Money and banking in a New Keynesian model*

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October 2019

Abstract

This paper studies a New Keynesian model with a banking system. The central bank targets the interest rate on short safe bonds that are held by banks to back inside money and hence earn convenience yield for their safety or liquidity. Central bank operating procedures matter. In a floor system, the reserve rate and the quantity of reserves are independent policy tools that affect banks' cost of safety. In a corridor system, increasing the interbank rate by making reserves scarce increases banks' cost of liquidity and generates strong pass-through to other rates of return, output and inflation. In either system, policy rules that do not respond aggressively to inflation – such as an interest rate peg – need not lead to self-fulfilling fluctuations. The stabilizing effect from an endogenous convenience yield is stronger when there are more nominal rigidities in bank balance sheets.

*Email addresses: piazzesi@stanford.edu, ciaran@stanford.edu, schneidr@stanford.edu. We thank Roc Armenter, Eric Swanson, Tommaso Monacelli, as well as seminar and conference participants at the Bank of Canada, Chicago, ECB, Kellogg, Lausanne, Macro-Finance Society, NYU, Princeton, UC Santa Cruz, the RBNZ Macro-Finance Conference, SF Fed, Stanford, and the NBER SI Impulse and Propagations meeting for helpful comments and suggestions.

1 Introduction

Models of monetary policy typically assume that the central bank sets the nominal interest rate on household savings. In the presence of nominal rigidities, the central bank then has a powerful lever to affect households' stochastic discount factor and hence intertemporal decisions such as savings and investment. In practice, however, central banks target interest rates on short safe bonds that are predominantly held by intermediaries. At the same time, the behavior of such interest rates is not well accounted for by asset pricing models that fit expected returns on other assets such as long terms bonds or stocks: this "short rate disconnect" has been attributed to a convenience yield on short safe bonds.[1]

This paper studies a New Keynesian model with a banking system that is consistent with key facts on holdings and pricing of short safe bonds. A short rate disconnect arises because short safe bonds are held by banks to back inside money. As a result, the "plumbing" of the economy, that is, the structure of the banking system, bank liquidity management, and central bank operating procedures matter for the transmission of policy. In particular, in a corridor system with scarce reserves interest rate policy generates stronger pass-through from the policy rate to other rates of return and the real economy. In a floor system, interest rate policy is weaker, but the quantity of reserves serves as an independent policy instrument. The short rate disconnect further implies that policy need not respond aggressively to inflation, without inviting self-fulfilling fluctuations.

Our results follow from three familiar assumptions. First, inside money issued by banks earns a convenience yield for its liquidity, measured by the spread between the interest rate on savings and the interest rate on money, that is, households' cost of liquidity. Second, banks face leverage constraints: inside money must be backed by collateral. Short safe bonds are good collateral and earn a convenience yield for their safety, measured by the spread between the interest rate on household savings and the interest rate on short safe bonds held by banks, that is, banks' cost of safety.[2] Finally, pass-through from the policy rate to other rates occurs because *total* risk-adjusted expected returns – pecuniary expected returns plus convenience yields – on all assets are equated in equilibrium.

To see how an endogenous convenience yield affects the transmission of interest rate policy, suppose the central bank raises the interest rate on short safe bonds: it raises the target for the

[1] The short rate disconnect has been a stylized fact in the empirical literature on the term structure of interest rates since Duffee (1996). Lenel, Piazzesi and Schneider (2019) provide evidence of its connection to bank balance sheets.

[2] Both spreads are distinct from banks' cost of liquidity, measured by the spread between the interbank rate and the interest rate on reserves: in particular, both typically remain positive in floor systems when banks' cost of liquidity is essentially zero.

interbank rate in a corridor system, or the interest rate on reserves in a floor system. Standard New Keynesian logic says that sticky prices imply a higher real short rate and lower nominal spending. However, lower nominal spending lowers the convenience yield on inside money and hence on short safe bonds that back inside money, be they interbank loans or reserves. In other words, the central bank lowers banks' cost of safety. A lower total return on the policy instrument bonds in turn implies a lower risk-adjusted expected return on other assets. The response of the convenience yield to spending thus dampens the policy impact on output and inflation. This effect works through banks' cost of safety, and is thus present in both floor and corridor systems.

A corridor system differs from a floor system in how banks manage liquidity. Our model assumes that banks face liquidity shocks and leverage constraints that limit overnight borrowing. Reserves are more useful for handling liquidity shocks than other assets: if they are sufficiently scarce, they earn a higher convenience yield than other short safe bonds. The central bank can thus choose to implement a corridor system with scarce reserves: it targets a positive spread between the interbank rate and the reserve rate and elastically supplies reserves to achieve that spread. Since the central bank effectively targets two interest rates, the quantity of reserves is not an independent policy instrument. Moreover, tighter interest rate policy now increases banks' cost of liquidity.

The alternative is a floor system with abundant reserves: the central bank sets the interest rate on reserves and supplies enough reserves so that banks can manage liquidity without borrowing reserves overnight from other banks. The overnight interbank credit market ceases to operate and banks' cost of liquidity is constant at zero. Since reserves are no longer supplied elastically to target two interest rates, their quantity becomes an independent policy instrument: it affects the supply of safe collateral available to banks. In a floor system, both interest rate policy and the supply of reserves matter for banks' cost of safety. The supply of reserves is not a technicality handled by a trading desk that implements an interest rate spread. Even if the central bank chooses to formulate interest rate policy, agents' expectations about the future path of reserves are relevant.

Interest rate policy is more powerful in a corridor system. Indeed, an increase in the policy rate, say, not only lowers banks' cost of safety, but also increases their cost of liquidity. As liquidity management becomes more costly, banks lower the supply of inside money which increases households' cost of liquidity. In a floor system, in contrast, a higher reserve rate only lowers banks' cost of safety and thus always lowers households' cost of liquidity. In other words, the convenience yields on money and short bonds fall by less in a corridor system, and may in fact increase if liquidity management is sufficiently important for bank costs. In a corridor system, there is therefore stronger pass through from the policy rate to other rates

of return. We show that the difference can be quantitatively relevant, in particular when the model allows for a simple "cost channel" – inside money and consumption are complements in utility.

While a discussion of the "plumbing" requires a model with an explicit banking sector, it is interesting to ask whether a simpler model without banks can still serve as a useful guide to policy. We argue that it is misleading to use the standard model to think about a floor system: that model cannot capture the role of the quantity of reserves as a policy tool and that a higher policy rate lowers the cost of liquidity. We propose instead a simple setup where the central bank directly sets the interest rate on money. One interpretation is that the central bank issues a central bank digital currency (CBDC), for which it controls both the quantity and the interest rate. The CBDC model shares key features of our model that capture policy transmission in a floor system. At the same time, our results suggest that the standard model captures quite well how policy works in a corridor system. At relevant parameters, our model of the corridor system shares the feature of the standard model that households' cost of liquidity increases with the policy rate.

We emphasize nevertheless that the model with a corridor system does not reduce to the standard model even though reserves are supplied elastically to implement the target rate. This is because reserves are only one type of collateral used by banks to back inside money. A corridor system makes the supply of inside money more elastic, but it does not make it perfectly elastic. In particular, the quality and denomination of bank assets matters for the creation of inside money and hence for output and inflation. A negative shock to bank balance sheets – for example, a sudden reduction in the quality of existing loans – lowers the supply of inside money and hence increases its convenience yield. It raises the expected real rate of return on other assets and is contractionary.

In contrast to the standard model, our model says that interest rate rules that do not aggressively respond to inflation need not make the economy susceptible to self-fulfilling fluctuations. Consider for example an interest rate peg. Can there be a self-fulfilling recession? If agents believe that output is temporarily low, inflation slows as firms anticipate lower cost. With a pegged nominal rate, the real rate increases. In the standard model, the expected real return on all assets increases: lower demand makes the recessionary belief self-fulfilling. In our model, in contrast, lower spending lowers the convenience yield, which in turn keeps the expected real return on other assets low. Put differently, the Taylor principle – lower inflation should lead to a lower real interest rate – can hold for the interest rate on savings, even if it does not hold for the policy rate of the central bank. Endogenous adjustment of the convenience yield substitutes for policy as a stabilizing force.

Endogenous convenience yields move more when there are nominal rigidities in bank ba-

lance sheets. As a simple example, suppose the central bank runs a floor system and commits to a path for reserves. The quantity of reserves then works like a nominal anchor for the economy. An increase in the reserve rate lowers convenience yields not only by reducing real output, but also by lowering the price level and hence increasing the real quantity of reserves. As a result, conditions for local determinacy tend to be weaker when there is a nominal anchor. At the same time, our comparison of rules for the policy rate and reserve supply shows that a nominal anchor is not necessary for determinacy. The key force we emphasize is the endogenous convenience yield, which happens to move more when there is a nominal anchor.

Our paper adds to a growing literature on New Keynesian models with financial frictions, dating back to Bernanke, Gertler and Gilchrist (1999). Recent work has focused on financial frictions in the banking system; see for example Cúrdia and Woodford (2010), Gertler and Karadi (2011), Gertler et al. (2012), Christiano, Motto and Rostagno 2012, Ireland (2014), Del Negro et al. (2017), Brunnermeier and Koby (2018) or Wang (2019). In these models, banking also matters for transmission and there can be imperfect pass-through from the policy rate to deposit or loan rates. These papers nevertheless share the feature of the standard model that there is direct pass-through from the policy rate to the short rate, and so the households' nominal stochastic discount factor. They do not speak to the short rate disconnect, the key fact that motivates our analysis.[3]

Diba and Loisel (2019) study the determinacy properties of a New Keynesian model with banks at the zero lower bound. In their setup, reserves are an input into bank lending, and the government commits to a nominal path of reserves. They establish local determinacy under the assumption that reserves remain scarce at the zero lower bound. In our model, in contrast, determinacy properties follow from the convenience yield of bank liabilities. It is not essential that reserves are scarce, that the government commits to a nominal path of reserves or that the policy rate is the reserve rate. In fact, our comparison of operating procedures focuses on times away from the zero lower bound when either (i) reserves are scarce and the central bank targets an interbank rate – the US policy regime before the financial crisis – or (ii) reserves are abundant and the central bank sets the reserve rate – the US regime after the crisis.

There is recent work on New Keynesian models with convenience yields on other assets. In particular, Hagedorn (2018) studies a HANK model with uninsurable income risk and a riskfree asset. Some consumers are not on their intertemporal Euler equation, so that their

[3] Much recent work on New Keynesian models has been motivated by the zero lower bound on interest rates, and various "puzzles" such as large fiscal multipliers or strong impact of forward guidance. In this paper, we do not focus on a lower bound. Instead, our goal is to extend the New Keynesian model in a way that is consistent with data on interest rates as well as holdings of short safe bonds. From this perspective, 2008 is a watershed because the Fed adopted a floor system that made liquidity cheap for banks. That decision is still relevant now that the level of interest rates has risen again.

marginal rate of substitution is not equated to the interest rate. Michaillat and Saez (2018) assume that wealth is a separate argument in utility, in addition to consumption. In both cases, a convenience yield is priced into assets that serve as a store of value for households. Our perspective here is different: we emphasize the convenience yield on assets held by banks that drives a wedge between the policy rate and the rate at which households save, as we see in the data. Our mechanism is thus complementary to the above effects. For example, a HANK model with banks might feature weak pass-through from the policy rate to the rate on household savings.

More generally, our model builds on a long tradition of asset pricing with investors who face liquidity or collateral constraints, dating back at least to Lucas (1990), Kiyotaki and Moore (1997) and Geanakoplos (2003). Recent work has emphasized the role of constrained intermediaries, see for example Brunnermeier and Pedersen (2009), He and Krishnamurthy (2013), Brunnermeier and Sannikov (2016) or Bocola (2016), as well as, in monetary economies, Drechsler, Savov and Schnabl (2018), Brunnermeier and Sannikov (2016) or Di Tella and Kurlat (2018). Our model also features "intermediary asset pricing" but differs from much of the literature in that banks are firms that maximize shareholder value and can costlessly recapitalize. The mechanism we emphasize does not require frictions in equity markets, and does not rely on financial accelerator dynamics.

A convenience yield on short bonds is often captured by making bonds an argument in utility, see for example Bansal and Coleman (1996), Krishnamurthy and Vissing-Jorgensen (2012) or Nagel (2016). Williamson (2012) derives a convenience yield in a model of decentralized exchange. Lenel, Piazzesi and Schneider (2019) take a closer look at the quantitative asset pricing implications of the approach we follow here. They show that bank optimization implies an observable pricing kernel based on bank balance sheet ratios that accounts well for the short rate disconnect, especially at business cycle frequencies.

We also build on a growing literature that studies macroeconomic effects of the structure of the banking system. In particular, several authors have emphasized the importance of market power in deposits markets; see for example Yankov (2014), Driscoll and Judson (2013), Duffie and Krishnamurthy (1996), Egan, Hortacsu and Matvos (2017), Drechsler et al. (2018) or Corbae and D'Erasmo (2013). In addition, there has been recent interest in bank liquidity management, for example Bianchi and Bigio (2014), De Fiore, Hoerova and Uhlig (2018) or Piazzesi and Schneider (2018). Both features matter for the quantitative relevance of our mechanism; our results suggest that studying them further is important for understanding the transmission of monetary policy.

A key feature of our model is the distinction between several payment instruments and their potential scarcity, in our case, reserves and deposits. The link between scarcity of pay-

ment instruments and convenience yields is well established in monetary theory. In particular, Williamson (2019) and Venkateswaran and Wright (2014) have shown how assets that back payment instruments can inherit their convenience yields, an effect that is also central to our mechanism. The literature has typically studied the coexistence of multiple payments used by households, for example currency and various types of deposits; see also Rocheteau, Wright and Xiao (2018), Andolfatto and Williamson (2015), Lucas and Nicolini (2015) and Ennis (2018)). We abstract from currency and emphasize instead a layered payment system in which households only pay with inside money, and only banks pay with outside money directly issued by the government.

Our focus on macro outcomes leads us to abstract from several institutional details. In particular, we do not explicitly distinguish between banks and money market mutual funds. From our perspective, the key feature of money market funds is that they are also payment intermediaries: unlike plain vanilla mutual funds, they provide payment services – this is why their shares are included in broad measures of money.[4] Williamson (2015) and Begenau and Landvoigt (2018) have studied models where banks and shadow banks compete in the market for payment instruments. Moreover, bank heterogeneity in our model is stark and serves only to create an aggregate demand for liquidity. Whitesell (2006), Keister et al. (2008), Afonso and Lagos (2015) and Afonso et al. (2018) provide more detailed data and modeling on its role under scarce and abundant reserves. Our results suggest that these details should inform the transmission of monetary policy.

Our results show that the nature of nominal assets in the economy is important for the transmission of policy. Our setup shares this feature with the fiscal theory of the price level. In particular, Sims (2013) and Cochrane (2018) have studied the role of the maturity structure of government debt. A key difference between our approach and the fiscal theory is that the nominal assets that matter for us are those available to banks in order to back inside money. While government debt can be part of those assets, private contracts such as loans are also relevant, and their payoffs can affect the transmission of policy. Moreover, our results do not assume a non-Ricardian fiscal regime.

The paper is structured as follows. Section 2 presents the simple model of central bank digital currency to introduce the key effects. Section 3 studies banks under a floor system. Section 4 considers banks under a corridor system. Proofs and derivations are collected in the Appendix.

[4] In order to provide payment services, money market funds contract with their custodian banks to gain access to interbank payments arrangements. Our model can be viewed as consolidating money market funds with their custodian banks.

2 Monetary policy with a convenience yield: a minimal model

In this section we study a minimal model of a central bank targeting an instrument with a convenience yield: money earns a convenience yield because it enters the utility function. Households and firms solve the same problems as in textbook treatments of the New Keynesian model. The only difference is that the central bank sets the quantity as well as the interest rate on money, as opposed to the short rate of the representative agent's stochastic discount factor. A special case of the setup is thus a New Keynesian model with a money growth rule. The model is more general, however, because it explores a larger set of rules for both interest rates and the money supply.

Our interpretation is that there is a central bank digital currency (CBDC): everyone has deposit accounts at the central bank, which controls both the nominal quantity and the interest rate. The short rate, like nominal rates of return on all other assets, adjusts to clear markets. Our interest in this model stems from its formal similarity to the banking models in Sections 3 and 4. We will show that the same mechanisms are at work both when the central bank makes reserves abundant – hence controlling their price and quantity – and when the central bank elastically supplies reserves to hit a fed funds rate target. Details of the banking system can be understood as altering the coefficients of policy rules in the model of this section.

2.1 Setup with central bank digital currency

Every period, the representative household chooses consumption goods C_t, nominal money balances D_t and labor N_t. Preferences are time separable with discount factor β and felicity

$$\frac{1}{1-\frac{1}{\sigma}}\left(C_t^{1-\frac{1}{\eta}}+\omega\left(D_t/P_t\right)^{1-\frac{1}{\eta}}\right)^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}}-\frac{\psi}{1+\varphi}N_t^{1+\varphi}, \tag{1}$$

where P_t is the price level, that is, the price of consumption goods in terms of money. Moreover, η is the intratemporal elasticity of substitution between consumption and real balances and σ is the intertemporal elasticity of substitution between bundles at different dates. If $\sigma=\eta$, utility is separable in consumption and real balances.

The New Keynesian model is usually derived by assuming separable utility. Most of our theoretical results – in particular on determinacy and the dampening of policy effects – already obtain in this case. We nevertheless develop the model for general nonseparable utility. We then emphasize below the case $\sigma>\eta$, where consumption and real balances are complements (that is, the cross partial derivative of the utility function is strictly positive). Complementarity helps fit the response of velocity to interest rates in the data. Moreover, it introduces a "cost

channel" – marginal cost increases with the opportunity cost of money – which has interesting theoretical effects, as discussed in Section 2.3 below.[5]

Money is provided by the central bank which issues a digital currency that pays the nominal interest rate i_t^D. The household can also invest in other short safe assets that pay the nominal interest rate i_t^S. The cost of liquidity $i_t^S - i_t^D$ is the convenience yield on digital currency. We refrain from calling i_t^S the interest rate on short bonds. The banking models below introduce short bonds explicitly; in equilibrium, they are held by banks whose valuation pushes the bond rate below i_t^S. Instead we refer to i_t^S as the *shadow rate*. It represents the (nominal) short rate in the household's stochastic discount factor and hence the first-order term in the nominal rate of return on any asset *held directly by households*. Since we linearize the model below and abstract from higher order terms, i_t^S is the relevant rate of return for all intertemporal decisions, as well as for the valuation of firms by shareholders.

The household budget constraint at date t is

$$P_t C_t + D_t + S_t = W_t N_t + T_t + \Pi_t + D_{t-1}(1 + i_{t-1}^D) + S_{t-1}(1 + i_{t-1}^S). \tag{2}$$

Income on the right-hand side consists of labor income at the nominal wage W_t, government transfers T_t, profits Π_t from firms, as well as payoffs from money and other assets that earn the rate i^S. The income is spent on consumption expenditure on the left-hand side and a new portfolio of money and other assets. Our timing convention is that money D_t chosen at date t provides liquidity services at that date – that is, it facilitates shopping for consumption C_t.

First order conditions. Households' optimal choices for consumption, money and bonds satisfy the standard Euler equations. First, the marginal rate of substitution of consumption for real balances must be equal to the relative price of liquidity services provided by money, or the opportunity cost of money. This intratemporal Euler equation describes a "money demand" relationship often studied in the empirical literature:

$$D_t = P_t C_t \, \omega^\eta \left(\frac{i_t^S - i_t^D}{1 + i_t^S} \right)^{-\eta}. \tag{3}$$

Since utility is homogenous of degree one in consumption and money, households hold money in proportion to nominal spending. Moreover, money holdings are decreasing in the opportunity cost of money, here the spread between other assets and money $i_t^S - i_t^D$. The elasticity of substitution η works like an interest elasticity of money demand.

[5] We focus on utility that is homogeneous of degree one in consumption and money in order to obtain a unitary income elasticity of money demand. Some derivations of the standard model instead work with separable utility that allows for different curvature parameters. It will become clear below how to extend our results to this case.

When consumption and money are complements, an increase in the opportunity cost of money lowers the marginal utility of consumption. To clarify the effect on labor supply as well as savings, we write the ideal price index for a bundle of consumption and liquidity services from money as

$$Q_t := \left(1 + \omega^{\eta} \left(\frac{i_t^S - i_t^D}{1 + i_t^S}\right)^{1-\eta}\right)^{\frac{1}{1-\eta}}. \tag{4}$$

This ideal price index is measured in units of consumption. Since the household cares about bundles, as opposed to only consumption goods, labor supply depends on the real wage measured in units of bundles, W_t/P_tQ_t. A higher spread $i_t^S - i_t^D$ thus not only increases the price of liquidity services, but also lowers the price of leisure. At the same time, it affects the household's savings decision by increasing the real return on assets in units of bundles, that is, $(1 + i_t^S)P_tQ_t/P_{t+1}Q_{t+1}$: future consumption bundles become relatively cheaper.

When consumption and money are complements, an increase in the opportunity cost of money lowers labor supply relative to consumption. Indeed, the first-order conditions imply a second intratemporal Euler equation that links the marginal rate of substitution of labor for consumption to the real wage:

$$Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} \psi N_t^{\varphi} = \frac{W_t}{P_t}. \tag{5}$$

In the separable case, the optimal choice of labor relative to consumption depends only on the relative price between these two goods: the real wage in units of consumption. When money and consumption are complements, in contrast, an increase in the opportunity cost of money makes consumption less attractive and leads households to take more leisure. Relative to the standard model, there is a "labor wedge" that is increasing in the opportunity cost of money.[6] This *cost channel* was emphasized in early flexible price DSGE models, but has received less attention in the new Keynesian literature.

