A Model of Intermediation, Money, Interest, and Prices*

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Saki Bigio† and Yuliy Sannikov‡

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Abstract

A model integrates a modern implementation of monetary policy (MP) into an incomplete-markets monetary economy. Policy sets corridor rates and conducts open-market operations and fiscal transfers. These tools grant independent control over credit spreads and inflation. We study the implementation of spreads and inflation via different MP instruments. Through the influence on spreads, MP affects the evolution of real credit, interests, output, and the wealth distribution (both in the long and short-run). We decompose effects through different transmission channels. We study the optimal spread management and find that the active management of spreads is a desirable target.

Keywords: Monetary Economics, Monetary Policy, Credit Channel.

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†Department of Economics, University of California, Los Angeles and NBER, email sbigio@econ.ucla.edu

‡Stanford Business School and NBER, email sannikov@gmail.com

1 Introduction

In modern economies, monetary policy (MP) operates through the provision of reserves and a corridor of policy rates.[1] A popular view among academics is that these tools implement a desired nominal interest rate, which grants control over inflation, and this is ultimately what matters for MP (Woodford, 1998). A bank-centric view has it that these tools influence bank credit and spreads, and thus, impact real activity through their influence on the financial system (Bernanke and Blinder, 1988, 1992). Although this view is widely held by practitioners, and has a strong empirical support (Kashyap and Stein, 2000; Drechsler et al., 2017), its theoretical foundations are still being laid out. This paper presents an incomplete-markets economy where credit is intermediated by banks that hold reserves to manage liquidity. MP is implemented through a corridor system and open-market operations (OMO). The paper articulates how these tools affect credit, monetary balances, borrowing and lending rates, inflation, and output, in the context of an incomplete-markets economy.

In the environment, operating a corridor system grants MP enough tools to implement inflation and manage credit spreads as independent targets. Whereas the control over inflation relates to well-traveled transmission mechanisms, the control over credit spreads is a notion of the credit channel.[2] This feature allows for the positive analysis of the credit channel within an incomplete-markets economy. Studying the credit channel in an incomplete-markets economy is important. During booms, policy circles debate whether *MP is sowing the seeds of crises,* but during busts, that it is akin to *pushing on a string.* Should MP tighten credit during booms, but stimulate it during busts is at the core of historical and contemporary policy debates (Bagehot, 1873; Stein, 2018). The normative insight of the paper is that the active management of credit spreads is important to reduce productive inefficiencies and mitigate the extent of crises, but this power comes at the expense repressing credit, which hurts insurance. This message is particularly pertinent now, that countries are considering the replacement of corridor systems for floor systems. For this paper, this change means surrendering an important policy tool.

We build this case through the study of a canonical continuous-time incomplete-markets environment. This is an endowment economy where households face idiosyncratic risk, as in Huggett (1993). To speak about productive efficiency though, we let households choose between a safe endowment process and a risky, but more profitable process. Here, it is never efficient to chose the safe endowment, because risk is only idiosyncratic, but the market incompleteness

[1] A corridor system is a rate on discount-window loans and interest on reserves. The discount rate is the rate at which a central bank lends reserves to banks that are below their reserve requirement. The rate on reserves is the rate at which banks are remunerated by holding reserve balances at the central bank.

[2] A narrative description of different transmission channels of MP is found in Ben S. Bernanke (1995)'s "Inside the Black box". Kashyap and Stein (2000) presented evidence on the credit channel by exploiting differences in the cross-section of liquidity ratios across banks. Bindseil (2014) describes the modern implementation of MP through banks across countries.

can lead to that inefficient choice. Although stylized, the mechanism captures the idea that financial stress leads to inefficient choices. Mechanically, it produces a map from the fraction of debt-constrained agents to output.

Credit is nominal and intermediated by a fringe of competitive banks. In addition to deposits and loans, banks hold reserves to manage liquidity. The power to influence spreads stems from an institutional feature. Whereas loans are permanently held by the issuer bank, deposits circulate. Thus, banks use reserves to settle deposit transfers. A potential shortage of reserves by some banks, opens the door for interbank credit. The interbank market, however, operates with matching frictions (á la Ashcraft and Duffie, 2007; Afonso and Lagos, 2015). As a result, not all reserves deficits can be tapped with private credit and some are forced to borrow at a penalty rate set by MP. The overall quantity of reserves and the corridor rates set by MP, translates into an intermediation cost. Ultimately, banks are a pass-through from a policy corridor spread to actual credit spreads.

A similar implementation of the credit channel already appears in work by Bianchi and Bigio (2017a), and in related works by Piazzesi and Schneider (2016); De Fiore et al. (2018); Chen et al. (2017); Drechsler et al. (2017). Here, bank decisions are simplified, and the pass-through from policy rates to spreads is immediate. The emphasis is not on the banking sector, but on the effects within an incomplete-markets economy. The latter delivers a broad set of implications for changes in credit spreads. Notably, the real effects of MP are driven by the precautionary motive. Because MP indirectly affects the distribution of wealth, it influences the mass of agents that choose the inefficient endowment, and this impacts on productive efficiency. Because this mechanism is independent of inflation, the model connects transparently with other transmission mechanisms.

The paper first delves into the details of implementation. It presents closed-form expressions for nominal deposits and loans interests. These nominal rates carry different premia over the rate on reserves. The difference of these premia is a real credit spread which, in turn, is expressed as a function of a liquidity ratio and the policy corridor spread set by MP. The implementation is explicit about a reserve satiation regime (a floor system), and a zero-lower bound on deposit rates (DZLB). Away from either regime, OMO operations and/or reductions in policy corridor spread, implement a reduction in the credit spread. Another tool, the interest on reserves, grants direct control over inflation, without an effect on the spread. In a satiation regime, all rates equal the interest on reserves, so MP can control inflation, but not spreads.[3] In a DZLB, OMO are irrelevant, but then changes in the interest on reserves produces a joint movement in spreads

[3]Different from Woodford (1998), the control over nominal rates is achieved without open-market operations, but by setting the interest on reserves. Changes on inflation are neutral, but we are explicit that with additional frictions, a control over nominal rates can produce effects through the interest-rate, inflation-cost, and debt-deflation channels, all of which can be thought of operating independently. In each case, the model would need an additional ingredient: nominal rigidities, cash transactions, and long-term debt, respectively.

and inflation.

After the implementation, the paper presents an analysis of the real effects of MP. The economy can be solved in real terms; deposit, money and loans markets collapse into a single market for real claims. In turn, clearing in the real claims market is influenced by a spread target. As a result, through its control over spreads, MP influences the real rates at which real wealth clears, both, in the short and the long run. Since these rates impact the mass of agents in debt, it also impacts output at any horizon. The effects of spreads can be decomposed into a direct and an indirect fiscal effect. The direct effect is the induced reduction in the real deposit rate and the increase in the real loans rate. Because savers are more interest sensitive than borrowers, to satisfy market clearing, the loans rate reacts more to MP than the deposit rate (as found empirically by Drechsler et al., 2017). The fiscal effect emerges because, the operations that engineer a spread generate income to central bank, which then has to redistribute them. In an incomplete-markets economy, this produces non-Ricardian effects. This fiscal indirect effect is, nevertheless, swamped by the direct effect.

Temporary and permanent changes in spreads have different effects. A permanent increase in the spread has non-monotone effect on output. Because spreads primordially impact the loans rate, the effect on efficiency is driven by the impact on borrowers. With higher loans rates, borrowers have stronger incentives to repay debts, on the one hand. On the other hand, debt roll over is greater with higher rates. As a result, mild increases in spreads, concentrate the distribution of wealth toward the middle as borrowers repay their debts faster. Since less agents hit their debt limits, this promotes efficiency. Further increases in spreads, makes it harder to repay debt, to the point that borrowers give up. This effects fans out the distribution of debt, and more agents hit their debt limits. This triggers more inefficient behavior. When the effect is temporary, the debt accumulation effect is stronger. Therefore, a greater mass of agents reaches is debt limits, efficiency falls, while spreads are raised temporarily.

The model also has implications for the statistical relationship between monetary aggregates and inflation. Whereas it is entirely consistent with the quantity theory of money, it can also produce a liquidity effect. For example, a temporary open-market operation can produce a reduction in inflation. The effect of the operation is a reduction in spreads and an output expansion in which increase the real deposit rate. If the policy keeps the rate on reserves constant, the monetary expansion is deflationary.

Turning to the normative analysis, the optimal spread is governed by trade off. The ends with a study of the problem of an egalitarian planner, following the approaches in Nuno and Moll (2018) and Nuno and Thomas (2017). The planner chooses a path for spreads, for a fixed asset-position of the central bank. The optimal steady-state spread is positive, for a lower net asset position. The optimal spread balances distributional against efficiency considerations. In terms

of redistribution: spreads hurt everyone, specially the very poor and very rich who care the most about rates. The indirect fiscal effect helps mostly the poor, so the net balance is an improvement of the middle class. In terms of efficiency, higher spreads translate into worse ex-ante insurance, but improve output efficiency, which through general equilibrium forces, benefit everyone. The case for positive steady-state spreads is enhanced when the endowment choice spills over to other household's income—for example, with a labor demand externality. Finally, to analyze the benefits of a counter-cyclical spread policy, the paper studies an aggregate credit crunch episode. A numerical example illustrates the advantages of a policy that features positive spreads during booms, but eliminates the spread during a crunch. This is implemented with a corridor system that satiates banks with reserves during crises, but runs through a standard corridor system in normal times.

The organization is as follows. Section 3 lays out the core model. Section 4 describes the determination of credit, interest and prices and the implementation of MP. Section 5 presents a study on MP regimes. Section 6 studies the optimal use of spreads. Section 7 concludes. We connect with the next, in Section 2.

2 Connection with the Literature

The title emphasizes the connection with the two most common frameworks for MP analysis. One approach emphasizes the connection between *money and prices* and the other between *interest and prices*. In the first approach, money plays a transactions role (Lucas and Stokey, 1987; Lagos and Wright, 2005) and there is a tight connection between prices and the quantity of (outside) money. The real rate is fixed, so any real effects follow because inflation is a transactions tax. The other approach is the new-Keynesian approach where the important connection is between *interest and prices*. Under that framework, MP controls real rates directly because prices are rigid. There is no role for monetary balances. Neither framework emphasizes the effect of MP on credit, at least not directly. The model here establishes a meaningful connection between *intermediation, money, interest and prices*. Because the credit-channel here can be studied independently from the control of inflation, it only complements the inflation-tax or interest-rate channels in those approaches.

After 2008, there's been an increased interest on how MP interacts with credit markets. That

gap is being filled and incomplete-market models are a natural starting point.[4] In fact, the first generation of heterogeneous agent models, Lucas (1980) and Bewley (1983), were about MP, and not interested in heterogeneity per se. However, neither model established how MP affects credit.[5] Credit, of course, has a tradition in heterogeneous agent models (see the early work of Huggett, 1993; Aiyagari, 1994), but literature evolved abstracting away from it initial interest on MP.

A recent generation of works has introduced nominal rigidities into heterogeneous agent models. To replicate the credit crunch of 2008, Guerrieri and Lorenzoni (2017) studies the tightening of borrowing limits in a Bewley economy with nominal rigidities.[6] These models are appealing because, as an artifact of heterogeneity, MP responses depend on the distribution of wealth and borrowing constraints. Auclert (2016) decomposes the response to policy changes, into different forces that appear in that class of models. Kaplan et al. (2016) introduce illiquid assets, which produces high income elasticities among rich agents, something that changes the nature of propagation in the new-Keynesian model.[7] In that new generation of works, MP operates exclusively through the interest-rate channel of the new-Keynesian model. Instead, here MP operates through the credit-channel, by affecting spreads.

Another set of recent works in the *money and prices* tradition, allows for credit in models where money plays a transactions role. When credit (inside money) is an imperfect substitute for outside money, the inflation-tax channel spills over to the supply of credit (see for example Berentsen et al., 2007; Williamson, 2012; Gu et al., 2015). Rocheteau et al. (2016) bring the insights of money-search transactions into a heterogeneous agent environment. The model here abstracts from the inflation-tax channels, but can naturally be adapted to feature transactions, following the methodology in Rocheteau et al. (2016).

By introducing long-term debt, another set of works, Gomes et al. (2016) for example, recognizes that MP affects the distribution of wealth through debt deflation. Nuno and Thomas (2017) take

[4]Models that feature credit must provide a motive for credit. One way is to endow agents with different technologies as in Bernanke and Gertler (1989) and the other is make them subject to idiosyncratic risk. To establish a connection between MP and credit markets, models must have features by which MP impacts credits. A first such model is Bernanke et al. (1999) which incorporated nominal rigidities into the two-sector economy of Bernanke and Gertler (1989). In Bernanke et al. (1999) MP was capable of moving real rates because of nominal rigidities. In that model, and models that follow it, Christiano et al. (2009), credit imperfections amplify the effects of the interest-rate channel—through the financial accelerator. However, the effect on credit spreads is not an independent instrument, as it is here.

[5]In both models, there was a constant supply of outside money. Lucas (1980) studied a stable price equilibrium. Bewley (1983) focused on the case where money earned an interest rate financed with lump-sum taxes, so interest rate had redistributive consequences as it was funded with lump-sum transfers. Ljungqvist and Sargent (2012, Chapter 18) describes shows how policies in Bewley (1983) models are akin to changes in borrowing limits in economies with pure credit. Lippi et al. (2015) introduce aggregate shocks into a pure currency economy, and study the optimal helicopter drops.

[6]Following up on that work, McKay et al. (2015) compare the effects of forward-guidance policies in representative agent new-Keynesian models and incomplete markets economies.

[7]Greenwald (2016) and Wong (2016) study interest rate sensitivities to mortgage refinancing options.

that insight to a heterogeneous agent environment, and study optimal MP in a heterogeneous agent environment with nominal rigidities and possible debt deflation.

The credit-channel in this paper is not new. The implementation is inherited from Bianchi and Bigio (2017a). That paper articulates a notion of the credit channel and how MP functions through corridor rates. In contrast to this paper, that presents a rich description of bank decisions and studies shocks that impact the interbank market, whereas the non-financial side is static. In that paper, any dynamic effects of MP follow from the evolution of bank net worth. Here, the banking side is simplified, but the dynamics depend on the evolution of household wealth. Piazzesi and Schneider (2016) also feature a similar implementation of MP. The focus of that paper is on the connection between interbank settlements and asset prices. Our model also shares common elements with and Brunnermeier and Sannikov (2012). In Brunnermeier and Sannikov (2012), agents face undiversified investment risk so a demand for currency emerges due to market incompleteness.[8]

The focus on incomplete-markets economies leaves room for normative analysis. The methodology employed in the normative study here follows directly from Nuno and Thomas (2017), which together with Bhandari et al. (2019), are the first papers to study optimal MP under incomplete markets. In both works, MP balances aggregate-demand stabilization with insurance considerations. Instead, here the problem is to design the optimal management of the credit channel, wheighing financial stability with insurance considerations. Seeing financial stability as a crucial element of MP is discussed formally in Stein (2012), for example. The normative message, that MP should actively target credit spreads, is controversial.Curdia and Woodford (2016) and Arce et al. (2019), for example, study whether the control over spreads is a useful tool in economies with nominal rigidities. Their answer is no, and that suggests that there are no costs from switching to a floor system. Instead, we take the sides of Stein (2012) and Kashyap and Stein (2012), and the control of spreads is crucial for financial stability. A corridor system is a way to achieve so, and moving to a floor system is a mistake.

[8]Other related work includes Silva (2016), that focuses on open-market operations and the effects of expected inflation. In Buera and Nicolinni (2016), the identity between borrowers and lenders is determined by a threshold interest rate. Furthermore, there is an explicit role for outside money as a transactions instruments and MP has real effects by affecting the stock of risk-free bonds which, in turn, affects the threshold identity of borrowers and lenders.

3 Environment

3.1 From Policy Spreads to Real Credit Spreads

In the model that follows, we embed financial intermediation (by banks) in an environment where money holdings, prices and rates are determined in general equilibrium. In this introductory section, we present the banking block. We derive a simple formula that maps a MP corridor spread, into a real intermediation spread, for given monetary aggregates. Later, we show how real spreads determine monetary aggregates, and thus, how the CB has the ability to control real spreads.

Banks. There is free entry and perfect competition among banks.[9] We consider the static portfolio decision of a bank within an interval of time of length Δ—which we later take to zero. We assume banks are owned by households. Because there are no aggregate shocks within the Δ period, the bank's objective is simply to maximize expected profits. Competition leads to zero bank profits in expectation.

At the start of the Δ interval, banks choose their supply nominal deposits, a, nominal loans, l, and reserve holdings, m. The aggregate supply of deposits and loans, and holdings of reserves are denoted by A^b, L^b, and M^b, respectively. Deposits, loans, and reserves earn corresponding rates i^a, i^l, and i^m. Whereas the loan and deposit rates are equilibrium objects, i^m is a policy instrument.

After the portfolio decision is made, banks face random payment shocks, as in Bianchi and Bigio (2017a); Piazzesi and Schneider (2016). In particular, within the interval, payment shocks take one of two values, $\omega \in \{-\delta, +\delta\}$ and occur with equal probability and are i.i.d across banks. If $\omega = \delta$, a bank receives δa deposits and is credited δa reserves from other banks. If $\omega = -\delta$, the bank transfers δa deposits and δa debited to other banks. Naturally, if a bank receives a deposit, it absorbs a liability of another bank. If it loses a deposit, another bank absorbs its liability. As a result of the transfer of liabilities, assets need to be transferred to settle the transfer. A key assumption is that within the time Δ interval, loans are illiquid in the sense that they must stay with banks. Therefore, net deposit flows must be settled with reserves which are cleared at the CB. After the payment shock to a bank, its net reserve balance at the CB is:

$$b = m - \varrho a + \omega a.$$

[9] Banks operate without equity. The introduction of a role for bank equity (via restrictions like capital requirements or limited participation) would produce bank profits and induce would make equity an aggregate state variable. For simplicity, we abstract away from this dimension in this paper. See den Heuvel (2002) for an early model of bank equity capital and monetary policy. Recent work by Wang (2019) studies the pass-through of MP as function of bank equity.

The coefficient, $\varrho \in [0,1]$ represents a constant liquidity requirement coefficient.[10] Clearly, since ω is random, the reserve balance is not entirely under the control of a bank. For that reason, it is possible that the bank ends with a negative balance, $b < 0$. In that case, the bank with a negative balance must close this gap, either by borrowing reserves from banks with surplus, or from the CB. Figure A.1 in the Appendix presents the corresponding T-accounts for the scenarios that can emerge within the Δ interval. For the rest of the paper, we work with policies that guarantee aggregate excess liquidity: $M - \varrho A > 0$.

Interbank Market. After the reserve positions are determined, an interbank market opens and banks borrow and lend reserves to each other. For a balance b, a fraction of those balances, are lent (or borrowed if negative) in the interbank market. In particular, if a bank has a surplus b, it lends the fraction ψ^+ to other banks and, hence, $b - \psi^+ b$ remains idle in a CB account. If the bank has a deficit of $-b$, it borrows only the fraction ψ^- from other banks, and the remainder deficit $-(b - \psi^- b)$ is borrowed from the CB at a discount window rate i^{dw}. The discount rate is also a policy choice. By convention, borrowed reserves from the CB earn the interest on reserves i^m. Thus, the effective borrowing cost is the policy spread $\iota \equiv i^{dw} - i^m$. The trading probabilities $\{\psi^+, \psi^-\}$ are meant to capture trading frictions in the interbank market.

Integrating b across banks yields expressions for aggregate surplus and deficit positions.

$$B^- \equiv -\int_{-\infty}^{0} b dG(b) = \max\left\{(\delta + \varrho) A^b - M^b, 0\right\} \quad \text{and} \quad B^+ \equiv \int_0^{\infty} b dG(b) = M^b + (\delta - \varrho) A^b,$$

where G is a measure of reserve balances induced by individual bank decisions. These expressions follow directly from the independence assumption on ω.

Clearing in the interbank market requires that the total amount of reserve balances lent is equal to the amount borrowed:

$$\psi^- B^- = \psi^+ B^+. \tag{1}$$

Trading frictions, a well documented empirical feature (see Ashcraft and Duffie, 2007; Afonso and Lagos, 2014), are key in the model to have a pass-through from policy to credit spreads. There are many ways to induce trading frictions. Here, we assume that interbank market is an over-the-counter (OTC) market in the spirit of Afonso and Lagos (2015), but we adopt the formulation in Bianchi and Bigio (2017b) that renders analytic expressions. The interbank market works as follows: The market operates in a sequence of n trading rounds. Given the initial positions $\left\{B_0^-, B_0^+\right\} \equiv \{B^-, B^+\}$, surplus and deficit positions are matched randomly. When a match

[10] We can interpret ϱ as reserve requirements, imposed either by the CB, or other forms of regulation, or self imposed by the bank for cash-management purposes. Notice that we model the balance in terms of the pre-transfer deposits. If we make this a function of after-payment balances, the balance is defined differently as by $b = m + \omega a - \varrho(1 + \omega) a$. There are no material differences.

is formed, the two banks agree on an interbank-market rate for the transaction. The remainder of surplus and deficit positions define an new balance, $\left\{B_1^-, B_1^+\right\}$. New matches are formed, and a new interbank-market rate emerges. The process is repeated n times, defining a sequence $\left\{B_j^-, B_j^+\right\}_{j\in 1:n}$ until a final round is reached. Whatever deficit remains is then borrowed from the CB at a cost given by ι.

The interbank-market rate at a given trading round is determined by a bargaining problem in which banks take into consideration the matching probabilities and trading terms of future rounds. This produces an endogenous average interbank rate, $\bar{i}^f$. Given trading probabilities, the policy rates and the average rate $\bar{i}^f$, the average rates earned on negative and positive positions are respectively:

$$\chi^- = \psi^- \left(\bar{i}^f - i^m\right) + (1-\psi^-)\cdot\iota, \text{ and } \chi^+ = \psi^+ \left(\bar{i}^f - i^m\right).$$

Banks take into account these costs and benefits when forming their portfolios. To express $\{\chi^-, \chi^+\}$, Bianchi and Bigio (2017b) assume that matches are formed on a per-position basis and according to a Leontief matching technology $\frac{\lambda}{n}\min\left\{B_j^-, B_j^+\right\}$ where λ captures the trading efficiency. Let $\theta = B^-/B^+ \leq 1$ define an initial interbank "market tightness." In the limit as $n \to \infty$, trading probabilities across all trading rounds, $\{\psi^+, \psi^-\}$, converge to $\psi^+(\theta) = \theta(1-\exp(-\lambda))$ and $\psi^-(\theta) = 1-\exp(-\lambda)$, two expressions consistent with market clearing. Then, the average interbank market rate $\bar{i}^f$ that results when both bargaining weights are equal is

$$\bar{i}^f(\theta, i^m, \iota) \equiv i^m + \iota\cdot\frac{\left((\theta+(1-\theta)\exp(\lambda))^{1/2}-1\right)}{(1-\theta)(\exp(\lambda)-1)}. \tag{2}$$

The corresponding expressions for the average cost functions are:

$$\begin{aligned}\chi^+(\theta,\iota) &= \iota\cdot\frac{\left(\theta(\theta+(1-\theta)\exp(\lambda))^{1/2}-\theta\right)}{(1-\theta)\exp(\lambda)} \text{ and,} \\ \chi^-(\theta,\iota) &= \iota\cdot\frac{\left((\theta+(1-\theta)\exp(\lambda))^{1/2}-\theta\right)}{(1-\theta)\exp(\lambda)}.\end{aligned} \tag{3}$$

These coefficients are independent of i^m and only depend on the total gains from trade, $\iota = i^{dw} - i^m$. Of course, i^m affects the direct return of holding reserves. If the CB has the ability to control χ, it will have control over credit spreads.

The Bank Problem. We turn to the banks optimal portfolio choice. The average benefit (cost) of

an excess (deficit) reserve balances, b, is:

$$\chi(b;\theta,\iota) = \begin{cases} \chi^{-}(\theta,\iota)\, b & \text{if } b \leq 0 \\ \chi^{+}(\theta,\iota)\, b & \text{if } b > 0 \end{cases}. \tag{4}$$

We label χ the *liquidity yield* function. With this function, we are ready to present the bank's problem:

Problem 1 *[Bank's Problem] A bank maximizes its instantaneous expected profits:*

$$\pi^b = \max_{\left\{l^b, m^b, a^b\right\} \in \mathbb{R}^3_+} \left(i^l l^b + i^m m^b - i^a a^b + \mathbb{E}\left[\chi\left(b;\theta,\iota\right)\right] \right)$$

subject to the budget constraint $l + m = a$ and the law of motion for reserve balances.

$$b\left(a,m\right) = \begin{cases} m + (\delta - \varrho)\, a \text{ with probability } 1/2 \\ m - (\delta + \varrho)\, a \text{ with probability } 1/2 \end{cases}.$$

At the individual level, the bank objective is linear. As in any model with linear firms, banks earn zero (expected) profits in equilibrium. However, the kink in $\chi(b)$ introduces concavity (at the individual asset level) that is necessary to pin down the portfolio. This feature is akin to what occurs with competitive firms that operate a Cobb-Douglas production technology with two inputs—whereas firms earn zero-profits and the individual scale is indeterminate, the ratio of inputs and factor prices is determined in equilibrium.

Equilibrium Credit Spreads. Next, we explain how a ratio of money aggregates determines the equilibrium loan and deposit rates. To that end, define the aggregate *liquidity ratio* as $\Lambda \equiv M^b / A^b$. The interbank market tightness can be in terms of this ratio:

$$\theta\left(\Lambda\right) \equiv \max\left\{ \frac{\delta + \varrho - \Lambda}{\delta - \varrho + \Lambda}, 0 \right\}. \tag{5}$$

The tightness θ is decreasing in the liquidity ratio, because with more liquidity, there is less need to borrow. The function is also bounded: $\theta = 1$ when $\Lambda = \varrho$, and $\theta = 0$ for any $\Lambda = \varrho + \delta$. If we substitute (5) into (4), we can express χ as a function of the policy corridor, ι, and the liquidity ratio, Λ and do not depend on the level of $\left\{M^b, A^b\right\}$.

The linearity of the bank's problem, coupled with free-entry condition, yield corresponding equilibrium nominal rates and spread:

Proposition 1 *[Nominal Rates and Real Spread] Consider an aggregate liquidity ratio Λ . Then, for given $\{i^m, \iota\}$, any equilibrium with finite loans and deposits must feature the following loans and deposit*

rates:

$$i^l \equiv i^m + \underbrace{\frac{1}{2}\left[\chi^+\left(\theta\left(\Lambda\right),\iota\right)+\chi^-\left(\theta\left(\Lambda\right),\iota\right)\right]}_{\text{liquidity value of reserves}} \tag{6}$$

$$i^a \equiv i^m + \underbrace{\frac{1}{2}\left[\left(1-\varrho+\delta\right)\chi^+\left(\theta\left(\Lambda\right),\iota\right)+\left(1-\varrho-\delta\right)\chi^-\left(\theta\left(\Lambda\right),\iota\right)\right]}_{\text{liquidity value of reserves-liquidity cost of deposits}}. \tag{7}$$

The equilibrium credit spread, $i^l - i^a$, is given by,

$$i^l - i^a = \frac{\delta}{2}\left(\chi^- - \chi^+\right) + \frac{\varrho}{2}\left(\chi^+ + \chi^-\right) = \frac{\delta}{2}\left(\chi^- - \chi^+\right) + \frac{\varrho}{2}\left(i^l - i^m\right)\cdot. \tag{8}$$

Furthermore, if $\Lambda \geq \varrho + \delta$, then, $i^l = i^a = i^m$ and if $\Lambda = \varrho$, then, $i^l = i^a = i^m + \frac{1}{2}\iota$. In all cases, banks earn zero expected profits.

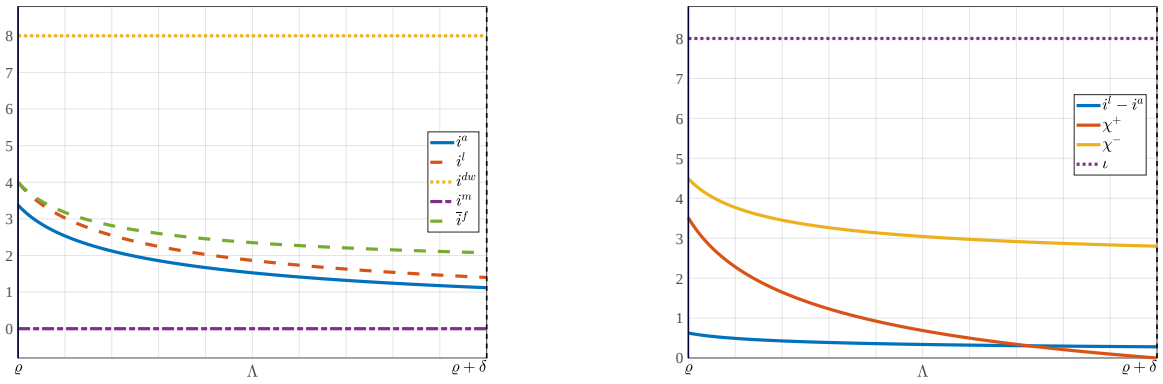
Proposition 1 establishes that the interest on reserves is a base rate for both the nominal borrowing and lending rates. The credit spread is positive when $\Lambda < \varrho + \delta$ is positive. Both rates carry a different liquidity premium relative to the rate on reserves. To understand this, consider first the loans liquidity premium. Loans earn a premium over reserves because, on the margin, an additional reserve earns χ^+ if the bank is in surplus or spares the bank χ^- if the bank is in deficit—each scenario occurs with equal probability. The deposit liquidity premium reflects that an additional deposit produces a marginal increase in reserve balances δ or decrease $-\delta$ if the balance is negative. The deposit premium is thus the sum of the expected marginal increase in the interbank payments produced by an additional reserves minus the that of a marginal deposit. The spread (8) directly follows from subtracting the deposit rate from the loans rate. The following lemma shows necessary and sufficient condition large ρ/δ ratio, the spread is always decreasing in the liquidity ratio Λ:

Lemma 1 *The spread is decreasing in $\Lambda \in [\rho, \rho+\delta])$ if and only if,*

$$\frac{\rho}{\delta} \geq \frac{1+\exp\left(-\lambda/2\right)}{3-\exp\left(-\lambda/2\right)}. \tag{9}$$

We work with the assumption that parameters satisfy (9) for the rest of this paper. Figure 1 depicts the formulas in Proposition 1 for nominal rates and the spread as functions of Λ. The figure summarizes the implementation we have discussed so far: The left panel plots $\left\{i^l, \bar{i}^f, i^a\right\}$ as functions of (6) and (7) for fixed policy rates $\{\iota, i^m\}$. Both rates lie in between i^m and i^{dw}.[11]

[11] Credit risk or illiquidity is enough to produce rates above those bands.



