How Automation that Substitutes for Labor Affects Production Networks, Growth, and Income Inequality

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Abstract

We study the impact of technological change on GDP growth, income inequality, and the interconnectedness of the economy. We do this in a model of production networks that contrasts the effects of improvements in intermediate goods that substitute for labor with those that complement and enhance labor. Technological changes that complement labor increase productivity and change neither the interconnectedness between sectors nor the relative wages between high-skilled and low-skilled labor. In contrast, technological advances that (directly or indirectly) affect goods that substitute for labor (e.g., robots, AI) have impacts that depend on the state of the economy. As a good substitutes for labor, improvements in its productivity push labor into other less productive processes, but wages also adjust and slow that displacement. The growth resulting from the technology improvement is attenuated and income inequality between low- and high-skilled workers grows. At the same time, the production network becomes denser and interconnectedness grows, changing the centralities of different sectors and enhancing the impact of future technological changes. Eventually, automation fully substitutes for labor and then all further technological advances translate directly into productivity gains. Our findings imply that growth effect of recent technological developments in automation technologies should emerge gradually, and at an initial cost of increased income inequality.

Keywords: Production Networks, Input-Output Analysis, Growth, Inequality, Reallocation of Labor, Technological Advances

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1 Introduction

The production of goods and services has become increasingly complex and networked. Many involve multiple tasks or parts, some even hundreds or hundreds of thousands. Moreover, some intermediate goods are not only complements in production, but can substitute for or completely replace others. For instance, Kevlar and other synthetic materials substitute for metals and fabrics, and robots and AI substitute for labor in manufacturing (e.g., assembly lines), distribution (e.g., warehouses and drivers), and services (e.g., computer-based ordering and data collection systems). In this paper, we study the effects of technological advances in such a world that is extensively networked and in which goods not only complement each other in production, but can also substitute for other inputs – in particular for labor.

Despite the complexity of such a world, we show that there are tractable formulas that describe the impacts of different technological advances (cost reducing or quality improving) that can enhance the use of certain type of goods. Including the possibility of goods that substitute for others is important for two reasons. One is that many technological advances change the mix of labor and other inputs, both historically as technological advances reduced the use of labor in agriculture and manufacturing, and at present as automation and AI are displacing labor in the production of a wide variety of goods and services. The second is that the effects of technological advances on the production network in the presence of substitution effects fundamentally differ from the case of pure complements. Since traditional input-output analysis has focused on the case of complements, it misses many important effects which change the ways that technological improvements and other shocks move through the economy.

In particular, a main feature of our analysis is a general equilibrium effect of changes in wages in reaction to technological advances. These effects counteract the impact of technological advances that substitute for labor. The extent to which they mitigate such advances depends on where the labor displaced by new technologies ends up, which depends on the full production network in a way that we fully characterize. We focus on substitute effects between various inputs and labor, but the analysis could be extended to include other substitution between goods.

Let us briefly describe the intuition behind the effects before more fully describing our analysis and the contribution relative to the existing literature.

Consider the production of some good Y that uses high-skilled labor H, low-skilled labor L, an input good that can substitute for low-skilled labor X. Suppose that it takes the form

$$Y = (L + AX)^{\alpha} H^{1-\alpha},$$

where $\alpha \in (0, 1)$ tracks the relative shares of the high and low-skilled inputs. X substitutes for low-skilled labor at a rate A. A change in A reflects a technological advance in the input X that makes it a better substitute for labor. Let us examine the effect of a change in A on

the value of production Y:[1]

$$\frac{\partial Y}{\partial A} = \alpha X(L + AX)^{\alpha-1} H^{1-\alpha} = \alpha X \frac{Y}{L + AX}.$$

This expression is decreasing in L and increasing in X. Early in the substitution/automation process, L is high and X is low, and so the impact of an advance A is low. As, substitution takes place, L falls and X increases, and thus the derivative increases as well. This shows the basic force at work: the impact changes depending on the stage of automation and the values of L and X. As substitution is just beginning $X = 0$ and $L > 0$, and so is beginning with a 0 effect. As substitution continues, L drops and X increases, and so does $\frac{\partial Y}{\partial A}$ (as a function of the level of Y, which is also increasing). Eventually, when substitution is complete and $L = 0$, then $\frac{\partial Y}{\partial A} = \alpha \frac{Y}{A}$, which then looks like a complementary input.

Thus, understanding the impact of a change in A, and how this depends on the level of A, requires knowing what the levels of L and X are. The key is that the levels of L and X depend on how productive L is in other sectors of the economy. If L is relatively productive elsewhere, then as A increases, L decreases rapidly and X rises rapidly.[2] If instead, L is not so productive elsewhere, then the main change is a drop in the wage and only a slight decrease in L and increase in X. Thus, the levels of L and X depend critically on the more general production network, and hence so does the relative level of the derivative $\frac{\partial Y}{\partial A}$.

Although the level of Y is a general equilibrium term also for complements, there is an added difference here as the adjustments has an opposite sign from the usual effect. If X is a complement to L, then as L becomes more productive elsewhere, the overall level of Y drops and so the derivative of Y with respect to A decreases. In contrast, here in the case of substitutes, as L becomes more productive elsewhere, this decreases its use in production and increases the use of X and increases the derivative - at any given level of Y - producing a counter-acting force.

With this intuition in hand, there are three main contributions of our study. First, we provide a network model of production that enables us to contrast the effects of changes in intermediate goods that substitute for labor with those that complement labor. Second, we show how these different technological changes have very different implications for GDP growth, income inequality, and sector level consumption and labor productivity; and how that depends on the phase of substitution. Accordingly, we develop a network centrality measure of the impact of productivity changes in different sectors, and discuss how network centralities evolve as technological progresses occur.

In particular, we build an input-output model where intermediate goods that are complements for or substitutes to labor are produced within the economy. To analyze the substitutability of labor and intermediate goods, we consider two different types of labor:

[1] L, H, and X adjust with A, but by the Envelope Theorem, their effect washes out of the derivative.

[2] The effect is not discontinuous, as one might superficially expect given the linear substitution, since the low-skilled wage drops as L shifts – as it is necessarily moving into sectors where it was initially marginally less productive. As we show below, the shift is continuous.

high- and low-skilled. The difference between these types of labor is that low-skilled labor performs routine and repetitive tasks that can be substituted by "automation goods" (e.g., robots, software, driverless trucks, drones...), while high-skilled labor instead complements all other goods used in the production process. Dividing labor into these two classes allows us both to derive predictions about how different types of labor are affected by cost-reducing or quality improving technological changes, and also to identify new types of network effects in how change ripples through the economy.

We first analyze a three-sector model that consists of a final good sector, an intermediate good sector that is a complement to labor in production of the final good - a "resource sector", and an intermediate good sector that can substitute for labor in the production of the final good - an "automation sector". In our first result, we focus on how technological advances affect low- and high-skilled labor wages and the relative wage (income inequality). The impact of technological advances on wages and income inequality depends on whether the labor in some sectors are being substituted for by the technology, either directly or indirectly, or whether that substitution is already complete.

During the substitution (or transition) phase, labor is replaced by automation. In this phase, the demand for low-skilled labor, and hence low-skilled wages, decrease as the productivity of the automation sector rises. In this stage, the productivity gains that arise due to the substitution to a more productive factor are captured by high-skilled labor and the high-skilled labor wage rises following a technology improvement in the automation sector. As a result, this phase leads to increased interconnectivity in the economy and also the increased income inequality due to the replacement of low-skilled labor with automation good. The transition phase is not abrupt, but can be prolonged since the wage adjusts, and this adjustment depends on the alternative uses of low-skilled labor.

Eventually, the substitution phase is complete. Further technological improvements have a classical input-output effect (i.e. wages and consumption rise) and have no impact on the relative wage. Moreover, the growth rate of wages in the post-substitution phase is always larger than the growth rate of wages in the substitution phase due to the stronger interconnectivity and network effects.

Following the three-sector model, we extend our analysis to N-sector economy. Similar to the previous part, a firm i uses labor in two forms: high-skilled and low-skilled. Different than the three-sector model, here we consider substitutable and non-substitutable tasks in each sector. In the general model, substitution can be quite indirect. For example, a technological advance in the production of a material like Kevlar can replace metal, and that is then used to make robots lighter and more efficient, and thus spurs their use in warehouses. So, any good in a long supply chain can end up affecting the substitution.

Our initial result in this part shows that a modified version of Hulten's Theorem [37] extends to our setting, and the impact of a shock to any sector, including the automation good sectors, on GDP is summarized by its Domar [28] weight. The key is that the Domar weights depend on which of the phases the economy is in. During the substitution phase, the Domar weights change. Still we can develop expressions for the Domar weights and the

first-order impact of shocks during the transition period can be explained only by the Domar weights.[3]

The Domar weight of a sector can thus be interpreted as its network influence. We then refine our understanding of network influence. In particular, our next result focuses on the evolution of the production networks and sectors' network influences. We analyze how an increase in the rate of automation changes the sectoral Domar weights. We show that an increase in the rate of automation for a given set of tasks results in increased Domar weights of the producer of that set of automation goods and their direct and indirect suppliers. This result implies that the network influence or Domar weight of automation good producers and their direct and indirect suppliers increase over time, whereas sectors that are not suppliers of automation goods, including the consumers of automation goods (e.g., IT intensive industries) do not have increased network influences following the substitution effects in the economy. As a result, as productivities of the producers of automation goods and/or their suppliers rise, the production network becomes denser and the interconnectedness between sectors gets stronger until the substitution phase is over for each automatable task. Furthermore, the evolution of the production network in this defined direction enhances the absolute and relative impacts of future technological changes on GDP growth.

