Optimal Campaign Finance

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Abstract

We analyze a general model of campaign finance in which a planner knows each type of candidate's ability to raise money and their campaigning ability. The planner assigns electoral 'scores', which are then transmitted to an electorate that picks a winner stochastically. These scores characterize all possible campaign-finance regimes. For instance, a system of pure public funding allocates scores that are independent of candidates' fundraising ability; fundraising capped at $1 million dollars would involve assigning scores such that all candidates with the potential to fundraise over $1 million receive the same score, and all those with fundraising potential below $1 million dollars received gradated scores. We characterize the first best in a manner that is not dependent on preferences or the solution concept, and can compare different campaign-finance regimes to each other and the first best.

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1 Introduction

"I would like to have an ample fund to spread the light of Republicanism, but I am willing to undergo the disadvantage to make certain that in the future we shall reduce the power of money in politics for unworthy purposes."—William Howard Taft [(1)]

There is a long tradition in the social sciences–dating at least to the work of 18th Century mathematicians the Marquis de Condorcet and Jean-Charles de Borda–of analyzing how the design of electoral systems affect election outcomes. A more recent body of work focuses on how money in politics affects such outcomes, and how campaign-finance regimes can be structured to maximize social welfare.

There are many possible campaign-finance regimes, making it challenging to make global comparisons of different regimes.

To date, attempts to model the welfare impact of various campaign finance regimes almost always proceed in a "bottom-up" manner. That is, micro-founded preferences for candidates, voters and perhaps special interest groups (SIGs) are specified, then equilibria are characterized for a particular solution concept. In order to make statements as to the efficiency of a given campaign-finance reform, the model is solved again for some alternative campaign finance regime, usually a corner case most popular in current public discourse such as purely publicly funded elections. The welfare outcomes of these two (or sometimes three) models are compared, and implications for campaign finance reform are drawn.

In these models there is no formal sense of "the space" of all possible campaign finance regimes / elections. There are particular cases, often corner cases, and occasionally a spectrum of regimes[(2)] is analyzed. But there is no sense in which qualitatively different CF regimes can be embedded in the same space. These comparisons are only ever between a handful of regimes, and as such conclusions can only ever be of the form "campaign finance regime X is better than Y", and one can never speak to the optimality of a regime in a global sense.

This paper responds to these challenges in the following way: We introduce a parsimonious way to model a much larger space of potential campaign finance regimes. We use a variant of a score auction which was first introduced in Che (1993) to model auctions such as government procurement auctions where bids consist of a price and a measure of quality of the services to be provided.

In our context, a planner knows each candidate-type's potential fundraising capacity, their potential campaigning ability, and so on. The planner then assigns electoral 'scores', to be thought of as competitiveness in general, which are then put to an 'electorate' who picks a winner stochastically.

The first virtue of this approach is that it nests a large space of potential campaign finance regimes. For example, if the planner wanted a cap of fundraising at 1 million dollars, they would assign scores such that everyone with the potential to fundraise over 1 million dollars nevertheless received the same score, and all those with fundraising potential below one million dollars received

[(1)] Pruitt (2015)

[(2)] e.g. A spectrum of the form: Institute a cap of γ for private fundraising, and then give public matching funds in proportion αF to the privately raised funds F, as found in Coate (2004).

gradated scores. If the planner wanted to implement a system of pure public grants, they would allocate scores that would not depend on fundraising propensity at all. If the planner thought that all fundraising under 1 billion dollars was equally (un)informative, and only use of fundraising was to identify super-star candidates, they would allocate scores accordingly. The space of all possible score allocations, given fundraising and campaigning abilities, nests basically all conceivable campaign finance regimes. This solves the problem that has plagued much of the literature up to this point: each paper only compares a handful or narrow sliver of different possible regimes. Here we can compare across all regimes.

Secondly, by assigning electoral outcomes (at least stochastically), we are able to offer a statement about the first-best or optimal outcome that is not restricted by whatever preference or equilibrium concept is chosen. Given this first-best, one can look at micro-founded papers in a new light: As more or less plausible mechanisms for achieving an electoral outcome. The task of campaign finance design is then to induce behavior in the players that brings about this first-best regime.

Given the novelty of our approach, in this paper we focus on valence only elections with competition on, for example, policy dimensions.[(3)] This is first so that the approach can be demonstrated parsimoniously, but more importantly, that in many electoral settings this is a reasonable assumption, especially from the planner's (e.g. constitutional designer's) point of view.

For example, the Democratic Party's presidential primary might have as it's overarching goal to select a candidate that can beat the Republican opponent. A constitutional designer might want judges, if they are to be elected, to be of the highest quality in their ability to administer justice, explicitly ignoring partisan constraints. Local non-partisan offices such as police chief or fire chief should ideally select only for ability.

In section 2, we first explain the set-up of our modeling approach. We then interpret two well-known models, Prat (2002a) and Coate (2004), in terms of our model and make clear how their models and campaign finance comparisons can be thought of as special cases of this approach where more structure is imposed on the problem.

In section 3 we study the structure of the optimal campaign finance regime, making sharp predictions as to when the "boundary " regimes (pure public and pure private/oligarchic) are optimal. Broadly speaking, a purely public regime that does not allow for private fundraising is optimal if and only if the potential fundraising abilities contain no valence signal, positive or negative.

Given this core result, in section 4 we study how the optimal regime changes as some of (sparse) parameters of the model change. For example, how does the optimal election differ between the planner having more cardinal, as opposed to more ordinal, preferences between candidates. And, as the level of uncertainty that the electorate introduces is changed, how does this alter the planner's incentives and subsequent choice of the optimal regime.

Finally, in section 5, we demonstrate that our core results are robust to three realistic extensions:

[(3)] Although simple measures of policy outcomes, such as a one dimensional distance of candidate's policy from that of the median voter's ideal can be incorporated directly, as will be clear from the setup.

Elections with many (≥ 2) candidates; elections with incumbents who may have an incumbency advantage; and electorates who are reactive (to approximate equilibrium dynamics in a reduced form sense) to the structure of an election when distinguishing between two candidates.

Of particular interests are results in subsection 5.1 that study the welfare implications of an incumbency benefit. A more complete explanation is contained within that section, but by way of summary we find that incumbency effects are downstream from the design of optimal regimes in the following selection sense: If the election and campaign finance regime that is implemented is optimal, then those who become incumbents will be better on average, from the planner's point of view, than those who are challengers. Given this, even an incumbency advantage that uniformly boosts all incumbents, good and bad, will turn out to be welfare improving since it flows to good candidates more often.

1.1 Existing Literature

The issue of campaign finance is inextricably linked to the large literature on elections. Our goal here is not to provide an exhaustive overview of that literature but point out the key dimensions of elections fit into our framework and affect the optimal campaign-finance regime.

An early paper in which money is traded solely for access to politicians such that information relevant to political decisions can be provided is Austen-Smith (1998). The novelty of this paper is that politicians are price setters in the market for access. In this market's equilibrium, SIGs with similar preferences to the politician provide the most information but are willing to pay the least for the access, and conversely extreme groups pay more to provide more biased information. In this setting, limiting campaign contributions limits the politician's market power to price discriminate, and hence the difference in contribution she can extract from extremists versus moderates is smaller, thereby increasing the informational quality that is passed to the candidate.

A key paper that focuses on valence-only elections is Meirowitz (2008). In that paper, there is no policy competition, and effort is undertaken by candidates convincing voters to vote for them. Campaign finance reform is interpreted in that setting as changing the marginal costs of candidates campaigning. The voter's payoff is increasing in the effort spent charming them, and as such reform that lowers the amount of resources candidates can deploy typically hurts voters. Incumbency implications are discussed below. Cole et al. (2018) adds uncertainty to the model of Meirowitz (2008), as well as allowing splits the single cost parameter of Meirowitz (2008) into a fundraising and campaigning cost parameters, and reaches largely similar conclusions.

Within the literature that combines both valence and policy dimensions of particular interest, and discussed at greater length in section 2.3, is Coate (2004). In that model, interest groups trade donations for favors and those donations are used to inform voters about the candidates qualifications for office. Voters rationally update their beliefs given the advertisements they see, and consequently they understand that when a candidate has a lot of money for advertising they must have promised many favors. Thus, campaign finance reform can offer welfare improvements by restricting advertisements to make them more persuasive to voters.

Again of particular interest and discussed at length subsequently are the models in Prat (2002a) and Prat (2002b). Each paper offers a different model of the organization of the lobbying market but in each case SIG money is able to reveal valence information about the candidate. In both cases, a ban on lobbying / contributions can improve the welfare of the median voter specifically when the policy distortion that comes from SIG money outweighs the valence signaling benefit.

More recently, Boleslavsky and Cotton (2015) study elections where candidates have both a policy choice and valence level. But crucially, these are assumed to be orthogonal. As such, more informative campaigns allows for more extreme, yet high quality, candidates to be elected. Consequently, limiting campaign contributions limits the informativeness of campaigns and and encourages policy moderation, and such reform is welfare improving if the latter effect dominates the former.

In another vein, Dahm and Porteiro (2008) models SIGs as providing policy relevant information in exchange for favors, and points out that the natural side effect of limiting campaign contributions is a decrease in policy relevant information and therefore potentially worse policy outcomes.

Daley and Snowberg (2009) analyzes a model in which candidates need to choose effort levels toward fundraising and policymaking, both of which are costly. If there is a positive correlation between fundraising ability and policymaking ability, and if voters cannot punish candidates for past policy (which is assumed by looking for Markovian strategies), then candidates use fundraising to signal ability at a cost of worse policy outcomes. Thus, banning fundraising increases voter welfare.

There have been numerous studies of the effects of campaign finance regimes or reform in policy-only elections without a valence characteristic. Austen-Smith (1987) assumes strategically unsophisticated SIGs, and finds that campaign contributions may distort policy outcomes, although only in one direction. Grossman and Helpman (1994) and then Grossman and Helpman (1996) assume that SIGs trade contributions for policy commitments. In equilibrium, policy outcomes are distorted to favor SIGs, and since SIGs only get any use from their donations if their favored candidate is elected, the contributions and thus the policy distortion are tilted in favor of the candidate seen as more competitive. Congleton (1989) notes that whether or not there is a distortionary effect on policy depends wholly on how the distribution of donor preferences compare to the distribution of preferences among the voters as a whole. Naturally, if donors have markedly different preferences from the median voter, then policy positions that candidates will choose to adopt will diverge from the bliss point of the median voter.

Ashworth (2006) studies a model where SIGs provide campaign funding in exchange for favors, without any effect on policy, and shows that the welfare effects of public financing turn crucially on the strength of the incumbency effect: If the incumbency advantage is very large or very small then public financing offers an improvement, but typically it leads to bad candidates having incentives to mount socially useless campaigns.

Finally, the perceived incumbency advantage has received quite substantial theoretical attention. In Grossman and Helpman (1996) the SIGs prefers to give money to the competitive candidate,

and in their model the incumbency advantage in terms of competitiveness translates into receiving greater contributions from SIGs, and in exchange a greater distortion of policy than if the race were more evenly balanced.

In Ashworth (2006) the incumbent is assumed to be, ex ante, more likely to be moderate than their challenger, owing to "selection effects in repeated elections" and voter's preferences for moderates. Ashworth and de Mesquita (2008) fixes symmetrical policy positions of the candidates and study the effect of entry costs and candidate quality on the incumbent advantage. The conclusions there are similar in spirit to our study of incumbency: The incumbency advantage exists but it is driven by the fact that high quality candidates win elections more often, and thus the pool of incumbents is of higher quality than the general pool of candidates, and that these high quality incumbents deter challengers. A similar, purely selection based effect arises in Daley and Snowberg (2009).

Caselli et al. (2014) allows incumbents to signal their type at a cost that increases if they are truly of lower type. In equilibrium the distribution of signals sent by incumbents is higher than for the candidates at large, and since voters value higher types this generates an endogenous incumbency advantage. Meirowitz (2008) shows that when campaigns are just contests to raise and spend money, campaign finance reform that affects both incumbents and challengers equally typically ends up benefiting the incumbent. This is because if incumbents have an electoral advantage then they will typically campaign less than disadvantaged challengers. Thus limiting the sources of campaign funds typically affects the challenger more than the incumbent.

In a different vein, Prato and Wolton (2018) introduces a behavioral component to elections and concludes that a major source of the incumbency advantage is that voter's have a lower cost of paying attention to them.

This paper offers justification for the *optimality* in a global sense, not just in a finite comparison of a small number of electoral systems, of an incumbency advantage in valence only elections. While most other studies offer incumbency advantages due to selection effects as an outcome of the model, we ask and answer the following question. If incumbency advantages could simply be chosen, would a planner wish for them to exist? The answer is yes, so long as the elections are designed optimally, in which case the welfare improvements are due to a selection effect.

But it is important to note that in most other models the selection effect is endogenous, and at best that outcome can be compared with another model that has many changes, of which a different incumbency effect is perhaps just one. Here we can vary exogenously just the incumbency advantage, and show that this exogenous advantage is welfare improving given optimality of the regime design in general.

2 Model

There is a P(lanner) designing a campaign finance regime and election structure. We think of these as intertwined, and may refer to either the "regime" or the "CFR" or the "election" throughout. All

of these refer to the same choice that the planner is to make. We may think of P as, for example, a constitution writer designing elections for an office, or perhaps the Republic or Democratic National Committee setting the rules for their presidential primaries.

There is a finite universe of potential candidate types that might ever plausibly compete in this election. We denote this entire set by $\mathcal{N}$, which has size $|\mathcal{N}| = N < \infty$, and a typical candidate from this set is denoted by $n_i \in \mathcal{N}$. Each candidate type is completely described by two characteristics. Firstly, candidate n_i has a level of fundraising capacity denoted by f_i which represents how much money they might be able to maximally raise if the planner permits fundraising. Secondly, candidate n_i has a level of campaigning skill or alternatively, personal charisma) c_i that represents everything else, apart from money, that contributes to a politician winning an election.

The planner wants a certain type of candidate to win the election. We denote the true type, that the planner ultimately cares about, of candidate n_i, by t_i. Formally, we define the **type evaluation function**

$$TE : \mathcal{N} \to \mathbb{R},$$

that, for the planner, maps observable characteristics to 'true types', and we use t_i as shorthand for the output

$$t_i = TE(f_i, c_i).$$

This may be interpreted in various ways but is generically a valence characteristic. For example, when the RNC designs their presidential primary, they might first and foremost care that the candidate that emerges victorious has the best possible chance to win a general election against the eventual Democratic challenger. Or, alternatively, in a very local setting, if counties are to elect sheriffs, judges or fire department chiefs, then naturally the designer of those contests wants an election structure that elects the candidate most capable of administering the law or fighting fires, as the case may be.

2.1 The election and campaign finance

Ideally the planner would like to assign probability 1 to the candidate with the highest expected true type, but this first-best outcome is not achievable given underlying randomness in the election.

We model this degree of control as follows. The planner determines the electoral competitiveness score, s_i of any potential candidate. We assume the planner has a total number of competitiveness 'points', $\overline{s}$ that they have to allocate among all potential candidate types. The cap is completely arbitrary, but represents the obvious fact that electoral competitiveness is relative and, between players, a substitute good.

The planner makes this allocation, and then given two actual candidates with electoral competitiveness scores s_1 and s_2, we model the underlying randomness of elections by assuming that

player 1 wins with probability prescribed by the function

$$p_1 = W(s_1, s_2), \quad p_2 = 1 - W(s_1, s_2) = W_2(s_1, s_2).$$

In our core model, we abstract from the participation decision and incumbency effects. These will be explored in section 5. Thus here we simply assume the process is as follows: the planner makes the election / campaign finance choice, two players are drawn without replacement from the set of all possible candidate types $\mathcal{N}$, and then these two players mechanically compete in the election according to the given win function. Thus the *probability of each potential candidate pair that might compete in an election is equal*, and thus omitted from the maximization problem. One can model a greater frequency of a certain class of candidates with an arbitrary number of very similar, but not identical candidates.