The intertemporal Euler equation for the shadow rate relates the marginal utilities of con-

[6]With elasticities below one ($\eta < \sigma \leq 1$), competing income effects determine the labor wedge. These income effects dominate both the choice between consumption and liquidity services, and the choice between bundles and labor. A higher spread today makes liquidity services more expensive and, with a strong income effect, reduces consumption. A higher price for liquidity services also makes leisure cheaper and, with a strong income effect, increases demand for the bundle which includes more consumption. With separable utility, the two forces exactly cancel, and we obtain the Euler equation for labor from the standard model. Complementarity between money and consumption ($\eta < \sigma$) makes the income effect from the cost of liquidity stronger: a higher spread today thus leads the agent to consume relatively less and take relatively more leisure.

sumption at different dates to interest rates:

$$\beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^S \right) = 1. \tag{6}$$

Optimal savings implies that the discounted gross rate of return on assets is equal to one. In the nonseparable case, discounting by the marginal rate of substitution reflects the expected change in the opportunity cost of money. In particular, when money and consumption are complements the household acts as if he discounts the future more when the opportunity cost of money is temporarily lower: cheap liquidity today encourages consumption today.

Combining (3) and (6), we can write an analogous intertemporal Euler equation for money. It clarifies that money is valued not only for its payoff, but also earns a convenience yield:

$$\beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] \left(1 + i_t^D \right) + \omega \left(\frac{P_t C_t}{D_t} \right)^{\frac{1}{\eta}} = 1. \tag{7}$$

The total return on money on the left hand side now consists not only of the pecuniary rate of return (again appropriately discounted) but also adds a nonpecuniary benefit that is increasing in the velocity of money $V_t := P_t C_t / D_t$: if spending is high relative to money, shopping is more of a hassle and the convenience yield – the marginal benefit of additional money – is higher. The response of the convenience yield to velocity is stronger if the interest rate elasticity of money demand η is lower.

Firms. The supply side of the model is standard. Competitive firms make the consumption good from a continuum of intermediate goods; their production function is CES with elasticity of substitution ϵ. Monopolistically competitive firms make intermediate goods from labor using the linear production function $Y_t = N_t$. We assume Calvo price setting: the opportunity for an intermediate goods firm to reset its nominal price is an i.i.d. event that occurs with probability $1 - \zeta$. The firm commits to satisfy demand at its posted price every period.

Government. The government has two policy tools: the interest rate on money i_t^D and the money supply D_t , the total size of the household's digital currency account. Below, we allow policy to either use feedback rules or fix exogenous paths for these instruments. In either case, we consolidate the central bank and Treasury, and assume that the government levies lump sum taxes T_t to satisfy its budget constraint $D_t + P_t T_t = (1 + i_{t-1}^D) D_{t-1}$. An equilibrium then consists of sequences for consumption, labor, lump sum taxes, output of the various goods as well as the nominal interest rates i_t^S, i_t^D, the wage and the price level such that households and firms optimize, the government budget constraint and policy rules are satisfied, and the markets for goods, labor and money clear.

We consider Taylor rules for the interest rate, that is, i_t^D is a function of current inflation and output.[7] For the money supply, we consider rules of the form

$$D_t = \mu_t D_{t-1} + P_t G_t. \tag{8}$$

Mechanically, at date t, the government increases or shrinks the nominal money supply by a factor μ_t, and then issues new currency worth G_t consumption goods. A simple special case is commitment to a path for the nominal money supply (that is, $G_t = 0$). The government then provides a "nominal anchor" for the economy.

More generally, we allow for rules that do not provide a nominal anchor. For example, suppose that $\mu_t = 0$ and $G_t > 0$. The government then commits to a path for *real* balances. A motivation could be that the government desires a certain size of the balance sheet of the central bank relative to, say, long run output of the economy. An intermediate case obtains for $\mu \in (0,1)$: the government gradually moves real balances by retiring a share $1-\mu$ of the nominal money supply and adding money worth G_t goods. The transition law for *real* balances becomes

$$\frac{D_t}{P_t} = \mu \frac{D_{t-1}}{P_{t-1}} \frac{P_{t-1}}{P_t} + G_t. \tag{9}$$

We can view μ as a measure of *nominal rigidity* in the money supply process: inflation erodes real balances more with higher μ. This perspective will be useful below to discuss the role of nominal rigidities on bank balance sheets.

Equilibrium. Regardless of the details of the policy rule, characterization of equilibrium is routine and relegated to Appendix A.2. The equilibrium paths of output, the shadow rate i_t^S, and the price level satisfy a system of difference equations: a New Keynesian Phillips curve – derived from firms' optimal price setting – together with the intertemporal Euler equation (6) and money market equilibrium (3). A convenient way to describe equilibrium dynamics is to linearize the difference equations around a steady state – this is how we proceed below.

Steady state. To obtain an equilibrium with constant real quantities and rates of return, we assume that the government chooses constant policy parameters μ, G and i^D. Let π denote the steady state rate of inflation, which must equal the rate of nominal money growth. From the Euler equation (6), the steady state shadow rate is $i^S = \delta + \pi$, where $\delta = 1/\beta - 1$ is the household's discount rate. The real rate of return on money is $r^D = i^D - \pi$. Both the velocity of money and the price index for a bundle of consumption and liquidity services are constant

[7] We are particularly interested in this case because in the banking models of Sections 3 and 4, a Taylor rule on short bonds held by banks – as currently used by many central banks – will work effectively like a Taylor rule for i_t^D.

in steady state:

$$V = \omega^{-\eta} \left(\frac{\delta + \pi - i^D}{1 + \delta + \pi} \right)^{\eta}; \quad Q = \left(1 + \omega^{\eta} \left(\frac{\delta + \pi - i^D}{1 + \delta + \pi} \right)^{1-\eta} \right)^{\frac{1}{1-\eta}}. \tag{10}$$

Long run output is also constant at

$$Y = \left(\frac{\epsilon - 1}{\epsilon} \frac{1}{\psi} Q^{-(1-\frac{\eta}{\sigma})} \right)^{\frac{1}{\varphi + \frac{1}{\sigma}}}. \tag{11}$$

In the separable case, $\eta = \sigma$, firms' marginal cost does not depend on the cost of liquidity, but only on the disutility of labor and the markup. When money and consumption are complements, $\eta < \sigma$, in contrast, cheaper liquidity lowers firms' marginal cost and hence increases output. If it were costless to produce real balances, then it would be optimal to drive the cost of liquidity to zero. In this paper, we are interested in the response of the economy with standard preferences and interest rate policies. We thus maintain preferences that preclude the possibility of satiation with money.

What determines steady state inflation? There are two cases. With a nominal anchor, that is, if the government commits to a nominal path of money, then we simply have $\pi = \mu - 1$. If instead $G > 0$, we can solve for steady state real balances from (9) and express velocity as

$$V = \left(1 - \frac{\mu}{1 + \pi} \right) \frac{Y}{G}. \tag{12}$$

Inflation, output, the price index Q and velocity are then jointly determined by (10)-(12). Moreover, for any given inflation rate π, and any policy parameters $i^D < \delta + \pi$ and $\mu < 1 + \pi$, there is a G such that steady state inflation is indeed π.

What about the steady state price level? If the government commits to a path for nominal money, the quantity equation determines the price level. Without a nominal anchor, in contrast, the price level is indeterminate. Indeed, for $\mu < 1$, the policy parameters G and i^D only determine inflation and the long run level of real balances. For any price level, agents believe that the government supplies nominal money so that the long run level of real balances is met. From (10), fixing G and i^D is the same as fixing i^S and i^D. In the long run, the economy thus behaves as if the government pegs both the interest rate on money and the interest rate i^S.

Linearized model To study the dynamics of the model, we follow the standard approach of log-linearizing around a steady state with zero inflation. The inflation rate is $\Delta p_t = \log P_t / P_{t-1} = p_t - p_{t-1}$. We indicate log deviations from steady state by hats. We arrive

at a system of linear difference equations for output, the interest rate and the price level. Derivations are provided in Appendix A.2. In particular, the New Keynesian Phillips Curve and Euler equation take the standard form

$$\Delta\hat{p}_t = \beta\Delta\hat{p}_{t+1} + \lambda\left(\left(\varphi + \frac{1}{\sigma}\right)\hat{y}_t + \left(1 - \frac{\eta}{\sigma}\right)\frac{\chi}{\delta - r^D}\left(i_t^S - \delta - (i_t^D - r^D)\right)\right), \tag{13a}$$

$$\hat{y}_t = E_t\hat{y}_{t+1} - \sigma\left(i_t^S - \Delta\hat{p}_{t+1} - \delta\right) + (\sigma - \eta)\frac{\chi}{\delta - r^D}\left(\Delta i_{t+1}^S - \Delta i_{t+1}^D\right). \tag{13b}$$

In the separable case, $\eta = \sigma$, the last term in both equations is zero and we arrive at the standard three equation model. As usual, the parameter $\lambda = (1-\zeta)(1-\beta\zeta)/\zeta$ measures the response of inflation to marginal cost.

With complementarity, $\eta < \sigma$, there is a cost channel: a temporarily higher cost of liquidity $i_t^S - i_t^D$ increases firm's marginal cost and lowers output. The strength of the cost channel depends on the parameter χ, the elasticity of the price of a bundle of consumption and money (4) with respect to the cost of liquidity:

$$\chi = \left(1 + \omega^{-\eta}\left(\frac{\delta - r^D}{1 + r^D}\right)^{\eta - 1}\right)^{-1}. \tag{14}$$

The elasticity χ is positive and increasing in households' preference for liquidity as captured by the utility weight ω. In the relevant case of a strong income effect ($\eta < 1$), it is also increasing in the steady state price of liquidity chosen by the central bank: a higher price of liquidity increases the expenditure share on liquidity.

Equilibrium in the money market is summarized by the intratemporal Euler equation (3)

$$i_t^S - \delta = i_t^D - r^D + \frac{\delta - r^D}{\eta}\left(\hat{p}_t + \hat{y}_t - \hat{d}_t\right). \tag{15}$$

The general principle here is that, to first order, expected returns on all assets are equated. The return on money has a pecuniary component, the interest rate i_t^D on money, as well as a convenience yield. The coefficient in front of velocity is the inverse semielasticity of money demand with respect to the cost of liquidity. It depends both on the elasticity η and on the steady state spread $\delta - r^D$. If money demand is less elastic, then fluctuations in velocity have a stronger effect on the return on money.

We can now define equilibria for different policy regimes. If policy is specified as an exogenous path for both the nominal quantity of money and the interest rate on money, an equilibrium corresponds to a solution (y_t, i_t^S, p_t) to the system of linear difference equations

given by (13) and (15). In this system, the price level is an endogenous state variable. A special case has the interest rate on money pegged at zero – this version is discussed in the literature as the New Keynesian model with a money supply rule. We allow instead for general exogenous paths for the interest rate on money.

If interest rate policy is specified as a Taylor rule, we add the equation

$$i_t^D = r^D + \phi_\pi \Delta \hat{p}_t + \phi_y \hat{y}_t + u_t, \tag{16}$$

where u_t is a monetary policy shock. The interest rate i^D now also becomes an endogenous variable of the system comprised of (13), (15) and (16). We do not claim that this policy rule is optimal or otherwise desirable for the rate on a CBDC. We are interested in it only because it is a simple rule that has been widely studied. Our goal is to describe what happens if the central bank targets an asset with a convenience yield in this way. The bank models below will show that this is a useful way to think of postwar monetary policy.

If the money supply is specified by the feedback rule (9) with $\mu < 1$, the local dynamics of money are

$$\hat{d}_t - \hat{p}_t = \mu \left(\hat{d}_{t-1} - \hat{p}_{t-1} - \Delta \hat{p}_t \right). \tag{17}$$

With endogenous money, it is helpful to work with real balances as a state variable rather than the price level. For a fixed path for the interest rate on money, an equilibrium now consists of sequences for inflation, the shadow rate i_t^S, output and real balances $\hat{d}_t - \hat{p}_t$ that satisfy (13), (15) and (17). Given such sequences and some initial steady state level of prices (together with an associated initial money supply), we also obtain paths for the money supply and the price level. We can again accommodate a Taylor rule by making i_t^D endogenous and adding (16).

2.2 The separable case

In this section, we study the CBDC model when utility is separable in consumption and money. The behavior of the private sector is then exactly the same as in the standard three equation New Keynesian model: the New Keynesian Phillips curve and Euler equation are given by (13) with $\eta = \sigma$. Moreover, money market equilibrium (15) is the same here as in the derivation of the standard model in Woodford (2003) and Gali (2008).

The only difference between the CBDC model and the standard New Keynesian model is in the specification of policy. The standard model adds an interest rate rule for the shadow rate i_t^S and sets the interest rate i_t^D to zero. Since the central bank targets two interest rates, there cannot be an exogenous path or rule for the money supply. Instead, money is elastically supplied to achieve the desired interest rates i_t^S and i_t^D. The CBDC model, in contrast, does

not impose a policy rule for the shadow rate, and it replaces the peg of i_t^D at zero with a policy path or rule for i_t^D. Since it drops one equation, it has to add one as well: this is the path of feedback rule for money which thus becomes a policy instrument together with i_t^D. Next, we consider the implications of this change for price level determinacy as well as the transmission of policy.

Price level determinacy. When interest rate policy is specified as a path for the shadow rate i^S, the standard model is known to permit multiple equilibria, even when attention is restricted only to bounded paths for output and inflation. It is helpful to recall the intuition for this result. We focus on the case where the central bank pegs the nominal shadow rate i^S to some fixed number. One equilibrium is always that inflation and output are constant at their steady state values, so the price level remains at its initial condition. However, there are other equilibria with self-fulfilling booms and inflation.

To construct such an alternative equilibrium, suppose agents believe that output is high today and gradually falls back towards steady state. According to the New Keynesian Phillips curve, paths of high output imply paths of marginal cost above steady state, and hence inflation. However, with a nominal interest rate peg for i^S, a path with high inflation is a path of low real expected returns on savings. According to the Euler equation, agents respond to low expected returns by intertemporally substituting consumption toward the present. High demand for goods in turn calls for high equilibrium output: the initial belief in high output is thus self-fulfilling.

A Taylor rule with a high coefficient on inflation breaks the argument: in response to high inflation, the central bank aggressively raises the nominal shadow rate and hence the real return on savings. It thereby discourages consumption today – this is what rules out a self-fulfilling inflationary boom. The central bank can achieve a similar stabilizing effect if it increases the nominal rate in response to high output. Both features of policy implement the *Taylor principle*: the response of the nominal return on savings to inflation should be larger than one.

In the CBDC model, the Taylor principle can be satisfied even if the central bank pegs the policy rate. This is because the nominal return on savings is not controlled by the central bank but moves endogenously with the convenience yield. The movement is stabilizing: for example, an inflationary boom implies higher spending and hence a higher convenience yield. It thus also raises the shadow rate as returns are equated in equilibrium according to (15). It remains to assess when the convenience yield effect is strong enough to rule out multiple equilibria. The key issue is whether an increase in spending generated by an inflation boom sufficiently increases velocity.

We say that equilibrium is locally determinate if the relevant difference equation describing it has a unique bounded solution for any initial condition. We characterize local determinacy with feedback rules for the policy rate and the money supply by

Proposition 2.1: Suppose utility is separable in consumption and money ($\sigma = \eta$). If $\mu = 1$, the system of difference equations (13) and (15) -(17) has a unique bounded solution for any initial level of real balances ($\hat{d}_{-1} - \hat{p}_{-1}$). For $\mu < 1$, the system has a unique bounded solution if and only if

$$LR(i^S, \Delta p) := \frac{\delta - r^D}{\eta}\left(\frac{\mu}{1-\mu} + \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})}\right) + \phi_\pi + \phi_y \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})} > 1. \tag{18}$$

The proof is in Appendix A.1. It is not essential for the argument that policy follows feedback rules; the condition also covers the case of a peg $\phi_\pi = \phi_y = 0$ and a constant money supply $\mu = 1$. It thus extends to any bounded exogenous interest rate or money supply paths because it relies only on the eigenvalues of the homogenous part of the difference equation. The condition formally generalizes the Taylor principle to the case of an endogenous convenience yield: it ensures that the long run response $LR(i^S, \Delta\hat{p})$ of the shadow rate to inflation is larger than one. Without a convenience yield, the first term is zero and the condition reduces to that from Bullard and Mitra (2002): sufficiently strong reaction of the central bank to either inflation or output is necessary and sufficient to stabilize the economy. The term multiplying ϕ_y is the long run response of output to inflation: according to the Phillips curve, higher inflation must be driven by higher cost and hence requires higher output.

The new element here is that the return on savings (15) reflects the convenience yield. Higher inflation goes along with a higher convenience yield for two reasons. First, higher output, or higher real spending, increases velocity as captured by the second term in the first bracket. With some rigidity in the money supply, $\mu > 0$, high inflation further increases the convenience yield by decreasing real balances. As money becomes more scarce in real terms, its convenience yield rises. From (17), the long run response of real balances to inflation is $\mu/(1-\mu)$. The convenience yield effect is thus overwhelming with a nominal anchor, $\mu = 1$. In that case, the long run response of the interest rate to inflation is infinite and local determinacy is guaranteed regardless of the other parameters of the economy.

At the same time, the convenience yield effect reduces the scope for multiple equilibria even in the extreme opposite case of fixed real balances ($\mu = 0$). It now works only through changes in output – its strength thus depends on the slope of the Phillips curve. In particular, there is less scope for multiple equilibria if prices are less flexible or preferences over labor and consumption are such that marginal cost responds less to output. In either case, lower inflation implies a larger long run drop in output and hence in the convenience yield and

the return on savings. From (15), the strength of the convenience yield effect also increases with the inverse semielasticity of money demand $(\delta - r^D)/\eta$, which determines by how much lower output lowers the convenience yield.

More generally, the proposition clarifies that nominal rigidities in money supply are a stabilizing force. For $\mu \in (0,1)$, the government does not commit to a path for money going forward. However, there is always a legacy amount of nominal money in the economy. If the price level falls, then this legacy money is revalued and the convenience yield declines. As in part (a), we then have a stronger stabilizing force as the convenience yield responds to the price level. We view this case as especially relevant since it suggests that simply the use of nominal money as a medium of exchange induces a stabilizing force. In other words, what matters is only that the money supply is partly predetermined from the past; it is not essential that it will not respond to future inflation.

Monetary policy transmission. We emphasize two differences between policy transmission in the CBDC model versus the standard New Keynesian model. Consider first the role of money. In the standard model, there is a strong sense in which money doesn't matter: for a given interest rate rule, money demand shocks have no effect on inflation, output and the shadow rate. Formally, the result follows because a system consisting of (13) with $\eta = \sigma$, (15) and a Taylor rule for i^S is block recursive: we can solve for output, inflation and the shadow rate independently of the parameters and any shifters of the money market equilibrium condition. The latter only determines how much money needs to be endogenously supplied in order to achieve the target interest rate i^S.

In the CBDC model with a policy rule for the interest rate on money, money matters even if utility is separable. Indeed, the system consisting of (13) with $\eta = \sigma$, (15) and (16) is not block recursive. A shock to money demand, such as a change in the weight on money in utility, would enter as an additive shock in (15). If the central bank sticks to its interest rate rule, such a shock affects the shadow rate i_t^S and hence the allocation. At the same time, a change in the exogenous quantity of digital currency supplied by the central bank has real effects for a given interest rate rule. In the banking models studied below, this property carries over to the quantity and quality of collateral assets used by banks to back inside money.

Second, consider interest rate policy. In the CBDC model, changes in the policy rate have weaker real effects than in the standard New Keynesian model. The reason is the imperfect pass-through from the policy rate to the shadow rate, and hence to intertemporal decisions, as described by (15). Indeed, consider a positive monetary policy shock, say, that increases the nominal rate on money. With sluggish price adjustment, the real rate on money also increases, which entails lower output and lower inflation on impact, as in the standard model. However, lower spending also reduces the convenience yield on money. As returns on all assets are

equated according to (15), the effect of the policy shock on the shadow rate i^S is lower than in the standard model. In this sense, interest rate policy is weaker. We quantify the effect in Section 2.4.

2.3 Nonseparable utility and the cost channel

In the CBDC model, the pass-through (15) from the policy rate to the shadow rate depends importantly on the elasticity of money demand η. Since standard estimates of η are lower than conventional numbers for the intertemporal elasticity of substitution σ, the separable case is overly restrictive. In this section, we thus explore the nonseparable case with $\eta < \sigma$, where money and consumption are complements in utility. A key new feature is then that the cost channel terms in the Phillips curve and Euler equation become relevant: a temporarily higher cost of liquidity for households $i_t^S - i_t^D$ increases firms' marginal cost and hence inflation; at the same time, it makes consumption more expensive and hence lowers output.

The introduction of a cost channel accentuates the difference between interest rate policy in the CBDC model versus the standard model. To see this, consider again an increase in the policy rate in the CBDC model. A drop in spending and hence a lower convenience yield now feeds back to output and inflation: a lower cost of liquidity amplifies the fall in inflation but further dampens the fall in output. Interestingly, the cost channel effects here are the opposite of those in the standard model: if the central bank can increase the shadow rate holding fixed the rate on money, the cost of liquidity for households increases. In the standard model, the cost channel thus dampens the fall in inflation and amplifies the fall in output.