(a) Equilibrium nominal rate given ι and Λ

(b) Equilibrium spread given ι and Λ

Figure 1: Interest Rates and Spread as Functions of Λ

Note: Panel (a) plots the nominal deposit, loan, average interbank rate, and policy rates as function of liquidity yield and spread as functions of liquidity ratio Λ. Panel (b) shows the components of the liquidity yield and the equilibrium spread. The figure is constructed using parameters from the calibration presented in section 5.

Both rates feature a spread when $\Lambda \in (\varrho, \varrho + \delta)$. We also see how the credit spread decreases with the liquidity ratio.

The next section, embeds bank intermediation into the incomplete markets economy, in the spirit of the early monetary model of Bewley (1983). Before we proceed, we discuss the assumptions encountered so far.

Digression: discount-window loans. The discount-window rate and the size of payment shocks stand in for features missing from the model. In practice, the cost of reserve shortages can be much larger than the discount-window rate set by the CB. One reason for this is that discount-window loans require high quality collateral. If collateral is scarce and a bank cannot close its position, the bank that cannot close a negative balance can be intervened (for a related bank model with collateralized discount loans see De Fiore et al., 2018). Another issue is that discount window loans can bear a stigma (as in Ennis and Weinberg, 2013). Since in the model, discount-window loans are uncollateralized and , the discount-window rate may be too low compared to the actual cost of a reserve shortage. For this reason, the discount-winder rate in the model must be treated as a much larger cost than the discount-rate seen in the data.

The withdrawal shock in the model is i.i.d. In the data, payment shocks are likely to be persistent. Thus, to precisely capture the cost of withdrawals, we must increase the size of shocks to compensate for the lack of persistence in the withdrawal shock. Adding persistence makes the model more realistic, but comes at the expense of tractability. The value of ϱ can be interpreted literally as a reserve requirement, but since the introduction of sweep accounts, the effective requirement is small. However, current bank regulation imposes minimum liquid-asset holdings. In addition, if banks may self-impose minimum liquidity holdings to avoid runs. These features are left out of the model.

3.2 General Equilibrium

We now embed financial intermediation into the general equilirium model. We take a continuous time limit of the banks problem. Within a time interval of size Δ, average profits are $\Delta \cdot \pi^b$—all rates are scaled by Δ and the objective is linear. Since bank policy functions are independent of Δ, the equilibrium rates of Proposition 1 also scale with Δ, even as $\Delta \to 0$. Next, intermediation, into the continuous-time limit of the general equilibrium. To do so, we work with a $\Delta \to 0$ limit.[12]

The non-financial sector of the economy features a measure-one continuum of heterogeneous households. From there perspective, time is indexed by some $t \in [0, \infty)$. The price of the good in terms of money is P_t. Banks intermediate between borrower and lender households, but since they make zero profits, they are simple pass-through entities. The CB determines the policy corridor rates, conducts open-market operations and makes/collects (lump-sum) transfers/taxes to/from households. Households attempt to smooth idiosyncratic income shocks, via the insurance provided by the intermediation sector. However, a social inefficiency arises when households hit their borrowing limits.

Notation. Individual-level variables are denoted with lower case letters. Aggregate nominal state variables are denoted in capital letters. Aggregate real variables are written in calligraphic font. For example, a_t^h will denote nominal household deposits, A_t^h the aggregate level of deposits, and $\mathcal{A}_t^h$ real household deposits.

Households. Households face a consumption-saving problem. Household preferences are described by:

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} U\left(c_t\right) dt\right]$$

where $U\left(c_t\right) \equiv \left(c_t^{1-\gamma} - 1\right) / \left(1-\gamma\right)$ is their instantaneous utility.

Households receive an flow of real income given by:

$$dw_t = \left(y\left(u\right) + T_t\right) dt + \sigma(u) dZ_t.$$

Income is the sume of transfers T_t and an endowment income process. To generate endowment

[12]Note that the balance by the end of a time interval b_t, is a random variable. If we were to track b_t as a function of time, this stochastic process would not be well defined—the sum of coin tosses in continuous time is not well defined. However, treating $b_{t+\Delta}$ as the single realization of a random variable in a single instance is a perfectly well defined object.

income, households chose among two processes, $u \in \{L, H\}$; respectively the low (L) and high (H) intensity processes. The choice of u is done every instant. By assumption, $y(H) > y(L) > 0$, but $\sigma(H) > \sigma(L) = 0$. The term dZ_t is a noise associated with an idiosyncratic Brownian motion process: households partially control that risk because u affects their risk exposure $\sigma(u)$. Naturally, the process trades of risk for return.

All financial assets are nominal. Although all claims are nominal, the individual state variable is, s_t, the stock of real financial claims. Households store wealth in bank deposits, a_t^h, or currency, m_t^h, and borrow loans against banks, l_t^h. By convention, $\{a_t^h, m_t^h, l_t^h\} \geq 0$. The real rates of return on deposits and liabilities are $r_t^a \equiv i^a - \dot{P}_t/P_t$ and $r_t^l \equiv i^l - \dot{P}_t/P_t$. Currency earns no nominal interest, and thus, its real return is $-\dot{P}_t/P_t$. A The law of motion of real wealth follows

$$ds_t = \left(r_t^a \frac{a_t^h}{P_t} - \frac{\dot{P}_t}{P_t} \frac{m_t^h}{P_t} - r_t \frac{l_t^h}{P_t} - c_t \right) dt + dw_t \tag{10}$$

and the balance-sheet identity:

$$\frac{a_t^h}{P_t} + \frac{m_t^h}{P_t} = s_t + \frac{l_t^h}{P_t}.$$

From a household's perspective, there is no distinction between holding deposits or currency beyond their rates of return. Hence, currency is only held when the nominal deposit rate is zero, and both assets yield the same return. This feature is introduced into the model only to articulate a deposit zero-lower bound (DZLB) as an implementation constraint. Another observation is that households will never hold deposits and loans if there's a positive spread between them. Combining these insights, (10) can be written more succintly as:

$$ds_t = (r_t(s)\, s - c_t)\, dt + dw_t \text{ where } r_t(s) \equiv \begin{cases} r_t^a \text{ if } s_t > 0 \\ r_t^l \text{ if } s_t \leq 0 \end{cases}. \tag{11}$$

There are two important additional assumptions. First, endowment risk cannot be diversified due to incomplete markets. In particular, households face a *debt limit* $s_t \geq \bar{s}$ where $\bar{s} \leq 0$ is exogenous and constant. Technically, this means that at $s = \bar{s}$, it must be that $ds_t \geq 0$. The second assumption is that consumption is a function of current savings and the the intensity choice, but cannot be contingent on dw_t. This is the continuous-time analogue to the assumption that consumption is chosen prior to the arrival of an income shock in the discrete-time represetation, and implies that households cannot perfectly control theirs savings. These two assumptoins are important. Because the endowment risk is only idiosyncractic, any choice $u = L$ is a social waste induced by market incompleteness. Households will be forced to make that wasteful choice when they reach their borrowing limits. Since, households canot borrow more at the debt limit,

$c_t dt \leq dw_t$ when at $s = \bar{s}$, but this can only be guaranteed if $u_t = L$ at $s = \bar{s}$.

The corresponding Hamilton-Jacobi-Bellman (HJB) equation of the household's problem is:

Problem 2 *[Household's Problem] The household's value and policy functions are the solutions to:*

$$\rho V(s,t) = \max_{\{c\} \geq 0, u \in \{L,H\}} U(c) + V_s \cdot (r_t(s)\, s - c + (y(u) + T_t)) + \frac{1}{2} V_{ss} \sigma^2(u) + \dot{V}. \quad (12)$$

subject to $s_t \geq \bar{s}$.

At each instant, there's a distribution $f(s,t)$ of real financial wealth across households. The law of motion of this distribution satisfies a Kolmogorov-Forward Equation: Let $c(s,t)$, $u(s,t)$ and $m^h(s,t)$ be the solutions to the household's problem. The drift of real wealth is

$$\mu(s,t) \equiv r_t(s) \left(s - m^h(s,t) / P_t \right) - \dot{P}_t / P_t \cdot m^h(s,t) / P_t - c(s,t) + y(u(s,t)) + T_t.$$

The volatility of real wealth is $\sigma_s^2(s,t) \equiv \sigma^2(u(s,t))$. The path of the distribution of real wealth , $f(s,t)$, satisfies the following Kolmogorov-Forward equation:

$$\frac{\partial}{\partial t} f(s,t) = -\frac{\partial}{\partial s} \left[\mu(s,t) f(s,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial s^2} \left[\sigma_s^2(s,t) f(s,t) \right]. \quad (13)$$

The solution to the household problems produces a drift and volatility for real wealth. In turn, these objects affect the law of motion of the wealth distribution. Output can be written as:

$$Y_t = y(H) - \int_{\bar{s}}^{\infty} \underbrace{(y(H) - y(L))}_{\text{Efficiency Loss}} \mathbb{I}_{[u(s,t)=L]} f(s,t)\, ds.$$

Central Bank. The CB has a nominal net asset position, E_t, defined as:

$$E_t \equiv L_t^f - M_t.$$

The net-asset position is the difference between loans held by the CB, L_t^f, and the CB liabilities, i.e., the monetary base, M_t. The monetary base is divided into the aggregate holdings of reserves by banks M_t^b and the currency holdings of currency by households, $M0_t$. The monetary base is always positive. The CB can issue or purchase loans L_t^f: when negative L_t^f is understood as a stock of government bonds, when positive, as the loan purchases of the CB.[13] An open-market

[13]There is no distinction between private and public loans. In fact, whenever $L_t^f < 0$, an increase in L_t^f is interpreted as conventional open-market operation. Instead, when $L_t^f > 0$, an increase L_t^f is an unconventional open-market operation. The assumption is that government bonds are as illiquid as private loans from the point of view of banks.

operation (OMO) is a simultaneous increase or decrease in M_t and L_t^f. Finally, the CB chooses T_t.

In addition to these operations, the CB sets the discount window rate i_t^{dw} at which banks can borrow reserves and the interest on reserves, i_t^m, that we presented earlier.[14] In the analysis, we the think of i_t^m and the corridor spread $\iota_t = i_t^{dw} - i_t^m$ as independent instruments, so for example, when we study changes in i_t^m holding fixed ι_t, we are effectively moving both rates in a parallel shift.

The nominal income flow of the CB is:

$$\pi_t^f = i_t^l L_t^f - i_t^m \left(M_t - M0_t\right) + \iota_t \left(1 - \psi_t^-\right) B_t^-. \tag{14}$$

The CB earns i_t^l on L_t^f, and pays i_t^m on the portion of the money supply held as reserves. The third term, $\iota_t \left(1 - \psi_t^-\right) B_t^-$, is the income earned from discount window loans to banks. The net-asset position evolves according to

$$dE_t = \underbrace{\pi_t^f dt - P_t T_t dt}_{\text{undistributed income}} = - \underbrace{\left(dM_t - dL_t^f\right)}_{\text{unbacked transfers}}.$$

The CB accumulates a nominal claim on the private sector as undistributed income. The net-asset position decreases with the difference between the monetary base and the loan purcahses of the CB. In real terms, the CB's net-asset position is $\mathcal{E}_t \equiv E_t / P_t$ and its loan holdings are $\mathcal{L}_t^f \equiv L_t^f / P_t$.

Markets. Outside money is held as bank reserves or currency. The aggregate currency stock is

$$M0_t \equiv \int_{\bar{s}}^{\infty} m_t^h(s) f(s,t) ds.$$

Equilibrium in the outside-money market is:

$$M0_t + M_t^b = M_t. \tag{15}$$

The credit market has two sides, a deposit and a loans market. In the deposit market, households hold deposits supplied by banks. In the loans market, households obtain loans supplied by banks. The distinction between the loans and deposits is that they clear with different interest

[14] The CB faces a solvency restriction, $i_t^{dw} - i_t^m \geq 0$ and also $i^{dw} \geq 0$. The spread $i_t^{dw} - i_t^m \geq 0$ because a negative corridor spread would enable banks to borrow from the discount-window and lend back the CB making arbitrage profits. If $i^{dw} < 0$,banks could borrow reserves and lend reserves as currency to households swapping the currency for deposits at zero rates. This operation would produce another arbitrage for the bank.

rates. The deposit market clears when:

$$A_t^b = \int_0^\infty a_t^h(s) f(s,t) ds, \tag{16}$$

where $a_t^h(s) \equiv P_t s - m_t^h(s)$, for a positive s. The left of this equation is the bank supply of deposits. The loans market clears when:

$$L_t^b + L_t^f = \int_{\bar{s}}^0 l_t^h(s) f(s,t) ds, \tag{17}$$

where $l_t^h(s) \equiv -P_t s$ for negative s. Finally, the goods market clears when:

$$\int_{\bar{s}}^\infty y(u(s,t)) f(s,t) ds \equiv Y_t = C_t \equiv \int_{\bar{s}}^\infty c(s,t) f(s,t) ds. \tag{18}$$

Equilibrium. A price path-system is the vector function $\{P(t), i^l(t), i^a(t)\} : [0,\infty) \to \mathbb{R}_+^3$. A policy path is the vector function $\left\{L_t^f, M_t, , i_t^{dw}, i_t^m, T_t\right\} : [0,\infty) \to [0,\infty) \to \mathbb{R}_+^5$. Next, we define an equilibrium path.

Definition 1 *[Perfect Foresight Equilibrium.] Given initial condition for the distribution of household wealth $f_0(s)$, an initial net-asset position, E_0, and an initial price level P_0, and a policy path $\left\{\mathcal{L}_t^f, \mathcal{E}_t, i_t^m, \iota_t, T_t\right\}$, a perfect-foresight equilibrium (PFE) is (a) a price system, (b) a real wealth distribution path $f(s,t)$, (c) a path of aggregate bank holdings $\left\{L_t^b, M_t^b, A_t^b\right\}_{t\geq 0}$, and (d) household's policy and value functions, $\left\{c(s,t), u(s,t), m^h(s,t), V(s,t)\right\}_{t\geq 0}$ such that:*

1. *the path of aggregate bank holdings solves the static bank's problem (1) at each t,*
2. *the household's policy rule and value functions solve the household's problem (2),*
3. *the law of motion for $f(s,t)$ is consistent with Kolmogorov-Forward equation (13),*
4. *the government's policy path satisfies the governments budget constraint (14),*
5. *all the asset markets and the goods market clear, (1,15-18)*

Next, we characterize the equilibrium dynamics. A **steady state** occurs when $\frac{\partial}{\partial t} f(s,t) = 0$ and $\left\{r_t^a, r_t^l\right\}$ are constant. We use subscripts ss denote variables at steady state. An important assumption is that we treat P_0 in the definition of equilibrium as given to avoid issues of multiplicity. Like in any model with an nominal assets, time-zero price determines the real distribution of wealth and thus an equilibrium path. The approach here is to think of P_0 as determined from past policies. The idea is to think of the time-zero price as a the price level at steady state consistent with a steady state given a nominal monetary base of M_t that was committed a priori.

This approach circumvents the need for refinements that pin-down time zero prices such as the fiscal theory of the price level. We do want to take a stance on whether those are reasonable assumptions.

Digression: Model Assumptions. The financial architecture in the model captures a fundamental feature of banking. In practice, banks issue deposits in two transactions. The first is a swap of liabilities with the non-fnancial sector. When banks make loans, they effectively credit borrowers with deposits, a bank liability is exchanged for a household liability. This swap is the process of inside-money creation. Deposits then circulate as agents exchange deposits for goods. This circulation gives rise to the settlement positions. The second transaction is the exchange deposits (a bank liability) for currency (a government liability).

A missing element are government bonds. In practice, central banks conduct open-market operations by purchasing government bonds. Here, negative holdings of L^f are interpreted as goverment bonds. The implicit assumption is that bonds are as illiquid as private loans. Bianchi and Bigio (2017a) introduce government bonds that are more liquid than loans, but less so than reserves because they cannot be used for settlements.

The endowment choice produces a new mechanism for monetary policy. The goal of this formulation is to capture that when agents in an economy are close to borrowing limits, they undetertake inefficient choices. By focusing on endowments, we are silent who makes that choice. One interpretation is that households conduct entrpreneurial activities. As a result, near their borrowing constraints, households act more cautiously in their entrepreneurial activities. This interpretation is explicit in section 6 where we introduce entrepreneurs and workers.

4 Implementation

Any nominal spread between loans and deposits equals a the spread in real terms. We label the equilibrium real spread by $\Delta r_t = i_t^l - i_t^a$. As shown in Proposition 1, the real spread is governed by Λ and ι. We now explain how a CB can implement a desired real credit spread. First, we show that given an equilibrium spread Δr_t, market clearing in real financial claims is consistent with an equilibrium real deposit rate r_t^a. This real equilibrium rate is the one that solves a single clearing condition, which implies clearing in all asset markets, as demonstrated by the following proposition:

Proposition 2 *[Real Wealth Clearing] Let nominal rates be given by (6) and (7), and let the liquidity ratio be given by Λ_t. Then, market-clearing in real terms,*

$$-\int_{\bar{s}}^{0} sf(s,t)\,ds = \int_{0}^{\infty} sf(s,t)\,ds + \mathcal{E}_t \quad \textit{for } t\in[0,\infty), \tag{19}$$

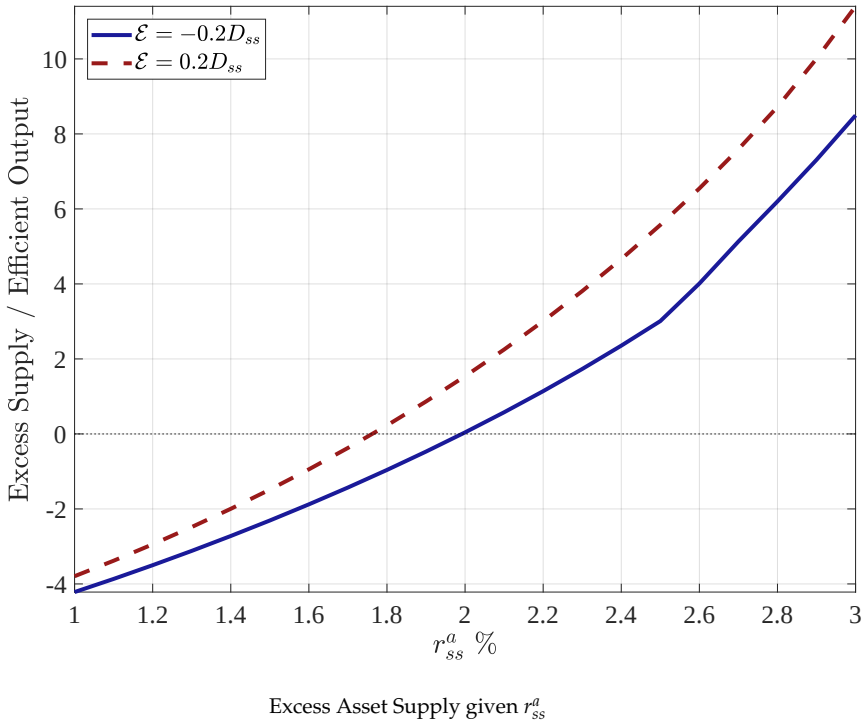


Figure 2: Excess Savings as a function of Deposit Rate in Steady State

Note: The figure depicts the between excess savings supply as function of the real deposit rate (in steady state) fro two CB net-asset positions. The spread is constant taking the real spread as given. The excess supply is the difference between aggregate deposits and aggregate loans as a percentage of efficient output, i.e. $(A_{ss} + \mathcal{E} - L_{ss})/Y(H)$. The figure is constructed using parameters from the calibration presented in section 5.

implies market clearing in all asset markets. Furthermore, if (19) and the Kolmogorov-Forward equation (13) hold, then, the goods market clearing condition (18) also holds.

The proposition thus shows that all clearing conditions are summarized by a single clearing condition in real wealth. If we obtain the real deposit rate, we also obtain the real value of deposits. Now, if the CB induces a real spread, and banks earn zero profits, the revenues from the spread must go somewhere. Figure 2 plots the relationship between deposit-output ratio and equilibrium deposit rate in steady state under different government net-asset position. A more negative net-asset position raises up aggregate private deposit.

In this model, the CB earns the revenues obtained from the spread, which actually equal the revenues from discount-window loans. The next proposition exploits this observation to express the law of motion of the real net-asset position $\mathcal{E}_t$.

Proposition 3 *[Real Budget Constraint] Consider equilibrium in all asset markets, then $\mathcal{E}_t$ satisfies:*

$$\dot{\mathcal{E}}_t = \underbrace{(r^a_t + \Delta r_t)}_{\text{return on CB balance sheet}} \mathcal{E}_t + \Delta r_t \underbrace{\int_0^\infty s f(s,t)\, ds}_{\text{discount-window profits}} - T_t,\ \mathcal{E}_0 \text{ given.} \tag{20}$$

The first term in 14 is the portfolio returns earned by CB earns (losses) which equal the real lending rate times the net asset position. The second term captures that in net, the CB all the intermediation profits. Finally, transfers subtract from the real asset position. A policy path is constrained by solvency conditions: An important restriction is a long-run solvency constraint for the CB. In particular, there's limit $lim_{t\to\infty}\mathcal{E}_t \geq \underline{\mathcal{E}}$, for some minimum $\underline{\mathcal{E}}$ that guarantees that the CB can raise enough revenues and satisfy $d\underline{\mathcal{E}} = 0$. It must be the case that at $\underline{\mathcal{E}}$ discount-window revenues cover any balance sheet costs. This condition is equivalent to assuming that the the CB's liabilities are not worth zero in equilibrium. The model features a Laffer curve for curve for CB revenues. Although we don't solve for $\underline{\mathcal{E}}$ explicitly, in the exercises we analyze in the following section, we impose that all policy paths lead to a convergent stable government net-asset position and $lim_{t\to\infty}d\mathcal{E}_t = 0$. Another restriction in the opposite direction is that $\mathcal{E}_t \leq -\bar{s}$, which is equivalent to saying that the CB claim on the public cannot exceed the public's debt limit. Figure 16 in Appendix B plots the components of the CB's profits (relative to private savings) for various values of Λ_t.

Implementation. From equation (8), we know that the real spread Δr_t is a function of the liquidity ratio and the corridor spread, $\{\Lambda_t, \iota_t\}$. The policy corridor $\{\iota_t\}$ is directly controlled by the CB. A natural question is to what extent does the CB control the real spread? To answer that, we need to understand its control over the the liquidity ratio. The main result of this section is that OMO affects the liquidity ratio, unless the economy unless the economy reaches a DZLB or unless the economy is satiated with reserves.

We first characeterize the DZLB. The DZLB emerges because households can convert deposits into currency but banks cannot.[15] Hence, although the CB can set $i_t^m < 0$, the deposit rate is always $i_t^a \geq 0$. To characterize the DZLB, we define Λ_t^{zlb} as the threshold liquidity such that for any liquidity ratio above that point, the equilibrium deposits rate, as determined by equation (7) would be negative:

$$\Lambda^{zlb}\left(i_t^m, \iota_t\right) \equiv \inf\left\{\Lambda | 0 > i_t^m + \frac{1}{2}\left[\left(1-\varrho+\delta\right)\chi^+\left(\Lambda,\iota_t\right) + \left(1-\varrho-\delta\right)\chi^-\left(\Lambda,\iota_t\right)\right]\right\}.$$

Because a negative deposit rate cannot occur in equilibrium, we know that $\Lambda_t \leq \Lambda^{zlb}\left(i_t^m, \iota_t\right)$. If the CB attempts to increase Λ_t, beyond Λ^{zlb}, the increment in the money supply must immediately translate into an increase $M0_t$, but not in M_t^b! Within DZLB, the CB loses the ability to influence spreads through an increase in OMO. Furthermore, because $\chi^- \geq \chi^+ \geq 0$, we know that Λ^{zlb} is a finite only if $i_t^m < 0$. Thus, the DZLB is relevant only when the rate on reserves is negative.

[15] We assume that banks can't hold currency whether by regulation, taxation, or physical costs.

We can define *a monetary-base liquidity ratio*, Λ_t^{MB} as

$$\Lambda_t^{MB} \equiv \frac{M_t}{A_t}.$$

Different from the liquidity ratio of banks, Λ_t, which is the relevant object to determine bank spreads, the the monetary-base liquidity ratio Λ_t^{MB} is defined in terms of the total monetary base, which includes reserves but also currency. We express Λ_t^{MB} in terms of the composition of the CB's balance sheet in real terms:

$$\Lambda_t^{MB} = \frac{\left(L_t^f - E_t\right)/P_t}{A_t/P_t} = \frac{\mathcal{L}_t^f - \mathcal{E}_t}{\int_0^\infty sf\left(s,t\right)ds} \equiv \Lambda^{MB}\left(\mathcal{E}_t, f_t, \mathcal{L}_t^f\right).$$

Collecting results we obtain:

Proposition 4 *[Implementation Conditions] Consider an equilibrium path for* $\left\{r_t^a, \Delta r_t, f_t, \dot{P}_t/P_t\right\}_{t\geq 0}$. *To implement the equilibrium path, the CB chooses* $\left\{i_t^m, \iota_t, \mathcal{L}_t^f, T_t\right\}$ *subject to the following restrictions:*

1. *The equilibrium liquidity ratio is* $\Lambda_t = \min\left\{\Lambda^{zlb}\left(i_t^m, \iota_t\right), \Lambda^{MB}\left(\mathcal{E}_t, f_t, \mathcal{L}_t^f\right)\right\}$.
2. *The real spread* Δr_t *is given by (8) and* $\{\Lambda_t, \iota_t\}$.
3. *The real rate* r_t^a *is consistent with real-wealth clearing (19).*
4. *The real net-asset position,* $\mathcal{E}_t$, *evolves according to (20).*
5. *The inflation rate is:*

$$\dot{P}_t/P_t = i_t^m + \frac{1}{2}\left[\chi^+\left(\theta\left(\Lambda_t\right), \iota\right) + \chi^-\left(\theta\left(\Lambda_t\right), \iota\right)\right] - r_t^a - \Delta r_t. \tag{21}$$

Proposition 4 describes the dynamic allocations that can be induced by the CB. Allocations are affected by the CB because it controls the real spread either through changes in corridor rates or through OMO. The real spread, given a distribution of wealth and a net-asset position, pins the real deposit rate. Through market clearing, the size of $\mathcal{E}_t$ also influences the real interest rate. In addition, the CB can select a rate on reserves to target an inflation rate.

In addition to the DZLB, OMO are also irrelevant when banks are satiated with reserves. This occurs when the CB satiates banks with reserves and as a result $\theta = 0$. This regime occurs when $\Lambda_t \geq \varrho + \delta$, because in that case, banks have enough liquidity to cover against a withdrawal. When, $i^m < 0$, we know that $\Lambda^{zlb}\left(i_t^m, \iota_t\right) < \delta$,because otherwise $\chi^- = \chi^+ = 0$, and the deposit rate would be negative. This means that the satiation regime occurs only when rates on reserves

are positive. This observation, allows us to organize the effects of policy tools into a single proposition:

Proposition 5 *[Properties of Equilibrium Rates and Spreads] Consider a CB policy given by* $\left\{\mathcal{L}_t^f, i_t^m, \iota_t\right\}$.

1. *[Corridor Regime] If* $\Lambda_t < \min\left\{\delta + \varrho, \Lambda^{zlb}\left(i_t^m, \iota_t\right)\right\}$ *then* $i^{dw} > i^l > i^a > i^m$ *and* $0 < \Delta r < \iota$ *and,*

$$\left\{\frac{\partial i^l}{\partial \mathcal{L}_t^f}, \frac{\partial i^a}{\partial \mathcal{L}_t^f}, \frac{\partial \Delta r}{\partial \mathcal{L}_t^f}\right\} < 0, \left\{\frac{\partial i^l}{\partial i_t^m}, \frac{\partial i^a}{\partial i_t^m}, \frac{\partial \Delta r}{\partial i_t^m}\right\} = \{1,1,0\}, \left\{\frac{\partial i^l}{\partial \iota_t}\frac{\iota_t}{i^l}, \frac{\partial i^a}{\partial \iota_t}\frac{\iota_t}{i^a}, \frac{\partial \Delta r}{\partial \iota_t}\frac{\iota_t}{\Delta r}\right\} = \{1,1,1\}.$$

2. *[Floor Regime] If* $\Lambda_t \geq \delta + \varrho$ *then* $i^l = i^a = i^m$ *and* $\Delta r = 0$ *and*

$$\left\{\frac{\partial i^l}{\partial \mathcal{L}_t^f}, \frac{\partial i^a}{\partial \mathcal{L}_t^f}, \frac{\partial \Delta r}{\partial \mathcal{L}_t^f}\right\} = 0, \left\{\frac{\partial i^l}{\partial i_t^m}, \frac{\partial i^a}{\partial i_t^m}, \frac{\partial \Delta r}{\partial i_t^m}\right\} = \{1,1,0\}, \left\{\frac{\partial i^l}{\partial \iota_t}, \frac{\partial i^a}{\partial \iota_t}, \frac{\partial \Delta r}{\partial \iota_t}\right\} = \{0,0,0\}.$$

3. *[DZLB and negative* i^m *regime] If* $\Lambda_t = \Lambda^{zlb}\left(i_t^m, \iota_t\right)$ *then* $i^l > i^a = 0$ *and* $\Delta r > 0$,

$$\left\{\frac{\partial i^l}{\partial \mathcal{L}_t^f}, \frac{\partial i^a}{\partial \mathcal{L}_t^f}, \frac{\partial \Delta r}{\partial \mathcal{L}_t^f}\right\} = 0, \frac{\partial i^l}{\partial i_t^m} > 0, \frac{\partial i^a}{\partial i_t^m} = 0, \frac{\partial \Delta r}{\partial i_t^m} > 0, \frac{\partial i^l}{\partial \iota_t} > 0, \frac{\partial i^a}{\partial \iota_t} = 0, \frac{\partial \Delta r}{\partial \iota_t} > 0.$$

Proposition 5 establishes the direction of policy effects. There are three regimes. In the first regime, $\Lambda_t < \min\left\{\varrho + \delta, \Lambda^{zlb}\left(i_t^m, \iota_t\right)\right\}$ so liquidity is scarce enough to promote interbank lending. This regime is referred a *corridor system*. It features a positive credit spread. Open-market operations reduce the spread. Increases in i^m, induce a parallel increase on both nominal rates—and inflation—but not on the spread. An increase in the policy corridor, induces linear increase in all rates and the spread. The neutrality of i^m on the spread implies that the CB can control inflation independently.

When $i^m > 0$, and the liquidity ration exceeds $\varrho + \delta$, banks are satiated with reserves. In that case, all nominal rates equal i^m. This regime is referred to as a *floor system*. Increases in the policy corridor is neutral. Furthermore, OMO have no effects; they satisfy classic Wallace irrelevance, (Wallace, 1981). Thus, in floor system the CB loses the ability to affect spreads, and handles inflation with i^m.