Given all of the above, we write the planner's optimization problem as follows:

$$\begin{aligned} &\max_{s_n, \forall n \in \mathcal{N}} \sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} \left(g\left(t_n, t_{n'}\right) W_1\left(s_n, s_{n'}\right) + g\left(t_{n'}, t_n\right) W_2\left(s_n, s_{n'}\right) \right) \\ &\text{such that} \\ &s_n \geq 0 \qquad \text{for all } n \in \mathcal{N}, \\ &\sum_{n \in \mathcal{N}} s_n = \overline{s}. \end{aligned} \tag{2.1}$$

Denote, for convenience, the planners payoff as $PP(\boldsymbol{s}) = \sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} (g\left(t_n, t_{n'}\right) W_1\left(s_n, s_{n'}\right)$ $+ g\left(t_{n'}, t_n\right) W_2\left(s_n, s_{n'}\right))$.The function $g : \mathbb{R}^2 \to \mathbb{R}$ flexibly incorporates many different objectives the planner might plausibly have. We require the following:

Assumption 1. *For arbitrary $t, t', t'' \in \mathbb{R}^+$, if $t > t'$ then $g(t, t') > g(t', t)$, $g(t, t'') \geq g(t', t'')$, $g(t'', t) \geq g(t'', t')$. Further if $t = t'$ then $g(t, t') = g(t', t)$ and $g(t, t'') = g(t', t'')$.*

We will explore functional forms of g that are of particular interest in the subsection below.

2.2 Special cases of the model

This model encompasses a number of specific functional forms for the planner's preferences, a number of different actual campaign finance regimes, and a spectrum of representations for the 'voters', or 'electorate'.

The planner's preferences We refer to the function $g(\cdot)$ as the planner's preference function, or just preferences. This is appropriate, given $g(\cdot, \cdot)$ measures the planner's payoff from one candidate-type winning relative to another. The only restriction placed on such preferences is assumption 1, which is very loose.

If $g(a, b) = \mathbb{1}[a \geq b]$ so that $g(a, b) = 1$ iff $a \geq b$ is an indicator function, then we can interpret the planner's preferences as maximizing the probability that the better candidate wins in every possible

election pairing, without regard to the magnitude of difference between candidates. We will often study these preferences because they represent a planner with **ordinal**, but not cardinal, preferences between types. We would expect optimal elections to be such that medium candidate types are quite competitive, relative to preferences that distinguish between candidate types cardinally. The intuition for this is below.

If $g(a, b) = a$, then we can interpret $PP(\boldsymbol{s})$ as maximizing the average quality of the winning candidate. This function is an example of explicitly **cardinal** preferences within an election, in contrast to the ordinal preferences in the preceding paragraph. Such a planner is more sensitive to the magnitudes of difference between candidates. There are two intuitive conjectures we might make here.

First, that relative to the preferences represented by the indicator function above, the planner is more likely to privilege high types over medium types in the optimal election. The reason for this is that, over all possible opponents there is a greater chance a high type is the better type in any given election than a medium type. But for this type of cardinal preference, as opposed to the ordinal indicator function, the high type will be preferred by more than the medium type might be, or disfavored by less for any given opponent. Thus, the incentives that favor giving high types (relative to medium types) are accentuated, and the disincentives mitigated, in the case of cardinal preferences.

Second, when preferences are cardinal, we expect candidates with similar types to be treated close to equally. Or, put another way, we expect the score difference between two types to be continuous and monotonic in their type difference. However, when preferences are effectively an ordering, as in the indicator function case, then candidates with similar types need not receive very similar scores, since what matters is how close candidates are in a rank sense, not the magnitude of difference in true types.

Actual campaign finance regimes In practice there are many varieties of campaign finance regime. Here we summarize some key types that are nested within our model. Broadly,a type of campaign finance regime that is outside our model is a regime in which funding allocated in the current election cycle depends on past election results. Because our model does not provide for linkages between elections in general (although a version of this is analyzed in section 5), such regimes are not possible in our framework.

Leaving that to one side, our model captures a whole spectrum of possible and actual campaign finance regimes. A brief description of these regime types is offered below, as well as how they would be implemented in our model in terms of scores allocated.

First, and at one extreme of the spectrum, are elections financed through a lump sum public grant with a ban on private contributions. We will return to this more formally in section 3, but this is the theoretical implementation of a *pure public* campaign finance-regime in which fundraising capacity plays absolutely no role in determining the outcome of an election. In our model, this type of regime would be one in which scores are allocated based only on c_i, regardless of f_i, for each candidate.

A special case of this is now largely abandoned "Clean Elections" movement in the US. Clean Elections required a candidate to voluntarily forego all donations except the public lump sum, conditional on receiving some number of signatures to demonstrate the basic viability of their candidature. Voluntary participation was necessary for constitutional purposes. A stylized version of this form of Clean Elections in our model, without the issue of voluntary participation, would be a regime in which all scores are allocated based on c_i, regardless of f_i, but that below some threshold in competitiveness $\bar{c}$ all candidates received a score of zero.

Second, there were versions[(4)] of the Clean Elections program in which, to get around the issues relating to voluntary participation, if publicly funded candidates were outspent by privately funded candidates, then the publicly funded ones would receive funds to match the private candidates, thereby nullifying any private fundraising advantage. Although this looks like a purely public system in which the outcome is that the amount of funds raised is independent of fundraising capacity, in fact this system is best interpreted as a regime in which fundraising capacity is *punished.* This is because if two candidates end up with the same amount of money to spend, but one actually had to go and raise it while the other was simply given it, likely the latter is in a more competitive campaigning position. This would be due to the extra time the latter had available that they didn't spend fundraising, and the advantageous perception of not being as corruptible that might come from not taking any private money. One can conceive of gradations of this in which publicly funded candidates are given matching funds to keep the funds gap between them and their privately funded opponents to a threshold, where the scheme described above is the special case when the threshold is zero. In our model this regime would be one in which, at least for some levels of fundraising capacity, the allocated score s_i is in fact decreasing in f_i.

Next, a relatively widespread (e.g. in primary and general elections for the US Presidency, although these are very rarely opted into FEC (2019)) form of campaign finance regime is the combination of public grant and private fundraising. In such regimes, candidates (usually with some demonstration of viability) are given public funds, but they can also raise private money potentially subject to a cap. This nests a number of common regimes. If we call the the public grant amount γ and the cap κ, then a pure public grant case would be when $\kappa = 0$, and the US presidential elections are $\kappa = \infty$ and γ small. But the spectrum of regimes when $\kappa = \infty$ (and thus private fundraising is constrained only by candidate capacity, not by law) and γ varies from 0 to ∞ are of particular interest. These represent the hybrid systems that we will find to commonly be optimal, with the relative size of γ representing the convexity of the regime. That is, when γ is very small, then private fundraising capacity is very important. When γ grows private fundraising lessens in relative importance to campaigning skill. And when γ is theoretically very large (each candidate who collects 100 signatures gets 1 billion dollars to spend on the race for local fire chief) then the difference between raising one thousand or ten thousand private dollars is essentially null. This system asymptotically approximates pure public regimes, albeit through a different mechanism to one that bans private fundraising. In our model this spectrum would be a score allocation that

[(4)] Before they were declared unconstitutional in Arizona Free Enterprise Club's Freedom Club PAC v. Bennett.

increases in both c_i and f_i, but the convexity or concavity with which it increases in f_i is determined by the size of γ. A cap to fundraising capacity above some threshold would assign scores based on both c_i and f_i to an upper limit on f_i, beyond which any further improvement in fundraising ability would not translate into a higher score.

Finally, there are a variety of regimes that differ in their treatment of different types of private money. These are important, as we will discuss in section 5.4, when the planner has a different view of the informativeness of different types of private money. For example, there are regimes which allow both individual donations and corporate donations with possibly different caps. Disclosure of the identities of donors may or may not be required. There are regimes in which individual donations are permitted but corporate donations are not. Theoretically the opposite could be implemented although it never has been. As an extreme case, the current post-Citizens United regime in the US in which Super PACs can raise and spend unlimited amounts, and the identities of donors can be partially hidden before the election by only reporting their identities quarterly. In our model this would be a score allocation in which those with higher fundraising capacities f_i are highly advantaged compared with those with a lower score. In section 5.4 we distinguish between various types of private money and proclivities to raise these different types of money. In that framework, the differences between these regimes is made transparent.

2.3 Existing models of valence competition

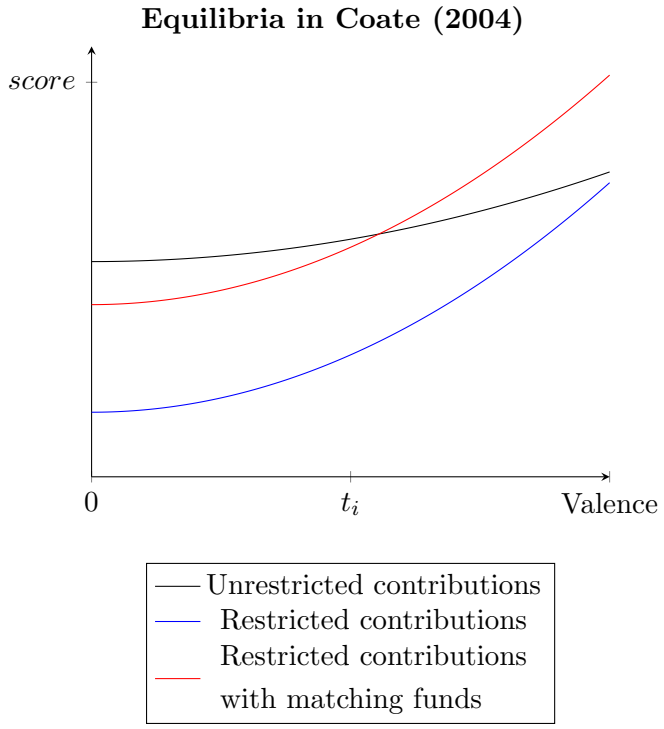
Many papers have studied valence competition, sometimes in isolation but often in combination with policy or effort dimensions. In each case they propose micro-foundations for the voters, candidates and special interest groups (if applicable) which, combined with an equilibrium concept, restrict the type of election outcomes that are possible. Within each model–i.e. within a restricted class of elections–statements about the welfare of various campaign finance regimes are comparisons between two 'points' in the restricted 'election space'. These papers typically do not make precise what the set of 'feasible' elections is, since in such papers typically only two points (e.g. one with unrestricted private fundraising and one with some restriction on fundraising) are compared. Typically there is no abstract sense in which something in middle of these elections could be constructed.

To clarify the purpose of this paper, we will first loosely translate some of the notable models that include valence competition into the language of our model.

2.3.1 Coate (2004)

In Coate (2004), policy favors are provided in exchange for money to be spent on advertising that gives information as to a candidate's valence. The informativeness of advertisements depend both on their gross amount and the level of policy distortion inferred by the voter in equilibrium. The more advertising there is, the more likely a voter is to see such advertising and then internalize the valence information contained within. But voter's discount large quantities of advertising if, in equilibrium, this signals unfavorable policy positions have been committed to by the candidate.

Thus, more advertising need not translate into a higher chance of winning. Indeed, in the equilibrium when there are no restrictions on contributions, advertising is high but largely disregarded so that the difference between more and less qualified candidates winning is small. This is illustrated by the black curve in the figure below. While the gross amount of advertising is high, such that the election is decided to a large extent by advertising, the differentiation between candidates is minimal.



Coate considers two possible types of campaign finance restriction. First, he considers a limit on campaign contributions. This has two counterveiling effects. On one hand the voters now infer more from a given level of advertisements, as they understand that in equilibrium fewer favors have been promised. On the other, the total quantity of advertisements has decreased, which means fewer voters see this advertising, and therefore the decisiveness of valence in the election is lessened. This is represented by the blue curve, in which candidates are more differentiated but the overall impact of valence on score is generally lower. Which of these two effects dominates cannot in general be decided.

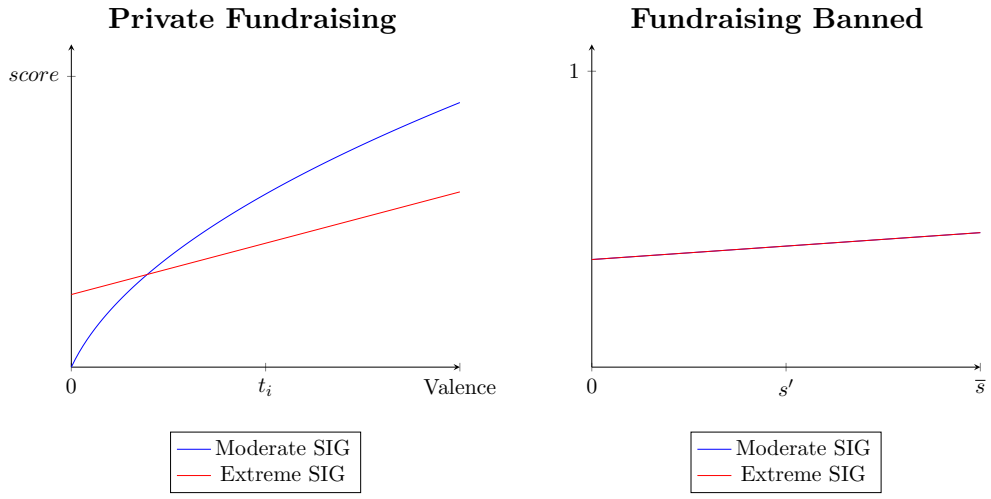
Second, Coate considers campaign finance regimes that consist of imposing a limit l on private funds and then matching these privately funds with a proportionality factor s. The attraction of this type of regime is clear: One can potentially utilize the valence information that comes from private money and supplement this with large amounts of matched public money so that the favors promised can be small. Indeed, the optimal such policy consists of a low limits l and a generous matching factor s so that, roughly, the informativeness of private money is increased as in the case without matching funds above, but the total amount of advertising is unchanged. That is, with this

type of policy, all the benefits of a limit can be obtained without any of the costs. This is the red curve in the figure above.

Thus Coate's paper shows that within his model, the best of these three campaign finance regimes is a particular form of restricted contributions with matching funds, as shown in the red curve above.

2.3.2 Prat (2002a)

In Prat (2002a), a special interest group (SIG) has private information about the valence of a candidate. They offer candidates money to spend on non-informative advertising in exchange for favorable policy choices. In equilibrium, candidates with a higher valence receive more money and are voted for with a weakly higher probability. Only two choices are considered for the planner: unrestricted donations and policy promises, and a complete ban on fundraising.



We focus on the separating equilibrium of Prat's model in which better candidates (those whose signal observed by the SIG is higher) receive more money from the SIG, and thus effectively have a higher electoral score. When SIG money is allowed, the informational benefit depends most crucially on how extreme the SIG's policy preferences are relative to the median voter. In the left panel of the figure above, we have drawn a stylized representation of how informative the electoral score is, in Prat's equilibrium, when the SIG is relatively moderate, and when it is extreme. At the limit, when the SIG is sufficiently extreme, this curve is identical to the public signal (right panel) and there is no additional valence signal from the SIG signal that gets incorporated into the electoral score . This is because, at this limit, the SIG makes the strong candidates participation constraint binding by balancing out the extra advertising money they get with a penalty from forcing them to adopt a more extreme policy, and as a result their chance of winning is exactly as it would have been without the advertising. On the right panel, somewhat uninformatively, we note that when SIG donations are banned there can be no advertising, and as there is minimal differentiation between candidates on the valence dimension, driven only by a public signal.

Thus the campaign finance section in Prat (2002a) is effectively a comparison between the elections that ensue in the case of moderate SIG's, (comparing the blue curve on the left to the blue curve on the right) or in the case of extreme SIG's (red curves). Prat concludes: First, that when the SIG is moderate, it is ambiguous as to whether the informational cost of banning SIG money outweighs the policy benefit; and second, when the SIG is sufficiently extreme, there is no informational cost of banning SIG money, only a policy benefit, and thus the ban is welfare improving.

3 Analysis

In this section we characterize the general structure of the optimal campaign finance regime. In the same way as the papers cited above suffer from optimizing over a far too narrow space of regime, a fair criticism of this approach is that we optimize over far too large a space. That is, once our model declares a regime to be optimal, it is not clear if there exists a particular election and campaign finance regime with preferences for voters, candidates, SIGs and other players that actually implements this optimal regime.