The presence of a cost channel in the CBDC model also introduces a new source of fragility if the central bank responds too strongly to output. Indeed, suppose that agents believe in a path of high expenditure by households on *bundles* of consumption goods and liquidity. Along such a path, cost is high for firms which translates into high inflation. With a low enough return on savings, the path is self-fulfilling. The new feature is that such a path need not exhibit high output. Instead, spending by household and firms' cost could be high because liquidity is expensive, while output is actually below steady state. We thus have self-fulfilling stagflation.

With a strong cost channel, an interest rate policy that responds positively to output can be destabilizing. To rule out multiplicity, we would like to follow the Taylor principle and increase the nominal return on savings when inflation is high. However, with a threat of stagflation, it does not help to lower the policy rate when output falls. If interest rate policy responds too strongly to output, then the above dynamics can be explosive and no bounded equilibrium exists even if there is a nominal anchor. To rule out this case, we assume in what

follows that

$$\phi_y \left(\frac{1}{\eta} - \frac{1}{\sigma}\right) \frac{\chi}{\varphi + \frac{1}{\sigma}} < \frac{\eta}{\delta - r^D}. \tag{19}$$

The condition is always satisfied if $\phi_y = 0$ or there is no cost channel. More generally, it restricts the product of ϕ_y and the long run effect of the policy rate on output.

The key to local determinacy is again the long run response of the return on savings to inflation. With a cost channel, it becomes

$$\begin{aligned} LR(i^S, \Delta\hat{p}) = \frac{\delta - r^D}{\eta} \left(\frac{\mu}{1-\mu} + \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})}\right) + \phi_\pi + \phi_y \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})} \\ + \left(\frac{1}{\eta} - \frac{1}{\sigma}\right) \frac{\chi}{\varphi + \frac{1}{\sigma}} \left(\phi_\pi - 1 - \phi_y \frac{\mu}{1-\mu}\right). \end{aligned}$$

The first line takes the same form as in the separable case (18), although now we now have $\eta < \sigma$. The second line shows that, with a cost channel, an aggressive interest rate response to inflation still helps avoid multiplicity, whereas an aggressive response to output now hurts. A peg leaves more room for multiplicity since the effect of inflation on output is weaker and hence the convenience yield effect is reduced. Finally, (19) implies that more nominal rigidity (higher μ) contributes to stability as before.

The determinacy properties with feedback rules are summarized by:

Proposition 2.2: Suppose consumption and money are complements in utility ($\eta < \sigma$) and (19) *holds. If $\mu = 1$, then the system of difference equations (13) and (15)-(17) has a unique bounded solution for any initial level of real balances ($\hat{d}_{-1} - \hat{p}_{-1}$). If $\mu < 1$, the system has a unique bounded solution for any initial condition if and only if $LR(i^S, \Delta p) > 1$.*

2.4 Numerical example

In this section, we present a numerical example to show that the differences between the standard model and the CBDC model can be quantitatively large. Throughout we focus on a version of the CBDC model with constant money supply. Moreover, we use a version of the standard model where money pays a constant interest rate r^D. This nonstandard assumption has no effect on dynamics. It permits a cleaner model comparison in the sense that the average interest rate on money and the average cost of liquidity for households are the same across the two models.

Calibration. The model period is a quarter. We select a standard parameter for the discount factor, $\beta = 0.99$, which implies a 4 percent discount rate δ per year. To calibrate the discount rate and the opportunity cost of money, we need measures of the interest rate on money as well

as the shadow rate in the households' stochastic discount factor. For the former, we choose the interest rate on Money of Zero Maturity (MZM), a broad measure of money constructed by the St. Louis Fed. For the latter, we want a short rate that is not contaminated by the convenience yield effects we study in our bank models below. We thus use the 3 month rate of the yield curve constructed by Gurkaynak, Sack and Wright (2007) using only Treasury bonds, leaving out T-bills that are predominantly held by payment intermediaries. The resulting average deposit spread is 2.4% per year, so we work with an average deposit rate $r^D = .004$.

We follow standard practice to identify the elasticity of money demand η from the time series relationship between the velocity of money and its opportunity cost. In particular, we find the semielasticity $\eta / \left(\delta - r^D\right)$ by regressing log velocity of MZM on the spread between the 3 month T-bill rate and the MZM own rate – the average rate on instruments in MZM. The coefficient on the spread is 8.1 which implies an elasticity of $\eta = .22$. This number is similar to what has been used in past studies. We identify the final preference parameter $\omega = 0.14$, the weight on money in utility from (10), to match an average velocity of $1/2$.

Other parameters take standard values from the New Keynesian literature. We set both the intertemporal elasticity of substitution σ and the Frisch elasticity φ equal to one. The probability of resetting prices is $1 - \zeta = .75$, so the response of inflation to marginal cost is $\lambda = .085$. Without a cost channel, this response only consists of the response of inflation to output, given by $\lambda \left(\varphi + 1/\sigma\right) = .17$. The strength of the cost channel is then measured by the parameter $\chi = .0118$; in other words, a one percentage point increase in the cost of liquidity has about the same effect on inflation as a 70bp increase in output.

Dampening. We now study contractionary monetary policy shocks that increase the policy rate by 25bps, or 1 percentage point per year. Figure 1 displays responses to an unanticipated one time shock when the central bank follows a Taylor rule with a coefficient of inflation $\phi_\pi = 1.5$ and no weight on output. The top three panels report percentage deviations from steady state in the price level, output and nominal money. The bottom three panels report percentage *point* deviations from steady state in inflation, the policy rate and households' cost of liquidity, that is, the spread between the shadow rate and the deposit rate. In all panels, light gray and black lines represent the standard New Keynesian model and the CBDC models, respectively.

The impact effects illustrate the dampening of interest rate policy when the policy instrument earns a convenience yield. While contractionary policy causes a recession and deflation in both models, output and inflation responses in the CBDC model are only about half the size of those in the standard model. There are two reasons, illustrated in the bottom right panel. First, pass-through is imperfect in the CBDC model: the spread between the policy rate and the shadow rate declines. This effect is quantitatively relatively small. Second, the cost of

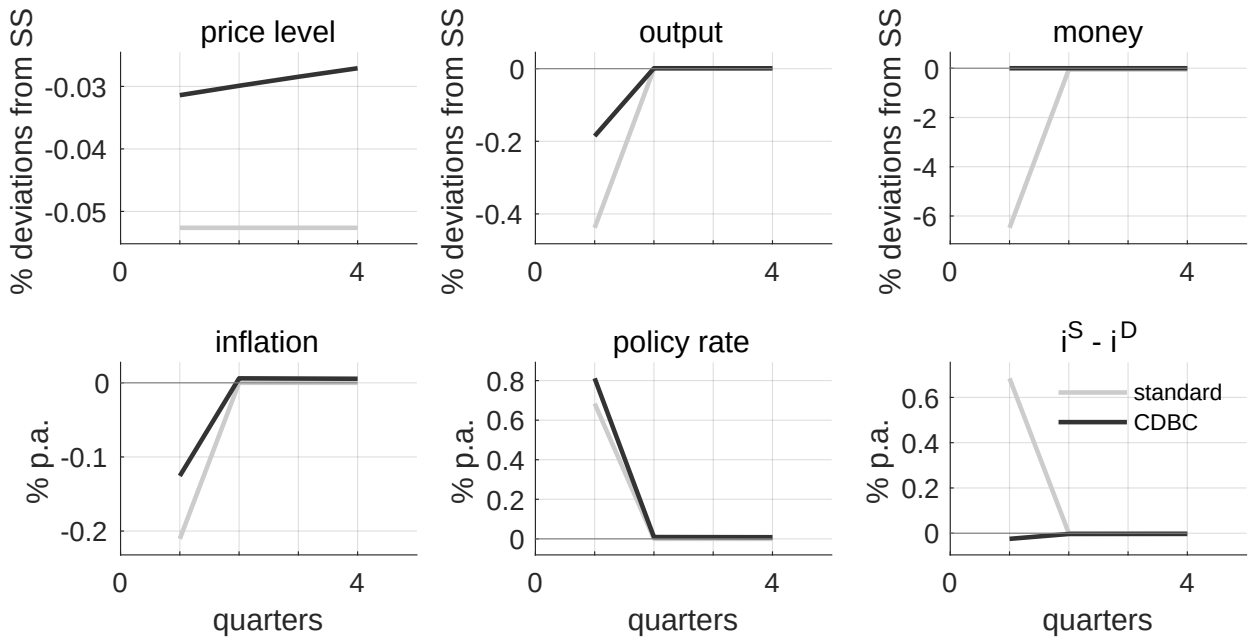


Figure 1: Impulse responses to a one time 25bp monetary policy shock; Taylor rule with coefficient 1.5 on inflation only. Top three panels: percent deviations from steady state; bottom three panels: percentage point deviations from steady state. Spreads are differences between shadow rate and policy rate.

liquidity in the standard model moves in the opposite direction from the CBDC model. This is an important force that makes output fall much more in the standard model.

The figure also shows that the CBDC model features internal propagation, whereas the standard model does not. Indeed, after the deflationary impact effect, the price level in the CBDC model gradually returns to the original steady state. Inflation thus turns positive after the initial shock. Output also turns positive, although the effect is very small. The result follows because the central bank in the CBDC model with constant money supply is effectively targeting the price level. We can see this when we substitute the Taylor rule for the rate on money (16) into the money market equilibrium equation (15). The resulting policy rule for the shadow rate i_t^S effectively engages in a version of nominal income targeting, and hence price level targeting. In contrast, the price level in the standard model jumps to a new steady state right away and inflation is back at its steady state rate of zero from the second period on.

As a complementary way to think about propagation, consider the evolution of money. Since the money supply in the CBDC model is constant, the initial price level decline increases real balances. After the initial shock, households thus find themselves in a world with too much money: the economy works *as if* there had been an unanticipated increase in the money supply. With sticky prices, output and inflation rise and gradually return to the steady state.

In the standard model, in contrast, the central bank withdraws money on impact in order to return the economy to steady state immediately in the second period.

Figure 2 considers a Taylor rule with interest rate smoothing: the current policy rate depends not only on inflation but also on the last policy rate. Most estimations in the literature find some persistence. The CBDC Taylor rule is

$$i_t^D = .5 i_{t-1}^D + 1.5\pi_t + v_t. \tag{20}$$

The results are qualitatively quite similar to those in Figure 1, but more gradual because of interest-rate smoothing. Comparing magnitudes across figures further shows that smoothing leads to stronger inflation responses in both models, making those responses more similar. However, the output response in the CBDC model is still about half its standard size. We conclude that for common policy rules, responses differ significantly across the two models.

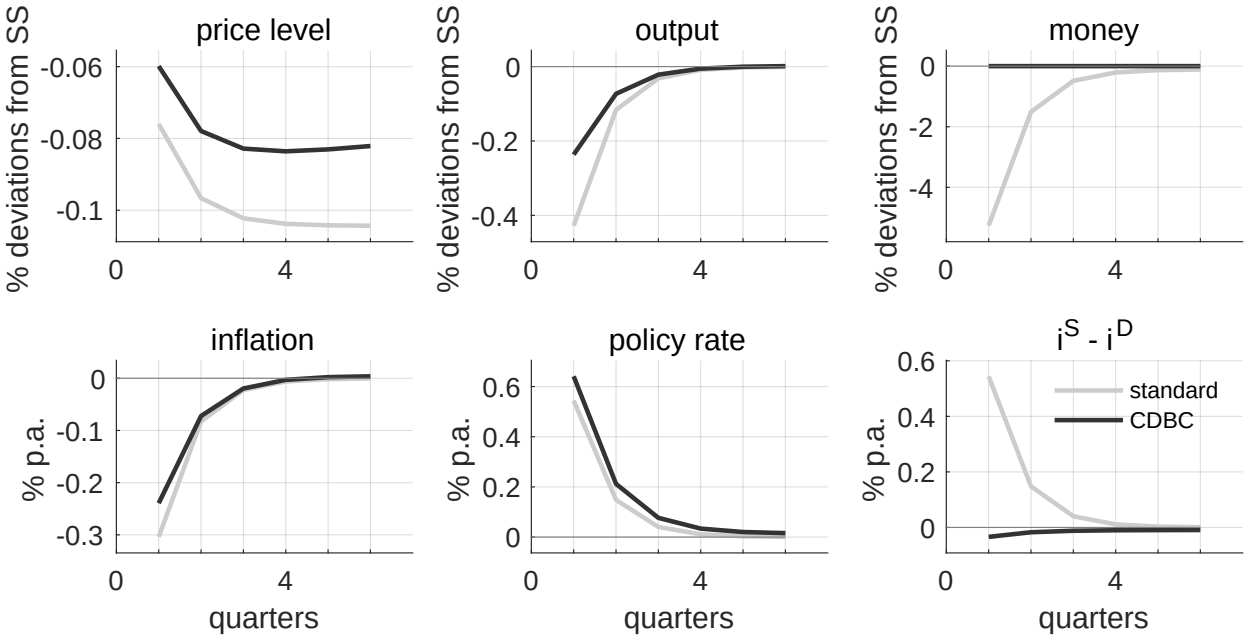


Figure 2: Impulse responses to a one time 25bp monetary policy shock; Taylor rule with coefficient 1.5 on inflation and .5 on past interest rate. Top three panels: percent deviations from steady state; bottom three panels: percentage point deviations from steady state. Spreads are differences between shadow rate and policy rate.

3 Banking with abundant reserves

In this section, we study banking when the central bank operates a floor system with abundant reserves. We consider a model with banks that issue inside money, labeled "deposits", and face a leverage constraint. To back deposits, banks can invest in high quality assets called "reserves" or in "other assets" that are of lower quality. From the perspective of the bank, the only difference between reserves and other assets is collateral quality. In particular, reserves play no special role in managing liquidity. We can therefore think of reserves broadly as short safe bonds, including Treasury bills. Other bank assets are subject to credit or interest rate risk: they include loans, longer term government bonds, as well as mortgage backed securities.

The setup of this section is designed to capture the policy environment in the United States since late 2008, when the initial round of quantitative easing made reserves abundant. As reserves lost their liquidity benefit, the spread between reserves and T-bills declined to essentially zero, and the fed funds market for borrowing and lending reserves between banks disappeared. The observed negative spread between the fed funds and reserve rate has been traced to a peculiarity of US money markets, namely that institutions such as GSEs have reserve accounts but are not legally banks who earn interest on reserves. We abstract from this feature here since we view it as minor and gradually eliminated by the Fed via the reverse repo program.

We emphasize that, in a layered payment system, abundance of reserves is not the same as interest rates reaching the zero lower bound. Indeed, while *banks'* cost of liquidity – measured by the spread between the Fed funds rate and the reserve rate – is zero with abundant reserves, the cost of liquidity paid by the private sector – measured by the difference between the shadow rate i_t^S and the deposit rate – remains positive. At the same time, reserves can be abundant both at and away from the zero lower bound. Recent tightening by the Fed has increased the reserve rate as the key policy rate, while maintaining the floor system with abundant reserves.

3.1 Setup

Banks hold reserves M_t, outside money issued by the government that earns the nominal interest rate i_t^M. They issue deposits D_t, inside money held by households that earns the interest rate i_t^D. The balance sheet of the typical bank is

Assets		Liabilities	
M	Reserves	Money	D
A	Other assets	Equity	

Other assets A_t available to banks earn the nominal interest rate i_t^A. Banks maximize shareholder value. We assume that bank equity can be adjusted every period at no cost. In this section, we further assume perfect competition among banks; market power in deposit markets is introduced below.

Bank i's nominal cash flow at date t reflects changes in deposits, reserves, deposit and other asset positions as well as interest on those positions:

$$M_{t-1}^i \left(1 + i_{t-1}^M\right) - M_t^i - D_{t-1}^i \left(1 + i_{t-1}^D\right) + D_t^i$$
$$+ A_{t-1}^i \left(1 + i_{t-1}^A\right) - A_t^i.$$

An individual bank maximizes the present value of cash flow, discounted by the shadow rate i_t^S. Since the model is deterministic, i_t^S represents the household stochastic discount factor and hence the banks' cost of capital, or the required rate of return on bank equity. It is convenient to work with nominal cash flows discounted by nominal rates to avoid extra notation.

Banks can issue deposits only if they have sufficient collateral to back them, as described by the leverage constraint

$$D_t^i \leq \ell \left(M_t^i + \rho_A A_t^i\right), \tag{21}$$

where $\ell \leq 1$ and $\rho_A < 1$. The parameter ρ_A captures the idea that reserves are better collateral than other assets. The parameter ℓ serves as a bound on leverage, defined as the ratio of debt to quality-weighted assets. One interpretation of the constraint is as a capital requirement: required equity must be higher if assets are lower quality. Even without regulation, a leverage constraint can be viewed as a simple way to model an increasing marginal cost of debt.[8]

We focus on the case of a positive deposit spread $i_t^S - i_t^D > 0$. We already know from the household Euler equation (3) that deposits provide a convenience yield whenever the supply of real balances is finite. It follows that, from the perspective of the bank, deposits represent a source of funding that is strictly cheaper than equity, which must earn the shadow rate i_t^S. Without a leverage constraint, it would be optimal to fund the bank entirely with deposits. The leverage constraint will thus bind in equilibrium. A limited quantity of collateral implies a limited quantity of deposits, which in turn justifies a positive deposit spread.[9]

[8] In a more general model, such costs might be derived from deadweight costs of bankruptcy. Collateral quality can then be derived from the riskiness of bank assets. While the resulting tradeoffs that determine leverage are similar to the ones here, adding portfolio choice under risk yields additional testable predictions on balance sheet ratios, explored for example in Lenel, Piazzesi and Schneider (2019).

[9] A hard leverage constraint simplifies the analysis, but is not essential for our results. In Piazzesi and Schneider (2018), optimal leverage follows from a smooth tradeoff between the marginal cost of leverage and the liquidity benefit of deposits. The key point both here and in that model is that the liquidity benefit works like the tax advantage of debt in the standard tradeoff theory of capital structure – combined with an increasing marginal cost of debt, it generates a determinate optimal leverage ratio.

Consider bank first-order conditions. Given the linear objective, a bank holds an asset (or issues a liability) if and only if its rate of return is equal to the cost of capital i_t^S. Here the appropriate "rate of return" incorporates both the pecuniary return – that is, the interest rate – and a nonpecuniary component introduced by the Lagrange multiplier on the leverage constraint. We focus on banks who hold both reserves and other assets and who issue deposits. The first order conditions are then

$$\begin{aligned} i_t^S &= i_t^M + \ell \gamma_t^i \left(1 + i_t^S\right), \\ i_t^S &= i_t^A + \rho_A \ell \gamma_t^i \left(1 + i_t^S\right), \\ i_t^S &= i_t^D + \gamma_t^i \left(1 + i_t^S\right). \end{aligned}$$

A binding leverage constraint induces *spreads* between bank assets and liabilities and banks' cost of capital. Mechanically, the presence of the leverage constraint implies that interest rates on reserves, loans and deposits are all below the cost of capital. For example, on the asset side, the reserve spread $i_t^S - i_t^M$ indicates that banks value reserves not only for their interest rate, but also as collateral that allows issuing more cheap deposits. At the same time, the spread on other assets $i_t^S - i_t^A$ is lower than the reserve spread since the collateral quality of loans (measured by $\rho_A < 1$) is lower than that of reserves. Similarly, on the liability side, banks pay depositors a lower rate of return than shareholders because issuing deposits incurs an additional leverage cost.

Combining bank first-order conditions clarifies the pricing of liquidity in a layered payment system

$$\begin{aligned} i_t^S - i_t^D &= \frac{M_t^i}{D_t^i}\left(i_t^S - i_t^M\right) + \frac{A_t^i}{D_t^i}\left(i_t^S - i_t^A\right), \\ &= \ell^{-1}\left(i_t^S - i_t^M\right). \end{aligned} \tag{22}$$

The cost of liquidity for households, captured by the deposit spread, reflects a weighted average of spreads on the two collateral assets used to back deposits. Since the bank can substitute freely between reserves and other assets, the deposit spread is proportional to the reserve spread. From the bank's perspective the formula describes marginal cost pricing of household liquidity: leverage makes the deposit spread higher than the reserve spread. Put differently, competition between banks for collateral assets implies that those assets inherit part of the liquidity benefit conveyed by deposits.

To close the model, we describe policy and the supply of assets. In contrast to the digital

currency model of Section 2, government policy now controls reserves, while the creation and valuation of deposits is endogenous. The government thus sets paths for the quantity of reserves M_t and the interest rate on reserves i_t^M. We further assume that the *real* supply of other assets is given by an exogenous path A_t^r, so in equilibrium $A_t = P_t A_t^r$. Concretely, we can think of firms or the government issuing a fixed amount of debt in real terms. The only other element of the model that is affected is profits in the household budget constraint, which add up firm and bank profits. Since households and firms operate in frictionless equity markets, their marginal conditions are unaffected.[10]

In equilibrium, all banks choose the same balance sheet ratios, so we can directly aggregate. An equilibrium consists of prices and quantities such that households, firms and banks optimize and asset and goods markets clear. In the system of difference equations characterizing equilibrium, the equations characterizing nonbank private sector are unchanged: we still have a New Keynesian Phillips curve, and Euler equation, and a market clearing condition for deposits. There are three new equations: the binding leverage constraint (21) and the pricing of deposits and other assets (22). These three equations help determine the three new endogenous variables D_t, i_t^D and i_t^A.