Now consider $i_t^m < 0$. This opens the possibility of a DZLB. A DZLB occurs when the liquidity ratio is above Λ_t^{zlb}. In that region, OMO are irrelevant because any increase in CB liabilities translates into an increase in currency, not reserves. However, the spread is still positive, even though OMO has no effects. The reason is that negative rates on reserves tax deposits. Since the deposit rate is fixed at zero, banks require a higher lending rate—its as if at zero, deposits have

an infinite price elasticity. As a result, changes in i_t^m produce a joint effect on the real spread and inflation; something that does not occur in a corridor system. A similar result was independently obtained in Brunnermeier and Koby (2019). Figure 17 in Appendix B presents additional figures for the case of negative rate on reserves.

Policy Discussion: Tools and Targets. At any point, the CB here has four tools: $\left\{i^m, \iota, \mathcal{L}_t^f, T_t\right\}$. We saw that i^mcontrols inflation and T_t has direct redistributional effects. We also so that $\left\{\iota, \mathcal{L}_t^f\right\}$ can produce a desired spread. To understand whether the latter are redundant assets, we must ask if the implementation of a credit spread have fiscal consequences? In the model, the spread can be obtained by moving the corridor spread ι, or via OMO. We are tempted to argue that these instruments have different fiscal consequences, so the choice of tools matters for the ability to redistribute wealth. In fact, this intuition is wrong in light of Proposition 3:

Corollary 1 *[No Fiscal Consequence of an implementation choice] Consider two policies $\{\iota_t, \Lambda_t\}$ that implement the same real spread target, Δr_t. Both are consistent with the same discount window profits and, hence, produce the same fiscal revenues.*

Because $\left\{\iota, \mathcal{L}_t^f\right\}$ have the same effect on households through the spread, and have the same effect on fiscal revenues, both instruments are redundant. However, ι can be increased to achieve any spread. Instead, OMO can produce spreads within the bound $\Delta r \in \left\{\frac{\iota}{2}, \iota\right\}$.

Policy Discussion: Alternative Implementations. It is worth discussing MP implementations used in practice (Bindseil, 2014, reviews cross-country practices.). In the model, one alternative way to the control the real spread directly through OMO while keeping ι constant, is to target an interbank-market rate $\bar{i}^f$: given ι, we can find a Λ consistent that delivers $\bar{i}^f$. Because there is also map from $\bar{i}^f$ to Δr_t, a target for the interbank rate also implements a spread.

In practice, most CBs have explicit interbank target, but restrict they way in which they do so. So set corridor systems with a constant corridor width ι and move Λ and target an interbank market at the middle, $\bar{i}^f = i^m + \frac{1}{2}\iota$. Other countries, keep the rate on reserves at zero but move ι, and simultaneously target $\bar{i}^f$ at a constant distance from ι. Our analysis suggests that under either system, CB will simultaneously move spreads and inflation, perhaps inadvertently. However, in doing so, CBs lose an instrument: they can target inflation and spreads within a given mix, but not as independent targets. If CBs are open to move i^m and ι, the can reach both targets. This paper argues that targeting spreads is desirable.

Policy Discussion: the DZLB The effects of policy at the zero lower bound here are different from those that emerge in cash-in-advance constraints. In those models, a ZLB emerges if the CB floods the public with savings instruments so that the asset clears at negative rates. That opens the door to an unrealistic arbitrage: in which the households borrow at negative rates from the

	Instrument				Channel		
Regime	i_t^m	T_t	ι_t	L_t^f	Fisherian	Non-Ricardian	Credit
Floor system (sec 5.1)	✓				✓		
Transfers (sec 5.2)	✓	✓			✓	✓	
Corridor system (sec 5.3)	✓	✓	✓	✓	✓	✓	✓

Table 1: Instruments and Transmission Channels

CB to hold currency.[16] Here, it is important to note that the ZLB applies to the nominal deposit rate, but not on the policy instruments. The policy conclusion is that main implication is that a CB that reduces i_t^m, perhaps as an attempt to increase inflation, will cause an increase in credit spreads.

Policy Discussion: Fiscal-Monetary Interactions. The model inherits classical monetary properties in Bewley economies (Bewley (1983); Ljungqvist and Sargent (2012), , Chapter 18.11)). First, a version of the quantity theory holds. If we fix a path for T_t, and Δr_t, then we can scale every nominal variable by a scalar and obtain the same equilibrium. Second, changes in the growth rate nominal transfers produce an increase in steady-state inflation. If i^m increases in that same rate, again the policy is neutral.[17] Third, because the economy is neutral, there is potentially a continuum of equilibrium indexed by time-zero prices. By normalizing P_0, we fix the initial distribution of wealth.

5 Positive Analysis: From Instruments to Channels

This section discusses three MP rules. First, a regime where MP eliminates any credit spread (a floor system). Second, a regime where MP operates only through transfers. Third, a regime that opens spreads (a corridor system). Table 1 presents a summary of the instruments employed under each regime and the channels that it activates. **Calibration.** Next, we present a calibration to produce the computations. The paper has many missing elements, but we attempt to get quantitative sense of the transmission mechanisms. The calibration, is also a guide to where the model needs realism. The calibration is summarized in Table 2 and inspired by the US economy. Risk-aversion, which coincides with the inverse inter-temporal elasticity, γ, is set to 2. The time discount, ρ, is set to 4%, which yields a real steady-state deposit rate of approximately 1.0%. The high intensity endowment normalized. The low intensity endowment is set to 0.8, this produces

[16] In Rognlie (2016), negative rates are possible because there are costs of holding physical currency.

[17] For illustration, assume a policy where from t onwards, the CB increases the growth rate of nominal transfers. Then inflation rate will increase at a constant rate as long as the CB, but the CB must also increase i_t^m and i^{dw} by the new rate of inflation. Thus, if the CB moves its policy rates accordingly, monetary policy is super neutral. If instead, the CB increases transfers but keeps the real rate constant, it effectively changes the real value of transfers and thus also the real rate. In that case, monetary policy is not super neutral.

Parameter	Description	Reference/Target
$\gamma = 2$	risk aversion	standard
$\rho = 0.04$	time discount	1.0% real risk-free rate
$y(H) = 1$	high intensity endowment drift	normalization
$y(L) = 0.8$	low intensity endowment drift	max loan-to-income ratio of 12
$\sigma(H) = 1$	high intensity endowment volatility	savings to GDP ratio $A_{ss}/Y_{ss} = 3.5$
$\mathcal{E}_{ss} = -0.2 \cdot A_{ss}$	net asset position	70% public debt to GDP
$\bar{s} = -12 \cdot y(L)$	debt limit	max loan-to-income ratio of 12
$\tilde{s}_{ss} = \bar{s}$	leverage constraint	no shock at steady state
$\mathcal{L}_t^f = 0$	CB assets	normalization
λ=2.1	interbank-market match efficiency	follows Bianchi and Bigio (2017a)
$i^m = r_{ss}^a$	reserve rate	inflation target 0%
$i^{dw} = 5\%$	discount rate	real spread of 2%

Table 2: Parameter Values

Note: The table lists the calibrated values of parameters and the corresponding reference/target of calibration.

an output drop of 10% during a severe credit crunch episode, which we study in Section 6. This output decline of the scale seen during severe financial crises (see Cerra and Saxena, 2008; Barro and Ursua, 2010). The volatility under the high intensity endowment is set to $\sigma(H) = 1$. This yields a private-savings to output ratio of about 4—a number in line with the capital-to-output ratio in the US. We think of the net-asset position in terms of the consolidated US Government, not only the Federal Reserve. For that reason, we set $\mathcal{E}_{ss}$ to -20% of private assets. This figure yields a level of public-debt-to-output ration of 70%, the pre Great Recession level in the US. The debt limit is $\bar{s}$ is 12 times the low endowment income. This parameter produces a debt-to-income ratio of 12 for the poorest households, a number in line with the literature. The amount of real assets held by the CB, $\mathcal{L}_t^f$, is set to zero.[18]

The interbank-market efficiency, λ, is set to 2.1. This number is directly taken from Bianchi and Bigio (2017a), who calibrate it to match the size of discount-window loans. The rate on reserves is set to $i_{ss}^m = r_{ss}^a$, so steady-state inflation is zero; a normalization. The discount-window rate is set to produce a steady-state spread of 2%. The required spread between the discount-window rate and the rate on reserves is much higher than in the data, but as we argued above, this is a stand in for missing elements such as collateral and stigma (De Fiore et al., 2018).

Steady-state Moments. To get a sense of quantitative fit, we report steady-state moments in Table 3. The model produces a 20.2% share of agents at their debt limit and 43.1% that hold positive debt. The corresponding output efficiency loss is 4.0%. The CB's operational profits are 5.6% of output. In the US, the transfers of the Federal Reserve to the Federal Government are similar to corporate tax revenues, about 1.8% of GDP. Since the model does not have operational

[18]Prior to 2008, the assets held by the Federal Reserve where small relative to liabilities. The choice is, however, a normalization. As showed earlier, a spread can be implemented in many ways.

Moment	Value
Fraction of households at debt limit	20.2%
Fraction of households in debt	43.1%
Output efficiency loss (relative to potential)	4.0%
CB operational revenue / GDP	5.6%
CB interest-rate expense / GDP	1.4%
Wealth quantiles/per-capita GDP $\{Q_{10}, Q_{25}, Q_{50}, Q_{75}, Q_{90}\}$	$\{-10.0, -5.87, 1.43, 6.09, 10.2\}$

Table 3: Additional Moments (Not Targetted)

Note: The table reports the untargetted moments of the calibrated model.

costs for banks nor for the CB, this figure which is three times as high as in the data, is reasonable. The interest expense on the CB's position is 1.4% of GDP. Finally, we report levels of wealth over GDP measured as wealth over per-capita income, at different quantiles. The model misses the return shocks needed to produce the concentration ratios at the top, but does a fair job at the bottom of the distribution. Since as we show, most of the dynamics stem from the behavior of the poor, missing the wealth concentration at the top, should not have an important effect on the quantitative responses.

5.1 Nominal Rate Target and the Fisherian Channels

We begin with a policy that neutralizes the spread, but has a nominal rate target, i.e. a floor system. We have the following corollary:

Corollary 2 *[Floor System] Let the CB either set $\iota_t = 0$ or set $i^m \geq 0$ and $\Lambda \geq \rho + \delta$, then, $\Delta r_t = 0$, the evolution of $\{r_t^a, f(s,t)\}$ is unaffected by policy, inflation is controlled by i_t^m as given by (21).*

Corollary 2 is a special case of Proposition 4. If the CB satiates banks with reserves or eliminates the policy corridor, monetary policy is neutral. However, the CB controls inflation: it effectively resets the unit of account every period and increases the money supply accordingly. This showcases that the CB can control nominal rates, without OMO.[19] The control inflation with a single instrument connects with three well-travelled transmission mechanisms: the New-Keynesian model, the inflation-tax and debt deflation which were reviewed earlier. In the model, incorporating nominal rigidities would produce an interest-rate channel like in recent heterogeneous-agent new-Keynesian models (Guerrieri and Lorenzoni, 2017; Kaplan et al., 2016; Auclert, 2016). One can also incorporate sporadic currency transactions as in Rocheteau et al. (2016), that would produce an interesting interaction between currency holdings, the distribution of wealth, and productive efficiency that would be affected by the inflation tax channel. One can also make

[19] In the Proposition, we set $E_t = 0$ for the sake of exposition, but the result holds with little loss in generality. Without a policy corridor, the CB controls inflation even if there are no reserves. Woodford (2001) advocates for this policy and calls it the Wicksellian doctrine.

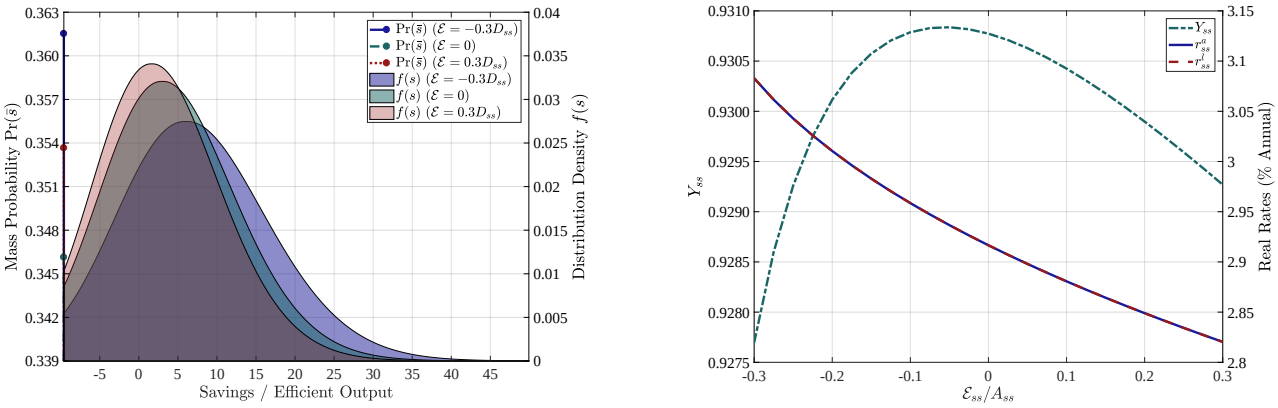
loans long-termed, so that unexpected changes in i_t^m are non-neutral. As discussed in Gomes et al. (2016); Auclert (2016); Nuno and Thomas (2017), this would generate surprise inflation which would compresses the distribution of real wealth, and affect productive efficiency.

5.2 Fiscal Transfers and the Non-Ricardian Channel

This section presents the effects of transfers. There reasons is twofold. Although the focus is on the credit channel, any increase in spreads generates fiscal revenues that have to be rebated. Studying transfers in isolation, allows us to isolate the direct from the fiscal effect spreads. Second, fiat money transfers are classic exercises in monetary economics. We begin with the steady-state. We maintain the implementation of $\Delta r = 0$, but allow the net-asset position $\mathcal{E}_{ss}$ to differ from zero—transfers are $T_{ss} = r_{ss}^a \mathcal{E}_{ss}$. The next corollary to Proposition 4 is a classic result:

Corollary 3 *[Non-Ricardian Effects] An economy with steady-state net asset position $\mathcal{E}_{ss}$ induces the same equilibrium allocation as an alternative economy with zero-net $\mathcal{E}_{ss}^{(a)} = 0$ with a borrowing $\bar{s}^{(a)} = \bar{s} + \mathcal{E}_{ss}$.*

Corollary 3 shows that a reduction in the $\mathcal{E}_{ss}$ is akin to a more relaxed debt limit. Figure 3 reports the corresponding real rate, output, and the real wealth distributions for different levels of $\mathcal{E}_{ss}$. Panel (a) depicts the distribution of wealth, Panel (b) output and the real rate. We see that a lower net-asset position, is associated with a more dispersed distribution, but a higher mean. A more negative net-asset position produces higher real interest rate; a more spread-out distribution and higher rates indicate better ex-ante insurance. Panel (b) shows a non-montonic relation between the net-asset position and output. When the net-asset position is very low, the wealth distribution is very dispersed, so a larger mass of agents concentrate at the debt limit. This causes greater output inefficiency. As $\mathcal{E}_{ss}$ improves from low values, the distribution tends to concentrate and the mass at the debt limit falls. This improves output. However, there's a counteracting force that kicks when the net-asset position improves above a certain point. As $\mathcal{E}_{ss}$ increases, the interest rate falls, and this attracts more debt. In fact, at the limit where, $\mathcal{E}_{ss} = -\bar{s}$, all agents must be at their debt limits—output is at its lowest possible $Y(L)$. These two forces, better risk sharing and the crowding out of private savings, explain the non-monotonic behavior of output. This non-monotonic effect is similar to the effect that appears in Aiyagari and McGrattan (1998). One important note is that once the CB pays interest on reserves, the level of real transfers does not affect inflation. Instead, the money supply adjusts to finance i^m, but does not alter the real rate (see the discussion in Ljungqvist and Sargent, 2012, Ch. 18). A final, important clarification is that transfers here have non-Ricardian effects only because households cannot borrow against future taxes. In fact, if me modify the economy and let the borrowing depend on the net-present value of taxes, then transfers have no effects:



(a) Wealth Distribution different $\mathcal{E}_{ss}/A_{ss}$

(b) Output different $\mathcal{E}_{ss}/A_{ss}$

Figure 3: Steady State Effects of Transfer/Net Asset Position.

Note: The figure depicts the output, real deposit and loan rates, and real wealth distributions for different levels of the government net-asset position. For panel (a), the measure of households with assets $\bar{s}$ is in mass probability (left scale), and the measure of households with $s > \bar{s}$ is in probability density (right scale). Efficient output is expressed as the endowment income flow of high intensity process $y(H)$. For panel (b), Y_{ss} is the flow of households' aggregate income in steady state. Deposit and loan rates are expressed in annual percentage terms. For both panels, the government net-asset position is expressed in a fraction of aggregate deposit in steady state.

Proposition 6 *[Conditions for Ricardian Equivalence] Consider an alternative economy such that agents now agents face a time varying borrowing limit equal to* $\bar{s}^{(a)}(t) = \bar{s} + h(t)$*, where is the present value of transfers* $h(t) \equiv \int_t^\infty \exp\left(-\int_t^\tau r^a(z)\,dz\right) T_\tau d\tau = \mathcal{E}_{ss}$*. Consider a policy that sets* $\iota_t = 0$ *or satiate banks with reserves. Then, any path of fiscal transfers* $\{T_t, \mathcal{E}_t\}$ *does not alter allocations.*

The next experiment considers a one-time decrease transfers, holding fixed the spread at $\Delta r_{ss} =$ 2%. The program is announced at time zero, but carried out a year later, and lasts for two years. We study the effects of anticipated policies because even if the policy is anticipated, once it takes place, the effect is similar to when the policy is unanticipated. Yet, with an unanticipated policy, we can also study the effect of the anticipation of the policy. Thus, the exercises allow to highlight anticipation effects, but also the impact effect, both of announced and unannounced policies.

The economy initiates at the steady-state. This initial condition produces a level of steady-state lump-sum taxes. When the policy is announced, taxes increase by $\Delta r_{ss} \cdot L_{ss}/2$, and slowly revert to steady state by the end of the program. The idea is to make the exercise comparable to the fiscal effect of a reduction in spreads, which we study next. Two years into the program, the policy is reverses so that the net-asset position transitions back to steady state—this is done so that the annual reversal rate is approximately 10. We have to be careful not to return the net-asset position too quickly though. Hence, we adopt a fiscal rule: denote by t^{start} the time of the policy program (i.e. t^{start} = month 12) and t^{end} the end of the program (i.e. t^{post} = month 36).

The transition path for T_t is given by[20]

$$T_t = \begin{cases} T_{ss}, & \text{if } t < t^{start}, \\ \Delta r_t L_t + r^a_{ss}\mathcal{E}_{ss} - \tau_t, & \text{if } t \in \left[t^{start}, t^{post}\right], \\ \Delta r_t L_t + \left[1 - \left(\delta^{trans}\right)^{t-t^{post}}\right]\left[r^a_t \mathcal{E}_t + \delta^{ef}\left(\mathcal{E}_t - \mathcal{E}_{ss}\right)\right] & \text{if } t > t^{post}, \\ + \left(\delta^{trans}\right)^{t-t^{post}} r^a_{ss}\mathcal{E}_{ss}, & \end{cases} \tag{22}$$

where

$$T_{ss} = \Delta r_{ss} L_{ss} + r^a_{ss}\mathcal{E}_{ss}$$

is the steady-state aggregate fiscal transfers, and

$$\tau_t = \frac{t^{post} - t}{t^{post} - t^{start}} \cdot \frac{\Delta r_{ss} \cdot L_{ss}}{2}$$

represents the magnitude of increase in taxes. Thus, the law of motion for $\mathcal{E}_t$ is

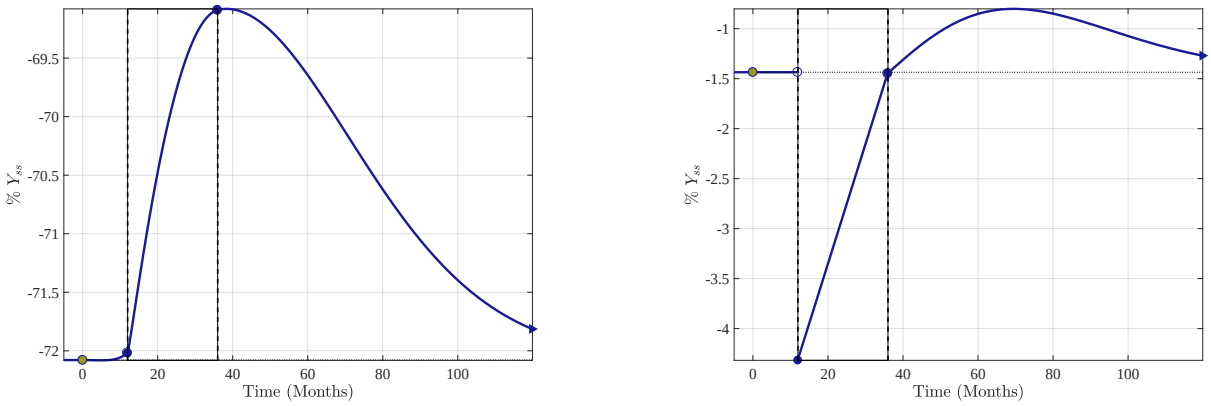
$$\begin{aligned} \dot{\mathcal{E}}_t &= r^a_t \mathcal{E}_t + \Delta r_t L_t - T_t \\ &= \begin{cases} r^a_t \mathcal{E}_t + \Delta r_t L_t - T_{ss}, & \text{if } t < t^{start}, \\ r^a_t \mathcal{E}_t - r^a_{ss}\mathcal{E}_{ss} + \tau_t, & \text{if } t \in \left[t^{start}, t^{post}\right], \\ - \left[1 - \left(\delta^{trans}\right)^{t-t^{post}}\right]\delta^{ef}\left(\mathcal{E}_t - \mathcal{E}_{ss}\right) & \text{if } t > t^{post}. \\ + \left(\delta^{trans}\right)^{t-t^{post}}\left(r^a_t \mathcal{E}_t - r^a_{ss}\mathcal{E}_{ss}\right), & \end{cases} \end{aligned} \tag{23}$$

The parameters δ^{trans} and δ^{ef} control the speed of transition after the policy program. For the rest of the paper, we employ the same fiscal rule and we set $\delta^{trans} = 0.9$ and $\delta^{ef} = 0.1$. The policy paths for this exercise are depicted in Figure 4.

Figure 5 reports the responses of macroeconomic variables. Upon announcement, agents anticipate the change in T_t. In the anticipation phase, we observe (Panel c) a mild decrease in both, borrowing and lending rates. We also observe (Panel b) a decrease in credit and an increase in output (Panel d). The anticipation of future taxes increase the precautionary behavior. With this expectation, falling into debt is seen as more painful. This reduces the credit demand in the anticipation phase. Although credit contracts, the announcement effect is slightly expansionary becuase the mass of agents at the debt limit falls.

Once the policy is enacted, the effects reverse. We see an increase in real rates and credit, and a

[20] The derivations of the fiscal transfer paths are provided in Appendix C.



(a) Net Government Position $\mathcal{E}_t$

(b) Fiscal Transfers T_t

Figure 4: Fiscal Policy Path.

Note: This figure plots the time path of fiscal transfers and government's net-asset position in response to an anticipated one-time increase in lump-sum taxes. The net-asset positions and fiscal transfers are in percentage of contemporaneous aggregate output. The policy change program is announced at time zero, takes place one year later (at the first vertical dashed line), and lasts for two years (between the two vertical dashed lines). During the program, the lump-sum transfer is raised up to close the gap between current value and steady-state value at a fixed rate. After the program (after the second vertical dashed line), the lump-sum transfer is adjusted to close the gap of net-asset position between current value and steady-state value at an exponential rate of 10% annually. The detailed transition paths of net government position and fiscal transfers are given by (22) and (23).

contraction in output. This pattern occurs because the taxes make debt repayments more difficult. On impact, real rates decrease despite a lower stock of debt. The reason is that with greater taxes, more agents hit the constraint. By the termination of the program, taxes gradually reverse to steady state; output and credit variables mirror the behavior during the program. The study of transfers is also important because "helicopter drops" have been advocated as a stimulus tool in recent year—implemented as tax rebates funded by the CB. An increase in transfers would reverse the sign of the transitions in the experiment. Thus, the exercise here also warns against the use of helicopter drops. The risk is not inflation, which cab be controlled by i^m, and the growth in the future money supply. Rather, the risk is that the anticipation of transfers can lead to an ex-ante expansion in credit, that can lead to further productive inefficiency.

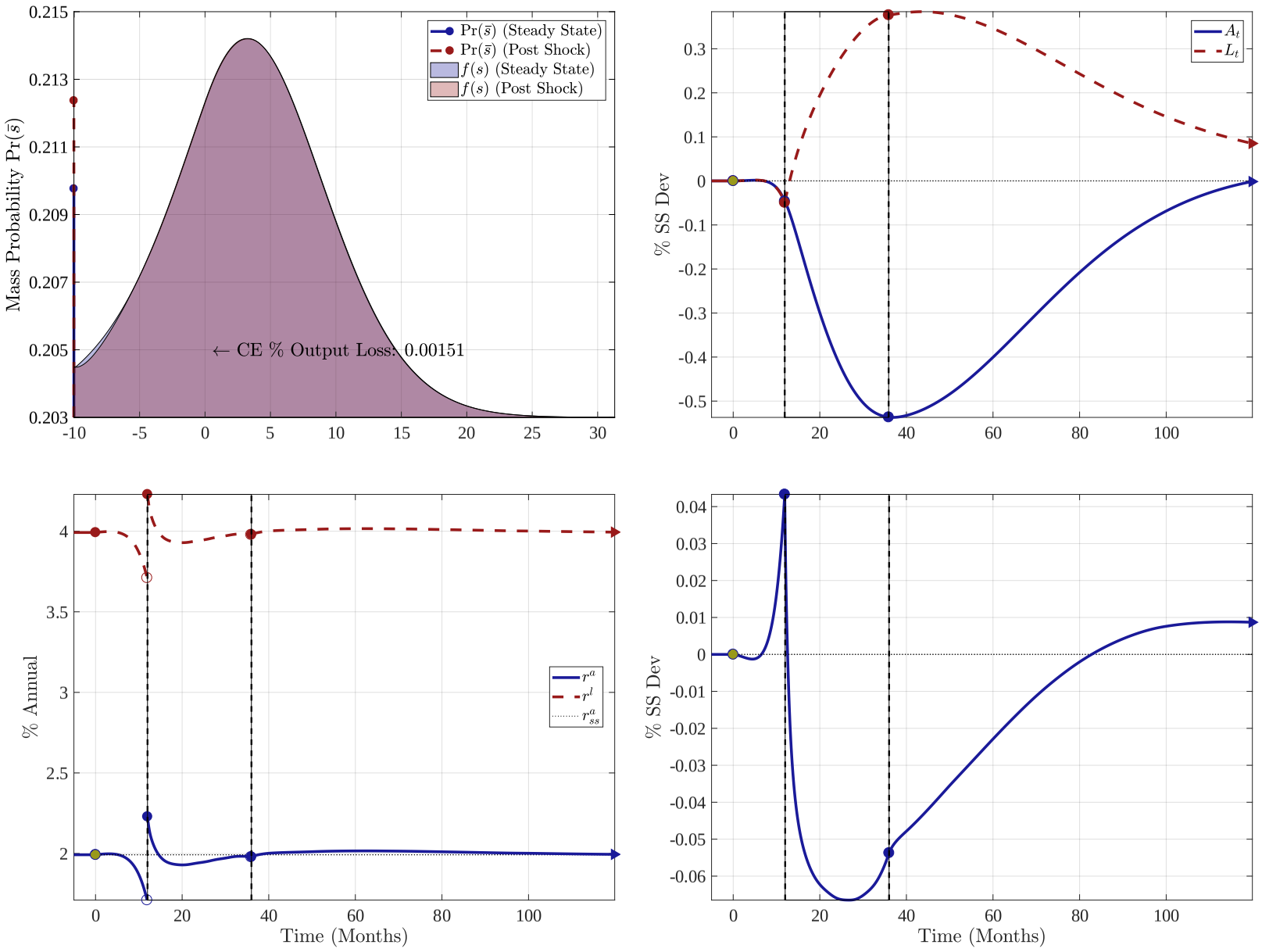


Figure 5: Transitional Dynamics following Fiscal Policy Path

Note: The figure reports the real wealth distribution, and the responses to credit, rates and output after the anticipation of an increase in lump-sum taxes. In panel (a), the measure of households with assets $\bar{s}$ is in mass probability (left scale), and the measure of households with $s > \bar{s}$ is in probability density (right scale). The CE % loss is expressed in the percentage deviation of aggregate certainty equivalent after the announcement. In panels (b) and (d), the aggregate deposits, loans and output are expressed in percentage deviations from steady state. In panel (c), the real rates are expressed in annual percentages.

5.3 Credit-Spread Target and the Credit Channel

We now move to consider the effects of a spread target, Δr. We start with the steady-state effect. Figure 6 reports the real wealth distribution (panel a) and output and real interests (panel b) for different values of Δr. Above, we saw how higher lump-sum taxes shift the wealth distribution to the right, and spread it out. The steady-state effect of spreads is different. Higher spread compresses the wealth distribution while total wealth remains constant—for any spread, private wealth integrates to $\mathcal{E}_{ss}$. Thus, the spread is a tax on intermediation and, like any tax, it has an incidence on both, borrowers and lenders. Thus, a higher spread reduces the real deposit rate and raises the real loans rates. As a result, the spread makes saving less attractive to both borrowers and savers. This is an indication that higher spreads produce worse risk-sharing.

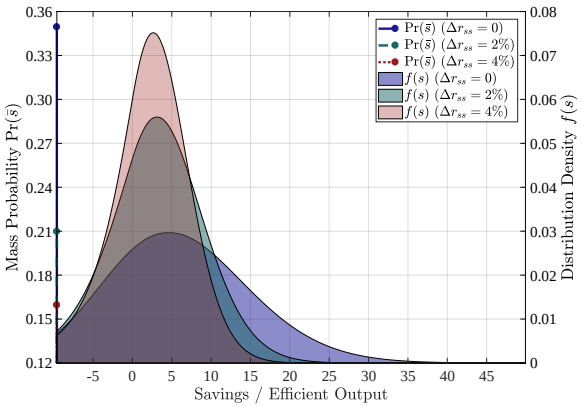
The effect of the real spread on steady-state output is non-monotone. At a zero spread, an increase in the spread increases output. This is because up to a first-order, it makes borrowing less attractive and, therefore, less agents hit their debt limit. There is a force in the opposite direction that dominates for higher spreads not seen from the figure. Higher loan rates make it more difficult to repay debt. At some point, the mass of households at the debt limit begins to increase again. As borrowing rates continue to increase, households switch to the inefficient endowment process.[21] In our calibration, output increases with the spread until the spread reaches 8%. The main takeaway of this exercise is that for small values, spreads can increase output efficiency, at the expense of risk-sharing.