Given that in our model productivity shocks end up having different effects on different types of labor, we examine how those effects work in more detail. In particular, we examine the changes in households' consumption and labor productivities at sector level. First, we show that an increase in automation has ambiguous effects on households' consumption at sector level. Even though the GDP growth in response to the technological improvements is always positive, we show that some sectors might experience a decrease in household consumption. This is due to the fact that the high-skilled labor wage increases, and low-skilled labor wage decreases, with automation during substitution. Correspondingly, the price of a good rises if the firm's production process is heavily on high-skilled labor but is not benefitting from automation in its processes, and similarly can decrease if uses low-skilled labor even without any automation in its production. As a result, in contrast to input-output studies that involve only complements (e.g., Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi [3]), our results imply that the propagation of supply-side shocks is not limited to downstream (direct and indirect customer) sectors. Technological changes affect the demand for fdifferent types of labor, and, therefore, ultimately affect the wages and level of production in all sectors including ones not upstream or downstream from those with technological changes.

[3]Given that Hulten's Theorem provides a first order approximation, it might be misleading during the transition phase due to the changes in the Domar weights in this phase, while the theorem applies with no approximation error in other phases. The discussion of the higher-order effects in the three sector model is in the Appendix. See Baqaee and Farhi [16] for the analysis on the higher order effects.

1.1 Relation to the Literature

Understanding how productivity shocks can ripple through an economy is more important than ever, and has been an area of renewed research.[4] This has been studied both theoretically and empirically. For example, Carvalho, Nirei, Saito, and Tahbaz-Salehi [25] show how the supply chain disruptions in Japan after the Great East Japan Earthquake of 2011 led to wider disruption, both downstream and upstream. Similarly, Barrot and Sauvagnat [17] and Acemoglu, Akcigit, and Kerr [1] show evidence of network-based propagation of idiosyncratic shocks. Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi [3] showed how idiosyncratic shocks can actually become amplified through a production network.

Our advance, as mentioned above, is to extend an input-output analysis to include substitutes while still producing a tractable analysis. We show how there are countervailing wage-adjustment forces that slow the impact of technological advances on the economy. We also show how these depend on the alternative uses for labor, while also showing how a version of Hulten's Theorem and expressions for the Domar weights extend to the substitute setting.

As a by product of our analysis, we provide expressions for exactly how inequality grows in response to technological change, as well as how growth is attenuated by wage adjustments. The same technological change can have very different impacts depending on the rest of the economy and production network.

The role of complements versus substitutes in production have appeared in important studies from Griliches [33] and Stokey [45] to Krusell, Ohanian, Rios Rull, and Violante [40], Autor and Dorn [10], and Hémous and Olsen [34].[5] For example, Krusell et al. [40] show how this difference can help explain the growth in income inequality and the growing skill premium observed over past decades. Our analysis helps show how that depends on the alternative uses of various forms of labor in the overall economy and wider production network. If low-skilled labor is easily and productively absorbed in new tasks, or can be easily retrained, then the inequality effect will be small, while if labor is pushed into less productive roles, then inequality will grow, which seems to be the modern case. This contrasts with the movement of labor from agriculture to manufacturing that took place in the mid twentieth century. Now low-skilled labor (and middle-skilled) is being moved from relatively productive

[4] Following Leontief [41], the early (e.g., Hulten [37], Long and Plosser [42], Basu [19], Dupor [28], Horvath [35, 36], Basu and Fernald [20], and Shea [44]) and recent literatures (e.g., [31], Carvalho and Gabaix [27], Jones [38, 39], Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi [3], Acemoglu, Ozdaglar and Tahbaz-Salehi [4], Boehm, Flaaen and Nayar [**?**], Atalay [9], Bartleme and Gorodnichenko [18], Bigio and La'O [22], Baqaee [13], Fadinger, Ghiglino and Teteryatnikova [30], Baqaee and Farhi [14, 15], Bernard, Dhyne, Magerman, Manova, and Moxnes [21]) on the macroeconomic consequences of interconnectedness have made it clear that the productivity changes in one part of an economy can ripple through the economy and have a wide impact, and that idiosyncratic shocks do not all cancel out, but some can be magnified via the network (e.g., Acemoglu, Carvalho, Ozdaglar and Tahbaz-Salehi [3] and Acemoglu, Ozdaglar and Tahbaz-Salehi [4]).

[5] Beyond these papers, there is also a broader literature on the impact of automation that includes Autor, Katz and Krueger [12], Acemoglu and Restrepo [5, 6, 7], Aghion, Jones, and Jones [8]; but which is less directly related to our paper.

tasks to less valuable ones and the wage level is falling.

Our model also helps shed light on the Solow Paradox and slow growth in response to technological advances that seemingly should have a large effect on the economy. For instance Brynjolfsson, Rock, and Syverson [24] detail the modern version of this paradox, and examine four explanations. A leading one that they emphasize is that it takes time to develop complementary technologies that can take advantage of new advances and inventions. Our model provides a fifth explanation that differs from the four they offer. Ours is that many recent technological advances substitute in some way for labor or other inputs, and the wage/price adjustments due to the general equilibrium effects attenuates the impact of a technological advance.

Similarly, the productivities in different sectors depend on how productive various forms of labor are in various sectors. And our results on the evolution of the input-output network provides additional insights into how technologies are adopted in different sectors (e.g., see Carvalho and Voigtländer [26]).

In our model, we use a Cobb-Douglas production technology, with the twist that some tasks involve the possibility of substituting one good (e.g., technology) for another (e.g., low-skilled labor). This sort of perfect substitutability of machines (or automation) for low-skilled labor and the unit elasticity of substitution between these two and another type of labor is used by Autor, Levy, and Murnane [11], who provide a detailed discussion of this specific modeling choice by giving examples on the characteristics of tasks (i.e. routine vs non-routine tasks). In a setup capturing different elasticities of substitution between factors, Krusell et al. [40] find that the key elasticity of substitution between low-skilled workers and capital is higher than the elasticity of substitution between high-skilled workers and capital.

In addition to the papers already mentioned above, our paper is also related to Baqaee and Farhi [16]. Both papers focus on the impacts of technology shocks that can be divided into two main categories: a pure technology effect and a reallocative effect. However, our model can be thought of as a generalization that provides explanations for how given shocks have different implications depending on the source of the shock, the substitution versus complementarities in various parts of the downstream production network, the level of productivities, and the weights various inputs in different production functions throughout the economy. In other words, we investigate the factors that places one type of shock into one category, and another shock into another category. For instance, our model provides explanations for how the technological improvements that result in exactly the same productivity change in a resource good instead of the automation sector might have different implications. In particular, our model allows us to interpret the changes in the Domar weights due to the substitution effects. A more technical difference is that, differently from Baqaee and Farhi [16], (and also Baqaee [13], Grassi [32], and Bigio and La'O [22]), we focus on competitive equilibrium, and Hulten's Theorem (appropriately extended) holds in our set-up.[6]

[6]One could extend our analysis to an imperfectly competitive model or with other sorts of production functions, but both topics are beyond the scope of the present paper.

Lastly, our study is also loosely related to the literature on the endogenous formation of production networks, such as Acemoglu and Azar [2], Carvalho and Voigtländer [26], and Oberfield [43]. In our model, firms do not choose their set of suppliers, yet the production network changes following technological advances, and so there is a form of endogenous network. For instance, as the productivity of an automation good sector rises, it can start to supply goods to other sectors, forming new links and increasing the interconnectedness of the economy.

The remainder of the paper is organized as follows. In Section 2, we introduce the production network model and discuss the network interactions in our setup. Section 3 includes our analysis of the impact of technology shocks and automation decisions on wages and income inequality in a three-sector model. In Section 5, we extend our analysis to an n-sector model and characterize the implications of technological change. In Section 6 we provide results on how the impacts of technological changes on households' consumption and labor productivity differ by sector. In Section 4 we discuss the network characteristics and transition to automation. In Section 7 we conclude.

2 The Model

2.1 Production Processes

We consider a perfectly competitive economy consisting of a set of $N = \{1, ..., n\}$ sectors/firms, with a representative firm denoted by i.

In what follows we use the terms firm and sector interchangeably, although clearly one can distinguish them if one prefers – it will not make a difference in the analysis of the model. We also abstract away from the use of capital in our analysis. It can be added but is of no particular consequence in our model as will become clear. Instead, we focus entirely on the interactions of labor with other goods in production processes.

In particular, a firm uses labor in two forms: high-skilled and low-skilled. We denote the amount of high and low skilled labor used by firm i by H_i and L_i, respectively. The difference is that low-skilled labor can be substituted for by the goods produced in automation goods sectors (i.e. robots, software, etc), while high-skilled labor does not have a direct substitute. One can simply think of defining high and low skills in this way - the words "high" and "low" have no other particular meaning in our model. We use the terminology since they do correspond to higher and lower skills in the data - as new technologies tend to enhance high skilled labor while replacing more routine tasks that are associated with lower skills. For instance, high skilled labor might include management, R&D, and some engineering, while low-skilled labor would include warehouse workers, drivers, manufacturing line workers, various secretarial workers, customer service workers, and so forth. Moreover, there are tasks which are performed by low-skilled labor in each firm, but do not have a direct substitute.