3.2 Linearized model

Equilibrium with banks is characterized by a system of linear difference equations that has the same structure as that for the digital currency model in Section 2. Indeed, the nonbank private sector equations (13) and (15) continue to hold. What is new is that the deposit spread depends on policy via the pricing equation (22). The cost channel coefficient χ defined in (14) thus depends on the policy rate and bank leverage via steady state deposit pricing. Moreover, we can substitute for the deposit spread in (15) to arrive at a new equation for pass-through from the policy rate i_t^M to the shadow rate:

$$i_t^S - \delta = i_t^M - r^M + \frac{\delta - r^M}{\eta}\left(\hat{p}_t + \hat{y}_t - \hat{d}_t\right). \tag{23}$$

The second term on the right hand side is the convenience yield on reserves: just like the convenience yield on deposits in (15), it moves with the velocity of deposits. The bank model here differs from the CBDC model in that the policy rate is no longer the rate on money itself, but instead the rate on a collateral asset used by banks to back money. Nevertheless,

[10] A richer model would make the demand for bank loans endogenous, and possibly responsive to the state of the economy. We choose work with exogenous rules to maximize transparency. Fixed debt is a baseline scenario motivated by the fact that bank assets tend to adjust slowly to shocks. We discuss other assumptions on both policy and the supply of loans below.

competition by banks for collateral assets implies that pass-through works the same way. In particular, as in the CBDC model, households' elasticity of money demand η is a key determinant of variation in the convenience yield and hence the strength of pass-through.

To complete the linear system, we add equations for the endogenous production and pricing of real deposits by banks as well as a feedback rule for reserves:

$$\hat{d}_t - \hat{p}_t = \alpha_m(\hat{m}_t - \hat{p}_t) + (1 - \alpha_m)\hat{a}^r_t, \tag{24a}$$

$$i^S_t - i^D_t = \ell^{-1}\left(i^S_t - i^M_t\right), \tag{24b}$$

$$\hat{m}_t - \hat{p}_t = \mu(\hat{m}_{t-1} - \hat{p}_{t-1} - \Delta\hat{p}_t) \tag{24c}$$

where $\alpha_m := M/(M + \rho_A PA^r)$ is the steady state quality-weighted share of reserves on banks' balance sheets. The first row is the loglinearized leverage constraint: it relates the quantity of deposits to the quantity of collateral. It implies in particular that velocity in (23) is the ratio of spending to an exogenous nominal quantity, as in (15). The second row shows that again households' cost of liquidity is proportional to the spread between the shadow rate and the policy rate. The final row applies (17) to reserves.

A key difference between the bank model and the CBDC model is that the private sector cost of liquidity $i^S_t - i^D_t$ is no longer the same as the spread between the shadow rate and the policy rate. Instead, it includes a markup determined by bank leverage, as shown in (24b). Since the policy spread is lower than the deposit spread, fluctuations in velocity have a smaller effect on the short interest rate in the bank model – in other words, pass-through from the policy rate to the shadow rate becomes stronger. Intuitively, reserves inherit the convenience yield from deposits because they serve as collateral. If banks are not very levered, the effect is weaker. A drop in spending that lowers the convenience yield on deposits then has a smaller effect on the yield on reserves.

Nevertheless, we conclude that the bank model differs from the standard New Keynesian model in the same way as the CBDC model: there is imperfect pass-through and the cost of liquidity is decreasing in the policy rate. In fact, with separable utility, the bank model is formally equivalent to a CBDC model with a higher semielasticity of money demand. With nonseparable utility, this is not true, however: the strength of the cost channel as captured by the coefficient χ continues to reflect only the average cost of liquidity for *households* $\delta - r^D$. For the cost channel, it is not relevant how money is produced and what policy rate banks face; all that matters is the private sector cost of liquidity.

The formal similarities between the CBDC and bank model clarify the equivalence of superficially distinct institutional features. In particular, in an environment with abundant reserves,

monetary tightening makes liquidity cheaper. Indeed, raising the reserve rate reduces the tax imposed by the government on the production of inside money. Its impact is thus analogous to an increase in the deposit rate in the CBDC model. It is not the same as raising the shadow rate in the household stochastic discount factor. In order to understand policy with abundant reserves, the CBDC model is thus a better reduced form analogy than the standard model.

Determinacy of equilibrium. An equilibrium is a solution to the system of difference equations consisting of (13), (23) and (24). Appendix A.1 shows that Propositions 2.1 and 2.2 also hold for this system. The general condition for determinacy is again that the long run response of the nominal shadow rate to inflation is larger than one, here that response is

$$LR(i^S, \Delta\hat{p}) = \frac{\delta - r^M}{\eta}\left(\frac{\alpha_m \mu}{1-\mu} + \frac{1-\beta}{\lambda(\varphi+\sigma^{-1})}\right) + \phi_\pi + \phi_y \frac{1-\beta}{\lambda(\varphi+\sigma^{-1})} + \left(\frac{1}{\eta} - \frac{1}{\sigma}\right)\frac{\chi}{\varphi + \frac{1}{\sigma}}\left(\phi_\pi - 1 - \phi_y \frac{\alpha_m \mu}{1-\mu}\right).$$

There are two notable differences relative to the CBDC case. First, the parameter χ defined in (14) is higher for a given policy spread, $\delta - r^M$, since it incorporates banks' markup, which strengthens the cost channel. If policy sets a peg, markups thus increase the scope for multiplicity; they further make a response to inflation more effective and a response to output less effective.

A second difference is that nominal rigidities in the money supply rule are now relevant only to the extent that banks hold reserves on the balance sheet: with low α_m, the stabilizing effect of the convenience yield is reduced. The more general point here is that any nominal rigidity can contribute to stability. For example, we have assumed so far that all bank assets are real and exogenous. If instead banks loans or government debt held by banks, say, are nominal assets that follow a rule similar to our money supply rule, then the scope for multiplicity would be reduced further. In the stark case were the entire asset supply happens to follow the same rule as the reserve supply, we would have $\alpha_m = 1$.

Bank assets and loan shocks. Consider a shock to the supply of other assets – say because bank borrowers become more constrained – works like a contraction of the money supply. It increases the convenience yield on money, and thereby also the convenience yield on reserves: as other assets become more scarce, reserves become more valuable as collateral to back broad money. From (23), pass-through increases the shadow rate even if the central bank does not change the policy rate. Negative loan shocks thus generate a recession with deflation. While we have varied only the quantity of other assets A_t here, an exogenous change in their *quality* as measured by ρ_A would work in much the same way. For example, an announcement that ratings of bank assets are worse than expected, would reduce quality-adjusted collateral

supply, thereby reducing deposit supply and so increasing the convenience yield on money.

Quantitative easing. We can also use the model to think about QE, or more generally unconventional balance sheet policies of the central bank. Consider two examples. First, a swap of high quality reserves for other nominal assets of lower quality on bank balance sheets is described by $dA = -dM$ and hence $\hat{a}_t = -(M/A)\,\hat{m}_t$ so the change in the money supply from (24) is

$$\hat{d}_t = \alpha_m \hat{m}_t - (1-\alpha_m)\,\frac{M}{A}\hat{m}_t = (1-\rho_A)\,\alpha_m \hat{m}_t.$$

The substitution of good for bad collateral thus increases the money supply and stimulates the economy, and more so if the collateral purchased by the central bank is of worse quality.

As a second example, consider a central bank purchase of assets *not* held by banks. In terms of our model, such bonds are held directly by households. The purchase of such bonds thus works mechanically like a "helicopter drop" of reserves: there is an increase in M not accompanied by a drop in other bank assets A. The central bank intervention effectively increases the collateral available to back inside money. The policy thus stimulates the economy even more than a purchase of assets held by banks. We recognize that to draw stronger conclusions here requires a more explicit model of why some assets are held within the banking system while others are not.[11] We can already see however, that even in a richer model a key determinant of the power of unconventional policy is in how it changes bank collateral assets and their convenience yield.

Bank market power

Before calibrating the model, we provide a simple extension to bank market power in deposit markets. For tractability, we assume monopolistically competitive banks that offer varieties of deposits. We thus modify preferences so households care about a CES aggregate of different varieties D_t^i, each produced by a different bank i:

$$D_t = \left(\int \left(D_t^i \right)^{1-\frac{1}{\eta_b}} \right)^{\frac{1}{1-\frac{1}{\eta_b}}},$$

where η_b measures the elasticity of substitution between varieties. One interpretation is that the household sector works like a large "family" with members in different regions, and for historical reasons banks exert local market power. The key effect we are after is that deposits

[11] Such a model might add additional institutions or intermediaries such as pension funds, insurance companies, or foreign central banks that value certain assets more than banks, and hence bid down their prices, making them unattractive as collateral to back inside money. The unconventional policy provides a way to circumvent a situation with endogenously segmented markets.

are a cheap funding source for banks not only because of their liquidity benefit to households, but also because of market power.

Consider deposit demand faced by an individual bank. Bank i supplies liquidity to households at the price $(i^S - i_t^{D,i})/(1 + i_t^S)$, where $i_t^{D,i}$ is the deposit rate promised by bank i. CES preferences imply an ideal price index that aggregates the individual liquidity prices. We define the average deposit rate i_t^D such that the spread $(i^S - i_t^{D,i})/(1 + i_t^S)$ achieves that aggregate price of liquidity. We can then write deposit demand as

$$D_t^i = \left(\frac{i_t^S - i_t^{D,i}}{i_t^S - i_t^D} \right)^{-\eta_b} D_t. \tag{25}$$

The derivation is familiar from monopolistic competition in the goods market and relegated to Appendix A.3. The only unusual feature is that prices take the form of spreads since the relevant good is liquidity.

In equilibrium, individual banks maximize profits, taking as given aggregate deposit demand. The quantity of nominal deposits still follows from banks' binding leverage constraint. However, market power increases the price of liquidity by a constant markup:

$$i_t^S - i_t^D = \frac{\eta_b}{\eta_b - 1} \ell^{-1} \left(i_t^S - i_t^M \right). \tag{26}$$

Since liquidity is more expensive with market power, households reduce demand and the equilibrium *real* quantity of deposits is lower. With given nominal collateral, this is achieved by a higher average price level. The overall scale of the banking system is thus smaller the higher is market power.

The dynamics of the model are qualitatively unchanged once market power is introduced. There are however two key changes to the system of difference equations. First, the cost channel coefficient χ now incorporates the markup via the steady state version of (26) Second, we replace (24b) by (26). With separable utility, these changes affect only the deposit rate – there is no direct effect on the dynamics of the convenience yield on reserves. More generally, when a cost channel is present ($\eta < \sigma$), then a larger markup increases the sensitivity of firms' marginal cost to households' cost of liquidity. It follows that market power accentuates the difference between interest rate policy in our bank model versus the standard New Keynesian model.

3.3 Numerical example

Our numerical example is again designed to show that deviations from the standard model can be potentially significant. We assume that the central bank runs a Taylor rule with interest rate smoothing (20) with a coefficient 1.5 on inflation and .5 on the last interest rate. We also assume that other bank assets are real and constant; this means that the only relevant new equations are (23) and (26). We thus need to pick two new parameters: the average spread $\delta - r^M$ between the policy rate and the shadow rate, and the markup factor that links the reserve and deposit spreads.

We assume that the average short term rate targeted by the central bank is the same as the historical average of the policy rate of 4.6% per year. As before, we identify the shadow rate with the average short rate from Gurkaynak, Sack and Wright (2007) which is 4.9% per year, so the average spread $\delta - r^M$ is 30 basis points. With an MZM own rate of 2.5% per year, the markup factor $\eta_b / [(\eta_b - 1)\ell]$ is about 8. In the current exercise, we cannot identify the extent to which the markup is due to market power as opposed to leverage, but it is plausible that ℓ is relatively close to one, so that a large component must be due to bank market power.

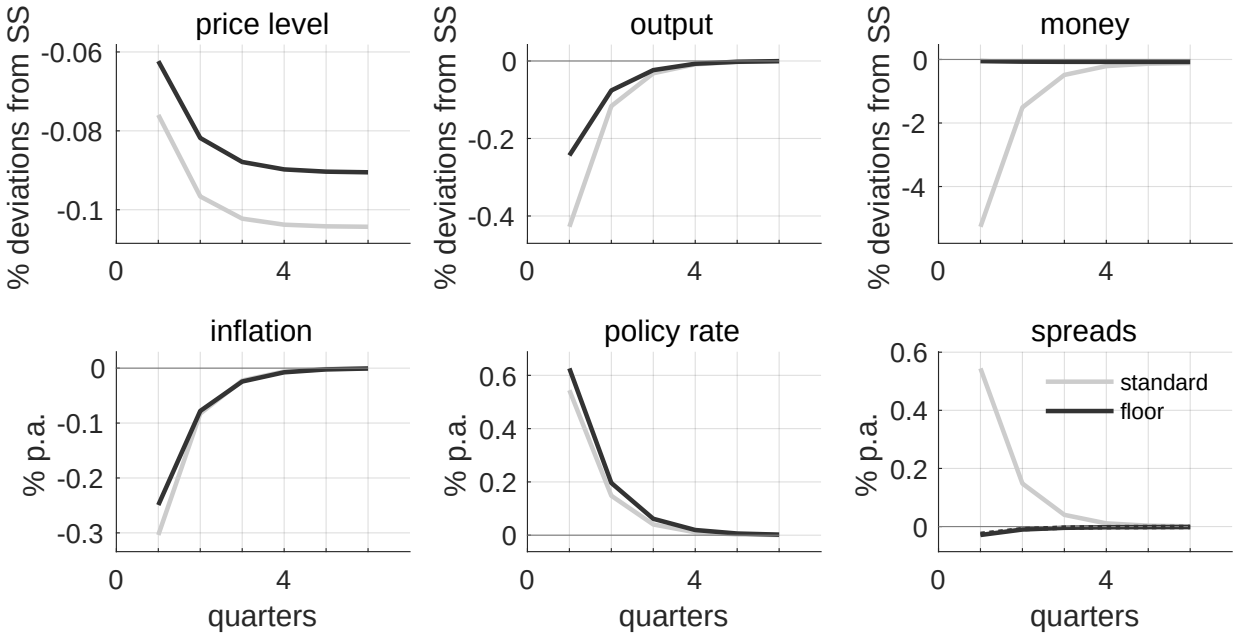


Figure 3: Impulse responses to a one time 25bp monetary policy shock; Taylor rule with coefficient 1.5 on inflation and .5 on past interest rate. Top three panels: percent deviations from steady state; bottom three panels: percentage point deviations from steady state. Spreads are differences between shadow rate and policy rate (solid lines) and difference between policy rate and deposit rate (dotted line).

Figure 3 shows responses to a one time contractionary monetary policy shock that increases the policy rate by 25bps. The panels look essentially the same as those for the CBDC model in Figure 1. This is even though the pass-through coefficient in (23) is much smaller in size. The reason is that the strength of the cost channel has not changed: it continues to be driven by households' cost of liquidity. The smaller policy spread is therefore not important for magnitudes. As long as the policy spread is positive, the convenience yield channel is active and the dampening effects explained above are relevant.

4 Banking with scarce reserves

In this section we study banking when the central bank operates a corridor system with scarce reserves. We extend the simple banking model of the previous section by adding liquidity shocks, thus creating a motive for banks to hold reserves even if they earn a lower rate than the rate on other short safe assets. We abstract entirely from reserve requirements: reserve holdings in the model correspond to excess reserves in the data. Our mechanism thus remains relevant today where reserve requirements have essentially disappeared. Indeed, when we integrate the banking system of this section with the New Keynesian model, we go to a "reserveless limit" where the quantity of reserves is negligible on bank balance sheets.

Corridor systems typically work with three rates: a lower bound at which banks can deposit funds at the central bank, an upper bound at which banks can borrow from the central bank, and a target for the interbank overnight rate within the corridor. Since the market for overnight interbank loans is an over-the-counter market, the target is typically an average of recorded trades, such as the US federal funds rate. The trading desk of the central bank can steer the overnight rate towards the target by changing the supply of reserves via open market policy that alters the scarcity of reserves. In addition to the three overnight rates, systems with scarce reserves often allow for intraday credit from the central bank, such as the Fed's overdraft facility in the US.

Our model focuses on two features of corridor systems that distinguish them from floor systems. First, liquidity is costly for banks in the sense that the interest rate on reserves is below the interest rate on overnight loans as well as other short bonds. Second, reserves are supplied elastically by the central bank in order to meet the interest rate target. We can capture both features by assuming that (i) there is a perfectly competitive overnight interbank market, and (ii) overnight interbank loans are slightly worse collateral to back inside money than reserves. Assumption (ii) generates an incentive for banks to economize on reserves, and assumption (i) allows for the central bank to elastically supply reserves to set the spread between overnight and reserve rates. While a richer model might generate more detailed

predictions of interbank interactions, our goal is to highlight two features that shape the impact of policy, and distinguish corridor and flow systems in that regard.

4.1 Setup

We start from the bank model with market power in deposit markets from Section 3. To generate a liquidity benefit for reserves, we now introduce bank level liquidity shocks, motivated by banks' provision of liquid deposits. Formally, suppose every period has two subperiods. In the first subperiod, bank i selects a portfolio of reserves M_t^i and other assets A_t^i and issues deposits D_t^i. In the second subperiod an individual bank must transfer $\tilde{\lambda}_t^i D_t^i$ funds to other banks. If $\tilde{\lambda}_t^i$ is negative, then the bank receives funds and thus increases its debt. We assume that the shocks are iid across banks with a continuous cdf $G\left(\tilde{\lambda}\right)$ that is strictly increasing on the interval $[-\bar{\lambda}, \bar{\lambda}]$. We also assume that the shocks have mean zero. With a continuum of identical banks, this means that all flows in the second subperiod remain within the banking system.

Once liquidity shocks have been realized, a fed funds market opens. Interbank loans are traded competitively at the rate i_t^F; they are repaid in the first subperiod of the subsqeuent period. Markets for deposits, other assets or equity remain closed. The bank budget constraint in the second subperiod is therefore

$$M_t^i - \tilde{\lambda}_t^i D_t^i = \tilde{M}_t^i + F_t^{i+} - F_t^{i-},$$

where $\tilde{M}_t^i$ denotes reserves held overnight (carried over to period $t+1$), while F_t^{i+} and F_t^{i-} are funds lent and borrowed in the fed funds market, respectively.

The bank leverage constraint must now hold after the second subperiod; it is given by

$$F_t^{i-} + D_t^i \left(1 - \tilde{\lambda}_t^i\right) \leq \ell \left(\tilde{M}_t^i + \rho_F F_t^{i+} + \rho_A A_t^i\right).$$

Bank debt issued on the left-hand side now consists of both interbank borrowing plus deposits. At the same time, bank collateral on the right-hand side includes not only reserves and other assets A, but also interbank lending. Since interbank loans are private, we assume that they are worse collateral than reserves: the weight ρ_F is less than one. This assumption makes it worthwhile for banks hold reserves even if the fed funds rate i_t^F is above the reserve rate.

Optimal liquidity management

A bank's problem in the second subperiod is to choose $\tilde{M}_t^i, F_t^{i+}$ and F_t^{i-} to maximize next period's cash

$$\tilde{M}_t^i \left(1 + i_t^M\right) + \left(1 + i_t^F\right) \left(F_t^{i+} - F_t^{i-}\right),$$

subject to the budget and collateral constraints as well as nonnegativity constraints on all three variables. If $i_t^F = i_t^M$, then banks are indifferent between holding reserves or lending them out, and the optimal policy is indeterminate. The interesting case is that of "scarce reserves" when the fed funds rate is strictly above the reserve rate:

Proposition 4.1. Suppose $i_t^F > i_t^M$ and a bank's optimal policy $\left(M_t^i, A_t^i, D_t^i\right)$ in the first subperiod satisfies

$$\left(1 + \bar{\lambda}\left(1 - \ell\right)\right) D_t^i \leq \ell M^i + \ell \rho_A A^i. \tag{27}$$

The bank's optimal reserve holdings in the second subperiod are then

$$\tilde{M}_t^i = \max\left\{\lambda_t^{i*} - \tilde{\lambda}_t^i, 0\right\} \frac{1 - \rho_F \ell}{\ell\left(1 - \rho_F\right)} D_t^i, \quad \text{where } \lambda_t^{i*} := \frac{D_t^i - \rho_F \ell M_t^i - \rho_A \ell A_t^i}{\left(1 - \rho_F \ell\right) D_t^i}. \tag{28}$$

When the fed funds rate is higher than the reserve rate, banks strictly prefer to lend out reserves. However, counterparty risk in the fed funds market, captured by the assumption $\rho_F < 1$, implies that lent out reserves are worse collateral than reserves held directly with the Fed. As a result, banks that receive a sufficiently large inflow of deposits – that is, they must end the day with particularly high leverage – do not have enough other collateral to lend out all reserves. Instead, they keep some reserves, which are the highest quality collateral, on their balance sheets.

Banks' optimal response to liquidity shocks in the second subperiod depends on their initial balance sheet composition. Condition (27) says that a bank that experiences the largest possible deposit inflow $\tilde{\lambda} = -\bar{\lambda}$ can satisfy its collateral constraint if it keeps all its reserves on the balance sheet. It thus represents a constraint on banks' choice problem in the first subperiod. If banks' initial portfolio does not satisfy this condition, then there are shocks in the second subperiod such that the bank cannot continue to operate.