Next, we consider a transition after a reduction Δr_t. Again, the policy is announced at time zero, implemented after a year, and lasts for two years. The policy halves the spread from a steady-state level of 200bps. Again, we split the description into the ex-ante and ex-post phases. As explained earlier, the policy has a direct effect through the reduction in spreads and an indirect, fiscal effect.

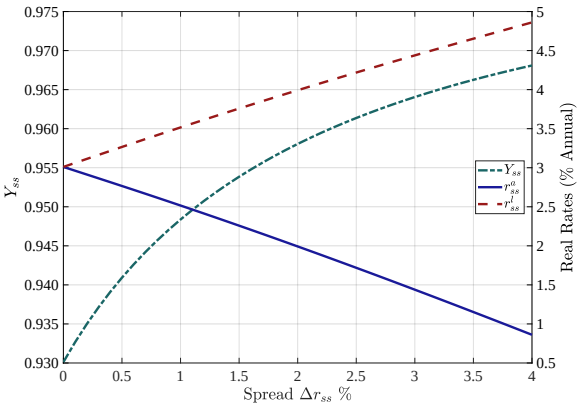
The effects are described in Figure 7. In the ex-ante phase, the announcement stimulates credit. The reason is the easing of the precautionary motive. The credit market clears with higher rates because because savers are more interest-rate elastic; as a result, both the real deposit and loans rates increase . Also, deposits increases more than loans because the CB's net-asset position worsens. Because borrowers expect an easing of credit spreads, they hit the debt limit more often which explains the output decline.

Once the reduction in spreads takes place, the easing of credit spreads reverses the effects. The reduction in the spread has an incidence on both, a lower loans rate and a higher deposit rate. A

[21] In fact, at the limit with infinite spreads, all households have $s \geq 0$, only save in currency and switch to the safe technology at $s = 0$.



(a) Wealth Distribution different Spreads



(b) Output different Spreads

Figure 6: Steady State Effects of Real Spreads.

Note: This figure depicts the real wealth distributions, output and real deposit and loan rates for different values of spread. In panel (a), the measure of households with assets $\bar{s}$ is in mass probability (left scale), and the measure of households with $s > \bar{s}$ is in probability density (right scale). Efficient output is expressed as the endowment income flow of high intensity process $y(H)$. In panel (b), Y_{ss} is the flow of households' aggregate income in steady state. Deposit and loan rates are expressed in annual percentage terms. For both panels, the real spread is expressed in annual percentage terms.

lower loans rate allows borrowers to abandon their debt limit faster. The expectation of higher future borrowing rates, is an additional force to reduce debt. The overall effect is an output expansion that overcomes the initial drop. As the policy is reverted, real rates are normalized and the distribution of wealth returns to steady state. With lower rates, there is a continued expansion in quantity of deposits, but private borrowing declines—a lower net-asset position crowds out private debt.

An important observation is that the dynamics effects after the decrease in spread have the opposite sign as the tax increase studied above, although the fiscal effect effect of the spread reduction is of similar scale. The only variable that has the same behavior, is the real deposit rate, whose response on impact is twice as high after the reduction in the spread. On impact, the responses of output is an order of magnitude larger in absolute terms. All in all, this means that the direct effect of spreads overshadows the fiscal effect. Another important observation is that the loans rate is more sensitive than the deposit rate. Recent work by Drechsler et al. (2015) documents this difference in sensitivity. Because they can control for measures of local competition, the authors attribute the effect to market power behavior by banks. Here, the deposit rate is less responsive, simply because savers are more interest-rate elastic than borrowers in an incomplete markets economy. Intuitively, borrowers are closer to their borrowing constraints, so their prudence effect makes them respond in addition to intertemporal substitution which is equally strong for the rich. Since wealth is slow moving, and real wealth has to clear, most of the incidence of the spread is on the loans rate.[22]

[22]We can again use the analogy of a spread as a tax on intermediation and interpret the result in terms of a tax incidence. Translated into the logic of tax incidence, this means that the to obtain an equilibrium in the credit market, loans rates must respond more than the deposit rate. Tax incidence is higher on the most inelastic side.

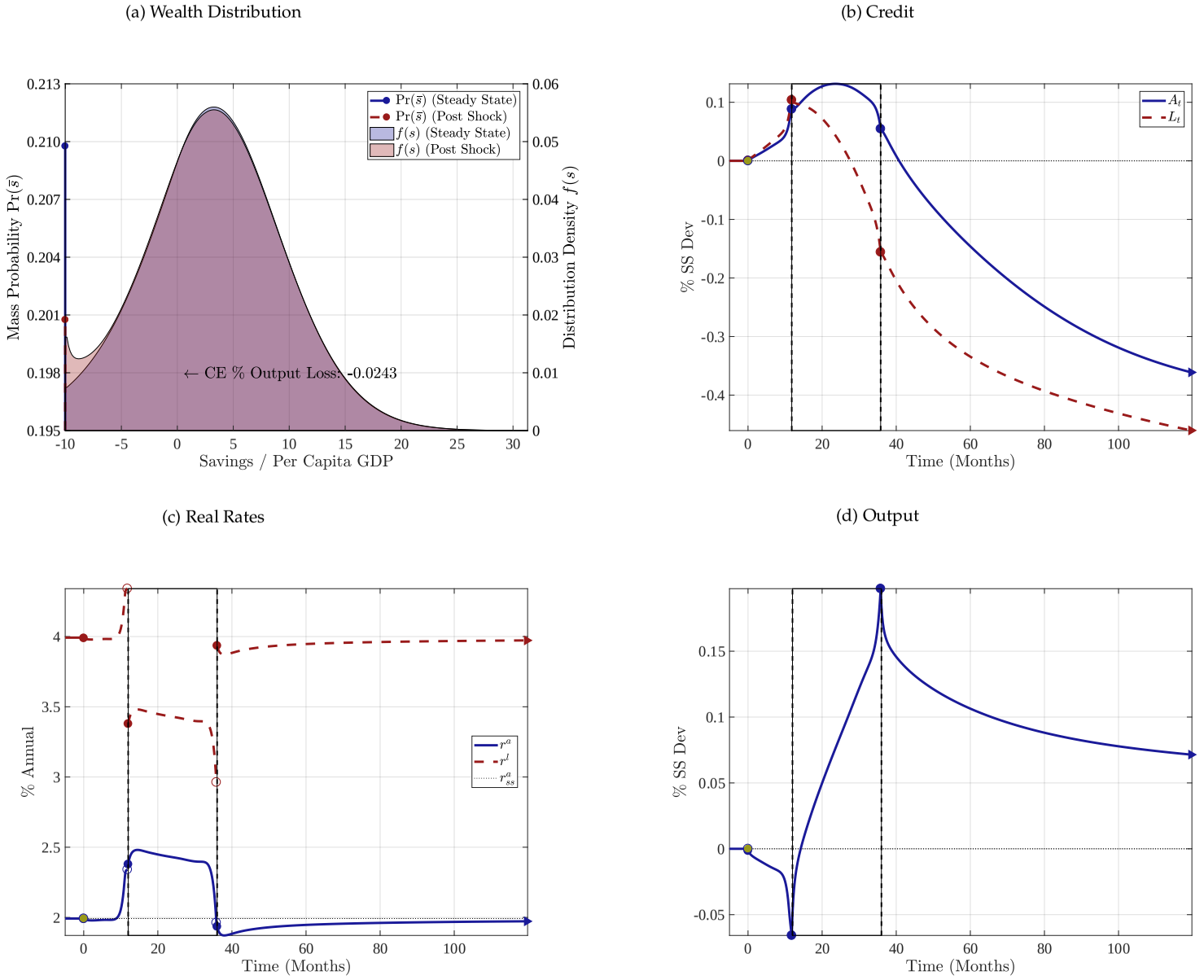


Figure 7: Transition Dynamics after a Credit Spread Reduction (Credit Target).

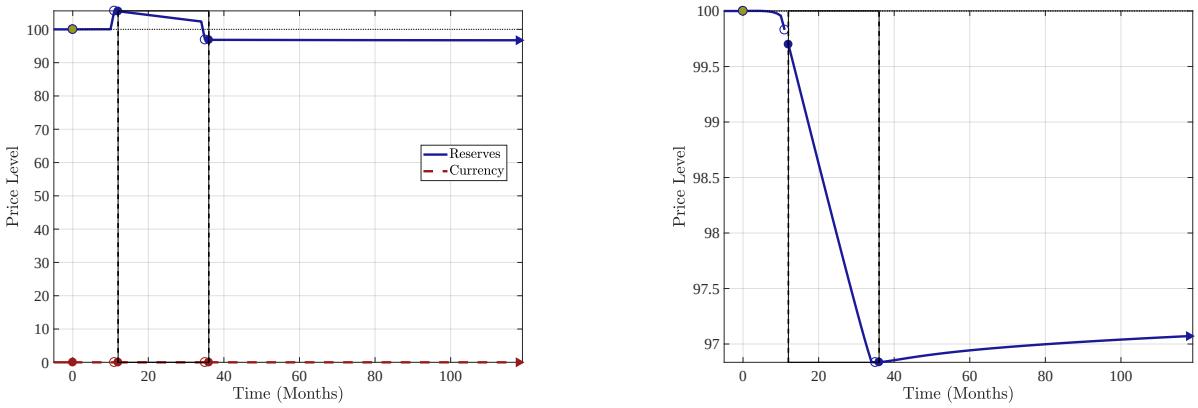
Note: The figure reports the real wealth distribution, and the responses to credit, rates and output after the anticipation of a credit crunch. In panel (a), the measure of households with assets $\bar{s}$ is in mass probability (left scale), and the measure of households with $s > \bar{s}$ is in probability density (right scale). The CE % loss is expressed in the percentage deviation of aggregate certainty equivalent after the announcement. In panels (b) and (d), the aggregate deposits, loans and output are expressed in percentage deviations from steady state. In panel (c), the real rates are expressed in annual percentages. The reduction in the spread is announced at time 0, takes place one year later (at the first vertical dashed line), and lasts for two years (between the two vertical dashed lines). The credit spread is initially fixed at 2% annual, reduces by 50% during the reduction periods, and increases back to initial value after the shock. The time path of fiscal transfers (lump-sum taxes) and government's net-asset position is as follows. During the credit reduction periods, the lump-sum transfer is fixed at steady-state value, and the net-asset position is updated according to (20). After the shock periods (after the second vertical dashed line), the lump-sum transfer is adjusted to close the gap of net-asset position between current value and steady-state value at an exponential rate of 10% annually. The detailed transition paths of net government position and fiscal transfers are given by (22) and (23) with $\tau_t \equiv 0$.

Spread Implementation and the correlation between Inflation and Monetary Aggregates. We have been silent about the implementation in the exercises. As we showed earlier, there are various ways to implement the policy, and these have different effects on the statistical correlation between inflation and monetary aggregates. Next, we discuss an implementation of a reduction in spreads via OMO, while keeping the corridor spread constant. The details of the implementation are presented in Figure 8. Panel (a) in Figure 8 shows the increase in money aggregates. An OMOs that increases reserves is carried in 12-th month and reversed two years later. The OMO are slowly reversed during the program and strongly by the end to track the evolution of deposits that shrink. The goal is to have the constant liquidity ratios, Λ, during and after the program. In the example, reserves increase by the start of the program, 5% followed by a similar decline at the end. Panel (b) depicts inflation. The lesson is that the policy leads to deflation, although the quantity of both, outside money (M0) and deposits (M1) increase! The reason is that the increase in the liquidity ratio of banks produced by the OMO, stimulates credit. The implied reduction in spreads leads to an increase in the real rate deposit rate while it also leads to a reduction in nominal deposit rate. Following Fisher's equation, the economy must converge to a lower price level, consistent with the path of inflation and the reduction in the supply of reserves. Figure 19 in Appendix E presents the decomposition of the path of the real deposit rate.

The reduction in inflation that follows an OMO is known as the liquidity effect. This effect is documented by Alvarez et al. (2009) which uses a segmented market model to rationalize the effect. The result here is also related to the unpleasant monetarist arithmetic (Wallace and Sargent, 1981). In that analysis, an OMO causes permanent deflation, because it generates capital gains that that alter the required seigniorage to balance the budget. The source of the effects here are different. The effect liquidity effect here is produced by the reduction in credit spreads. Since budget balance is not altered here, the effect on inflation is not permanent as in the unpleasant arithmetic. The lesson from the exercise is the same though, it is important to be explicit about the MP instruments used, before we can establish an empirical connection between monetary aggregates and prices. As we argued, this model is consistent with quantity theory, which presents a sharp relation between money and prices, and consistent with OMO that break that relationship. Appendix E extends this discussion to implementations with changes in the corridor rates and negative interest on reserves within a DZLB.

6 Normative Analysis: Optimal use of the Credit-Channel

This section studies the optimal spread target. The section is silent about the implementation with the understanding that MP has enough instruments to reach a desired target. In the section,



(a) Currency and Reserve Holdings

(b) Price Index

Figure 8: Monetary Aggregates and Price Level after an OMO

Note: The figure presents the responses of monetary aggregates and price index after the implementation of reducing credit spread via OMO. The OMO implements the path of credit spread in Figure 7. In panel (a), the initial levels of reserves and currency are normalized to 100 and 0, respectively. In panel (b), the initial level of price is normalized to 100.

we postulate and solve a Ramsey planner problem where the instrument is the spread.[23] We first study the steady-state solution of the Ramsey problem and its comparative statics. We also extend the framework to allow for a labor-demand externality, a feature that strengthens the case for positive spread. We then introduce a credit-crunch episode and study the benefits of reducing spreads during the creunch.

6.1 Optimal Spreads: with and without a demand externality

Aggregate Labor Demand Externality. An optimal spread turns out to be optimal even without a demand externality, but here we introduce a parameter that captures an aggregate labor demand externality (DE), a feature that strengthens the case for optimal spreads. We now endow households with a labor supply of $\bar{n}$ hours. Like in Hansen (1985) and Rogerson (1988), labor endowments are indivisible—a unit of labor is active or inactive. This supply is perfectly inelastic, there is no disutility. Also, the labor endowment of one household must be employed by others.

We also adapt the endowment process. Now, it requires a specific amount of hours $n(u)$. We let $n(H) > n(L)$ be the labor requirements of the high and low intensities. We normalize $n(H) = \bar{n}$, so if all entrepreneurs operate with high intensity, all hours are used. If any household chooses the low intensity, there is unemployment. The labor market suffers from a labor hold-up problem as in Caballero and Hammour (1998). Once a household hires hours, output is held up by the worker. In particular, workers can threat households to divert the fraction $(1 - \eta_l)$ of the

[23] Although we can directly compute steady-state welfare for different values Δr_{ss}, the value that maximizes steady-state welfare is not the steady-state solution of a planner problem. By analogy, the golden-rule consumption that maximizes steady-state welfare does not coincide with the steady-state consumption of a Ramsey growth model.

output $y(u)$. Thus, the wage is bargained after the production of output.[24] As a result, ex-post output is split into η_l to the owner of the endowment and $1-\eta_l$ to the worker. We assume that workers are diversified across all households. More importantly, η_l captures the extent of a demand externality: If $\eta_l = 1$, the choice of utilization has only an incidence on the owner of the endowment, but if $\eta_l < 1$, the it has an incidence on other households. Since the real wage given u is determined ex–post, it is non-contractable. Instead, the owner of the endowment chooses a intensity unilaterally.[25]

Since labor income is perfectly diversified, each household receive a common labor income flow:

$$w_t^l = (1-\eta_l)\int_{\bar{s}}^{\infty} y(u(s,t))\, f(s,t)\, ds.$$

The only change in the household's problem is that now its real income flow depends on the choice of others:

$$dw_t = \left(\eta_l y(u_t) + w_t^l + T_t\right) dt + \sigma(u_t)\, dZ_t$$

where $\eta_l y_t(u)$ is the own endowment flow. We see from the expression, that as others choose the low intensity, this will affect the intensity choice of the household.

Optimal Policy. The Ramsey planner solves the following problem:

Problem 3 *[Ramsey Problem] An egalitarian planner's problem maximizes $\{\Delta r(t)\}$ to maximize*

$$W(f_0) = \max_{\{\Delta r(t)\}\geq 0} \int_{\bar{s}}^{\infty} V(s,0)\, f(s,0)\, ds \tag{24}$$

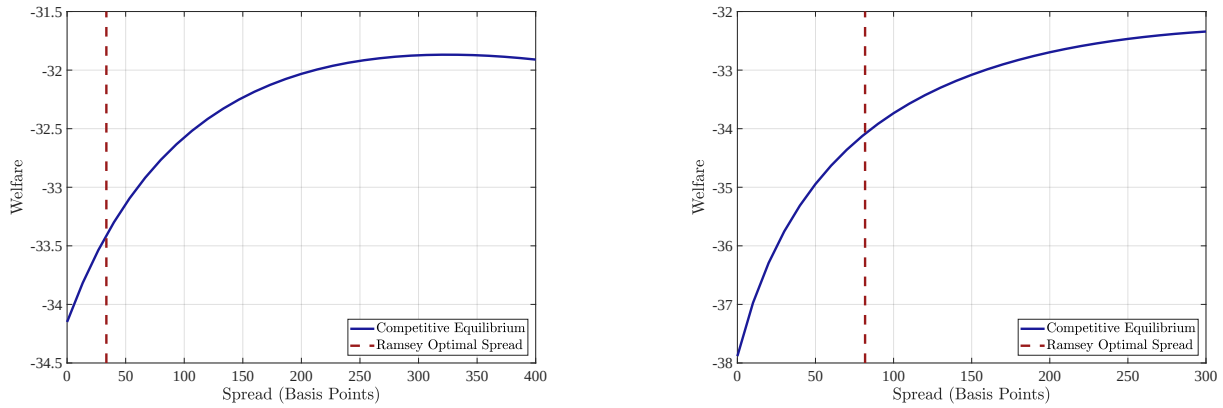
subject to the household's problem 2, the law of motion of wealth (13), a path for $\mathcal{E}_t$, and the resource constraint (18).

The solution follows Nuno and Moll (2018) and Nuno and Thomas (2017) and the derivations and a discussion is relegated to appendix G. Next, we discuss the numerical results of the solution.

[24] This construction can be approximated by a limit. Suppose that technologies are fixed over specific time intervals $\Delta t, 2\Delta t$, ... For every interval, assume that once the technology is chosen and workers are hired, contracts are negotiated on the spot and according to a bargaining problem. Presumably, this hold-up problem leads to an output split according to some Nash-bargaining problem. In that case, output is divided in η and $(1-\eta)$ shares to entrepreneurs and workers correspondingly.

[25] Labor market-clearing must be consistent with a level of unemployment:

$$\Upsilon(t) = \int_0^{\infty} \left[1 - (\mathbb{I}[u(s,t) = H] - 1)\frac{n(L)}{\bar{n}}\right] f_t(s)\, ds.$$



(a) Without DE

(b) With DE ($\eta_L = 0.9$)

Figure 9: Steady State Welfare

Note: The figure plots the aggregate steady-state welfare of the competitive equilibrium as a function of credit spreads under the scenarios without DE (panel a) and with DE (panel b). In both panels, welfare is expressed as aggregate steady-state value function across households, and the credit spread is expressed in annual basis points. The vertical dashed red line is located at the optimal Ramsey credit spread that corresponds to each scenario.

Optimal Steady-State Spread. Figure 9 plots two objects. The solid (blue) curves are the steady-state welfare obtained by varying a steady-state spread in the horizontal axis. Panel (a) reports the results without the externality ($\eta_l = 1$) and panel (b) with an externality ($\eta_l = 0.9$). The second object is the vertical dashed line (red). That line is located at the value of the spread that coincides with the steady-state (the asymptotic value) spread path chosen by the Ramsey planner. Steady welfare is increasing in a steady-state spread up to a spread of 300bps. The steady-state spread of the Ramsey problem is lower, close to 50bps. The Ramsey steady-state spread is lower because, as in a Ramsey growth model, steady-state consumption does not equal the constant consumption that maximizes welfare, an artifact of time discounting. Here, the force that causes the steady-state of the optimal spread under the constant spread that maximizes steady-state welfare is also impatience. The distribution of wealth moves slowly with the spread, but the reduction in insurance (despite the increase in transfers) occurs immediately. Hence, the planner discounts the steady-state benefits of an increase in spreads, and is concerned with contemporaneous insurance. Nevertheless, the he does not go all the way, and allows a positive spread at the limit of each policy path.

To unpack the forces that drive a positive spread at steady-state, we can decompose the effect. Consider the value in (24) at steady state. It can be represented as

$$W_{ss}\left(\Delta r_{ss}\right) \equiv \int_{\bar{s}}^{\infty} V_{ss}\left(s;\Delta r_{ss}\right) f_{ss}\left(s;\Delta r_{ss}\right) ds.$$

Thus, small changes in Δr_{ss} are decomposed into:

$$\frac{\partial W_{ss}\left(\Delta r_{ss}\right)}{\partial \Delta r_{ss}} = \underbrace{\int_{\bar{s}}^{\infty} \frac{\partial V_{ss}\left(s;\Delta r_{ss}\right)}{\partial \Delta r_{ss}} f_{ss}\left(s;\Delta r_{ss}\right) ds}_{\text{Value Effect}} + \underbrace{\int_{\bar{s}}^{\infty} V_{ss}\left(s;\Delta r_{ss}\right) \frac{\partial f_{ss}\left(s;\Delta r_{ss}\right)}{\partial \Delta r_{ss}} ds}_{\text{Composition Effect}}.$$

The first term, is a value effect effect measures how the spread changes the value function of each agent, while holding the distribution of wealth fixed. The second term, the composition effect, captures the induced change in the wealth distribution. The value effect can be further decomposed into three forces that we encountered earlier. A direct effect (1) occurs because, holding the deposit rate fixed, the increase in the spread lowers the borrowing rate. Thus, the direct effect is always negative and it hurts borrowers. The fiscal effect (2) emerges because the spread produces revenues that translate into transfers. A general equilibrium effect (3) occurs because the policy causes a parallel shift in the real deposit and loans rates, holding fixed the spread. This effect encodes the potential increase in output. Typically, it amplifies the reduction rates—to promote consumption and, hence, market clearing when the inefficiency falls when less agents hit their debt limits. This effect tends to help borrowers, but hurts the wealthy.

Figure 10 presents a numerical decomposition of how the different effects benefit agents across the distribution of wealth—abstracting from the externality for now. To give a sense of magnitude, we plot each agent's certainty equivalent welfare relative to efficient output. Numerically, we increase the spread by 10bps departing from the Ramsey optimal. We decompose the change in the value function of each agent according to the three forces—higher-order effects are negligible. We observe that the direct effect is negative for every agent, as anticipated. The fiscal effect is positive because transfers are received by all households. The GE effect is positive for all borrowers but hurts almost all savers—not all because savers with little wealth are likely to become borrowers and the reduction in rates is a form of insurance. The overall value effect is a positive improvement of the middle class.[26]

[26]The rich care little about the transfers, but do care about the reduction in the real deposit rate. The very poor suffer from a higher loans rate. The middle benefits from lower rates and the transfers.

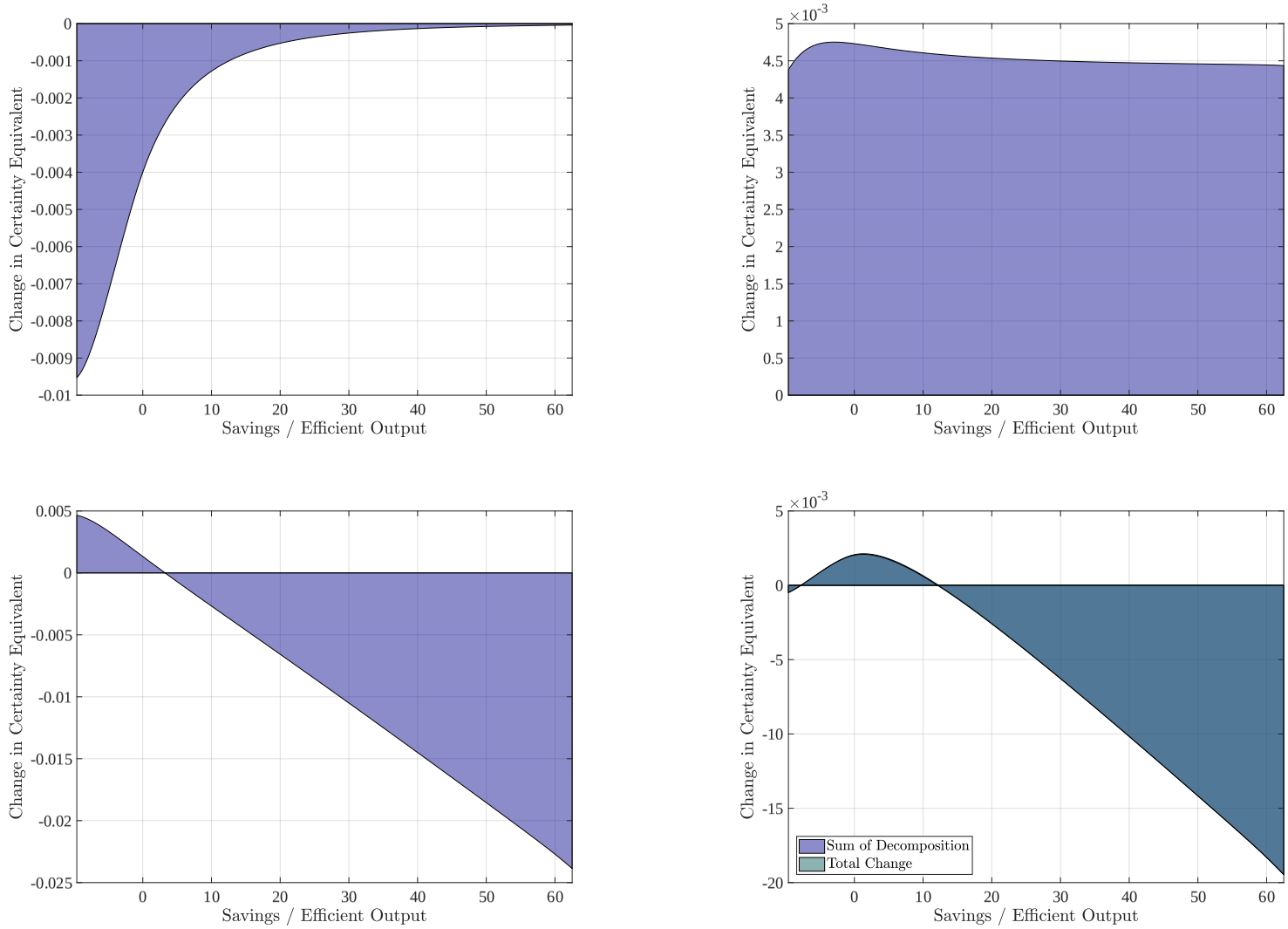


Figure 10: Steady-State Decomposition of Value Effect by Types

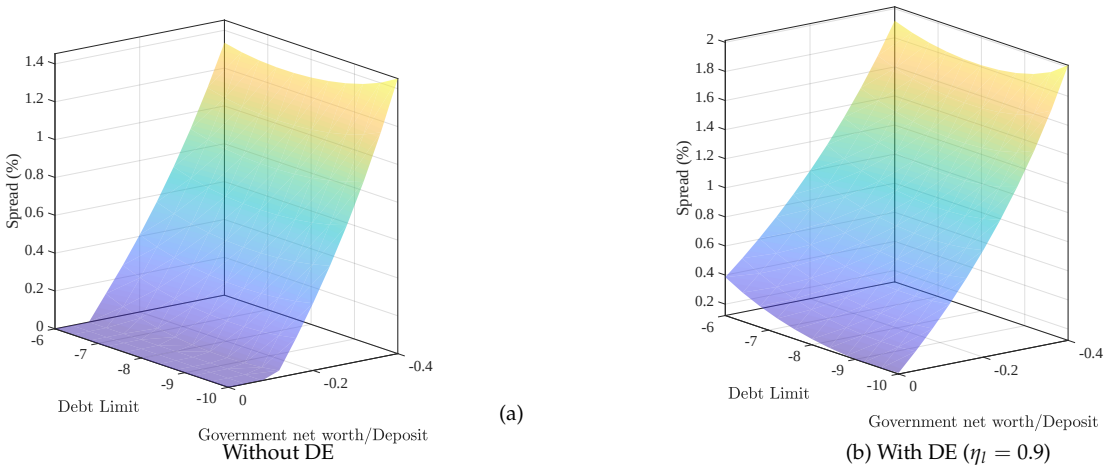
Note: The figure depicts the decomposition of the steady-state value effect of credit spread for agents with different savings under the scenario without DE. The value effect is measured and decomposed after increasing the spread 0.1% from the Ramsey optimal steady-state level. The welfare change is expressed in certainty equivalent. Efficient output is $y(H)$. In panel (d), "Sum of Decomposition" is computed by adding up the three effects reported in panel (a)-(c), and "Total Change" is computed directly from the agents' value functions.

Change in Spread	w/o DE 0.34% → 0.44%	w/ DE 0.82% → 0.92%
	Decomposition (% of Efficient Output)	
Agg. CE before Change	88.53	88.25
Agg. CE after Change	88.59	88.32
Change in CE	0.061	0.069
Value Effect	0.037	0.033
Composition Effect	0.019	0.031
Value plus Composition Effect	0.056	0.064
Higher-Order Effect	0.005	0.005
	Decomposition (% of Value Effect)	
Agg. Direct Effect of Spread	-1347.05	-1511.94
Agg. Fiscal Effect	1235.11	1307.45
Agg. GE Effect	217.81	310.98
Sum of First-Order Decomposition	105.88	106.49
Second-Order Effect	-5.88	-6.49

Table 4: Decomposition of Certainty Equivalent Change in Planner's Welfare

Table 4 reports summary of the aggregate effects of the 10bps increase—the last column are the welfare changes with the externality. Welfare increases with the perturbation because the Ramsey steady-state is below the spread that maximizes welfare. To obtain a quantitative sense, numbers are reported in planner certainty equivalent relative to efficient output. The highest CE possible for the planner is efficient output—the value obtained under complete markets and no inequality. The aggregate change CE is about 0.06%. Of this amount, the value effect accounts for more than half, and a smaller portion from the composition effect. The decomposition of the value effects summarizes the information conveyed by Figure 10. We can see that the GE effect compensates the losses from the direct effect, which mitigated by the fiscal effect. The Ramsey planner does not increase the spread, because, the benefits of the GE effect can take longer to kick in a transition, than the reduction in insurance produced by the direct effect.

