reports a subset of her donors and still receives the same amount of blood, then the second condition in Assumption 4 implies that her donors do not donate less blood. Therefore, by Theorem 2 we have the following result.

Proposition 1. *Under Assumptions 1, 2 and 4, every sequential targeting mechanism is strongly donor monotonic.*

We give an example of a feasible schedule function inducing endogenous exchange rates and making sequential targeting mechanisms strongly donor monotonic in Figure 5.

[30] Note that every patient's demand would be fully satisfied if $(2, 2)$ were a feasible schedule for Patient 1.

[31] By similar arguments, it can be shown that Patient 1 is able to manipulate in the same way under any sequential targeting mechanism in which the first target set includes only Patient 2. Thus a serial dictatorship mechanism may not be donor monotonic under such feasible schedule functions.

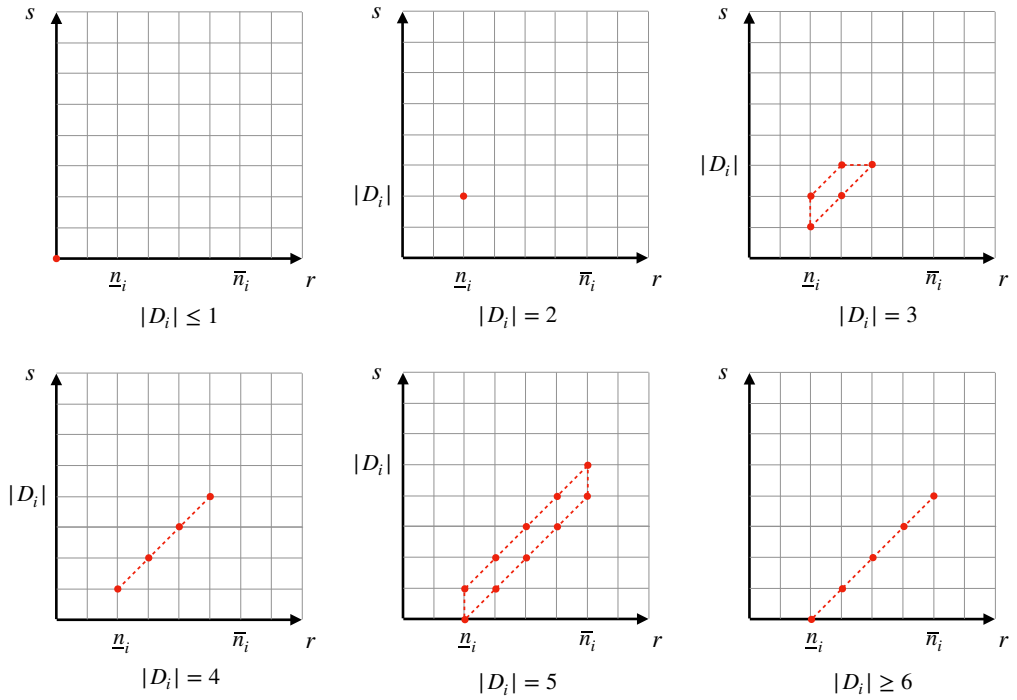


Figure 5: A policy that involves endogenous exchange rates can also guarantee strong donor monotonicity. Exchange rates are endogenous when the patient has three or five donors. In either case, the upper bound includes the feasible schedule set for any donor set with one less donor, and the lower bound is included in the feasible schedule set for any donor set with one more donor. The feasible schedule function satisfies Assumptions 1, 2 and 4.

One important circumstance under which strong donor monotonicity can be achieved is when the feasible schedule functions feature exogenous exchange rates, in the sense that for every possible amount of blood received in a feasible schedule set, there is a unique amount of supply associated with it. That is, for every $i \in I$, $D_i \in \mathcal{D}_i$ and $(r, s) \in \mathcal{S}_i(D_i)$, there does not exist $s' \neq s$ such that $(r, s') \in \mathcal{S}_i(D_i)$. In this case, Assumption 3 and Assumption 4 are equivalent.

Remark 1. *Consider any $i \in I$ such that $D_i \neq \emptyset$ for some $D_i \in \mathcal{D}_i$. The feasible schedule function $\mathcal{S}_i$ involves only exogenous exchange rates and satisfies Assumptions 1, 2 and 3 if and only if*

- *for every $D_i \in \mathcal{D}_i$ such that $\mathcal{S}_i(D_i) \neq \{(0,0)\}$, there exist $\underline{s}_i(D_i), \overline{r}_i(D_i) \in \mathbb{Z}_+$, where $\underline{s}_i(D_i) \leq |D_i|$, $\underline{s}_i(D_i) = 0$ if $\underline{n}_i = 0$, and $\overline{r}_i(D_i) \leq \overline{n}_i$, such that*

$$\mathcal{S}_i(D_i) = \{(r, s) \in \mathbb{Z}_+^2 : s - \underline{s}_i(D_i) = r - \underline{n}_i \quad \text{and} \quad s \leq |D_i|, r \leq \overline{r}_i(D_i)\},$$

- *for every $D_i \in \mathcal{D}_i$ and $D_i' \subseteq D_i$ such that $\mathcal{S}_i(D_i) \neq \{(0,0)\}$ and $\mathcal{S}_i(D_i') \neq \{(0,0)\}$, $\underline{s}_i(D_i) \leq \underline{s}_i(D_i')$ and $\overline{r}_i(D_i) \geq \overline{r}_i(D_i')$, and*

- *for every $D_i \in \mathcal{D}_i$ and $D_i' \subseteq D_i$, $\mathcal{S}_i(D_i) = \{(0,0)\}$ implies $\mathcal{S}_i(D_i') = \{(0,0)\}$.*

We refer to the feasible schedule functions defined in the Remark as **two-part tariffs**. We illustrate an example of a two-part tariff in Figure 6.

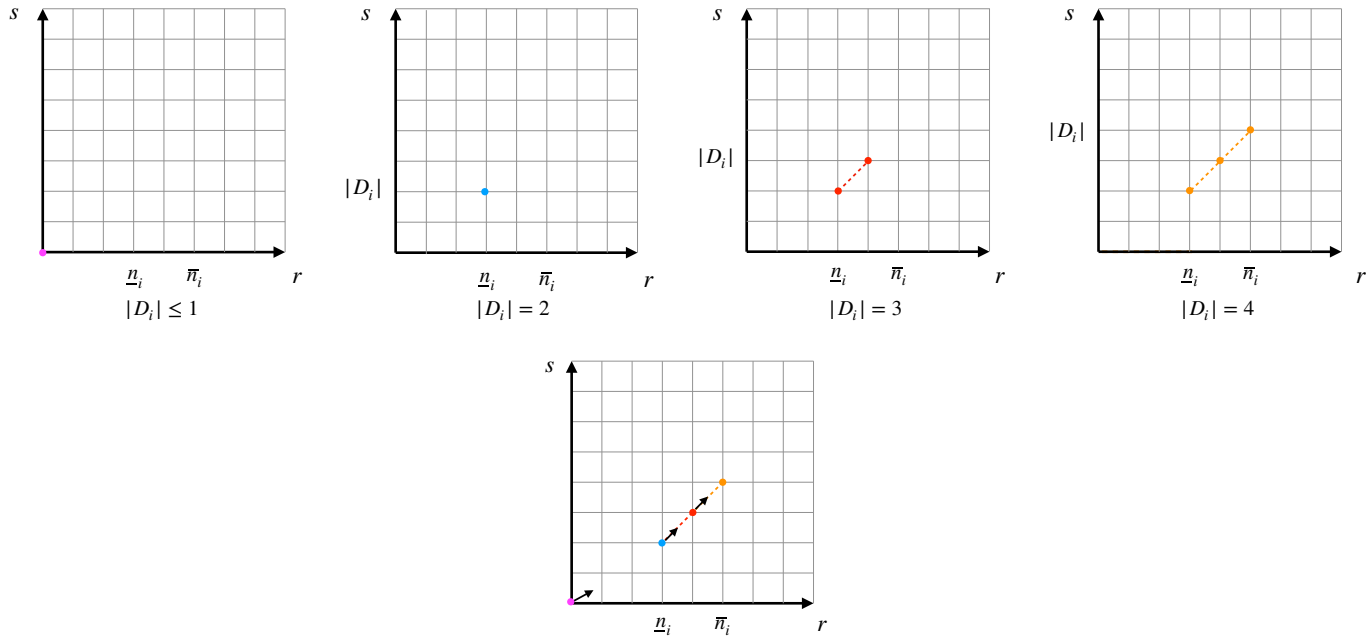


Figure 6: An illustration of the two-part tariff policy. The patient needs to bring forward at least two donors to receive her minimum guarantee of $\underline{n}_i = 3$ units of blood. The top graphs illustrate $\mathcal{S}_i(D_i)$ when $|D_i|$ changes from 0 to 4. The bottom graph illustrates how the feasible schedule sets change as the number of donors increases.

4.4 Comparative Statics of Policy Changes

As the sequential targeting mechanisms have different policy variables, it is important to understand how each of these levers affects the welfare of different patients. To this end, we conduct comparative static exercises to understand how changing the feasible schedule function or the target sets affects the welfare of a patient in the sequential targeting mechanisms.

We start with changes in the target sets affecting a group of patients that are *moved down* in the sequence of the target sets. We show that sequential targeting mechanisms behave in an intuitive manner, in the sense that the earlier a group of patients are processed, the weakly better off they are.

Proposition 2. *Let f be a sequential targeting mechanism with respect to target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ, and f' be a sequential targeting mechanism with respect to target sets $\{N'_k\}_{k=1}^{\bar{k}'}$ and target function τ' such that for some $\ell \in \{1, \ldots, \bar{k}-1\}$ and set of patients $N \subseteq N_\ell$,*

- *if $k < \ell$, then $N'_k = N_k$ and $\tau'(k) = \tau(k)$,*
- *if $N \neq N_\ell$, then $N'_\ell = N_\ell \setminus N$ and $\tau'(\ell) = \tau(\ell)$.*

Then for any problem $D \in \mathcal{D}$,

- *if $\tau(\ell) = max$, then*

$$\sum_{i \in N} f(D)(i) \geq \sum_{i \in N} f'(D)(i), \text{ and}$$

- *if* $\tau(\ell) = min$, *then*

$$\sum_{d \in \cup_{i \in N} D_i} f(D)(d) \leq \sum_{d \in \cup_{i \in N} D_i} f'(D)(d).$$

The proposition says that, after moving down a group of patients $N \subseteq N_\ell$ without changing the target sets and target objective before Step ℓ, if the target objective of Step ℓ is maximization, then the patients in N receive weakly less blood after the change, and if the target objective of Step ℓ is minimization, then the donors of the patients in N donate weakly more blood after the change.

Although the above result is straightforward to show, the proof of the next result is much more involved. We consider changes in the feasible schedule function of a patient. Changing the target sets is about adopting a different sequential targeting mechanism. On the other hand, a change in a feasible schedule function is about the underlying fundamentals of the problem independent of which mechanism is adopted. Nevertheless, we inspect how the outcome of a given sequential targeting mechanism changes when we make the feasible schedule set of a patient weakly more favorable.

We introduce a notation to denote the possibility of changing the underlying feasible schedule functions. For a given profile $\mathcal{S} = (\mathcal{S}_i)_{i \in I}$ of feasible schedule functions and a sequential targeting mechanism f, let $f\big(D \,|\, \mathcal{S}\big)$ be the outcome of f for any problem $D \in \mathcal{D}$.

Proposition 3. *Suppose that Assumptions 1 and 2 are satisfied for all feasible schedule functions considered. Consider any patient $i \in I$, any problem $D \in \mathcal{D}$ and any sequential targeting mechanism f. If $\mathcal{S}$ and $\mathcal{S}'$ are two profiles of feasible schedule functions such that $\mathcal{S}_j = \mathcal{S}'_j$ for all $j \in I \setminus \{i\}$, and $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i, then*

$$f\big(D \,|\, \mathcal{S}\big)(i) \geq f\big(D \,|\, \mathcal{S}'\big)(i).$$

The proposition states that if a patient is given a weakly more favorable feasible schedule set, then she always receives weakly more blood under the same sequential targeting mechanism, when Assumptions 1 and 2 are satisfied. The proof of of this proposition is similar to that of Theorem 2 with certain modifications.

5 Policy Discussion

In this section, we first discuss how certain practical challenges in designing blood allocation policies can be addressed using our framework. Then we explain practical implementation details regarding possible day-to-day functioning of a blood bank that adopts our mechanisms.

5.1 Policy Design with Feasible Schedules and Sequential Targeting Mechanisms

Our framework has two main tools that can be used to satisfy different objectives in blood allocation. The sequential targeting mechanisms allow lexicographic optimization of the objectives to be implemented. On the other hand, feasible schedule functions can be used to impose exchange rate policies and achieve more nuanced objectives regarding fairness, efficiency, and incentives. We start our discussion with the policy goals that can be reached by the flexibility that the latter tool brings to design.

5.1.1 Equitable Blood Allocation with Feasible Schedule Design

An important flexibility of our proposal is that exchange rates can be determined endogenously. This can be especially useful when some patients may potentially have few or no paired donor candidates. We can design policies that accommodate for the patients without donors and with donors equitably as much as possible, keeping incentive and efficiency properties of the sequential targeting mechanisms intact.

An example is provided in Figure 7. In this example, a patient i can receive the minimum guarantee of $\underline{n}_i = 1$ unit of blood even if she does not have any donor. She can also receive up to her maximum need of $\overline{n}_i = 3$ units in this case. As she brings forward more donors, her chances of receiving more units of blood beyond $\underline{n}_i = 1$ weakly increase by donor monotonicity. Moreover, under such a policy her donors never donate more blood than what she receives.

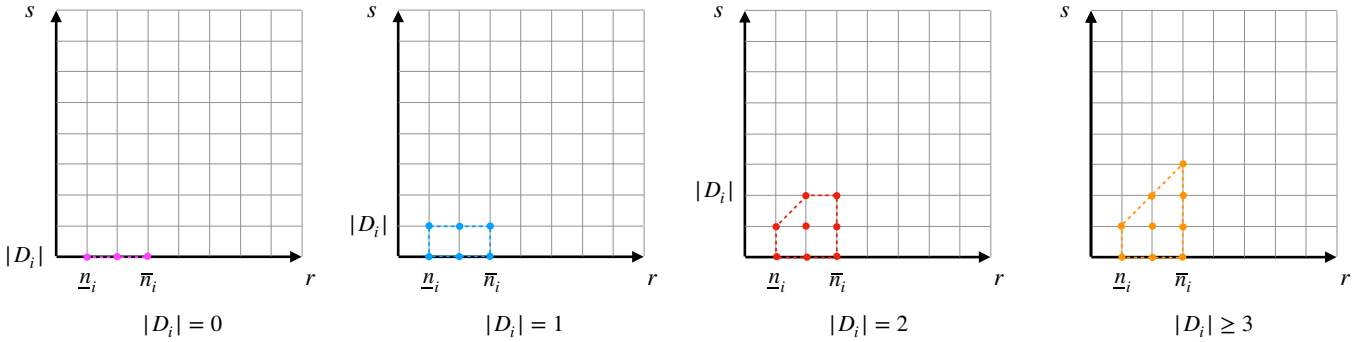


Figure 7: An equitable feasible schedule function policy for patients who may have few or no donors.

As mentioned before, we can also differentiate patients based on the urgency of the procedures they need, and design feasible schedule functions differently for urgent care patients and patients in need of elective procedures.

Our proposals are also compatible with other equitable replacement donor policies that are used around the world. For example, in leading Chinese hospitals, patients whose hometowns are away from the city where the hospital is located are often not required to supply as many donors as those patients from the city. The rationale behind this policy is that, relatives of patients from other cities are usually not readily available to donate on behalf of the patients. Similarly, in Cambodia, replacement donor requirements are waived for a patient if she has no next-of-kin (Davies, 2004). Thus, patient specific nature of our feasible schedule functions can accommodate such fairness considerations as well.

5.1.2 Blood Type Targeting with Feasible Schedule Design

Blood banks occasionally fall short in blood components of certain blood types while others are aplenty. Although ABO-identical compatibility is required for certain blood components, this can be relaxed for others. Components of rare blood types such as AB Rh D$-$, will always be in short supply in blood bank inventories. On the other extreme, blood type O Rh D$-$ is considered to be a universal type for red blood cells. Therefore, it may be important for blood banks to target its wide-spread collection. This goal can be achieved in multiple ways in our framework. One is using the multi-step targeting functions in a smart way. A more direct way is incentivizing the patients to provide donors

of desired blood types through feasible schedule policies. In Figure 8, we provide an example of a schedule design that favors bringing forward more type O Rh D− donors: the patient can receive the same amount of blood by supplying less if she has donors of blood type O Rh D−.