Bank portfolios and capital structure

Consider now the bank's portfolio and capital structure choice in the first subperiod. The objective function is

$$\begin{aligned} & E[\left(1 + i_t^A\right) A_t^i - \left(1 + i_t^{D,i}\right) D_t^i + \left(1 + i_t^F\right)\left(M_t - \tilde{\lambda}_t D_t^i\right) \\ & - \left(i_t^F - i_t^M\right) \frac{1 - \rho_F \ell}{\ell\left(1 - \rho_F\right)} \int_{-\bar{\lambda}}^{\lambda_t^*} \left(\lambda_t^* - \tilde{\lambda}\right) dG\left(\tilde{\lambda}\right)] D_t^i \\ & - \left(A_t^i + M_t^i - D_t^i\right)\left(1 + i_t^S\right), \end{aligned} \tag{29}$$

where expectations are taken over liquidity shocks $\tilde{\lambda}_t^i$ and the threshold shock λ_t^{i*} is given by

(28). Banks anticipate that they will typically trade liquid funds at the rate i_t^F, either borrowing or lending in the fed funds market. For liquidity shocks below λ^*, however, they will hold reserves overnight. If either $i_t^F = i_t^M$ or there are no liquidity shocks ($\bar{\lambda}$ is zero), the problem reduces to that of Section 3.

Banks maximize (29) subject to the leverage constraint (27) and the demand function (25). They take as given aggregate deposits and rates charged by other banks, as well as the Fed funds rate that will prevail in the second subperiod – since there is no aggregate shock, they can perfectly foresee that rate. The FOC for reserves and other assets A are

$$i_t^S = i_t^F - \left(i_t^F - i_t^M\right) G\left(\lambda_t^{i*}\right) + \left\{\left(i_t^F - i_t^M\right) \frac{1}{1-\rho_F} G\left(\lambda_t^{i*}\right) + \ell \gamma_t^i\right\}, \tag{30a}$$

$$i_t^S = i_t^A + \rho_A \left\{\left(i_t^F - i_t^M\right) \frac{1}{(1-\rho_F)} G\left(\lambda_t^{i*}\right) + \ell \gamma_t^\iota\right\}, \tag{30b}$$

where γ_t^i is the Lagrange multiplier on (27).

The structure of the first order conditions is analogous to the case of abundant reserves: banks value assets not only for their pecuniary payoffs but also for their convenience yield. The pecuniary return on reserves is now stochastic. With probability $1 - G(\lambda^*)$, the bank is unconstrained so that the rate of return is the fed funds rate i_t^F. With probability $G(\lambda^*)$, the collateral constraint binds and the bank must hold reserves overnight at i_t^M. The term in braces is the marginal collateral benefit from an extra unit of reserves. The collateral benefit from other assets is a share ρ_A of that from reserves, due to the lower collateral quality of other assets.

The marginal collateral benefit of reserves has two parts. More reserves (i) imply that the collateral constraint binds less often and more reserves that flow in can be lent out, as well as (ii) may relax the worst case constraint (27). The collateral benefit from reserves is larger when Fed funds are better collateral (higher ρ_F). The spread between the shadow rate and the fed funds rate can be written as

$$i_t^S - i_t^F = \rho_F \left(i_t^S - \left[i_t^F - \left(i_t^F - i_t^M\right) G\left(\lambda_t^{i*}\right)\right]\right) + (1-\rho_F)\, \ell \gamma_t^i.$$

If the worst case constraint does not bind, banks are on the margin between fed funds and reserves, and the spread on fed funds differs from that on reserves only because of the collateral quality. We further have from (30) that $i_t^S - i_t^A = \frac{\rho_A}{\rho_F}\left(i_t^S - i_t^F\right)$ – as in Section (3) bank optimization implies that the spread on the policy rate is proportional to the rate on other bank collateral.

The bank's first order condition for deposits is

$$i_t^S - i_t^{D,i} = \frac{\eta_b}{\eta_b - 1}\left\{\frac{i_t^F - i_t^M}{\ell\left(1-\rho_F\right)}\left(\left(1-\rho_F\ell\right)\int_{-\bar{\lambda}}^{\lambda_t^{i*}}\left(\lambda_t^* - \tilde{\lambda}\right)dG\left(\tilde{\lambda}\right) + G\left(\lambda_t^{i*}\right)\left(\rho_F\ell\frac{M_t^i}{D_t^i} + \rho_A\ell\frac{L_t^i}{D_t^i}\right)\right)\right.$$
$$\left. +\gamma_t^i\left(1+\bar{\lambda}\left(1-\ell\right)\right)\right\}. \tag{31}$$

As in (26), banks price liquidity at a markup over marginal cost, which now depends on conditions for liquidity management, in particular the spread between the fed funds rate and the reserve rate as well as the distribution of liquidity shocks. If banks have stronger balance sheets and liquidity is cheaper for banks (lower $i_t^F - i_t^M$) then deposits are cheaper to produce and the deposit spread is lower.

Equilibrium

Banks are ex ante identical and face a choice problem with constant returns to scale: at given interest rates, they all choose the same ratios M_t/D_t and A_t/D_t. In equilibrium, the supply of reserves provided by the central bank to implement its interest rate target must satisfy the demand of banks with shocks below λ^*:

$$\frac{1-\rho_F\ell}{\ell\left(1-\rho_F\right)}\int_{-\bar{\lambda}}^{\lambda_t^*}\left(\lambda_t^* - \tilde{\lambda}\right)dG\left(\tilde{\lambda}\right) = \frac{M_t}{D_t}. \tag{32}$$

By Walras' law, this "reserve market clearing" condition implies that the Fed funds market also clears. It says that the money multiplier is negatively related to the threshold shock λ^* and hence the probability that banks have to hold cash overnight. Indeed, the derivative of the bracket on the left hand side is $G\left(\lambda^*\right) > 0$. Intuitively, if banks hold fewer reserves relative to deposits, then less cash is available for the sector overall to withstand liquidity shocks. As a result, the equilibrium probability of holding cash overnight must decline.

We have now described a "banking module" that integrates easily into our New Keynesian setup: bank optimization and reserve market clearing determine bank balance sheet ratios and interest rates on bank instruments (deposits and other bank assets A) for given policy rates targeted by the central bank. Mechanically, for given i_t^F and i_t^M, we can solve the five equations (27)-(32) for the ratios M_t/D_t and A_t/D_t, the interest rates i_t^A and i_t^D as well as the multiplier γ_t. With an exogenous path for the quantity of other assets as in Section 3, we then obtain an endogenous quantity and interest rate on deposits. We show next that the role of banks can again be summarized by equations for interest rate pass-through, deposit supply and the cost of liquidity, as before. An important difference to the model with abundant reserves is that reserves are supplied elastically by the government; we no longer have to specify a path or feedback rule for reserves.

Steady state

Consider liquidity management in steady state. The central bank fixes interest rates r^F and r^M and supplies reserves elastically to achieve those rates. Given these two policy rates as well as the shadow rate δ, the five equations (27)-(32) determine balance sheet ratios, rates on bank instruments and a multiplier. It is helpful to distinguish two types of equilibria. In an *elastic supply* equilibrium, banks' worst case leverage constraint (27) does not bind. In other words, banks choose initial leverage low enough that even the worst case deposit inflow does not require holding all reserves overnight to satisfy the leverage constraint. In an *inelastic supply* equilibrium, the worst case constraint does bind.

The following proposition shows that the government can choose the type of equilibrium by choosing banks' cost of liquidity $r^F - r^M$:

Proposition 4.2. There is a threshold level $r^{F*} \in \left(r^M, \delta\right)$ for the steady state federal funds rate such that there is a unique elastic supply equilibrium if $r^F > r^{F*}$ and there is a unique inelastic supply equilibrium if $r^F \leq r^{F*}$.

If liquidity is more expensive – that is, $r^F - r^M$ is high enough – banks choose lower initial leverage ratios so the worst case constraint remains slack. In contrast, cheap liquidity encourages leverage. An inelastic supply equilibrium works very much like an equilibrium in the model with abundant reserves studied in Section 3 – in particular, balance sheet ratios and rates are determined separately.[12] In other words, once liquidity is sufficiently cheap, the model behaves as if reserves are not scarce.

The purpose of this section is to study dynamics when the average share of reserves in bank balance sheets is small and the average spread between the fed funds and reserve rates is high, as was the case in the United States before 2007. We show now that this environment can be described by an elastic supply equilibrium with "small" liquidity shocks. In particular, for any target balance sheet ratios and interest rates, we can find a liquidity shock distribution such that those calibration targets are met by an elastic supply equilibrium. Moreover, in the relevant case where bank reserve shares are negligible, we can make liquidity shocks arbitrarily small.

Proposition 4.3. (a) For any interest rates r^F and r^M and weight ρ_F such that $r^F > (1-\rho_F)\,\delta +$

[12]Indeed, the ratios M/D and A/D follow from the binding worst case constraint together with market clearing. In particular, the exogenous quantity of other assets directly pins down the quantity of deposits. The ratios further imply λ^* and the interest rate on other assets follows as

$$i^S - i^A = \rho_A \left(i^S - i^F + G\left(\lambda^*\right)\left(i^F - i^M\right)\right).$$

Finally, the deposit interest rate follows from (31).

$\rho_F r^M$, and for any balance sheet ratios M/D and A/D such that $(M + \rho_A A)/D > 1$, there is a leverage constraint parameter $\ell < 1$ and a distribution of liquidity shocks G such that there exists a steady state equilibrium with elastic supply.

(b) As the ratio of reserves to deposits goes to zero, we can choose the liquidity shock distribution such that the support bound $\bar{\lambda}$ also goes to zero.

4.2 The linearized model with elastic money supply

The key new effect in an elastic supply equilibrium is that interest rate policy affects bank leverage and the money multiplier – the supply of real balances to households is interest elastic. Indeed, suppose the central bank tightens by increasing the fed funds rate. As banks face a higher liquidity cost, they reduce deposits so as to become constrained less often – the threshold shock λ_t^* declines. The reduction in deposits allows banks to economize on reserves, which carry a high opportunity cost. The central bank thus reduces the supply of reserves in order to implement the higher fed funds rate. In fact, a decline in the thresholds λ^* lowers the ratio of reserves to deposits and increases the ratio of other assets to deposits – banks become less liquid and better collateralized.[13]

To clarify the response of deposit supply to interest rates, we linearize the first order condition for reserves (30a), substitute for λ^* from its definition and for the endogenous reserve deposit ratio from market clearing. The dynamics of the ratio of deposit to other assets is given by

$$\hat{d}_t - \hat{a}_t = \varepsilon \left(\frac{i_t^S - i_t^F}{\delta - r^F} - \frac{i_t^F - i_t^M}{r^F - r^M} \right); \qquad \varepsilon := \frac{(1 - \rho_F \ell) D}{\rho_A A} \frac{(1 - \rho_F + \rho_F G(\lambda^*))}{1 - \rho_F} \frac{G(\lambda^*)}{g(\lambda^*)}, \tag{33}$$

where the parameter ε can be interpreted as an interest elasticity of deposit supply. Banks respond both to the cost of collateral and to the cost of liquidity: a higher spread between the shadow rate and the fed funds rate means that collateral is more costly, which leads banks to increase leverage. At the same time, a higher spread between the fed funds and reserve rates means that liquidity management is more costly, which lowers leverage. Both forces imply that a higher fed funds rate – other things equal – lowers the supply of deposits.[14]

In addition to the quantity of deposits, the banking module determines the interest rate on deposits. To first order, bank optimization and reserve market clearing imply a pricing

[13] Formally, the optimal threshold λ^* is determined from (30) with $\gamma_t = 0$. For the market to clear, (32) requires that the ratio M/D declines. From the definition of λ^*, A/D must increase in order for λ^* and M/D to both decline.

[14] The elasticity ε depends on steady state balance sheet ratios and hence ultimately on steady state policy rates. Indeed, at a higher average fed funds rate r^F, both λ^* and D/A are lower and the money supply is less elastic.

equation analogous to (22):

$$\frac{\eta_b - 1}{\eta_b}\left(i_t^S - i_t^D\right) = \left(i_t^S - i_t^M\right)\frac{M}{D} + \left(i_t^S - i_t^A\right)\frac{A}{D}$$
$$= \frac{\rho_F M + \rho_A A}{\rho_F D}\left\{i_t^S - i_t^F + \tilde{\alpha}_m\left(i_t^F - i_t^M\right)\right\}; \quad \tilde{\alpha}_m = \frac{\rho_F M}{\rho_F M + \rho_A A}. \tag{34}$$

As in the case of abundant reserves, the deposit spread reflects the weighted spreads on collateral used to back deposits. Moreover, it can again be written as a simple markup over a "policy spread". The difference is that liquidity management changes the relevant concept of leverage as well as the relevant policy rate, which is now a weighted average between the fed funds and reserve rates.[15]

We can now combine (34) and (15) to derive an interest rate pass-through equation for the model with scarce reserves:

$$i_t^S - \delta = i_t^F - r^F - \tilde{\alpha}_m\left(i_t^F - r^F - \left(i_t^M - r^M\right)\right) + \frac{\delta - r^F + \tilde{\alpha}_m\left(r^F - r^M\right)}{\eta}\hat{v}_t. \tag{35}$$

The structure of the equation is the same as in (15): the shadow rate in households' stochastic discount factor equals a policy rate plus a convenience yield on the policy instrument "inherited" from the liquidity benefit of deposits.

The reserveless limit

In the typical policy environment with scarce reserves, the share of excess reserves on bank balance sheets is negligible. For example, excess reserves at US banks before 2007 averaged less than one basis point of total bank assets. In what follows we simplify formulas by setting $\tilde{\alpha}_m = 0$. This approximation considerably simplifies the notation and is accurate for the relevant episode we want to study. We have shown in Proposition 3.3 that for any small target ratio M/D, there is an elastic supply equilibrium that gives rise to that target ratio.

The dynamics of the model in the "reserveless limit" are given by three equations for pass-

[15]The weight $\tilde{\alpha}_m$ depends on the endogenous supply of reserves and hence on steady state policy rates. In particular, a higher fed funds rate implies a lower ratio M/A and a lower $\tilde{\alpha}_m$.

through, deposits and the cost of liquidity that are analogous to (23)-(24).

$$i_t^S - \delta = i_t^F - r^F + \frac{\delta - r^F}{\eta}\left(\hat{p}_t + \hat{y}_t - \hat{d}_t\right), \tag{36a}$$

$$\hat{d}_t - \hat{p}_t = \frac{\eta}{\eta+\varepsilon}\hat{a^r}_t + \frac{\varepsilon}{\eta+\varepsilon}\left((\hat{p}_t + \hat{y}_t) - \frac{\eta}{r^F - r^M}\left(\left(i_t^F - i_t^M\right) - (r^F - r^M)\right)\right), \tag{36b}$$

$$i_t^S - i_t^D = \frac{\eta_b}{\eta_b - 1}\frac{\rho_A A}{\rho_F D}\left(i_t^S - i_t^F\right). \tag{36c}$$

The structure of the pass-through and liquidity cost equations is exactly the same as with abundant reserves: the shadow rate equals the policy rate plus a convenience yield proportional to velocity, and the deposit spread – households' cost of liquidity – is proportional to the policy spread.

The equilibrium quantity of deposits now reflects the response of deposit supply to interest rates. For very small $\varepsilon = 0$), real balances are effectively pinned down by real collateral. With large ε however, the role of collateral is weaker, and deposits increase with spending (the first term in the bracket) and decrease with banks' cost of liquidity (the second term). Since a decline in spending is associated with a drop in nominal deposit supply, it entails a smaller increase in the convenience yield of the policy instrument. The direct impact of banks' cost of liquidity further implies that the convenience yield can in principle *increase* with the policy rate.

The deposit equation describes equilibrium in the deposit market. With $\varepsilon = 0$, interest elastic household demand meets inelastic bank supply: a drop in spending that lowers money demand has no effect on quantities and is met only by price adjustment, that is, a lower convenience yield on deposits and hence also on the policy instrument. With positive ε, (33) shows that a lower convenience yield on the policy instrument reduces bank leverage and deposit supply. As ε becomes very large, lower spending is eventually met by a one-for-one reduction in deposits.

Substituting for velocity in the pass-through equation, we can view pass-through alternatively as

$$i_t^S - \delta = i_t^F - r^F + \frac{\delta - r^F}{\eta + \varepsilon}\left(\hat{y}_t - \hat{a}_t^r\right) + \frac{\varepsilon}{\eta+\varepsilon}\frac{\delta - r^F}{r^F - r^M}\left(i_t^F - i_t^M - (r^F - r^M)\right).$$

This equation shows that, for given demand elasticity η, the elasticity of deposit supply ε locates the model somewhere on a spectrum between the model with abundant reserves of Section 3 and the standard New Keynesian model. Indeed, if ε is close to zero, then the model reduces to the model with abundant reserves. In contrast, as ε becomes large, we have that the

government directly controls the short rate in the household stochastic discount factor.

Determinacy of equilibrium and interest rate policy

Equilibrium with a Taylor rule is a solution $\left(\hat{p}_t, \hat{y}_t, i_t^D, i_t^S, i_t^F\right)$ to the system of difference equations consisting of (13) and (36) as well as the Taylor rule for i_t^F. The only modification to (13) is that the cost channel coefficient χ defined in (14) now depends on exogenous parameters through steady state deposit pricing, as in (36c). The structure of the system is the same as that of the bank model in the previous section. Appendix A.1 shows that Propositions 2.1 and 2.2. for the case $\mu = 0$ carry through to the model of this section: we have determinacy if and only if the long run response of the shadow rate to inflation is larger than one. The response here is

$$LR(i^S, \Delta\hat{p}) = \frac{\delta - r^F}{\eta + \varepsilon} \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})} + \left(\phi_\pi + \phi_y \frac{1-\beta}{\lambda(\varphi + \sigma^{-1})}\right)\left(1 + \frac{\delta - r^F}{\eta + \varepsilon} \frac{\epsilon}{\delta - r^M}\right) + \left(\frac{1}{\eta} - \frac{1}{\sigma}\right) \frac{\chi}{\varphi + \sigma^{-1}} \frac{\eta}{\eta + \varepsilon} \left(\phi_\pi - 1 - \phi_y \frac{\varepsilon}{\delta - r^M}\right).$$

Since we have assumed that bank assets are real, there is no stabilizing effect from nominal rigidities in bank balance sheets.[16] Nevertheless, the convenience yield effect is active and alters the condition for determinacy. Consider the case of separable utility shown in the first line. There are two forces. First, inflation increases real spending and hence the convenience yield and the shadow rate. Second, if inflation pushes the central bank to increase the Fed funds rate, this reduces deposits and further increases the convenience yield and the shadow rate. In a corridor system, the convenience yield effect thus makes central bank policy more effective at preventing instability.

4.3 Numerical example

We provide a numerical example to show that a model with a corridor system is quantitatively closer to the standard New Keynesian model than the model with a floor system of the previous section. We again assume that the central bank runs a Taylor rule with interest rate smoothing (20) with a coefficient 1.5 on inflation and .5 on the last interest rate. We also assume that other bank nominal assets A_t are constant. This no longer implies constant nominal deposits, since reserves are endogenous and deposit supply is elastic. The new equations (36) contain three new parameters: the average spread $\delta - r^F$ between the interbank rate and the shadow rate, the markup factor that links the interbank and deposit spreads and the elasticity

[16] Appendix A.1 actually also covers the case in which all assets are nominal and evolve according to a feedback rule analogous to ??. It serves to show that nominal rigidity in the stock of non-reserve government debt or even private nominal debt can help ensure stability.

of deposit supply ε.

We set the policy spread $\delta - r^F$ to 30 basis points per year, so the average difference between the shadow rate and the policy rate is the same as in the previous section. The idea is that the central bank is always interested in achieving the same average level of the policy rate; it just uses different operating procedures, setting r^F with scarce reserves and r^M with abundant reserves. The reserve rate in this section is set to zero. Banks' cost of liquidity $r^F - r^M$ is thus equal to the average policy rate of 4.6% per year. We maintain a deposit rate of 2.5% per year. The calibration is consistent with the fact that banks' cost of liquidity was typically above households' cost of liquidity of $\delta - r^D$ of 2.4% per year in the regime with scarce reserves before 2007.

We choose the markup to capture the same ratio of deposit spread to policy spread as in the previous section. Again we do not need to take a stand on whether it is due to market power, leverage, or here the collateral quality of federal funds – all that matters for dynamics is the composite coefficient in (36). Finally, the elasticity ε cannot be identified from steady state moments alone. We choose the value $\varepsilon = .24$ based on the properties of the impulse response: we require that a one percent increase in the policy rate goes along with a 50bp increase in the deposit rate. This order of magnitude is consistent with the numbers reported by Drechsler, Savov and Schnabel (2017).

Figure 4 shows responses to a one time contractionary monetary policy shock that increases the interbank rate by 25bps. Qualitatively, the shape of responses for output and inflation are now hard to distinguish from those of the standard model. Moreover, the money response is also similar as banks reduce deposits. The calibrated interest elasticity is thus high enough so as to make bank liquidity cost important. At the same time, there is still some dampening in the impulse response for output – the cost channel remains strong. The bottom left panel reports the spreads $i^S - i^D$ as a solid line as well as $i^F - i^D$ as a dashed line. Due to the small shadow spread, the two are almost identical. Calibrating to larger increases in the deposit spread – that is, more inert behavior of the deposit rate – would increase ε and drive the corridor model closer to the standard model.