Comparative Statics. The steady-state optimal Ramsey spread is not always positive and can be zero depending on parameters. Figure 11 depicts in the verical axis the optimal steady-state spread of the Ramsey problem as a function of debt limit ($\bar{s}$) and the net-asset position relative to deposits $\varepsilon_f \equiv \mathcal{E}_{ss}/\mathcal{A}_{ss}$. The left panel is constructed with $\eta_l = 1$ and the right with $\eta_l = 0.9$. The first observation is that the spread is higher as the net-asset position deteriorates. When the net-asset position is negative, the government is issuing debt. Debt is paid by everyone, but the interest rate is only received by the wealth. The spread is a way to redistribute the tax burden toward the poor. The optimal spread is non-monotonic in the debt limit. The intuition is that the debt limit governs the extent of the output inefficiency, which is also non-monotonic in the debt limit. In fact, the Ramsey spread almost mirrors the behavior out the output without a spread. A corroboration of this result is that when we activate the labor demand externality, the optimal



(a) Without DE

(b) With DE ($\eta_l = 0.9$)

Figure 11: Optimal Steady-State Spread of Ramsey Problem

Note: The figure plots the Ramsey optimal steady-state spread as a function of debt limit, $\bar{s}$, and government net-asset position relative to deposits, $\mathcal{E}_{ss}/A_{ss}$, for the scenarios without DE (panel a) and with DE (panel b). The spread is expressed in annual percentage terms.

spread increases.

6.2 Spread Management during a Credit Crunch

Credit Crunch. We now study the benefits of relaxing spreads during a credit crunch. To introduce a credit crunch, we modify the model. In addition to the *debt limit* $\bar{s}$, we introduce a potentially time-varying *borrowing limit*, $\tilde{s}_t$. The borrowing limit is triggered before the household reaches its debt limit, $\bar{s} \leq \tilde{s}_t \leq 0$. The idea is that if households reach their borrowing limit, they cannot take on more debt principle, but they can roll it over. That is, in $s \in [\bar{s}, \tilde{s}_t]$,households can refinance their interest payments, but not take more debt. Formally, this means that $ds_t \geq r_t s_t dt$ in $s \in [\bar{s}, \tilde{s}_t]$. Thus, tthe earlier constraint now reads $c_t dt \leq r_t s_t dt + dw_t$ in $s \in [\bar{s}, \tilde{s}_t]$ and thus, the safe endowment $u_t = L$ is forced in $s \in [\bar{s}, \tilde{s}_t]$. The household's HJB is modified to take into account these new constraints.

Intuitively, $\tilde{s}_t$ triggers the inefficient choice earlier. We interpret an increase in $\tilde{s}_t$ as a credit crunch. This distinction between borrowing and debt limits has technical and economic motivations. The technical motivation is that it allows to study an unexpected credit crunch—an unexpected jump in the debt limit is now well defined mathematically.[27] The economic motivation is that if banks wants to cut back on credit, it is convenient to tighten the borrowing limit,

[27]Suppose we want to study a credit crunch by an unexpected tightening of the debt limit. If there is an unexpected change change in the debt limit, there would be a positive mass of households violating their debt limits because income flows continuously. This inconvenience does not apply when the borrowing limit $\tilde{s}_t$ moves unexpectedly. In the latter case, households now face a problem insuring risk, but are not forced to reduce their debt stock immediately. This is a technical assumption to circumvent an issue faced by models with debt limits. For example, Guerrieri and Lorenzoni (2017) must study a gradual shock to debt limits precisely to leave agents enough time to abandon their borrowing limits.

but not necessarily the debt limit.[28]

A Credit Crunch. Lets first discuss the transitions produced by a credit crunch, for now holding a fixed spread. We introduce a temporal expected increase in $\tilde{s}_t$, starting from $\tilde{s}_{ss} = \bar{s}$. The borrowing limit is known to tighten to $\tilde{s}_t = 0.8 \cdot \bar{s}$ in a year, and that the effect will lasts two years. Figure 12 shows the dynamics after the crunch. The anticipation of the crunch leads to a reduction in credit, because it is known that being in debt in the future will be painful. Naturally, borrowers want to payoff their debts, but then savers must hold less deposits. Panel (b) shows how both real deposits and loans fall along the transition. As a result, real deposit rates must fall, to discourage savers from savings. The borrowing rate also falls, because the spread is constant, and borrowers are less interest rate sensitive—Panel (c). In the ex-ante phase, output actually expands ass the mass of agents in the debt limit falls. Once the crunch takes place, a large mass of agents is suddenly in the borrowing-constrained region, $s_t \in [\bar{s}, \tilde{s}_t]$. This forces households in that region to the inefficient choice. The consequence is an immediate output collapse. Output falls continuously, as more households are dragged into the borrowing constrained region. The expectation of a recovery produces an increasing path of real interest rates—because borrowing constrained households roll over a greater stock of debt. Credit continues to decrease until it reverses as the end of the crunch approaches.

[28] When a bank extends a loan principal, it increases its liabilities. This is not true about a roll over. In the case of a rollover, banks earn interests which increase equity, but not liabilities. During financial crises, banks will want to roll over debt, although they are unwilling to extend principal since the latter consumes regulatory capital. In addition, if loan repayment is suddenly forced, it can trigger default. Defaults are costly for banks, because they lead to underwritings that also subtract regulatory capital. The formulation here is motivated by these observations, although their explicit modeling is outside the scope of the paper. This phenomenon is called evergreening. We do not model this explicitly, but we are guided by this economic interpretation. Our constraint is consistent with the interpretation. Caballero et al. (2008) present a model of evergreening.

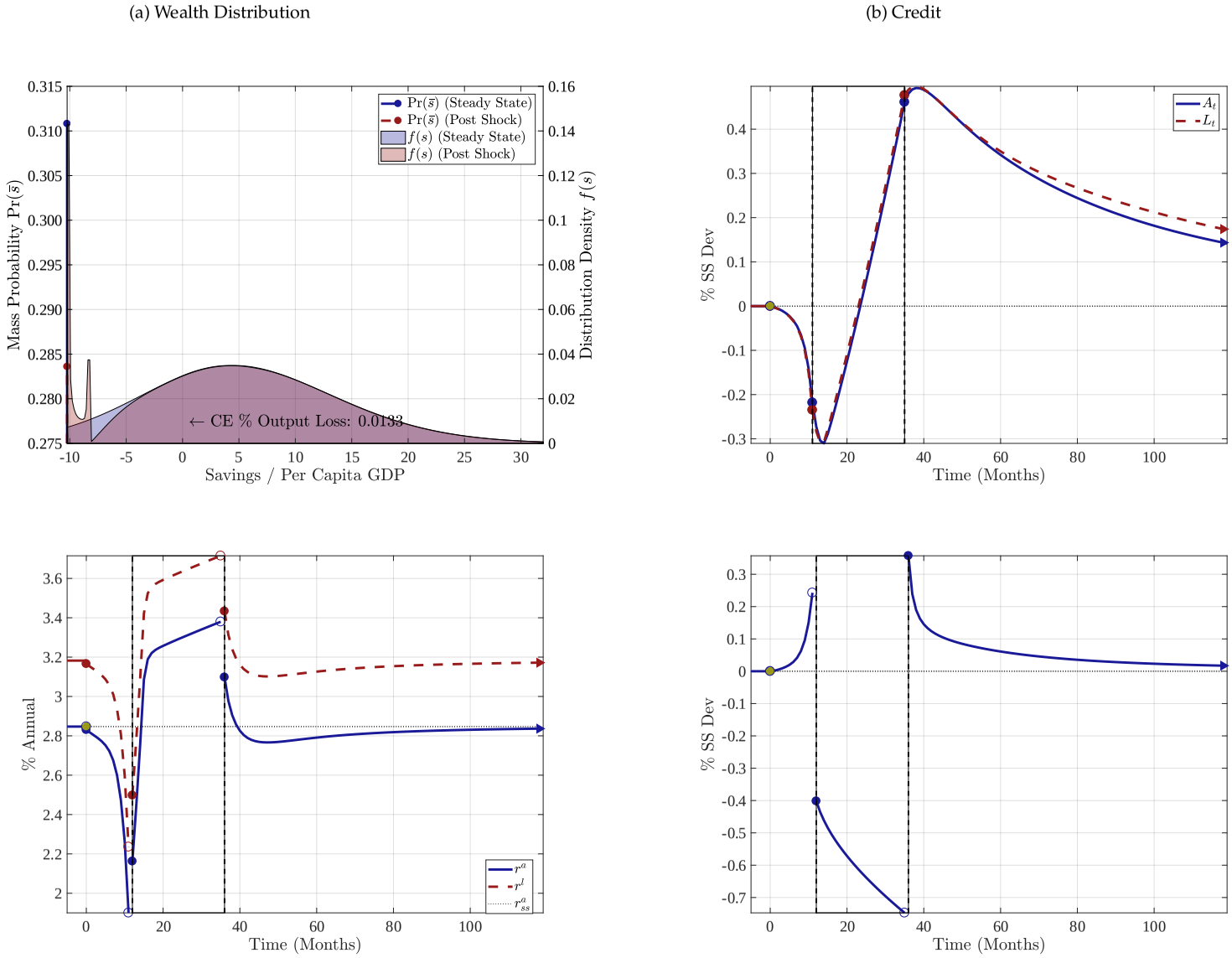


Figure 12: Transition Dynamics after a Credit Crunch (Passive Policy).

Note: The figure reports the real wealth distribution, and the responses to credit, rates and output after the anticipation of a credit crunch. In panel (a), the measure of households with assets $\bar{s}$ is in mass probability (left scale), and the measure of households with $s > \bar{s}$ is in probability density (right scale). The CE % loss is expressed in the percentage deviation of aggregate certainty equivalent after the announcement. In panels (b) and (d), the aggregate deposits, loans and output are expressed in percentage deviations from steady state. In panel (c), the real rates are expressed in annual percentages. The credit crunch is anticipated at time 0, takes place one year later (at the first vertical dashed line), and lasts for two years (between the two vertical dashed lines). The borrowing limit $\bar{s}$ is initially equal to $\bar{s}$, increases to $0.8 \cdot \bar{s}$ during the credit crunch periods, and decreases back to initial value after the shock. The time path of fiscal transfers (lump-sum taxes) and government's net-asset position is as follows. During the crunch, lump-sum transfers are at steady-state, and the net-asset position is updated according to (20). After the crunch (second vertical dashed line), lump-sum transfers are adjusted to let the net-asset position return to steady-state at an annual exponential rate of 10%. The detailed transition paths of net government position and fiscal transfers are given by (22) and (23) with $\tau_t \equiv 0$.

Spread	CE w/o DE	CE w/ DE
Floor	88.209	86.929
Passive Corridor	88.496	88.225
Active Corridor	88.497	88.233

Table 5: Aggregate (% of Efficient Output) Welfare in Transition Path

A Countercyclical Spread. We now consider how welfare varies with three different policies. The crunch is now unexpected. In the first scenario (floor), the steady-state is produced with a zero spread, and the spread is kept at zero during the crunch, as if the central bank runs a permanent floor system. In the second scenario (passive corridor), the spread is kept constant at the steady-state Ramsey value, starting from the steady state and throughout the transition. In the third, the steady-state starts from the optimal Ramsey spread, but is dropped to zero during the duration of the crunch, and reverts to back the Ramsey optimum when the crunch is over. The latter scenario is implemented with a large OMO that satiates banks with reserves during the crunch. Table 6.2 summarizes reports the time-zero CE for the value, relative to efficient output. The floor scenarior yields the lowest value by fare. This is because a zero spread produces a low steady-state welfare for the planner. A passive corridor does much better. The policy that satiates banks with reserves during the crunch, the active corridor, does better. The reason is that with a lower spread during a crunch, agents in debt repay their debt faster. The right columns verifies again that the externality magnifies the results, and builds a stronger case for the active management of spreads.

Finally, Figure 13 presents the effect on output under the three scenarios. A first comparison is the floor system with and without the externality (labeled floor w/ DE and floor w/o DE). Output with the demand externality is lower for the entire path. A passive corridor boosts output, to the point that it more than offsets the externality. However, the active corridor is the best policy. Most of the benefit from the active policy is seen after the crunch is over. The active policy cannot compensate for the large mass of agents at the borrowing limit because wealth is a slow moving object. However, the active corridor induces a stronger recovery because once the crunch is over, lower spreads enable borrowers to repay their debts faster during the crunch, and this enhance ex-post efficiency. During the 2008 financial crisis, the Federal Reserve flooded banks with reserves. The analysis suggest that that policy was correct, and rationalizes why it had no impact on inflation. The analysis is also suggestive that it may be time tor return to a corridor system.

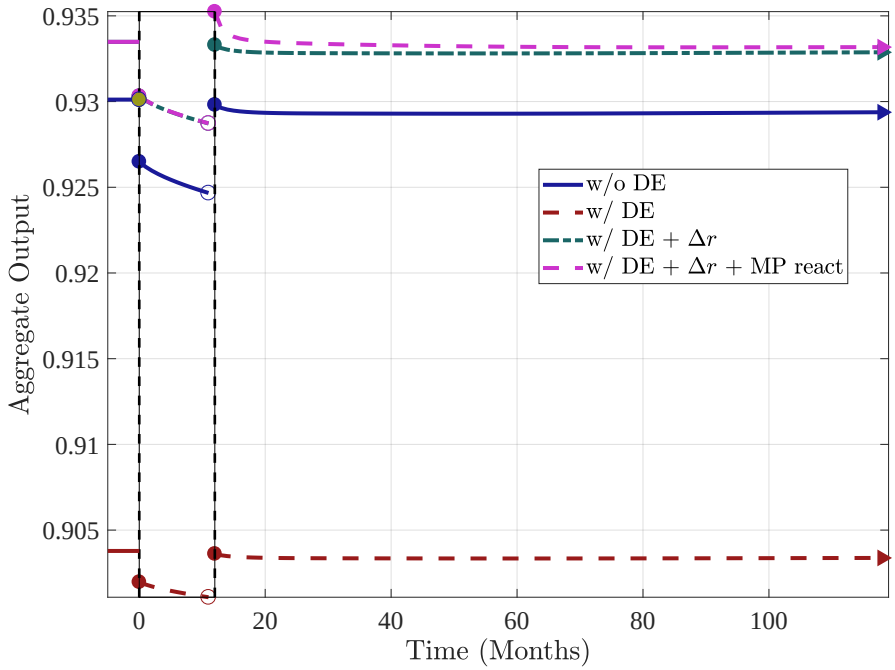


Figure 13: Transition Paths after Credit Crunch and Policy Responses

Note: The figure reports the response of real output after an unanticipated credit crunch for different scenarios of policy responses and demand externalities. The label "w/o DE" refers to a case without DE and a zero spread, "w/ DE" with DE and zero spread, "w/ DE + Δr" means with DE and a positive spread, "w/ DE + Δr + MP react" means with DE, a positive spread and closing spread to 0 during credit crunch. The credit crunch takes place at time 0 (at the first vertical dashed line), and lasts for one year (between the two vertical dashed lines). The borrowing limit $\tilde{s}$ is initially equal to $\bar{s}$, increases to $0.8 \cdot \bar{s}$ during the credit crunch periods, and decreases back to initial value after the shock. The time path of fiscal transfers (lump-sum taxes) and government's net-asset position is as follows: During the credit reduction periods, the lump-sum transfer is fixed at steady-state value, and the net-asset position is updated according to (20). After the shock periods (after the second vertical dashed line), the lump-sum transfer is adjusted to close the gap of net-asset position between current value and steady-state value at an exponential rate of 10% annually. The detailed transition paths of net government position and fiscal transfers are given by (22) and (23) with $\tau_t \equiv 0$.

7 Conclusion

In the final paragraph of the introduction to his collected works on monetary economics, Lucas (2013), Robert E. Lucas writes: "Now, toward the end of my career as at the beginning, I see myself as a monetarist. My contributions to monetary theory have been to incorporate the quantity theory into modern modeling. For the empirically well established predictions —long-run links— this job has been accomplished. On the harder questions of monetary economics — the real effects of monetary instability, the roles of inside and outside money, this work contributes examples but little in empirically successful models. It is understandable that in the leading operational macroeconomic models today— the RBC and the New Keynesian models—money as a measurable magnitude plays no role at all, but I hope we can do better than this in the future."

The model here is actually a descendant of on one of Lucas's early monetary models, Lucas (1980). Here, outside money (reserves) is an input for inside money creation (deposits and loans). The current attempt tries to bring a modern implementation of MP to an economy with incomplete markets. The novelty is that MP operates by controlling spreads. If we are open to accept that idea, we challenge some traditional views. For example, we may challenge the idea that MP is long-run neutral[29], and that inflation and monetary aggregates are tied together, two working restrictions in conventional empirical work. The model, rationalizes several empirical regularities. For example, the presence of a liquidity effect and a higher loan than deposit rate elasticity to policy changes. A normative message is that managing spreads is desirable: although spreads limit risk sharing, they may improve efficiency. In the case of a credit crunch, counter-cyclical spreads implemented via open-market operations, are a desirable policy that do not compromise inflation. Hence, the advice to remain with active corridor systems.

[29]This feature is also true in other incomplete-market economies with money, the reason is not the spread, but the effect of real monetary balances on credit markets.

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Appendix

A Accounting in the Model

A.1 Balance Sheets

Household Balance Sheet. The household's balance sheet in in nominal terms is structured as:

Assets	Liabilities
m_t^h	l_t^h
a_t^h	$P_t s_t$

Bank Balance Sheet. The balance sheet of an individual bank is structured as:

Assets	Liabilities
m_t^b	a_t^b
l_t^b	

CB Balance Sheet. The balance sheet of the CB is structured as:

Assets	Liabilities
L_t^f	M_t
	E_t

Accounting of OMO. To interpret OMO as purchases of government debt, consider F_t as an outstanding amount of nominal bonds issued by a fiscal authority. Let $F_t^{cb} < F_t$ be the stock of bonds held at the CB. In that case, the balance sheet of the consolidated government is

Assets	Liabilities
F_t^{cb}	$M_t + F_t$
	E_t

$=$

Assets	Liabilities
$F_t^{cb} - F_t$	M_t
	E_t

Thus, $L_t^f = F_t^{cb} - F_t < 0$ is the stock of government bonds held by banks and E_t is the stock of government liabilities net of CB purchases. A conventional open-market operation is simply an increase in F_t^{cb} funded with an increase in M_t. From the government's income flow, we can see that this operation would yield profits to the CB if there's a spread $i_t^l > i_t^m$.

Monetary Aggregates. The monetary aggregates are given by, M_t, the monetary base, $M0_t$, the currency and $M1_t \equiv A_t^b + M0_t$, the highest monetary aggregate.

Timeline of Interbank transactions. Figure A.1 presents the accounting for banks, within a Δ time interval. Unlucky banks get hit by negative withdrawal shocks, which can lead them to a negative balance of reserves in the

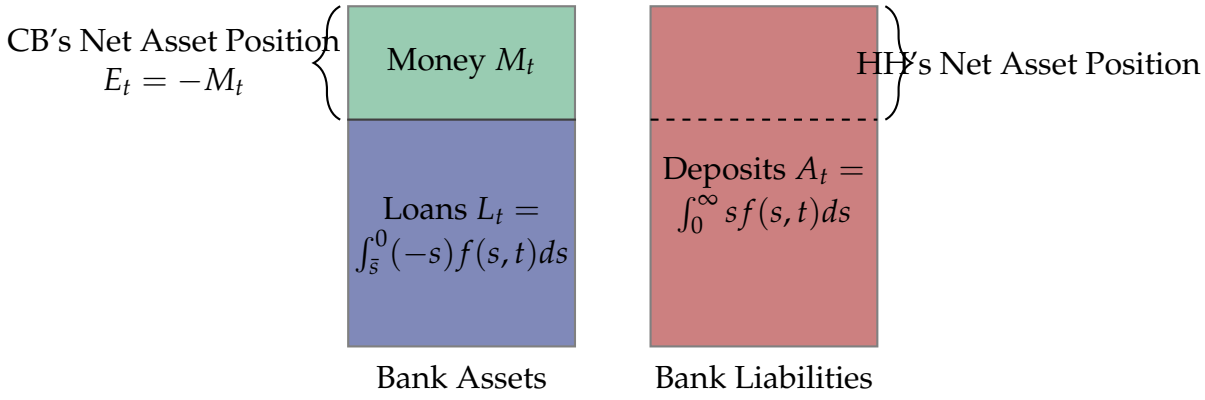


Figure 14: Baseline Bank Balance Sheet

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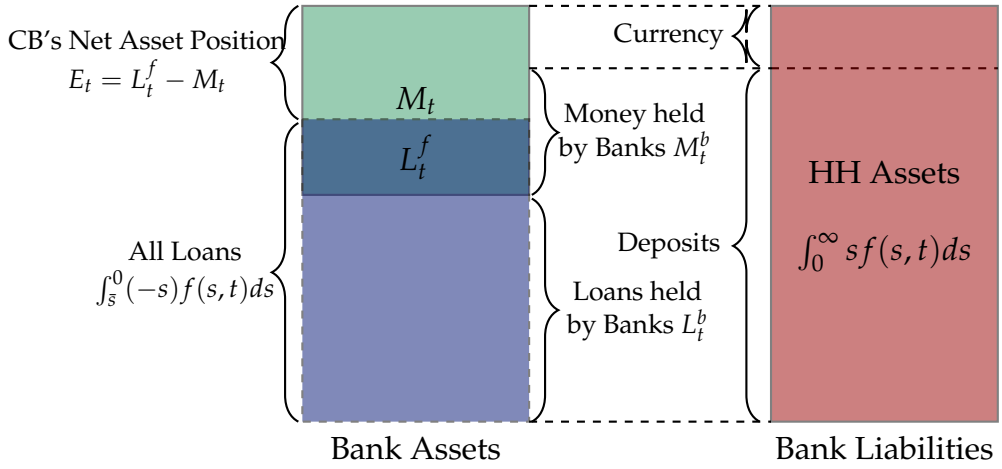
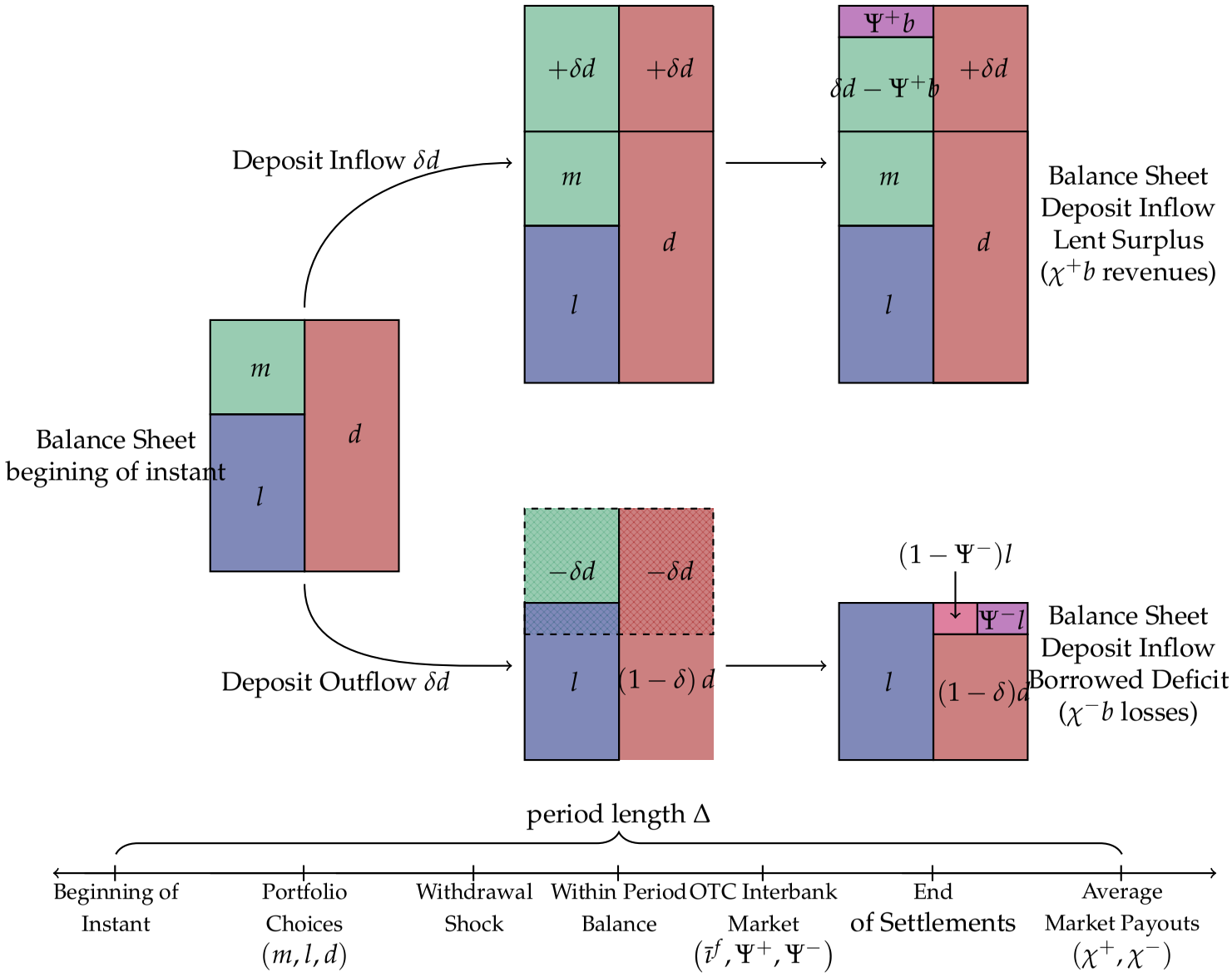


Figure 15: Bank Balance Sheet under ZLB/OMO

.

period. That bank mus cover the position by the end of the interval by borrowing funds from other banks, or from the discount window.


Balance Sheet
begining of instant
m
l
d
Deposit Inflow δd
$+\delta d$
$+\delta d$
m
l
d
$\Psi^+ b$
$\delta d - \Psi^+ b$
$+\delta d$
m
l
d
Balance Sheet
Deposit Inflow
Lent Surplus
($\chi^+ b$ revenues)
Deposit Outflow δd
$-\delta d$
$-\delta d$
l
$(1-\delta)\, d$
$(1-\Psi^-)l$
$\Psi^- l$
l
$(1-\delta)d$
Balance Sheet
Deposit Inflow
Borrowed Deficit
($\chi^- b$ losses)
period length Δ
Beginning of Instant
Portfolio Choices (m, l, d)
Withdrawal Shock
Within Period Balance
OTC Interbank Market $(\bar{\imath}^f, \Psi^+, \Psi^-)$
End of Settlements
Average Market Payouts (χ^+, χ^-)

A.2 Endowment Choice

The household's optimal policy is easy to characterize. The choice between risky and safe endowments is separable from the consumption and portfolio choices. The only portfolio choice is the currency-deposit composition when wealth is positive. This choice depends only on the nominal deposit rate: they hold only deposits when the nominal deposit rate is positive, they are indifferent between currency and deposits only if the nominal deposit rate is zero, and they strictly prefer currency if deposits are negative. The latter case never occurs in equilibrium. Consumption is given by a simple first-order condition: $U'(c) = V_s$. Finally, the risky endowment is chosen whenever:

$$\frac{Y(H) - Y(L)}{\frac{1}{2}\sigma^2(H)} \geq -\frac{V_{ss}}{V_s} = \gamma \frac{c_s(s,t)}{c(s,t)}.$$

The interpretation of this rule is that as long as the precautionary motive is not too strong, households prefer the risky endowment.

A.3 Flow of Funds Identities

Lemma 2 *If the deposit, loans and money markets clear, then:*

$$P_t \int_0^\infty s f(s,t) ds = -P_t \int_{\bar{s}}^0 s f(s,t) ds - E_t. \tag{25}$$

Proof. The deposits and loans markets clearing condition requires:

$$A_t^b = \int_0^\infty a_t^h(s) f(s,t) ds \tag{26}$$

$$L_t^b + L_t^f = \int_{\bar{s}}^0 l_t^h(s) f(s,t) ds, \tag{27}$$

and clearing in the money market requires:

$$M_t^b + M0_t = M_t \tag{28}$$

We also have that the budget constraint (balance sheet) of banks satisfies the following identity:

$$A_t^b = L_t^b + M_t^b. \tag{29}$$

Real household assets are held as nominal deposits or currency, hence:

$$P_t \int_0^\infty s f(s,t) ds = \int_0^\infty a_t^h(s) f(s,t) ds + M0_t. \tag{30}$$

and, similarly for liabilities:

$$-P_t \int_{\bar{s}}^0 s f(s,t) ds = \int_{\bar{s}}^0 l_t^h(s) f(s,t) ds. \tag{31}$$

Once we combine (26), (27) and (29), we obtain a single condition:

$$\int_0^{\infty} a_t^h(s) f(s,t) ds = \int_{\bar{s}}^{0} l_t^h(s) f(s,t) ds - L_t^f + M_t^b. \tag{32}$$

This condition can be expressed in terms of real household wealth, with the use of (30) and (31):

$$P_t \int_0^{\infty} s f(s,t) ds = -P_t \int_{\bar{s}}^{0} s f(s,t) ds - L_t^f + M_t^b + M0_t.$$

If we use the money market clearing-condition, (15), and employ the definition of net-asset position of the CB, we obtain (25). **QED.**

B Interbank-Market Equilibrium and Implementation Figures

The parameter λ captures the matching efficiency of the interbank market.[30] According to Bianchi and Bigio (2017b), the corresponding trading probabilities for surpluses and deficit positions along a trading session are:

$$\psi^{+}(\theta) \equiv \begin{cases} 1-e^{-\lambda} & \text{if } \theta \geq 1 \\ \theta\left(1-e^{-\lambda}\right) & \text{if } \theta < 1 \end{cases}, \qquad \psi^{-}(\theta) \equiv \begin{cases} \frac{1-e^{-\lambda}}{\theta} & \text{if } \theta > 1 \\ 1-e^{-\lambda} & \text{if } \theta \leq 1 \end{cases}.$$

The resulting average interbank market rate is determined by the average of Nash bargaining over the positions and is given by:

$$\bar{i}^{f}(\theta, i^{m}, \iota) \equiv \begin{cases} i^{m}+\iota-\left(\left(\frac{\bar{\theta}(\theta)}{\theta}\right)^{\eta}-1\right)\left(\frac{\theta}{\theta-1}\right)\left(\frac{\iota}{e^{\lambda}-1}\right) & \text{if } \theta > 1 \\ i^{m}+\iota(1-\eta) & \text{if } \theta = 1 \\ i^{m}+\iota-\left(1-\left(\frac{\bar{\theta}(\theta)}{\theta}\right)^{\eta}\right)\left(\frac{\theta/\bar{\theta}(\theta)}{1-\theta}\right)\left(\frac{\iota}{e^{\lambda}-1}\right) & \text{if } \theta < 1 \end{cases}, \tag{33}$$

and the average liquidity-yield functions are

$$\chi^{+}(\theta, \iota) = \iota\left(\frac{\bar{\theta}(\theta)}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}(\theta)^{1-\eta}-\theta}{\bar{\theta}(\theta)-1}\right) \text{ and } \chi^{-}(\theta, \iota) = \iota\left(\frac{\bar{\theta}(\theta)}{\theta}\right)^{\eta}\left(\frac{\theta^{\eta}\bar{\theta}(\theta)^{1-\eta}-1}{\bar{\theta}(\theta)-1}\right), \tag{34}$$

where η is a parameter associated with the bargaining power of banks with reserve deficits, and $\bar{\theta}(\theta)$ is the end-of-day market tightness:

$$\bar{\theta}(\theta) = \begin{cases} 1+(\theta-1)\exp(\lambda) & \text{if } \theta > 1 \\ 1 & \text{if } \theta = 1 \\ \left(1+\left(\theta^{-1}-1\right)\exp(\lambda)\right)^{-1} & \text{if } \theta < 1 \end{cases}. \tag{35}$$

Thus, the path for $\left\{\psi_t^{+}, \psi_t^{-}, \bar{i}_t^{f}, \chi_t^{+}, \chi_t^{-}\right\}$ is given by $\psi_t^{+} \equiv \psi^{+}(\theta_t)$, $\psi_t^{-} \equiv \psi^{-}(\theta_t)$, $\bar{i}_t^{f} \equiv \bar{i}^{f}(\theta_t, i_t^{m}, \iota_t)$, $\chi_t^{+} \equiv \chi^{+}(\theta_t, \iota_t)$ and $\chi_t^{-} \equiv \chi^{-}(\theta_t, \iota_t)$. In the paper, we set $\eta = 1/2$. By replacing $\bar{\theta}(\theta)$ with (35) and setting $\theta < 1$, equations (33) and (34) reduce to (2) and (3).