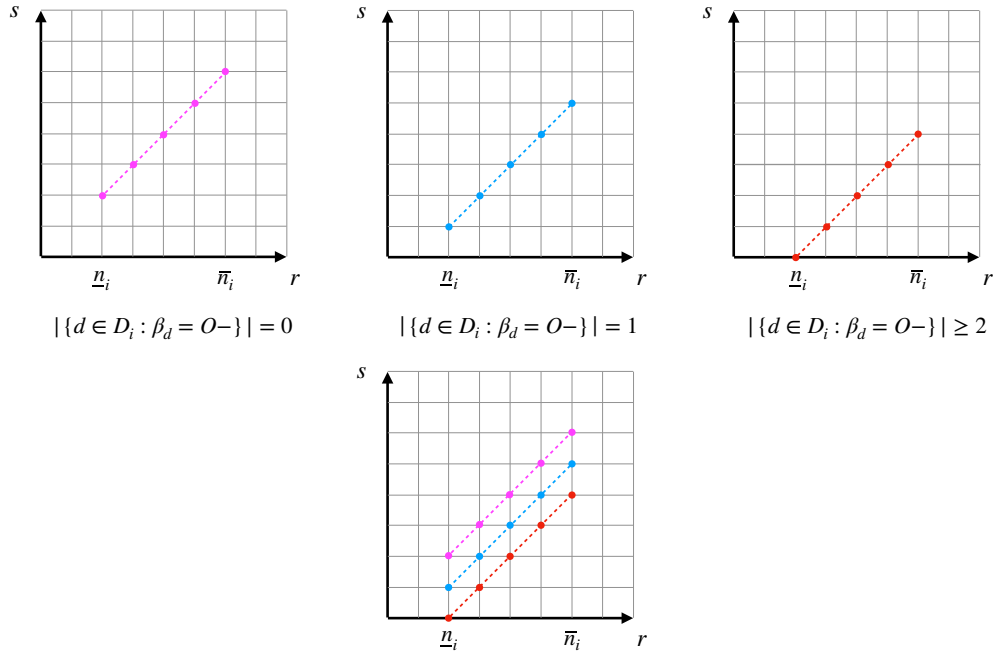


Figure 8: A policy that provides stronger incentives to reveal type O Rh D− donors (denoted as $O-$). In each case, the patient's feasible schedule set consists of those schedules on the graph in which the amount supplied does not exceed her number of donors, and only $(0,0)$ if there is no such schedule. This feasible schedule function satisfies Assumptions 1, 2 and 4. Assume that Assumption 1 is satisfied for the other patients. Then if Patient i has one or two type $O-$ donors, concealing a type $O-$ donor leads to a strictly worse outcome for her.

5.1.3 Beyond One-to-one Exchange

As mentioned before, some countries in Africa (e.g., Congo and Cameroon) and Mexico use exchange rates higher than one. All of these countries, for various reasons, use two-to-one exchange rate: two units of blood need to be supplied for each unit received. The following example shows that under such an exogenous exchange rate policy, a sequential targeting mechanism may not be donor monotonic.

Example 4. Suppose that the set of patients is $I = \{1,2,3,4\}$. For each $i \in I$, $\underline{n}_i = 0$, and for every possible donor set $D_i \in \mathcal{D}_i$,

$$\mathcal{S}_i(D_i) = \left\{(r,s) \in \mathbb{Z}_+^2 : s = 2r \text{ and } r \leq \min\left\{\overline{n}_i, \left\lfloor |D_i|/2 \right\rfloor\right\}\right\}.$$

Each patient's blood type, maximal need and donor set are given as follows.

- $\beta_1 = A$, $\overline{n}_1 = 2$, and Patient 1 has two type B donors and four type O donors.
- $\beta_2 = B$, $\overline{n}_2 = 2$, and Patient 2 has four type O donors.

- $\beta_3 = O$, $\overline{n}_3 = 4$, and Patient 3 has one type A donor and seven type AB donors.
- $\beta_4 = A$, $\overline{n}_4 = 1$, and Patient 4 has two type AB donors.

In addition, the blood bank only has one unit of type A blood in its inventory. Assume ABO-identical transfusion. Let f be a sequential targeting mechanism with respect to target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ such that $N_1 = I$, $N_2 = \{4\}$ and $\tau(2) = \max$. Then f selects the following allocation.

- Patient 1 receives one unit of type A blood and her two type B donors donate.
- Each $i \in \{2, 3, 4\}$ receives $\overline{n}_i$ units of type β_i blood and all the donors of i donate.

If Patient 1 conceals her two type B donors, then f selects the following allocation.

- Patient 1 receives two units of type A blood and her four type O donors donate.
- Patient 3 receives four units of type O blood and all of her donors donate.
- Patient 2 and Patient 4 do not receive any blood and their donors do not donate.

Therefore, Patient 1 successfully manipulates. When she conceals her type B donors, Patient 2 can no longer receive any blood and hence cannot provide type O blood to Patient 3. To maximize total transfusions, the mechanism selects Patient 1's four type O donors to provide blood for Patient 3. Then the two-to-one exchange rate requires Patient 1 to receive the two units of type A blood, despite that the other type A Patient, Patient 4, has higher priority, i.e., $4 \in N_2$.[32] ∎

However, we can generate endogenous exchange rate policies that closely approximate the two-to-one exchange rate. Under these policies the sequential targeting mechanisms are donor monotonic. See Figure 9 for an example.

5.1.4 Sequential Targeting Policy Design and Explicit Objectives of the Blood Bank

The sequential targeting mechanisms can be tailored to accommodate many policy objectives besides maximizing the allocated volume of blood. A plausible multistep approach can first maximize the blood given to urgent care patients in Step 1, and then remaining patients in Step 2 among the allocations that maximize the blood allocated to the urgent care patients. Depending on the shortages faced by the blood bank, the blood bank can easily maximize its remaining inventory in Step 3 (equivalently, maximize the donations made by the patients' donors) or minimize its remaining inventory (equivalently, minimize the donations made by the patients' donors), which can potentially depend on the remaining shelf life of the blood components in stock. To this end, we extend our model and the sequential targeting mechanisms to cover explicit objectives of the blood bank.

We include the blood bank b as an agent in the model and specify in an allocation α the amount of type X blood the bank receives, $\alpha_X(b)$, for each $X \in \mathcal{B}$. Assume that the bank has a complete and

[32] By similar arguments, it can be shown that f is not donor monotonic if we change the first target set such that $N_1 = \{3\}$. Therefore, a serial dictatorship mechanism may not be donor monotonic under the two-to-one exchange rate policy.

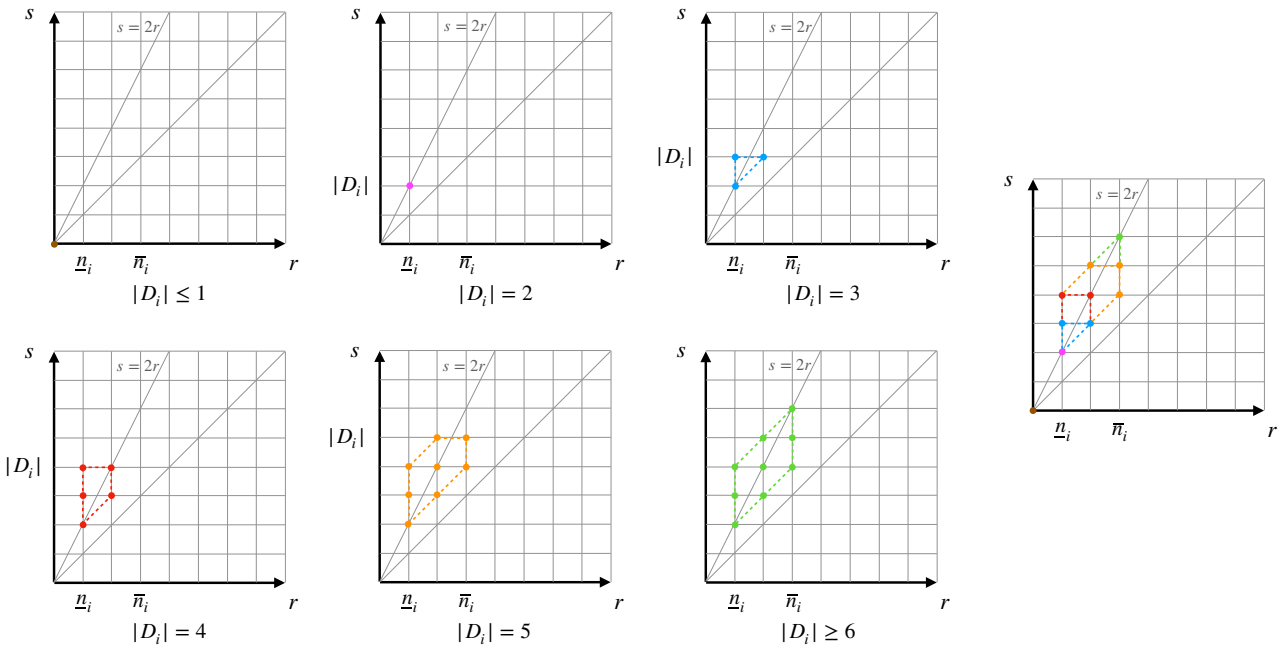


Figure 9: A feasible schedule function is designed to approximate the two-to-one exchange rate. Given that $\underline{n}_i = 1$ and $\overline{n}_i = 3$, if the patient has at least $s \in \{2, \dots, 6\}$ donors, then $(\frac{1}{2}s, s)$ should be a feasible schedule when s is an even number, and we consider $(\lfloor \frac{s}{2} \rfloor, s)$ and $(\lceil \frac{s}{2} \rceil, s)$ as feasible schedules when s is an odd number. Then the above graphs illustrate the feasible schedule function that assigns the smallest set of schedules that include these feasible schedules in each case so that Assumptions 1, 2, and 3 are satisfied for Patient i.

transitive (but not necessarily antisymmetric) preference relation $\succsim_b$ over blood bundles $(z_X)_{X \in \mathcal{B}} \in \mathbb{Z}_+^{|\mathcal{B}|}$ that it keeps in its inventory. For each $X \in \mathcal{B}$, let $\mathbf{I}^X \in \mathbb{Z}_+^{|\mathcal{B}|}$ be the bundle that includes only 1 unit of type X blood, i.e., $\mathbf{I}^X_X = 1$ and $\mathbf{I}^X_Y = 0$ for every $Y \neq X$. We assume that the bank's preference is **responsive**: for every $z = (z_X)_{X \in \mathcal{B}} \in \mathbb{Z}_+^{|\mathcal{B}|}$ and $X, Y \in \mathcal{B}$,

- $(z_X + 1, z_{-X}) \succsim_b (z_Y + 1, z_{-Y}) \iff \mathbf{I}^X \succsim_b \mathbf{I}^Y$,
- $(z_X + 1, z_{-X}) \succsim_b z \iff \mathbf{I}^X \succsim_b \mathbf{0}$, and
- $z \succsim_b (z_X + 1, z_{-X}) \iff \mathbf{0} \succsim_b \mathbf{I}^X$.

We modify the definition of a sequential targeting mechanism with respect to $\{N_k\}_{k=1}^{\bar{k}}$ and τ such that (i) there exists a unique target set N_k with $b \in N_k$, and (ii) $N_k = \{b\}$ and $\tau(k) = \max$, where "max" means the target is to choose the best allocations for the bank in $\mathcal{A}_{k-1}$, i.e.,

$$\mathcal{A}_k = \max_{\succsim_b} \mathcal{A}_{k-1}.$$

We also modify the definition of Pareto efficiency to include the blood bank's welfare. Then it is straightforward to extend the proofs of Theorems 1 and 2 to include the blood bank as an agent with responsive preference. In particular, responsiveness is only needed in the proof of Lemma 4.

Given this setup, an objective of maximizing the remaining inventory of some certain blood types in a step can be achieved by the responsive preference relation that only finds these blood types

acceptable, and minimizing the remaining inventory of some certain blood types can be achieved by the responsive preference relation that only finds these types unacceptable.

5.1.5 Voucher and Blood Assurance Programs and Minimum Guarantees

In China and United States, blood assurance programs are still being used. In such programs, if a healthy person altruistically donates blood, she gets vouchers for herself or her immediate family members to redeem in case they need blood transfusion in the near future. We can tailor our feasible schedule policies to function along with such programs in tandem. The vouchers brought by a patient i can be counted toward her minimum donor requirements to receive her minimum guarantee $\underline{n}_i$. Moreover, blood banks can further give priority to such patients in the target sets of the mechanisms, or more favorable feasible schedules to encourage future good Samaritan behavior.

5.2 Practical Implementation

Replacement donor programs do not possess certain simultaneity of donation and transplantation requirements that solid organ exchanges possess. As mentioned earlier, this gives us flexibility to schedule donations and transfusions separately. Moreover, the donated blood should be tested and processed for safety reasons, which makes it unsuitable for immediate transfusion. Thus, blood banks have to function through slack inventory.

Our proposal also carries this important practical implementation feature. Blood banks already use certain internal prioritization of patients, for example due to urgency of the medical procedures at the hospitals, and their needs. This can be extended to incorporate our optimization approach since the basis is already in play.

In order to apply our proposal, one needs to leverage the non-simultaneity flexibility between donation and transfusion. Indeed, in places such as Argentina, patients are not required to bring their intended donors up front at the time they need transfusions. Moreover, issues of reneging on donation promises are reported to be insignificantly small.[33] Our proposal is more conveniently applicable through a donor registry system that allows patients to register information about their potential donors at the time they are seeking blood. A potential donor registered into the system may later be utilized depending on the type and units he is willing to provide or the amount of blood the patient ends up receiving. Whenever the bank receives a request for blood, an urgent-care patient is allocated her full need whereas a non-urgent patient is provisionally allocated her minimum need using the existing stock.[34] When a certain threshold of potential donor count is reached,[35] the mechanism is implemented to determine the actual blood assignment of non-urgent patients together with which potential donors would be requested to donate blood.

6 Related Literature

The literature on market design for living-donor kidney exchange spanned by Roth, Sönmez, and Ünver (2004, 2005, 2007) in economics is related to the current paper, although most of this literature

[33] Based on personal communication with the director of Tucuman Blood Bank, Dr. Felicitas Agote, on July 7, 2020.

[34] In many places, blood centers are generally mandated to keep a certain stock of every blood type in their inventories to serve urgent needs, see e.g., Indian Department of Health (2016).

[35] The optimal threshold would naturally depend on other characteristics such as the arrival rate of patients and donors, and the distribution of blood types in the population. This is beyond the scope of the current model.

is about exchanging one transplant organ for one donor's organ with the following notable exception. The complementarities in the initial blood units supplied are similar to the complementarities in dual organ exchange in Ergin, Sönmez, and Ünver (2017). However, one-to-one exchange rate is not crucial in our model while it is important in the latter study. The differences in institutional details between solid organ exchange applications and our main application are explained in Section 2. Our two donor monotonicity notions would reduce to the donor monotonicity notion introduced in Roth, Sönmez, and Ünver (2005) if the demand were unit-valued and exchange rate were one-to-one.

Incentives in blood and blood-component donation are first studied by Lacetera, Macis, and Slonim (2012, 2013). Lacetera, Macis, and Slonim (2013) question the position and guidelines of the World Health Organization and several national blood collection agencies that have been based on the view that offering economic incentives to blood donors is detrimental to the quantity and safety of the blood supply. Lacetera, Macis, and Slonim (2012) provide evidence from a natural field experiment showing that economic incentives have a positive effect that increases with the incentive's economic value. Slonim, Wang, and Garbarino (2014) argue that blood donation is heavily influenced by sta ndard economic forces such as supply and demand, economies of scale, and moral hazard. They discuss promising directions to increase supply and improve the supply and demand balance even in the absence of market prices. Pay-it-forward and pay-it-backward incentive schemes for encouraging COVID-19 convalescent blood plasma donation have recently been proposed by Kominers, Pathak, Sönmez, and Ünver (2020) in a market design context. They propose issuing vouchers for the convalescent plasma donation of patients who recover from COVID-19 that can be used by the family of this donor who may become sick in the future to gain prioritized access to plasma therapy. They also propose issuing vouchers for patients who pledge to donate after recovery in return for their own prioritized access to plasma therapy. Since one donor can donate plasma that can treat more than one patient, with appropriate number of vouchers and willingness to donate, the system can collect enough plasma to treat all patients. Their paper inspects the steady-state analysis of a stylized large-market model, while ours is on mechanism design in a finite environment.

There are not many papers on mechanism or market design (see Haeringer, 2018 for an introductory survey) for multi-unit exchange of indivisible goods even under the restriction of one-to-one exchange rate. Besides Ergin, Sönmez, and Ünver (2017), two notable exceptions are Andersson, Cseh, Ehlers, and Erlanson (2020), who consider the design of time banks or barter markets to be cleared by centralized clearinghouses, and Manjunath and Westkamp (2019), who study shift exchanges for medical doctors and other professionals as a novel market design problem. These papers consider multi-unit consumption of indivisible goods under one-to-one exogenously set exchange rate. Our paper as well as their paper considerably generalize the priority mechanism introduced for only bilateral kidney exchanges, i.e., one-to-one donor exchanges among two patients with unit demand, proposed by Roth, Sönmez, and Ünver (2005).[36] Another recent paper on the multi-unit exchange model with one-to-one exchange rate, Aziz (2019) derives a sufficient condition for strategy-proofness of a mechanism.

[36]Unit-demand compatibility-based dichotomous preference matching model has been studied in the context of graph theory – for example see Lovasz and Plummer (1986) for an excellent survey of this discrete math literature. Incentive and fairness properties of mechanisms on such graphs were first analyzed by Bogomolnaia and Moulin (2004) in an economic model of a two-sided matching market.

Although the setup of Manjunath and Westkamp (2019) slightly differs from ours as goods are classified as strongly acceptable, weakly acceptable, and unacceptable, the lack of trade of weakly acceptable goods make the exchange domain similar to ours. They consider priority mechanisms that are greedy in the sense of serial dictatorships. On the other hand, the setup of Andersson, Cseh, Ehlers, and Erlanson (2020) is also very similar to ours. They consider mechanisms that maximize the volume of exchanges and choose among them using a priority ordering over agents. Both papers show that their respective mechanisms are strategy-proof and efficient in their domains. Our donor monotonicity axiom is similar to strategy-proofness in these domains: the only difference is we do not consider expansion of donor sets as feasible manipulations, thus, strategy-proofness implies donor monotonicity. On the other hand, their feasibility requirement is a special case of ours, as they both require only one-to-one exchange rate. Moreover, their mechanisms are members of our more general class of sequential targeting mechanisms. Our paper's main theoretical innovation is going beyond one-to-one exchange, which was standard in the literature of all indivisible goods exchange papers and unifying a new mechanism class that includes all previously studied mechanisms and beyond.