Nevertheless, even if the implementability of the optimum is an open question, taking this general and abstract approach allows one to make negative statements quite sharply. For example, in valence-only elections such as we study, we will be able to make sharp characterizations of when boundary cases such as a solely publicly funded election might be optimal. This makes a useful distinction between when a particular micro-founded model predicts the Pareto dominance or relative optimality of a campaign finance regime, and when one can say in general that no other possible induced election/campaign finance regime can do better.

The model we present below is sparse, with minimal restrictions on the win function, and on the planner's preferences between candidates. Nevertheless, we briefly sketch extensions to this model that preserve the qualitative structure of our analysis. These extensions are explored formally in sections 5. The qualitative structure of the main results, propositions 1 and 3, is preserved if we make any of the following alterations:

- There is an incumbent, perhaps with an incumbency advantage;
- There are $m \leq n$ players in every election, instead of just two;
- As a reduced form for generalized equilibrium dynamics, we allow the win function $W(s_1, s_2)$ to depend on the campaign finance regime chosen, $\boldsymbol{s}$;
- There are multiple types of fundraising abilities such as fundraising from individuals, corporations, Super PACs and so on.

3.1 Analysis of the Optimal Campaign Finance Regime

We make the following assumptions on the win function:

Assumption 2. *The win function, W, is continuous and continuously differentiable on $[0, \overline{s}]$ in both of its arguments. It is strictly increasing in its first argument, and strictly decreasing in its second. Moreover, $W_1(s, s) = W_2(s, s) = \frac{1}{2}$.*

The existence of a solution to the program (2.1) then follows immediately from the compactness of the feasible region and Assumption 2.

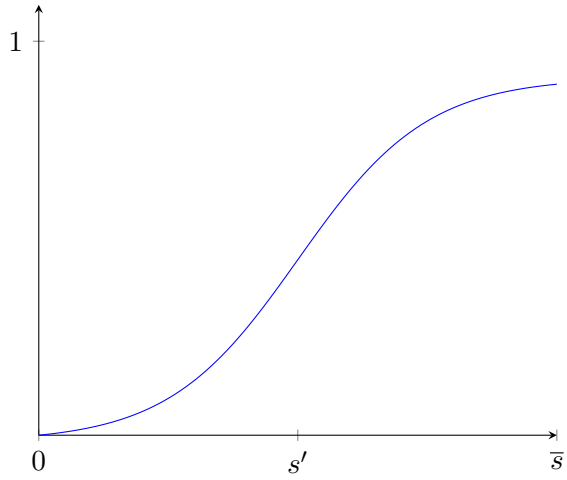
The core result of this paper follows. It says that these geometry of the election should coincide with the geometry of the planner's beliefs over candidate types in the following sense.

Proposition 1. *Under assumptions 1 and 2, in any optimal allocation that solves* (2.1)*, for any two players $n, n' \in \mathcal{N}$ with a non-zero score total, $s_n + s_{n'} > 0$, the planner chooses $s_n > s_{n'}$ if $t_n > t_{n'}$.*

That is, a planner should optimally construct an election / campaign-finance regime that gives, among the types that are differentiated, a player a strictly higher level of competitiveness if they are strictly better in terms of expected true type. Proposition 1 is almost an if and only if statement: it is true that $s_n > s_{n'}$ only if $t_n \geq t_{n'}$. However, without further assumptions, what happens in the case of $t_n = t_{n'}$ cannot be perfectly pinned down. Thus proposition 1 determines optimal election structure at least up to some variation within each set of candidate types that have the same true type. For more structure, we require will at times impose one of the following assumptions on the second order structure of the win function.

Assumption 3. *The derivative of the win function $W(s, s')$ has: $\frac{\partial^2 W(s,s')}{\partial s^2} > 0$ on $[0, s')$, $\frac{\partial^2 W(s,s')}{\partial s^2} = 0$ at $s = s'$, and $\frac{\partial^2 W(s,s')}{\partial s^2} < 0$ on $(s', \overline{s}]$.*

That is, Assumption 3 is assuming the win function looks, from the perspective of player 1, when we fix player two's score at s', like the following:



Economically we are assuming that:

1. The more competitive a player is, naturally the higher their chance of winning,

2. The player lagging in competitiveness benefits from increasing returns to scale,

3. The player leading in competitiveness suffers from decreasing returns to scale.

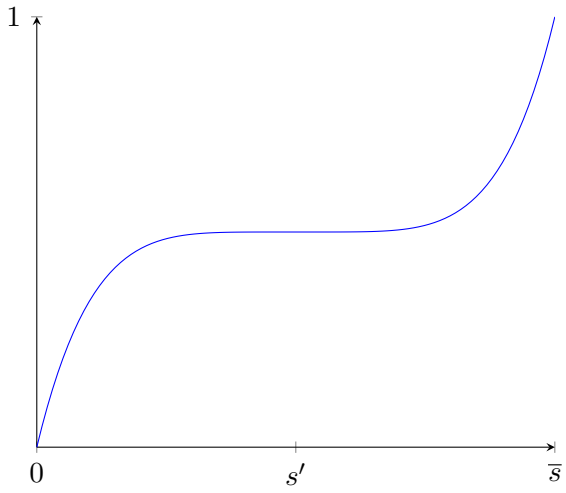
Given these assumptions, we can pin down precisely the geometry of the optimal election / campaign finance regime.

Proposition 2. *Under assumptions 1, 2 and 3, in any optimal allocation that solves* (2.1)*, for any two players* $n, n' \in \mathcal{N}$ *that have a non-zero score total* $s_n + s_{n'} > 0$*, the planner chooses* $s_n > s_{n'}$ *iff* $t_n > t_{n'}$*,* $s_n = s_{n'}$ *iff* $t_n = t_{n'}$ *and* $s_n < s_{n'}$ *iff* $t_n < t_{n'}$*.*

An alternative second-order assumption, which is more compatible with some types of elections, is essentially an inverse to Assumption 3.

Assumption 4. *The derivative of the win function* $W(s, s')$ *has:* $\frac{\partial^2 W(s,s')}{\partial s^2} < 0$ *on* $[0, s')$*,* $\frac{\partial^2 W(s,s')}{\partial s^2} = 0$ *at* $s = s'$*, and* $\frac{\partial^2 W(s,s')}{\partial s^2} > 0$ *on* $(s', \overline{s}]$*.*

This type of electorate described by this assumption is, loosely, an electorate that is very poor at distinguishing between more and less competitive candidates. They are able to distinguish both superstar candidates and extremely poor candidates, but for the large mass of candidates in the middle the electorate selects the more competitive one very noisily. Geometrically, we are assuming here that the win function takes the following form.



This type of electorate is one that is very effectively able to differentiate extremely good and extremely bad candidates from mediocre candidates, but cannot very effectively differentiate candidates nearer the middle. Although this is not a sufficient condition, this type of shape is the most likely to lead to a planner optimally treating candidates with the same true type differently. To see this suppose we have two types with the same t value, say $t_1 = t_2$. Against some candidates the planner wants these candidates to win. In such situations, when these two candidates would be stronger, there is convexity in the score payoff for these scores s_1 and s_2. Thus, against some third party perhaps with $s_3 = 1$, it would be better to assign the two players with $t_1 = t_2$ unequal

scores, say $s_1 > s_2 > 1$ rather than equal scores. Similarly when considering the possible matchups in which these two players would be disfavored by the planner, the planner's payoff is be concave in the scores s_1 and s_2. Hence to minimize this concave function (because here the planner wants these two players to lose) the scores should also be spread.

3.2 Implications for Campaign Finance

The propositions above, although abstract, have sharp implications for campaign finance. Given the dependence of second-order assumptions for deciding behavior when a planner is indifferent between two candidate-types, in this section we assume all types t_i are **distinct.** Given this, for brevity we define the rank function applied to a vector $\boldsymbol{x} \in \mathbb{R}^n$ by setting the ith entry equal to

$$(rank(\boldsymbol{x}))_i = |\{x_j, j = 1, \ldots, n : x_j \geq x_i\}|.$$

Thus the rank function just assigns to each entry in a vector that entry's rank position in the vector, which will be an integer between 1 and n inclusive. Given this we can say that two vectors are ordered identically, which we denote by "$=_o$" as follows:

$$\boldsymbol{x} =_o \boldsymbol{y} \iff rank(\boldsymbol{x}) = rank(\boldsymbol{y}).$$

We can now make clear what it means for a campaign finance regime to be, ordinally, as independent of one of the observable parameters as practicable.

Definition 1. *A **maximally public** (private) campaign finance regime is one in which* $\boldsymbol{c} =_o \boldsymbol{s}$ *(*$\boldsymbol{f} =_o \boldsymbol{s}$*).*

That is, if campaigning ability is all that matters to know the relative competitiveness scores of candidates, we say such a regime is as ignorant of fundraising ability as possible, and we call this a maximally publicly funded system. And conversely for maximally private.

Naturally, we should think of any score allocation that takes into account both c_i and f_i as a mixed system. Corresponding to these definitions of campaign finance regimes are an intuitive notion of what it means for an observable characteristic, f_i or c_i, to be broadly predictive of the underlying true types that the planner cares about electing.

Definition 2. *Campaigning (fundraising) types* c_i *(*f_i*) do not **ordinally predict** *t_i *if* $\boldsymbol{f} =_o \boldsymbol{t}$ *(*$\boldsymbol{c} =_o \boldsymbol{t}$*). Otherwise, we say that campaigning (fundraising) types have some predictive power for* t_i*.*

These definitions make clear what a planner needs to consider when they are evaluating the relative predictiveness of the various observables on which a campaign finance regime might select. The planner must care, in an ex ante sense, whether particular observables are linked with (by whatever mechanism they choose in their TE function) true type. And, for an observable type to not predict a true type it must be that the ranking on the other observable trait is identical to the true type, and thus knowing the original true type changes nothing.

As such, we can place intuitive constraints on exactly when maximally public or private campaign finance regimes might be optimal.

Proposition 3. *A maximally public (private) campaign finance regime is optimal only if f_i (c_i) does not ordinally predict t_i.*

This is the intuitive, but core result of this subsection. It is worth discussing at more length.

For example, it is plausible that donations by individuals of a small amount really are predictive of electability (which the designer wants) but large donations by large corporations just select for malleable and corruptible candidates (which the designer does not want). In the language of this model, that might mean that TE is increasing in f_i for small f_i, and then constant or decreasing for large f_i. As per the proposition, in this case a purely public system would not be optimal. The optimal system would perhaps cap the size of donations, restrict donations to individuals, cap the total amount of donations, or a similar mechanism to include the positively predictive donations and exclude the corrupting large donations.

This proposition says something stronger: that for a purely public system of campaign finance to be optimal at least it must be the case that fundraising propensity says very little about true valence.

Of the 24 candidates for the Democratic Party's 2020 presidential nomination, all of them declared they would refuse corporate PAC money, and 11 said they would refuse Super PAC money from any donor (see Scott (2019) and Wikipedia contributors (2019)). Moreover, all 24 supported a constitutional amendment to overturn the Citizens United decision and thus limit donations substantially. The justifications typically given focus on the policy outcomes that are potentially interfered with by donors and their special interests. But the proposition above makes a sharp claim about when this is optimal from a valence point of view.

As we show in section 5.4, when we break up fundraising ability into multiple types, for example the ability to raise money from corporations, money raised directly, money raised through Super PACs, and so on, that if the optimal election is to completely ban one of these types of fundraising, then that type of fundraising must similarly have very little or no predictive power for valence. Specifically, if banning Super PAC money is to be optimal from a valence point of view, it must be the case that there is no valence signal in a candidate's potential to raise money from Super PACs.

While such bans may still be justified on policy distortion grounds, this analysis still shows that such a ban will likely not be costless. For it to be truly costless one would need evidence that the banned money contains no valence signal, above and beyond the allowed money. This is formalized in section 5.4.

4 Comparative Statics of the Optimal Campaign Finance Regime

Within the structure imposed by propositions 1 and 2, a planner's choice of optimal election is sensitive to the exogenous nature of the electorate in question, to their chosen function $g(\cdot, \cdot)$, and more. In particular, a planner might understand the electorate to be particularly discriminating,

in that candidates with small differences in their competitiveness will have large differences in their probabilities of winning, or conversely. A planner might wish to optimize over different functions $g(\cdot, \cdot)$ and the resulting optimal regime might be qualitatively different.

In this section, we explore how the geometry of the optimal regime changes in response to these various factors. For transparency and tractability, we will *sometimes* analyze the optimization problem under the particular functional form

$$W(s_1, s_2) = \frac{(s_1 + 1)^p + 1}{(s_1 + 1)^p + (s_2 + 1)^p + 2}. \tag{4.1}$$

It is easy to check that (4.1) satisfies Assumption 2 but not 3, although that latter fact will not be consequential in our analysis.

This particular functional form allows chiefly for an easy encapsulation of how discriminating an election is. When $p = 0$, the election is a random coin toss regardless of the competitiveness scores of the various candidates. As p increases, fixing $s_1 > s_2$, the probability that player 1 wins goes to 1. Thus the limiting case of $p \to \infty$ is a perfectly discriminating electorate, who will perfectly choose between two candidates after observing their electoral competitiveness. Thus comparative statics in p are a proxy for a discriminating electorate.

To be precise, we define the following

Definition 3. *An electorate (defined by their win function) W_1 is more discriminating than another electorate W_1' iff, for $s_1 > s_2$, $W_1(s_1, s_2) > W_1'(s_1, s_2)$ and if $s_1 < s_2$ then $W_1(s_1, s_2) < W_1'(s_1, s_2)$.*

We define the partial order $W \succ_D W'$ to represent this.

It is straightforward to show the following:

Lemma 1. *The functional form (4.1) is increasing in discriminitiveness in p. That is, for $p > p'$*

$$\frac{(s_1 + 1)^p + 1}{(s_1 + 1)^p + (s_2 + 1)^p + 2} \succ_D \frac{(s_1 + 1)^{p'} + 1}{(s_1 + 1)^{p'} + (s_2 + 1)^{p'} + 2}.$$

A proposition that refers to the generalized optimization program (2.1) does not assume this particular functional form. Abusing notation slightly, a proposition that refers to the program labeled (4.1) is a reference to the full optimization program (2.1) with the general W_1 replaced with the particular function given above.

4.1 More discriminating electorates are better

The first comparative static result concerns the relative optimality of different electorates. That is, fixing all other aspects of the optimization program, which electorate would a planner rather face. The answer, in line with intuition, is that a more discriminating electorate is preferred.

Proposition 4. *Fixing all other aspects of (2.1), the optimized value of the objective function is strictly increasing in the discriminativeness of the electorate.*

If we think of an electorate as implementing, with some probability, the rankings of candidate types that a planner has chosen, then it is intuitive to see why a more discriminating electorate is better for the planner: the planner's implicit rankings over candidates will be translated in to realized outcomes with less noise the more discriminating the electorate.

4.2 More discriminant electorates lead to more gradated optimal elections

Our analysis up to this point has given ordinal characterizations of the optimal campaign finance regime *for the candidate-types who receive a positive level of points.* In particular, there is a non-zero[(5)] number of candidate-types that receive zero points from the planner. Thus, despite the planner having a preference over these candidate-types, if any such candidates actually entered the election they would be indistinguishable in terms of their probability of winning.

This is driven by the natural tradeoff between distinguishing between some higher types and some lower types, and distinction between or among these groups of broadly high or broadly low types. Thus, under certain circumstances the planner is choosing to give up precision in their implicit ranking of lower candidate-types so as to ensure the best types are sufficiently advantaged over low types. As we will see, when this tradeoff tends in one direction or the other is intimately connected with how discriminate the electorate is.

First we must describe the potential sets of candidate types that the planner might plausibly give non-zero scores to. The following is an immediate consequence of Propositions 1 and 2.

Lemma 2. *If a candidate type with true type t is assigned a score of zero in the optimal regime, then so are all candidates with true type $t' \leq t$.*

This proposition says that if the planner does not find it optimal to give non-zero score to a particular candidate-type, then it cannot be optimal to give a positive score to a lower candidate-type. The intuition for this is as follows: suppose two candidate types are both currently receiving zero. Index these candidates by their true types $t_1 > t_2$. Consider the pros and cons of giving either t_1 or t_2 a score of ϵ instead of zero.