B.1 Additional Implementation Figures

[30] This can be shown very easily using a differential form.

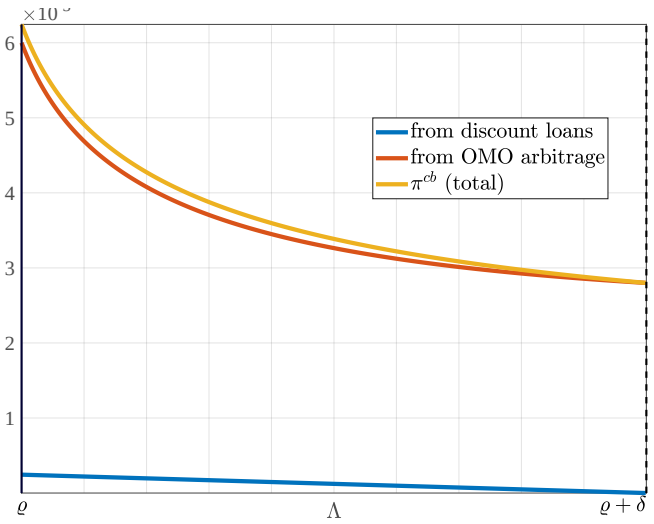
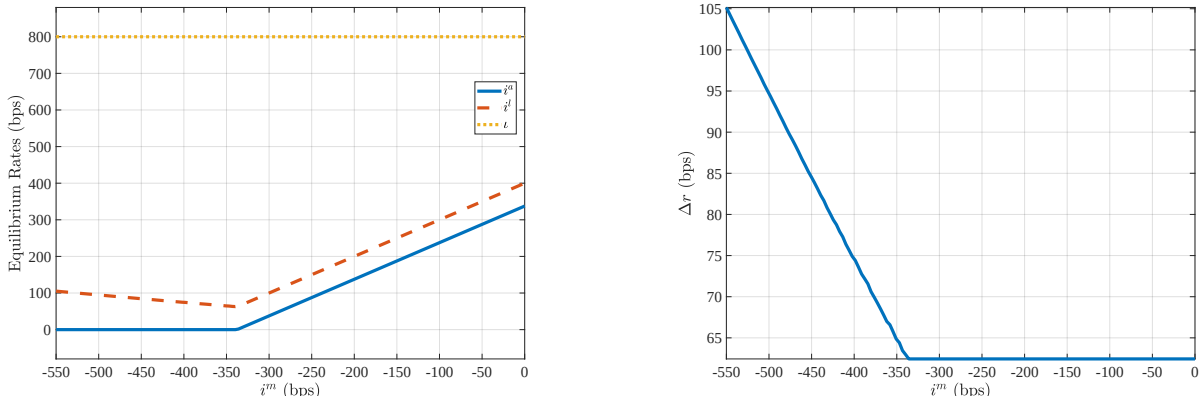


Figure 16: Composition of CB profit margins given Λ

Note: This figure plots the components of CB's profits over deposits as a function of liquidity ratio.

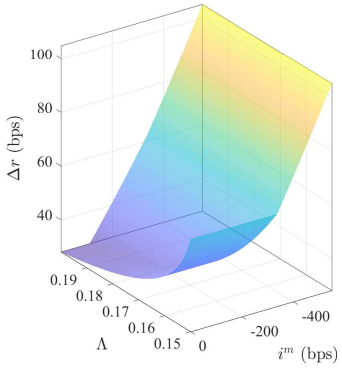


(a) Equilibrium Rates

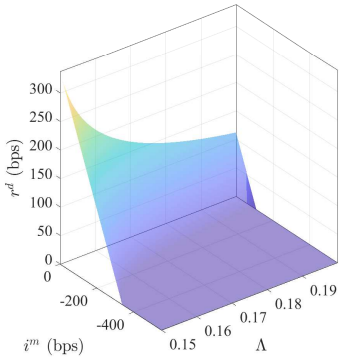
(b) Equilibrium Spread

Figure 17: Negative Interest on Reserves and the DZLB.

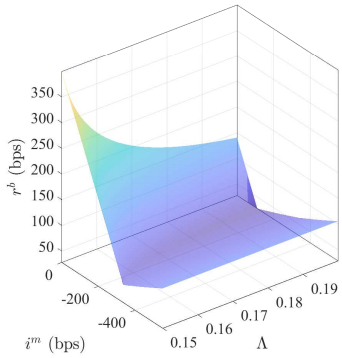
Note: This figure depicts the equilibrium rates and spread as a function of interest on reserves under DZLB. All the rates and spread are expressed in basis points.



(a) Equilibrium Spread



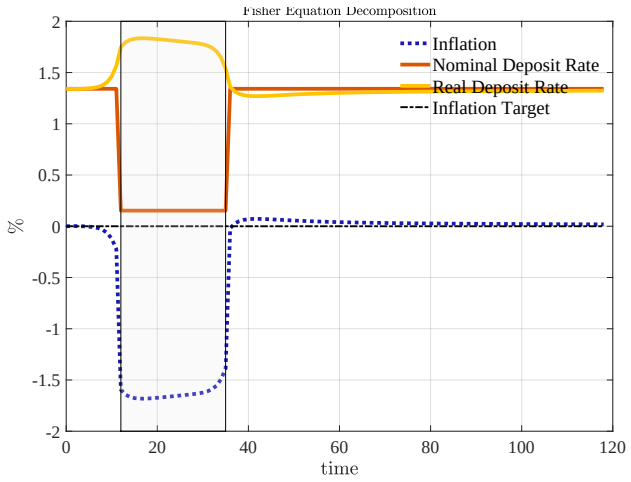
(b) Equilibrium Deposit Rate



(c) Equilibrium Loan Rate

Figure 18: Negative Interest on Reserves, Liquidity Ratio and the DZLB.

Note: This figure presents the equilibrium spread, deposit rate and loan rate as functions of liquidity ratio and interest on reserves under DZLB. All the rates and spread are expressed in basis points.



(a) Decomposition of Fisher Equation

Figure 19: Transition Dynamics of Fisher Equation Components under the Implementation of a Spread Reduction via OMO

Note: This figure reports the responses of inflation, nominal deposit rate, real deposit rate and inflation target according to Fisher equation decomposition, after the credit spread reduction implementation via OMO in Section 5.3.

C Transition Path of Fiscal Transfer In Response to Shock

The transition path of fiscal transfer T_t during shock periods is designed to close the gap between steady-state value due to shock at a constant rate. In particular, given any increase $\hat{\tau}$ in lump-sum tax $t = t^{start}$, we assume the value of lump-sum tax will decrease to steady-state value at a constant rate within the shock periods. That is, we solve the parameter ν such that

$$\dot{\tau}_t = \nu, \text{ with } t \in \left[t^{start}, t^{post}\right], \tau_{t^{start}} = \hat{\tau}, \text{ and } \tau_{t^{post}} = 0.$$

The solution is

$$\tau_t = \frac{t^{post} - t}{t^{post} - t^{start}} \hat{\tau}, t \in \left[t^{start}, t^{post}\right].$$

For the case in (22) $\hat{\tau}$ is set to be $-\frac{\Delta r_{ss}}{2} \int_{\bar{s}}^{0} s f_{ss}(s)\, ds$.

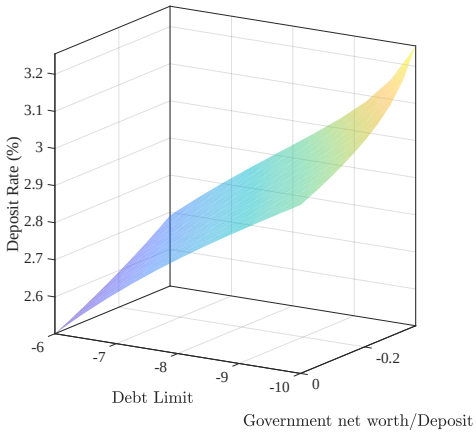
In post-shock periods, the fiscal transfer T_t is assumed to close the gap of $\mathcal{E}_t$ with steady state value at a constant exponential rate. In particular, given any $\mathcal{E}_t$ at $t > t^{post}$ and any time interval dt, define the auxiliary net-asset position $\mathcal{E}_t^{aux}$ and the auxiliary fiscal transfer T_t^{aux} that satisfy

$$\mathcal{E}_t^{aux} = \mathcal{E}_{ss} + \exp\left(-\delta^{ef} \cdot dt\right)\left(\mathcal{E}_t - \mathcal{E}_{ss}\right),$$

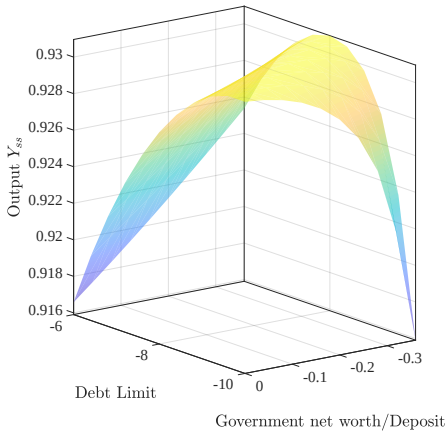
and

$$\mathcal{E}_t^{aux} = \left[1 + \left(r_t^a + \Delta r_t\right) \cdot dt\right] \mathcal{E}_t + \Delta r_t \cdot dt \cdot \int_0^{\infty} s f(s,t)\, ds - T_t^{aux} \cdot dt.$$

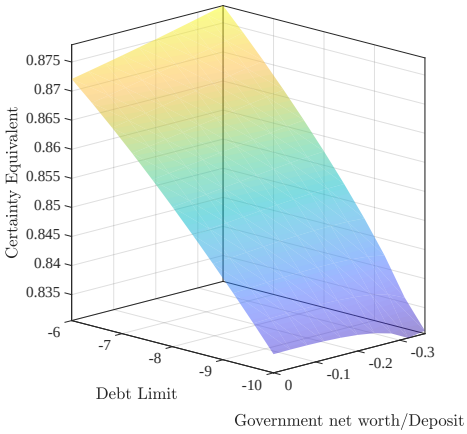
(a) Deposit Rate



(b) Output



(c) Aggregate Certainty Equivalent



(d) Aggregate Welfare

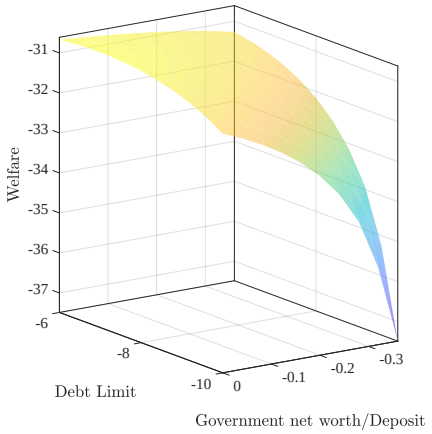


Figure 20: Steady State with Zero Spread under Different Debt Limit and Government Net-Asset Position (w/o DE)

Note: This figure plots the real deposit rate, real output, aggregate certainty equivalent and aggregate welfare in steady state with zero creadit spread and without demand externality, as functions of debt limit and government net-asset position.

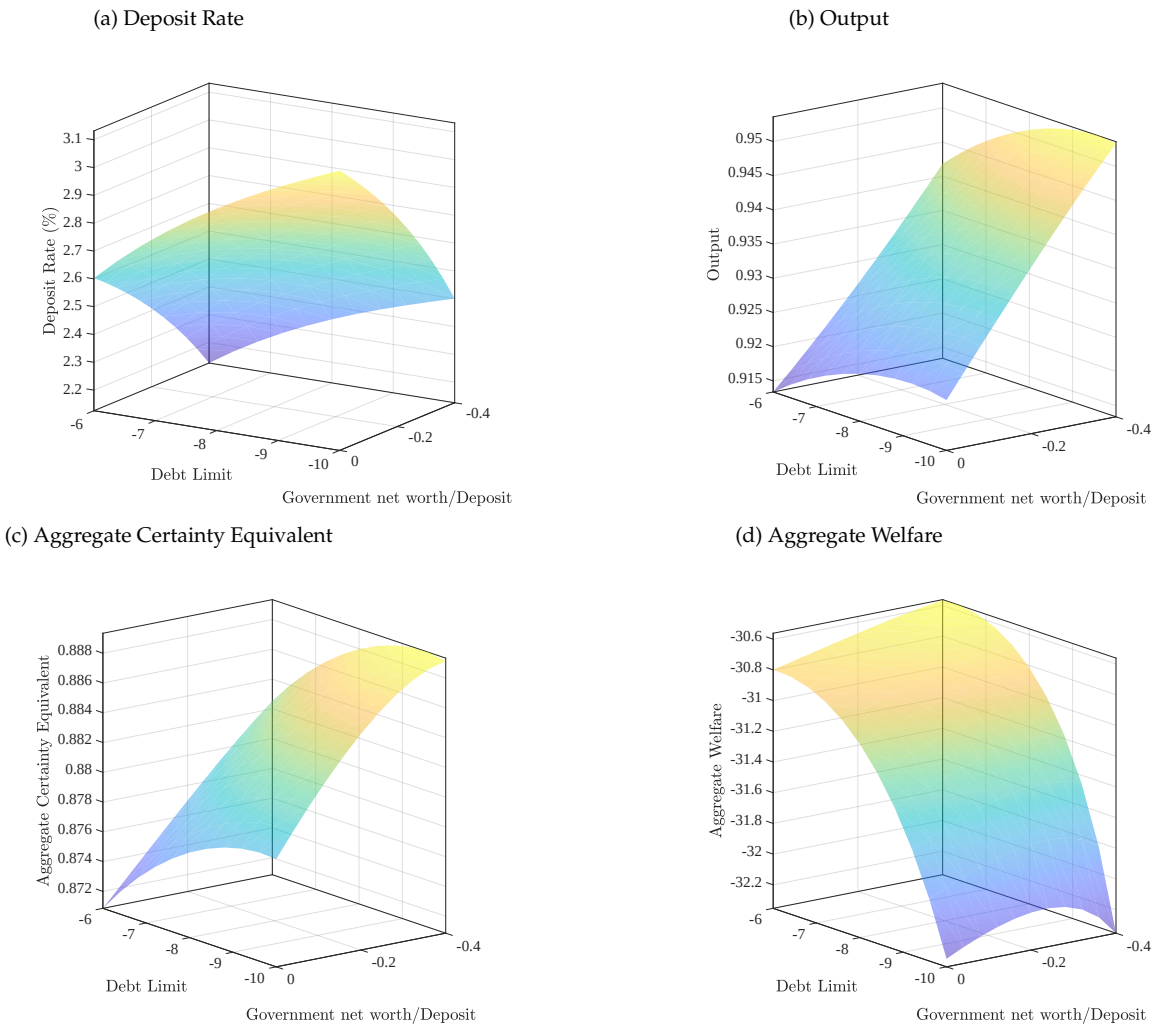


Figure 21: Steady State with Optimal Spread under Different Debt Limit and Government Net-Asset Position (w/o DE)

Note: This figure plots the real deposit rate, real output, aggregate certainty equivalent and aggregate welfare in steady state with Ramsey optimal creadit spread and without demand externality, as functions of debt limit and government net-asset position.

Taking $dt \to 0$, the solution to T_t^{aux} is

$$T_t^{aux} = r_t^a \cdot \mathcal{E}_t - \Delta r_t \cdot \int_{\bar{s}}^{0} sf\left(s,t\right) ds + \delta^{ef}\left(\mathcal{E}_t - \mathcal{E}_{ss}\right).$$

Then the fiscal transfer T_t is given by a weighted combination of T_t^{aux} and $T_{t^{post}}$, such that

$$\begin{aligned} T_t &= \left[1 - \left(\delta^{trans}\right)^{t-t^{post}}\right] T_t^{aux} + \left(\delta^{trans}\right)^{t-t^{post}} \cdot T_{t^{post}} \\ &= -\Delta r_t \cdot \int_{\bar{s}}^{0} sf\left(s,t\right) ds + \left[1 - \left(\delta^{trans}\right)^{t-t^{post}}\right] \left[r_t^a \cdot \mathcal{E}_t + \delta^{ef}\left(\mathcal{E}_t - \mathcal{E}_{ss}\right)\right] + \left(\delta^{trans}\right)^{t-t^{post}} r_{ss}^a \mathcal{E}_{ss}, \end{aligned}$$

where $T_{t^{post}} = -\Delta r_t \cdot \int_{\bar{s}}^{0} sf\left(s,t\right) ds + r_{ss}^a \mathcal{E}_{ss}$ by the design of T_t for $t \in \left[t^{start}, t^{post}\right]$. Note that if $\delta^{trans} = 0$, the fiscal transfer is equal to the auxiliary one, which means the fiscal transfer closes the gap of net-asset position with its steady state value at a constant exponential rate δ^{ef}.

D Solution Algorithm

Propositions 1, 2 and 3 are the objects we need to solve the model. They allow us to solve the model entirely by solving for the equilibrium path of a single price. For example, we can solve the model by solving the path for a real deposit rate r_t^a. The spread Δr_t follows immediately from Proposition 1 if we know the path for ι_t and Λ_t set by the CB. The real spread gives us r_t^l. To solve the household's problem, we need the path for $\left\{r_t^a, r_t^l, T_t\right\}$. The path for T_t is must be consistent with (20) and this yields a path for real government liabilities, $\mathcal{E}_t$. Then, $\mathcal{E}_t$ together with the evolution of $f(s,t)$ obtained from the household's problem, yield two sides of one equation enters (19). The rate equilibrium rate r_t^a must be the one that solves (19) implicitly.

Note that given the real credit spread Δr and government's net-asset position $\mathcal{E}$, the HJB equation (12), KF equation (13) and the real market clearing condition (19) imply that the equilibrium solution to the real markets is independent of implementation variables. Thus we divide the solution algorithm into two parts: the part of real market and the part of implementation. For the part of real market, the path of credit spread is taken as given. For the part of implementation, we simply use the equations (8) in Proposition 1 to show that the target credit spread is within the range of our calibration. Our algorithm closely follows the finite difference in Achdou et al., 2017.

D.1 Solution Algorithm: Stationary Equilibrium in Real Markets

We need to compute the value of deposit rate that satisfies the real market clearing condition (19) in steady state. We use an iteration algorithm that proceeds as follows. First, we take the real credit spread Δr as given, consider an initial guess of deposit rate $r^{a,0}$, and set the interation index $l := 0$. Then:

1. **Individual household's problem.** Given $r^{a,l}$, solve the household's value function $V^l(s)$ from HJB equation (12) using a finite difference method. Calculate the saving policy function $\mu^l(s)$ and production technology choice $u^l(s)$.
2. **Aggregate distribution.** Given $\mu^l(s)$ and $u^l(s)$, solve the KF equation (13) for $f^l(s)$ using a finite difference method.
3. **Equilibrium deposit rate.** Given $f^l(s)$, compute the net supply of real financial claims $S\left(r^{a,l}\right) = \int_{\bar{s}}^{\infty} s f^l(s)\, ds + \mathcal{E}_{ss}$ and update the interest rate: if $S\left(r^{a,l}\right) > 0$, decrease it to $r^{a,l+1} < r^{a,l}$ and vice versa.

The above iteration stops when $S\left(r^{a,l}\right)$ is close enough to 0. Then we call $\left(r^{a,l}, V^l(s), f^l(s)\right)$ a stationary equilibrium.

D.1.1 Solution to the HJB equation

The household's HJB equation is solved using an upwind finite difference scheme similar to Achdou et al., 2017. It approximates the value function $V(s)$ on a finite grid with step $\Delta s : s \in \{s_1, ..., s_N\}$, where $s_i = s_{i-1} + \Delta s = s_1 + (i-1)\Delta s$ for $2 \leq i \leq N$. The bounds are $s_1 = \bar{s}$ and $s_N = s^{max}$, such that $\Delta s = (s^{max} - \bar{s}) / (N-1)$. The upper bound s^{max} is an arbitrarily large number such that $f(s,t) = 0$ for all $s > s^{max}$. We use the short-hand notation $V_i \equiv V(s_i)$, and similarly for the policy function u_i and c_i.

Note that the HJB involves the first and secon derivates of the value function, $V_i' = V'(s_i)$ and $V_i'' = V''(s_i)$. The first derivative is approximated with either a forward (F) or a backward (B) approximation,

$$V_i' \approx \partial_F V_i \equiv \frac{V_{i+1} - V_i}{\Delta s},$$
$$V_i' \approx \partial_B V_i \equiv \frac{V_i - V_{i-1}}{\Delta s}.$$

The second-order derivative is approximated by a central difference:

$$V_i'' \approx \partial_{ss} V_i \equiv \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta s)^2}.$$

Let the superscript n be the iteration counter. The HJB equation is approximated by the following upwind scheme,

$$\frac{V_i^{n+1} - V_i^n}{\Delta} + \rho V_i^{n+1} = U(c_i^n) + \partial_F V_i^{n+1} \cdot (\mu_{i,F}^n)^+ + \partial_B V_i^{n+1} \cdot (\mu_{i,B}^n)^- + \frac{1}{2}(\sigma_i^n)^2 \partial_{ss} V_i^{n+1},$$

where

$$\mu_{i,F}^n = r(s_i) \cdot s_i - (\partial_F V_i^n)^{-1/\gamma} + y(u_i^n) + T,$$
$$\mu_{i,B}^n = r(s_i) \cdot s_i - (\partial_B V_i^n)^{-1/\gamma} + y(u_i^n) + T,$$

and $(\sigma_i^n)^2 = \sigma^2(u_i^n)$.

c^n

The choice of production technology u_i^n is such that $u_i^n = H$ if and only if

D.2 Solve KFE

We use finite difference method to solve the stationary KF equation (). Add the math here

D.3 Solution Algorithm: Transition Dynamics

The equilibrium transition path is solved in finite horizon $[0, T]$, assuming that the terminal state of the economy is steady state. We use an interation algorithm as follows. Given the initial distribution of real wealth $f_0(s)$ and the path of exogenous shocks (e.g. equation (22) for a fiscal transfer shock, or the path of real credit spread Δr_t), guess a function $r_t^{a,0}$ and set the iteration index $l := 0$. Then

1. **Individual household's problem.** Given $r_t^{a,l}$ and the terminal condition $V^l(s,T) = V_{ss}(s)$, solve the HJB equation backward in time to compute the path of $V^l(s,t)$. Calculate the production technology choice $u^l(s,t)$ and saving policy function $\mu^l(s,t)$.
2. **Aggregate distribution.** Given $\mu^l(s,t)$ and $u^l(s,t)$, solve the Kolmogorov Forward equation () with initial condition $f^l(s,0) = f_0(s)$ forward in time to compute the path for $f^l(s,t)$.

3. **Equilibrium deposit rate.** Given $f^l(s,t)$, calculate $S\left(r_t^{a,l},t\right) = \int_{\bar{s}}^{\infty} s f^l(s,t)\,ds + \mathcal{E}_t$ and update $r_t^{a,l+1} = r_t^{a,l} - \xi \frac{\partial S\left(r_t^{a,l},t\right)}{\partial t}$ for each t, where $\xi > 0$ is a parameter of update.

The above iteration stops when $\max_t \left\{\left|S\left(r_t^{a,l},t\right)\right|\right\}$ is close enough to 0.

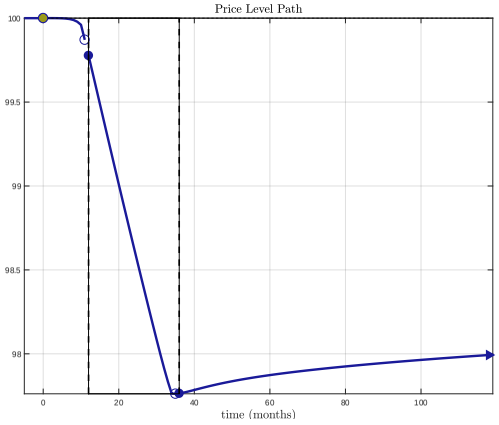
E Supplementary Section - Alternative Implementations

Components of Fisher Equation - Implementation of a Spread via OMO. Figure 19 shows the decomposition of inflation and the real and nominal deposit rates produced by the implementation of the spread in Figure 8. The increase in real rates follows from the dynamics of the real credit spread. Deposit rates are constant until the OMO is actually carried out. Inflation follows the difference between both paths. The rate on reserves is set to implement a zero inflation target in the long-run.

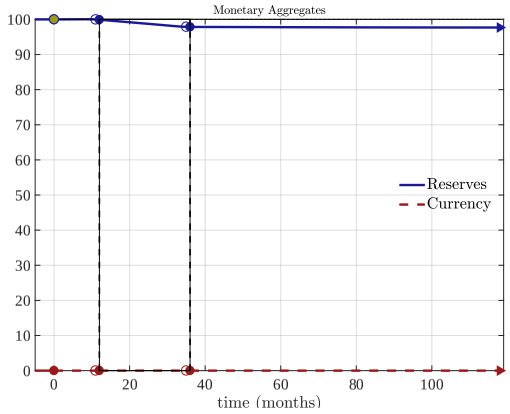
Implementation of a Spread via Reduction in Corridor Rates. Figure 22 describes the details of a reduction in ι that implements the same spread as the OMO in Figure 8. The figure also reports the decomposition in Figure 19. The qualitative pattern is almost identical, although the quantities are not the same. Since real spreads are independent of inflation, the real deposit rate is the same. However, the nominal deposit rate decreases by slightly more than with an OMO. Notice how in Panel (b) there is no increase in the quantity of reserves.

Implementation of a Spread via Increase of i^m at the DZLB. Figure 23 describes the details of an increase in i^m that implements the same spread as the OMO in Figure 8 and the reduction in ι in Figure 22. The qualitative pattern is now different. First, for the implementation to work at all, the economy must be at the DZLB, because only in this region do changes in i^m that keep ι constant have real effects. At the DZLB currency holdings—Panel (b)—are positive. Since i^m is negative, but deposit rates are positive, as in the previous example, this implementation features steady-state deflation.

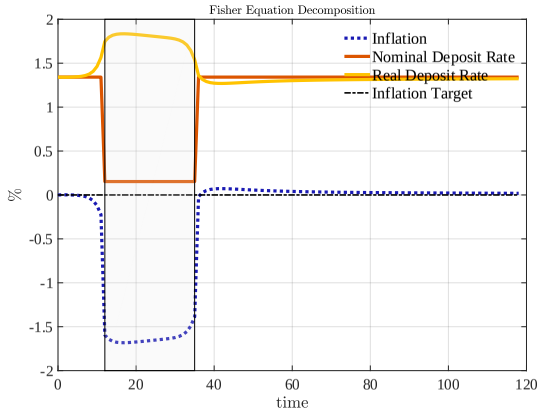
Here, the increase in the interest on reserves, once on negative territory, also produce a deflation since the pattern for real rates is the same. Different from the previous examples, at the DZLB, the deposit rate is flat at zero. The increase in interest on reserves reduces the loans rate, because it acts like a reduction the tax-like effect of negative reserve rates. We see also that the currency ratio of the economy falls.



(a) Price Index



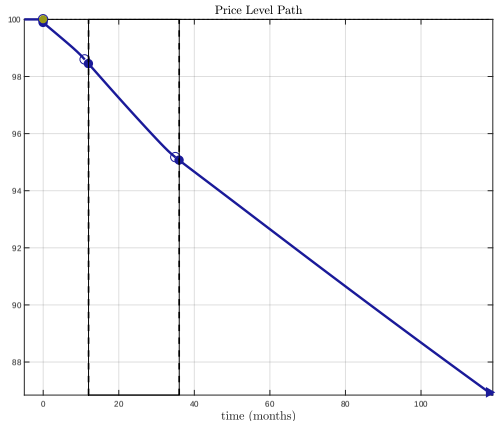
(b) Currency and Reserve Holdings



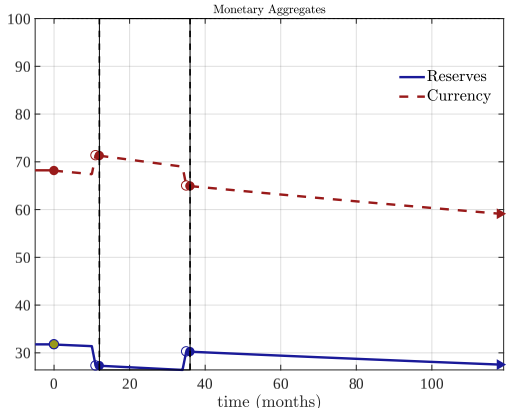
(c) Decomposition of Fisher Equation

Figure 22: Transition Dynamics of Fisher Equation Components under the Implementation of a Spread Reduction via Reduction in Corridor Rates

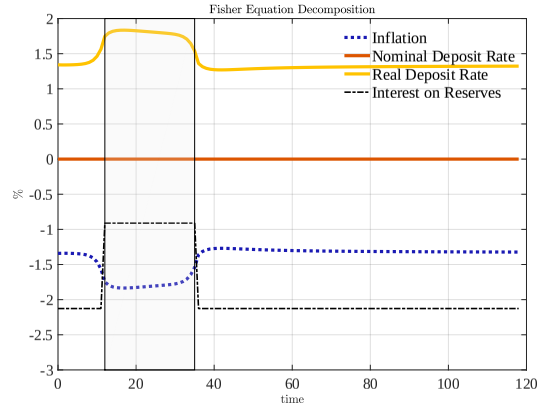
Note: This figure reports the responses of price index, monetary aggregates and Fisher equation components, after a reduction in corridor rate ι that implements the same spread reduction as the OMO in Figure 8.