They only consider one-to-one exchange while we have endogenous exchange rates defined through feasible schedule sets. As far as we know, this is the first consideration of variable exchange rates with incentive axioms in the literature. Moreover, our sequential targeting mechanisms substantially generalize both greedy priority mechanisms and maximal priority mechanisms using multi-step targeting approach. This multi-step optimization approach is also new in mechanism design.

Similar to our main insight in the blood allocation context, Agarwal et al. (2019) underline and calculate the welfare loss in the US kidney exchange due to inefficient mechanisms and agency problems. They argue that while that the number of transplants that can be performed crucially depends on the marginal product of each patient-donor pair, current platform rules largely ignore this variation in the social value of submissions, much like the inefficiency caused by fixed exchange rates in blood allocation.

Price discovery and Pareto efficient allocation through endogenously determined exchange rates are the main features of competitive equilibrium. In allocation of indivisible goods, this approach was first utilized by Hylland and Zeckhauser (1979) using "fake" monetary incomes, by using pseudo-market equilibrium. This approach generally fails to guarantee the existence of a competitive equilibrium with endowments and no money income – as in our model – even with single-unit demand under dichotomous preferences and probabilistic assignment possibility (see Garg, Tröbst, and Vazirani, 2020). Moreover, competitive equilibrium as a mechanism is not incentive compatible in small markets.[37]

The comparison of feasible schedule sets based on favorability is similar in spirit as the weak order of sets in Che, Kim, and Kojima (2019) used in establishing weak monotone comparative statics in games of strategic complementarities and other social choice and mechanism design models, although our comparative static exercises are not related to theirs.

[37] A related market was inspected by Budish (2011) with mult-unit demand and deterministic allocation without any endowments leading to the existence of an approximate equilibrium.

Appendix A Proofs

A.1 Proof of Theorem 1

Let f be a sequential targeting mechanism with respect to some sequence of target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ. Assume to the contrary, for some problem $D \in \mathcal{D}$, $f(D) = \alpha$ is Pareto dominated by an allocation α'. Let

$$S_1 = \big\{k \in \{1, ..., \bar{k}\} : \tau(k) = \max \text{ and } \alpha(i) \neq \alpha'(i) \text{ for some } i \in N_k\big\},$$

and

$$S_2 = \big\{k \in \{1, ..., \bar{k}\} : \tau(k) = \min \text{ and } \sum_{d \in D_i} \alpha(d) \neq \sum_{d \in D_i} \alpha'(d) \text{ for some } i \in N_k\big\}.$$

Then $S_1 \cup S_2 \neq \emptyset$, since otherwise α' is never eliminated, i.e., $\alpha' \in \mathcal{A}_{\bar{k}}$ and α' is welfare equivalent to α. Let $k = \min(S_1 \cup S_2)$. Then $\alpha' \in \mathcal{A}_{k-1}$. We have two cases regarding k:

Case 1. $k \in S_1$: So $\tau(k) = \max$ and there exists $i \in N_k$ such that $\alpha(i) \neq \alpha'(i)$. Since α' Pareto dominates α, $\alpha'(i) > \alpha(i)$, and $\alpha'(j) \geq \alpha(j)$ for every $j \in N_k \setminus \{i\}$. Therefore, $\sum_{j \in N_k} \alpha'(j) > \sum_{j \in N_k} \alpha(j)$, contradicting the fact that α maximizes the amount of blood received by N_k among the allocations in $\mathcal{A}_{k-1}$.

Case 2. $k \in S_2$: So $\tau(k) = \min$ and there exists $i \in N_k$ such that $\sum_{d \in D_i} \alpha(d) \neq \sum_{d \in D_i} \alpha'(d)$. By the definition of a sequential targeting mechanism, for each $j \in N_k$ there exists $k' < k$ such that $\tau(k') = \max$ and $j \in N_{k'}$. Since $k = \min(S_1 \cup S_2)$, we have $\alpha(j) = \alpha'(j)$ for every $j \in N_k$. As α' Pareto dominates α, $\sum_{d \in D_j} \alpha'(d) \leq \sum_{d \in D_j} \alpha(d)$ for every $j \in N_k$. Then it follows from $\sum_{d \in D_i} \alpha(d) \neq \sum_{d \in D_i} \alpha'(d)$ that $\sum_{d \in \cup_{j \in N_k} D_j} \alpha'(d) < \sum_{d \in \cup_{j \in N_k} D_j} \alpha(d)$, contradicting the fact that α minimizes the amount of blood donated by N_k among $\mathcal{A}_{k-1}$.

A.2 Proof of Theorem 2

We first show that Assumptions 1, 2 and 3 imply the following two assumptions on the feasible schedule function.

Assumption 1′. For every $i \in I$, $D_i \in \mathcal{D}_i$, and $(r, s), (r', s') \in \mathcal{S}_i(D_i)$,

1. If $r' > r$ and $s' > s$, then

$$(r + 1, s + 1) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r' - 1, s' - 1) \in \mathcal{S}_i(D_i).$$

2. If $r' > r$ and $s' \leq s$, then

$$(r + 1, s) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r' - 1, s') \in \mathcal{S}_i(D_i).$$

3. If $s' > s$ and $r' \leq r$, then

$$(r, s + 1) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r', s' - 1) \in \mathcal{S}_i(D_i).$$

Assumption 2′. For every $i \in I$, $D_i, D_i' \in \mathcal{D}_i$ with $D_i' \subseteq D_i$, $(r,s) \in \mathcal{S}_i(D_i)$ and $(r',s') \in \mathcal{S}_i(D_i')$, we have

1. If $r' > r, s' > 0$ and $s < \left|D_i\right|$, then

$$(r+1, s+1) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1, s'-1) \in \mathcal{S}_i(D_i').$$

2. If $r' > r$ and $s' \leq s$, then

$$(r+1, s) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1, s') \in \mathcal{S}_i(D_i').$$

Lemma 1. *Assumption 1′ and Assumption 2′ are satisfied.*

Proof of Lemma 1. Consider any $i \in I$ and $D_i \in \mathcal{D}_i$. Denote $\mathcal{S}_i(D_i) = F$.

For any $x, y \in \mathbb{Z}_+^2$, where $x = (r,s)$ and $y = (r',s')$, denote $\text{dis}(x,y) = r' - r + s' - s$, and $y > x$ if $r' > r$ and $s' > s$. Suppose that $x = (r,s) \in F$, $y = (r',s') \in F$, and $y > x$. We want to show that $(r+1, s+1) \in F$. If $\text{dis}(x,y) = 2$, then we are done. If $\text{dis}(x,y) > 2$, then consider $z = \left\lceil \frac{x+y}{2} \right\rceil > x$. By L-convexity (Assumption 1), $z \in F$. It follows from $\text{dis}(x,y) > 2$ that either $\left\lceil \frac{r+r'}{2} \right\rceil < r'$ or $\left\lceil \frac{s+s'}{2} \right\rceil < s'$. Hence $\text{dis}(x,z) < \text{dis}(x,y)$. If $\text{dis}(x,z) > 2$, we can repeat the argument and find $z' \in F$ such that $z' > x$ and $\text{dis}(x,z') < \text{dis}(x,z)$. Therefore, it must be the case that $(r+1, s+1) \in F$. By symmetric arguments, it can be shown that $(r-1, s-1) \in F$. So Condition 1 in Assumption 1′ is satisfied.

Next we show Condition 2. Suppose that $x = (r,s) \in F$, $y = (r',s') \in F$, $r' > r$ and $s' \leq s$. First, we argue that there exists $s'' \leq s$ such that $(r+1, s'') \in F$. If $r' = r+1$, we are done. If $r' > r+1$, then consider $\left\lceil \frac{x+y}{2} \right\rceil = (r_1, s_1)$. We have $r' > r_1 > r$ and $s_1 \leq s$. By L-convexity, $(r_1, s_1) \in F$. If $r_1 > r+1$, we can repeat the argument and find $(r_2, s_2) \in F$ such that $r_1 > r_2 > r$ and $s_2 \leq s$. Therefore, eventually we have $(r+1, s'') \in F$ for some $s'' \leq s$. Denote $z = (r+1, s'')$. If $s'' < s$, consider $\left\lceil \frac{x+z}{2} \right\rceil = (r+1, s_3)$. Then $s'' < s_3 \leq s$. By L-convexity, $(r+1, s_3) \in F$. If $s_3 < s$, we can repeat the argument and find some s_4 such that $(r+1, s_4) \in F$ and $s_3 < s_4 \leq s$. Therefore, we must have $(r+1, s) \in F$. By symmetric arguments, it can be shown that $(r'-1, s') \in F$. Finally, Condition 3 in Assumption 1′ can be shown in a similar way as the proof of Condition 2.

To show Assumption 2′, let $D_i' \subseteq D_i$, $(r,s) \in \mathcal{S}_i(D_i)$ and $(r',s') \in \mathcal{S}_i(D_i')$.

Suppose that $r' > r, s' > 0$ and $s < \left|D_i\right|$. Since $r' > 0$, by Assumption 3, there exists s_1 such that $(r', s_1) \in \mathcal{S}_i(D_i)$. Since $r' > r$ and $s < |D_i|$, by Assumption 2, there exists $s_2 > s$ such that $(r', s_2) \in \mathcal{S}_i(D_i)$. Then it follows from Condition 1 in Assumption 1′ that $(r+1, s+1) \in \mathcal{S}_i(D_i)$. This also implies that $\mathcal{S}_i(D_i) \neq \{(0,0)\}$, and hence $r \geq \underline{n}_i$. Then given that $r' > r \geq \underline{n}_i$, $\mathcal{S}_i(D_i') \neq \{(0,0)\}$ and there exists s_3 such that $(\underline{n}_i, s_3) \in \mathcal{S}_i(D_i')$. Since $r' > \underline{n}_i$ and $s' > 0$, by Assumption 2, there exists $s_4 < s'$ such that $(\underline{n}_i, s_4) \in \mathcal{S}_i(D_i')$. Then it follows from Condition 1 in Assumption 1′ that $(r'-1, s'-1) \in \mathcal{S}_i(D_i')$.

It remains to show Condition 2 in Assumption 2′. Suppose that $r' > r$ and $s' \leq s$. By Assumption 3, there exists $s_1 \leq s'$ such that $(r', s_1) \in \mathcal{S}_i(D_i)$. It follows from Condition 2 in Assumption 1′ that $(r+1, s) \in \mathcal{S}_i(D_i)$. Then, we want to show that $(r, s') \in \mathcal{S}_i(D_i)$. This is true if $s' = s$. Suppose that $s' < s$. Then consider $(r', s_1) \in \mathcal{S}_i(D_i)$ and $(r,s) \in \mathcal{S}_i(D_i)$, where $r' > r$ and $s_1 < s$. By

repeated applications of Condition 3 in Assumption 1′, we have $(r, s') \in \mathcal{S}_i(D_i)$. Finally, it can be easily shown as before that $r' > r \geq \underline{n}_i$ and hence there exists s_2 such that $(r, s_2) \in \mathcal{S}_i(D'_i)$. Given that $(r, s') \in \mathcal{S}_i(D_i)$ and $s' \leq |D'_i|$, by Assumption 3, there exists $s_3 \geq s'$ such that $(r, s_3) \in \mathcal{S}_i(D'_i)$. Since $r' > r$ and $s' \leq s_3$, it follows from Condition 2 in Assumption 1′ that $(r' - 1, s') \in \mathcal{S}_i(D'_i)$. ■

We introduce new machinery to prove this theorem. In particular, we will construct an *extended problem* in which each blood type has a replica and there are some new dummy agents. This extended market with dummy agents and blood types will be helpful for accounting purposes as it will eliminate the need to use the market clearing conditions as an exogenous constraint, which are used in the definition of an allocation.

First, we treat the blood bank b as if it were a *pseudo patient* in the extended problem by introducing a donor set for it. It has a set of (volunteer non-remunerated) donors D_b, where for each blood type $X \in \mathcal{B}$ the blood bank has v_X donors.

Then, for each blood type $X \in \mathcal{B}$, we construct a *dummy blood type* $\hat{X}$. Define $\hat{\mathcal{C}}(\cdot)$ as follows: For each $X \in \mathcal{B}$,

$$\hat{\mathcal{C}}(X) = \mathcal{C}(X) \cup \{\hat{Y} : Y \in \mathcal{C}(X)\} \qquad \text{and} \qquad \hat{\mathcal{C}}(\hat{X}) = \{X\}.$$

For each $X \in \mathcal{B}$, we construct a *dummy patient* $i_{\hat{X}}$ and her set of *dummy donors* $D_{i_{\hat{X}}}$, such that

$$\begin{aligned} &\beta_{i_{\hat{X}}} = \beta_d = \hat{X} \quad \text{for every } d \in D_{i_{\hat{X}}}, \\ &\overline{n}_{i_{\hat{X}}} = \left|D_{i_{\hat{X}}}\right| = \sum_{i \in I} \overline{n}_i, \text{ and} \\ &\underline{n}_{i_{\hat{X}}} = 0. \end{aligned}$$

Moreover, let her feasible schedule set be

$$\mathcal{S}_{i_{\hat{X}}}(D_{i_{\hat{X}}}) = \left\{(r, s) \ : \ 0 \leq r \leq \overline{n}_{i_{\hat{X}}} \quad \text{and} \quad s = r\right\}.$$

For any problem $\mathcal{P} = \langle I, \beta_I, \underline{n}, n, D, \beta_D, v \rangle$, which has been simply denoted as $D = (D_i)_{i \in I}$, we use $\hat{D} = \left(D, D_b, (D_{i_{\hat{X}}})_{X \in \mathcal{B}}\right)$ to denote the **extended problem** after the blood bank is added as a pseudo patient, its inventory is added as a donor set, and dummy blood types, patients, and donors are added to the problem D.

Let $\hat{I} = I \cup \{b\} \cup \{i_{\hat{X}}\}_{X \in \mathcal{B}}$. From now on in this proof, we refer to each $i \in \hat{I}$ as a **patient** (in reality it can be a real patient, a dummy patient, or the blood bank) and each $d \in \cup_{i \in \hat{I}} D_i$ as a **donor**, who can be a real donor, dummy donor, or just a unit of blood component supplied by the blood bank. Let $\hat{\mathbf{D}} = \cup_{i \in \hat{I}} D_i$ be the set of donors in the extended problem.

Given an extended problem $\hat{D}$, a(n) **(auxiliary) matching** is a function $M : \hat{I} \to 2^{\hat{\mathbf{D}}}$, where the match of patient i, $M(i)$, is denoted as M_i for every $i \in \hat{I}$ by a slight abuse of notation, such that

1. $M_i \cap M_j = \emptyset$ for every $i, j \in \hat{I}$ with $i \neq j$ and $\cup_{i \in \hat{I}} M_i = \hat{\mathbf{D}}$,
2. for every $i \in \hat{I} \setminus \{b\}$ and $d \in M_i \setminus D_i$, $\beta_d \in \hat{\mathcal{C}}(\beta_i)$,
3. for every $i \in \hat{I} \setminus \{b\}$, $\left(\left|M_i \setminus D_i\right|, \left|D_i \setminus M_i\right|\right) \in \mathcal{S}_i(D_i)$.

In a matching M, each patient $i \in \hat{I}$ is matched with a donor set M_i that we refer to as the **match** of i. Every allocation $\alpha \in \mathcal{A}$ in a problem D is associated with a matching M in its extended problem $\hat{D}$ and vice versa as we prove in a claim in the proof of Lemma 4 below. In particular, the match of patient $i \in \hat{I} \setminus \{b\}$ consists of two parts.

- The first part $M_i \setminus D_i$ is the set of donors that she receives blood from. These donors necessarily belong to other patients or the blood bank in our definition of a matching. Thus, they have to possess one of the compatible blood types with patient i (Condition 2 in the definition of a matching).
- The second part $M_i \cap D_i$ is the set of her own donors who end up not donating to anybody. We assume that they are matched back with the patient in a matching.

This definition implies that a patient never receives blood from her own donors in a matching of an extended problem. Although in an allocation this may not be the case, we introduced the dummy patients and their donors to account for this possibility: if in an allocation a patient $i \in \hat{I} \setminus \{b\}$ is receiving blood from one of her own donors, this is represented in a matching in the following manner:

- this donor $d \in D_i$ is matched with the dummy patient induced by her blood type, $i_{\hat{\beta}_d}$,
- her patient i is matched with one of the dummy donors of this dummy patient, i.e., with some $\hat{d} \in D_j$ where $j = i_{\hat{\beta}_d}$, in return.

As a result, the donors of patient $i \in \hat{I} \setminus \{b\}$ who actually donate in a matching M are $D_i \setminus M_i$. Therefore, $(|M_i \setminus D_i|, |D_i \setminus M_i|)$ has to be a feasible schedule in $\mathcal{S}_i(D_i)$ (Condition 3 in the definition of a matching).

Two matchings M and M' are **welfare equivalent** if $|M_i \setminus D_i| = |M'_i \setminus D_i|$ and $|D_i \setminus M_i| = |D_i \setminus M'_i|$ for every $i \in \hat{I} \setminus \{b\}$.

The concept of a matching will be useful for simplicity and tractability in our proofs, as we will be able to convert the match of a patient alone to the feasible schedule the patient is allocated without needing to check how many of her donors are matched. Moreover, we eliminate the need to check whether a matching satisfies market clearing condition: by representing an allocation through a matching, we can know keep track of which patient is matched which donor, and by construction of a matching, the matches have to be represented through feasible schedules.