We thus think of different inputs having different roles in the production process. In

particular, some tasks that low-skilled labor perform can be replaced by some input good - e.g., a box packer can be replaced by a robot. While there are other input goods, such as the boxes, that are used in the production process but do not substitute for labor. We thus divide the inputs in the production by firm i by whether they can substitute for some low skilled labor, or whether they do not:

- $j \in a_i$: "automation" inputs, which can substitute for low-skilled labor in some tasks (e.g., software, industrial robots),
- $j \in n_i$: "non-automation" inputs, the goods from another sector that do not replace labor (e.g., electricity, raw materials).

We let Y_i be the total production of each $i \in N$, and A_i^P be a productivity multiplier. We let X_{ij} denote the amount of input from $j \in a_i \cup n_i$ that i uses in production, where the sets of a_i and n_i are sector specific. We let L_{i0} denote the amount of low-skilled labor that firm i employs outside of automatable tasks, and L_{ij} denote the amount of low-skilled labor that is used that can be replaced by the automation input j. In addition, A_j^Q represents the productivity of good j. The production function of each representative firm $i \in N$ has the form:

$$Y_i = A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j \in a_i} [L_{ij} + A_j^Q X_{ij}]^{\alpha_{ij}^L} \right] \prod_{j \in n_i} (A_j^Q X_{ij})^{\alpha_{ij}^n} \tag{1}$$

Production exhibits constant returns to scale and we take the exponents to be non-negative and to sum to 1:

$$\alpha_i^H + \alpha_{i0}^L + \left(\sum_{j \in a_i} \alpha_{ij}^L \right) + \left(\sum_{j \in n_i} \alpha_{ij}^n \right) = 1. \tag{2}$$

We further assume that there is always some use for low and/or high skilled labor in each sector. Thus, $\left(\sum_{j \in a_i} \alpha_{ij}^L \right) + \left(\sum_{j \in n_i} \alpha_{ij}^n \right) < 1$ holds, which implies that $\alpha_{i0}^L > 0$ and/or $\alpha_i^H > 0$ holds $\forall i \in N$.[7] We denote the set producers for which their goods can be potential substitutes for labor somewhere in a given economy by $K = \bigcup_{i \in N} a_i$, so that $K \subset N$.

2.2 Labor Supply

Each type of labor is supplied perfectly inelastically. The total available supply of low-skilled and high-skilled labor are constant and denoted by L and H, respectively.[8]

[7] This assumption implies that all elements of the Leontief inverse of a given input-output matrix is non-negative.

[8] A dynamic environment for the labor market where the supply of L and H lves over time via skill training is a future research direction.

2.3 Consumption

In a given economy, the good produced in any firm i can be used for consumption or can be used as an intermediate good in other firms.

Letting C_i denote the amount of production of firm i used for consumption, the total production of firm i satisfies:

$$Y_i = \sum_{j \in N} X_{ji} + C_i.$$

The final consumption goods are evaluated by a total utility, or equivalently aggregated into a single final good, that takes a Cobb-Douglas form:

$$C = \prod_{i \in N} \left(A_i^Q C_i \right)^{\beta_i},$$

where $\beta_i > 0$ for all $i \in N$ and $\sum_{i \in N} \beta_i = 1$.

C^L and C^H denote the consumption of the final good by low- and high-skilled labor, respectively. Thus, total consumption is given by:

$$C = C^L + C^H$$

2.4 Competitive Equilibrium

At competitive equilibrium, the representative firm in each sector maximizes profit, and market clearing conditions hold for each good and each type of labor.

In particular:[9]

DEFINITION 1 *A competitive equilibrium is a set of prices* $\{p_i\}_{i \in N}$*, wages* $, w_L$ *and* w_H*, and quantities* $\left\{ Y_i, \{L_{ij}\}_{j \in K \cup 0}, H_i, C_i^L, C_i^H, \{X_{ij}\}_{j \in N} \right\}_{i \in N}$ *such that*

1 *Firms maximize profits: For each* $i \in N$*,* $\{L_{ij}\}_{j \in K \cup 0}, H_i, \{X_{ij}\}_{j \in N}$ *solve*

$$\max_{\{L_{ij}\}_{j \in K \cup 0}, H_i, \{X_{ij}\}_{j \in N}} \quad p_i \left(A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j \in a_i} \left(L_{ij} + A_j^Q X_{ij} \right)^{\alpha_{ij}^L} \right] \left[\prod_{j \in n_i} (A_j^Q X_{ij})^{\alpha_{ij}^n} \right] \right) - \left(\sum_{j \in K \cup 0} w_L L_{ij} + w_H H_i + \sum_{j \in N} p_j X_{ij} \right).$$

2 $\left\{ C_i^L \right\}$*, and similarly for* $\left\{ C_i^H \right\}$*, solve the utility maximization problem of the representative worker*

$$\max_{\left\{ C_i^L \right\} : \sum_i p_i C_i^L \leq w_L L} \prod_{i \in N} \left(A_i^Q C_i^L \right)^{\beta_i}.$$

[9] With constant returns to scale, profit maximization in equilibrium implies that there are 0 profits, and so we do not specify who earns the profits, as those shares are irrelevant and would just add more notation.

3 Markets clear:

- *goods:* $Y_i = \sum_{j \in N} X_{ji} + C_i^L + C_i^H$,
- *and labor markets:* $L = \sum_{i \in N} \sum_{j \in K \cup 0} L_{ij}$ *and* $H = \sum_{i \in N} H_i$.

We remark that 2 is exactly equivalent to having a representative firm in a perfectly competitive "final goods" market sell bundles of goods that solve

$$\max_{\{C_i\}} p_f \prod_{i \in N} \left(A_i^Q C_i \right)^{\beta_i} - \sum_{i \in N} p_i C_i,$$

and then having the low and high-skilled workers consume the bundled final goods that are multiples of $\prod_{i \in N} \left(A_i^Q C_i \right)^{\beta_i}$ such that they exhaust their budgets: $p_f C^L = w_L L$ and $p_f C^H = w_H H$.

This alternative formulation allows us to let the price of the final good Q be the numeraire ($p_f = 1$), which enables us to highlight relative changes of the low-skilled labor wage, w_L, and high-skilled labor wage, w_H.

There exists a competitive equilibrium in our model by standard arguments.

In fact, if $\beta_i > 0$ for each i, then given that all goods are consumed in equilibrium, and must be in relative proportions that are determined by the β_is and relative prices, it can be shown that there exists a unique equilibrium set of prices $\{p_i\}_{i \in N}$, wages w_L and w_H, and quantities $\{C_i, \}_{i \in N}$. So, for Sections 5 -4 we maintain the assumption that $\beta_i > 0$ for each i, while in the next section we allow for some 0's to simplify some examples.

2.5 The Equilibrium Level of Automation and the Input-Output Network

Let us now define some basic notation that will be used to track the input-output network in this world. That network depends on how much automation has taken place in all of the automatable tasks - the $j \in a_i$ across i's.

In equilibrium, a firm i uses a combination of automation good and low-skilled labor in each of its automatable tasks $j \in a_i$. Accordingly, we let t_{ij} denote the equilibrium share of expenditures on automation good $j \in a_i$ in sector $i \in N$, where $t_{ij} = \frac{p_j X_{ij}}{w_L L_{ij} + p_j X_{ij}}$. Therefore, $t_{ij} \in [0, 1]$ might be thought of as the rate of automation of task j in sector i. The equilibrium share of expenditures on labor in an automatable task $j \in a_i$ in sector $i \in N$ is then equal to $1 - t_{ij} = \frac{w_L L_{ij}}{w_L L_{ij} + p_j X_{ij}}$.

We then define two different input-output matrices. One considers all of the possible structural relationships if automation were complete in the economy, and the other represents a current equilibrium (or actual) input-output network.

The "potential" or "structural" input-output network is denoted by $\Omega^S_{n \times n} = \Omega^n_{n \times n} + \Omega^L_{n \times n}$. First, $\Omega^n_{n \times n}$ is the component of the structural input-output network that summarizes the

input-output linkages in the economy via the non-automatable tasks. The ij^{th} entry of the $\Omega^n_{n\times n}$ is the weight of non-automatable task $j \in n_i$ in the production function of firm i ($[\Omega^n]_{ij} = \alpha^n_{ij}$). Second, $\Omega^L_{n\times n}$ is the other component of the structural input-output network $\Omega^S_{n\times n}$ that summarizes the potential linkages via automatable tasks, where the ij^{th} entry of the $\Omega^L_{n\times n}$ is equal to α^L_{ij}.

The structural input-output network and the equilibrium input-output network might differ, depending on the incompleteness of automation. The equilibrium input-output network is denoted by $\Omega_{n\times n} = \Omega^n_{n\times n} + \Omega^a_{n\times n}$, where the second component $\Omega^a_{n\times n}$ is determined at equilibrium such that if good j replaces some of the low-skilled labor in task j in firm i at equilibrium, then $[\Omega^a]_{ij} = t_{ij}\alpha^L_{ij}$.

Therefore, one might consider the structural input-output network as the extreme case network where all substitution is complete and only automation goods are used in each automatable task in the economy. On the other hand, the actual level interconnectedness in the economy is summarized by the Leontief inverse matrix at equilibrium, which is given by:

$$(I - \Omega)^{-1} = (I - \Omega^n - \Omega^a)^{-1} \tag{3}$$

The Leontief inverse matrix represents the dependencies across sectors at equilibrium. Broadly, if there exists a directed path between industry i and industry j at equilibrium, then the ij^{th} entry of the Leontief inverse matrix is positive, and it is zero otherwise. Thus, switching to automation in certain tasks increase the connectivity among industries and creates additional direct and indirect network effects.