(a) Price Index



(b) Currency and Reserve Holdings



(c) Decomposition of Fisher Equation

Figure 23: Transition Dynamics of Fisher Equation Components under the Implementation of a Spread Reduction via Increase of Interest on Reserves

Note: This figure reports the responses of price index, monetary aggregates and Fisher equation components, after an increase in i^m that implements the same spread reduction as the OMO in Figure 8.

F Proofs

F.1 Proof of Proposition 1

Take $\{\Lambda_t, i_t^m, \iota_t\}$ as given. By equations (3) and (5), $\{\chi^+, \chi^-\}$ are also given. Consider an individual bank's problem:

$$\max_{a,l\geq 0} i_t^l l + i_t^m(a-l) - i_t^a a + \mathbb{E}\left[\chi_t[b(a,a-l)]\right]$$
$$\max_{a,l\geq 0} i_t^l l + i_t^m(a-l) - i_t^a a + \frac{1}{2}\Big[\chi^+(\Lambda_t,\iota_t)(a-l-\varrho a+\delta a) + \chi^-(\Lambda_t,\iota_t)(a-l-\varrho a-\delta a)\Big]$$
$$\max_{a,l\geq 0} \Big[i_t^l - i_t^m - \frac{1}{2}\left(\chi^+(\Lambda_t,\iota_t)+\chi^-(\Lambda_t,\iota_t)\right)\Big]l - \Big[i_t^a - i_t^m - \frac{1}{2}\left((1-\varrho+\delta)\chi^+(\Lambda_t,\iota_t) + (1-\varrho-\delta)\chi^-(\Lambda_t,\iota_t)\right)\Big]a$$

The problem is linear. Thus, a necessary condition for a positive and finite supply of loans and deposits are conditions (6) and (7). Since in equilibrium the demand of deposits and loans is finite, the result follows. Substitute (6) and (7), and the bank earns zero expected profits from any choice of $\{a, l\}$. Now, observe that by, definition of real rates, $r_t^l = i_t^l - \dot{P}_t/P_t$ and $r_t^a = i_t^a - \dot{P}_t/P_t$. Hence, $\Delta r_t = i_t^l - i_t^a$. Thus, the expression for the real spread follows immediately from subtracting the right-hand side of (7) from the right-hand side of (6). This concludes the proof of Proposition 1. **QED.**

F.2 Proof of Lemma 1

We use the method of change of variable. Denote $x = \left(\theta + (1-\theta)\exp(\lambda)\right)^{1/2}$, which implies $\theta = \frac{\exp(\lambda)-x^2}{\exp(\lambda)-1}$. Observe that for $\theta \in [0,1]$, the domain of x is $[1, \exp(\lambda/2)]$ and x is strictly decreasing in θ. Therefore, to prove the lemma, it is equivalent to derive the necessary and sufficient condition of spread monotonically decreasing in x for any $x \in [1, \exp(\lambda/2)]$.

Replace θ with its expression in x in (3), the liquidity-yield functions are rewritten as

$$\chi^+ = \frac{\iota}{\exp(\lambda)} \cdot \frac{\exp(\lambda) - x^2}{x+1}$$

and

$$\chi^- = \frac{\iota}{\exp(\lambda)} \cdot \frac{x + \exp(\lambda)}{x+1}.$$

One can observe that both χ^+ and χ^- are decreasing in x, which implies they are both decreasing in Λ. Replacing χ^+ and χ^- with the above expressions in (8), the spread is rewritten as

$$i^l - i^a = \frac{\iota}{2\exp(\lambda)}\left[\frac{2\varrho\left(\exp(\lambda)-1\right)}{x+1} - (\varrho-\delta)x + 2\varrho\right].$$

Taking derivative with respect to x gives

$$\frac{\partial\left(i^l - i^a\right)}{\partial x} = -\frac{\iota}{2\exp(\lambda)}\left[\frac{2\varrho\left(\exp(\lambda)-1\right)}{(x+1)^2} + (\varrho-\delta)\right].$$

Note that the derivaive is monotonically increasing in x. Thus the necessary and sufficient condition is

$$\left.\frac{\partial\left(i^{l}-i^{a}\right)}{\partial x}\right|_{x=\exp(\lambda/2)} \leq 0,$$

which is equivalent to

$$\rho \geq \frac{1+\exp\left(-\lambda/2\right)}{3-\exp\left(-\lambda/2\right)}\delta.$$

QED.

F.3 Proof of Proposition 3

The profits of the CB are given by:

$$\pi_t^f = i_t^l L_t^f - i_t^m\left(M_t - M0_t\right) + \iota_t\left(1-\psi_t^-\right)B_t^-.$$

Note that the earnings from discount-window loans equal the average payment in the interbank market, and thus:

$$\iota_t\left(1-\psi_t^-\right)B_t^- = -\mathbb{E}\left[\chi_t\left(b\left(A_t, A_t - L_t\right)\right)\right]. \tag{36}$$

By Proposition 1, banks earn zero profits in expectation. Thus,

$$-\mathbb{E}\left[\chi_t\left(b\left(A_t, A_t - L_t\right)\right)\right] = i_t^l L_t^b + i_t^m M_t^b - i_t^a A_t^b. \tag{37}$$

Thus, substituting (36) and (37) into the expression for π_t^f above yields:

$$\begin{aligned}\pi_t^f &= i_t^l L_t^f - i_t^m\left(M_t - M0_t\right) + i_t^l L_t^b + i_t^m M_t^b - i_t^a A_t^b.\\ &= i_t^l L_t^h - i_t^a A_t^h,\end{aligned}$$

where we used the clearing condition in the money market, $M_t^b + M_t^0 = M_t$, the deposit market, $A_t^b = A_t^h$, and the loans market, $L_t^h = L_t^b + L_t^f$. Now, observe that:

$$\pi_t^f = -i_t^l P_t \int_{\bar{s}}^{0} s f(s,t)ds - i_t^a\left(P_t\int_0^{\infty} s f(s,t)ds - M0_t.\right),$$

but we know from the household's problem that $i_t^a M0_t$=0. Hence, profits are given by:

$$\pi_t^f = -i_t^l P_t \int_{\bar{s}}^{0} s f(s,t)ds - i_t^a P_t \int_0^{\infty} s f(s,t)ds.$$

Divide (25) by the price level to obtain:

$$-\int_{\bar{s}}^{0} s f(s,t)ds = \int_0^{\infty} s f(s,t)ds + \mathcal{E}_t.$$

and thus:

$$\pi_t^f = \left(i_t^l - i_t^a\right) P_t \int_0^\infty sf(s,t)ds + i_t^l E_t = \Delta r_t P_t \int_0^\infty sf(s,t)ds + i_t^l E_t.$$

Dividing both sides by the price level leads to:

$$\frac{\pi_t^f}{P_t} = \Delta r_t \int_0^\infty sf(s,t)ds + i_t^l \mathcal{E}_t = \Delta r_t \int_0^\infty sf(s,t)ds + \left(r_t^a + \Delta r_t + \frac{\dot{P}_t}{P_t}\right)\mathcal{E}_t. \tag{38}$$

Then, note that:

$$d\mathcal{E}_t = \frac{dE_t}{P_t} - \frac{\dot{P}_t}{P_t}\mathcal{E}_t = \frac{\pi_t^f}{P_t} - T_t - \frac{\dot{P}_t}{P_t}\mathcal{E}_t.$$

But, a substitution of (38) yields:

$$d\mathcal{E}_t = \left((r_t^a + \Delta r_t)\mathcal{E}_t + \Delta r_t \int_0^\infty sf(s,t)ds - T_t\right)dt.$$

This proves Proposition 3. **QED.**

F.4 Proof of Proposition 2

(1) The accounting identities in Section 3.2 and Lemma 2, show that if all markets clear, the real market clears. Then, by dividing (25) by the price level, we obtain:

$$-\int_{\bar{s}}^0 sf(s,t)ds = \int_0^\infty sf(s,t)ds + \mathcal{E}_t, \text{ for } t \in [0,\infty).$$

The proposition establishes that if this condition holds, all asset markets clear. To proceed with the proof, argue that if the condition holds, but one of the markets doesn't clear, we reach contradiction.

To see that, observe that real household's assets position equations (30) and (31), and (25) imply

$$\int_{\bar{s}}^0 l_t^h(s)f(s,t)ds = \int_0^\infty a_t^h(s)f(s,t)ds + M0_t + L_t^f - M_t, \text{ for } t \in [0,\infty). \tag{39}$$

Combining with banks' budget constraint (29), the above equation (39) implies that if two asset markets clear, the third one must clear as well. For example, suppose the deposit market and loans market clear. Then by (39) and (29),

$$\begin{aligned} M_t &= \int_0^\infty a_t^h(s)f(s,t)ds + M0_t + L_t^f - \int_{\bar{s}}^0 l_t^h(s)f(s,t)ds \\ &= A_t^b + M0_t + L_t^f - L_t^b - L_t^f \\ &= M_t^b + M0_t, \end{aligned}$$

this establishes money market clearing condition. The same proof follows for the other two markets.

(2) To prove that the goods market clearing condition also holds, observe that if condition (19) holds, then taking time derivatives we obtain:

$$0 = \frac{\partial}{\partial t}\left[\int_{\bar{s}}^{\infty} sf(s,t)ds\right] + \frac{\partial}{\partial t}\left[\mathcal{E}_t\right],$$

Then we have:

$$0 = \int_{\bar{s}}^{\infty} s\frac{\partial}{\partial t}\left[f(s,t)\right]ds + \frac{\partial}{\partial t}\left[\mathcal{E}_t\right],$$

but recall that if the KFE equation holds, then:

$$0 = \int_{\bar{s}}^{\infty} s\left[-\frac{\partial}{\partial s}\left[\mu\left(s,t\right)f\left(s,t\right)\right] + \frac{1}{2}\frac{\partial^2}{\partial s^2}\left[\sigma_s^2\left(s,t\right)f\left(s,t\right)\right]\right]ds + \frac{\partial}{\partial t}\left[\mathcal{E}_t\right].$$

Now, observe that, if we employ the integration by parts formula:

$$-\int_{\bar{s}}^{\infty} s\frac{\partial}{\partial s}\left[\mu\left(s,t\right)f\left(s,t\right)\right]ds = -s\mu\left(s,t\right)f\left(s,t\right)|_{\bar{s}}^{\infty} + \int_{\bar{s}}^{\infty}\mu\left(s,t\right)f\left(s,t\right)ds.$$

We know that

$$-s\mu\left(s,t\right)f\left(s,t\right)|_{\bar{s}}^{\infty} = 0$$

and that

$$\int_{\bar{s}}^{\infty}\mu\left(s,t\right)f\left(s,t\right)ds = \int_{\bar{s}}^{\infty}\left[r_t\left(s\right)\left(s - m^h\left(s,t\right)/P_t\right) - \dot{P}_t/P_t\cdot m^h\left(s,t\right)/P_t - c\left(s,t\right) + h\left(u\left(s,t\right),t\right)\right]f\left(s,t\right)ds.$$

First, note that:

$$\int_{\bar{s}}^{\infty} r_t\left(s\right)sf\left(s,t\right)ds = \int_{\bar{s}}^{\infty} r_t^l\cdot sf\left(s,t\right)ds - \int_0^{\infty}\Delta r_t\cdot sf\left(s,t\right)ds.$$

Second, the household's problem solution implies $i_t^a\cdot m^h\left(s,t\right) = 0$ for any $\left(s,t\right)$, and $m^h\left(s,t\right) = 0$ for any $s \leq 0$. Then we have

$$\begin{aligned}
&\int_{\bar{s}}^{\infty}\left(r_t\left(s\right) + \dot{P}_t/P_t\right)\left(m^h\left(s,t\right)/P_t\right)f\left(s,t\right)ds\\
=&i_t^l\int_{\bar{s}}^{0}\frac{m^h\left(s,t\right)}{P_t}f\left(s,t\right)ds + i_t^a\int_0^{\infty}\frac{m^h\left(s,t\right)}{P_t}f\left(s,t\right)ds\\
=&0.
\end{aligned}$$

Third, by definition,

$$\int_{\bar{s}}^{\infty}\left(-c\left(s,t\right) + h\left(u\left(s,t\right),t\right)\right)f\left(s,t\right)ds = Y_t - C_t + T_t.$$

Finally, the term:

$$\frac{1}{2}\int_{\bar{s}}^{\infty} s\cdot\frac{\partial^2}{\partial s^2}\left[\sigma_s^2(s,t)\,f(s,t)\right]ds = \frac{1}{2}\,s\cdot\frac{\partial}{\partial s}\left[\sigma_s^2(s,t)\,f(s,t)\right]\Big|_{\bar{s}}^{\infty} - \frac{1}{2}\int_{\bar{s}}^{\infty}\frac{\partial}{\partial s}\left[\sigma_s^2(s,t)\,f(s,t)\right]ds$$
$$= 0 - \frac{1}{2}\,\sigma_s^2(s,t)\,f(s,t)\Big|_{\bar{s}}^{\infty} = 0.$$

Thus, we are left with:

$$r_t^l\int_{\bar{s}}^{\infty} sf(s,t)\,ds - \Delta r_t\int_0^{\infty} sf(s,t)\,ds + Y_t - C_t + T_t + \frac{\partial}{\partial t}\left[\mathcal{E}_t\right] = 0.$$

But then, given the law of motion for real equity (20),

$$\frac{\partial}{\partial t}\left[\mathcal{E}_t\right] + r_t^l\int_{\bar{s}}^{\infty} sf(s,t)\,ds - \Delta r_t\int_0^{\infty} sf(s,t)\,ds + T_t = 0.$$

This implies the goods market clearing condition. **QED.**

F.5 Proof of Proposition 4

It suffices to show the equations for real credit spread and inflation rate. Along an equilibrium path for $\{r_t^a, \mathcal{E}_t, f_t, \Delta r_t, T_t\}$ the set of implementable nominal interbank rates and inflation rates is the set of $\{\dot{P}_t/P_t, \bar{i}_t^f\}$ where

$$\frac{\dot{P}_t}{P_t} = i_t^l - (\Delta r_t + r_t^a) = i_t^m + \frac{1}{2}\Big[\chi^+(\Lambda_t,\iota_t) + \chi^-(\Lambda_t,\iota_t)\Big] - \Delta r_t - r_t^a \tag{40}$$
$$\bar{i}_t^f = \chi^+(\Lambda_t,\iota_t)/\psi^+(\theta(\Lambda_t)) + i_t^m \tag{41}$$

for any $\{i_t^m, \iota_t, \mathcal{L}_t^f\}$ such that

$$\Delta r_t = r_t^l - r_t^a = i_t^l - \dot{P}_t/P_t - i_t^a + \dot{P}_t/P_t$$
$$= \Delta i_t = \varrho\frac{\chi^+(\Lambda_t,\iota_t) + \chi^-(\Lambda_t,\iota_t)}{2} + \delta\frac{\chi^-(\Lambda_t,\iota_t) - \chi^+(\Lambda_t,\iota_t)}{2},$$
$$\mathcal{L}_t^f \leq -\int_{\bar{s}}^0 sf(s,t)ds, \quad (\iota_t, i_t^m) \in \mathbb{R}_+^2.$$

Equations (40) and (41) steams form definitions for nominal, real and interbank rate. The implementation constraint $\mathcal{L}_t^f \leq -\int_{\bar{s}}^0 sf(s,t)ds$ simply tells that there must be enough private liabilities to set $\mathcal{L}_t^f$. **QED.**

F.6 Proof of Corollary 2

It suffices to show that $\Delta r_t = 0$ when $i^m \geq 0$ and $\Lambda \geq \rho + \delta$. Note that the interbank market is satiated with reserves if $\Lambda_t \geq \bar{\Lambda} = \varrho + \delta$. Then the interbank market tightness is $\theta(\Lambda_t) = 0$ for any $\Lambda_t \geq \bar{\Lambda} = \varrho + \delta$. First, we must take the following limit

$$\lim_{\theta\to 0}\frac{\bar{\theta}(\theta)}{\theta} = \lim_{\theta\to 0}\frac{1}{\theta[1 + (\theta^{-1} - 1)\exp(\lambda)]} = \lim_{\theta\to 0}\frac{1}{\theta + (1-\theta)\exp(\lambda)} = \exp(-\lambda),$$

where $\bar{\theta}(\theta)$ is given by (35) in Appendix B. Then, given (η, λ), for any $\Lambda_t \geq \bar{\Lambda}$, (34) implies:

$$\chi^+(\Lambda_t, \imath_t) = \lim_{\theta \to 0} \imath_t \theta \left(\frac{\overline{\theta}(\theta)}{\theta} \right)^{\eta} \left(\frac{[\overline{\theta}(\theta)/\theta]^{1-\eta} - 1}{\overline{\theta}(\theta) - 1} \right) = 0,$$

$$\chi^-(\Lambda_t, \imath_t) = \lim_{\theta \to 0} \imath_t \left(\frac{\overline{\theta}(\theta)}{\theta} \right)^{\eta} \left(\frac{\theta[\overline{\theta}(\theta)/\theta]^{1-\eta} - 1}{\overline{\theta}(\theta) - 1} \right) = \imath_t \exp(-\eta\lambda).$$

Although $\chi_t^- > 0$, there are not banks with reserves deficit, thus

$$\mathbb{E}\left\{ \chi_t [b(a, a-l)] | \theta_t \right\} = \chi^+(\Lambda_t, \imath_t)\,(a - l - \varrho a) = 0$$

Hence, the bank's problem becomes

$$\pi_t^b = \max_{a,l} (i_t^l - i_t^m) l_t - (i_t^a - i_t^m) a_t$$

and by FOCs we obtain that $i_t^m = i_t^a = i_t^l = \bar{i}_t^f$. **QED.**

F.7 Proof of Proposition 5

1. [Corridor Regime] In this case, $\Lambda_t = \Lambda^{MB}\left(\mathcal{E}_t, f_t, \mathcal{L}_t^f\right)$, $\theta(\Lambda_t) \in (0,1)$, $\left\{ i^l, i^a, \Delta r \right\}$ is given by (6), (7) and (8), and $\{\chi^+, \chi^-\}$ is given by (3). Since $\Lambda^{MB}\left(\mathcal{E}_t, f_t, \mathcal{L}_t^f\right)$ is increasing in $\mathcal{L}_t^f$, then the proof of Lemma 1 in Appendix F.2 implies $\left\{ \frac{\partial i^l}{\partial \mathcal{L}_t^f}, \frac{\partial i^a}{\partial \mathcal{L}_t^f}, \frac{\partial \Delta r}{\partial \mathcal{L}_t^f} \right\} < 0$. By (6), (7) and (8) one can observe that $\frac{\partial i^l}{\partial i_t^m} = \frac{\partial i^a}{\partial i_t^m} = 1$ and $\frac{\partial \Delta r}{\partial i_t^m} = 0$. By (3), both χ^+ and χ^- are proportional to $\imath$, thus the elasticities of $\left\{ i^l, i^m, \Delta r \right\}$ with respect to $\imath_t$ are all equal to 1.
2. [Floor Regime] In this case, $\theta(\Lambda_t) = 0$ and the proof of Corollary 2 establishes all the results.
3. [DZLB and negative i^m regime] In this case, the definition of Λ^{zlb} implies that $i^a \equiv 0$ and Λ_t is independent of $\mathcal{L}_t^f$. Thus $\mathcal{L}_t^f$ has no impact on $\left\{ i^l, i^a, \Delta r \right\}$. The equilibrium $\left\{ i^l, \Delta r \right\}$ are still given by (6) and (8). To prove the effects of $\{i_t^m, \imath_t\}$ on $\left\{ i^l, \Delta r \right\}$, it suffices to show the sign of $\frac{\partial i^l}{\partial i^m}$ and $\frac{\partial i^l}{\partial \imath}$. We take total differentiation of (7). This gives

$$\begin{aligned} 0 =& di^m + \frac{1}{2}(1 - \varrho + \delta)\left(\frac{\partial \chi^+(\theta(\Lambda), \imath)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^+(\theta(\Lambda), \imath)}{\partial \imath} d\imath \right) \\ &+ \frac{1}{2}(1 - \varrho - \delta)\left(\frac{\partial \chi^-(\theta(\Lambda), \imath)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^-(\theta(\Lambda), \imath)}{\partial \imath} d\imath \right), \end{aligned} \tag{42}$$

which implies

$$\left\{ \frac{\partial \Lambda}{\partial i^m}, \frac{\partial \Lambda}{\partial \imath} \right\} > 0,$$

and

$$\begin{aligned} di^l &= di^m + \frac{1}{2}\left(\frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \iota} d\iota\right) + \frac{1}{2}\left(\frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \iota} d\iota\right) \\ &= \frac{\varrho-\delta}{2}\left(\frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \iota} d\iota\right) + \frac{\varrho+\delta}{2}\left(\frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \iota} d\iota\right). \end{aligned}$$

Let $di^m > 0$ and $d\iota = 0$. Then by Lemma 1 we have

$$\frac{\partial i^l}{\partial i^m} = \frac{\varrho-\delta}{2} \cdot \frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \Lambda}\frac{\partial \Lambda}{\partial i^m} + \frac{\varrho+\delta}{2} \cdot \frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \Lambda}\frac{\partial \Lambda}{\partial i^m} < 0.$$

Let $di^m = 0$ and $d\iota > 0$. The proof of Lemma 1 in Appendix F.2 and equation (3) imply that

$$\frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \Lambda} < \frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \Lambda} < 0$$

and

$$\frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \iota} > \frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \iota} > 0.$$

Then equation (42) implies

$$\begin{aligned} &\left(\frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^+(\theta(\Lambda),\iota)}{\partial \iota} d\iota\right) \\ =& -\frac{1-\varrho-\delta}{1-\varrho+\delta}\left(\frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \iota} d\iota\right) \\ <& 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial i^l}{\partial \iota} &= \left(\frac{\varrho+\delta}{2} - \frac{1-\varrho-\delta}{1-\varrho+\delta}\frac{\varrho-\delta}{2}\right) \cdot \left(\frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \iota} d\iota\right) \\ &= \frac{\delta}{1-\rho+\delta} \cdot \left(\frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \Lambda} d\Lambda + \frac{\partial \chi^-(\theta(\Lambda),\iota)}{\partial \iota} d\iota\right) \\ &> 0 \end{aligned}$$

QED.

F.8 Proof of Corollary 1

The discount window profits are equal to $\Delta r_t \int_0^\infty s f(s,t)\, ds$ since banks are competitive and earn zero profits. Given the same real credit spread Δr_t, the equilibrium real wealth distribution $f(s,t)$ is also same. Thus Corollary 1 is established. **QED.**

F.9 Proof of Corollary 3

The proof is established by relabelling. Note that in Problem 2 with $r_t^a = r_{ss}^a$, $\Delta r_t = 0$ and $T_{ss} = r_{ss}^a \mathcal{E}_{ss}$, the households' problem is

$$\begin{aligned}\rho V(s,t) &= \max_{\{c\}\geq 0, u\in\{L,H\}} U(c) + V_s \cdot (r_{ss}^a \cdot s - c + y(u) + T_{ss}) + \frac{1}{2} V_{ss}\sigma^2(u) + \dot{V} \\ &= \max_{\{c\}\geq 0, u\in\{L,H\}} U(c) + V_s \cdot (r_{ss}^a \cdot (s + \mathcal{E}_{ss}) - c + y(u)) + \frac{1}{2} V_{ss}\sigma^2(u) + \dot{V}\end{aligned}$$

subject to

$$s_t \geq \bar{s} \;\Leftrightarrow\; s_t + \mathcal{E}_{ss} \geq \bar{s} + \mathcal{E}_{ss}.$$

Denote $s_t^{(a)} = s_t + \mathcal{E}_{ss}$, $\bar{s}^{(a)} = \bar{s} + \mathcal{E}_{ss}$ and $V^{(a)}\left(s^{(a)}, t\right) = V\left(s^{(a)} - \mathcal{E}_{ss}, t\right)$. Then the households' problem can be written as

$$\rho V^{(a)}\left(s_t^{(a)}, t\right) = \max_{\{c\}\geq 0, u\in\{L,H\}} U(c) + V_{s^{(a)}}^{(a)} \cdot \left(r_{ss}^a \cdot s_t^{(a)} - c + y(u)\right) + \frac{1}{2} V_{s^{(a)}s^{(a)}}^{(a)} \sigma^2(u) + \dot{V}^{(a)}$$

subject to

$$s_t^{(a)} \geq \bar{s}^{(a)}.$$

This economy has the same equilibrium allocation as the original one. **QED.**

F.10 Proof of Proposition 6

The proof is similar to Corollary 3 and is also established by relabelling. Taking differentiation of $h(t)$ with respect to t gives us

$$0 = \dot{h}(t) = -T_t + r_t^a \cdot h(t),$$

which implies $T_t = r_t^a \cdot h(t)$. Note that a policy that sets $\iota_t = 0$ or satiate banks with reserves imply $\Delta r_t = 0$. Thus denote $s_t^{(a)} = s_t + h(t) \equiv s_t + \mathcal{E}_{ss}$, $\bar{s}_t^{(a)} = \bar{s} + h(t) \equiv \bar{s} + \mathcal{E}_{ss}$ and $V^{(a)}\left(s^{(a)}, t\right) = V\left(s^{(a)} - \mathcal{E}_{ss}, t\right)$, the households' problem 2 with $\Delta r_t = 0$ and $T_t = r_t^a \cdot h(t)$ can be written as

$$\rho V^{(a)}\left(s_t^{(a)}, t\right) = \max_{\{c\}\geq 0, u\in\{L,H\}} U(c) + V_{s^{(a)}}^{(a)} \cdot \left(r_t^a \cdot s_t^{(a)} - c + y(u)\right) + \frac{1}{2} V_{s^{(a)}s^{(a)}}^{(a)} \sigma^2(u) + \dot{V}^{(a)}$$

subject to

$$s_t^{(a)} \geq \bar{s}_t^{(a)}.$$

This economy has the same equilibrium allocation as the original one, and the allocation is independent of $\{T_t, \mathcal{E}_t\}$. **QED.**

G Ramsey Problem

We first summarize the main results of this section, namely restating the Ramsey problem, setting up the Lagrangian, and presenting the necessary conditions for optimality. We then move on to the proofs in further detail.

G.1 Summary of Main Results

For Problem 3 it is useful to rewrite the planner's objective function as follows:

Lemma 3 *The welfare criterion (24) can be expressed as*

$$W(f_0) = \int_0^\infty e^{-\rho t} \int_{\bar{s}}^\infty U(c(s,t)) f(s,t) \, ds dt. \tag{43}$$

We consider the case where the planner credibly commits to a path of credit spread $\{\Delta r(t)\}_{t\in[0,\infty)}$, but takes a steady-state value for the net asset position as given. The optimal credit spread path is then a function of the initial distribution $f_0(s)$ and time. Thus, we can restate the Ramsey problem as the following value functional of planner:

$$W^R(f_0) = \max_{\left\{\Delta r_t, r_t^a, f(\cdot,t), V(\cdot,t), c(\cdot,t), u(\cdot,t)\right\}_{t\in[0,\infty)}} \int_0^\infty e^{-\rho t} \int_{\bar{s}}^\infty U(c(s,t)) f(s,t) \, ds dt, \tag{44}$$

subject to the law of motion of wealth distribution (13), asset market clearing condition (19), households' HJB equation (12) and two first-order conditions on the choice of $c(s,t)$ and $u(s,t)$:

$$U'(c(s,t)) - \frac{\partial V(s,t)}{\partial s} = 0 \tag{45}$$

and

$$u = H \text{ if and only if } \frac{Y(H) - Y(L)}{\frac{1}{2}\sigma^2(H)} \geq -\frac{V_{ss}}{V_s} \text{ and } s > \tilde{s}_t. \tag{46}$$

Note that by Walras' law the goods market clears if and only if the asset market clears. We use the latter since under this condition the numerical solution is more stable.