The analogue of a mechanism in extended problems is a rule: A **rule** is a function F that maps each extended problem $\hat{D}$ to a matching $F(\hat{D})$. A rule F is **donor monotonic** if for any $D, D' \in \mathcal{D}$ and $i \in I$ such that $D'_i \subseteq D_i$ and $D'_j = D_j$ for every $j \in I \setminus \{i\}$, we have[38]

$$|F_i(\hat{D}) \setminus D_i| \geq |F_i(\hat{D}') \setminus D'_i|.$$

We define the sequential targeting rules for extended problems by extending sequential targeting mechanisms. Let $\{N_k\}_{k=1}^{\bar{k}}$, a sequence of nonempty subsets of $\hat{I} \setminus \{b\}$, $\bar{k} \geq 2$, be the **target sets**, and $\tau : \{2, ..., \bar{k}\} \to \{\max, \min\}$ be the **target function**. We define a similar **sequential targeting**

[38]Note that we do not consider manipulations by the dummy patients as their donor sets are fixed.

procedure as before, for any extended problem $\hat{D}$: Let $\mathcal{M}_0$ be the set of all the matchings for $\hat{D}$; At Step 1, let $\mathcal{M}_1 \subseteq \mathcal{M}_0$ be the subset of matchings that maximize the amount of blood received by the patients in N_1; generally, at Step k, let $\mathcal{M}_k \subseteq \mathcal{M}_{k-1}$ be the subset of matchings in $\mathcal{M}_{k-1}$ that maximize the amount of blood received by the patients in N_k if $\tau(k) = \max$, and $\mathcal{M}_k \subseteq \mathcal{M}_{k-1}$ be the subset of matchings in $\mathcal{M}_{k-1}$ that minimize the amount of blood supplied by the patients in N_k if $\tau(k) = \min$. As before, we make the same two assumptions jointly on the target sets and target function. First, for every $k \in \{2, ..., \bar{k}\}$, if $\tau(k) = \min$, then for any $i \in N_k$ there exists $k' < k$ such that $i \in N_{k'}$ and $\tau(k') = \max$. Second, $\bar{k} \geq 2\left|\hat{I} \setminus \{b\}\right|$ and the last $2\left|\hat{I} \setminus \{b\}\right|$ sets, $\{N_k\}_{k=\bar{k}-2\left|\hat{I}\setminus\{b\}\right|+1}^{\bar{k}}$, are each a singleton such that each patient $i \in \hat{I} \setminus \{b\}$ appears exactly twice as $N_k = N_\ell = \{i\}$ for some distinct $k, \ell \geq \bar{k} - 2\left|\hat{I} \setminus \{b\}\right| + 1$ with one target as $\tau(k) = \max$ and the other target as $\tau(\ell) = \min$.

We say that F is a **sequential targeting rule**, with respect to $\{N_k\}_{k=1}^{\bar{k}}$ and τ, if for each $\hat{D}$, F selects a matching $F(\hat{D})$ from the outcome set $\mathcal{M}_{\bar{k}}$ of the sequential targeting procedure for $\hat{D}$.

Given any problem D, we say an allocation α for D and a matching M for $\hat{D}$ are **welfare equivalent** if for every $i \in I$, $\alpha(i) = \left|M_i \setminus D_i\right|$ and $\sum_{d \in D_i} \alpha(d) = \left|D_i \setminus M_i\right|$. The following result implies that to prove Theorem 2, it is sufficient to show that the sequential targeting rules are donor monotonic.

Lemma 2. *For every sequential targeting mechanism f, there is a sequential targeting rule F such that for every $D \in \mathcal{D}$, $f(D)$ and $F(\hat{D})$ are welfare equivalent.*

Proof of Lemma 2. We first prove the following claim:

Claim. For every allocation α for D, there is a matching M for $\hat{D}$ such that α and M are welfare equivalent. For every matching M for $\hat{D}$, there is an allocation α for D such that M and α are welfare equivalent.

Proof. We prove the claim in two parts.

Part 1. Let α be an allocation for D. Consider the extended problem $\hat{D}$, and any $X \in \mathcal{B}$. Since $\left|D_{i_{\hat{X}}}\right| = \sum_{j \in I} \overline{n}_j$, there exists a collection of disjoint donor sets $\left\{M_j^{\hat{X}}\right\}_{j \in I: X \in \mathcal{C}(\beta_j)}$ such that for every $j \in I$ with $X \in \mathcal{C}(\beta_j)$,

1. $M_j^{\hat{X}} \subseteq D_{i_{\hat{X}}}$, and
2. $\left|M_j^{\hat{X}}\right| = \alpha_X(j)$.

Since $\sum_{j \in I: X \in \mathcal{C}(\beta_j)} \alpha_X(j) \leq v_X + \sum_{d \in \cup_{j \in I} D_j : \beta_d = X} \alpha(d)$, there exists a set of donors $M_{i_{\hat{X}}}^X \subseteq \cup_{j \in I \cup \{b\}} D_j$ such that

1. $\beta_d = X$ for every $d \in M_{i_{\hat{X}}}^X$,
2. $\alpha(d) = 1$ for every $d \in M_{i_{\hat{X}}}^X \setminus D_b$, and
3. $\left|M_{i_{\hat{X}}}^X\right| = \sum_{j \in I: X \in \mathcal{C}(\beta_j)} \left|M_j^{\hat{X}}\right|$.

Then we construct a matching M for $\hat{D}$ as follows:

- For each $j \in I$, $M_j = \left(\cup_{X \in \mathcal{C}(\beta_j)} M_j^{\hat{X}}\right) \cup \{d \in D_j : \alpha(d) = 0\}$,

- for each $X \in \mathcal{B}$, $M_{i_{\hat{X}}} = M^X_{i_{\hat{X}}} \cup \Big(D_{i_{\hat{X}}} \setminus \big(\cup_{j\in I: X\in\mathcal{C}(\beta_j)} M^{\hat{X}}_j\big)\Big)$, and
- $M_b = \hat{\mathbf{D}} \setminus \big((\cup_{j\in I} M_j) \cup (\cup_{X\in\mathcal{B}} M_{i_{\hat{X}}})\big)$

Therefore, each patient $j \in I$ is matched with $\alpha_X(j)$ dummy donors of type $\hat{X}$ for every $X \in \mathcal{C}(\beta_j)$, and j's own donor d is matched with j if and only if $\alpha(d) = 0$. Moreover, for each dummy patient $i_{\hat{X}}$, the number of X donors from $I \cup \{b\}$ matched with her is equal to the number of her $\hat{X}$ donors that are not matched with her. Hence, M is a well-defined matching for $\hat{D}$ and it is welfare equivalent to α.

Part 2. On the other hand, let M be a matching for $\hat{D}$. Construct α as follows:

- For each $j \in I$ and $X \in \mathcal{C}(\beta_j)$, let $\alpha_X(j) = \big|\big\{d \in M_j \setminus D_j : \beta_d \in \{X, \hat{X}\}\big\}\big|$, and
- for each $j \in I$ and $d \in D_j$, let $\alpha(d) = 0$ if $d \in M_j$, and $\alpha(d) = 1$ if $d \notin M_j$.

If α is an allocation for D, then it is straightforward to show that it is welfare equivalent to M. To show that α is a well-defined allocation, we only need to verify Condition 1 in the definition of an allocation: for any blood type $X \in \mathcal{B}$,

$$\begin{aligned}
\sum_{j\in I: X\in\mathcal{C}(\beta_j)} \alpha_X(j) &= \sum_{j\in I: X\in\mathcal{C}(\beta_j)} \big|\{d \in M_j \setminus D_j : \beta_d = X\}\big| + \sum_{j\in I: X\in\mathcal{C}(\beta_j)} \big|M_j \cap D_{i_{\hat{X}}}\big| \\
&\leq \sum_{j\in I: X\in\mathcal{C}(\beta_j)} \big|\{d \in M_j \setminus D_j : \beta_d = X\}\big| + \big|\{d \in M_{i_{\hat{X}}} : \beta_d = X\}\big| \\
&\leq \sum_{j\in I} \big|\{d \in D_j \setminus M_j : \beta_d = X\}\big| + v_X \\
&= \sum_{d\in\cup_{j\in I} D_j : \beta_d = X} \alpha(d) + v_X
\end{aligned}$$

where the first inequality follows from the construction of $\mathcal{S}_{i_{\hat{X}}}(D_{i_{\hat{X}}})$, as well as the fact that $\hat{\mathcal{C}}(\hat{X}) = \{X\}$. ∎

Let f be a sequential targeting mechanism with respect to $\{N_k\}_{k=1}^{\bar{k}}$ and τ. Then there exists a sequential targeting rule F with respect to $\{N_1, ..., N_{\bar{k}}, N_{\bar{k}+1}, ..., N_{\bar{k}+t}\}$ where $t = 2(\big|\hat{I}\setminus\{b\}\big| - \big|I\big|) = 2\big|\mathcal{B}\big|$ accounts for the dummy patients introduced in the extended problem who appear themselves twice at the end of the rule as target sets, and a target function τ', where $\tau'(k) = \tau(k)$ for every $k \in \{2, ..., \bar{k}\}$. We want to show that for any $D \in \mathcal{D}$, $f(D)$ and $F(\hat{D})$ are welfare equivalent.

Assume to the contrary, for some $D \in \mathcal{D}$, $f(D)$ and $F(\hat{D})$ are not welfare equivalent. By the Claim, there is an allocation α for D that is welfare equivalent to $F(\hat{D})$, and there is a matching M for $\hat{D}$ that is welfare equivalent to $f(D)$. Then α and $f(D)$ are not welfare equivalent. It follows that at some step $k \in \{1, ..., \bar{k}\}$ of the sequential targeting procedure for D, $\alpha \in \mathcal{A}_{k-1}$ and $\alpha \notin \mathcal{A}_k$. But this implies that at step k of the sequential targeting procedure for $\hat{D}$, $M \in \mathcal{M}_{k-1}$ and $F(\hat{D}) \notin \mathcal{M}_k$, contradiction. ■

The proof of donor monotonicity of sequential targeting rules relies on comparing two matchings for two extended problems and constructing two new ones based on the differences of the matches

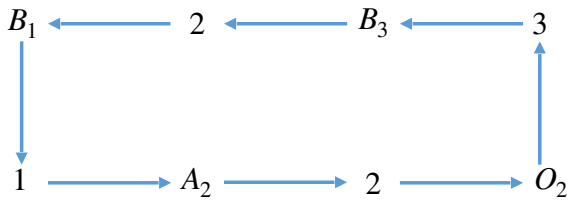


Figure 10: Suppose $I = \{1, 2, 3\}$ with $\beta_1 = A$, $\beta_2 = B$ and $\beta_3 = O$. Suppose that $\hat{D} = \hat{D}'$, and the donor sets are given by $D_1 = \{B_1\}$, $D_2 = \{A_2, O_2\}$, $D_3 = \{B_3\}$ and $D_b = \emptyset$. Every patient's maximal need is one unit and the exchange rate is one-for-one. Consider the following two matchings M and M': $M_1 = \{B_1\}$, $M_2 = \{A_2, B_3\}$, $M_3 = \{O_2\}$ and $M_b = \emptyset$; $M'_1 = \{A_2\}$, $M'_2 = \{O_2, B_1\}$, $M'_3 = \{B_3\}$ and $M'_b = \emptyset$. The above graph gives a cycle C from M to M', and we have $M + C = M'$ and $M' - C = M$. Note that Patient 2 cannot form a cycle with only Patient 1 or Patient 3.

of patients feasible for the two problems, respectively. We introduce the following graph theoretical concepts based on finding these differences:

Let $\hat{D}$ and $\hat{D}'$ be two extended problems such that $D'_i \subseteq D_i$ for every real patients $i \in I$, $D'_{i_{\hat{X}}} = D_{i_{\hat{X}}}$ for every $X \in \mathcal{B}$ and $D'_b = D_b$. Given a matching M for $\hat{D}$ and a matching M' for $\hat{D}'$, a **cycle from M to M'** is a directed graph of patients and donors in which each patient/donor points to the next donor/patient and is denoted as a list $C = (i_1, d_1, ..., i_{\bar{t}}, d_{\bar{t}})$, $\bar{t} \geq 2$, such that for each $t \in \{1, ..., \bar{t}\}$ (let $i_{\bar{t}+1} = i_1$ and $d_0 = d_{\bar{t}}$):

1. $i_t \in \hat{I}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.

2. If $i_t \neq b$, $d_{t-1} \in D_{i_t}$, and $d_t \notin D_{i_t}$, then

$$\left(\left|M_{i_t} \setminus D_{i_t}\right| + 1, \left|D_{i_t} \setminus M_{i_t}\right| + 1\right) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad \left(\left|M'_{i_t} \setminus D'_{i_t}\right| - 1, \left|D'_{i_t} \setminus M'_{i_t}\right| - 1\right) \in \mathcal{S}_{i_t}(D'_{i_t}).$$

3. If $i_t \neq b$, $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then

$$\left(\left|M_{i_t} \setminus D_{i_t}\right| - 1, \left|D_{i_t} \setminus M_{i_t}\right| - 1\right) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad \left(\left|M'_{i_t} \setminus D'_{i_t}\right| + 1, \left|D'_{i_t} \setminus M'_{i_t}\right| + 1\right) \in \mathcal{S}_{i_t}(D'_{i_t}).$$

4. If $i_t = i_{t'} = i$ for some $t' \neq t$, then either

   - $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.

In a cycle C, each patient points to a donor she is matched with under M' but not under M, while each donor points to the patient that she is matched with under M. Note that each donor in a cycle must be both in the extended problems $\hat{D}$ and $\hat{D}'$. Starting from the base matching M, we assign each patient in the cycle the donor she points to (who is one of her M' matches) instead of the donor she is pointed by (who is one of her M matches): That is for each t, add d_t to M_{i_t} and remove d_{t-1} from M_{i_t}. Condition 1 above guarantees that this leads to a well-defined function, which we denote as $M + C$ and satisfies Conditions 1 and 2 in the definition of a matching. The patients involved in the

cycle may not be distinct. But Condition 4 above says that if one patient appears twice in the cycle, then her schedule is not affected by the exchanges, i.e., the amount of blood received and the amount of blood supplied remain the same. Note that this condition also implies that a patient cannot appear more than twice in a cycle. Finally, given that every patient who is matched with a different schedule under $M + C$ than under M either receives one more unit and supplies one more unit, or receives one less unit and supplies one less unit. In particular, Conditions 2 and 3 above imply Condition 3 in the definition of a matching. Therefore $M + C$ is a matching for $\hat{D}$. Similarly, we could instead start from M' and assign each patient in the cycle the donor she is pointed by (one of her M matches) instead of the donor she points to (one of her M' matches): That is for each t, add d_{t-1} to M'_{i_t} and remove d_t from M'_{i_t}. These exchanges also lead to a well-defined matching for $\hat{D}'$, denoted as $M' - C$. See Figure 10 for an example of a cycle and how new matchings are constructed using this cycle.

It is wise to note that the cycle addition or removal operations do not necessarily make all patients involved better off or worse off together. They will be used in the proof to construct new matchings that are closer to each other in terms of matches of patients.

Another concept similar to a cycle is a chain. A **chain from M to M'** is a directed graph of patients and donors such that each patient/donor points to the next donor/patient in the chain and is represented as a list $C = (i_1, d_1, ..., i_{\bar{t}-1}, d_{\bar{t}-1}, i_{\bar{t}})$, $\bar{t} \geq 2$, such that

1. For every $t \in \{1, 2, \ldots, \bar{t}\}$, $i_t \in \hat{I}$ such that if $i_t = b$ then $t \in \{1, \bar{t}\}$, and $i_1 \neq i_{\bar{t}}$.
2. For every $t \in \{1, 2, \ldots, \bar{t} - 1\}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.
3. For every $t \in \{2, \ldots, \bar{t} - 1\}$, if $d_{t-1} \in D_{i_t}$ and $d_t \notin D_{i_t}$, then

$$\left(\left|M_{i_t} \setminus D_{i_t}\right| + 1, \left|D_{i_t} \setminus M_{i_t}\right| + 1\right) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad \left(\left|M'_{i_t} \setminus D'_{i_t}\right| - 1, \left|D'_{i_t} \setminus M'_{i_t}\right| - 1\right) \in \mathcal{S}_{i_t}(D'_{i_t}).$$

4. For every $t \in \{2, \ldots, \bar{t} - 1\}$, if $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then

$$\left(\left|M_{i_t} \setminus D_{i_t}\right| - 1, \left|D_{i_t} \setminus M_{i_t}\right| - 1\right) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad \left(\left|M'_{i_t} \setminus D'_{i_t}\right| + 1, \left|D'_{i_t} \setminus M'_{i_t}\right| + 1\right) \in \mathcal{S}_{i_t}(D'_{i_t}).$$

5. If $i_{\bar{t}} \neq b$, then

$$\left(\left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|, \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right| + 1\right) \in \mathcal{S}_{i_{\bar{t}}}(D_{i_{\bar{t}}}) \quad \text{and} \quad \left(\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right|, \left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| - 1\right) \in \mathcal{S}_{i_{\bar{t}}}(D'_{i_{\bar{t}}})$$

when $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, and

$$\left(\left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right| - 1, \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|\right) \in \mathcal{S}_{i_{\bar{t}}}(D_{i_{\bar{t}}}) \quad \text{and} \quad \left(\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| + 1, \left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right|\right) \in \mathcal{S}_{i_{\bar{t}}}(D'_{i_{\bar{t}}})$$

when $d_{\bar{t}-1} \notin D_{i_{\bar{t}}}$.