In summary, the equilibrium input-output network has the following properties:

i) $\Omega_{ij} = t_{ij}\alpha^L_{ij} \in [0, \alpha^L_{ij}]$ for all ij s.t. $j \in a_i$.

ii) $\Omega_{ij} = \alpha^n_{ij}$ for all ij s.t. $j \in n_i$.

iii) $\Omega_{ij} = 0$ for all ij s.t. $j \in N \setminus a_i \cup n_i$.

As a result, Ω is the equilibrium input-output network, where Ω_{ij} is the reliance of good i on good j in its production process. Therefore, as the share of expenditures on automation good rise, the production network becomes denser and the interconnectedness in the economy gets stronger. We explore the equilibrium levels of the t_{ij}s below.

3 Technological Changes, Growth, and Income Inequality in a Three-Sector Economy

In this section, we study how technological changes affect automation decisions, GDP growth, and income inequality in a three-sector economy consists of a non-automation good sector, an automation good sector, and a "final good" sector; denoted by n, a, and f, respectively. The difference between how the resource goods and the automation goods enter into the final good production relies on the substitutability of inputs. While the good produced

in the automation sector is a substitute for the low-skilled labor in final good sector, the non-automation good is not.

The Cobb-Douglas production functions are:

$$Y_a = A_a^P (L_a)^{\alpha_a^L} (H_a)^{\alpha_a^H} \tag{4}$$

$$Y_n = A_n^P (L_n)^{\alpha_n^L} (H_n)^{\alpha_n^H} \tag{5}$$

$$Y_f = A_f^P (L_f)^{\alpha_f^L} (H_f)^{\alpha_f^H} [L_{fa} + A_a^Q X_{fa}]^{\alpha_{fa}^L} (A_n^Q X_{fn})^{\alpha_{fn}^n}. \tag{6}$$

In this three sector setting, we simplify things and set $\beta_n = \beta_a = 0$ while $\beta_f = 1$, and so the only good that is directly consumed is the "final good". We also normalize $A_f^Q = 1$, and hence we have $Y_f = C = C^L + C^H$.

3.1 Prolonged Automation, Wage Adjustments, and Inequality

First, we analyze how technological improvements end up causing a general equilibrium effect in terms of wage adjustments, which provides for a continuous and prolonged substitution, and also increases wage inequality along the way.

We start with an example with just two sectors to make things transparent, so we are dropping n for now, so that the final good sector uses only the automation good as an intermediate input.

Example 1.

The production functions are as follows:
$Y_a = A_a^P L_a^{0.5} H_a^{0.5}$
$Y_f = H_f^{0.5} (L_{fa} + A_a^Q X_{fa})^{0.5}$
We set $\alpha_{a0}^L = \alpha_a^H = \alpha_{fa}^L = \alpha_f^H = 0.5$, and $A_f^P = L = H = 1$.

As shown in Figure 1.a, for initial very small productivity values for the automation sector ($A_a^P A_a^Q$), there is no automation in firm f and, thus, firm a produces zero output at equilibrium. Here automation is just not worthwhile yet.

Eventually, following sufficient technological improvements, so that the productivity of the automation sector reaches to a sufficiently high level ($A_a^P A_a^Q \geq A^*$), firm f starts to use automation good and firm a starts to produce positive amounts of output.

Given the Cobb-Douglas production function with constant returns to scale and zero profit conditions, the final good producer's total spending for task a at equilibrium is always equal to $\alpha_a^L Y_f$, where the final good price is normalized to 1. Accordingly, for the intermediate levels of productivity of automation sector such that $A^* < A_a^P A_a^Q < A^{**}$, firm f spends $0 < t_{fa} < 1$ fraction of its total cost for task a on automation and $(1 - t_{fa})$ fraction of its total cost for task a on low-skilled labor. Accordingly, firm f spends $t_{fa} \alpha_a^L Y_f$ on automation goods.

In this intermediate range for productivity, as the automation sector becomes more productive and/or the quality of the automation good rises, firm f's demand for the automation

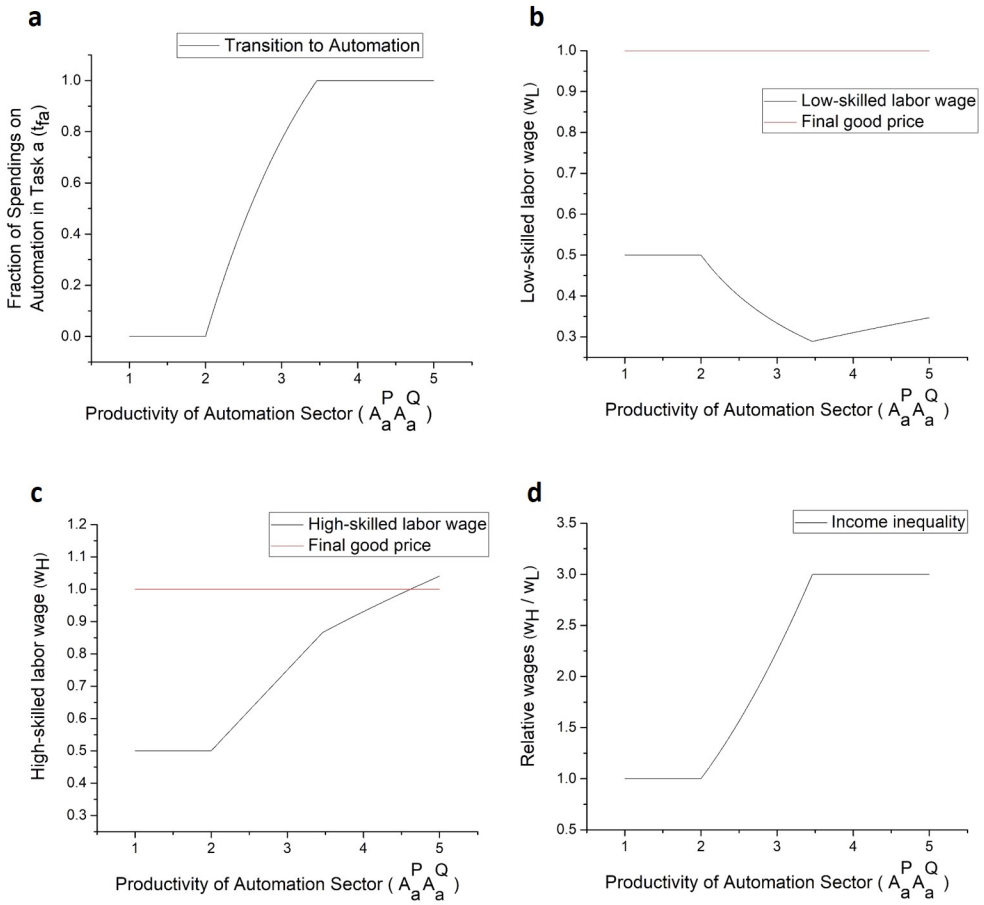


Figure 1: Automation in final good sector and its impact on wages in Example 1

good rises and its demand for low-skilled labor falls. This is partly offset by a falling low-skilled wage, and a rising high-skilled wage. The change in the fraction of its spending on automation, t_{fa}, shows us firm f's transition to automation.

As the productivity increases further, firm f eventually switches to automation in full and uses zero labor in task a once the productivity of the automation sector reaches a threshold level $A_a^P A_a^Q \geq A^{**}$.

Figure 1.a. summarizes the transition to automation in firm f and depicts how the parameter t_{fa} changes, and Figure 1.b, 1.c, and 1.d depicts how absolute and relative wages change.

As depicted in Figure 1,[10] there are essentially two key phases of automation (beyond a degenerate one where the automation good is so inefficient not to be used in the automatable

[10]Given the economy in Example 1, we consider that the initial equilibrium before any shock hits is such that the final good producer uses only labor in the automatable task. Accordingly, we take a sufficiently low initial value for $A_a^P A_a^Q$, and set it to $A_a^P A_a^Q = 1$ and see how transition to automation occurs and wages change as $A_a^P A_a^Q$ rises in between $1 \leq A_a^P A_a^Q \leq 5$.

task). In that degenerate pre-automation phase, there is no reallocation of labor, and the relative wage $\left(\frac{w_H}{w_L}\right)$ remains constant. The first key phase is when automation takes place and the final good producer uses both the automation input and low-skilled labor in combination. As this phase progresses, the demand for low-skilled labor decreases and the productivity gains that arise due to automation are captured by the high-skilled labor. Eventually, the economy has fully automated, and then any technological shock has only the classical input-output effect (i.e., wages and consumption rise), and the shocks have no impact on relative wages.

Phases	w_L	w_H	$\frac{w_H}{w_L}$
$A_a^P A_a^Q \leq 2$ (pre-automation)	$\frac{1}{2}$	$\frac{1}{2}$	1
$2 < A_a^P A_a^Q < 2\sqrt{3}$ (automation phase)	$\frac{1}{A_a^P A_a^Q}$	$\frac{A_a^P A_a^Q}{4}$	$\frac{\left(A_a^P A_a^Q\right)^2}{4}$
$A_a^P A_a^Q \geq 2\sqrt{3}$ (post-automation phase)	1	3	3

Table 1. The changes in automation, wages and income inequality in Example 1

Next, we provide a result about the impact of technological changes on wages and income inequality, that formalize the numerical example above in the more general case.