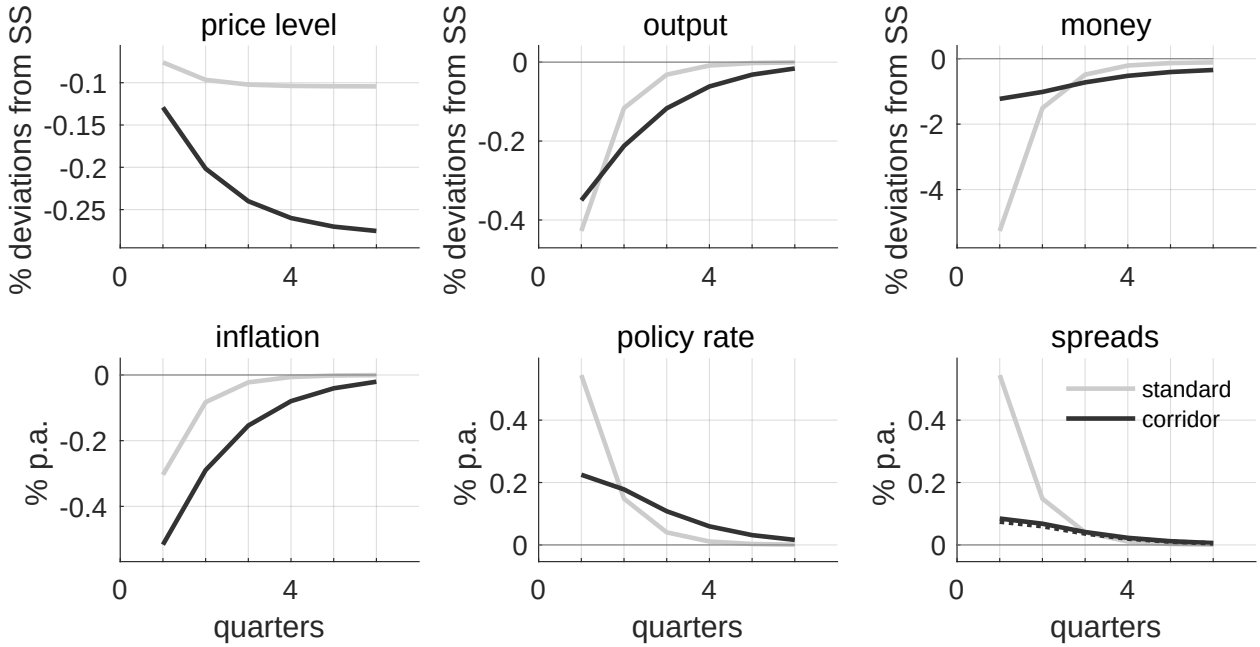


Figure 4: Impulse responses to a one time 25bp monetary policy shock; Taylor rule with coefficient 1.5 on inflation and .5 on past interest rate. Top three panels: percent deviations from steady state; bottom three panels: percentage point deviations from steady state. Spreads are differences between shadow rate and policy rate (solid lines) and difference between policy rate and deposit rate (dotted line).

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A Appendix

A.1 Determinacy properties

In this section we study a general system of difference equations that nests all versions of our model. After introducing notation to write the system in matrix form, we state Proposition A.1 that nests Propositions 2.1 and 2.2 in the text, and also shows that Proposition 2.2 continues to hold in the bank models of Section 3 and 4.

To set up the general system, we denote by $\hat{v}_t$ the log deviation of velocity from the steady state. We also write i_t^P for a generic policy interest rate, i_t^S for the shadow rate and $\hat{n}_t$ for exogenous nominal assets. We then consider the following system in $(\hat{y}_t, \hat{v}_t, i_t^P, i_t^S, \hat{n}_t - \hat{p}_t)$:

$$\Delta\hat{p}_t = \beta E_t \Delta\hat{p}_{t+1} + \lambda\left(\left(\varphi + \frac{1}{\sigma}\right)\hat{y}_t + \left(\frac{1}{\eta} - \frac{1}{\sigma}\right)\chi\hat{v}_t\right) \tag{A.1}$$

$$\hat{y}_t = E_t\hat{y}_{t+1} - \sigma(i_t^S - E_t\Delta\hat{p}_{t+1} - \delta) + \sigma\left(\frac{1}{\eta} - \frac{1}{\sigma}\right)\chi E_t\Delta\hat{v}_{t+1} \tag{A.2}$$

$$i_t^S - \delta = i_t^P - r^P + \frac{\delta - r^P}{\eta}\hat{v}_t \tag{A.3}$$

$$i_t^P = r^P + \phi_y\hat{y}_t + \phi_\pi\Delta\hat{p}_t + u_t \tag{A.4}$$

$$\hat{n}_t - \hat{p}_t = \mu\left(\hat{n}_{t-1} - \hat{p}_{t-1}\right) - \mu\alpha\Delta\hat{p}_t \tag{A.5}$$

$$\hat{v}_t = \frac{\eta}{\eta+\epsilon}\left(\hat{p}_t + \hat{y}_t - \hat{n}_t\right) + \frac{\eta}{\eta+\epsilon}\frac{\epsilon}{\delta - r^M}\left(i_t^P - r^P\right) \tag{A.6}$$

We are interested in bounded solutions given some initial condition for the real value of nominal assets $\hat{n}_{-1} - \hat{p}_{-1}$.

All models in the paper are special cases of this system. They differ in some of the coefficients as well as in what interest rate represents the policy rate and what quantity represents exogenous nominal assets (if any). The bank models further describe other endogenous variables such as the deposit rate or the interest rate on other assets, but those variables are simple functions of i_t^S, $\hat{p}_t$ and $\hat{y}_t$ which are not important for characterizing determinacy.

In particular, the system of difference equations in the CBDC model, given by (13) and (15)-(17), is a special case of the (A.1) - (A.6), where the policy rate is the deposit rate $i_t^P = i_t^D$, nominal assets are deposits $\hat{n}_t = \hat{d}_t$, and we have $\alpha = 1$ and $\epsilon = 0$. The system of difference equations for the model with a floor system from Section 3, given by (13), (23) and (24), is a special case with the policy rate is the reserve rate $i_t^P = i_t^M$, nominal assets are reserves

$\hat{n}_t = \hat{m}_t$, and we have $\alpha = \alpha_m$ and $\epsilon = 0$. Finally, the system of difference equations for the model with a corridor system from Section 4, given by (13), (16) and (36) is the special case where the policy rate is the interbank rate $i_t^P = i_t^F$, and the only nominal assets are loans $\hat{n}_t = \hat{l}_t$ and $\alpha = 1$.

Substituting out for velocity $\hat{v}_t$ and the two interest rates, we have a three equation system for inflation, the real value of nominal assets, and output. In matrix notation, it is

$$\begin{bmatrix} E_t \Delta p_{t+1} \\ E_t \hat{y}_{t+1} \\ \hat{n}_t - \hat{p}_t \end{bmatrix} = A \begin{bmatrix} \Delta p_t \\ \hat{y}_t \\ \hat{n}_{t-1} - \hat{p}_{t-1} \end{bmatrix} + b_t \tag{A.7}$$

with initial condition $\hat{n}_{-1} - \hat{p}_{-1}$ and where b_t is a vector of exogenous variables.

To ease notation, we define the non-negative coefficients

$$\begin{aligned} B &= \left(\frac{1}{\eta} - \frac{1}{\sigma}\right)\chi, \quad \gamma = \delta - r^P, \quad \kappa = \lambda\left(\varphi + \frac{1}{\sigma}\right), \\ A_V &= \frac{\eta}{\eta + \epsilon}, \quad B_V = A_V \frac{\epsilon}{\delta - r^M}, \quad \Gamma = \beta\left(\sigma^{-1} + BA_V + BB_V\phi_y\right). \end{aligned}$$

We write A_{ij} for the element in the ith row and jth column of A. We thus have $A_{31} = -\mu\alpha$, $A_{32} = 0$, $A_{33} = \mu$, and the elements in the first two rows are:

$$\begin{aligned} A_{11} &= \left(\frac{1 - \mu\alpha A_V \lambda B - \lambda B B_V \phi_\pi}{\beta}\right), A_{12} = -\left(\frac{\kappa + B\lambda A_V + B\lambda B_V \phi_y}{\beta}\right), A_{13} = \frac{\mu \tilde{B} A_V}{\beta}, \\ A_{21} &= \frac{\beta}{\Gamma}\left[\phi_\pi + BB_V\phi_\pi + \frac{\gamma\mu\alpha}{\eta}A_V + \frac{\gamma}{\eta}B_V\phi_\pi + \mu\alpha(1-\mu)BA_V\right], \\ &\quad - \frac{\beta}{\Gamma}\left[(1 + \mu\alpha BA_V + BB_V\phi_\pi)(\frac{1 - \alpha\mu A_V B\lambda - B\lambda B_V\phi_\pi}{\beta})\right], \\ A_{22} &= 1 + \frac{\beta}{\Gamma}\left[\phi_y + \frac{\gamma}{\eta}A_V + \frac{\gamma}{\eta}B_V\phi_y + (1 + \mu\alpha BA_V + BB_V\phi_\pi)(\frac{\kappa + A_V B\lambda + B\lambda B_V\phi_y}{\beta})\right], \\ A_{23} &= -\frac{\beta}{\Gamma}\left[BA_V\mu(1-\mu) + \frac{\mu\gamma}{\eta}A_V + (1 + \mu\alpha BA_V + BB_V\phi_\pi)(\frac{\mu B\lambda A_V}{\beta})\right]. \end{aligned}$$

To state the proposition, we define the long run responses to a change in inflation. From the law of motion for the real value of nominal assets, we have

$$LR(\hat{n} - \hat{p}, \Delta\hat{p}) = -\frac{\mu}{1-\mu}\Delta\hat{p}$$

From the Phillips curve, the response of output to inflation is

$$LR(\hat{y}, \Delta\hat{p}) = \left(\frac{(1-\beta) - \tilde{B}B_V\phi_\pi - \tilde{B}A_V\left(\frac{\alpha\mu}{1-\mu}\right)}{\kappa + \tilde{B}A_V + \tilde{B}B_V\phi_y} \right) \Delta\hat{p}$$

Finally, using the Taylor rule, (A.4), and the pass-through equation, (A.3), we have that:

$$\begin{aligned} LR\left(i^S, \Delta\hat{p}\right) &= LR(i^P, \Delta\hat{p}) + LR(i^S - i^P, \Delta\hat{p}) \\ &= \left(\phi_\pi + A_V\frac{\gamma}{\eta}\frac{\alpha\mu}{1-\mu} + \frac{\gamma}{\eta}B_V\phi_\pi\right)\Delta\hat{p} + \left(\phi_y + \frac{\gamma}{\eta}(A_V + B_V\phi_y)\right) LR(\hat{y}, \Delta\hat{p}). \end{aligned}$$

We impose throughout (19), written in the notation here as

Condition 1: $\phi_y B\lambda < \kappa\gamma/\eta$.

Proposition A1: Suppose Condition 1 holds. If $\mu = 1$, *the system of difference equations (A.1) - (A.6) has a unique bounded solution for any initial condition* $(\hat{n}_{-1} - \hat{p}_{-1})$. *If* $\mu < 1$, *the system has a unique bounded solution for any initial condition if and only if*

$$\frac{LR(i^S, \Delta\hat{p})}{\Delta\hat{p}} > 1 \tag{A.8}$$

Proof. We define the condition

$$(1-\mu)\left(\frac{LR(i^S, \Delta\hat{p})}{\Delta\hat{p}} - 1\right) > 0 \tag{A.9}$$

Condition 1 is simply (A.9) evaluated at $\mu = 1$. Otherwise, for $\mu < 1$, (A.9) is equivalent to (A.8). We will show that, for $\mu < 1$, if the matrix A in (A.7) has exactly one eigenvalue inside the unit circle, then (A.9) holds. We then show that for any $\mu \in [0, 1]$, (A.9) implies that A has exactly one eigenvalue inside the unit circle. Finally, we check the rank condition on A in Blanchard and Kahn (1980). It then follows that, for any $\mu \in [0, 1]$, (A.9) guarantees a unique bounded solution to (A.7) for any μ. Since Condition 1 is the special case of (A.9) for $\mu = 1$, we have in particular that a unique bounded solution obtains for $\mu = 1$.

The characteristic polynomial of A

The eigenvalues of A are the roots of its characteristic polynomial

$$p(\lambda) = \lambda^3 - a_2\lambda^2 + a_1\lambda - a_0$$

where the coefficients take the form

$$\begin{aligned}
a_2 &= \left(1 + \frac{1}{\beta} + \mu\right) + \left(\frac{1}{\Gamma}\right)\left[\beta\frac{\gamma}{\eta}A_V + \kappa + B\lambda A_V(1 + \mu\alpha\phi)\right] \\
&\quad + \left(\frac{\phi_y}{\Gamma}\right)\left[\beta(1 + \frac{\gamma}{\eta}B_V) + B\lambda B_V\right] + \left(\frac{\phi_\pi}{\Gamma}\right)B\lambda B_V\phi > 2 \qquad \text{(A.10)} \\
a_1 &= \frac{1 + \mu + \mu\beta}{\beta} + \left(\frac{1}{\Gamma}\right)\left[(1 + \mu\beta)\frac{\gamma}{\eta}A_V + \mu\kappa(1 + \frac{\alpha\gamma}{\eta}A_V) + \mu A_V B\lambda(1 + \alpha\phi)\right] \\
&\quad + \left(\frac{\phi_y}{\Gamma}\right)\left[(1 + \mu\beta)(1 + \frac{\gamma}{\eta}B_V) - \alpha\mu A_V B\lambda + \mu B\lambda B_V\right] \\
&\quad + \left(\frac{\phi_\pi}{\Gamma}\right)\left[\kappa + \kappa\frac{\gamma}{\eta}B_V + B\lambda B_V(1 + \mu)\phi + B\lambda A_V\right] \qquad \text{(A.11)} \\
a_0 &= \frac{\mu}{\beta} + \frac{\mu}{\Gamma}\left[\frac{\gamma}{\eta}A_V\right] + \frac{\mu\phi_y}{\Gamma}\left[1 + \frac{\gamma}{\eta}B_V\right] + \frac{\mu\phi_\pi}{\Gamma}\left[\kappa(1 + \frac{\gamma}{\eta}B_V) + B\lambda A_V + B\lambda B_V\phi\right] \qquad \text{(A.12)}
\end{aligned}$$

We note that $a_2 > 2$ and $a_0 \geq 0$, with strict inequality if and only if $\mu > 0$. Moreover, Condition 1 implies that $a_1 > 1$. The characteristic polynomial thus has a root at zero if and only if $\mu = 0$. For $\mu > 0$, Descartes' rule of signs implies that the polynomial has either one or three positive real roots and no negative real roots. We thus always have one positive real root. In addition, there could be two more positive real roots, or there could be a pair of complex conjugates.

We have $p(0) = -a_0$ and the values of the polynomial at plus and minus one are:

$$\begin{aligned}
p(-1) &= -\frac{2}{\beta}(1 + \mu)(1 + \beta) - \left(\frac{1}{\Gamma}\right)\left[\frac{\gamma}{\eta}A_V((1 + \mu)(1 + \beta) + \mu\kappa) + \kappa(1 + \mu) + B\lambda A_V(1 + \mu + 2\alpha\mu\phi)\right] \\
&\quad - \left(\frac{\phi_y}{\Gamma}\right)\left[(1 + \frac{\gamma}{\eta}B_V)(1 + \mu)(1 + \beta) + B\lambda B_V(1 + \mu)\right] \\
&\quad - \left(\frac{\phi_\pi}{\Gamma}\right)\left[(1 + \mu)(\kappa + B\lambda A_V) + \kappa B_V\frac{\gamma}{\eta}(1 + \mu) + B\lambda B_V\phi(2 + 2\mu)\right] - \left(\frac{1}{\Gamma}\right)\alpha\mu A_V\left(\frac{\kappa\gamma}{\eta} - \phi_y B\lambda\right)
\end{aligned}$$

$$\begin{aligned}
p(1) &= \left(\frac{1}{\Gamma}\right)\left[(1 - \mu)(\kappa + B\lambda A_V + B_V\kappa\frac{\gamma}{\eta})(\phi_\pi - 1) + (1 - \mu)(1 - \beta)(\frac{\gamma}{\eta}A_V + \phi_y(\frac{\gamma}{\eta}B_V + 1))\right] \\
&\quad + \left(\frac{1}{\Gamma}\right)\left[(\alpha\mu A_V + (1 - \mu)B_V)(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda)\right]
\end{aligned}$$

Condition (A.9) is equivalent to $p(1) > 0$. It further implies that $p(-1) < 0$ for all μ.

Necessity of A.9 for $\mu < 1$.

We now establish necessity of (A.9). If the difference equation has a unique bounded solution, exactly one eigenvalue of A is inside the unit circle. In other words, exactly one root of $p(\lambda)$ is inside the unit circle. We know that $p(-1) < 0$. If $p(1) \leq 0$, then the polynomial crosses the horizontal axis within $(-1, 1)$ either twice or never. Thus, it is necessary that $p(1) > 0$ i.e. (A.9) holds.

To establish sufficiency, we first show that (A.9) ensures a unique stable eigenvalue, that is, the condition $p(1) > 0$ ensures exactly one root of $p(\lambda)$ lies inside the unit circle. It is convenient to do this part of the proof in two steps, first for $\mu = 0$ and then for $\mu \in (0, 1]$.

(A.9) *implies a unique stable eigenvalue for $\mu = 0$.*

The case $\mu = 0$ is special because $a_0 = 0$. The three roots $(\lambda_1, \lambda_2, \lambda_3)$ are

$$\lambda_1 = 0; \lambda_2 = \frac{a_2 - \sqrt{a_2^2 - 4a_1}}{2}; \lambda_3 = \frac{a_2 + \sqrt{a_2^2 - 4a_1}}{2}$$

The roots λ_2, λ_3 are either both real or both complex. If they are both complex or they are real and equal, then both must lie outside the unit circle since $a_2 > 2$. Assume instead they are both real and distinct, with $\lambda_2 < \lambda_3$. We know that $\lambda_2 > 1$ if and only if:

$$\begin{aligned} a_2^2 - 4a_1 &< a_2^2 - 4a_2 + 4 \\ \text{i.e. } 1 - a_2 + a_1 &> 0 \end{aligned}$$

If $\mu = 0$, (A.9) can be written as $p(1) = 1 - a_2 + a_1 > 0$. We therefore have $\lambda_2 > 1$ and hence also $\lambda_3 > 1$. It follows that exactly one root, $\lambda = 0$, lies within the unit circle.

(A.9) *implies a unique stable eigenvalue for $\mu > 0$.*

We show next that for any $\mu \in (0, 1]$, (A.9) also ensures that there is a unique root of $p(\lambda)$ inside the unit circle. We know that there can be either one or three roots inside the unit circle. Indeed, $p(1) > 0$, given by (A.9), and $p(0) < 0$ imply that the polynomial has either one or three real roots in the interval $(0, 1)$. Moreover, if there is a pair of complex roots, those roots have the same modulus. We thus want to rule out that there are three roots inside the unit circle. The following result provides restrictions on a cubic polynomial that allows this case:

Lemma A1: Suppose $p(\lambda) = \lambda^3 - b_2\lambda^2 + b_1\lambda - b_0$ is a cubic polynomial with strictly positive real-valued coefficients b_2, b_1, b_0 that satisfies $p(1) > 0$. If all roots lie within the unit circle, then the coefficients satisfy

(a) $b_0 < 1$,

(b) $b_0^2 - b_2 b_0 + b_1 - 1 < 0$,

(c) $b_2 < 2 + b_0$.

Proof.

Part (a): Denote the three roots by $(\lambda_1, \lambda_2, \lambda_3)$, where λ_1 is the smallest real root, and (λ_2, λ_3) are either both real roots or both complex roots. Since we can write the polynomial as $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$, we have $b_0 = \lambda_1\lambda_2\lambda_3$. Suppose (λ_2, λ_3) are real roots. We know they must both be positive. Since $p(0) < 0$ and $p(1) > 0$, we have $\lambda_2\lambda_3 \in (0, 1)$ and hence $b_0 < \lambda_1 < 1$. If instead (λ_2, λ_3) are complex roots, then $\lambda_2\lambda_3 = |\lambda_2|^2 = |\lambda_3|^2 \in (0, 1)$ and again $b_0 < \lambda_1 < 1$.

Part (b): We have

$$\begin{aligned} p(b_0) &= b_0^3 - b_2 b_0^2 + b_1 b_0 - b_0 \\ &= b_0 \left(b_0^2 - b_2 b_0 + b_1 - 1 \right) \end{aligned}$$

We show that $p(b_0) < 0$. Since $b_0 > 0$, condition (b) then follows.

To show $p(b_0) < 0$, suppose first that (λ_2, λ_3) are both real roots. then both turning points of $p(\lambda)$ must be larger than λ_1. It follows that $p(\lambda) < 0$ for any $\lambda < \lambda_1$. From the proof of part (a), we have that $b_0 < \lambda_1$ and hence $p(b_0) < 0$. If instead (λ_2, λ_3) are both complex roots, $p(\lambda)$ only crosses the horizontal axis once at $\lambda = \lambda_1$. Since $p(0) < 0$, then $p(\lambda) < 0$ for any $\lambda \in (0, \lambda_1)$. As $b_0 < \lambda_1$, again $p(b_0) < 0$.