6. If $i_1 \neq b$, then

$$\left(\left|M_{i_1} \setminus D_{i_1}\right|, \left|D_{i_1} \setminus M_{i_1}\right| - 1\right) \in \mathcal{S}_{i_1}(D_{i_1}) \quad \text{and} \quad \left(\left|M'_{i_1} \setminus D'_{i_1}\right|, \left|D'_{i_1} \setminus M'_{i_1}\right| + 1\right) \in \mathcal{S}_{i_1}(D'_{i_1})$$



Figure 11: Suppose $I = \{1, 2, 3\}$ with $\beta_1 = \beta_2 = A$ and $\beta_3 = B$. The donor sets in two extended problems $\hat{D}$ and $\hat{D}'$ are given by $D_1 = \{B_1\}$, $D'_1 = \emptyset$, $D_2 = D'_2 = \emptyset$, $D_3 = D'_3 = \{A_3\}$ and $D_b = \{A_b, A'_b, B_b\}$. Every patient's maximal need is two units and the feasible schedules are such that the amount supplied does not exceed the amount received. Let M be a matching for $\hat{D}$ such that $M_1 = \{A'_b\}$, $M_2 = \{A_b\}$, $M_3 = \{A_3, B_1, B_b\}$ and $M_b = \emptyset$. Let M' be a matching for $\hat{D}'$ such that $M'_1 = \{A_b, A'_b\}$, $M'_2 = \{A_3\}$, $M'_3 = \{B_b\}$ and $M'_b = \emptyset$. In this case, there does not exist a cycle from M to M', but the above graph gives a chain C from M to M'. Then $M + C$ is a matching for $\hat{D}$, where $(M + C)_1 = \{A_b, A'_b\}$, $(M + C)_2 = \{A_3\}$, $(M + C)_3 = \{B_1, B_b\}$ and $(M + C)_b = \emptyset$. Moreover, $M' - C$ is a matching for $\hat{D}'$, where $(M' - C)_1 = \{A'_b\}$, $(M' - C)_2 = \{A_b\}$, $(M' - C)_3 = \{A_3, B_b\}$ and $(M' - C)_b = \emptyset$.

when $d_1 \in D_{i_1}$, and

$$\left(\left|M_{i_1} \setminus D_{i_1}\right| + 1, \left|D_{i_1} \setminus M_{i_1}\right|\right) \in \mathcal{S}_{i_1}(D_{i_1}) \quad \text{and} \quad \left(\left|M'_{i_1} \setminus D'_{i_1}\right| - 1, \left|D'_{i_1} \setminus M'_{i_1}\right|\right) \in \mathcal{S}_{i_1}(D'_{i_1})$$

when $d_1 \notin D_{i_1}$.

7. If $i_t = i_{t'} = i$ for some t, t' such that $1 < t < t' < \bar{t}$, then either

   - $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or,
   - $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.

   If $i_{\bar{t}} = i_t = i$ for some t such that $1 < t < \bar{t}$, then either

   - $d_t, d_{t-1} \in D_i$ and $d_{\bar{t}-1} \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_{\bar{t}-1} \in D_i$.

   If $i_1 = i_t = i$ for some t such that $1 < t < \bar{t}$, then either

   - $d_t, d_{t-1} \in D_i$ and $d_1 \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_1 \in D_i$.

A chain differs from a cycle as the last element of a chain is a patient and she does not point to any donor. We refer to this patient, $i_{\bar{t}}$, as the **head** of the chain. As a result there is no donor pointing back to i_1 whom we refer to as the **tail** of the chain. The blood bank b can appear only as the tail or the head of a chain and cannot be anywhere else (Condition 1).

The definition of a chain takes care of special considerations for the tail and head agents, while the other conditions are similar to that of a cycle (Conditions 2, 3, 4, and 7). The other conditions make sure that if we remove a donor without adding one from the match of the head patient under M, then the resulting match will be feasible (or add a donor without removing one under M') (Condition 5). The corresponding condition for the tail makes sure that if we add a donor without removing one from the match of the tail patient under M, then the resulting match will be feasible (or remove a donor

without adding one under M') (Condition 6). Condition 7 also ensures that the tail/head patient can also appear at most twice in the chain (but not both as tail and head simultaneously).

Similar to the case of a cycle, given a chain C, we can construct a new matching, denoted by $M+C$, for $\hat{D}$ as follows: starting from M, for each t such that $1 \leq t \leq \bar{t}-1$, remove d_t from $M_{i_{t+1}}$ and add it to M_{i_t}. We can also construct a new matching, denoted by $M'-C$, for $\hat{D}'$ as follows: starting from M', for each $1 \leq t \leq \bar{t}-1$, remove d_t from M'_{i_t} and add it to $M'_{i_{t+1}}$. See Figure 11 for an example of a chain and how new matchings are constructed using this chain.

Unlike in a cycle addition or removal, the number of donors that a patient is matched with does only stay the same for patients who are neither the tail nor head of a chain. The head gains an additional donor after a chain addition and loses one donor after a chain removal. On the other hand, the tail patient loses one matched donor after a chain addition while she gains an additional matched donor after a chain removal.

Cycle operations would be all we needed if we were dealing with exogenous exchange rate of one unit donated per one unit received. However, as our model is more general with endogenously determined exchange rates, chain operations will play an important role in our proof.

The following observation is straightforward to show from the construction.

Observation 1. *Let C be a cycle or a chain from matching M to matching M'. For every $i \in \hat{I}$, we have*

$$\left|(M+C)_i \setminus D_i\right| - \left|M_i \setminus D_i\right| = \left|M'_i \setminus D'_i\right| - \left|(M'-C)_i \setminus D'_i\right|,$$

and

$$\left|D_i \setminus (M+C)_i\right| - \left|D_i \setminus M_i\right| = \left|D'_i \setminus M'_i\right| - \left|D'_i \setminus (M'-C)_i\right|.$$

We prove three lemmata which prove Theorem 2. The first lemma is the most crucial one that derives the technical result behind the proof of the theorem.

Lemma 3. *Consider any $D, D' \in \mathcal{D}$ and $i \in I$ such that $D'_i \subseteq D_i$, $\left|D_i \setminus D'_i\right| = 1$, and $D'_j = D_j$ for every $j \in I \setminus \{i\}$. If M is a matching for $\hat{D}$, M' is a matching for $\hat{D}'$, and $\left|M'_i \setminus D'_i\right| > \left|M_i \setminus D_i\right|$, then there is a cycle or a chain from M to M'.*

The proof of this lemma is rather involved. We illustrate the ideas behind the proof using an example first. The example only demonstrates substantially different *cases* in the proof of the lemma used in the construction of a cycle or a chain, as some of the considered cases use similar constructions.

Example 5. Suppose that the set of patients is $I = \{1, ..., 7\}$. The first row in Table 1 gives the blood type of each patient. The second row gives the donor set of each patient as well as the blood bank, where $X_i \in D_i$ (or $X'_i \in D_i$) is a type-X donor of patient i, and X_b (or X'_b) is a type-X donor from the bank. All blood types are Rh D+. We consider ABO-identical transfusion. Suppose that $n_i = 3$ for every patient i, $\underline{n}_2 = 1$ and $\underline{n}_i = 0$ for every patient $i \neq 2$.

Suppose that Patient 1 conceals his donor O_1.[39] Let

$$D'_1 = D_1 \setminus \{O_1\},$$

[39] Assume that the patients are male, and the donors are female in this example.

Patients/Bank:	1 (A)				2 (A)		3 (B)	4 (O)		5 (AB)		6 (A)	7 (O)	Bank		
Donors:	B_1	B'_1	AB_1	$\boldsymbol{O_1}$	B_2		A_3	A_4		A_5	O_5	AB_6	A_7	A_b	A'_b	O_b
$(\underline{n_i}, n_i)$:	$(0,3)$				$(1,3)$		$(0,3)$	$(0,3)$		$(0,3)$		$(0,3)$	$(0,3)$			
$\boldsymbol{M}$	B_1	AB_1	A_7	A'_b	A_b		A_3	A_4	O_b	A_5	O_5	AB_6	O_1	B'_1	B_2	
$\boldsymbol{M'}$	B'_1	A_7	A'_b	A_b	B_2	A_3	A_4	B_1		O_b		AB_1	AB_6	A_5	O_5	$\emptyset$
$\boldsymbol{M''}$	B_1	O_1	A_7	A'_b	A_b		A_3	A_4	O_b	A_5	AB_1	AB_6	O_5	B'_1	B_2	

Table 1: The patients, blood bank, their donors, and the patients' minimum and maximum demand for Example 5 . Each patient is denoted by his index number, we also list his blood type next to his index number. Each donor is denoted by her blood type and her paired patient is denoted by a subscript next to her blood type. Each blood type is Rh D+. We only consider ABO-identical transfusion. When Patient 1 is truthful about his donor set matching M is obtained. When he conceals his donor O_1, matching M' is obtained and he receives more blood. M'' is another matching that we explain in the example.

and $D'_i = D_i$ for every patient $i \neq 1$. Finally, for each patient i and each donor set D''_i, let

$$\mathcal{S}_i(D''_i) = \{(r,s) : \underline{n}_i \leq r \leq n_i, 0 \leq s \leq |D''_i|, s \leq r\}.$$

The last three rows in the table specify three matchings M, M', and M'', where M and M'' are matchings for $\hat{D}$ and M' is a matching for $\hat{D}'$.[40] Given that Patient 1 receives more units of blood under M' than under M, we discuss how to find a cycle or a chain from M to M' using an iterative "pointing procedure from M to M'" that we formally define in the proof of Lemma 3. At each step of the procedure, a patient points to a donor that he is matched with under M' but not under M, then this donor points to the patient that she is matched with under M.

We start with the patient who shrank his donor set, Patient 1. Since he receives more blood at M', there is a donor in $M' \setminus M$ that is not his own. He points to one arbitrary donor, who is not his own, in this set. We assume Patient 1 first points to A_b. Then A_b points to Patient 2, who she is matched with under M.

Although B_2 is a donor matched with Patient 2 at M' but not at M, we don't let Patient 2 point to B_2, because

$$\left(\left|M_2 \setminus D_2\right| - 1, \left|D_2 \setminus M_2\right| - 1\right) = (0,0) \notin \mathcal{S}_2(D_2),$$

that is if we eventually execute this supposed swap of donors to obtain a new matching from M, the outcome will not be a feasible matching (i.e., the schedule of Patient 2 is not in his feasible schedule set). Generally, when a patient is pointed by a donor that is not his own (respectively, his own donor), we always first check whether this patient can point to a donor that is not his own (respectively, his own donor), such that the exchanges in the cycle or chain would not change the patient's schedule. Therefore, we let Patient 2 point to A_3 or A_4. Suppose that Patient 2 points to A_3, then A_3 points to Patient 3. As discussed before, generally, when a patient is pointed by his own donor, we check whether he can point to his own donor. If this is not possible, then he must donate more blood at M' and there are two possible cases:

[40] For simplicity dummy patients and dummy donors are omitted.

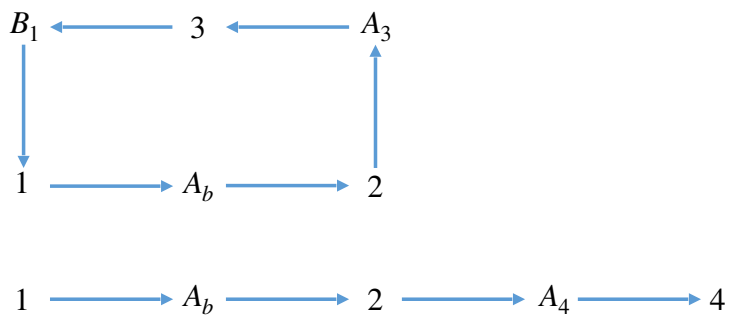


Figure 12: A cycle and a chain from matching M to M' found using the pointing procedure from M to M' (illustrating Case 2 and Case 3 in the proof of Lemma 3, respectively).

- If he also receives more blood under M', then we let him point to a donor that is not his own so that, by Assumption 1′, Condition 2 in the definition of a cycle, or Condition 3 in the definition of a chain is satisfied.
- If he does not receive more blood under M', then we stop here and by Assumption 1′, he will be the head of a chain (i.e., Condition 5 in the definition of a chain is satisfied).

In the example, Patient 3 cannot point to his own donor and he receives more blood under M', so we let him point to B_1. Then B_1 points to Patient 1, and a cycle is found: see the cycle in Figure 12 - this construction corresponds to Case 2 in the proof of Lemma 3.

Recall that Patient 2 could also point to A_4. If Patient 2 points to A_4, then A_4 points to Patient 4. Given that Patient 4 cannot point to his own donor and he does not receive more blood at M', we have to stop at Patient 4. In this case, a chain is identified as in the graph in Figure 12 - this construction corresponds to Case 3 in the proof of Lemma 3. Note that Condition 6 in the definition of a chain is satisfied for Patient 1. This follows from Assumption 2′ and the fact that his schedules under M and M' are $(2, 2)$ and $(3, 2)$ respectively.

Generally, according to Assumption 2′, the fact that the manipulating patient receives more blood but does not donate more blood under M' implies that he can sometimes be the tail of a chain. However, if he both receives and donates more blood under M', then Condition 6 in the definition of a chain may not be satisfied. To further discuss this case, we modify the example slightly: Suppose that B_1' is matched with the bank instead of Patient 1 under M', and we change the feasible schedules of Patient 1 such that

$$\mathcal{S}_1(D_1'') = \{(r, s) : 0 \leq r \leq n_1, 0 \leq s \leq |D_1''|, s = r\}$$

for any D_1''. Then the list $(1, A_b, 2, A_4, 4)$ in Figure 12 is no longer a chain from M to M' since $(|M_1 \setminus D_1| + 1, |D_1 \setminus M_1|) = (3, 2) \notin \mathcal{S}_1(D_1)$.

In this case we have to invoke a "backward" pointing procedure. That is, pointing occurs from M' to M, and at each step a patient points to a donor that he is matched with under M but not under M', while the donor points to the patient that she is matched with under M'. Then the edge orientation will be reversed to construct a chain or a cycle from M to M'. This corresponds to Case

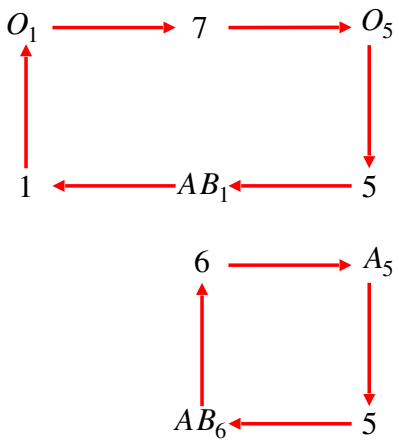


Figure 13: A cycle from M to M' and another directed graph that is a *pseudo-cycle* from M to M' in the modified example found using the pointing procedure from M' to M. The orientation of the edges are reversed at the end so that the cycle and pseudo-cycle are from M to M' (illustrating Case 4.1 and Case 4.5 in the proof of Lemma 3, respectively).

4 in the proof of Lemma 3.

We still start the procedure with Patient 1. As he donates less blood under M, there is a donor in $M_1 \setminus M_1'$ that is his own donor. At the beginning of the pointing procedure from M' to M, Patient 1 points to such a donor. Assume that Patient 1 points to AB_1. Then AB_1 points to Patient 5. Similar to the previous construction, when a patient is pointed by a donor that is not his own, we first check whether he can point to a donor that is not his own. If this is not possible, then he must receive less blood under M and there are two cases:

- If he also donates less blood under M, let him point to a donor of his own.
- If he does not donate less blood under M, we stop here.

In the example, Patient 5 cannot point to a donor that is not his own and he donates less blood under M. So Patient 5 points to A_5 or O_5. Suppose that Patient 5 points to A_5, then A_5 points to Patient 6, Patient 6 points to AB_6, and AB_6 points to Patient 5. After reversing the edge orientation, a cycle from M to M' is found: see the first cycle in Figure 13 - this construction corresponds to Case 4.1 in the proof of Lemma 3.

On the other hand, if Patient 5 points to O_5, then O_5 points to Patient 7, who points to the concealed donor O_1. Let O_1 point to Patient 1. After reversing the edge orientation, we obtain a list $(1, O_1, 7, O_5, 5, AB_1)$, which is the second graph in Figure 13. However, this is not a cycle from M to M', we refer to it as a pseudo-cycle. Case 4.5 in the proof of Lemma 3 deals with this type of situation. We can still carry out the exchanges in the list based on M, and this leads to the matching M'' for $\hat{D}$. Since Patient 1's schedules under M'' and M are the same, we can repeat the previous pointing procedures and identify a cycle or a chain C from M'' and M'. Note that as the donor O_1 is matched with Patient 1 under M'', she will not appear in the pointing procedures. Finally, we conclude the proof by showing that C is also a cycle or a chain from M to M'. ∎

We are ready to state the proof of Lemma 3.

Proof of Lemma 3. Consider two problems $D, D' \in \mathcal{D}$ such that for some patient $i_1 \in I$, $D'_{i_1} \subseteq D_{i_1}$, $\left|D_{i_1} \setminus D'_{i_1}\right| = 1$, and $D'_i = D_i$ for every $i \in I \setminus \{i_1\}$. Suppose that M is a matching for $\hat{D}$, M' is a matching for $\hat{D}'$ and $\left|M'_{i_1} \setminus D'_{i_1}\right| > \left|M_{i_1} \setminus D_{i_1}\right|$. Then there exists a donor $d_1 \notin D_{i_1}$ such that $d_1 \in M'_{i_1} \setminus M_{i_1}$.

We iteratively construct a finite directed graph using matchings M and M' of patients and donors starting with patient i_1, which is denoted as $(i_1, d_1, i_2, d_2, \ldots)$, ending either with a patient or a donor and each node in the list points to the next node.

We refer to this as the *pointing procedure from M to M'*:

Step 1: Let i_1 point to d_1, and d_1 point to $i_2 \in \hat{I}$ such that $d_1 \in M_{i_2}$. If $i_2 = b$ then we stop at i_2 in Step 1, otherwise we continue.

Step $t \geq 2$: At the end of Step $t-1$, patient $i_t \in \hat{I} \setminus \{i_1, b\}$ is pointed by d_{t-1} where $d_{t-1} \in M_{i_t} \setminus M'_{i_t}$.