Candidate t_1 is preferred to weakly more candidate types than t_2, and worse than weakly fewer candidate types than t_2. Thus, the benefits of giving ϵ to t_2 are weakly worse than giving it to t_1, and the costs weakly higher. Note that in the probability the election turns out to be precisely t_1 versus t_2, we would strictly prefer t_1 to have ϵ and t_2 zero than vice versa.

The question remains as to what might lead a planner to optimally give a non-zero score to a greater set of candidate types. To summarize this, we define:

Definition 4. *Fixing the set of candidate-types $\mathcal{N}$, an campaign finance regime $\boldsymbol{s}$ is weakly (strictly) more **gradated** than the regime $\boldsymbol{s'}$ if the number of players receiving non-zero scores is weakly (strictly) higher in $\boldsymbol{s}$ than in $\boldsymbol{s'}$.*

Now, we are able to pin down the relationship between the discriminitiveness of an electorate and the gradation of the resulting optimal campaign finance regime.

[(5)] The least favored candidate type always receives zero points in the optimum.

Proposition 5. *For any player m except those of the lowest type, for a sufficiently discriminating election player m is optimally allocated a non-zero score, $s_m > 0$.*

In particular, labeling the functional form 4.1 as W_p, there exist $0 = p_1 \leq p_2 \leq p_3 \cdots \leq p_n = \infty$ such that player m receives a non-zero score, $s_m > 0$, for all electorates W_p with $p > p_m$.

The intuition for this proposition is as follows: when an electorate is not very discriminating, if a planner wishes to give two candidate types, say two quite high types, a markedly different chance of winning they must expend many points to differentiate these scores. Points spent on these relatively high types are thus unable to be spent on differentiating low types as much. And when many points are spent on high types, even spending a few points on middle-low types so as to differentiate them from very low types is suboptimal, since such middle-low types would not be sufficiently differentiated from middle-high types.

On the other hand, when the electorate is more discriminate, fewer points need to be spent to differentiate the top few candidates. This leaves more points to be spent differentiating the middle or bottom class of candidates. Moreover, the downside of spending points on weaker candidates - that this makes them more competitive against stronger candidates than one might otherwise like–is less of an issue as the electorate becomes more discriminate, because one can have smaller yet equally decisive gaps between higher and lower types.

For example, a two party system (or a constitution that leads to it) is a very gradated system. It might be optimal if you pessimistic about the ability of the electorate to decide on the best option without noise. On the other hand a plurality system is noisier in that how competitiveness is translated into winning is much more random.

4.3 Planner preferences and the convexity of the election

Although there are many different functions $g(\cdot)$ that both satisfy our assumptions and are defensible in their plausibility, there are two key classes on which we will focus. In particular, a planner can have **ordinal** preferences within an election, which is to say that given any two candidates they care only to maximize the probability that the better candidate wins, without regard to how much better that candidate is than the other. Or, conversely, the planner can have **cardinal** preferences, such that only the magnitude of that difference matters, and thus candidates that are very close in type will be treated very similarly.

We should pause here to note that preferences over the entire universe of candidates, as embodied by the type evaluation function $TE(\cdot)$, need not bear a structural similarity to the planner's preferences within an election, as embodied by the $g(\cdot)$ function. Although they are obviously related: if a planner has ordinal preferences within an election, then clearly the only outcome that matters from the TE function is ordinal, such that they can be much less careful in their type evaluation, since only the ranking of candidate types matters. Nevertheless, one can still have a view as to the magnitude of difference between candidate types even if the only preferences they operationalize are ordinal.

Here we will take as the representative of ordinal preferences the function $g(a,b) = \mathbb{1}[a \geq b]$. And as representative of cardinal preferences we will take the function $g(a,b) = a$ so that the planner with these preferences is effectively maximizing the average type of the winning candidate. Although we will be mostly interested in comparing these extreme cases, for the purposes of proof and stronger monotonic statements, we will analyze a planner who has preferences given by a weighted sum of these two extreme preference types, which we denote by:

$$g_\alpha(a,b) = \alpha \mathbb{1}[1 \geq b] + (1-\alpha)a, \qquad \text{where } \alpha \in [0,1].$$

The main quantity in which we will be interested is the following measure of the 'convexity' of the optimal election. When there are m players who receive non-zero scores, and we order players in order of those scores $s_1, \ldots, s_m$ then we say:

Definition 5. *An election $\boldsymbol{s}$ with m players receiving non-zero scores is more **convex** than another election $\boldsymbol{s'}$ if $s_1 > s'_1$ and $s_m < s'_m$.*

That is, the strongest type receives more points, and the weakest non-zero type less points, in the more convex election. What happens to the middle types cannot in general be determined, but these extremal players provide an intuitive measure of this convexity, or 'spread', of an election. The following proposition makes precise the relationship between ordinality/cardinality of the planner's preferences within an election and the convexity of the optimal election.

To briefly comment on the assumptions: we deal with only preferences with the second-order qualities given by Assumption 4. Those preferences represent electorates that look for superstars, so to speak. That is, when it comes to stronger players, the electorates are convex in their decision. If they were concave, as in the alternate assumption, loosely speaking the concavity in the electorate would interfere with the convexity we are studying here, and the result would be indeterminate. Hence, we impose some underlying convexity on the electorate and study how the different preferences lead to degrees of convexity in the optimal election. Second, we make either of two assumptions solely to discipline the set of optima. The assumption of quasi concavity is to guarantee uniqueness of equilibria such that when we trace the path of the arguments of the maximum it is in fact a function, not a correspondence, and we don't need to worry about non-local jumps. The assumption of a function that is not flat[6] at any point guarantees that if there are multiple equilibria, they are isolated from one another and we don't need to worry about a 'ridge' of equilibria. Finally, the assumption that the differences between the true types is no smaller than one is innocuous, since this is just to ensure that the difference $t_i - t_j$ is greater than $\mathbb{1}[t_i > t_j]$ when indeed $t_i > t_j$. But since the indicator function seeks only to represent ordinal preferences, it could be scaled arbitrarily without loss of generality. In particular, it could be scaled to whatever the smallest difference between any two different true types are.

Proposition 6. *Take $\alpha, \alpha' \in [0,1]$ with $\alpha > \alpha'$, and $|t_i - t_j| \geq 1$ for all $t_i \neq t_j$. Throughout we*

[6] In the technical sense of having all derivatives vanish at a point.

*make assumption 4. For a win function W such that (2.1) is quasi-concave in $\boldsymbol{s}$, we have that the optimal election at α is less **convex** than the optimal election at α'.*

When we only assume that (2.1) is nowhere flat, then as α falls to α' the optima with lowest value of s_1 increases in s_1, and the optima with highest value of s_m decreases in s_m.

That is, what we have shown is that when a planner's preferences become more cardinal and less ordinal than before, the optimal election becomes more convex, all else held equal. This confirms the intuitive hypothesis we discussed. When a planner becomes more sensitive to differences between players, the planner especially cares about elections with a large gap in types between the candidates. This happens more often and to a larger degree to stronger candidates and conversely hurts candidates the weaker they are. For intermediate candidates this pulls in both directions, but as established above the extremal players in any election are stretched apart as the planner's preferences move in the cardinal direction.

5 Extensions

5.1 Incumbents

The model we have worked with thus far assumes that which candidates compete in each election is independent of which candidates competed in the prior election. Or alternatively, we can interpret that model as imposing a one-term limit on each candidate.

However, a valid concern is that incumbency offers electoral advantages, and that this is bad because of the policy implications and/or because it props up otherwise weak incumbents. In the former case, there is concern that private fundraising complements incumbency dangerously, in that special interest groups want to give money to incumbents simply because they are in office and can make policy changes to the advantage of the special interest groups and detriment to the electorate. In the latter case, we might worry that incumbents have a fundraising and/or publicity advantage that makes them a more effective campaigner, which is specifically detrimental when the candidate is fundamentally a poor one who is artificially advantaged by incumbency. In this section we analyze an extension of the model above that will allow us to comment on the veracity of some of these concerns.

In this new model, there are two periods with an election to take place in each. In the first period, there is no incumbent, and candidates are chosen at random as in the model we have studied up to this point and the election proceeds in that same way. In the second period, the winner from the first period is an incumbent, a challenger is selected at random and the election proceeds as normal except for the fact that the incumbent has an advantage β that increases their probability of winning, regardless of their underlying type. For simplicity, we assume this maximum β is small enough to ever give anyone a chance of winning an election greater than one or less than zero.

Formally, the planner now solves the program:

$$\max_{\beta, s_n, \forall n \in \mathcal{N}} \Bigg[\sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} \left(g\left(t_n, t_{n'}\right) W_1\left(s_n, s_{n'}\right) + g\left(t_{n'}, t_n\right) W_2\left(s_n, s_{n'}\right) \right) \\ \sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} P_n^{Inc}(\boldsymbol{s}) \left(g\left(t_n, t_{n'}\right) \left(W_1\left(s_n, s_{n'}\right) + \beta \right) + g\left(t_{n'}, t_n\right) \left(W_2\left(s_n, s_{n'}\right) - \beta \right) \right) \Bigg]$$

with

$$P_n^{Inc}(\boldsymbol{s}) = \frac{1}{|\mathcal{N}||\mathcal{N}-1|} \sum_{n' \in \mathcal{N} \setminus \{n\}} W_1(s_n, s_{n'}) \tag{5.1}$$

such that

$$s_n \geq 0 \text{ for all } n \in \mathcal{N}, \qquad \sum_{n \in \mathcal{N}} s_n = \overline{s}, \qquad \beta \in [0, \overline{\beta}], \qquad W_1(\overline{s}, 0) < 1 - \overline{\beta}.$$

That is, whereas all candidate-type pairs are equally likely in the first period, in the second period some incumbents are more likely than others owing to the fact that types with a higher score will win with a greater probability in the first period. And in the second period, whomever is the incumbent gets a boost of β to their probability of winning.

There are two reasons why we have chosen to incorporate an incumbency advantage in this uniform way.

The first is that the empirical evidence (see for example Fouirnaies and Hall (2014)) generally shows that the incumbent advantage is independent of the office holder's experience, the challenger's quality or on the potential office-holding benefits. Thus, in the context of our model, where the main variables are candidate and challenger quality, a uniform incumbent advantage has empirical support.

Second, a uniform incumbent advantage, if anything, underestimates (in the sense of the planner's welfare in our model) the true benefits of an incumbency advantage. As we will see, because this uniform advantage accrues with greater probability to better candidates, it ex-post functions to benefit better types more often. This will drive the positive welfare effect of the incumbency advantage. But if the story we have in mind is that the benefit of incumbency is increased fundraising capacity due to the rent-seeking of special interest groups, then office-holders best poised to take advantage of this benefit are those who are better fundraisers. But in such a world with private fundraising, they are also stronger candidates to begin with. Thus, in this conception of an incumbency advantage, the benefit will accrue to stronger candidates explicitly, as well as through the implicit mechanism our model includes. Hence if our uniform incumbent advantage is good because it benefits good candidates more often, then an incumbent advantage that explicitly gives further benefit to strong candidates will also certainly raise welfare.

Given this, we can first show that the general proposition that structures the optimal election holds true in this extended model. Formally,

Proposition 7. *Under Assumptions 1 and 2, in any optimal allocation that solves* (5.1)*, for any*

two players $n, n' \in \mathcal{N}$ that have some chance of winning $s_n \neq s_{n'}, s_n + s_{n'} > 0$, the planner chooses $s_n > s_{n'}$ if $t_n > t_{n'}$.

The above confirms that the typical structure of the optimal election does not vary even with an incumbency advantage. The proposition above refers to the optimal incumbency advantage (which is as yet unknown) but the proof actually demonstrates it to be true for any incumbency advantage β within the constraints of (5.1). With this structure in hand, we can prove the main proposition of this section.

Proposition 8. *Considered as exogenous, the planner's welfare increases monotonically in β. Therefore, if the planner can choose β, they choose it to be maximal.*

Note that this shows a uniform incumbency advantage to be optimal, when *chosen.* Specifically, even though weak incumbents may occasionally be propped up by this uniform advantage, in the optimal election and campaign finance regime it flows more often to strong incumbents, and hence is welfare improving. That is to say that when campaign finance and election is optimally designed, a selection based positive incumbency effect is downstream. That is not so say that bad election structures can't be made worse by incumbency effects, but simply that good ones are indeed made better.

Although the interpretation is clear is β is exogenous, for conceptual consistency we will take it as endogenous. That is because we are looking for the first-best for the planner, unconstrained by the player's preferences. As such, whether or not it is achievable, we want to consider what the planner would choose given maximal control, and the discussion above shows that the incumbent advantage is optimal even when chosen independently. That is a stronger statement than an incumbency advantage being optimal in a particular equilibrium, which we highlight.

5.2 Multi-player elections

Many elections, even if there are functionally only two competitive candidates (especially in systems with two entrenched parties and/or a majority win rule), are in fact fought between more than two candidates. In this section we demonstrate that with assumptions analogous to those in section 3, our main proposition goes through in this more general setting.

In this setting, we cannot simply work with one win function to represent all of the electorate's underlying randomness. Thus, in an election with m candidates, who are allocated scores $(s_{n_1}, \ldots, s_{n_m}) \equiv \boldsymbol{s}$, we write:

$$W_i(\boldsymbol{s}), \quad \sum_{i=1,\ldots,m} W_i(\boldsymbol{s}) = 1$$

to be the probability that candidate n_i wins. But we restrict to win functions that are anonymous, such that if we swap the first player's score with the second, then their probabilities of winning swap and all others remain the same. Given this, we are able to define the planner's generalized optimization program when there are m players:

$$\max_{s_n, \forall n \in \mathcal{N}} \sum_{n_1 \in \mathcal{N}} \cdots \sum_{n_m \in \mathcal{N} \setminus \{n_1, \ldots, n_{m-1}\}} \left[\sum_{i=1}^{m} g_i \left(t_{n_1}, \ldots, t_{n_m} \right) W_i \left(s_{n_1}, \ldots, s_{n_m} \right) \right]$$

such that

$$s_n \geq 0 \qquad \text{for all } n \in \mathcal{N}, \tag{5.2}$$

$$\sum_{n \in \mathcal{N}} s_n = \overline{s}.$$

First we can write down the generalizations of Assumptions 1 and 2.

Assumption 1-M. *Label $\boldsymbol{t} = (t_1, t_2 \ldots, t_m) \in \mathbb{R}^m$ and $\boldsymbol{t}'_i = (t_1, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots, t_m)$. For all $t_i > t_j$ we have $g_i(\boldsymbol{t}) \geq g_j(\boldsymbol{t})$ with not all holding with equality. If $t_i > t'_i$ then $g_i(\boldsymbol{t}) \geq g_i(\boldsymbol{t}'_i)$ and $g_j(\boldsymbol{t}) \leq g_j(\boldsymbol{t}'_i)$ for all $j \neq i$. If $t_i = t_j$ then $g_i(\boldsymbol{t}) = g_j(\boldsymbol{t})$.*

Assumption 2-M. *Each win function W_i, for $i = 1, \ldots, m$ is anonymous, strictly increasing in the ith argument, and strictly decreasing in all but the ith argument. Further, for every vector of scores $\boldsymbol{s}$, we have $\sum_i W_i(\boldsymbol{s}) = 1$.*

It is clear that these assumptions reduce to their counterparts in Section 3 if there are only two players. In Section 3 we used this anonymity to just write $g_2(t_1, t_2) = g_1(t_2, t_1) \equiv g(t_2, t_1)$ in which case it is clear that Assumption 1-M reduces to Assumption 1-M. The equivalence of Assumptions 2 and 2-M is clear once one notes than in the 2 player case we dispensed with W_2 and instead just wrote $1 - W$.

There is a further complication one encounters when attempting to generalize from the case of two candidate elections. When there were two players, the function g could always distinguish two candidates, since there existed the possibility of an election being fought between these two candidates, in which case assumption 1 requires that the planner have a strict preference between the two. However, when there are many candidates, there exist plausible functions g such that, for a particular pair of candidates, even though the planner might assign different true type values to these two candidates, that difference is never registered in the planner's optimization problem.