Let $s_L = \frac{w_L L}{w_L L + w_H H}$ and $s_H = \frac{w_H H}{w_L L + w_H H}$ be the share of low- and high-skilled labor, respectively.

PROPOSITION **1** *There exists two threshold levels of productivity of automation sector A^* and A^{**} such that*

i) (pre-automation) for $A_a^P A_a^Q < A^*$, low-skilled labor wage and high-skilled labor wage grow at the same rate, and hence the income inequality $(\frac{w_H}{w_L})$ remains constant,

ii) (transition to automation) for $A^* \leq A_a^P A_a^Q \leq A^{**}$, high-skilled labor wage rises at a higher rate than the low-skilled labor wage, and hence the income inequality increases,

iii) (post-automation) for $A_a^P A_a^Q > A^{**}$, low-skilled labor wage and high-skilled labor wage grow at the same rate, and hence the income inequality remains constant. Moreover, the constant growth rate of wages within the productivity range $A_a^P A_a^Q > A^{**}$ is weakly higher than the constant growth rate of wages within the productivity range $A_a^P A_a^Q < A^*$.

The growth of low-skilled labor wage, high-skilled labor wage, and income inequality (or relative wage) in response to technological changes satisfy the following conditions in pre-automation, transition to automation, and post-automation phases, respectively:

$$dlnw_L =: \begin{cases} dlnA_f^P + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) \\ dlnA_f^P \ + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) \ + t_{fa}\alpha_{fa}^L dln\left(A_a^P A_a^Q\right) \ - \frac{s_H}{\alpha_a^H} dln\left(A_a^P A_a^Q\right) \\ dlnA_f^P \ + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) + \alpha_{fa}^L dln\left(A_a^P A_a^Q\right) \end{cases}$$

$$dlnw_H =: \begin{cases} dlnA_f^P + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) \\ dlnA_f^P \ + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) + t_{fa}\alpha_{fa}^L dln\left(A_a^P A_a^Q\right) \ + \frac{s_L}{\alpha_a^H} dln\left(A_a^P A_a^Q\right) \\ dlnA_f^P \ + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) + \alpha_{fa}^L dln\left(A_a^P A_a^Q\right) \end{cases}$$

$$dln\left(\frac{w_H}{w_L}\right) =: \begin{cases} 0 \\ \frac{dln\left(A_a^P A_a^Q\right)}{\alpha_a^H} \\ 0 \end{cases}$$

where $s_L = \frac{w_L L}{w_L L + w_H H}$ *and* $s_H = \frac{w_H H}{w_L L + w_H H}$ *are the share of low- and high-skilled labor, respectively; and* $t_{fa}\alpha_{fa}^L$ *is the Domar weight of the automation good producer firm a* $\left(t_{fa}\alpha_{fa}^L = \frac{p_a X_{fa}}{p_f Y_f}\right)$.

Proposition 1 provides various predictions for the change in labor income.

First, as we discussed earlier, Proposition 1 shows that the income inequality remains same during the pre-automation and post-automation phases. It follows from the fact that any shock in these stages does not cause reallocation of labor. However, income inequality rises if and only if the economy is in the automation phase. Second, during the automation phase of the economy, under the numeraire final good price ($p_f = 1$), high-skilled labor wage always increases, yet low-skilled labor wage might increase or decrease depending on whether productivity effect or substitution effect dominates. In this phase, the technology shock to automation sector decrease the wage of low-skilled labor while increasing the wage of high-skilled labor. So, following only a shock to the automation-sector, not only the income inequality rises but also the actual low-skilled wage decreases. Moreover, the key parameters determining the change in wage gap are the weights of the high-skilled labor tasks and low-skilled labor tasks in each sector. As α_a^H rises, the wage gap increases more since the good that replaces low-skilled labor is more high-skilled intensive in that case. In addition, as α_a^H rises, the growth in relative wage in this stage decreases. These together imply that the transition period takes longer for higher values of α_a^H.

Following this result, in the next section, we analyze how technology shocks in different sectors have different implications for net-output growth.

3.2 Technological Changes and Growth

We now bring the non-automation sector back and analyze the implications of technological improvements on net-output growth. This allows us to contrast how technological improvements in the non-automation sector affect final output compared to how the automation input affects output.

We start with Example 2 below. In this example, we set the productivity of the final good sector to $A_f^P = 1$ and weight of tasks in each firm to $\frac{1}{2}$. We also set $L = H = 1$.

Example 2.

The production functions are as follows:

$Y_n = A_n^P L_n^{0.5} H_n^{0.5}$,

$Y_a = A_a^P L_a^{0.5} H_a^{0.5}$,

$Y_f = (L_f + A_a^Q X_{fa})^{0.5} \left(A_n^Q X_{fn}\right)^{0.5}$.

Example 2 illustrates how the net-output (or total consumption) changes as the technology in automation sector or resource good sector improves. As we discussed earlier, before

any automation occurs in firm f ($A_a^P A_a^Q \leq A^*$), firm a produces zero output. On the other hand, during automation, technology shocks to the automation sector have additional network effects in the economy such as changes in relative wage, which alter the growth rate of consumption. Following the transition period, the impact of technology shocks on consumption growth in the post-automation phase can be explained by the classical input-output effect such that the wages and net-output increase at the same rates and relative wage remains constant. As a result, the GDP growth rate due to an increase in productivity of automation sector depends on the phase of the economy when the shock hits.

Another observation from Example 2 is that the macroeconomic consequences of TFP shock also depend on which sector is hit by the given shock.

i) in the pre-automation phase, any shock to automation sector has no implications on total consumption, whereas any shock to the non-automation good sector increases the consumption level,

ii) during automation, the same shocks hitting both sectors have different implications for final output growth,

iii) in the post-automation phase, the same shocks hitting both sectors have exactly the same final output growth effect.

Figure 2.a, 2.b, and 2.c provide the comparison of growth levels following technological improvements in resource good sector and automation sector.

Figure 2.d and 2.e show the level of percentage change in consumption following changes in productivity $A_a^P A_a^Q$ or $A_n^P A_n^Q$, for everything else constant. In addition, Figure 2 captures three different cases for the constant productivity parameters. As shown in Figure 2, the change in the selected level of the constant technology parameter $A_n^P A_n^Q$ do not alter the threshold level productivity A^* and A^{**}, and have only the level effects on total consumption. This can be seen in Figure 2.d and 2.e where the growth rate is equal for different constant values for productivity. For this reason, we select the same constant parameters for $A_a^P A_a^Q$ and $A_n^P A_n^Q$ in order to investigate the cases in a symmetric way.

As a result, Example 2 illustrates a case where the exactly the same changes in productivities in different sectors have different implications on total consumption, depending on whether the goods are complements or substitutes for labor.

Next we provide a result for the more general case on the implications of technological changes on consumption growth in three-sector model.

PROPOSITION **2** *In a given three-sector economy with constant weights, labor supply, the growth in total consumption (or net-output) satisfies the following as a function of changes in productivities:*

$$\begin{cases} dlnC = dln(A_f^P) + \alpha_{fn}^n dln(A_n^Q A_n^P) & \text{if } A_a^Q A_a^P < A^* \\ dlnC = dln(A_f^P) + \alpha_{fn}^n dln(A_n^Q A_n^P) + t_{fa}\alpha_{fa}^L dln(A_a^Q A_a^P) & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ dlnC = dln(A_f^P) + \alpha_{fn}^n dln(A_n^Q A_n^P) + \alpha_{fa}^L dln(A_a^P A_a^Q) & \text{if } A_a^Q A_a^P > A^{**} \end{cases}$$

where $0 < t_{fa} < 1$ is the ratio of firm f's spending on automation good to its total spending on task a at equilibrium, and $t_{fa}\alpha_{fa}^L$ is the Domar weight of the automation good