1. If $d_{t-1} \in D_{i_t}$: We have two cases:
   (a) If there exists $d \in D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then at Step t, let i_t point to $d_t = d$, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.[41]
   (b) If there does not exist $d \in D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then $\left|D'_{i_t} \setminus M'_{i_t}\right| > \left|D_{i_t} \setminus M_{i_t}\right|$. We have two subcases:
      i. If $\left|M'_{i_t} \setminus D'_{i_t}\right| > \left|M_{i_t} \setminus D_{i_t}\right|$: Then there exists $d_t \notin D_{i_t}$ such that $d_t \in M'_{i_t} \setminus M_{i_t}$. At Step t, let i_t point to d_t, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.
      ii. If $\left|M'_{i_t} \setminus D'_{i_t}\right| \leq \left|M_{i_t} \setminus D_{i_t}\right|$: Then i_t does not point and stop at i_t at Step $t-1$.
2. If $d_{t-1} \notin D_{i_t}$: We have two cases:
   (a) If there exists $d \notin D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then at Step t, let i_t point to $d_t = d$, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.
   (b) If there does not exist $d \notin D_{i_t}$ such that $d \in M'_{i_t} \setminus M_{i_t}$: Then $\left|M'_{i_t} \setminus D'_{i_t}\right| < \left|M_{i_t} \setminus D_{i_t}\right|$. We have two subcases:
      i. If $\left|D'_{i_t} \setminus M'_{i_t}\right| < \left|D_{i_t} \setminus M_{i_t}\right|$: Then there exists $d_t \in D_{i_t}$ such that $d_t \in M'_{i_t} \setminus M_{i_t}$. At Step t, let i_t point to d_t, and d_t point to i_{t+1} such that $d_t \in M_{i_{t+1}}$.
      ii. If $\left|D'_{i_t} \setminus M'_{i_t}\right| \geq \left|D_{i_t} \setminus M_{i_t}\right|$: Then i_t does not point and stop at i_t at Step $t-1$.

If d_t is constructed, $i_t = i_{\underline{t}} \notin \{i_1, b\}$ for some $\underline{t} < t$ and neither

- $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$, nor
- $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$

holds, then stop at donor d_t at Step t and remove i_{t+1} from the graph construction.

If patient $i_{t+1} \in \{i_1, b\}$, then stop at i_{t+1} at Step t.

Otherwise, continue with Step $t+1$.

[41] Generally for each $t \geq 1$, such i_{t+1} always exists, since d_t is a donor in the extended problem $\hat{D}$.

Note that, according to the above construction, $i_t \neq i_{t+1}$ for any t. Moreover, the procedure stops under four circumstances:

- when some $i \notin \{i_1, b\}$ has appeared before, and she is pointed by and points to her own donors in one instance, and is pointed by and points to donors who are not her own in the other instance,
- when i_1 is pointed to,
- when b is pointed to,
- when some $i \notin \{i_1, b\}$ does not point.

The last circumstance implies that any patient can be pointed at most three times in the procedure. Hence, the procedure always stops in a finite number of steps.

We consider the following four cases based on these circumstances. Case 1 and Case 2 cover the first two circumstances in order and show the existence of a cycle in each case. Case 3 covers the third and the fourth circumstances together when i_1 does not supply more blood under M' than M and shows the existence of a chain. Finally, Case 4 covers the third and the fourth circumstances together when i_1 supplies more blood under M' than M and shows the existence of a cycle or a chain. This is the most involved case and we will handle it the last.

Case 1. The procedure stops at $d_{\bar{t}}$ at Step $\bar{t}$.

Then for some $\underline{t} < \bar{t}$, $i_{\underline{t}} = i_{\bar{t}} \notin \{i_1, b\}$ and neither of the following is true:

1. $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\bar{t}}, d_{\bar{t}-1} \notin D_{i_{\underline{t}}}$.
2. $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_{\underline{t}}}$ and $d_{\bar{t}}, d_{\bar{t}-1} \in D_{i_{\underline{t}}}$.

We show that $(i_{\underline{t}}, d_{\underline{t}}, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M'.

First, for any t such that $\underline{t} < t \leq \bar{t} - 1$, $i_t \notin \{i_1, b\}$, since otherwise the procedure stops at i_t at Step $t-1$. It follows that $D_{i_t} = D'_{i_t}$ for every t such that $\underline{t} \leq t \leq \bar{t} - 1$. By the construction, Condition 1 in the definition of a cycle is satisfied. Next, we show Condition 2 and Condition 3. First, consider any t such that $\underline{t} < t \leq \bar{t} - 1$. If $d_{t-1} \in D_{i_t}$ and $d_t \notin D_{i_t}$, then by the construction, we have $\left|M'_{i_t} \setminus D'_{i_t}\right| > \left|M_{i_t} \setminus D_{i_t}\right|$ and $\left|D'_{i_t} \setminus M'_{i_t}\right| > \left|D_{i_t} \setminus M_{i_t}\right|$. Since

$$\left(\left|M_{i_t} \setminus D_{i_t}\right|, \left|D_{i_t} \setminus M_{i_t}\right|\right) \in \mathcal{S}_{i_t}(D_{i_t}) \text{ and } \left(\left|M'_{i_t} \setminus D'_{i_t}\right|, \left|D'_{i_t} \setminus M'_{i_t}\right|\right) \in \mathcal{S}_{i_t}(D'_{i_t}) = \mathcal{S}_{i_t}(D_{i_t}),$$

it follows from Assumption 1′ that

$$\left(\left|M_{i_t} \setminus D_{i_t}\right| + 1, \left|D_{i_t} \setminus M_{i_t}\right| + 1\right) \in \mathcal{S}_{i_t}(D_{i_t}) \text{ and } \left(\left|M'_{i_t} \setminus D'_{i_t}\right| - 1, \left|D'_{i_t} \setminus M'_{i_t}\right| - 1\right) \in \mathcal{S}_{i_t}(D'_{i_t}).$$

Similarly, if $d_{t-1} \notin D_{i_t}$ and $d_t \in D_{i_t}$, then by the construction we have $\left|M'_{i_t} \setminus D'_{i_t}\right| < \left|M_{i_t} \setminus D_{i_t}\right|$ and $\left|D'_{i_t} \setminus M'_{i_t}\right| < \left|D_{i_t} \setminus M_{i_t}\right|$. It follows from Assumption 1′ that

$$\left(\left|M_{i_t} \setminus D_{i_t}\right| - 1, \left|D_{i_t} \setminus M_{i_t}\right| - 1\right) \in \mathcal{S}_{i_t}(D_{i_t}) \text{ and } \left(\left|M'_{i_t} \setminus D'_{i_t}\right| + 1, \left|D'_{i_t} \setminus M'_{i_t}\right| + 1\right) \in \mathcal{S}_{i_t}(D'_{i_t}).$$

Second, consider $i_{\underline{t}}$. Suppose that $d_{\bar{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\underline{t}} \notin D_{i_{\underline{t}}}$. Then either $d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ or $d_{\bar{t}} \notin D_{i_{\underline{t}}}$, as the procedure stops at the donor $d_{\bar{t}}$. Since we have either

- $d_{\bar{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\bar{t}} \notin D_{i_{\underline{t}}}$, or,
- $d_{\underline{t}-1} \in D_{i_{\underline{t}}}$ and $d_{\underline{t}} \notin D_{i_{\underline{t}}}$,

by the construction,

$$\left|M'_{i_{\underline{t}}} \setminus D'_{i_{\underline{t}}}\right| > \left|M_{i_{\underline{t}}} \setminus D_{i_{\underline{t}}}\right| \text{ and } \left|D'_{i_{\underline{t}}} \setminus M'_{i_{\underline{t}}}\right| > \left|D_{i_{\underline{t}}} \setminus M_{i_{\underline{t}}}\right|.$$

Then by Assumption 1′,

$$\left(\left|M_{i_{\underline{t}}} \setminus D_{i_{\underline{t}}}\right| + 1, \left|D_{i_{\underline{t}}} \setminus M_{i_{\underline{t}}}\right| + 1\right) \in \mathcal{S}_{i_{\underline{t}}}(D_{i_{\underline{t}}}) \text{ and } \left(\left|M'_{i_{\underline{t}}} \setminus D'_{i_{\underline{t}}}\right| - 1, \left|D'_{i_{\underline{t}}} \setminus M'_{i_{\underline{t}}}\right| - 1\right) \in \mathcal{S}_{i_{\underline{t}}}(D'_{i_{\underline{t}}}).$$

That is, Condition 2 in the definition of a cycle is satisfied for $i_{\underline{t}}$. By similar arguments, it can be shown that Condition 3 is also satisfied for $i_{\underline{t}}$.

It remains to show Condition 4. If $i_t = i_{t'}$ and $\underline{t} < t < t' \leq \bar{t} - 1$, then either

- $d_t, d_{t-1} \in D_{i_t}$ and $d_{t'}, d_{t'-1} \notin D_{i_t}$, or
- $d_t, d_{t-1} \notin D_{i_t}$ and $d_{t'}, d_{t'-1} \in D_{i_t}$,

since otherwise the procedure stops at $d_{t'}$ at Step t'. Finally, suppose that $i_t = i_{\underline{t}}$ and $\underline{t}+1 < t < \bar{t}-1$. Since the procedure does not stop at d_t at Step t, we have either

(i) $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$, or,

(ii) $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$.

Consider (i) first. Recall that $i_t = i_{\underline{t}} = i_{\bar{t}}$. If $d_{\bar{t}-1} \notin D_{i_t}$, then by the construction of the pointing procedure from M to M', $d_t \notin D_{i_t}$ implies that there exists a donor in $M'_{i_t} \setminus M_{i_t}$ that is not her own, and thus, she should again point to such a donor when she repeats for the third time as $i_{\bar{t}}$: $d_{\bar{t}} \notin D_{i_t}$. So we have $d_{\bar{t}}, d_{\bar{t}-1} \notin D_{i_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \in D_{i_t}$, which contradicts Case 1's assumption. Therefore, $d_{\underline{t}}, d_{\bar{t}-1} \in D_{i_t}$ and $d_t, d_{t-1} \notin D_{i_t}$. Similarly, if (ii) is true, then $d_{\bar{t}-1} \notin D_{i_t}$, since otherwise $d_t \in D_{i_t}$ implies $d_{\bar{t}} \in D_{i_t}$, contradiction. Hence, $d_{\underline{t}}, d_{\bar{t}-1} \notin D_{i_t}$ and $d_t, d_{t-1} \in D_{i_t}$. This shows that Condition 4 holds, as well.

Case 2. The procedure stops at $i_{\bar{t}}$ at Step $\bar{t} - 1$ and $i_{\bar{t}} = i_1$.

To show that $(i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M', where $d_1 \notin D_{i_1}$, we verify Condition 2 in the definition of a cycle when $d_{\bar{t}-1} \in D_{i_1}$. Since $d_{\bar{t}-1} \in M_{i_1}$ and $d_{\bar{t}-1} \in M'_{i_{\bar{t}-1}}$, $\left|D_{i_1} \setminus M_{i_1}\right| < \left|D_{i_1}\right|$ and $\left|D'_{i_1} \setminus M'_{i_1}\right| > 0$. Then given that in the hypothesis of the lemma $\left|M'_{i_1} \setminus D'_{i_1}\right| > \left|M_{i_1} \setminus D_{i_1}\right|$, by Assumption 2′, we have

$$\left(\left|M_{i_1} \setminus D_{i_1}\right| + 1, \left|D_{i_1} \setminus M_{i_1}\right| + 1\right) \in \mathcal{S}_{i_1}(D_{i_1}) \text{ and } \left(\left|M'_{i_1} \setminus D'_{i_1}\right| - 1, \left|D'_{i_1} \setminus M'_{i_1}\right| - 1\right) \in \mathcal{S}_{i_1}(D'_{i_1}).$$

The other conditions are similar to Case 1 and can be shown as in Case 1.

Case 3. The procedure stops at $i_{\bar{t}}$ at Step $\bar{t} - 1$, $i_{\bar{t}} \neq i_1$, and $\left|D'_{i_1} \setminus M'_{i_1}\right| \leq \left|D_{i_1} \setminus M_{i_1}\right|$.

Then either $i_{\bar{t}} = b$ or in the procedure the patient $i_{\bar{t}} \in \hat{I} \setminus \{i_1, b\}$ does not point. We show that $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ is a chain from M to M'. First, $i_t \neq b$ for any $t \in \{2, \ldots, \bar{t}-1\}$ since otherwise the procedure stops at an earlier step. Second, we verify Condition 5 in the definition of a chain. Suppose that $i_{\bar{t}} \neq b$. If $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, then by the construction, $\left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| > \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|$ and $\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| \leq \left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|$. Since $D_{i_{\bar{t}}} = D'_{i_{\bar{t}}}$, by Assumption 1′,

$$\left(\left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|, \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right| + 1\right) \in \mathcal{S}_{i_{\bar{t}}}(D_{i_{\bar{t}}}) \text{ and } \left(\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right|, \left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| - 1\right) \in \mathcal{S}_{i_{\bar{t}}}(D'_{i_{\bar{t}}}).$$

The case that $d_{\bar{t}-1} \notin D_{i_{\bar{t}}}$ can be shown similarly. Next, Condition 6 follows from the fact that $\left|M'_{i_1} \setminus D'_{i_1}\right| > \left|M_{i_1} \setminus D_{i_1}\right|$ and $\left|D'_{i_1} \setminus M'_{i_1}\right| \leq \left|D_{i_1} \setminus M_{i_1}\right|$, as well as Assumption 2′. Finally, we verify Condition 7 for i_1 and $i_{\bar{t}}$. For any $t \in \{2, \ldots, \bar{t}-1\}$, $i_1 \neq i_t$, since otherwise the procedure stops at an earlier step. Suppose $i_{\bar{t}} = i_t$ for some t such that $1 < t < \bar{t}$. Then $i_t \neq b$. First consider the case that $d_{\bar{t}-1} \in D_{i_t}$. If $d_t \in D_{i_t}$, then $d_t \in M'_{i_t} \setminus M_{i_t}$ and $i_{\bar{t}} = i_t$ should point to this donor at Step $\bar{t}$, which contradicts with the fact that the pointing procedure from M to M' ends with $i_{\bar{t}}$. So $d_t \notin D_{i_t}$. Then $d_{t-1} \notin D_{i_t}$, since otherwise $i_{\bar{t}} = i_t$ should point to d_t at Step $\bar{t}$. In the case that $d_{\bar{t}-1} \notin D_{i_t}$, it can be similarly shown that $d_t, d_{t-1} \in D_{i_t}$. These are the crucial conditions to check; the other conditions are similar to Case 1 and can be shown as in Case 1.

Case 4. The procedure stops at $i_{\bar{t}}$ at Step $\bar{t}-1$, $i_{\bar{t}} \neq i_1$, and $\left|D'_{i_1} \setminus M'_{i_1}\right| > \left|D_{i_1} \setminus M_{i_1}\right|$.

In this case, we may not have $\left(\left|M_{i_1} \setminus D_{i_1}\right| + 1, \left|D_{i_1} \setminus M_{i_1}\right|\right) \in \mathcal{S}_{i_1}(D_{i_1})$, and hence $(i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ may not be a chain from M to M'. To find a cycle or a chain, we do the reverse of what we did before and we use the pointing procedure from M' to M. Since we are seeking a cycle or a chain from M to M', in end we reverse the orientations of the constructed edges. There will also be two slight complications in Step t 1.(b)i.A and Step t 2.(a)i that we will explain later.

Pointing procedure from M' to M constructs a second directed graph of patients and donors starting with $j_1 = i_1$ and denoted as $(j_1, c_1, j_2, c_2, \ldots)$. It ends either with a patient or a donor and each node in the list points at the next node in the list. Since $\left|D'_{j_1} \setminus M'_{j_1}\right| > \left|D_{j_1} \setminus M_{j_1}\right|$ starting donor $c_1 \in D'_{j_1}$ such that $c_1 \in M_{j_1} \setminus M'_{j_1}$ exists.

Similar to the pointing procedure from M to M', the pointing procedure from M' to M also stops in a finite number of steps. Recall that $\left|D_{i_1} \setminus D'_{i_1}\right| = 1$ and $D_{i_1} \supset D'_{i_1}$. We refer to the donor that is omitted from the donor set of i_1 in D'_{i_1} as the *concealed donor*. The procedure stops under five circumstances instead of four:

- when some $j \notin \{j_1, b\}$ has appeared before in the constructed graph and she is pointed by and points to her own donors in one instance, and is pointed by and points to donors who are not her own in the other instance,
- when b is pointed,
- when some $j \notin \{i_1, b\}$ does not point,
- when j_1 is pointed by a donor who is not concealed,
- when j_1 is pointed by the concealed donor (this can happen in Step $\bar{t}$ 1.b.(i)A or 2.a.(i)).

We consider the following five subcases based on these circumstances. Subcase 4.1 and Subcase 4.2 cover the first circumstance and show the existence of a cycle. Subcase 4.3 covers the second and the third circumstances together and shows the existence of a cycle or a chain. Subcase 4.4 covers the fourth circumstance and shows the existence of a cycle. Finally, Subcase 4.5 covers the last circumstance and shows the existence of a cycle or a chain.

Subcase 4.1. The procedure stops at c_t at Step t, for some $\underline{t} < t$, $j_t = j_{\underline{t}} \notin \{j_1, b\}$ and neither of the following is true:

- $c_t, c_{t-1} \in D_{j_t}$ and $c_{\underline{t}}, c_{\underline{t}-1} \notin D_{j_t}$.
- $c_t, c_{t-1} \notin D_{j_t}$ and $c_{\underline{t}}, c_{\underline{t}-1} \in D_{j_t}$.

Then reversing the edges we found in the second directed graph, $(j_t, c_{t-1}, \ldots, j_{\underline{t}+1}, c_{\underline{t}})$ is a cycle from M to M'.