Example 1. *Suppose the function $g_i(t_1, t_2, \ldots, t_m)$ takes value 1 if $t_i = \max\{t_1, t_2, \ldots, t_m\}$ and zero otherwise. A planner who has this g function is attempting to maximize the chance that the best candidate wins in each of the possible candidate combinations that might come to constitute an election. Suppose there are $n = 5$ possible candidate types, but an election is to be fought between $m = 3$ of them. Suppose the candidates are labeled in order of their true types $t_1 > t_2 > \cdots > t_5$. In no possible election is either t_4 or t_5 the best candidate, and the planner will never treat these candidates differently, even though in true type t_4 is strictly better than t_5*

This leads us to the general concept that is important when considering the set of candidates the planner functionally cares to differentiate among.

Definition 6. *Two players t_n and $t_{n'}$ with $t_n > t_{n'}$ are **differentiable** according to a planner with preference $g(\cdot)$ if there exists some vector of $m-2$ other players $\boldsymbol{t} \in \mathcal{N} \setminus \{t_n, t_{n'}\}$ such that $g_1(t_n, t_{n'}, \boldsymbol{t}) > g_2(t_n, t_{n'}, \boldsymbol{t})$.*

That is, we say that two candidate-types are differentiable according to a planner's preference function g if there exists a plausible election in which the planner has a strict preference between which of those two players comes first.

Example 2. *Suppose that $g_i(t_1, \ldots, t_m) = t_i$. That is, the planner is maximizing the average quality of the winning candidate. Then all candidates are differentiable.*

Given these analogous assumptions, we have the appropriate analogue of Proposition 1.

Proposition 9. *Under assumptions 1-M and 2-M, in any optimal allocation that solves* (5.2)*, for any differentiable two players $n, n' \in \mathcal{N}$ that have some score allocated $s_n + s_{n'} > 0$, the planner chooses $s_n > s_{n'}$ if $t_n > t_{n'}$.*

5.3 Campaign-finance dependent win functions

The core purpose of this paper was to analyze the optimal campaign finance regime without the particular (and somewhat arbitrary) constraints that come from the choice of micro-foundations and an equilibrium concept. Nevertheless, it is important to establish that the broad thrust of this paper is robust to a notion of equilibrium-type behavior by voters.

Specifically, we have assumed up to this point that the voters evaluate, in terms of the probability of winning, two candidates with given scores s_1 and s_2 solely based on these scores. Thinking of these scores as an amalgam of charisma and advertisements (and whatever else the planner cares about), we have effectively assumed that these scores are all that matters to voters, no matter the larger campaign finance structure (i.e. the whole of $\boldsymbol{s}$) from which these two particular players are drawn.

What we should worry about is the following: suppose there are two candidates, one of whom spends 1 million dollars and the other 900,000 dollars on advertisements, and who some arbitrary levels of charisma c_i. Suppose the overall campaign finance regime (i.e. all of $\boldsymbol{s}$) is such that privately funded advertising expenditure is capped at 1 million dollars. Alternatively suppose there is no cap, and there exist potential candidates who might spend 1 billion dollars, were they to enter into an election. The given levels of advertising spending have vastly different meanings in these two contexts. In the former case, these are candidates almost as attractive as possible to private donors. In the latter case, these are terrible candidates who are just lucky no stronger candidate came along.

As such, we write down an extended model below in which we allow for voters to treat these two cases differently. The program the planner solves is identical to 2.1 except for the win function now being dependent on $\boldsymbol{s}$ and as such labeled $W_{\boldsymbol{s}}(s_n, s_{n'})$.

Moreover we require that only the overall shape of the campaign finance regime matters, not the identity of a particular score holder. That is, if $\phi : \mathcal{N} \to \mathcal{N}$ is a permutation function (i.e. a

bijection) and we write $\boldsymbol{s}_\phi$ for the permutation of scores in the natural way, then we require that $W_{\boldsymbol{s}}(s_n, s_{n'}) = W_{\phi(\boldsymbol{s})}(s_n, s_{n'})$ for all permutation functions ϕ.

Define the mapping $H : \mathbb{R}^{|\mathcal{N}|} \to C_b(\mathbb{R}^2, \mathbb{R})$ that maps $\boldsymbol{s} \mapsto W_{\boldsymbol{s}}(\cdot, \cdot)$. We make some assumptions. When we endow the codomain of H with the uniform norm, we require that H is a continuous mapping. Next, for all $\boldsymbol{s}$, we require that $W_{\boldsymbol{s}}$ satisfies Assumption 2. Moreover, we require the following: As $\boldsymbol{s}$ changes to $\boldsymbol{s}'$ with respect to any two player's scores (say $s_i \to s_i + \epsilon$ and $s_j \to s_j - \epsilon$), consider how the win function in an election between one of these players, and a third player $k \neq i, j$, changes. There are two effects: the **direct effect** $W_{\boldsymbol{s}}(s_i + \epsilon\ , s_k) - W_{\boldsymbol{s}}(s_i, s_k)$ and the **indirect effect** that pertains to the movement in win function$W_{\boldsymbol{s}'}(s_i, s_k) - W_{\boldsymbol{s}}(s_i, s_k)$. We assume that the direct effect dominates the indirect effect. This is to rule out pathological examples and for tractability. We then have the following:

Proposition 10. *Under the assumptions above, in any optimal allocation that solves the program with $W_{\boldsymbol{s}}(\cdot, \cdot)$, for any two players $n, n' \in \mathcal{N}$ that have some score allocated $s_n \neq s_{n'}, s_n + s_{n'} > 0$, the planner chooses $s_n > s_{n'}$ if $t_n > t_{n'}$.*

This demonstrates that having an electorate that is reactive to the overall shape of an election does not affect the core geometry of the optimal regime, so long as for each $\boldsymbol{s}$, the electorate behaves in the natural way when deciding between scores. In particular, this dependence of the win function on $\boldsymbol{s}$ serves as a reduced form implementation of some potentially micro-founded equilibrium dynamics through which fixed levels of advertising can be interpreted differently in line with the larger campaign finance regime in place.

5.4 Multiple types of fundraising ability

Suppose the planner could separate out multiple different types of fundraising ability. So instead of just f_i we had $f_i^1, f_i^2 \dots f_i^k$ with perhaps the first representing fundraising ability from personal donations, the second being corporate money from normal PACs, the third being money from Super PACs, and so on.

Thus we reinterpret our full type evaluation function as the valence type the planner predicts given knowledge of all these parameters, and as before we label this as

$$t_i = TE(f_i^1, f_i^2 \dots f_i^k, c_i)$$

with the whole vector labeled $\boldsymbol{t}$. Conversely, imagine a planner who did not information as to one of these k fundraising abilities. Specifically, we write a planner's prediction for the valence type knowing all but the kth (the final one, for example) type of fundraising ability as

$$t_i^{\setminus k} = TE(f_i^1, f_i^2 \dots f_i^{k-1}, c_i)$$

with the whole vector labeled $\boldsymbol{t}^{\setminus \boldsymbol{k}}$.

Recalling the notation from section 3.2, we have these analogous definitions:

Definition 7. *We say that a campaign finance regime doesn't* ***strongly depend on*** f_k *if* $\boldsymbol{t}^{\backslash \boldsymbol{k}} =_o \boldsymbol{s}$. *Conversely, a regime does strongly depend when* $\boldsymbol{t}^{\backslash \boldsymbol{k}} \neq_o \boldsymbol{s}$.

That is, a regime effectively ignores the kth fundraising type when the regime ranks candidates identically to the predicted valences if the kth fundraising type was not known. Analogously to the main section we then have the following

Definition 8. *The kth fundraising type* f_k *does not* ***ordinally predict*** $\boldsymbol{t}$ *if* $\boldsymbol{t}^{\backslash \boldsymbol{k}} =_o \boldsymbol{t}$ *Otherwise, we say that the kth fundraising type does have some ordinal predictive power for* $\boldsymbol{t}$.

That is, we say that the marginal predictive power of knowing this kth fundraising characteristic and all others, relative to not knowing it yet knowing all others, is non-small when it changes the planner's ordinal ranking of candidates. Given these we can write down the proposition analogous to proposition 3. The proof is essentially identical and thus omitted.

Proposition 11. *A campaign finance regime that doesn't strongly depend on* $\boldsymbol{f^k}$ *is optimal only if* $\boldsymbol{f^k}$ *does not ordinally predict* $\boldsymbol{t}$.

This brings back to the core message of this paper. Any initiative to ban a certain type of money from elections cannot be said to be costless unless there is strong evidence that that type of money brings has almost no marginal valence signal. Even when the desire to prevent policy distortion is persuasive, one must still consider the likely decrease in valence that will follow from structuring an election to not utilize all the valence signal available.

6 Conclusion

In this paper we have offered a framework for analyzing campaign finance regimes that does not depend on the preference of players or a particular solution concept. This framework is general enough to nest a wide variety of possible campaign-finance restrictions–a fact we illustrated by reference to some of the more prominent models in this literature: Prat (2002a) and Coate (2004).

In addition to providing a technique for making global welfare comparisons among different campaign-finance regimes, we establish two main substantive results. First, a pure public funding regime is only optimal in the extreme setting where the candidates' fundraising ability contain no positive or negative valence signal.

Second, incumbency advantages–something that is generally thought to be problematic–can in fact be good if the election and campaign finance regime that is implemented is optimal. In that case, the candidates who become incumbents will be better on average, from the planners point of view, than those who are challengers. Given this, even an incumbency advantage that uniformly boosts all incumbents, good and bad, will turn out to be welfare improving since it flows to good candidates more often.

The issue of campaign-finance reform, at least in the United States, is a daunting one in the wake of the *Citizens United* decision. But a necessary first step toward reform is a more complete

understanding of the welfare properties of the full range of campaign-finance regimes. We hope that our framework will prove useful in that endeavor.

A Proofs

Notation

Define the following notations: write $\mathcal{N}_n^{\succ}, \mathcal{N}_n^{\sim}, \mathcal{N}_n^{\prec}$ for the set of all players in $\mathcal{N}$ that have, respectively, expected true types strictly less than, equal to, and strictly greater than the expected true type of player n. Similarly define $\mathcal{N}_{n'}^{\succ}, \mathcal{N}_{n'}^{\sim}, \mathcal{N}_{n'}^{\prec}$ for player n'.

Proof of Proposition 1

This is a special case of Proposition 9 with $m = 2$.

Proof of Proposition 2

Proof. Proposition 1 is clearly applicable here, and hence the 'if' part of the claims for strict inequality hold immediately. It suffices to show that, for players n and n' with $t_n = t_{n'}$ and $s_n + s_{n'}$, we have $s_n = s_{n'}$. This is sufficient because we will have then showed that $t_n > t_{n'} \implies s_n > s_{n'}, t_n = t_{n'} \implies s_n = s_{n'}$ and $t_n < t_{n'} \implies s_n < s_{n'}$. It will then follow that, for example, $s_n > s_{n'} \implies t_n > t_{n'}$ because if not, and $t_n \leq t_{n'}$ we would have $s_n \leq s_{n'}$, a contradiction. Thus we prove that claim now.

Take two players n and n' $t_n = t_{n'}$ and $s_n + s_{n'} > 0$. Clearly a necessary condition for optimality is that there does not exist an improving reallocation just between s_n and $s_{n'}$. So as before we write $s' = s_n + {}_{n'} > 0$ and consider the optimal choice of s_n such that player n gets s_n points and player n' gets $s' - s_n$ points. By a decomposition identical to the proof of proposition 1, except noting that here $\left(\mathcal{N}_{n'}^{\prec} \cap \mathcal{N}_n^{\succ}\right) = \varnothing$ and we need not consider the sets $\mathcal{N}_n^{\sim}$ and $\mathcal{N}_{n'}^{\sim}$ since the planner is indifferent over the victor in each, we can write the summands in the objective function that depend on s_n as

$$\sum_{k \in \mathcal{N}_{n'}^{\succ}} \left[W_1\left(s_n, s_k\right) + W_1\left(s' - s_n, s_k\right)\right] + \sum_{k \in \mathcal{N}_n^{\prec}} \left[\left(1 - W_1\left(s_n, s_k\right)\right) + \left(1 - W_1\left(s' - s_n, s_k\right)\right)\right]. \quad \text{(A.1)}$$

Consider the first summand in (A.1). By definition, all players $k \in \mathcal{N}_{n'}^{\succ}$ here have $t_k < t_n = t_{n'}$ and so by Proposition 1 we have $s_k < s_n$. Thus, by this fact and Assumption 3, $W_1(s_n, s_k)$ is strictly concave over the feasible region $[s_k, s' - s_k]$.

But note the general fact that if a function $\phi(\cdot)$ is increasing and strictly concave over $[a, b]$, the maximum of $\phi(a + x) + \phi(b - x)$ with $x \in [a, b]$ occurs at $(a + b)/2$. That is, when $a + x = b - x$.

In this case, that means that $\sum_{k \in \mathcal{N}_{n'}^{\succ}} \left[W_1\left(s_n, s_k\right) + W_1\left(s' - s_n, s_k\right)\right]$ is maximised at $s_n = s' - s_n \implies s_n = s_{n'}$. An identical argument shows that the also strictly concave function $\sum_{k \in \mathcal{N}_n^{\prec}} \left[\left(1 - W_1\left(s_n, s_k\right)\right) + \left(1 - W_1\left(s' - s_n, s_k\right)\right)\right]$ is similarly maximised at $s_n = s' - s_n \implies s_n = s_{n'}$.

This proves that if $t_n = t_{n'}$ then $s_n = s_{n'}$ and this completes the proof.

■

Proof of Proposition 3

Proof. We will deal with the proposition for a maximally public campaign finance regime. The argument for a maximally private campaign finance regime clearly follow with a simple replacement of symbols.

From proposition 1, and since in this section we have assumed all true types are distinct, it is easy to see that optimality of an election (represented by it's scores $\boldsymbol{s}$) requires at least that $\boldsymbol{s} =_o \boldsymbol{t}$. That is, scores and true types are ordinally matched.

If we had that the optimal regime was maximally public but that f_i did have some predictive power for t_i, we would then have $\boldsymbol{s} =_o \boldsymbol{c}$ because the optimal regime is maximally public, $\boldsymbol{c} \neq_o \boldsymbol{t}$ because of the predictive power, hence $\boldsymbol{s} \neq_o \boldsymbol{t}$ which contradicts optimality. Thus we have that if the optimal election is to be maximally public we must have that f_i has no predictive power for t_i, and contrapositively if f_i does have this predictive power the optimal election cannot be maximally public. This proves the proposition.

■

Proof of Lemma 1

Proof. Recall that the particular win function in question is given by

$$W_1(s_1, s_2) = \frac{(s_1+1)^p + 1}{(s_1+1)^p + (s_2+1)^p + 2}.$$

The derivative with respect to p is easily shown to be

$$\begin{aligned}\frac{\partial W_1}{\partial p} &= \frac{(s_1+1)^p\left((s_2+1)^p+1\right)\log(s_1+1) - \left((s_1+1)^p+1\right)(s_2+1)^p\log(s_2+1)}{\left((s_1+1)^p+(s_2+1)^p+2\right)^2} \\ &= \frac{(s_1+1)^p(s_2+1)^p\left(\log(s_1+1)-\log(s_2+1)\right)}{\left((s_1+1)^p+(s_2+1)^p+2\right)^2} \\ &+ \frac{(s_1+1)^p\log(s_1+1) - (s_2+1)^p\log(s_2+1)}{\left((s_1+1)^p+(s_2+1)^p+2\right)^2}\end{aligned}$$

Written this way, it is clear to see that $\frac{\partial W_1}{\partial p}$ is positive exactly when $s_1 > s_2$ and negative when $s_1 < s_2$. Thus when we increase p the win probability will go up for player 1 when they have $s_1 > s_2$ and will go down when $s_1 < s_2$. Thus the lemma is proven.

■

Proof of Proposition 4

Proof. Given two versions of the program (2.1) that differ only in the electorate, as represented by the win function, with W_1 more discriminating than W_1'. Denote the objective functions of these programs by, respectively, $PP(\boldsymbol{s})$ and $PP'(\boldsymbol{s})$. And thus the optimised values of these objective functions as $PP(\boldsymbol{s}^*)$ and $PP'(\boldsymbol{s'})$ respectively. We need to show that $PP(\boldsymbol{s}^*) > PP'(\boldsymbol{s'})$.