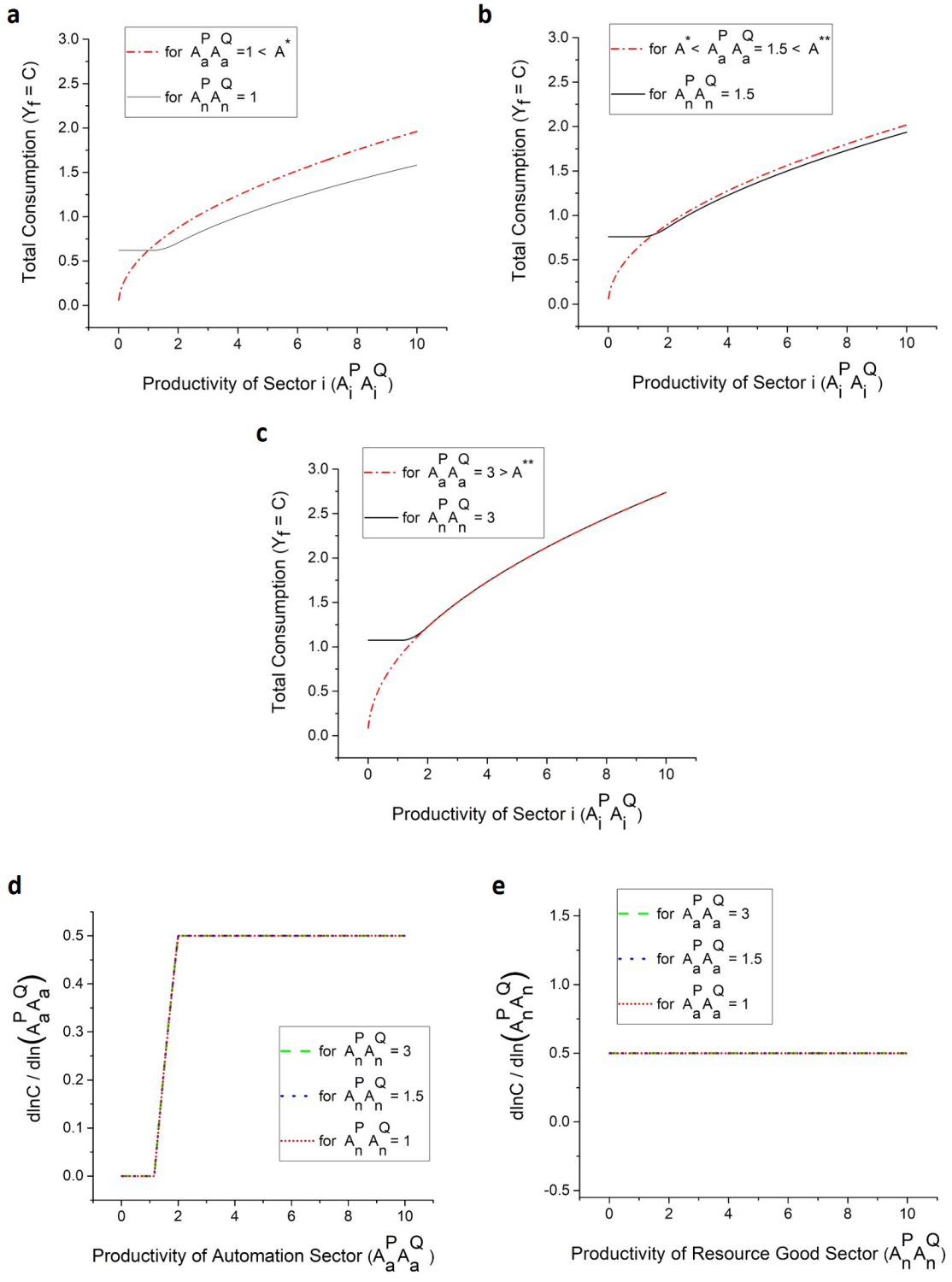


Figure 2: Technology shocks and the changes in total consumption in Example 2

producer firm a $\left(t_{fa}\alpha^L_{fa} = \frac{p_a X_{fa}}{p_f Y_f}\right)$.

Proposition 2 shows that the impact of technology shocks on consumption growth depends on both the sector hit by the shock and the phase of the economy. During the automation phase of the economy, the growth level is not constant and it instead is increasing in t_{fa}. Therefore, the same rate of increase in technology of automation sector results in lower growth in the early stages of automation and higher growth in the later stages of automation or after firm f switches to automation in full.

Hulten [37] provides a key result on the measurement of the macroeconomic impact of the microeconomic shocks. Hulten's Theorem shows that in competitive economies following a total factor productivity (TFP) shock to producer i (a shock to A_i), the following condition holds: $dlnC = m_i dlnA_i$. where $C = \sum_{i \in N} C_i$ is the total net-output in the economy, and A_i is the TFP of producer i. The term m_i is the Domar weight of producer i; that is, $m_i = \frac{p_i Y_i}{\sum_{i \in N} p_i C_i}$, where p_i is the price of good i, Y_i is the total production of firm i (so, $p_i Y_i$ is the total sales of producer i), and $\sum_{i \in N} p_i C_i$ is equal to the total GDP.

The key implication of Hulten's Theorem is that to a first-order approximation in logs, one can ignore the full details of the network structure and use the observable sales shares of each firm/industry to derive the effects of technology shocks on net-output.

Proposition 2 implies that a version of Hulten's Theorem extends to our setup even though the economy enters into a transition path with changing growth levels.

An important difference in our setting from the usual Hulten Theorem case is that, as one can see from Eq. (40) in the Appendix, when we consider the second order impacts of the technological changes – which are captured by the changes in the Domar weights – the second-order term following the productivity shock in automation sector would depend on the high-skilled and low-skilled labor supply, weights of tasks in each production process and the initial level of productivity in automation sector. Therefore, first order approximation might be misleading during the automation phase, while the theorem applies with no approximation error in other phases.

4 The Rate of Automation, Changing Domar Weights and Total Consumption

Besides the fact that a version of Hulten's Theorem extends to our setting, our model allows us to quantify the changes in the Domar weights following technology shocks, and also enables us to provide further predictions as to conditions under which the final good sector will switch to automation, how long the transition to automation phase lasts, and how the impact of technological advancements on total consumption depends on alternative uses for labor.

4.1 How the Impact of a Technological Advance Depends on Alternative Uses for Low-Skilled Labor

In this part, we discuss how total consumption changes during the automation phase while capturing the changes in Domar weights. In order to do that, we revisit Example 1 and consider discrete changes in the technology in different phases.

Example 3.

The production functions are as follows:

$Y_n = A_n L_n$

$Y_a = A_a H_a$

$Y_f = (L_f + X_{fa})^{\alpha} X_{fn}^{1-\alpha}$

We set $L = H = 1$, and $\alpha = \frac{1}{2}$

In this economy, the equilibrium can be understood by maximizing

$Y_f = (L_f + A_a)^{\alpha} (A_n(1 - L_f))^{1-\alpha}$

Then, maximizing level of L_f is given by:

$$\begin{cases} L_f = \frac{1-A_a}{2} & 0 < A_a < 1 \\ L_f = 0 & A_a \geq 1 \end{cases}$$

$$\begin{cases} Y_f = (\frac{1+A_a}{2})^{0.5} (A_n)^{0.5} & 0 < A_a < 1 \\ Y_f = (A_a)^{0.5} (A_n)^{0.5} & A_a \geq 1 \end{cases}$$

For constant A_n, the rate of change in Y_f does not depend on the level of A_n. We set $A_n = 1$. The Domar weight of the automation sector and the rate of change in total consumption are as follows:

$$\begin{cases} \Delta lnC = \frac{1}{2} ln \left(\frac{1+zA_a}{1+A_a} \right) & m_a = 0.5 t_{fa} = \frac{A_a}{1+A_a} \text{ for } 0 < A_a < 1 (automation) \\ \Delta lnC = m_a lnz & m_a = 0.5 \text{ for } 0 < A_a < 1 (post - automation) \end{cases}$$

Table 2 below provides a case for Example 3. In this case, we show that $\Delta lnC < m_a \Delta lnA_a$ holds during the automatin phase, whereas $\Delta lnC = m_a \Delta lnA_a$ holds at the post-automation phase.

Cases	$\Delta ln(A_a)$	Domar Weight of a (Initial)	ΔlnC
Case i) $A_a = 0.5$ rises to $A_a = 1$ (automation)	$ln2$	$m_a = \frac{1}{3}$	$0.1438 < \frac{ln2}{3} = 0.2310$
Case ii) $A_a = 2$ rises to $A_a = 4$ (post-automation)	$ln2$	$m_a = \frac{1}{2}$	$\frac{ln1.5}{2}$

Table 2. The rate of change in total consumption in Example 3

Case i is an illustration of automation phase and case ii is an illustration of post-automation phase. By keeping the rate of change in technology constant, we compare the rate of change in total consumption in these two cases.

As shown in Table 2, following technological changes in automation sector in the post-automation phase, the rate of change in total consumption is given by the rate of change in productivity in automation sector times the Domar weight of automation sector ($m_a \Delta ln(A_a)$). This effect is exactly the same effect as the growth effect for small changes in technology. The reason is that during the post-automation phase, the only effect is the classical input-output effect with no reallocation effect (or no change in labor allocation and Domar weights). The classical input-output effect is the influence of a sector on total consumption whenever the resource allocation is fixed. On the other hand, in this example, during the automation phase, the rate of change in total consumption is lower than the classical input-output effect, and the extra effect, which is negative in this case comes from the reallocation of resources.

Our analysis so far covers two facts on how technological changes affect total consumption. The first one is that the growth effect of certain type of technological changes increase over time as automation goods replace labor and are used extensively in the economy. The second one is that during the automation phase, factors are reallocated as automation replaces labor in certain tasks and Domar weights change, and thus, the classical input-output effects due to the technological changes are accompanied by an additional effect, called substitution effect.

Next, we consider production technologies different than the Cobb-Douglas form that we studied so far. We now illustrate how the impact of technological advances depends on how useful low-skilled labor is in other sectors for other forms of production functions. If low-skilled labor is very productive elsewhere, then the technological advances in the automation technology have a higher impact on total consumption.

Example 4.

The production functions are:
$Y_n = A_n L_n$
$Y_a = A_a$
$Y_f = A_f H_f^{\alpha} (L_{fa} + X_{fa})^{1-\alpha} + X_{fn}$
Also, $L = H = 1$.

Here, we simplify things by having X_{fn} enter into the production function of the final good in an additively separable way rather than in a Cobb-Douglas form. The expressions have all of the same signs with the Cobb-Douglas form and we report those in the appendix,, but this simplifies the expressions substantially.

We also simplify the automation process not to use any labor at all, so that its increase does not impact the production of the other goods other than via the technological advance.

In this simple economy, the equilibrium can be understood by maximizing[11]

$$Y_f = A_f (L_f + A_a)^{1-\alpha} + A_n (1 - L_f),$$

where we use that $L = H = 1$.

[11]It is straightforward that the competitive equilibrium in this simple economy is equivalent to a planner maximizing total final good production.