Subcase 4.2. The procedure stops at c_t at Step t, for some $\underline{t} \in \{2, \ldots, \bar{t} - 1\}$, $i_{\underline{t}} = j_t \notin \{j_1, b\}$ and neither of the following is true:

- $c_t, c_{t-1} \in D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \notin D_{j_t}$.
- $c_t, c_{t-1} \notin D_{j_t}$ and $d_{\underline{t}}, d_{\underline{t}-1} \in D_{j_t}$.

Now we construct a cycle using the first directed graph constructed by the pointing procedure from M to M', $(i_1, d_1, i_2, d_2, \ldots)$, and the second graph constructed by the pointing procedure from M' to M, $(j_1, c_1, j_2, c_2, \ldots)$, recalling that $j_1 = i_1$ and the orientation of the edges in the second graph should be reversed: Then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, i_{\underline{t}-1}, d_{\underline{t}-1})$ is a cycle from M to M'. Note that Condition 4 in the definition of a cycle must be satisfied, since otherwise either the pointing procedure from M to M' or the pointing procedure from M' to M stops at an earlier step.

Subcase 4.3. The procedure stops at j_t at Step $t - 1$, and $j_t \neq j_1$.

Then either $j_t = b$ or the patient j_t does not point.

If $j_t = i_{\bar{t}} = b$, then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, i_{\bar{t}-1}, d_{\bar{t}-1})$ is a cycle from M to M'.

If it is not true that $j_t = i_{\bar{t}} = b$, then $(j_t, c_{t-1}, \ldots, c_1, i_1, d_1, \ldots, d_{\bar{t}-1}, i_{\bar{t}})$ is a chain from M to M'. To see this, we verify $j_t \neq i_{\bar{t}}$ and Condition 6 in the definition of a chain. First, assume to the contrary, $j_t = i_{\bar{t}}$. Then $j_t = i_{\bar{t}} \in \hat{I} \setminus \{j_1, b\}$. If $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$, then $c_{t-1} \notin D_{i_{\bar{t}}}$, since otherwise in the pointing procedure from M' to M, j_t should point at $d_{\bar{t}-1}$ or a different donor at Step t. However, by the construction, $d_{\bar{t}-1} \in D_{i_{\bar{t}}}$ implies $\left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| > \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|$ and $\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| \leq \left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|$, while $c_{t-1} \notin D_{i_{\bar{t}}}$ implies $\left|M'_{i_{\bar{t}}} \setminus D'_{i_{\bar{t}}}\right| > \left|M_{i_{\bar{t}}} \setminus D_{i_{\bar{t}}}\right|$ and $\left|D'_{i_{\bar{t}}} \setminus M'_{i_{\bar{t}}}\right| \leq \left|D_{i_{\bar{t}}} \setminus M_{i_{\bar{t}}}\right|$, contradiction. A similar contradiction can be reached when $d_{\bar{t}-1} \notin D_{i_{\bar{t}}}$. Therefore, $j_t \neq i_{\bar{t}}$. Second, consider Condition 6. If $j_t \neq b$, and $c_{t-1} \in D_{j_t}$, then by the construction $\left|D'_{j_t} \setminus M'_{j_t}\right| < \left|D_{j_t} \setminus M_{j_t}\right|$ and $\left|M'_{j_t} \setminus D'_{j_t}\right| \geq \left|M_{j_t} \setminus D_{j_t}\right|$. It follows from Assumption 1′ that

$$\left(\left|M_{j_t} \setminus D_{j_t}\right|, \left|D_{j_t} \setminus M_{j_t}\right| - 1\right) \in \mathcal{S}_{j_t}(D_{j_t}) \text{ and } \left(\left|M'_{j_t} \setminus D'_{j_t}\right|, \left|D'_{j_t} \setminus M'_{j_t}\right| + 1\right) \in \mathcal{S}_{j_t}(D'_{j_t}).$$

The case that $c_{t-1} \notin D_{j_t}$ can be shown similarly.

Subcase 4.4. The procedure stops at j_t at Step $t-1$, $j_t = j_1$ and $c_{t-1} \notin D_{j_1} \setminus D'_{j_1}$.

Then $(j_t, c_{t-1}, \ldots, j_2, c_1)$ is a cycle from M to M'.

Subcase 4.5. The procedure stops at j_t at Step $t-1$, $j_t = j_1$ (recall that $j_1 = i_1$ is the patient who concealed her donor from D_{i_1} to D'_{i_1}) and $c_{t-1} \in D_{j_1} \setminus D'_{j_1}$ (thus, c_{t-1} is the concealed donor).

We have $j_{t'} \in \hat{I} \setminus \{j_1, b\}$ for every $t' \in \{2, \ldots, t-1\}$, since otherwise the procedure stops at an earlier step. As j_t points to the concealed donor $c_{t-1} \notin M'_{j_t}$, $(j_t, c_{t-1}, \ldots, j_2, c_1)$ is not a cycle from M to M'.

However, we can still carry out the exchanges in the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$, starting from M: add c_{t-1} to M_{j_t} and remove c_1 from $M_{j_t}, \ldots$, add c_1 to M_{j_2} and remove c_2 from M_{j_2}. This leads to a well-defined matching M'' for $\hat{D}$. Since $c_1, c_{t-1} \in D_{j_t}$, $\left|M''_{j_t} \setminus D_{j_t}\right| = \left|M_{j_t} \setminus D_{j_t}\right|$ and $\left|D_{j_t} \setminus M''_{j_t}\right| = \left|D_{j_t} \setminus M_{j_t}\right|$. That is, Patient j_t receives and supplies the same amounts of blood under M'' and M.

Given that $\left|M''_{j_t} \setminus D_{j_t}\right| < \left|M'_{j_t} \setminus D_{j_t}\right|$, we can repeat the previous analysis and identify a cycle or a chain from M'' to M', using the pointing procedure from M'' to M' and if Case 4 is reached again then pointing procedure from M' to M''.

Note that, $c_{t-1} \notin M'_i$ for any $i \in \hat{I}$ implies c_{t-1} is not pointed in the pointing procedure from M'' to M', and if Case 4 is reached again, $c_{t-1} \in M''_{j_t}$ implies c_{t-1} is not pointed in the pointing procedure from M' to M'', as we start either procedure with patient $j_t = j_1 = i_1$.

Hence c_{t-1} does not appear in either pointing procedure and a cycle or a chain C from M'' to M' can be found, as this recursive Case 4.5 is never reached again.

To finish the proof of Lemma 3, it only remains to show that C is also a cycle or a chain from M to M'. We only consider the case that C is a chain (the proof for the case that C is a cycle is similar and simpler). Let $C = (l_1, a_1, \ldots, l_{\bar{w}-1}, a_{\bar{w}-1}, l_{\bar{w}})$, where $\bar{w} \geq 2$ and $a_1, \ldots, a_{\bar{w}-1}$ are donors and $l_1, \ldots, l_{\bar{w}}$ are patients. We verify the conditions in the definition of a chain from M and M'.

Since C is a chain from M'' to M', Condition 1 and Condition 7 are trivially satisfied for C to be a chain from M to M'. Since C is a chain from M'' to M', for any $w \in \{1, \ldots, \bar{w}-1\}$, $a_w \in M'_{l_w} \setminus M''_{l_w}$ and $a_w \in M''_{l_{w+1}} \setminus M'_{l_{w+1}}$. Given that M'' is obtained from M by carrying out the exchanges in the list $(j_t, c_{t-1}, \ldots, j_2, c_1)$, we have $a_w \notin M_{l_w}$, since otherwise $a_w \in M_{l_w}$ and $a_w \in M'_{l_w}$ imply a_w is not in the list and $a_w \in M''_{l_w}$. Similarly, we have $a_w \in M_{l_{w+1}}$, since otherwise $a_w \notin M_{l_{w+1}}$ and $a_w \notin M'_{l_{w+1}}$ imply $a_w \notin M''_{l_{w+1}}$. Therefore, Condition 2 is satisfied.

To show Conditions 3-6, we need the following result, which follows from the construction of the (reverse of) list $(j_t, c_{t-1}, \ldots, j_2, c_1)$ in the pointing procedure from M' to M. It essentially says that any patient's schedule under M'' must be "between" her schedules under M and M'.

Observation 2. *For any $i \in \hat{I} \setminus \{b\}$, if $\left(\left|M''_i \setminus D_i\right|, \left|D_i \setminus M''_i\right|\right) \neq \left(\left|M_i \setminus D_i\right|, \left|D_i \setminus M_i\right|\right)$, then $i \neq j_t$, and either*

- $\left|M'_i \setminus D'_i\right| > \left|M_i \setminus D_i\right|$, $\left|D'_i \setminus M'_i\right| > \left|D_i \setminus M_i\right|$, *and* $\left(\left|M''_i \setminus D_i\right|, \left|D_i \setminus M''_i\right|\right) = \left(\left|M_i \setminus D_i\right| + 1, \left|D_i \setminus M_i\right| + 1\right)$,

  or

- $\left|M'_i \setminus D'_i\right| < \left|M_i \setminus D_i\right|$, $\left|D'_i \setminus M'_i\right| < \left|D_i \setminus M_i\right|$, *and* $\left(\left|M''_i \setminus D_i\right|, \left|D_i \setminus M''_i\right|\right) = \left(\left|M_i \setminus D_i\right| - 1, \left|D_i \setminus M_i\right| - 1\right)$.

Consider any $w \in \{2, \dots, \bar{w}-1\}$ such that $a_{w-1} \in D_{l_w}$ and $a_w \notin D_{l_w}$. Condition 3 is clearly satisfied if $(|M''_{l_w} \setminus D_{l_w}|, |D_{l_w} \setminus M''_{l_w}|) = (|M_{l_w} \setminus D_{l_w}|, |D_{l_w} \setminus M_{l_w}|)$. Suppose that this is not true. Then $l_w \neq j_t$. By the construction of the chain C from M'' to M', we have $|M'_{l_w} \setminus D'_{l_w}| > |M''_{l_w} \setminus D''_{l_w}|$ and $|D'_{l_w} \setminus M'_{l_w}| > |D''_{l_w} \setminus M''_{l_w}|$. Then by Observation 2, $|M'_{l_w} \setminus D'_{l_w}| > |M_{l_w} \setminus D_{l_w}|$ and $|D'_{l_w} \setminus M'_{l_w}| > |D_{l_w} \setminus M_{l_w}|$. Hence it follows from Assumption 1′ that Condition 3 is satisfied. Condition 4 can be shown in a similar manner.

Next, consider Condition 5. Suppose that $l_{\bar{w}} \neq b$ and $a_{\bar{w}-1} \in D_{l_{\bar{w}}}$. For simplicity, we denote

- $(|M_{l_{\bar{w}}} \setminus D_{l_{\bar{w}}}|, |D_{l_{\bar{w}}} \setminus M_{l_{\bar{w}}}|) = (r, s)$,
- $(|M''_{l_{\bar{w}}} \setminus D_{l_{\bar{w}}}|, |D_{l_{\bar{w}}} \setminus M''_{l_{\bar{w}}}|) = (r'', s'')$, and
- $(|M'_{l_{\bar{w}}} \setminus D'_{l_{\bar{w}}}|, |D'_{l_{\bar{w}}} \setminus M'_{l_{\bar{w}}}|) = (r', s')$.

Condition 5 is clearly satisfied if $(r, s) = (r'', s'')$. Suppose that $(r, s) \neq (r'', s'')$. Then $l_{\bar{w}} \neq j_t$. By the construction of the chain C from M'' to M', we have $s' > s''$ and $r' \leq r''$. Then by Observation 2, $r' > r$, $s' > s$ and $(r'', s'') = (r+1, s+1)$. Since $r' > r$ and $r' \leq r'' = r+1$, we have $r' = r+1$. By Assumption 1′ and the fact that $r' > r$ and $s' > s$, $(r'-1, s'-1) = (r, s'-1) \in \mathcal{S}_{l_{\bar{w}}}(D_{l_{\bar{w}}})$. Since $s'-1 \geq s'' > s$ and $(r, s) \in \mathcal{S}_{l_{\bar{w}}}(D_{l_{\bar{w}}})$, by Assumption 1′ again, we have $(r, s+1) \in \mathcal{S}_{l_{\bar{w}}}(D_{l_{\bar{w}}})$. Finally, $(r', s'-1) \in \mathcal{S}_{l_{\bar{w}}}(D'_{l_{\bar{w}}})$ since C is a chain from M'' to M'. The case that $a_{\bar{w}-1} \notin D_{l_{\bar{w}}}$ as well as Condition 6 can be shown similarly. ■

Lemma 4. *Every sequential targeting rule is donor monotonic.*

Proof of Lemma 4. We use Lemma 3 to prove Lemma 4. Let F be a sequential targeting rule with respect to some $\{N_k\}_{k=1}^{\bar{k}}$ and τ. Assume to the contrary, F is not donor monotonic. Then there exist $D, D' \in \mathcal{D}$ and $i \in I$ such that $D'_i \subseteq D_i$, $|D_i \setminus D'_i| = 1$, $D'_j = D_j$ for every $j \in I \setminus \{i\}$ and $|F_i(\hat{D}') \setminus D'_i| > |F_i(\hat{D}) \setminus D_i|$. By Lemma 2, there is a cycle or a chain C from $F(\hat{D})$ to $F(\hat{D}')$. We want to first show that $F(\hat{D})$ and $F(\hat{D}) + C$ are welfare equivalent. Suppose that this is not true. Let $\{\mathcal{M}_k\}_{k=1}^{\bar{k}}$ and $\{\mathcal{M}'_k\}_{k=1}^{\bar{k}}$ be the sequences of sets of matchings constructed in the sequential targeting procedure for $\hat{D}$ and $\hat{D}'$ respectively. Then at some step k of the sequential targeting procedure for $\hat{D}$, $F(\hat{D}) + C$ is not selected, i.e., $F(\hat{D}) + C \in \mathcal{M}_{k-1}$ but $F(\hat{D}) + C \notin \mathcal{M}_k$. It follows that for any $k' < k$,

$$\sum_{j \in N_{k'}} |F_j(\hat{D}) \setminus D_j| = \sum_{j \in N_{k'}} |(F(\hat{D}) + C)_j \setminus D_j| \text{ when } \tau(k') = \max, \text{ and}$$

$$\sum_{j \in N_{k'}} |D_j \setminus F_j(\hat{D})| = \sum_{j \in N_{k'}} |D_j \setminus (F(\hat{D}) + C)_j| \text{ when } \tau(k') = \min.$$

Then by Observation 1, we have

$$\sum_{j \in N_{k'}} |F_j(\hat{D}') \setminus D'_j| = \sum_{j \in N_{k'}} |(F(\hat{D}') - C)_j \setminus D'_j| \text{ when } \tau(k') = \max, \text{ and}$$

$$\sum_{j \in N_{k'}} |D'_j \setminus F_j(\hat{D}')| = \sum_{j \in N_{k'}} |D'_j \setminus (F(\hat{D}') - C)_j| \text{ when } \tau(k') = \min.$$

Therefore, $F(\hat{D}') - C \in \mathcal{M}'_{k-1}$. Since $F(\hat{D}) + C \notin \mathcal{M}_k$,

$$\sum_{j \in N_k} \left| F_j(\hat{D}) \setminus D_j \right| > \sum_{j \in N_k} \left| (F(\hat{D}) + C)_j \setminus D_j \right| \text{ when } \tau(k) = \max, \text{ and}$$

$$\sum_{j \in N_k} \left| D_j \setminus F_j(\hat{D}) \right| < \sum_{j \in N_k} \left| D_j \setminus (F(\hat{D}) + C)_j \right| \text{ when } \tau(k) = \min.$$

Then by Observation 1, we have

$$\sum_{j \in N_k} \left| F_j(\hat{D}') \setminus D'_j \right| < \sum_{j \in N_k} \left| (F(\hat{D}') - C)_j \setminus D'_j \right| \text{ when } \tau(k) = \max, \text{ and}$$

$$\sum_{j \in N_k} \left| D'_j \setminus F_j(\hat{D}') \right| > \sum_{j \in N_k} \left| D'_j \setminus (F(\hat{D}') - C)_j \right| \text{ when } \tau(k) = \min.$$

This implies that $F(\hat{D}') \notin \mathcal{M}'_k$, contradiction. Hence, $F(\hat{D})$ and $F(\hat{D}) + C$ are welfare equivalent. Then by Lemma 2 again, there is a cycle or a chain C' from $F(\hat{D}) + C$ to $F(\hat{D}')$. By the same arguments before, it can be shown that $(F(\hat{D}) + C) + C'$ and $F(\hat{D}) + C$ are welfare equivalent. Then $(F(\hat{D}) + C) + C'$ and $F(\hat{D})$ are welfare equivalent. We can continue this process and eventually we have $F(\hat{D})$ and $F(\hat{D}')$ are welfare equivalent, leading to a contradiction to that patient i can conceal a donor and benefit under $\hat{D}'$. ■

A.3 Proof of Proposition 2

Let f and f' be the two sequential targeting mechanisms specified in the statement of the proposition. Consider any problem $D \in \mathcal{D}$. Let $\{\mathcal{A}_k\}_{k=0}^{\bar{k}}$ and $\{\mathcal{A}'_k\}_{k=0}^{\bar{k}'}$ be the sequences of allocations constructed in the sequential targeting procedures under f and f' respectively.

If $N = N_\ell$, the conclusion of the proposition holds, since otherwise $f'(D) \in \mathcal{A}'_{\ell-1} = \mathcal{A}_{\ell-1}$ (equality follows as the first $\ell - 1$ steps are the same under both mechanisms) implies $f(D) \notin \mathcal{A}_\ell$.