Recall that $PP(\boldsymbol{s}) = \sum_{n\in\mathcal{N}}\sum_{n'\in\mathcal{N}}(g\left(t_n, t_{n'}\right)W_1\left(s_n, s_{n'}\right) + g\left(t_{n'}, t_n\right)W_2\left(s_n, s_{n'}\right))$. Now take an arbitrary candidate pair $s_1, s_2 \in \mathcal{N}$. If $t_1 = t_2$ then reallocations between these two candidates don't affect the value of the objective function, and so can be ignored. Thus take $t_1 > t_2$. From this, and assumption 1, it follows that $g(t_1, t_2) > g(t_2, t_1)$.

Now, we will evaluate the objective functions of both programs at the optimal bundle from the less discriminating program. That is, we will evaluate $PP(\boldsymbol{s'})$ and $PP'(\boldsymbol{s'})$.

We claim that $PP(\boldsymbol{s'}) - PP'(\boldsymbol{s'}) > 0$. That is, holding constant the optimal regime from program W_1' and swapping to the more discriminating electorate W_1, the planner does strictly better. To

see that, note by the discussion above, for every candidate pair, the difference is

$$\begin{aligned}
PP(\boldsymbol{s}') - PP'(\boldsymbol{s}') &= \sum_{n_1 \in \mathcal{N}} \sum_{n_2 \in \mathcal{N} \setminus \{n_1\}} \left(g\left(t_1, t_2\right)\left(W_1\left(s_1', s_2'\right) - W_1'\left(s_1', s_2'\right)\right) + g\left(t_2, t_1\right)\left(W_2\left(s_1', s_2'\right) - W_2'\left(s_1', s_2'\right)\right)\right) \\
&= \sum_{n_1 \in \mathcal{N}} \sum_{n_2 \in \mathcal{N} \setminus \{n_1\}} \left(g\left(t_1, t_2\right)\left(W_1\left(s_1', s_2'\right) - W_1'\left(s_1', s_2'\right)\right) + g\left(t_2, t_1\right)\left(W_1'\left(s_1', s_2'\right) - W_1\left(s_1', s_2'\right)\right)\right) \\
&= \sum_{n_1 \in \mathcal{N}} \sum_{n_2 \in \mathcal{N} \setminus \{n_1\}} \left[g\left(t_1, t_2\right) - g\left(t_2, t_1\right)\right]\left(W_1\left(s_1', s_2'\right) - W_1'\left(s_1', s_2'\right)\right) \\
&> 0,
\end{aligned}$$

where we use the fact that $W_1' = 1 - W_2'$, and the final inequality follows from proposition 1 which shows that $t_1 > t_2$ implies $s_1' > s_2'$, and so the fact that since W_1 is more discriminating than W_1' means the second set of parentheses is positive. The positivity of the first set of parenthesis has already been established. This inequality is true for every pair of players, and thus the inequality holds when we sum over all the possible pairs of players.

Finally, we note that simply by the definition of the optimum, we have $PP(\boldsymbol{s^*}) \geq PP(\boldsymbol{s'})$ and so we have the inequality

$$PP(\boldsymbol{s}^*) - PP'(\boldsymbol{s}') \geq PP(\boldsymbol{s}') - PP'(\boldsymbol{s}') > 0$$

from which the proposition directly follows.

■

Proof of Proposition 5

We must first prove a preliminary lemma that will be used in the proof of the main proposition.

Lemma 3. *Given a score s_2, a win function W' that satisfies assumption 2, and any $\epsilon, \delta > 0$, there exists another win function W also satisfying assumption 2, and with $W \succ_D W'$, with $\min_{s \in (s_2+\delta, \bar{s})} W(s, s_2) > 1 - \epsilon$.*

Proof. By construction, we show that a win function of the form

$$W(s_1, s_2) = \frac{(s_1+1)^p + 1}{(s_1+1)^p + (s_2+1)^p + 2}$$

suffices for sufficiently high p. We index these by p and suppress the arguments to just write W_p.

First, the second part of the condition. By construction W_p is increasing in it's first argument. Thus it's minimum value on $(s_2 + \delta, \bar{s})$ is precisely at $s_2 + \delta$. Such a value is given by

$$\frac{(\delta + s_2 + 1)^p + 1}{(\delta + s_2 + 1)^p + (s_2+1)^p + 2} = 1 - \frac{(s_2+1)^p + 1}{(\delta + s_2 + 1)^p + (s_2+1)^p + 2}$$

which goes to 1 as $p \to \infty$. Label the p that makes this value greater than $1 - \epsilon$ by p_ϵ.

It remains to show that by choosing a p large enough we can make $W_p \succ_D W'$. To do this we must make sure, for all $s_1 > s_2$, that $W_p(s_1, s_2) > W'(s_1, s_2)$. The argument for when $s_1 < s_2$ is analagous and omitted. Now, since $W'(\cdot, s_2)$ is continuous and continuously differentiable, it and it's derivative achieve a maximum on $[s_2, \bar{s}]$. Denote it's maximum value by M and it's maximum slope value by M'.

Consider the line through $(s_2, 1/2)$ with slope M'. Denote the point where this line achieves height 1 by $(s_2 + \delta_M, 1)$. Now run the preceeding argument with ϵ such that $1 - \epsilon = M$ and $\delta = \delta_M$.

The p that results, which we call p_M, makes sure that W_p achieves the maximum height of the win function W' (which is M) at the point where W' might have acheived that maximum height, had it grown at it's maximum slope constantly from $1/2$ onwards. This suffices to show W_{p_M} is higher than W', fixing s_2, for all $s_1 > s_2$. A similar argument can be run to make sure it is below W' for all $s_1 < s_2$. Take the maximum of that p, p_ϵ and p_M and such a W_p will suffice to complete this proof.

■

Proof of Proposition 5.

Proof. Lemma 2 establishes the order in which new candidates will be given non-zero scores: precisely in descending order of their true type, as decided by the planner.

Thus we establish first the following: for each candidate type, which we generically label with their true type t_m, $m \in \{1, 2, \ldots, n\}$, there exists a sufficiently discriminate election W_m so that for all elections W more discriminate than W_m, i.e. $W \succ_D W_m$, the candidate t_m optimally gets a non-zero score.

To see this, for it to be optimal to give candidate t_m a score of zero, this must dominate a transfer of small $\epsilon > 0$ from one of the other candidates to t_m. The benefits of such a transfer are candidate t_m beating all weaker candidates with probability $W(\epsilon, 0) > \frac{1}{2}$ instead of such elections being a coin flip. The benefits are thus given exactly by

$$B = \sum_{i \in \mathcal{N}_m^\succ} \left(W(\epsilon, 0) - \frac{1}{2} \right) (g(t_m, t_i) - g(t_i, t_m)) > 0.$$

As the electorates gets progressively more discriminate, $W(\epsilon, 0)$ increases to it's limit $\overline{W} > 1/2$, (which uniquely exists, as the sequence is bounded by 1 and monotone). Thus as electorates get progressively more discriminate, the benefits to this ϵ transfer get arbitrarily close to

$$B \to \overline{B} = \sum_{i \in \mathcal{N}_m^\succ} \left(\overline{W} - \frac{1}{2} \right) (g(t_m, t_i) - g(t_i, t_m)) > 0.$$

As to the costs, the ϵ transfer must come from somewhere, and so the costs have a (non-informative) upper bound over all possible ways we can take ϵ away from the preceding $m-1$ candidate types that still order the scores as per propositions 1 and 2. When we remove this ϵ, these strong candidates beat weaker candidates with potentially a slightly smaller probability. But in particular, we work with an upper bound on costs of this transfer. The upper bound in question is too assume that in every possible pairwise election, the ϵ in question is removed in the worst possible way for the planner: from the player with a higher true type. This is not a valid allocation, since it would be assuming players have different scores in different pairings, and the overall budget constraint is thus meaningless. But it is certainly an upper bound of the best way to remove ϵ from the first $m-1$ players.

For each pairwise possible election featuring the top $m-1$ candidates, where we denote the planner's payoff in an election between players i and j by

$$E_{i,j} = \left(g\left(t_i, t_j\right) W\left(s_i, s_j\right) + g\left(t_j, t_i\right) \left(1 - W\left(s_i, s_j\right)\right) \right.$$

There are $n(m-1) \equiv |E|$ such elections. For each election, the planner's new payoff, as a function of the electorate W, is given by

$$E_{i,j}^\epsilon = \left(g\left(t_i, t_j\right) W\left(s_i - \epsilon s_j\right) + g\left(t_j, t_i\right) \left(1 - W\left(s_i - \epsilon, s_j\right)\right) \right.$$

Thus the planner's loss on $E_{i,j}$ after these reallocations is given by

$$\Delta E_{i,j} = (g(t_i, t_j) - g(t_j, t_i)) \left(W\left(s_i, s_j\right) - W\left(s_i - \epsilon, s_j\right)\right).$$

By the lemma preceding, we can choose a sufficiently discriminating W to make the difference $W\left(s_i, s_j\right) - W\left(s_i - \epsilon, s_j\right)$ arbitrarily close to zero. We can do this for all the other (finite) i and j in question.

Thus with a sufficiently discriminating election, we can constrain this upper bounds on the costs of the ϵ tranfer to be as close to zero as desired. In particular, we can constrain it to be below the asymptotic benefit $\bar{B}$.

This shows that for a sufficiently discriminating election, this ϵ transfer will be improving, and thus that player m will optimally be given a non-zero score. This proves the first part of the proposition.

As to the second part, note the first part is true for all players. And in combination with Lemma 2, which defined the order in which it is optimal to change a player's score from zero to non-zero, we can see that the cutoffs for the various players to receive a non-zero score are weakly increasing in their true type and so we have $t_1 \leq t_2 \leq \ldots t_n = \infty$ noting that the player with the lowest true type always received zero points.

But the argument above shows that whenever giving a player a non-zero score is optimal for W_p, then it is also optimal for any higher p. This completes the proof of the second part.

■

Proof of Proposition 6.

Proof. To begin, we show that if we begin at the optimised scores for a given α, and then increase α slightly, then any reallocation of ϵ score to player 1 at the expense of any single one of the other players (who get non-zero scores) strictly improves payoffs. To see this write player one's score as $\bar{s} - s_2 - \cdots - s_m$ and the rest of the scores simply as $s_2, s_3, \ldots, s_m$. Thus, in this form, when we evaluate $\frac{\partial}{\partial s_i}$ for $i = 2, \ldots, m$, we are evaluating the slope in the direction of transferring a small amount of score away from player one and to player i. In particular we will evaluate $\frac{\partial}{\partial \alpha} \frac{\partial PP(\boldsymbol{s})}{\partial s_i}$ to see how this slope changes with α, recalling that a small increase in α represents planner preferences weighted a little more to the ordinal and a little less to the cardinal.

So we wish to evaluate $\frac{\partial}{\partial \alpha} \frac{\partial PP(\boldsymbol{s})}{\partial s_i}$. Consider how this re-allocation between players s_1 and s_i will affect both player's in the chance either of them fights an election against candidate $j \neq 1, i$. Candidate i will have a slightly higher change of beating j, and candidate 1 slightly lower. Formally, the terms in $\frac{\partial}{\partial \alpha} \frac{\partial PP(\boldsymbol{s})}{\partial s_i}$ that depend on s_j can be written in the following way. We use PP_j to denote the part of $PP(\boldsymbol{s})$ that depends on s_j. First in the case that $t_1 > t_j > t_i$ and thus $s_1 > s_j > s_i$ in equilibrium:

$$\begin{aligned}
\frac{\partial PP_j(\boldsymbol{s})}{\partial s_i} &= W'(s_i, s_j) \left[\alpha \left(\mathbb{1}[t_i > t_j] - \mathbb{1}[t_j > t_i]\right) + (1-\alpha)\left(t_i - t_j\right)\right] \\
&\quad - W'(1 - s_2 - \ldots s_m, s_j) \left[\alpha \left(\mathbb{1}[t_1 > t_j] - \mathbb{1}[t_j > t_1]\right) + (1-\alpha)\left(t_1 - t_j\right)\right] \\
&= W'(s_i, s_j) \left[-\alpha + (1-\alpha)\left(t_i - t_j\right)\right] \\
&\quad - W'(1 - s_2 - \ldots s_m, s_j) \left[\alpha + (1-\alpha)\left(t_1 - t_j\right)\right],
\end{aligned}$$

where we have only used the fact that $t_1 > t_j > t_i$. Now we differentiate with respect to α to calculate

$$\begin{aligned}\frac{\partial}{\partial\alpha}\frac{\partial PP_j(\boldsymbol{s})}{\partial s_i} &= W'(s_i, s_j)\left[(t_j - t_i) - 1\right] \\ &\quad + W'(1 - s_2 - \ldots s_m, s_j)\left[(t_1 - t_j) - 1\right] \\ &\geq 0,\end{aligned}$$

where the inequality follows from the assumption that $W(\cdot,\cdot)$ is always increasing in it's first argument and that $\|t_i - t_j\| \geq 1$.

Similarly with respect to the relative gains and losses when players 1 or i compete against player j when $t_1 > t_i > t_j$ a similar calculation yields

$$\begin{aligned}\frac{\partial}{\partial\alpha}\frac{\partial PP_j(\boldsymbol{s})}{\partial s_i} &= -W'(s_i, s_j)\left[(t_i - t_j) - 1\right] \\ &\quad + W'(1 - s_2 - \ldots s_m, s_j)\left[(t_1 - t_j) - 1\right] \\ &\geq 0,\end{aligned}$$

where now the inequality follows by assumption 4 which gives $W'(s_1, s_j) > W'(s_i, s_j)$ for $t_1 > t_i > t_j$ and thus in equilibrium $s_1 > s_i > s_j$.

Finally consider the election in which the two competing players turn out to be exactly players 1 and i. Similar working yields

$$\begin{aligned}\frac{\partial}{\partial\alpha}\frac{\partial PP_i(\boldsymbol{s})}{\partial s_i} &= \left(W'(s_i, s_j) + W'(1 - s_2 - \ldots s_m, s_j)\right)\left[(t_1 - t_j) - 1\right] \\ &\geq 0,\end{aligned}$$

but with the equality holding strictly for at least one such player i. By construction, since the derivative with respect to this reallocation is positive for each possible opponent the players with these reallocated scores might face, and for the election in which they compete against each other, it follows that the overall effect of the reallocation is strictly positive. That is,

$$\frac{\partial}{\partial\alpha}\frac{\partial PP(\boldsymbol{s})}{\partial s_i} = \sum_{j\neq i}\frac{\partial}{\partial\alpha}\frac{\partial PP_j(\boldsymbol{s})}{\partial s_i} + \frac{\partial}{\partial\alpha}\frac{\partial PP_i(\boldsymbol{s})}{\partial s_i} > 0.$$

Note that we have omitted the case where $t_i = t_j$ or $t_j = t_1$. Whenever this is the case, the relevant terms $\mathbb{1}[t_i > t_j] - \mathbb{1}[t_j - t_i]$ and $t_i - t_j$ will both be zero, thus eliminating the dependance on α. Hence $\frac{\partial}{\partial\alpha}\frac{\partial PP_j(\boldsymbol{s})}{\partial s_i} = 0$ here. This still satisfies the greater than or equal to requirement above, and since there must exist one player that has a different type, we will get the strict inequality from (at least) the election in which 1 and i directly compete. So the above is still true.

So we have shown that if we reallocate a small amount of effort from player 1 to player i, the payoff to this is strictly increasing in α. This is true for all player's $i = 2, \ldots, m$.

Next, we show that, when we write scores as $s_1, s_2, \ldots, s_{m-1}, 1 - s_1 - \cdots - s_{m-1}$, then any reallocation away from player m (who has the lowest score in equilibrium) to any other player $i = 1, \ldots, m-1$ has payoff decreasing in α. That is, where we use $\frac{\partial}{\partial s_i}$ to now mean this transfer from s_m to s_i, we claim that $\frac{\partial}{\partial\alpha}\frac{\partial PP(\boldsymbol{s})}{\partial s_i} < 0$. The arguments for this are largely identical to the above and thus are omitted for brevity.