The maximizing L_f is

$$\begin{cases} L_f = 1 & \text{if } A_a < \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \\ L_f = \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - A_a & \text{if } \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \leq A_a \leq \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} \\ L_f = 0 & \text{if } A_a > \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}. \end{cases}$$

and the corresponding Y_f is then

$$\begin{cases} Y_f = A_f(1+A_a)^{1-\alpha} & \text{if } A_a < \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \\ Y_f = A_f\left(\frac{(1-\alpha)A_f}{A_n}\right)^{(1-\alpha)/\alpha} + A_n\left(1 + A_a - \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}\right) & \text{if } \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \leq A_a \leq \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} \\ Y_f = A_f(A_a)^{1-\alpha} + A_n & \text{if } A_a > \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}. \end{cases}$$

It then follows that the corresponding $\frac{\partial Y_f}{\partial A_a}$ is:

$$\begin{cases} \frac{\partial Y_f}{\partial A_a} = (1-\alpha)A_f(1+A_a)^{-\alpha} & \text{if } A_a < \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1, \\ \frac{\partial Y_f}{\partial A_a} = A_n & \text{if } \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha} - 1 \leq A_a \leq \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}, \\ \frac{\partial Y_f}{\partial A_a} = (1-\alpha)A_f(A_a)^{-\alpha} & \text{if } A_a > \left(\frac{(1-\alpha)A_f}{A_n}\right)^{1/\alpha}. \end{cases}$$

Here, we see directly that during automation the rate at which overall production changes in response to technological advances in the automation sector are proportional to the usefulness of labor in the non-automation sector - exactly in this example. That is, $\frac{\partial Y_f}{\partial A_a} = A_n$.

next, we compare the changes in total consumption in different phases. Following the same rate of increase in A_a such that A_a becomes zA_a, the rate of change in Y_f depends on the phase of the economy. The rate of change in $C = Y_f$ during the automation and post-automation phases are given by:

$$(\Delta lnC)^{autom} = ln\left(\frac{1+4A_n^2(1+zA_a^{autom})}{1+4A_n^2(1+A_a^{autom})}\right)$$

$$(\Delta lnC)^{post-autom} = ln\left(\frac{(zA_a^{post-autom})^{\frac{1}{2}}+A_n}{(A_a^{post-autom})^{\frac{1}{2}}+A_n}\right)$$

Example 4 shows that the rate of change in total consumption rises as the productivity of labor on the alternative uses rises. This result is different than the Cobb-Douglas economy which we extensively study in our model. In a Cobb-Douglas economy, substitution effect depends on factor shares, or in other words the intensity of factors in production processes instad of productivities of other sectors.

These two examples provide an important lens into why it can be that substitution for labor can have very different effects depending on the alternative uses for labor. This provides new insights into the Solow Paradox and the findings of Brynjolfsson, Rock, and Syverson [24], for instance.

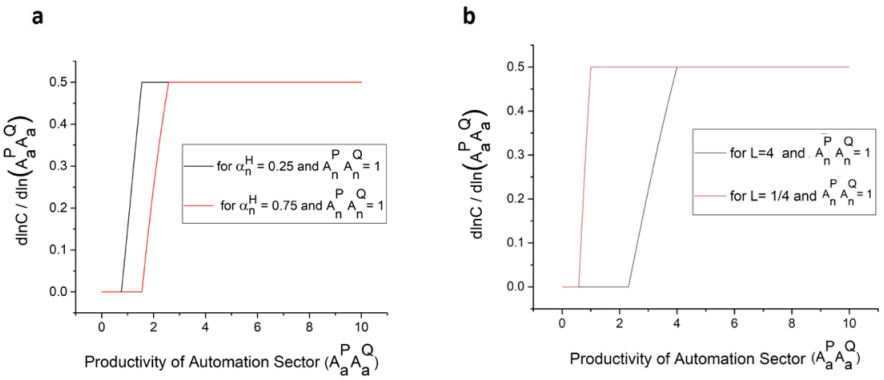


Figure 3: Transition to automation and net-output levels for different levels of $\frac{L}{H}$ or task weights in Example 2

4.2 The Rate of Automation

In this part, we analyze how the decision to automate and the rate of automation depends on the sector level task intensities and productivities.

We showed that there exists thresholds A^* and A^{**} which define the automation phase. In this part, we analyze how these thresholds A^* and A^{**} are determined and discuss how the firm characteristics such as labor intensity, and the network characteristics such as the intensity of each task in firm f, affect the automation decision of firm f.

First, we revisit Example 2 and consider two different values for $\frac{L}{H}$. As one can see from Table 3 below, the level of $\frac{L}{H}$ also plays a role on the growth level during automation.

Cases	Phases	$Y_f = C$
Case 1. $L = 4$ and $H = 1$	$A_a^P A_a^Q \leq \frac{4}{\sqrt{3}}$ (pre-automation phase)	$\left(\frac{4}{27}\right)^{\frac{1}{4}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} L^{\frac{3}{4}} H^{\frac{1}{4}}$
	$\frac{4}{\sqrt{3}} < A_a^P A_a^Q < 4$ (automation phase)	$\frac{1}{2\sqrt{2}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} \left(L \left(\frac{A_a^P A_a^Q}{2}\right)^{-\frac{1}{2}} + H \left(\frac{A_a^P A_a^Q}{2}\right)^{\frac{3}{2}}\right)$
	$A_a^P A_a^Q \geq 4$ (post-automation phase)	$\frac{1}{2} \left(A_n^P A_n^Q\right)^{\frac{1}{2}} \left(A_a^Q A_a^P\right)^{\frac{1}{2}} (LH)^{\frac{1}{2}}$
Case 1. $L = \frac{1}{4}$ and $H = 1$	$A_a^P A_a^Q \leq \frac{1}{\sqrt{3}}$ (pre-automation phase)	$\left(\frac{4}{27}\right)^{\frac{1}{4}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} L^{\frac{3}{4}} H^{\frac{1}{4}}$
	$\frac{1}{\sqrt{3}} < A_a^P A_a^Q < 1$ (automation phase)	$\frac{1}{2\sqrt{2}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} \left(L \left(\frac{A_a^P A_a^Q}{2}\right)^{-\frac{1}{2}} + H \left(\frac{A_a^P A_a^Q}{2}\right)^{\frac{3}{2}}\right)$
	$A_a^P A_a^Q \geq 1$ (post-automation phase)	$\frac{1}{2} \left(A_n^P A_n^Q\right)^{\frac{1}{2}} \left(A_a^Q A_a^P\right)^{\frac{1}{2}} (LH)^{\frac{1}{2}}$

Table 3. Total consumption (or net-output) in Example 2 under two different cases for the labor supply.

First of all, as shown in Figure 3, the threshold level of productivity in automation sector to initiate the transition to automation depends on the ratio of $\frac{L}{H}$. As $\frac{L}{H}$ increases, the threshold levels A^* and A^{**} increase.

Table 3 shows the changes in the threshold levels. The reason behind this change is that for higher levels of $\frac{L}{H}$, $\frac{w_L}{p_a}$ gets smaller and thus the ratio of $\frac{1}{A_a^Q}$ needs to be small enough to trigger firm f to switch to automation. In general, the threshold level of $A_a^P A_a^Q$ for automation depends on not only the ratio of $\frac{L}{H}$ but also on the weights of task in each sector.

Next, we analyze other sources that plays role in the transition to automation. As given in Appendix, the threshold levels A^* and A^{**} are as follows:

$$A^* = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}} \left(\frac{L}{H} \frac{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1 - \left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} \right)^{\alpha_a^H}$$

$$A^{**} = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}} \left(\frac{L}{H} \frac{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1 - \left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} \right)^{\alpha_a^H}$$

Moreover, the level of $t_f a$ is given by Equation (29), which enables us to calculate the rate of automation as well as the change in the rate of automation after a given technological change. Equation (29) implies that both depend on the weights of tasks in each sector and the productivity parameter for the automation sector, $A_a^P A_a^Q$. Importantly, the productivity parameters of other sectors that do not cause any substitution effect has no implications for the threshold level of technologies and for the rate of change in automation. In the next part, we discuss how this result can be different for different functional forms for the production. We provide cases where the productivites of other sectors matter in the magnitude of the rate of change in total consumption.

Proposition 3 provides the related comparative statics.

PROPOSITION **3**

- *A^* and A^{**} are increasing in $\frac{L}{H}$,*
- *A^* and A^{**} are increasing in α_n^H (decreasing in α_{n0}^L),*
- *For constant α_{fn}^n and α_{fa}^L, A^* and A^{**} are increasing in α_f^H (decreasing in α_{f0}^L),*

We already discussed the first part of Proposition 3. The other parts are about the relative weights of high-skilled labor task and non-substitutable low skilled labor task (low-skilled labor used in task 0). Hence, in a similar way, they follow from the fact that if the high-skilled labor task weight increases in the resource good sector, or increases in the final good sector for constant weights of the intermediate goods, then the ratio of $\frac{w_L}{p_a}$ increases due to the higher relative demand for the high skilled labor compared to the case where high-skilled intensity was lower. As a result, following any of these cases, both A^* and A^{**} rise. Figure 3 shows how the threshold levels of productivities and the automation and consumption growth changes in α_n^H and in ratio of $\frac{L}{H}$.

5 Technological Changes and Automation in an N-sector Economy

In this section, we extend our analysis to N-sector economy, where an increase in automation might occur via direct and/or indirect network effects.