Part (c): We start from condition (b) and use our assumption that $p(1) = 1 - b_2 + b_1 - b_0 > 0$ to obtain

$$\begin{aligned} 0 > b_0^2 - b_2 a_0 + b_1 - 1 &> b_0^2 - b_2 a_0 + (b_2 + b_0 - 1) - 1 \\ &= b_0^2 - (b_2 - 1)a_0 + (b_2 - 2) \\ &= (1 - b_0)(b_2 - 2 - b_0) \end{aligned}$$

Condition (c) follows because condition (a) ensures that $b_0 < 1$.■

We now show that Condition 1 does not allow conditions (a)-(c) of Lemma A1 to hold

jointly for our characteristic polynomial. It then follows that we cannot have three roots inside the unit circle, and thus have exactly one root inside the unit circle. We first note that there exists a threshold value $\bar{\mu} < \beta < 1$ such that condition (a) of Lemma A1 is violated for all $\mu > \bar{\mu}$. Indeed, we can always find $\bar{\mu}$ such that $a_0 = 1$. For the remaining case $\mu < \bar{\mu}$, we have the following Lemma:

Lemma A2: Assume that Condition 1 holds. Suppose the characteristic polynomial $p(\lambda) = \lambda^3 - a_2\lambda^2 + a_1\lambda - a_0$ *with coefficients* (A.10) - (A.12) *satisfies conditions (a) and (c) of Lemma A1. Then condition (b) of Lemma A1 does not hold.*

Proof. Condition (c) of Lemma A1 applied to our characteristic polynomial is given by

$$-(1/\beta - 1)(1-\mu) - \frac{1}{\Gamma}\left[(\beta-\mu)(\frac{\gamma}{\eta}A_V + \phi_y(1+\frac{\gamma}{\eta}B_V)) + B\lambda A_V\alpha\mu\phi + \phi_\pi(1-\mu)B\lambda B_V\phi\right]$$
$$+\frac{1}{\Gamma}\left[(\kappa + B\lambda A_V + \kappa\frac{\gamma}{\eta}B_V)(\mu\phi_\pi - 1) + B_V(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda)\right] > 0 \tag{A.13}$$

To check condition (b) of Lemma A1, we define the function $g(\mu) := a_0^2 - a_2a_0 + a_1 - 1$, where dependence of the coefficients on μ is given by (A.10) - (A.12). In particular, the coefficients are linear in μ, so the function $g(\mu)$ is quadratic in μ. We want to show that $g(\mu) > 0$ for all $\mu \in (0, \bar{\mu}]$, thus violating condition (b).

We know from (A.10) - (A.12) that $g(0) = a_1 - 1 > 0$. Since condition (c) of Lemma A1 is assumed to hold, we also know

$$\begin{aligned}
g(\bar{\mu}) &= a_1 - a_2 \\
&= \frac{1}{\Gamma}(1-\bar{\mu})\left[(\kappa + B\lambda A_V + \kappa\frac{\gamma}{\eta}B_V)(\phi_\pi - 1) + B_V(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda)\right] \\
&+ \frac{1}{\Gamma}\left[(1-\bar{\mu})(1-\beta)(\frac{\gamma}{\eta}A_V + \phi_y(\frac{\gamma}{\eta}B_V + 1)) + \alpha\bar{\mu}A_V(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda)\right] \\
&> \frac{1}{\Gamma}\left[(1-\bar{\mu})(1-\beta)(\frac{\gamma}{\eta}A_V + \phi_y(\frac{\gamma}{\eta}B_V + 1)) + \alpha\bar{\mu}A_V(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda)\right] \\
&> 0
\end{aligned}$$

where the third line uses (A.13) and $\bar{\mu} < \beta$, and the last line follows from Condition 1.

It remains to show that the function $g(\mu)$ is also positive in the interior of the interval $[0, \bar{\mu}]$. Since $g(\mu)$ is quadratic, it is either concave or convex everywhere. If it is concave, then $g(0) > 0$ and $g(\bar{\mu}) > 0$ imply that $g(\mu)$ is positive over the entire interval $[0, \bar{\mu}]$. Suppose therefore that $g(\mu)$ is convex. If the derivative of $g(\mu)$ at $\bar{\mu}$ is negative, then $g(\mu) > g(\bar{\mu}) > 0$ for all $\mu < \bar{\mu}$. If instead the derivative of $g(\mu)$ at $\bar{\mu}$ is positive, then $g(\mu)$ is bounded below by

the function

$$h(\mu) := g(\bar{\mu}) + (\mu - \bar{\mu})g'(\bar{\mu})$$

We proceed to show that $h(\mu) > 0$ for all $\mu \in (0, \bar{\mu})$, hence $g(\mu) > 0$ for all $\mu \in (0, \bar{\mu})$. As $g'(\bar{\mu}) > 0$, then $h'(\mu) > 0$, implying that if $h(0) > 0$, then $h(\mu) > 0$ for all $\mu \in (0, \bar{\mu})$. This is what we show.

The derivative of $g(\mu)$ at the point $\bar{\mu}$ is

$$\begin{aligned} g'(\bar{\mu}) = &-(\frac{1}{\beta} - 1)(\frac{1}{\bar{\mu}} - 1) \\ &- \left(\frac{1}{\Gamma}\right)\left(\frac{1}{\bar{\mu}} - 1\right)\left(\beta\frac{\gamma}{\eta}A_V + \kappa + B\lambda A_V + \phi_y(\beta(1 + \frac{\gamma}{\eta}B_V) + B\lambda B_V) + \phi_\pi B\lambda B_V\phi\right) \\ &- \left(\frac{1}{\Gamma}\right)\left(\alpha\phi_y A_V B\lambda - \alpha\kappa\frac{\gamma}{\eta}A_V + \alpha B\lambda A_V\phi\right) \\ &< \left(\frac{\alpha}{\Gamma}\right)A_V\left(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda\right) \end{aligned} \tag{A.14}$$

Substituting into the definition of h, we have that:

$$\begin{aligned} h(0) &= g(\bar{\mu}) - \bar{\mu}g'(\bar{\mu}) \\ &> g(\bar{\mu}) - \frac{\alpha\bar{\mu}}{\Gamma}A_V\left(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda\right) \\ &> \frac{1}{\Gamma}\left[(1-\bar{\mu})(1-\beta)(\frac{\gamma}{\eta}A_V + \phi_y(\frac{\gamma}{\eta}B_V + 1)) + \alpha\bar{\mu}A_V(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda)\right] - \frac{\alpha\bar{\mu}}{\Gamma}A_V\left(\kappa\frac{\gamma}{\eta} - \phi_y B\lambda\right) \\ &= \frac{1}{\Gamma}\left[(1-\bar{\mu})(1-\beta)(\frac{\gamma}{\eta}A_V + \phi_y(\frac{\gamma}{\eta}B_V + 1))\right] \qquad > 0 \end{aligned}$$

where the second line uses the bound from (A.14).■

Blanchard-Kahn Rank Condition

We have shown that (A.9) implies that the matrix A exhibits exactly one eigenvalue inside the unit circle. By Blanchard and Kahn (1980), this implies a unique bounded solution to (A.7) as long as a Rank Condition is satisfied. To check this rank condition, let B denote the matrix of left eigenvectors of A, sorted by their modulus in ascending order. We want to show that the block corresponding to the predetermined variables is nonsingular. In our context, this means showing that the top left element of B is different from zero.

Suppose this were not true, that is, we have a left eigenvector $(0, x, y)$ of A that satisfies:

$$\begin{bmatrix} 0 & x & y \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 & x & y \end{bmatrix}$$

where λ_1 is the unique eigenvalue in $(0,1)$. Consider the second column of the equation. Since $A_{23} = 0$, it reads $xA_{22} = \lambda_1$. It cannot hold since $A_{22} > 1$ and $\lambda_1 < 1$. ■

A.2 Characterization of equilibrium in the CBDC model

In this appendix, we collect derivations and proofs for the CBDC model of Section 2.

Household first-order conditions

The maximization problem of the household is:

$$\max_{\{C_t, D_t, N_t, S_t\}} \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\frac{1}{\sigma}} \left[C_t^{1-\frac{1}{\eta}} + \omega \left(\frac{D_t}{P_t} \right)^{1-\frac{1}{\eta}} \right]^{\frac{1-\frac{1}{\sigma}}{1-\frac{1}{\eta}}} - \psi \frac{N_t^{1+\varphi}}{1+\varphi}$$

s.t.

$$P_t C_t + D_t + S_t \leq W_t N_t + T_t + \Pi_t + (1 + i_{t-1}^D) D_{t-1} + (1 + i_{t-1}^S) S_{t-1}.$$

It is helpful to introduce notation for the bundle of consumption and liquidity services consumed by the household; we define

$$B_t := \left[C_t^{1-\frac{1}{\eta}} + \omega \left(\frac{D_t}{P_t} \right)^{1-\frac{1}{\eta}} \right]^{\frac{1}{1-\frac{1}{\eta}}}.$$

Denoting the Lagrange multiplier on the budget constraint by λ_t, the household first-order conditions for consumption, money, other assets and labor are

$$\begin{aligned} B_t^{(\frac{1}{\eta}-\frac{1}{\sigma})} C_t^{-\frac{1}{\eta}} &= \lambda_t P_t, \\ B_t^{(\frac{1}{\eta}-\frac{1}{\sigma})} \omega \left(\frac{D_t}{P_t} \right)^{-\frac{1}{\eta}} &= \lambda_t P_t - \beta (1 + i_t^D) P_t E_t \left[\lambda_{t+1} \right], \\ \lambda_t &= \beta E_t \left[\lambda_{t+1} (1 + i_t^S) \right], \\ \varphi N_t^{\varphi} &= \lambda_t W_t. \end{aligned}$$

To obtain the money demand equation (3), we simplify the money FOC by substituting out for $E_t \lambda_{t+1}$ from the bond FOC and for λ_t from the consumption FOC:

$$V_t = \frac{P_t C_t}{D_t} = \left(\frac{1}{\omega} \frac{i_t^S - i_t^D}{1 + i_t^S} \right)^{\eta}. \tag{A.15}$$

Substituting out for real balances D_t/P_t, we rewrite the bundle B_t of consumption and liquidity services as

$$\begin{aligned} B_t &= \left[C_t^{1-\frac{1}{\eta}} + \omega \left(\frac{D_t}{P_t} \right)^{1-\frac{1}{\eta}} \right]^{\frac{1}{1-\frac{1}{\eta}}} \\ &= \left[1 + \omega^{\eta} \left(\frac{i_t^S - i_t^D}{1+i_t^S} \right)^{1-\eta} \right]^{\frac{1}{1-\frac{1}{\eta}}} C_t \\ &= Q_t^{-\eta} C_t, \end{aligned}$$

where $Q_t := \left[1 + \omega^{\eta} \left(\frac{i_t^S - i_t^D}{1+i_t^S} \right)^{1-\eta} \right]^{\frac{1}{1-\eta}}$ is the ideal price index for the bundle.

The consumption FOC can now be rewritten as

$$Q_t^{\frac{\eta}{\sigma}-1} C_t^{-\frac{1}{\sigma}} = \lambda_t P_t. \tag{A.16}$$

Household labor supply (5) now follows by combining the consumption and labor FOCs to substitute out λ_t:

$$\varphi Q_t^{1-\frac{\eta}{\sigma}} C_t^{\frac{1}{\sigma}} N_t^{\varphi} = \frac{W_t}{P_t}.$$

Similarly substituting out λ_t from (A.16) further delivers the intertemporal Euler equations for other assets and money (6) and (7), respectively:

$$\beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] (1 + i_t^S) = 1$$

$$\beta E_t \left[\left(\frac{Q_{t+1}}{Q_t} \right)^{\frac{\eta}{\sigma}-1} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\sigma}} \frac{P_t}{P_{t+1}} \right] (1 + i_t^D) + \omega \left(\frac{P_t C_t}{D_t} \right)^{\frac{1}{\eta}} = 1.$$

Linearization. We follow the literature in writing log deviations from steady state in gross rates of return as deviations from steady state in net returns. For example, the gross return on money deposits is $1 + i_t^D$, and we write the log deviation from the steady state rate as

$$\log \left(1 + i_t^D \right) - \log \left(1 + i^D \right) \approx i_t^D - i^D.$$

This approximation is justified if rates of return are small, as is the case in our quarterly model with riskfree assets.

For money demand, we simplify notation by performing an additional approximation:

$$\hat{v}_t \approx \eta \frac{1+r^D}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D) \approx \frac{\eta}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D). \tag{A.17}$$

The first equality is justified by loglinearizing and expressing rates of return in net levels, as explained above. The second equality is justified by recognizing that the small steady state return r^D multiplies small spreads $i_t^S - i_t^D$ and so we treat the product as second order.

The derivation of the New Keynesian Phillips curve and Euler equation follow the textbook treatment by Gali (2008). The Phillips curve relates the growth rate of the price level to future price growth as well as marginal cost:

$$\Delta \hat{p}_t = \beta E_t \Delta \hat{p}_{t+1} + \lambda \hat{mc}_t.$$

Since labor is the only factor of production and we abstract from the productivity shock, marginal cost variation is only variation in wages, that is, $\hat{mc}_t = \hat{w}_t$.

To find the variation in wages, consider first the effect of the cost of liquidity on the price of a bundle of consumption and liquidity. We write $Z_t = \frac{i_t^S - i_t^D}{1+i_t^S}$ for the price of liquidity and find

$$\begin{aligned}
\hat{q}_t &= \frac{\omega^\eta Z^{1-\eta}}{1+\omega^\eta Z^{1-\eta}} \hat{z}_t \\
&= \frac{\omega^\eta \left(\delta - r^D\right)^{1-\eta}}{(1+\delta)^{1-\eta} + \omega^\eta \left(\delta - r^D\right)^{1-\eta}} \hat{z}_t \\
&= \frac{\omega^\eta \left(\delta - r^D\right)^{1-\eta}}{(1+\delta)^{1-\eta} + \omega^\eta \left(\delta - r^D\right)^{1-\eta}} \eta^{-1} \hat{v}_t \\
&= \frac{\omega^\eta \left(\delta - r^D\right)^{1-\eta}}{(1+\delta)^{1-\eta} + \omega^\eta \left(\delta - r^D\right)^{1-\eta}} \frac{1}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D) \\
&= \frac{\chi}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D),
\end{aligned}$$

where the second and third line substitute for the steady state price Z and the log deviation $\hat{z}_t$, respectively, from (A.15), the fourth line substitutes for $\hat{v}_t$ from (A.17) and the fifth line defines the parameter χ: it measures the response of the price of a bundle to the price of liquidity.

The loglinearized first order condition for labor is now

$$\begin{aligned}\hat{w}_t &= \left(1-\frac{\eta}{\sigma}\right)\hat{q}_t + \frac{1}{\sigma}\hat{y}_t + \varphi\hat{n}_t \\ &= \left(1-\frac{\eta}{\sigma}\right)\frac{\chi}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D)) + \frac{1}{\sigma}\hat{y}_t + \varphi\hat{n}_t \\ &= \left(1-\frac{\eta}{\sigma}\right)\frac{\chi}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D)) + \frac{1}{\sigma}\hat{y}_t + \varphi\hat{n}_t,\end{aligned}$$

where the third line follows from the production function and the fact that we abstract from productivity shocks, so $\hat{y}_t = \hat{n}_t$. Finally, substituting wages for marginal cost, the Phillips curve takes the form in (13):

$$\Delta\hat{p}_t = E_t\Delta\hat{p}_{t+1} + \lambda\left((\varphi + \frac{1}{\sigma})\hat{y}_t + \left(1-\frac{\eta}{\sigma}\right)\frac{\chi}{\delta - r^D}(i_t^S - i_t^D - (\delta - r^D))\right).$$

Proof of Proposition 2.1

Part (a) The CBDC model with separable utility is a special case of the general system (**??**) studied in Appendix A.1. The special case takes i_t^P as the deposit rate, $\hat{n}_t$ as the exogenous quantity of deposits, and sets the interest elasticity of money supply to $\varepsilon = 0$. Moreover, the coefficient χ is given by

$$\chi = \left(1 + \omega^{-\eta}\left(\frac{\delta - r^D}{1 + r^D}\right)^{\eta - 1}\right)^{-1}. \tag{A.18}$$

With separable utility, we further have $\eta = \sigma$. The sufficient condition for a unique bounded solution in Proposition A is trivially satisfied for any nonnegative Taylor rule coefficients.■

Part (b) If $\mu = 0$, output and inflation follow the two-dimensional system

$$\begin{pmatrix} \Delta\hat{p}_{t+1} \\ \hat{y}_{t+1} \end{pmatrix} = A\begin{pmatrix} \Delta\hat{p}_t \\ \hat{y}_t \end{pmatrix} + b_t$$

where b_t is exogenous and

$$A = \begin{bmatrix} 1/\beta & -\lambda\left(\varphi + \sigma^{-1}\right)/\beta \\ \sigma\phi_\pi - \sigma/\beta & 1 + \sigma\phi_y + \delta - r^D + \lambda\left(\sigma\varphi + 1\right)/\beta \end{bmatrix}.$$

There is a unique bounded solution if and only if both of the eigenvalues are outside the

unit circle. The characteristic polynomial is $p(\lambda) = \lambda^2 - c_1\lambda + c_0$ with coefficients

$$c_1 = \frac{1}{\beta} + 1 + \sigma\phi_y + \delta - r^D + \lambda\left(\sigma\varphi + 1\right)/\beta,$$
$$c_0 = \left(\frac{1}{\beta}\right)\left(1 + \sigma\phi_y + \delta - r^D + \lambda(\sigma\varphi + 1)\phi_\pi\right).$$

We distinguish real and complex roots. If the roots are real, the smaller root is

$$\lambda = \frac{c_1 - \sqrt{c_1^2 - 4c_0}}{2}$$

which is larger than one if and only if $1 - c_1 + c_0 > 0$ which delivers the condition.

If the roots are complex, they are outside the unit circle since the determinant of A is larger than one.■

Part (c) Consider the system of inflation, output and real balances $\hat{d}_t - \hat{p}_t$ comprised of (13), (15) and (17). In matrix form, we have

$$\begin{pmatrix} \Delta\hat{p}_{t+1} \\ \hat{y}_{t+1} \\ \hat{d}_t - \hat{p}_t \end{pmatrix} = A\left(\mu\right)\begin{pmatrix} \Delta\hat{p}_t \\ \hat{y}_t \\ \hat{d}_{t-1} - \hat{p}_{t-1} \end{pmatrix} + b_t, \tag{A.19}$$

where the coefficients of the matrix $A\left(\mu\right)$ are given by $A_{31}\left(\mu\right) = -\mu$, $A_{32}(\mu) = 0$, $A_{33}\left(\mu\right) = \mu$ and $A_{11}(\mu) = 1/\beta, A_{12}(\mu) = -\kappa/\beta, A_{13}(\mu) = 0$ as well as

$$A_{21}(\mu) = \mu\left(\delta - r^D\right) + \sigma\phi_\pi - \sigma/\beta,$$
$$A_{22}(\mu) = 1 + \sigma\phi_y + \delta - r^D + \sigma\kappa/\beta,$$
$$A_{23}(\mu) = -\mu\left(\delta - r^D\right).$$

The coefficients are continuous in μ and converge to the coefficients of (**??**) in Appendix A.1 as $\mu \to 1$ under the special case of the CBDC model where $A_V = 1$, $B_V = 0$ and $\sigma = \eta$. It then follows that the coefficients of the characteristic polynomial here are continuous in μ and converge to the polynomial in Appendix A.1 as $\mu \to 1$. It follows that there must be a threshold $\bar{\mu}$ s.t. Proposition A in Appendix A.1 applies for $\mu \geq \bar{\mu}$.

A.3 Derivations for the bank model with a floor system in Section 3

In this appendix we collect derivations for the bank model of Section 3 as well as the proof of Proposition 3.1.

Bank market power

In the setup with monopolistic competition, bank i supplies liquidity to households at the price $Z_t^i = (i_t^S - i_t^{D,i})/(1 + i_t^S)$, where $i_t^{D,i}$ is the deposit rate promised by bank i. The spread $i_t^S - i_t^{D,i}$ is interest foregone by investing in deposits as opposed to the shadow rate, discounted by $(1 + i_t^S)$ as the interest is received next period.

Households value different varieties of deposits according to a CES aggregator with elasticity of substitution η_b. For given individual bank deposit rates $i_t^{D,i}$ and hence liquidity prices Z_t^i, let Z_t denote the ideal CES price index that aggregates the individual bank liquidity prices Z_t^i. We then define the ideal average deposit rate i_t^D by

$$\frac{i_t^S - i_t^D}{1 + i_t^S} = Z_t.$$

Household maximization delivers bank i's deposit demand function

$$D_t^i = \left(\frac{Z_t^i}{Z_t}\right)^{-\eta_b} D_t = \left(\frac{i_t^S - i_t^{D,i}}{i_t^S - i_t^D}\right)^{-\eta_b} D_t. \tag{A.20}$$

Bank cash flow is as before, except that bank i now pays the deposit rate $i_{t-1}^{D,i}$ it has chosen:

$$\begin{aligned} &M_{t-1}^i \left(1 + i_{t-1}^M\right) - M_t^i - D_{t-1}^i \left(1 + i_{t-1}^{D,i}\right) + D_t^i \\ &+ A_{t-1}^i \left(1 + i_{t-1}^A\right) - A_t^i \end{aligned}$$

Bank i maximizes shareholder value by choosing M_t^i, A_t^i, and $i_t^{D,i}$ subject to (21) and (A.20).

Writing γ_t for the multiplier on the leverage constraint, the terms in the Lagrangian involving the date t deposit rate are

$$D_t^i - \frac{D_t^i}{1 + i_t^S} \left(1 + i_t^{D,i}\right) - \gamma_t D_t^i = \left(Z_t^i - \gamma_t\right) D_t^i.$$

Shareholder maximization thus works like profit maximization with constant marginal cost γ_t via choice of a price Z_t^i.