Now, returning to player 1, by the definition of an optimum, when we evaluate $PP(\boldsymbol{s})$ at the

optimum $\boldsymbol{s^*(\alpha)}$ (where we make explicit this is the optimum at α) we must have, in terms of these reallocations, that $\frac{\partial PP(\boldsymbol{s^*(\alpha)})}{\partial s_i} = 0$ for all $i = 2, \ldots, m$. But the above means that if we increase α very slightly to α' then this slope $\frac{\partial PP(\boldsymbol{s^*(\alpha)})}{\partial s_i} = 0$ strictly increases to be positive. That is, when α increases very slightly, but we remain at the old optimum $bms^*(\alpha)$, then in the direction of decreasing s_1 slightly at the expense of increasing any other s_i slightly, this strictly improves the objective function value.

Now,assume that the objective function is quasi-concave so that the optimum is always unique. We show that after α has increased, the maximum that at α was equal to $\boldsymbol{s^*(\alpha)}$ is close to the new optimum at α', written as $\boldsymbol{s^*(\alpha')}$. That is, for $\alpha' - \alpha > 0$ sufficiently small, $\|\boldsymbol{s^*(\alpha')} - \boldsymbol{s^*(\alpha')}\|$ is arbitrarily small.

To see this, recall our original optimisation problem with this particular α indexed g function, and the equivalent problem implicitly defined above:

$$\begin{aligned}
&\max_{s_n, \forall n \in \mathcal{N}} \sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} \left(g_\alpha \left(t_n, t_{n'} \right) W_1 \left(s_n, s_{n'} \right) + g_\alpha \left(t_{n'}, t_n \right) W_2 \left(s_n, s_{n'} \right) \right) \\
&\text{such that} \\
&s_n \geq 0 \qquad \text{for all } n \in \mathcal{N}, \\
&\sum_{n \in \mathcal{N}} s_n = \overline{s},
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&\max_{s_n, \forall n \in \mathcal{N}} \sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} \left(g_\alpha \left(t_n, t_{n'} \right) W_1 \left(s_n, s_{n'} \right) + g_\alpha \left(t_{n'}, t_n \right) W_2 \left(s_n, s_{n'} \right) \right) \\
&\text{with } s_1 = \overline{s} - s_2 - s_3 - \cdots - s_N \\
&\text{such that} \\
&s_n \geq 0 \qquad \text{for all } n \in \mathcal{N}.
\end{aligned} \tag{A.3}$$

Clearly the solution set is coincident, since these are simple reformulations of the same program. Studying the former program, (A.2), we consider this with the function being optimised as $PP : \mathbb{R}^N \times [0,1] \to \mathbb{R}$ which is continuous by assumption. Next, the constraint set correspondence $C : [0,1] \rightrightarrows \mathbb{R}^n$ is constant over all α, the output is the simplex which is closed, bounded and thus compact, the correspondence is constant and thus trivially upper and lower hemicontinuous and thus continuous.

Thus the conditions for Berge's Maximum Theorem are met and we conclude that the arg max correspondence $C^*(\alpha) = \arg\max_{\boldsymbol{s}} PP(\boldsymbol{s}; \alpha)$ is upper hemicontinuous. But under our assumption of quasi-concavity, the arg-max correspondence is in fact a function of α, in which case upper hemicontinuity reduces to normal continuity of this arg-max function. Thus, our claim above that by moving α to α' that is sufficiently close, we can keep the maxima arbitrarily close follows directly from the definition of continuity. But the set of maxima are identical between programs (A.2) and (A.3) and so the conclusion about their continuity as α changes carries over.

So we know that $\boldsymbol{s^*(\alpha)}$ and $\boldsymbol{s^*(\alpha')}$ are arbitrarily close. In fact, we can choose α' sufficiently close α to ensure that to go between $\boldsymbol{s^*(\alpha')}$ and $\boldsymbol{s^*(\alpha')}$ is monotonic [(7)] Recall that we have already shown that in all of the $m-1$ directions in which we can decrease s_1 and remain within

[(7)]This is the definition of a maximum" that when we get close enough it is only uphill to the top of the mountain, so to speak.

the constraints by increasing $s_i, i = 2, 3, \ldots, m$ accordingly, the slope at $\boldsymbol{s^*(\alpha)}$ is upward. Thus, it follows that for sufficiently close α', the new value of s_1 optimised at α' must be lower than the old s_1 value optimised at α.

Thus we have shown that $\frac{\partial s_1^*(\alpha)}{\partial \alpha} < 0$. That is, by making the planner's preferences slightly more ordinal, we decrease the score of the strongest player in the optimal regime. Conversely, when the preferences become slightly more optimal, this optimal score allocated to the strongest player increases in α.

An identical but reversed argument shows that $\frac{\partial s_m^*(\alpha)}{\partial \alpha} > 0$. That is, when the planner's preferences become more cardinal, the lowest player receiving a non-zero score receives an even lower non-zero score. This proves the first part of the proposition.

Finally, when we do not assume quasi-concavity but simply non-flatness, we now possibly have multiple, yet isolated maxima at a given α. But because of their isolation, we can find a compact domain around them in which we solve (A.2) or (A.3) restricted to this compact domain, within which the maxima will be unique. Run the above argument on each of these domain's to see that each of these (at least) local maxima will have s_1^* coordinate decreasing in α and s_m^* increasing in α. But it is possible that the multiple equilibria at α reduce to fewer equilibria at α'. Each locally decreases in s_1^*, but if we began at the maxima with lowest $s_1^*(\alpha)$ and increase to α', there is still a local maxima near $s^*(\alpha)$ with $s_1^*(\alpha') < s_1^*(\alpha)$, but this may no longer be the global maximum, which would therefore have a higher s_1^* value by the assumption that we started with the maxima with lowest $s^*(\alpha)$ value, didn't move far, and the maxima are isolated.

But then it is clear to see that if we begin at the maxima for α that has the highest $s_1^*(\alpha)$ value then when we increase α the new optimised value $s_1^*(\alpha')$ can only decrease, either through the local argument from above, or dicontinuously downward since we assumed we began at the highest. An analagous argument for s_m^* proves the second part of the proposition.

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■

Proof of Proposition 7.

Proof. Fix any β. Suppose for a contradiction that there exist two players $n, n' \in \mathcal{N}$ with $t_n > t_{n'}$ yet in the optimum with the high score s_n going to the low player $t_{n'}$ and the low score $s_{n'}$ going to the high player t_n. We will show that the planner's payoff is strictly improved by swapping these scores, such that the high player t_n receives the high score s_n and the low player receives the low score.

To that end, first consider all the election combinations in periods 1 and 2 where players t_n and $t_{n'}$ never compete. For simplicity we write $\Delta g_{i,j} = g_1(t_i, t_j) - g_1(t_j, t_i)$ for any players $i, j \in \mathcal{N}$. We can write out the part of the payoff that depends on s_n and $s_{n'}$ when the high player gets the high score and the low player gets the low score (the "deviation" payoff) PP_{dev} as follows:

$$\begin{aligned} PP_{dev} = \frac{1}{\mathcal{N}} \sum_{j \in \mathcal{N}\setminus\{n\}} [W_1(s_n, s_j)\Delta g_{n,j}] + P_n^{Inc}(\boldsymbol{s}_{dev}) \sum_{j \in \mathcal{N}\setminus\{n\}} [(W_1(s_n, s_j) + \beta)\Delta g_{n,j}] \\ + P_j^{Inc}(\boldsymbol{s}_{dev}) \sum_{j \in \mathcal{N}\setminus\{n\}} [(W_1(s_j, s_n) + \beta)\Delta g_{j,n}] + \frac{1}{\mathcal{N}} \sum_{j \in \mathcal{N}\setminus\{n'\}} \left[W_1(s_{n'}, s_j)\Delta g_{n',j}\right] \\ + P_{n'}^{Inc}(\boldsymbol{s}_{dev}) \sum_{j \in \mathcal{N}\setminus\{n'\}} \left[(W_1(s_{n'}, s_j) + \beta)\Delta g_{n',j}\right] + P_j^{Inc}(\boldsymbol{s}_{dev}) \sum_{j \in \mathcal{N}\setminus\{n'\}} \left[(W_1(s_j, s_{n'}) + \beta)\Delta g_{j,n'}\right], \end{aligned}$$

where the first term is the first period payoff if player n is involved, the second term the second period payoff if player n is the incumbent, the third term the second period payoff if player n is the challenger, and the fourth, fifth and sixth terms analogous for player n'

We can then tediously write out the analogous "original" expression when player n got the low score of $s_{n'}$ and low player n' the high score of s_n:

$$\begin{aligned}
PP_{org} = &\frac{1}{\mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{n\}} [W_1(s_{n'}, s_j)\Delta g_{n,j}] + P_n^{Inc}(\boldsymbol{s}_{org}) \sum_{j \in \mathcal{N} \setminus \{n\}} [(W_1(s_{n'}, s_j) + \beta)\Delta g_{n,j}] \\
&+ P_j^{Inc}(\boldsymbol{s}_{org}) \sum_{j \in \mathcal{N} \setminus \{n\}} [(W_1(s_j, s_{n'}) + \beta)\Delta g_{j,n}] + \frac{1}{\mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{n'\}} \left[W_1(s_n, s_j)\Delta g_{n',j}\right] \\
&+ P_{n'}^{Inc}(\boldsymbol{s}_{org}) \sum_{j \in \mathcal{N} \setminus \{n'\}} \left[(W_1(s_n, s_j) + \beta)\Delta g_{n',j}\right] + P_j^{Inc}(\boldsymbol{s}_{org}) \sum_{j \in \mathcal{N} \setminus \{n'\}} \left[(W_1(s_j, s_n) + \beta)\Delta g_{j,n'}\right],
\end{aligned}$$

Now we subtract these expressions, noting tha t$P_n^{inc}(\boldsymbol{s}_{org}) = P_{n'}^{inc}(\boldsymbol{s}_{dev})$ since between the original and deviating allocations we have just swapped s_n and $s_{n'}$, and similarly $P_{n'}^{inc}(\boldsymbol{s}_{org}) = P_n^{inc}(\boldsymbol{s}_{dev})$, and we label the result ΔPP, to get

$$\begin{aligned}
\Delta PP = &\frac{1}{\mathcal{N}} \sum_{j \in \mathcal{N} \setminus \{n\}} \left[(W_1(s_n, s_j) - W_1(s_{n'}, s_j))\left(\Delta g_{n,j} - \Delta g_{n',j}\right)\right] \\
&+ \sum_{j \in \mathcal{N} \setminus \{n\}} \left[\left(\Delta g_{n,j} - \Delta g_{n',j}\right)\left((W_1(s_n, s_j) + \beta)P_n^{Inc}(\boldsymbol{s}_{org}) - (W_1(s_{n'}, s_j) + \beta)P_{n'}^{Inc}(\boldsymbol{s}_{org})\right)\right] \\
&+ \sum_{j \in \mathcal{N} \setminus \{n\}} \left[\left(\Delta g_{n,j} - \Delta g_{n',j}\right) P_j^{Inc}(\boldsymbol{s}_{org})(W_1(s_n, s_j) - W_1(s_{n'}, s_j))\right].
\end{aligned}$$

Note that assumption 1 implies that the difference of difference term:

$$Deltag_{n,j} - \Delta g_{n',j} = g(t_n, t_j) - g(t_j, t_n) - g(t_{n'}, t_j) - g(t_j, t_{n'}) \geq 0$$

since $g(t_n, t_j) \geq g(t_{n'}, t_j)$ and $g(t_j, t_n) \leq g(t_j, t_{n'})$. Noting further that the probability of becoming a given player becoming an incumbent is increasing in that player's score, and by assumption 1, it can be seen that this quantity is weakly greater than zero. That is $\Delta PP \geq 0$.

Next consider the possibility of player's n and n' competing against each other in period 1, or in period 2, or both. The payoff that occurs under the deviation allocation in this head-to-head possibility is

$$\begin{aligned}
PP_{dev}^{H2H} = &\frac{1}{\mathcal{N}} \left[W_1(s_n, s_{n'})\Delta g_{n,n'}\right] + P_n^{Inc}(\boldsymbol{s}_{dev}) \left[(W_1(s_n, s_{n'}) + \beta)\Delta g_{n,n'}\right] \\
&+ \frac{1}{\mathcal{N}} \left[W_1(s_{n'}, s_n)\Delta g_{n',n}\right] + P_{n'}^{Inc}(\boldsymbol{s}_{dev}) \left[(W_1(s_{n'}, s_n) + \beta)\Delta g_{n',n}\right]
\end{aligned}$$

and similarly if we swap these scores to the original payoff profile (that we supposed was optimal

for a contradiction)

$$PP_{org}^{H2H} = \frac{1}{\mathcal{N}}\left[W_1(s_{n'}, s_n)\Delta g_{n,n'}\right] + P_{n'}^{Inc}(\boldsymbol{s}_{org})\left[(W_1(s_{n'}, s_n) + \beta)\Delta g_{n,n'}\right] + \frac{1}{\mathcal{N}}\left[W_1(s_n, s_{n'})\Delta g_{n',n}\right] + P_n^{Inc}(\boldsymbol{s}_{org})\left[(W_1(s_n, s_{n'}) + \beta)\Delta g_{n',n}\right].$$

We take the difference to get that

$$\Delta PP^{H2H} = \frac{1}{\mathcal{N}}\Delta g_{n,n'}\left[W_1(s_n, s_{n'}) - W_1(s_{n'}, s_n)\right] + \Delta g_{n,n'}(P_n^{Inc}(\boldsymbol{s}_{dev}) - P_{n'}^{Inc}(\boldsymbol{s}_{dev}))\left[(W_1(s_n, s_{n'}) + \beta) - (W_1(s_{n'}, s_n) + \beta)\right].$$

Since all these terms are strictly positive, we conclude that $\Delta PP^{H2H} > 0$. Thus we have shown that their is a strictly positive deviation away from the hypothesized allocation, and hence such an allocation cannot be optimal. This completes the proof.

■

Proof of Proposition 8.

Proof. First suppose we considered β as exogenous. For each $\beta \in [0, \overline{\beta}$ we can solve the maximisation problem. Considered as a function,

$$PP : \mathbb{R}^N \times [0, \overline{\beta}] \to \mathbb{R}$$

is continuous and differentiable by assumption. The constraint set correspondence: $C : [0, 1] \rightrightarrows \mathbb{R}^n$ is constant over all β, the output is the simplex which is closed, bounded and thus compact, the correspondence is constant and thus trivially upper and lower hemicontinuous and thus continuous.

And so, by Berge's maximum theorem, the value function $V(\beta) = \max_{s_n, \forall n \ in \mathcal{N}} PP(\boldsymbol{s}; \beta)$ is continuous. Thus there exists a maximising β, since we are maximising the continuous $V(\beta)$ over the compact domain $[0, \overline{\beta}]$.

Now fix any allocation $\boldsymbol{s}$ that satisfies proposition 7. We will show that $\frac{\partial PP(\boldsymbol{s})}{\partial \beta} > 0$. In particular this will be true at the optimising $\boldsymbol{s}$. Thus, when we increase β, we can do strictly better than before the increase of β. This will show that in particular $\frac{\partial PP(\boldsymbol{s}^*)}{\partial \beta} > 0$. Thus we just need to show $\frac{\partial PP(\boldsymbol{s})}{\partial \beta} > 0$.