Suppose that $N \neq N_\ell$, and $\tau(\ell) = \max$. Assume to the contrary,

$$\sum_{i \in N} f(D)(i) < \sum_{i \in N} f'(D)(i).$$

Since $N'_\ell = N_\ell \setminus N$, $\tau'(\ell) = \max$ and $f(D) \in \mathcal{A}_{\ell-1} = \mathcal{A}'_{\ell-1}$,

$$\sum_{i \in N_\ell \setminus N} f(D)(i) \leq \sum_{i \in N_\ell \setminus N} f'(D)(i).$$

The above two inequalities imply

$$\sum_{i \in N_\ell} f(D)(i) < \sum_{i \in N_\ell} f'(D)(i).$$

Given that $f'(D) \in \mathcal{A}'_{\ell-1} = \mathcal{A}_{\ell-1}$, this implies $f(D) \notin \mathcal{A}_\ell$, contradiction. The case where $\tau(\ell) = \min$ can be shown similarly.

A.4 Proof of Proposition 3

The proof of this comparative statics result uses the same techniques as those in the proof of Theorem 2. We explain how to modify the previous arguments to prove it. First, we present the following restriction regarding different feasible schedule functions, which is a counterpart of Assumption 2′.

Assumption 2″. For every $i \in I$ and $D_i \in \mathcal{D}_i$, if $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i, then for any $(r, s) \in \mathcal{S}_i(D_i)$ and any $(r', s') \in \mathcal{S}'_i(D_i)$, we have

1. If $r' > r, s' > 0$ and $s < |D_i|$, then

$$(r+1, s+1) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1, s'-1) \in \mathcal{S}'_i(D_i).$$

2. If $r' > r$ and $s' \leq s$, then

$$(r+1, s) \in \mathcal{S}_i(D_i) \quad \text{and} \quad (r'-1, s') \in \mathcal{S}'_i(D_i).$$

The same arguments in the proof of Lemma 1 can be used to show that when Assumptions 1 and 2 are satisfied for all feasible schedule functions, Assumptions 1′ and 2″ are satisfied.

Second, we use the same construction of extended problems as before. Moreover, since Lemma 2 holds for arbitrary feasible schedule functions, we know that for every sequential targeting mechanism f, there exists a sequential targeting rule F such that for any profile of feasible schedule functions $\mathcal{S}$ and $D \in \mathcal{D}$, $f(D \,|\, \mathcal{S})$ and $F(\hat{D} \,|\, \mathcal{S})$ are welfare equivalent. Therefore, it is sufficient to prove the comparative statics result for the sequential targeting rules. That is, we want to show the following result:

Lemma 5. *Consider a patient $i \in I$. Let $\mathcal{S}$ and $\mathcal{S}'$ be two profiles of feasible schedule functions such that $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i for every $D_i \in \mathcal{D}_i$. Then for any sequential targeting rule F and any problem $D \in \mathcal{D}$,*

$$\left| F_i\big(\hat{D} \,|\, \mathcal{S}\big) \setminus D_i \right| \geq \left| F_i\big(\hat{D} \,|\, (\mathcal{S}'_i, \mathcal{S}_{-i})\big) \setminus D_i \right|.$$

To prove this lemma, we need the cycle and chain operations as before. Recall that cycles and chains are defined respect to two different matchings corresponding to two different problems. We modify their definitions slightly such that they are defined with respect to two different matchings corresponding to the same problem but different feasible schedule functions. Below we give the modified definition of a cycle.[42]

Given a matching M for $\hat{D}$ and $\mathcal{S}$, and a matching M' for $\hat{D}$ and $\mathcal{S}'$, a cycle from M to M' is a directed graph of patients and donors in which each patient/donor points to the next donor/patient and is denoted as a list $C = (i_1, d_1, ..., i_{\bar{t}}, d_{\bar{t}})$, $\bar{t} \geq 2$, such that for each $t \in \{1, ..., \bar{t}\}$ (let $i_{\bar{t}+1} = i_1$ and $d_0 = d_{\bar{t}}$):

[42]The only changes we make are replacing $\hat{D}'$ with $\hat{D}$ and replacing $\mathcal{S}_i(D'_i)$ with $\mathcal{S}'_i(D_i)$ everywhere these two appear in the definition.

1. $i_t \in \hat{I}$, $d_t \in M'_{i_t} \setminus M_{i_t}$ and $d_t \in M_{i_{t+1}}$.

2. If $i_t \neq b$, $d_{t-1} \in D_{i_t}$, and $d_t \notin D_{i_t}$, then

$$\left(\left|M_{i_t} \setminus D_{i_t}\right| + 1, \left|D_{i_t} \setminus M_{i_t}\right| + 1\right) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad \left(\left|M'_{i_t} \setminus D_{i_t}\right| - 1, \left|D_{i_t} \setminus M'_{i_t}\right| - 1\right) \in \mathcal{S}'_{i_t}(D_{i_t}).$$

3. If $i_t \neq b$, $d_{t-1} \notin D_{i_t}$, and $d_t \in D_{i_t}$, then

$$\left(\left|M_{i_t} \setminus D_{i_t}\right| - 1, \left|D_{i_t} \setminus M_{i_t}\right| - 1\right) \in \mathcal{S}_{i_t}(D_{i_t}) \quad \text{and} \quad \left(\left|M'_{i_t} \setminus D_{i_t}\right| + 1, \left|D_{i_t} \setminus M'_{i_t}\right| + 1\right) \in \mathcal{S}'_{i_t}(D_{i_t}).$$

4. If $i_t = i_{t'} = i$ for some $t' \neq t$, then either

   - $d_t, d_{t-1} \in D_i$ and $d_{t'}, d_{t'-1} \notin D_i$, or
   - $d_t, d_{t-1} \notin D_i$ and $d_{t'}, d_{t'-1} \in D_i$.

The definition of a chain is modified in the same way. Then the following result is a counterpart of Lemma 3.[43]

Lemma 6. *Let $\mathcal{S}$ and $\mathcal{S}'$ be two profiles of feasible schedule functions. Consider any $D \in \mathcal{D}$ and $i \in I$. Suppose that $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ at D_i. If M is a matching for $\hat{D}$ and $\mathcal{S}$, M' is a matching for $\hat{D}$ and $(\mathcal{S}'_i, \mathcal{S}_{-i})$, and $\left|M'_i \setminus D_i\right| > \left|M_i \setminus D_i\right|$, then there is a cycle or a chain from M to M'.*

Using Assumptions 1′ and 2″, Lemma 6 can be proved in the same way as Lemma 3. Since there is no eliminated donor, Case 4.5 in the proof of Lemma 3 cannot happen.

Finally, by arguments similar to those in the proof of Lemma 4, Lemma 5 can be proved using Lemma 6. Specifically, we prove by contradiction. Suppose that for some sequential targeting rule F, $i \in I$, $D \in \mathcal{D}$, and two profiles of feasible schedule functions $\mathcal{S}$ and $\mathcal{S}'$, $\mathcal{S}_i(D_i)$ is weakly more favorable than $\mathcal{S}'_i(D_i)$ and

$$\left|F_i\left(\hat{D} \,|\, \mathcal{S}\right) \setminus D_i\right| < \left|F_i\left(\hat{D} \,|\, (\mathcal{S}'_i, \mathcal{S}_{-i})\right) \setminus D_i\right|.$$

Then by Lemma 6, there is a cycle or a chain C from $F\left(\hat{D} \,|\, \mathcal{S}\right)$ to $F\left(\hat{D} \,|\, (\mathcal{S}'_i, \mathcal{S}_{-i})\right)$. It can be shown that $F\left(\hat{D} \,|\, \mathcal{S}\right) + C$ is welfare equivalent to $F\left(\hat{D} \,|\, \mathcal{S}\right)$. Then there is a cycle or a chain from $F\left(\hat{D} \,|\, \mathcal{S}\right) + C$ to $F\left(\hat{D} \,|\, (\mathcal{S}'_i, \mathcal{S}_{-i})\right)$. We can continue this process and eventually we have $F\left(\hat{D} \,|\, \mathcal{S}\right)$ and $F\left(\hat{D} \,|\, (\mathcal{S}'_i, \mathcal{S}_{-i})\right)$ are welfare equivalent, contradiction.

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[43] Similarly, the only changes we make are replacing $\hat{D}'$ with $\hat{D}$ and replacing $\mathcal{S}_i(D'_i)$ with $\mathcal{S}'_i(D_i)$ everywhere these two appear in the definition.

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Online Appendices

Appendix B A Polynomial-time Method to Compute the Outcome of a Sequential Targeting Mechanism

Let $I = \{1, 2, \ldots, |I|\}$ be the set of patients. Consider a sequential targeting mechanism f induced by target sets $\{N_k\}_{k=1}^{\bar{k}}$ and target function τ. Consider a problem $D \in \mathcal{D}$. For each k, we define a function $W_k : \mathcal{A} \to \mathbb{Z}_+$: For every $\alpha \in \mathcal{A}$,

$$W_k(\alpha) = \begin{cases} \sum_{i \in N_k, X \in \mathcal{B}} \alpha_X(i) & \text{if } \tau(k) = \max \\ -\sum_{d \in \cup_{i \in N_k} D_i} \alpha(d) & \text{if } \tau(k) = \min \end{cases}.$$

Let $h \in \mathbb{Z}_{++}$. We consider the following function $W : \mathcal{A} \to \mathbb{R}$: For every $\alpha \in \mathcal{A}$,

$$W(\alpha) = \sum_{k=1}^{\bar{k}} h^{\bar{k}-k} W_k(\alpha) = \alpha^T \cdot \left(\sum_{k=1}^{\bar{k}} h^{\bar{k}-k} w_k \right) \tag{1}$$

where we write $W_k(\alpha) = \alpha^T \cdot w_k$ with the appropriately defined vector $w_k \in \mathbb{Z}^a$ for each k where the "dimension" of a feasible allocation as

$$a = |I| \cdot |\mathcal{B}| + |\cup_{i \in I} D_i|.$$

Observe that by choosing h sufficiently high, the outcome of the sequential targeting mechanism can be written as the allocation that maximizes the linear function defined in Eq.(1).

$$f(D) = \arg\max_{\alpha \in \mathcal{A}} W(\alpha) \tag{2}$$

Next we show that the constraint "α is an allocation," i.e., $\alpha \in \mathcal{A}$, is equivalent to $\alpha \in \mathbb{Z}_+^a$ and α satisfies a system of linear inequalities when Assumption 1 holds.

Suppose Assumption 1 holds for feasible schedule set $\mathcal{S}_i(D_i)$ for every patient $i \in I$.

Then $(r_i, s_i) \in \mathcal{S}_i(D_i)$ if and only if there exist some integer coefficient vector $b_i \in \mathbb{Z}^6$ such that

$$\begin{aligned} r_i + s_i &\leq b_{i,1} \\ -r_i - s_i &\leq b_{i,2} \\ r_i &\leq b_{i,3} \\ -r_i &\leq b_{i,4} \\ s_i &\leq b_{i,5} \\ -s_i &\leq b_{i,6} \end{aligned}$$

We rewrite these linear inequalities in matrix form using

$$r_i = \sum_{X \in \mathcal{B}} \alpha_X(i) \qquad \text{and} \qquad s_i = \sum_{d \in D_i} \alpha(d)$$

and defining

$$\alpha_i = \Big(\big(\alpha_X(i)\big)_{X \in \mathcal{B}}, \big(\alpha(d)\big)_{d \in D_i} \Big) \qquad \text{and} \qquad A_i = \begin{pmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad \forall\, i \in I,$$

and

$$\alpha = (\alpha_i)_{i \in I}, \qquad A_I = \begin{pmatrix} A_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & A_{|I|} \end{pmatrix}, \qquad b_I = (b_i)_{i \in I}$$

as follows:

$$\alpha \cdot A_I \leq b_I \tag{3}$$

We rewrite the market clearing condition of an allocation,

$$\alpha = \sum_{i \in I : X \in \mathcal{C}(\beta_i)} \alpha_X(i) \leq v_X + \sum_{d \in \cup_{i \in I} D_i : \beta_d = X} \alpha(d) \qquad \forall\, X \in \mathcal{B},$$

in matrix inequality form as

$$\alpha \cdot A_{\mathcal{B}} \leq v \tag{4}$$

where

$$A_{\mathcal{B}} = (A_X^T)_{X \in \mathcal{B}}$$

defined by $\forall\, X \in \mathcal{B}$,

$$A_X = \left(\Big(\big(A_X(i,Y)\big)_{Y \in \mathcal{B}}, \big(A_X(d)\big)_{d \in D_i} \Big)_{i \in I} \right)$$

such that

$$A_X(i,Y) = \begin{cases} 1 & \text{if } Y = X \text{ and } X \in \mathcal{C}(\beta_i) \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, i \in I,\ \ \forall\, Y \in \mathcal{B}$$

and

$$A_X(d) = \begin{cases} 1 & \text{if } X = \beta_d \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, d \in \cup_{i\in I} D_i.$$

Finally, we have

$$\alpha(d) \leq 1 \qquad \forall\, d \in \cup_{i\in I} D_i, \text{ and}$$

We rewrite this as

$$\alpha \cdot A_D \leq b_D = (1, \ldots, 1) \tag{5}$$

where $A_D = \big(A_D(r,c)\big)_{r\leq a,\, c\leq \cup_{i\in I}|D_i|}$ such that $A_D(r,c) = 1$ if both row r and column c refer to the same donor d, and $A_D(r,c) = 0$ otherwise.

Then for every $\alpha \in \mathbb{Z}_+^a$, vector α is an allocation (i.e., $\alpha \in \mathcal{A}$) if and only if Eq. (3), (4), and (5) hold.

This, Eq. (1) and (2) imply that the following integer linear program in cannonical form finds the outcome of f:

$$f(D) = \arg\max_{\alpha\in\mathbb{Z}_+^a} \alpha^T \cdot \left(\sum_{k=1}^{\bar{k}} h^{\bar{k}-k} w_k\right) \tag{6}$$

subject to

$$\alpha \cdot A \leq b \tag{7}$$

where

$$A = (A_I, A_{\mathcal{B}}, A_D) \qquad \text{and} \qquad b = (b_I, v, b_D)$$

such that α is a $1 \times a$ non-negative integer vector, A is $a \times (6|I| + |\mathcal{B}| + |\cup_{i\in I} D_i|)$ integer matrix of entries 0, 1 and -1, and b is a $1 \times (6|I| + |\mathcal{B}| + |\cup_{i\in I} D_i|)$ integer vector.

We consider its linear program relaxation replacing the search space to $\alpha \in \mathbb{R}_+^a$ instead of $\mathbb{Z}_+^a$.

A matrix is **totally unimodular** if all of its square submatrix determinants are either $-1, 0, 1$. The following result is well known and straightforward to prove using Cramer's rule in linear algebra (for example see Schrijver (1998)):

Lemma 7. *Vertices of the polyhendron defined by Eq. (7) are integer for any integer vector b if and only if A is a totally unimodular matrix.*

Thus, for any linearly independent basis for α the linear program relaxation of the problem in (6) has only integer solutions for any integer vector b if and only if A is totally unimodular.

The following lemma establishes a condition for checking total unimodularity of A:

Lemma 8 (Ghouila-Houri, 1962)**.** *A is totally unimodular if and only if there exists a partition of the columns of A as K and L such that for the column vector $\kappa = \sum_{c\in K} A^c - \sum_{c\in L} A^c$ where A^c is the c'th column vector of A, we have $\kappa_r \in \{-1, 0, 1\}$ for every row $r = 1, \ldots, a$.*

We prove that A is indeed totally unimodular using this result:

Lemma 9. *Matrix A is totally unimodular.*

Proof of Lemma 9. We show that columns of A has such a partition K, L as in Lemma 8 as follows:

- For every column c of A with $c \leq 6|I|$, i.e., c is a column of A_I, assign c to K. Now since L has no columns
$$\sum_{c \leq 6|I| \,:\, c \in K} A^c - \sum_{c \leq 6|I| \,:\, c \in L} A^c = \mathbf{0}.$$
- For every column c of A with $6|I| < c \leq 6|I| + |\mathcal{B}|$, i.e., c is a column of $A_{\mathcal{B}}$, assign c to K. Now since L has no columns for
$$\kappa' = \sum_{c \leq 6|I|+|\mathcal{B}| \,:\, c \in K} A^c - \sum_{c \leq 6|I|+|\mathcal{B}| \,:\, c \in L} A^c.$$
For each row $r \leq a$, $\kappa'_r = 1$ if r refers to a patient i and blood type X such that $X \in \mathcal{C}(\beta_i)$ and $\kappa'_r = 0$ if r refers to a patient i and blood type X such that $X \notin \mathcal{C}(\beta_i)$. On the other hand, if r refers to a donor d then $\kappa^1_r = -1$.
- For every column c of A with $c > 6|I| + |\mathcal{B}|$, i.e., c is a column of A_D, assign c to K. Since
$$\kappa = \sum_{c \in K} A^c - \sum_{c \in L} A^c,$$
For each row $r \leq a$, $\kappa_r = 1$ if r refers to a patient i and blood type X such that $X \in \mathcal{C}(\beta_i)$, and $\kappa_r = 0$ otherwise,

Thus for $K = \{1, 2, \ldots, 6|I| + |\mathcal{B}| + |\cup_{i \in I} D_i|\}$, i.e., the set of all columns of A, and $L = \emptyset$, by Lemma 8, A is totally unimodular. ∎

These results are used to prove the following proposition

Proposition 4. *Under Assumption 1, the outcome of a sequential targeting mechanism can be found in polynomial time.*

Proof of Proposition 4. By Lemmata 7 and 9, under Assumption 1, the linear program relaxation of the integer linear program in Eq. (6) with constraint Eq. (7) has all its basic solutions integer. Thus, any polynomial LP method, such as the simplex algorithm, finds the outcome of a sequential targeting mechanism in polynomial time. ∎