The first order conditions with respect to Z_t^i take the standard form

$$\left(Z_t^i - \gamma_t\right) \eta_b \left(Z_t^i\right)^{-\eta_b - 1} \frac{D_t}{Z_t^{-\eta_b}} + \left(\frac{Z_t^i}{Z_t}\right)^{-\eta_b} D_t = 0,$$

$$\eta_b \left(\frac{i_t^S - i_t^{D,i}}{i_t^S - i_t^D}\right)^{-\eta_b - 1} \frac{1}{i_t^S - i_t^D} \left(1 - \frac{1 + i_t^{D,i}}{1 + i_t^S} - \gamma_t\right) D_t - \frac{1}{1 + i_t^S} D_t^i = 0.$$

A higher price of liquidity lowers profits by decreasing the quantity of deposits, but increases profits by increasing revenue per dollar issues.

Substituting from the demand function and rearranging, we have

$$i_t^S - i_t^{D,i} = \frac{\eta_b}{\eta_b - 1} \left(1 + i_t^S\right) \gamma_t.$$

Bank i chooses a price that multiplies marginal cost by a constant markup.

Solving the bank problem results in the following first order conditions:

$$\begin{aligned} i_t^S - i_t^M &= \ell \gamma_t (1 + i_t^S), \\ i_t^S - i_t^A &= \rho_A \ell \gamma_t (1 + i_t^S), \\ i_t^S - i_t^D &= \left(\frac{\eta_b}{\eta_b - 1}\right) \gamma_t (1 + i_t^S). \end{aligned}$$

Combining the reserves and deposits first order condition, we arrive at equation (26):

$$i_t^S - i_t^D = \frac{\eta_b}{\eta_b - 1} \ell^{-1} \left(i_t^S - i_t^M\right).$$

Proof of Proposition 3.1

The bank model with a floor system is a special case of the general system (**??**) studied in Appendix A.1. The special case takes the policy rate i_t^P equal to the interest rate on reserves, and sets $\hat{n}_t = \alpha_m \hat{m}_t + (1 - \alpha_m) \hat{a}_t$ and $\varepsilon = 0$. The coefficient χ is given by (A.18) together with

$$\delta - r^D = \frac{\eta_b}{\eta_b - 1} \ell^{-1} \left(\delta - r^M\right). \tag{A.21}$$

The necessary and sufficient condition for determinacy is therefore that of Proposition A in Appendix A.1.■

A.4 Proofs of Proposition 4.1-4.3

In this appendix, we collect proofs for the propositions in Section 4. For easier notation we drop superscripts indicating individual banks.

Proof of Proposition 4.1

A bank's problem in the second subperiod is to choose M', F^+ and F^- to maximize next period cash

$$M'\left(1+i^M\right)+\left(1+i^F\right)\left(F^+-F^-\right),$$

subject to the budget and collateral constraints as well as nonnegativity constraints in all three variables.

The first order conditions are

$$\begin{aligned} 1+i^M+\gamma\ell &= \lambda-\nu_{M'},\\ 1+i^F+\gamma\rho_F\ell &= \lambda-\nu_{F^+},\\ 1+i^F+\gamma &= \lambda+\nu_{F^-}, \end{aligned}$$

where γ is the multiplier on the collateral constraint, λ is the multiplier on the budget constraint, and the νs are the multipliers on the three nonnegativity constraints.

We distinguish solutions with positive reserve holdings from those with zero reserves. Suppose first a bank holds no reserves overnight, that is, $M'=0$. The optimal policy is then

$$F^+-F^-=M-\grave{\lambda}D$$

In order for the collateral constraint to be satisfied, we must have $D-M<\ell\rho_A A$. The precise split into F^+ and F^- is not important in this case – only the net position is determinate.

Suppose instead a bank holds reserves overnight, that is, $M'>0$ and hence $\nu_{M'}=0$. We must have $\gamma>0$: otherwise the fed funds lending and reserves FOC cannot jointly hold. Indeed, these FOC imply

$$1+i^F+\gamma\rho_F\ell\leq\lambda=1+i^M+\gamma\ell,$$

which cannot hold for $\gamma>0$ since we have assumed $i^F>i^M$. From the fed funds borrowing FOC, we must then have $\nu_{F-}>0$ and hence $F^-=0$.

When the bank holds reserves, we can thus combine the binding collateral constraint and

the budget constraint to find optimal reserve holdings and fed funds lending

$$\begin{aligned}
M' &= \frac{\left(1-\tilde{\lambda}\left(1-\rho_F\ell\right)\right)D-\rho_F\ell M-\rho_A\ell A}{\ell\left(1-\rho_F\right)},\\
F^+ &= M-\tilde{\lambda}D-M'\\
&= \frac{\left(M-\tilde{\lambda}D\right)\ell\left(1-\rho_F\right)-\left(1-\tilde{\lambda}\left(1-\rho_F\ell\right)\right)D+\rho_F\ell M+\rho_A\ell A}{\ell\left(1-\rho_F\right)}\\
&= \frac{M\ell+\rho_A\ell A-\left(1-\tilde{\lambda}\left(1-\ell\right)\right)D}{\ell\left(1-\rho_F\right)}
\end{aligned}$$

We need for this case that M' is positive and F^+ is nonnegative. The first condition is equivalent to $\tilde{\lambda}<\lambda^*$. The second condition is satisfied at any value of $\tilde{\lambda}$ as long as it is satisfied at $\tilde{\lambda}=-\bar{\lambda}$. The condition assumed in the proposition says that the second condition is indeed satisfied at $\tilde{\lambda}=-\bar{\lambda}$.■

Proof of Proposition 4.2

We first derive necessary and sufficient conditions for an elastic equilibrium to exist. Consider the threshold liquidity shock λ^* implicitly defined by the first order condition for reserves – the first equation in (30) – with $\gamma=0$. Since the cdf G is strictly increasing, we can define a function

$$f\left(r^F\right):=G^{-1}\left(\frac{1-\rho_F}{\rho_F}\frac{\delta-r^F}{r^F-r^M}\right)$$

over the interval $\left[\left(1-\rho_F\right)\delta+\rho_F r^M,\delta\right]$. The function f is strictly decreasing and we have that $f\left(\left(1-\rho_F\right)\delta+\rho_F r^M\right)=\bar{\lambda}$ and $f\left(\delta\right)=-\bar{\lambda}$. Any $r^F\in\left[\left(1-\rho_F\right)\delta+\rho_F r^M,\delta\right]$ thus implies a liquidity threshold $\lambda^*\in\left[-\bar{\lambda},\bar{\lambda}\right]$.

To satisfy the worst collateral constraint, the threshold liquidity shock must be sufficiently small. We have

$$\begin{aligned}
\ell\frac{M}{D}+\rho_A\ell\frac{A}{D} &= 1-\left(1-\rho_F\ell\right)\lambda^*+\left(1-\rho_F\right)\ell\frac{M}{D}\\
&= 1-\left(1-\rho_F\ell\right)\lambda^*+\left(1-\rho_F\ell\right)\left(\lambda^*G\left(\lambda^*\right)-\int_{-\bar{\lambda}}^{\lambda^*}\tilde{\lambda}dG\left(\tilde{\lambda}\right)\right)\\
&= 1-\left(1-\rho_F\ell\right)\left(\lambda^*\left(1-G\left(\lambda^*\right)\right)+\int_{-\bar{\lambda}}^{\lambda^*}\tilde{\lambda}dG\left(\tilde{\lambda}\right)\right)\\
&=: h\left(\lambda^*\right)
\end{aligned}$$

where the first equality uses the definition of the threshold shocks λ^* and the second uses fed funds market clearing (32).

The function h is strictly decreasing: we have $h'(\lambda^*) = -(1-\rho_F\ell)(1-G(\lambda^*)) < 0$. Moreover, we have $h\left(-\bar{\lambda}\right) = 1 + (1-\rho_F\ell)\bar{\lambda}$ and $h\left(\bar{\lambda}\right) = 1$, the latter due to our assumption that the mean of $\tilde{\lambda}$ is zero. We now consider the composite function $h \circ f$ and define the threshold fed funds rate by

$$(h \circ f)\left(r^{F*}\right) = 1 + \bar{\lambda}(1-\ell).$$

The composite function is strictly increasing in r^F with $(h \circ f)\left((1-\rho_F)\delta + \rho_F r^M\right) = 1$ and

$$(h \circ f)(\delta) = 1 + (1-\rho_F\ell)\bar{\lambda} > 1 + \bar{\lambda}(1-\ell)$$

It follows that there is a unique threshold funds rate $r^* \in \left(r^M, \delta\right)$.

For given r^F, an elastic equilibrium exists if and only if

$$(h \circ f)\left(r^F\right) > 1 + \bar{\lambda}(1-\ell).$$

Indeed, in this case the threshold shock λ^* implies balance sheet ratios that are sufficiently large for (27) not to bind.

Moreover, an inelastic equilibrium exists if and only if the condition does not hold. Indeed, in an inelastic equilibrium, we determine M/D and A/D from (27) and (32). The implied ratio λ^* is then given by $h^{-1}\left(\left(1 + \bar{\lambda}(1-\ell)\right)\right)$, which is larger than the ratio $f\left(r^F\right)$ consistent with $\gamma = 0$. Since G is strictly increasing, the multiplier γ is positive.■

Proof of Proposition 4.3

Part (a). We need to choose a continuous cdf G that is restricted by

$$\begin{aligned}
\frac{\delta - r^F}{r^F - r^M} &= \frac{\rho_F}{1-\rho_F} G(\lambda^*), \\
\int_{-\bar{\lambda}}^{\lambda^*} G(\lambda)\, d\lambda &= \frac{(1-\rho_F)\,\ell}{1-\rho_F\ell}\frac{M}{D}, \\
\lambda^* &= \frac{1 - \rho_F\ell\frac{M}{D} - \rho_A\ell\frac{A}{D}}{1-\rho_F\ell}.
\end{aligned}$$

In addition, the parameters have to satisfy $\lambda^* \in \left[-\bar{\lambda}, \bar{\lambda}\right]$ and the worst case leverage constraint

$$1 + \bar{\lambda}(1-\ell) \leq \ell\frac{M}{D} + \ell\rho_A\frac{A}{D}. \tag{A.22}$$

By assumption on interest rates and the weight ρ_F, there exists a value $G^* := G(\lambda^*) < 1$ satisfying the first equality. The integral on the left hand side in the second equation is

bounded above by $\left(\lambda^* + \bar{\lambda}\right) G\left(\lambda^*\right)$ and bounded below by zero. Indeed, by choosing the density g, we can go arbitrarily close to cdfs with mass points of G^* at either $-\bar{\lambda}$ or λ^* – these cases provide the upper and lower bounds, respectively.

It follows that we can find a suitable G that achieves the value G^* at λ^* as long as there exists $\xi \in (0,1)$ such that

$$\xi \left(\lambda^* + \bar{\lambda}\right) G^* = \frac{(1-\rho_F)\,\ell}{1-\rho_F \ell} \frac{M}{D} \tag{A.23}$$

The share ξ indicates the area described by the integral as a share of the maximally possible rectangle described by a cdf with a mass point at $-\bar{\lambda}$. Any cdf that distributes the mass below λ^* to achieve this share ξ will deliver the desired result.

Suppose now that $\ell = 1$. By assumption on the balance sheet ratios, (A.22) is satisfied. Moreover, for any $\bar{\lambda}$ sufficiently large so $\lambda^* \in \left[-\bar{\lambda}, \bar{\lambda}\right]$, we can find ξ to satisfy (A.23). Since all inequalities are strict and the conditions depend continuously on ℓ at $\ell = 1$, we can also lower ℓ slightly below one to obtain the same result.

Part (b). Consider a sequence M_n that converges to zero such that the assumptions of the proposition remain satisfied. Since $(M_n + \rho_A A)/D > 1$ for all M_n, we must have $\rho_A A/D > 1$. We can therefore choose $\ell < 1$ such that $\rho_A \ell A/D = 1$. Since $\rho_F < 1$, the sequence of thresholds λ_n^* is strictly positive and converges to zero from below. Fix $\xi < 1$ and choose $\bar{\lambda}_n$ to satisfy (A.23). Any such $\bar{\lambda}_n$ satisfies $\lambda_n^* \in \left(-\bar{\lambda}_n, \bar{\lambda}_n\right)$ since λ_n^* is negative and the left hand side of (A.23).must be positive. As n becomes large, both λ_n^* and M_n/D converge to zero, and therefore $\bar{\lambda}_n$ converges to zero as well. ■

A.5 Linear approximation to the bank model with scarce reserves

In this appendix, we derive linear approximations to the equations of the bank model of Section 4 as well as the proof of Proposition 4.4.

We show that the key linearized equations (33) and (34) hold in an elastic supply equilibrium. With a slack leverage constraint, we have $\gamma_t = 0$ in (30), which thus simplifies to

$$\begin{aligned} i_t^S - i_t^F &= \left(i_t^F - i_t^M\right) \frac{\rho_F}{1-\rho_F} G\left(\lambda_t^*\right), \\ i_t^S - i_t^A &= \frac{\rho_A}{\rho_F} \left(i_t^S - i_t^F\right). \end{aligned} \tag{A.24}$$

Given the two policy rates, the five equations (A.24), (31) with $\gamma_t = 0$, (32) and the definition of the threshold shock

$$\lambda_t^* = \frac{1 - \rho_F \ell \frac{M_t}{D_t} - \rho_A \ell \frac{A_t}{D_t}}{(1-\rho_F \ell)}$$

determine five variables: the balance sheet ratios M_t/D_t and A_t/D_t, the threshold λ_t^* and the interest rates on bank instruments i_t^A and i_t^D.

To derive (33), we start by loglinearizing the definition of the threshold shock λ_t^* and the money market clearing condition:

$$\lambda^* \hat{\lambda}_t^* = \frac{\rho_A \ell}{1-\rho_F \ell} \frac{A}{D} \left(\hat{d}_t - \hat{a}_t \right) + \frac{\rho_F \ell}{1-\rho_F \ell} \frac{M}{D} \left(\hat{d}_t - \hat{m}_t \right)$$
$$\frac{\ell \left(1-\rho_F\right)}{1-\rho_F \ell} \frac{M}{D} \left(\hat{d}_t - \hat{m}_t \right) = -G\left(\lambda^*\right) \lambda^* \hat{\lambda}_t^*$$

We can substitute out the endogenous change in the ratio of reserves to deposits $\frac{M}{D}\left(\hat{d} - \hat{m}\right)$ to obtain

$$\frac{1-\rho_F+\rho_F G\left(\lambda^*\right)}{1-\rho_F} \lambda^* \hat{\lambda}_t^* = \frac{\rho_A \ell}{1-\rho_F \ell} \frac{A}{D} \left(\hat{d}_t - \hat{a}_t \right) \tag{A.25}$$

Next, we loglinearize the first order condition for reserves – the first equation in (A.24) – to find

$$\begin{aligned}
\frac{g(\lambda^\star)\lambda^\star}{G(\lambda^\star)} \hat{\lambda}_t^\star &= \frac{(1+r^F)(\delta - r^M)}{(\delta - r^F)(r^F - r^M)} \left(i_t^S - i_t^F \right) - \frac{1+r^M}{r^F - r^M} \left(i_t^S - i_t^M \right) \\
&= \left(\frac{(1+r^F)(\delta - r^M)}{(\delta - r^F)(r^F - r^M)} - \frac{1+r^M}{r^F - r^M} \right) \left(i_t^S - i_t^F \right) - \left(\frac{1+r^M}{r^F - r^M} \right) \left(\imath_t^F - i_t^M \right) \\
&= \frac{1+\delta}{\delta - r^F} \left(i_t^S - i_t^F \right) - \frac{1+r^M}{r^F - r^M} \left(i_t^F - i_t^M \right)
\end{aligned}$$

Again assuming that net rates of return are small decimal numbers we obtain the approximation

$$\frac{g\left(\lambda^*\right)\lambda^*}{G\left(\lambda^*\right)} \hat{\lambda}_t^* = \frac{i_t^S - i_t^F}{\delta - r^F} - \frac{i_t^F - i_t^M}{r^F - r^M}.$$

Substituting for $\lambda^* \hat{\lambda}_t^*$ from (A.25) now leads to

$$\frac{g\left(\lambda^*\right)}{G\left(\lambda^*\right)} \frac{1-\rho_F}{1-\rho_F+\rho_F G\left(\lambda^*\right)} \frac{\rho_A \ell}{1-\rho_F \ell} \frac{A}{D} \left(\hat{d}_t - \hat{a}_t \right) = \frac{i_t^S - i_t^F}{\delta - r^F} - \frac{i_t^F - i_t^M}{r^F - r^M},$$

and rearranging delivers the equation in the text.

To derive (34), we first rewrite (31) as

$$\begin{aligned}\frac{\eta_b-1}{\eta_b}\left(i_t^S-i_t^D\right) &= \left(i_t^F-i_t^M\right)\frac{1-\rho_F\ell}{\ell\left(1-\rho_F\right)}\int_{-\bar{\lambda}}^{\lambda_t^*}\left(\lambda_t^*-\tilde{\lambda}\right)dG\left(\tilde{\lambda}\right)\\ &\quad+\left(i_t^F-i_t^M\right)\frac{1}{\ell\left(1-\rho_F\right)}G\left(\lambda_t^*\right)\left(\rho_F\ell\frac{M_t}{D_t}+\rho_A\ell\frac{A_t}{D_t}\right)\\ &= \left(i_t^F-i_t^M\right)\frac{M_t}{D_t}+\left(i_t^S-i_t^F\right)\left(\frac{M_t}{D_t}+\frac{\rho_A}{\rho_F}\frac{A_t}{D_t}\right)\\ &= \left(i_t^S-i_t^M\right)\frac{M_t}{D_t}+\left(i_t^S-i_t^A\right)\frac{A_t}{D_t}\end{aligned}$$

Here the first equality follows by substituting for the first term on the right hand side from (32) and substituting for the spread in the second term from (A.24).

Loglinearization around the steady state delivers

$$(1+r^M)\left(\frac{M^\star}{D^\star}\right)\left(i_t^S-i_t^M\right)+(1+r^A)\left(\frac{A^\star}{D^\star}\right)\left(i_t^S-i_t^A\right)-(1+r^D)\left(i_t^S-i_t^D\right)+\hat{Z}_t=0$$

where

$$\begin{aligned}\hat{Z}_t &= (\delta-r^M)\left(\frac{M^\star}{D^\star}\right)\left(\hat{m}_t-\hat{d}_t\right)+\left(\delta-r^A\right)\left(\frac{A^\star}{D^\star}\right)\left(\hat{a}_t-\hat{d}_t\right)\\ &= (\delta-r^M)\left(\frac{M^\star}{D^\star}\right)\left(\hat{m}_t-\hat{d}_t\right)-(\delta-r^A)\left(\frac{\rho_F}{\rho_A}\right)\left(\frac{M^\star}{D^\star}\right)\left(\hat{m}_t-\hat{d}_t\right)-(\delta-r^A)\left(\frac{1-\rho_F\ell}{\rho_A\ell}\right)\lambda^\star\hat{\lambda}_t^\star\\ &= \left((\delta-r^M)-(\delta-r^L)(\frac{\rho_F}{\rho_A})\right)\left(\frac{M^\star}{D^\star}\right)\left(\hat{m}_t-\hat{d}_t\right)-(\delta-r^A)\left(\frac{1-\rho_F}{\rho_A G(\lambda^\star)}\right)\left(\frac{M^\star}{D^\star}\right)\left(\hat{m}_t-\hat{d}_t\right)\\ &= \left[r^F-r^M-(\delta-r^F)\left(\frac{r^F-r^M}{\delta-r^F}\right)\right]\left(\frac{M^\star}{D^\star}\right)\left(\hat{m}_t-\hat{d}_t\right)\\ &= 0\end{aligned}$$

where the second and third equality use the log-linearized threshold shock and money market clearing conditions, respectively, and the last three lines use the steady state versions of the first order conditions for other assets and reserves. We end up with

$$\frac{\eta_b-1}{\eta_b}\left(i_t^S-i_t^D\right)=(i_t^S-i_t^M)\frac{1+r^M}{1+r^D}\frac{M}{D}+(i_t^S-i_t^A)\frac{1+r^A}{1+r^D}\frac{A}{D}$$

Since the ratios of gross returns are close to one, we drop them and work with the simpler

approximate formula

$$\frac{\eta_b - 1}{\eta_b}\left(i_t^S - i_t^D\right) = \left(i_t^S - i_t^M\right)\frac{M}{D} + \left(i_t^S - i_t^A\right)\frac{A}{D}.$$

The second equation in the text follows by substituting for $i_t^S - i_t^A$ from (A.24).

Proof of Proposition 4.4

For any interest elasticity of deposit supply $\varepsilon > 0$ and steady state ratio of other assets to deposits A/D, an elastic equilibrium of the bank model with a corridor system and reserves negligible on bank balance sheets is a special case of the general system (**??**) studied in Appendix A.1. The special case takes the policy rate i_t^P to be the federal funds rate i_t^F and sets $\hat{n} = \hat{a}_t$. The coefficient χ is determined by (A.18) and

$$\delta - r^D = \frac{\eta_b}{\eta_b - 1}\ell^{-1}\left(\delta - r^F\right).$$

The necessary and sufficient condition for determinacy is therefore that of Proposition A in Appendix A.1.■