To that end, recall the second-period objective function (which is the only part that depends on β) is

$$\begin{aligned}
&= \sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} P_n^{Inc}(\boldsymbol{s}) \left[g\left(t_n, t_{n'}\right)\left(W_1\left(s_n, s_{n'}\right) + \beta\right) + g\left(t_{n'}, t_n\right)\left(W_2\left(s_n, s_{n'} - \beta\right)\right)\right] \\
&+ \sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} P_n^{Inc}(\boldsymbol{s}) \left[g\left(t_n, t_{n'}\right)\left(W_1\left(s_n, s_{n'}\right) + \beta\right) + g\left(t_{n'}, t_n\right)\left(1 - W_1\left(s_{n'}, s_n - \beta\right)\right)\right] \\
&+ \sum_{n \in \mathcal{N}} \sum_{n' \in \mathcal{N} \setminus \{n\}} P_n^{Inc}(\boldsymbol{s})(W_1\left(s_n, s_{n'}\right) + \beta)\Delta g_{n,n'} + P_n^{Inc}(\boldsymbol{s}) g\left(t_{n'}, t_n\right).
\end{aligned}$$

Now we can write this as a sum over pairs (i.e. combining the possibility of n being incumbent and n' challenger, and vice versa, to get, where we suppress everything that doesn't depend on β,

$$\begin{aligned} &= \sum_{(n,n')\in\mathcal{N}^2, n\neq n'} \beta \left[P_n^{Inc}(\boldsymbol{s})\Delta g_{n,n'} + P_{n'}^{Inc}(\boldsymbol{s})\Delta g_{n',n} \right] \\ &= \sum_{(n,n')\in\mathcal{N}^2, n\neq n'} \beta\Delta g_{n,n'} \left[P_n^{Inc}(\boldsymbol{s}) - P_{n'}^{Inc}(\boldsymbol{s}) \right]. \end{aligned}$$

Thus, the derivative of $PP(\boldsymbol{s})$ with respect to β is given by

$$\frac{\partial PP(\boldsymbol{s})}{\partial \beta} = \sum_{(n,n')\in\mathcal{N}^2, n\neq n'} \Delta g_{n,n'} \left[P_n^{Inc}(\boldsymbol{s}) - P_{n'}^{Inc}(\boldsymbol{s}) \right] > 0.$$

Where the inequality follows from the fact that, at the optimum, when we have $t_n > t_{n'}$ we get both $\Delta g_{n,n'} > 0$ and $P_n^{Inc}(\boldsymbol{s}) - P_{n'}^{Inc}(\boldsymbol{s}) > 0$. This proves the claim.

■

Proof of Proposition 9.

Proof. Take an arbitrary pair of candidates $t_n, t_{n'} \in \mathcal{N}$ such that, in the optimum, we have $s_n + s_{n'} > 0$. We will prove the following, which in total prove the proposition: first, that if $t_n > t_{n'}$ then $s_n \geq s_{n'}$ and second we rule out the case of equality.

First suppose $t_n > t_{n'}$ but, for a contradiction, that $s_n < s_{n'}$ in the optimum. We will show that in fact this is dominated by swapping these scores s_n and $s_{n'}$ between players t_n and $t_{n'}$.

First consider the groups of m players that feature either player t_n or $t_{n'}$ but not both. Writing, without loss of generality (by the anonymity of g and W), player t_n or $t_{n'}$ as the first player and $\boldsymbol{t} \in \mathbb{R}^{m-1}$ as the vector of the remaining players who get scores $\boldsymbol{s}$, and then summing over the possible choices of the $m-1$ remaining players, we have that the payoff when player t_n gets s_n and $t_{n'}$ gets $s_{n'}$ is

$$\begin{aligned} PP_1(t_n \leftrightarrow s_n, t_{n'} \leftrightarrow s_{n'}) = \sum_{\boldsymbol{t}} \Bigg[& g_1(t_n, \boldsymbol{t}) W_1(s_n, \boldsymbol{s}) + \sum_{i=2}^{m} g_i(t_n, \boldsymbol{t}) W_i(s_n, \boldsymbol{s}) \\ & + g_1(t_{n'}, \boldsymbol{t}) W_1(s_{n'}, \boldsymbol{s}) + \sum_{i=2}^{m} g_i(t_{n'}, \boldsymbol{t}) W_i(s_{n'}, \boldsymbol{s}) \Bigg]. \end{aligned}$$

We can write out a similar expression for the payoff where instead player t_n gets the higher score of $s_{n'}$ and $t_{n'}$ gets the lower score of s_n. We subtract these, and get a difference in payoffs, denoted ΔPP_1, of

$$\Delta PP_1 = \sum_{\boldsymbol{t}} \Big[g_1(t_n, \boldsymbol{t}) \left(W_1(s_n, \boldsymbol{s}) - W_1(s_{n'}, \boldsymbol{s}) \right) + \sum_{i=2}^{m} g_i(t_n, \boldsymbol{t}) \left(W_i(s_n, \boldsymbol{s}) - W_i(s_{n'}, \boldsymbol{s}) \right)$$
$$+ g_1(t_{n'}, \boldsymbol{t}) \left(W_1(s_{n'}, \boldsymbol{s}) - W_1(s_n, \boldsymbol{s}) \right) + \sum_{i=2}^{m} g_i(t_{n'}, \boldsymbol{t}) \left(W_i(s_{n'}, \boldsymbol{s}) - W_i(s_n, \boldsymbol{s}) \right) \Big].$$

Factorising we have

$$\Delta PP_1 = \sum_{\boldsymbol{t}} \Big[\left(g_1(t_n, \boldsymbol{t}) - g_1(t_{n'}, \boldsymbol{t}) \right) \left(W_1(s_n, \boldsymbol{s}) - W_1(s_{n'}, \boldsymbol{s}) \right)$$
$$+ \sum_{i=2}^{m} \left(g_i(t_n, \boldsymbol{t}) - g_i(t_{n'}, \boldsymbol{t}) \right) \left(W_i(s_n, \boldsymbol{s}) - W_i(s_{n'}, \boldsymbol{s}) \right) \Big].$$

Now, for every $\boldsymbol{t}$, by assumption 1-M the first term is weakly positive, the third term weakly negative. By assumption 2-M the second term is strictly positive, the fourth term is strictly negative. From this it is easy to see that this difference is weakly positive, $\Delta PP_1 \geq 0$.

Next consider the elections that have both t_n and $t_{n'}$ as participants. Without loss of generality we put t_n as the first player, $t_{n'}$ as the second, and we redefine $\boldsymbol{t} \in \mathbb{R}^{m-2}$ as the vector of the remaining $m-2$ players, with $\boldsymbol{s}$ their scores at the supposed optimum. Summing over these possible $m-2$ players, the payoff from allocating the low s_n to the 'high' player t_n , and to player $t_{n'}$ the high score of $s_{n'}$ is

$$PP_2(t_n \leftrightarrow s_n, t_{n'} \leftrightarrow s_{n'}) = \sum_{\boldsymbol{t}} \Big[g_1(t_n, t_{n'}, \boldsymbol{t}) W_1(s_n, s_{n'}, \boldsymbol{s})$$
$$+ g_2(t_n, t_{n'}, \boldsymbol{t}) W_2(s_n, s_{n'}, \boldsymbol{s}) + \sum_{i=3}^{m} g_i(t_n, t_{n'}, \boldsymbol{t}) W_i(s_n, s_{n'}, \boldsymbol{s}) \Big].$$

Now, we can write out the payoff when we deviate to paying the high $s_{n'}$ to t_n and the low s_n to $t_{n'}$. Subtract this from the expression above to get

$$\Delta PP_2 = \sum_{\boldsymbol{t}} \Big[g_1(t_n, t_{n'}, \boldsymbol{t}) \left(W_1(s_{n'}, s_n, \boldsymbol{s}) - W_1(s_n, s_{n'}, \boldsymbol{s}) \right)$$
$$+ g_2(t_n, t_{n'}, \boldsymbol{t}) \left(W_2(s_{n'}, s_n, \boldsymbol{s}) - W_2(s_n, s_{n'}, \boldsymbol{s}) \right)$$
$$+ \sum_{i=3}^{m} g_i(t_n, t_{n'}, \boldsymbol{t}) \left(W_i(s_{n'}, s_n, \boldsymbol{s}) - W_i(s_n, s_{n'}, \boldsymbol{s}) \right) \Big].$$

Now, by the anonymity of the win function, the third line is zero. That is, for any player except the first and second, it doesn't matter if we swap the first player's score with the second, and keep all else the same. Further, again by the anonymity of the win function $W_1(s_n, s_{n'}, \boldsymbol{s}) = W_2(s_{n'}, s_n, \boldsymbol{s})$ and $W_2(s_n, s_{n'}, \boldsymbol{s}) = W_1(s_{n'}, s_n, \boldsymbol{s})$. Thus the expression above reduces to

$$\Delta PP_2 = \sum_{\boldsymbol{t}} \left[\left(g_1(t_n, t_{n'}, \boldsymbol{t}) - g_2(t_n, t_{n'}, \boldsymbol{t})\right) \left(W_1(s_{n'}, s_n, \boldsymbol{s}) - W_1(s_n, s_{n'}, \boldsymbol{s})\right) \right]$$

Since $s_{n'} > s_n$, by assumption 2-M the second bracket is strictly positive for each $\boldsymbol{t}$. By assumption 1-M, since $t_n > t_{n'}$ the first bracket is weakly positive for all $\boldsymbol{t}$, and strictly positive for some $\boldsymbol{t}$, since by assumption these two players are differentiable.

It follows that $\Delta PP_2 > 0$. Since the only way that the choice of s_n and $s_{n'}$ enter into the planner's payoff is when one or both of t_n and $t_{n'}$ end up in the election, the total change in payoff from deviating from the supposed low score to the high type and high score to the low type is $\Delta PP_1 + \Delta PP_2 > 0$ by the arguments above. Thus the deviation is profitable, the original allocation is suboptimal, and we must in fact have $s_n \geq s_{n'}$ when $t_n > t_{n'}$.

It remains to rule out the possibility that $s_n = s_{n'}$. To do so note that a necessary condition for $s_n = s_{n'}$ to be optimal is that the slope of the payoff function in the direction of moving a small amount of score from s_n to $s_{n'}$ needs to be flat. So denote $s_n + s_{n'} = S$ and thus we consider the derivative with respect to s_n. First for the terms with only one of n or n' in the election, denoted PP_1 as above, we put $s_{n'} = s_n$ to get:

$$\begin{aligned} PP_1 = \sum_{\boldsymbol{t}} \Bigg[& g_1(t_n, \boldsymbol{t}) W_1(s_n, \boldsymbol{s}) + \sum_{i=2}^{m} g_i(t_n, \boldsymbol{t}) W_i(s_n, \boldsymbol{s}) \\ & + g_1(t_{n'}, \boldsymbol{t}) W_1(S - s_n, \boldsymbol{s}) + \sum_{i=2}^{m} g_i(t_{n'}, \boldsymbol{t}) W_i(S - s_n, \boldsymbol{s}) \Bigg]. \end{aligned}$$

Differentiating, where we use superscripts j to denote differentiation with respect to the jth input variable, and then putting $s_n = S - s_n = s_{n'}$ gives

$$\frac{\partial PP_1}{\partial s_n}|_{s_n = s_{n'}} = \sum_{\boldsymbol{t}} \left[(g_1(t_n, \boldsymbol{t}) - g_1(t_{n'}, \boldsymbol{t})) W_1^{(1)}(s_n, \boldsymbol{s}) + \sum_{i=2}^{m} (g_i(t_n, \boldsymbol{t}) - g_i(t_{n'}, \boldsymbol{t})) W_i^{(1)}(s_n, \boldsymbol{s}) \geq 0 \right.$$

by reasoning virtually identical to the discussion of ΔPP_1 above.

We proceed to elections in which both t_n and $t_{n'}$ are participating, with t_n listed first and $t_{n'}$ second without a loss of generality. The payoff in terms of reallocations just between s_n and $s_{n'} = S - s_n$ is given by

$$\begin{aligned} PP_2 = \sum_{\boldsymbol{t}} \Bigg[& g_1(t_n, t_{n'}, \boldsymbol{t}) W_1(s_n, S - s_n, \boldsymbol{s}) \\ & + g_2(t_n, t_{n'}, \boldsymbol{t}) W_2(s_n, S - s_n, \boldsymbol{s}) + \sum_{i=3}^{m} g_i(t_n, t_{n'}, \boldsymbol{t}) W_i(s_n, S - s_n, \boldsymbol{s}) \Bigg]. \end{aligned}$$

We differientiate with respect to s_n, then setting $s_n = S - s_n$, and finally note that by anonymity $W_1^{(1)}(s_n, s_n, \boldsymbol{s}) = W_2^{(2)}(s_n, s_n, \boldsymbol{s}), W_1^{(2)}(s_n, s_n, \boldsymbol{s}) = W_1^{(2)}(s_n, s_n, \boldsymbol{s})$ and that $W_i^{(1)}(s_n, s_n, \boldsymbol{s}) = W_i^{(2)}(s_n, s_n, \boldsymbol{s})$

for all $i \geq 3$. As a result, we recover that

$$\frac{\partial PP_2}{\partial s_n}|_{s_n=s_{n'}} = \sum_{\boldsymbol{t}} \Big[(g_1(t_n, t_{n'}, \boldsymbol{t}) - g_2(t_n, t_{n'}, \boldsymbol{t}))W_1^{(1)}(s_n, s_n, \boldsymbol{s}) + (g_2(t_n, t_{n'}, \boldsymbol{t}) - (g_1(t_n, t_{n'}, \boldsymbol{t}))W_2^{(1)}(s_n, s_n, \boldsymbol{s}) \Big].$$

Similarly to above, we note that $W_1^{(1)} > 0$ and $W_2^{(1)} < 0$ by assumption 2-M. Then by assumption 1-M the difference $g_1 - g_2$ is weakly positive, but acheiving strict positivity for at least one summand $\boldsymbol{t}$, and conversely $g_2 - g_1$ is weakly negative but at least once strictly so. All of this together implies that $\frac{\partial PP_2}{\partial s_n}|_{s_n=s_{n'}} > 0$, and hence that $\frac{\partial PP}{\partial s_n}|_{s_n=s_{n'}} = \frac{\partial PP_1}{\partial s_n}|_{s_n=s_{n'}} + \frac{\partial PP_2}{\partial s_n}|_{s_n=s_{n'}} > 0$. This contradicts the necessary condition for an optimum and thus we conclude we cannot in fact have $s_n = s_{n'}$ when $t_n > t_{n'}$. This completes the proof.

■

Proof of Proposition 10.

Proof. As above. take an arbitrary pair of candidates $t_n, t_{n'} \in \mathcal{N}$ such that, in the optimum, we have $s_n + s_{n'} > 0$. We will prove the following, which in total prove the proposition: first, that if $t_n > t_{n'}$ then $s_n \geq s_{n'}$ and second we rule out the case of equality.

The first part of the proof follows identically to the proof of proposition 9 after noting that by the anonymity property, swapping the scores s_n and $s_{n'}$ between players t_n and $t_{n'}$ doesn't change the win function. Thus this shows we have $s_n \geq s_{n'}$.

Now, to rule out the case of equality, using the same notation as the proof of proposition 9, we consider elections featuring just one of n and n' but not both and note that the direct effect of increasing s_n and decreasing $s_{n'}$ all other scores held equal, evaluates at $s_n = s_{n'}$ is given by

$$\frac{\partial PP_1}{\partial s_n}|_{s_n=s_{n'}} = \sum_{i \in \mathcal{N} \setminus \{n,n'\}} (g(t_n, t_i) - g_1(t_{n'}, t_i)W_{\boldsymbol{s}}^{(1)}(s_n, s_i) - (g(t_i, t_n) - g(t_i, t_{n'}))W_{\boldsymbol{s}}^{(1)}(s_n, s_i) \geq 0$$

by assumption 1. There is also the indirect effect that comes from the slight change in $\boldsymbol{s}$, but by assumption this is dominated by the direct effect and as such $\frac{\partial PP_1}{\partial s_n}|_{s_n=s_{n'}} \geq 0$.

Now consider the election in which both candidates n and n' participate.

The direct effect is given by

$$\begin{aligned}\frac{\partial PP_2}{\partial s_n}|_{s_n=s_{n'}} &= (g(t_n, t_{n'}) - g(t_{n'}, t_n))W_{\boldsymbol{s}}^{(1)}(s_n, s_n) \\ &\quad - (g(t_{n'}, t_n) - g(t_n, t_{n'}))W_{\boldsymbol{s}}^{(1)}(s_n, s_n) \\ &= 2(g(t_n, t_{n'}) - g(t_{n'}, t_n))W_{\boldsymbol{s}}^{(1)}(s_n, s_n) \\ &> 0\end{aligned}$$

for the standard reasons.

But then we note that there is no indirect effect, since $W_{bms}(s_n, s_n) = W_{bms'}(s_n, s_n) = \frac{1}{2}$ for all $\boldsymbol{s}, \boldsymbol{s}'$. It follows then that the overall slope of the objective function is positive in the direction

of raising s_n a little at the expense of $s_{n'}$ and as such their equality could not have been optimal. This completes the proof. ■

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