The above Ramsey problem is an optimal control problem in a suitable function space. We construct a Lagrangian to solve the problem. As shown in the proof, the Lagrangian is given by

$$\begin{aligned}\mathcal{L}\left(\Delta r, r^a, f, V, c, u; f_0\right) \equiv \int_0^\infty e^{-\rho t} \int_{\bar{s}}^\infty & \{U\left(c\left(s,t\right)\right) f\left(s,t\right) \\ & + \phi\left(s,t\right)\left[-\frac{\partial}{\partial t} f\left(s,t\right) - \frac{\partial}{\partial s}\left[\mu\left(s,t\right) f\left(s,t\right)\right] + \frac{1}{2}\frac{\partial^2}{\partial s^2}\left[\sigma_s^2\left(s,t\right) f\left(s,t\right)\right]\right] \\ & + \omega\left(s,t\right)\left[U\left(c\left(s,t\right)\right) + \frac{\partial V\left(s,t\right)}{\partial s} \cdot \left(r\left(s,t\right) s - c\left(s,t\right) + y\left(u\left(s,t\right)\right) + T_t\right)\right. \\ & \left. + \frac{1}{2}\sigma^2(u\left(s,t\right))\frac{\partial^2 V\left(s,t\right)}{\partial s^2} + \frac{\partial V\left(s,t\right)}{\partial t} - \rho V\left(s,t\right)\right] \\ & + \varphi\left(s,t\right)\left[U'\left(c\left(s,t\right)\right) - \frac{\partial V\left(s,t\right)}{\partial s}\right]\Big\} ds dt \\ & + \int_0^\infty e^{-\rho t}\xi\left(t\right)\left[\int_{\bar{s}}^\infty s f\left(s,t\right) ds + e_f \int_0^\infty s f\left(s,t\right) ds\right] dt,\end{aligned}$$

where $e^{-\rho t}\phi\left(s,t\right)$, $e^{-\rho t}\omega\left(s,t\right)$, $e^{-\rho t}\varphi\left(s,t\right)$ and $e^{-\rho t}\xi\left(t\right)$ are the Lagrange multipliers associated to equations (13), (12), (45) and (19). We do not incorporate (46) in Lagrangian since $u\left(s,t\right)$ is a discrete choice variable. The necessary conditions for optimality are first-order conditions of the Lagrangian with respect to the functions $\Delta r, r^a, f, V, c$ by taking *Gateaux derivatives*. The following proposition characterizes the solution to this problem based on the first-order conditions.[31]

Proposition 7 *In addition to equations (13), (12), (19), (45) and (46), if a solution to the Ramsey problem (44) exists, the credit spread path Δr_t and the Lagrange multipliers must satisfy*

$$\int_{\bar{s}}^\infty \left[\phi_s\left(s,t\right) f\left(s,t\right) + V_s\left(s,t\right)\omega\left(s,t\right)\right] \cdot \left[L_t + \mathbb{I}_{\{s<0\}} \cdot s\right] ds = 0, \tag{47}$$

$$\begin{aligned}\rho\phi\left(s,t\right) = & \frac{\partial\phi\left(s,t\right)}{\partial t} + U\left(c\left(s,t\right)\right) + \frac{\partial\phi\left(s,t\right)}{\partial s} \cdot \left[r\left(s,t\right) s + \eta_l \cdot y\left(u\left(s,t\right)\right) + \left(1-\eta_l\right) Y_t - c\left(s,t\right) + T_t\right] \\ & + \frac{1}{2}\sigma_s^2\left(s,t\right)\frac{\partial^2\phi\left(s,t\right)}{\partial s^2} + \xi\left(t\right)\left[1 + \mathbb{I}_{\{s\geq 0\}} e_f\right] s \\ & + \left[\left(1-\eta_l\right) y\left(u\left(s,t\right)\right) - \mathbb{I}_{\{s<0\}}\Delta r_t \cdot s + \mathbb{I}_{\{s\geq 0\}} r_t^a \cdot e_f \cdot s\right] \cdot \int_{\bar{s}}^\infty \left[\phi_s\left(s,t\right) f\left(s,t\right) + V_s\left(s,t\right)\omega\left(s,t\right)\right] ds,\end{aligned} \tag{48}$$

$$\frac{\partial\omega\left(s,t\right)}{\partial t} = -\frac{\partial}{\partial s}\left[\mu\left(s,t\right)\omega\left(s,t\right)\right] + \frac{1}{2}\frac{\partial^2}{\partial s^2}\left[\sigma_s^2\left(s,t\right)\omega\left(s,t\right)\right] + \frac{\partial\varphi\left(s,t\right)}{\partial s}, \tag{49}$$

$$0 = \left(U'\left(c\left(s,t\right)\right) - \frac{\partial\phi\left(s,t\right)}{\partial s}\right) f\left(s,t\right) + \varphi\left(s,t\right) U''\left(c\left(s,t\right)\right), \tag{50}$$

[31] We consider the general case that allows aggregate labor demand externality.

$$\int_{\bar{s}}^{\infty} \left[\phi_s(s,t) f(s,t) + V_s(s,t)\omega(s,t)\right] \cdot \left[s + e_f A_t\right] ds = 0, \tag{51}$$

for any $(s,t) \in [\bar{s},\infty) \times [0,\infty)$, *where* $Y_t = \int_{\bar{s}}^{\infty} y(u(s,t)) f(s,t)\, ds$, $T_t = \Delta r_t L_t + e_f r_t^a A_t$, $L_t = -\int_{\bar{s}}^{0} s f(s,t)\, ds$, $A_t = \int_0^{\infty} s f(s,t)\, ds$, *and transversality and boundary conditions*

$$\lim_{T\to\infty} \exp(-\rho T)\, \phi(s,T) = 0, \tag{52}$$

$$\omega(\cdot,0) = \lim_{T\to\infty} \exp(-\rho T)\, \omega(\cdot,T) = 0, \tag{53}$$

$$\varphi(\bar{s},\cdot) = \lim_{s\to\infty} \varphi(s,\cdot) = 0. \tag{54}$$

Interpretation: The KFE multiplier ϕ is the solution to an HJB different from the HJB of V. It considers the value of resources and value in insurance. This is a second-order PDE with a transversality condition. The multiplier ω for HJB of V is not always equal to 0, which means the household's HJB could be binding. The multiplier for φ is not zero, although it is also redundant in terms of the choice of optimal spread. The multiplier of market clearing condition ψ is given by 51, which requires the aggregate asset weighted by ϕ_s is equal to 0. Equation 47 is the first-order condition for optimal spread path. The choice of Δr balances a pair of trade-off. First, an increase in Δr raises up aggregate welfare through fiscal transfer, which is captured by $\langle \phi_s f + V_s \omega, L_t \rangle_{\mathcal{S}}$. ϕ_s measures the marginal social value of an individual and L_t measures the marginal increase in assets for each individual. Second, an increase in Δr decreases borrowers' welfare through a higher borrowing interest rate, which is captured by $\left\langle \phi_s f + V_s \omega, \mathbb{I}_{\{s<0\}} \cdot s \right\rangle_{\mathcal{S}}$. The loss of marginal social value of a borrower is weighted by her debt level s.

G.2 Proofs

G.2.1 Mathematical preliminaries

First we introduce some mathematical concepts to simplify the expression and formalize the math. An operator T is a mapping from one vector space to another. Given the stochastic process s_t in (10), define an operator $\mathcal{A}$,

$$\mathcal{A}V \equiv \mu(s,t) \frac{\partial V(s,t)}{\partial s} + \frac{1}{2}\sigma_s^2(s,t) \frac{\partial^2 V(s,t)}{\partial s^2}, \tag{55}$$

with

$$\mu(s,t) \equiv r(s,t)\, s + \eta_l \cdot y(s,t) + (1-\eta_l) \int_{\bar{s}}^{\infty} y(u(s,t)) f(s,t)\, ds - c(s,t) + \Delta r_t L_t + e_f r_t^a A_t \tag{56}$$

and $\sigma_s^2(s,t) \equiv \sigma^2(u(s,t))$. The HJB equation (12) can be expressed as

$$\rho V = \max_{c \in \mathbb{R}_+, u \in \{L,H\}} \{U(c) + \mathcal{A}V\} + \frac{\partial V}{\partial t}.$$

Let $\mathcal{S} \equiv [\bar{s}, \infty)$ and $\Omega \equiv \mathcal{S} \times [0, \infty)$ be the valid domain. The space of Lebesgue-integrable functions $L^2(\mathcal{S})$ with the inner product

$$\langle f, g \rangle_{\mathcal{S}} = \int_{\mathcal{S}} f \cdot g ds$$

for any $f, g \in L^2(\mathcal{S})$, are both Hilbert spaces.

Given an operator $\mathcal{A}$, its adjoint is an operator $\mathcal{A}^*$ such that $\langle f, \mathcal{A}g \rangle_{\mathcal{S}} = \langle \mathcal{A}^* f, g \rangle_{\mathcal{S}}$. In the case of the operator defined by (55) its adjoint operator is given by

$$\mathcal{A}^* f \equiv -\frac{\partial}{\partial s}[\mu(s,t) f(s,t)] + \frac{1}{2}\frac{\partial^2}{\partial s^2}\left[\sigma_s^2(s,t) f(s,t)\right] \tag{57}$$

such that the law of motion of wealth distribution (13) results in

$$\frac{\partial f}{\partial t} = \mathcal{A}^* f.$$

We may verify that $\mathcal{A}$ and $\mathcal{A}^*$ are adjoint in $\mathcal{S}$. Given any $u, g \in L^2(\mathcal{S})$

$$\begin{aligned}
\langle g, \mathcal{A}u \rangle_{\mathcal{S}} &= \int g \mathcal{A}u ds = \int g(s)\left[\mu(s,t)\frac{\partial u}{\partial s} + \frac{1}{2}\sigma^2(s,t)\frac{\partial^2 u}{\partial s^2}\right] ds \\
&= \int u\left[-\frac{\partial}{\partial s}(g\mu) + \frac{1}{2}\frac{\partial^2}{\partial s^2}\left(\sigma^2(s,t) g\right)\right] ds \\
&= \int u \mathcal{A}^* g ds = \langle \mathcal{A}^* g, u \rangle_{\mathcal{S}},
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$ and we have integrated by parts.

The rest of the proofs strong base on the methodology of Nuno and Moll (2018) and Nuno and Thomas (2017).

G.2.2 Proof of Lemma 3

Given the welfare criterion defined in equation (24), we have

$$\begin{aligned}
W(f_0) &= \int_{\bar{s}}^{\infty} V(s,0) f(s,0) ds \\
&= \int_{\bar{s}}^{\infty} \mathbb{E}_0\left[\int_0^{\infty} e^{-\rho t} U(c(s_t,t)) dt | s_0 = s\right] f_0(s) ds \\
&= \int_{\bar{s}}^{\infty}\left[\int_{\bar{s}}^{\infty}\int_0^{\infty} e^{-\rho t} U(c(\hat{s},t)) f(\hat{s},t;s) dt d\hat{s}\right] f_0(s) ds \\
&= \int_0^{\infty} e^{-\rho t}\int_{\bar{s}}^{\infty} U(c(\hat{s},t))\left[\int_{\bar{s}}^{\infty} f(\hat{s},t;s) f_0(s) ds\right] d\hat{s} dt \\
&= \int_0^{\infty} e^{-\rho t}\int_{\bar{s}}^{\infty} U(c(\hat{s},t)) f(\hat{s},t) d\hat{s} dt,
\end{aligned}$$

where $f(\hat{s}, t; s)$ is the transition probability from $s_0 = s$ to $s_t = \hat{s}$, and in the last equality we used the Chapman-Kolmogorov equation,

$$f(\hat{s}, t) = \int_{\bar{s}}^{\infty} f(\hat{s}, t; s) f_0(s) ds.$$

QED.

G.2.3 Proof of Proposition 7: Solution to the Ramsey problem

Following the proof of Proposition 1 in Nuno and Thomas (2017), the idea is to construct a Lagrangian in a Hilbert function space and to obtain the first-order conditions by taking the Gateaux derivatives.

Step 1: Statement of the problem. The planner wishes to optimize a path of real spreads by maximizing:

$$W[f_0(\cdot)] = \max_{\{\Delta r(t), r^a(t), V(s,t), c(s,t), f(s,t)\} \geq 0} \int_0^{\infty} e^{-\rho t} \int_{\bar{s}}^{\infty} U(c(s,t)) f(s,t) ds dt$$

subject to the Kolmogorov equation:

$$f_t(s,t) = \mathcal{A}^* f(s,t); \text{ given } f(s,0) = f_0(s) \tag{58}$$

the accounting identity of the value (the HJB)

$$\rho V(s,t) = U(c(s,t)) + \mathcal{A} V(s,t) + V_t(s,t) \tag{59}$$

the asset market clearing condition

$$\int_{s<0} s f(s,t) ds + \int_{s \geq 0} \left[1 + e_f\right] s f(s,t) ds = 0 \tag{60}$$

the first-order condition

$$U'(c(s,t)) = V_s(s,t)$$

and

$$u = H \text{ if and only if } \frac{Y(H) - Y(L)}{\frac{1}{2}\sigma^2(H)} \geq -\frac{V_{ss}}{V_s} \text{ and } s > \tilde{s}_t.$$

Next we perform some useful calculations:

$$\frac{\partial}{\partial r^a} \mathcal{A}[z] = z_s \cdot \left(s + e_f A_t\right) \text{ and } \frac{\partial}{\partial \Delta r} \mathcal{A}[z] = z_s \cdot \left(L_t + \mathbb{I}_{\{s<0\}} \cdot s\right).$$

and

$$\frac{\partial}{\partial r^a} \mathcal{A}^*[z] = -\frac{\partial}{\partial s}\left[z \cdot \left(s + e_f A_t\right)\right],$$

and

$$\begin{aligned}\frac{\partial}{\partial \Delta r}\mathcal{A}^*[z] &= -\frac{\partial}{\partial s}\left[z\left(\mathbb{I}_{\{s<0\}}\cdot s + L_t\right)\right] \\ &= -z\mathbb{I}_{\{s<0\}} - z_s\left[\mathbb{I}_{\{s<0\}}\cdot s + L_t\right].\end{aligned}$$

Finally, we have the following derivative with respect to transfers:

$$\frac{\partial}{\partial L_t}\mathcal{A}[z] = z_s\cdot\Delta r \text{ and } \frac{\partial}{\partial A_t}\mathcal{A}[z] = z_s\cdot e_f r^a,$$

$$\frac{\partial}{\partial L_t}\mathcal{A}^*[z] = -z_s\cdot\Delta r \text{ and } \frac{\partial}{\partial A_t}\mathcal{A}^*[z] = -z_s\cdot e_f r^a.$$

Step2: The Lagrangian. The corresponding Lagrangian of this problem is given by:

$$\begin{aligned}&\mathcal{L}(\Delta r, r^a, f, V, c; f_0)\\ =\ &\langle\exp(-\rho t)U(c), f\rangle_\Omega\\ +\ &\langle\exp(-\rho t)\phi, \mathcal{A}^* f - f_t\rangle_\Omega\\ +\ &\langle\exp(-\rho t)\omega, U(c) + \mathcal{A}V + V_t - \rho V\rangle_\Omega\\ +\ &\left\langle\exp(-\rho t)\xi, \left\langle\left[1+\mathbb{I}_{\{s\geq 0\}}e_f\right]s, f\right\rangle_{\mathcal{S}}\right\rangle_{[0,\infty)}\\ +\ &\langle\exp(-\rho t)\varphi, U'(c) - V_s\rangle_\Omega\end{aligned}$$

where $\exp(-\rho t)\phi(s,t)$, $\exp(-\rho t)\omega(s,t)$, $\exp(-\rho t)\varphi(s,t) \in L^2(\Omega)$ and $\exp(-\rho t)\xi(t) \in L^2[0,\infty)$ are the Lagrange multipliers associated to equations 58, 59 and **??**. The given pieces of the problem is an initial disribution of real wealth f_0. An optimal policy sets the functional derivative of the Lagrangian equal to zero. The Lagrangian can be expressed as

$$\begin{aligned}\mathcal{L}(\Delta r, r^a, f, V, c; f_0) = &\int_0^\infty \exp(-\rho t)\left\langle U(c) - \rho\phi + \phi_t + \mathcal{A}\phi + \xi\cdot\left[1+\mathbb{I}_{\{s\geq 0\}}e_f\right]s, f\right\rangle_{\mathcal{S}} dt\\ &+\int_0^\infty \exp(-\rho t)\left(\langle\omega, U(c)\rangle_{\mathcal{S}} + \langle\mathcal{A}^*\omega - \omega_t, V\rangle_{\mathcal{S}} + \langle\varphi, U'(c) - V_s\rangle\right)dt\\ &-\lim_{T\to\infty}\langle\exp(-\rho T)\phi(\cdot,T), f(\cdot,T)\rangle_{\mathcal{S}} + \langle\phi(\cdot,0), f(\cdot,0)\rangle_{\mathcal{S}}\\ &-\langle\omega(\cdot,0), V(\cdot,0)\rangle_{\mathcal{S}} + \lim_{T\to\infty}\langle\exp(-\rho T)\omega(\cdot,T), V(\cdot,T)\rangle_{\mathcal{S}}\end{aligned}$$

where we have applied

$$\langle\phi, \mathcal{A}^* f\rangle = \langle\mathcal{A}\phi, f\rangle, \langle\omega, \mathcal{A}V\rangle = \langle\mathcal{A}^*\omega, V\rangle$$

and integrated by parts

$$\begin{aligned}
& \langle \exp(-\rho t)\phi, -f_t\rangle_\Omega \\
= & -\langle \exp(-\rho t)\phi, f_t\rangle_\Omega \\
= & -\int_{\mathcal{S}} \exp(-\rho t)\phi(s,t) f(s,t) |_0^\infty ds + \langle \exp(-\rho t)(\phi_t - \rho\phi), f\rangle_\Omega \\
= & \langle \exp(-\rho t)(-\rho\phi + \phi_t), f\rangle_\Omega - \lim_{T\to\infty} \langle \exp(-\rho T)\phi(\cdot,T), f(\cdot,T)\rangle_{\mathcal{S}} + \langle \phi(\cdot,0), f(\cdot,0)\rangle_{\mathcal{S}},
\end{aligned} \tag{61}$$

and

$$\begin{aligned}
& \langle \exp(-\rho t)\omega, V_t - \rho V\rangle_\Omega \\
= & \left\langle \omega, \frac{\partial}{\partial t}[\exp(-\rho t) V]\right\rangle_\Omega \\
= & \int_S \exp(-\rho t)\omega(s,t) V(s,t) |_0^\infty ds - \langle \exp(-\rho t)\omega_t, V\rangle_\Omega \\
= & \lim_{T\to\infty} \langle \exp(-\rho T)\omega(\cdot,T), V(\cdot,T)\rangle_S - \langle \omega(\cdot,0), V(\cdot,0)\rangle_S \\
& - \langle \exp(-\rho t)\omega_t, V\rangle_\Omega
\end{aligned}$$

Step 3: Necessary conditions. We take the Gateaux derivatives with respect to the controls $f, V, c, \Delta r, r^a$.

- The Gateaux derivative with respect to $f(s,t)$ is:

$$\begin{aligned}
& \frac{d}{d\alpha}[\mathcal{L}(\Delta r, r^a, f+\alpha h, V, c; f_0)] \\
= & \int_0^\infty \exp(-\rho t)\left(\left\langle U(c) - \rho\phi + \phi_t + \mathcal{A}\phi + \xi\cdot\left[1 + \mathbb{I}_{\{s\geq 0\}} e_f\right] s, h\right\rangle_{\mathcal{S}}\right) dt \\
& + \int_0^\infty \exp(-\rho t)\left(\left\langle \frac{\partial \mathcal{A}\phi}{\partial\alpha}, f\right\rangle_{\mathcal{S}} + \left\langle \frac{\partial \mathcal{A}^*\omega}{\partial\alpha}, V\right\rangle_S\right) dt \\
& - \lim_{T\to\infty} \langle \exp(-\rho T)\phi(\cdot,T), h(\cdot,T)\rangle_{\mathcal{S}} + \langle \phi(\cdot,0), h(\cdot,0)\rangle_{\mathcal{S}}
\end{aligned}$$

  which should be equal to zero for any function $\exp(-\rho t) h \in L^2(\Omega)$ such that $h(\cdot,0) = 0$. Recall that

$$\begin{aligned}
\frac{\partial \mathcal{A}\phi}{\partial\alpha} &= \left.\frac{\partial \mathcal{A}\phi}{\partial[L_t + \alpha H_t]}\right|_{\alpha=0} \frac{d[L_t + \alpha H_t]}{d\alpha} \\
&= \phi_s(s,t)\cdot\left[(1-\eta_l)\langle y, h\rangle_{\mathcal{S}} + \Delta r(t)\cdot\left\langle \mathbb{I}_{\{s<0\}}(-s), h\right\rangle_{\mathcal{S}} + r^a(t) e_f\left\langle \mathbb{I}_{\{s\geq 0\}} s, h\right\rangle_{\mathcal{S}}\right] = -\frac{\partial \mathcal{A}^*\phi}{\partial\alpha}.
\end{aligned}$$

  Then we must have

$$\begin{aligned}
\left\langle \frac{\partial \mathcal{A}\phi}{\partial\alpha}, f\right\rangle_S &= \langle \phi_s, f\rangle_{\mathcal{S}}\left[(1-\eta_l)\langle y, h\rangle_{\mathcal{S}} + \Delta r(t)\cdot\left\langle \mathbb{I}_{\{s<0\}}(-s), h\right\rangle_{\mathcal{S}} + r^a(t) e_f\left\langle \mathbb{I}_{\{s\geq 0\}} s, h\right\rangle_{\mathcal{S}}\right] \\
&= \langle \phi_s, f\rangle_{\mathcal{S}}\cdot\left\langle (1-\eta_l) y + \Delta r(t)\mathbb{I}_{\{s<0\}}(-s) + r^a(t) e_f \mathbb{I}_{\{s\geq 0\}} s, h\right\rangle_{\mathcal{S}}
\end{aligned}$$

and

$$\begin{aligned}\left\langle \frac{\partial \mathcal{A}^{*}\omega}{\partial \alpha}, V \right\rangle &= -\left\langle \omega_s, V \right\rangle_{\mathcal{S}} \left[\left(1-\eta_l\right) \left\langle y, h \right\rangle_{\mathcal{S}} + \Delta r\left(t\right) \cdot \left\langle \mathbb{I}_{\{s<0\}}\left(-s\right), h \right\rangle_{\mathcal{S}} + r^{a}\left(t\right) e_f \left\langle \mathbb{I}_{\{s\geq 0\}} s, h \right\rangle_{\mathcal{S}} \right] \\ &= \left\langle V_s, \omega \right\rangle_{S} \cdot \left\langle \left(1-\eta_l\right) y + \Delta r\left(t\right) \mathbb{I}_{\{s<0\}}\left(-s\right) + r^{a}\left(t\right) e_f \mathbb{I}_{\{s\geq 0\}} s, h \right\rangle_{\mathcal{S}}.\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}&\frac{d}{d\alpha}\left[\mathcal{L}\left(\Delta r, r^{a}, V, c, f+\alpha h; f_0\right)\right]|_{\alpha=0} \\ = &\int_0^{\infty} \exp\left(-\rho t\right) \left(\left\langle U\left(c\right) - \rho\phi + \phi_t + \mathcal{A}\phi + \xi \cdot \left[1 + \mathbb{I}_{\{s\geq 0\}} e_f\right] s, h \right\rangle_{\mathcal{S}} \right) dt \\ &+ \int_0^{\infty} \exp\left(-\rho t\right) \left[\left(1-\eta_l\right) \left\langle y, h \right\rangle_{\mathcal{S}} + \Delta r\left(t\right) \cdot \left\langle \mathbb{I}_{\{s<0\}}\left(-s\right), h \right\rangle_{\mathcal{S}} + r^{a}\left(t\right) e_f \left\langle \mathbb{I}_{\{s\geq 0\}} s, h \right\rangle_{\mathcal{S}} \right] \left(\left\langle \phi_s, f \right\rangle_{S} + \left\langle V_s, \omega \right\rangle_{S} \right) dt \\ &- \lim_{T\to\infty} \left\langle \exp\left(-\rho T\right) \phi\left(\cdot, T\right), h\left(\cdot, T\right) \right\rangle_{\mathcal{S}} + \left\langle \phi\left(\cdot, 0\right), h\left(\cdot, 0\right) \right\rangle_{\mathcal{S}},\end{aligned}$$

which implies that the necessary conditions for f are

$$\rho\phi = U\left(c\right) + \phi_t + \mathcal{A}\phi + \underbrace{\xi \left[1 + \mathbb{I}_{\{s\geq 0\}} e_f\right] s}_{\text{value of resources}} + \underbrace{\left[\left(1-\eta_l\right) y\left(s,t\right) - \mathbb{I}_{\{s<0\}} \Delta r \cdot s + \mathbb{I}_{\{s\geq 0\}} r^{a} \cdot e_f \cdot s \right] \cdot \left(\left\langle \phi_s, f \right\rangle_{S} + \left\langle V_s, \omega \right\rangle_{S} \right)}_{\text{value in insurance}} \tag{62}$$

for $(s,t) \in \Omega$ with the transversality condition

$$\lim_{T\to\infty} \exp\left(-\rho T\right) \phi\left(s, T\right) = 0 \tag{63}$$

with the intuition that each individual is valued for his own utility flow, but also by the net resources it brings.

- For $c\left(s,t\right)$, the Gateaux derivative is

$$\begin{aligned}&\frac{d}{d\alpha}\left[\mathcal{L}\left(\Delta r, r^{a}, f, V, c+\alpha h; f_0\right)\right]\Big|_{\alpha=0} \\ = &\int_0^{\infty} \exp\left(-\rho t\right) \left\langle U'\left(c\right) \cdot h + \frac{\partial \mathcal{A}\phi}{\partial \alpha}, f \right\rangle_{\mathcal{S}} dt \\ &+ \int_0^{\infty} \exp\left(-\rho t\right) \left(\left\langle \omega, U'\left(c\right) h \right\rangle_{S} + \left\langle \frac{\partial \mathcal{A}^{*}\omega}{\partial \alpha}, V \right\rangle_{\mathcal{S}} + \left\langle \varphi, U'' h \right\rangle_{\mathcal{S}} \right) dt \\ = &\int_0^{\infty} \exp\left(-\rho t\right) \left\langle U'\left(c\right) \cdot h + \frac{\partial \mathcal{A}\phi}{\partial \alpha}, f \right\rangle_{\mathcal{S}} dt \\ &+ \int_0^{\infty} \exp\left(-\rho t\right) \left(\left\langle \omega, U'\left(c\right) h \right\rangle_{S} + \left\langle \omega, \frac{\partial \mathcal{A}V}{\partial \alpha} \right\rangle_{\mathcal{S}} + \left\langle \varphi, U'' h \right\rangle_{\mathcal{S}} \right) dt \\ = &\int_0^{\infty} \exp\left(-\rho t\right) \left\langle \left(U'\left(c\right) - \phi_s\right) h, f \right\rangle_{\mathcal{S}} dt \\ &+ \int_0^{\infty} \exp\left(-\rho t\right) \left(\left\langle \omega, \left(U'\left(c\right) - V_s\right) h \right\rangle_{S} + \left\langle \varphi, U'' h \right\rangle_{\mathcal{S}} \right) dt.\end{aligned}$$

The Gateaux derivative should be zero for any function $\exp\left(-\rho t\right) h \in L^{2}\left(\Omega\right)$. Since the first-order condition of

households at optimum is $U'(c) = V_s$, this yields

$$0 = \left(U' - \phi_s\right) f + \varphi U''. \tag{64}$$

- The Gateaux derivative of $V(s,t)$ is

$$\begin{aligned}
&\frac{d}{d\alpha}\left[\mathcal{L}\left(\Delta r, r^a, f, V + \alpha h, c; f_0\right)\right]\Big|_{\alpha=0} \\
&= \int_0^\infty \exp\left(-\rho t\right)\left(\left\langle \mathcal{A}^*\omega - \omega_t, h\right\rangle_{\mathcal{S}} + \left\langle \varphi, -h_s\right\rangle_{\mathcal{S}}\right) dt \\
&\quad - \left\langle \omega\left(\cdot,0\right), h\left(\cdot,0\right)\right\rangle_{\mathcal{S}} + \lim_{T\to\infty}\left\langle \exp\left(-\rho T\right)\omega\left(\cdot,T\right), h\left(\cdot,T\right)\right\rangle_S \\
&= \int_0^\infty \exp\left(-\rho t\right)\left\langle \mathcal{A}^*\omega - \omega_t + \varphi_s, h\right\rangle_{\mathcal{S}} dt \\
&\quad - \left\langle \omega\left(\cdot,0\right), h\left(\cdot,0\right)\right\rangle_{\mathcal{S}} + \lim_{T\to\infty}\left\langle \exp\left(-\rho T\right)\omega\left(\cdot,T\right), h\left(\cdot,T\right)\right\rangle_S \\
&\quad - \int_0^\infty \exp\left(-\rho t\right)\ \varphi\left(s,t\right) h\left(s,t\right)|_{\bar{s}}^{\infty} dt
\end{aligned}$$

The Gateaux derivative should be zero for any function $\exp\left(-\rho t\right) h \in L^2\left(\Omega\right)$. Then for ω we obtain the transversality condition

$$\lim_{T\to\infty} \exp\left(-\rho T\right)\omega\left(\cdot,T\right) = 0, \tag{65}$$

and a Kolmogorov forward equation in ω

$$\omega_t = \mathcal{A}^*\omega + \varphi_s \tag{66}$$

with boundary conditions

$$\omega\left(\cdot,0\right) = 0, \tag{67}$$

Note that the multiplier ω is not always equal to 0, implying that the HJB of V is binding at some (s,t). For φ we have the following transversality condition

$$\lim_{s\to\infty} \varphi\left(s,\cdot\right) = 0,$$

and the boundary condition

$$\varphi\left(\bar{s},\cdot\right) = 0.$$

- For Δr, the Gateaux derivative is

$$
\begin{aligned}
&\frac{d}{d\alpha}\left[\mathcal{L}\left(\Delta r+\alpha h, r^a, f, V, c; f_0\right)\right]\Big|_{\alpha=0} \\
&=\int_0^\infty \exp\left(-\rho t\right)\left(\left\langle \frac{\partial \mathcal{A}\phi}{\partial \Delta r}, f\right\rangle_{\mathcal{S}}+\left\langle \frac{\partial \mathcal{A}^*\omega}{\partial \Delta r}, V\right\rangle_{\mathcal{S}}\right) h\left(t\right) dt \\
&=\int_0^\infty \exp\left(-\rho t\right)\left(\left\langle \frac{\partial \mathcal{A}\phi}{\partial \Delta r}, f\right\rangle_{\mathcal{S}}+\left\langle \frac{\partial \mathcal{A}V}{\partial \Delta r}, \omega\right\rangle_{\mathcal{S}}\right) h\left(t\right) dt \\
&=\int_0^\infty \exp\left(-\rho t\right)\left\langle \phi_s f+V_s\omega, L_t+\mathbb{I}_{\{s<0\}}\cdot s\right\rangle_{\mathcal{S}} h\left(t\right) dt
\end{aligned}
$$

which should be equal to zero for any function $\exp\left(-\rho t\right) h \in L^2\left[0,\infty\right)$. Then the equation yields:

$$
\left\langle \phi_s f+V_s\omega, \left[L_t+\mathbb{I}_{\{s<0\}}\cdot s\right]\right\rangle_{\mathcal{S}}=0. \tag{68}
$$

- The Gateaux derivative of r^a is

$$
\begin{aligned}
&\frac{d}{d\alpha}\left[\mathcal{L}\left(\Delta r, r^a+\alpha h, f, V, c; f_0\right)\right]|_{\alpha=0} \\
&=\int_0^\infty \exp\left(-\rho t\right)\left(\left\langle \frac{\partial \mathcal{A}\phi}{\partial r^a}, f\right\rangle_{\mathcal{S}}+\left\langle \frac{\partial \mathcal{A}^*\omega}{\partial r^a}, V\right\rangle_{\mathcal{S}}\right) h\left(t\right) dt \\
&=\int_0^\infty \exp\left(-\rho t\right)\left(\left\langle \frac{\partial \mathcal{A}\phi}{\partial r^a}, f\right\rangle_{\mathcal{S}}+\left\langle \frac{\partial \mathcal{A}V}{\partial r^a}, \omega\right\rangle_{\mathcal{S}}\right) h\left(t\right) dt \\
&=\int_0^\infty \exp\left(-\rho t\right)\left\langle \phi_s f+V_s\omega, s+e_f A_t\right\rangle_{\mathcal{S}} h\left(t\right) dt.
\end{aligned}
$$

The Gateaux derivative should be equal to zero for any $\exp\left(-\rho t\right) h \in L^2\left[0,\infty\right)$. The optimality condition then results in

$$
\left\langle \phi_s f+V_s\omega, s+e_f A_t\right\rangle_{\mathcal{S}}=0. \tag{69}
$$

QED.