We now allow for fully general combinations of automatable and non-automatable tasks in each sector, where the production function of each sector has a form:

$$Y_i = A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j \in a_i} [L_{ij} + A_j^Q X_{ij}]^{\alpha_{ij}^L} \right] \prod_{j \in n_i} (X_{ij})^{\alpha_{ij}^n}$$

In what follows, we focus on changes in the basic productivity of various goods A_i^P's, and simply normalize the quality parameters, $A_i^Q = 1$, for all $i \in N$. The analysis of specific changes in A_i^Q is an easy extension, and the normalization saves on notation.

5.1 Indirect Automation

An increase in automation might occur via direct and/or indirect network effects. Direct network effects might be thought of as increase in share of automation expenditures in a given task a following the technological improvements in automation good producer a. For instance, cost-reducing technological advancements in the production of industrial robots increase the expenditures on these goods in other sectors. On the other hand, an automation might occur via indirect network effects as well, where an increase in productivity in a given sector might result in switches to automation in some other tasks in some other sectors. For instance, a productivity increase in software producers reduce the production costs of industrial robot producers as well, which eventually results in switches to usage of industrial robots in some sectors.

To illustrate this point, we consider a three-sector example below that illustrates a case where indirect network effects result in automation. Different than the previous examples, here we consider that the automation good producer (firm a) uses the resource good (produced by firm n) as an intermediate good.

Example 5.

The production functions are as follows.
$Y_n = A_n^P L_n$
$Y_a = H_a^{0.5} (X_{an})^{0.5}$
$Y_f = H_f^{0.5} (L_f + X_{fa})^{0.5}$
Also, $A_a^P = A_f^P = L = H = 1$.

Similar to the previous examples, we consider $\beta_f = 1$ and $\beta_a = \beta_n = 0$. In this example, following the technological improvements in firm n, product n gets cheaper, and hence product a becomes cheaper as well. This causes further ripple effects to firm f, and firm

f starts to use product a. More specifically, for $A_n^P < 4$, firm f uses no automation good, for $4 \leq A_n^P \leq 12$, good a becomes as cheap as the low-skilled labor and firm f starts to automate task a, and for $A_n^P > 12$ firm f uses no labor in task a. As a result, this example shows how indirect network effects result in automation.

5.2 Growth, Income Inequality, and the Evolution of the Input-Output Network

In this part, first, we provide the following definition on automation, where we let $F_i = fracY_iA_i^P$ denote the production process of firm i.

DEFINITION **2** *Consider two economies* $E = \left(\left\{A_i^P\right\}, \{F_i\}, L, H\right)$ *and* $E' = \left(\left\{A_i^P\right\}', \{F_i\}, L, H\right)$ *that have identical* $\{F_i\}$*,* L*, and* H*. Then, the economy* E' *is* weakly more automated *than the economy* E *iff the equilibrium share of expenditures on automation in each automatable task* $j \in a_i$ *in each sector* $i \in N$ *is weakly greater in* E' *than in* E*. And we say that it is* more automated *if in addition, the equilibrium share of expenditures on automation in some* $j \in a_i$ *for some* $i \in N$ *is strictly greater in* E' *than in* E*.*

Following Definition 2, the level of automation in a given economy is increasing in the share of automation expenditures in automatable tasks. As we discussed earlier, an increase in rate of automation occurs via direct and/or indirect network effects of technological changes in various sectors.

As illustrated in the examples above, the equilibrium input-output network evolves following the technological developments in various sectors. Here, we discuss the network influence of sectors. The influence of a sector on overall economy determines its impact on the overall economy.

Proposition 4 implies that a version of Hulten's Theorem extends to our general setup for the n-sector economy.

PROPOSITION **4** *Let* m_i *be the Domar weight of sector* i *at equilibrium; that is,* $[\vec{m_i}] = (I - \Omega')^{-1}\left[\vec{\beta_i}\right]$*. Then,*

- *the growth in final-good-output in response to the sectoral technological changes satisfies:*

$$dlnC = \sum_{i \in N} m_i dlnA_i^P,$$

- *and the changes in* w_L *and* w_H *in response to technological changes satisfy:*

$$dlnw_H = \sum_{i \in N} m_i dlnA_i^P + dln\left(\sum_{i \in N} \alpha_i^H m_i\right)$$

$$dlnw_L = \sum_{i \in N} m_i dlnA_i^P + dln\left(\sum_{i \in N} \alpha_i^L m_i\right),$$

where $\alpha_i^L = \alpha_{i0}^L + \sum_{j \in a_i} (1 - t_{ij})\, \alpha_{ij}^L$ *is the total share of low-skilled labor in sector* i *at equilibrium.*

Of course, the Domar weights m_i and all the equilibrium values depend on the full production network, which determines the levels of automation which are critical in determining how much of each input is being used where. Still, the implication of the theorem is that to see the impact of small productivity changes, one can simply look at the current equilibrium expenditure levels.

Next, we provide an analysis of network influences, which can be obtained by ordering the Domar weights as an economy changes.

DEFINITION **3** *Let the network influence of sector* i *be defined as* $\frac{dlnC}{dlnA_i^P}$.

The network influence of a sector measures the growth effect of a productivity shock to that sector. Following Proposition 4, the network influence of any sector is equal to its Domar weight. Our next result shows how technological changes resulting in increase in automation alter the network influences.

PROPOSITION **5** *Consider economies* $E = \left(\left\{A_i^P\right\}, \{F_i\}, L, H\right)$ *and* $E' = \left(\left\{A_i^P\right\}', \{F_i\}, L, H\right)$, *such that economy* E' *is more automated than economy* E. *Then:*

- *the network influence of each sector* i *is weakly higher in the economy* E' *than in the economy* E, *and*
- *the network influence of sector* i *is strictly higher in the economy* E' *than in the economy* E *if and only if* i *is one of the more automated tasks or there exists a directed upstream (supplier) path from* i *to at least one of the more automated tasks* $j \neq i$ *(*i *is either direct or indirect supplier of at least one of the more automated tasks* j*) in* E'.

Proposition 5 shows how interconnectedness in the economy change following technological changes that cause substitution effects in the economy. As the substitution of labor by automation goods occurs, the size of the interactions in the economy get larger due to the increasing share of expenditures on automation goods. Proposition 5 shows that following an increase in the level of automation in a given set of tasks, which would occur due to the technological changes, the Domar weights of the producers of those automation goods and their direct and indirect suppliers rise. Given that the Domar weights represent the network influences of sectors, this result implies that the automation good producers and their direct and indirect suppliers experience a growing network influence over time due to the substitution effects that results in increased connectivity in the economy.

Next, we discuss how the income inequality changes as the substitution occurs in the economy. Earlier results and Proposition 5 imply that the income inequality rises as the level of automation increases. In the next result, we provide a formula for the relationship between the income inequality and the sources of technological changes.

First, we provide a definition of the phases in an N-sector model.

DEFINITION **4**

- *An economy is in the pre-automation phase if* $\frac{p_j}{w_L} > 1$ *holds for each automation good* $j \in K$*, which implies that* $t_{ij} = 0$ *holds for each automatable task* $j \in K$ *in each sector* $i \in N$*.*
- *An economy is in the automation phase if there exists at least one automation good* $j \in K$ *and sector* $i \in N$ *such that* $\frac{p_j}{w_L} = 1$ *and* $0 < t_{ij} < 1$*.*
- *An economy is in the post-automation phase if* $\frac{p_j}{w_L} < 1$ *holds for each automation good* $j \in K$*, which implies that* $t_{ij} = 1$ *holds for each automatable task* $j \in K$ *in each sector* $i \in N$*.*

Since automation in any sector displaces labor which changes relative wages, the automation phase in an economy is essentially the relevant case, since there is almost always some part of the economy in which this is transpiring.

PROPOSITION **6** *The changes in relative wages (and income inequality) in response to technology shocks in a given economy satisfy:*

- $dln\left(\frac{w_H}{w_L}\right) = 0$ *if the economy is in the pre-automation phase or the post-automation phase*
- $dln\left(\frac{w_H}{w_L}\right) = \frac{\sum_{i \in N} D_{ji} \frac{dln A_i^P}{\alpha_i^H}}{\sum_{i \in N} D_{ji}}$ *if the economy is in the automation phase, where* $D = (I - \Omega)^{-1}$ *is the Leontief inverse matrix and* $j \in K$ *is an automation good such that* $\frac{p_j}{w_L} = 1$ *and* $0 < t_{ij} < 1$ *for all* $i \in N$*.*

Proposition 6 shows how income inequality changes in response to technological changes in an N-sector economy. First, part *i* shows that the income inequality does not change during pre- and post-substitution phases. Part *ii* provides a formula for the magnitude of the change in income inequality during the transition phase.

6 Consumption and Labor Productivity by Sector

6.1 Households' Consumption at Sector Level

Proposition 4 implies that as technology changes, then the change in the sector level consumption by households must equal the sector level gross-output in all sectors that are not

direct or indirect suppliers of any of the tasks in which automation is taking place. This follows from the fact that the Domar weights of such sectors do not change in response to automation. More specifically, as shown in the appendix:

$$Y_i/C_i = m_i/\beta_i. \tag{7}$$

For constant values for m_i and β_i, we get $dlnY_i = dlnC_i$.

In contrast, the direct or indirect suppliers of tasks undergoing automation have weakly higher gross-output growth rate than the growth rate of consumption by households for these sectors ($dlnY_i > dlnC_i$), which again follows from above when we have an increase in m_i but a constant value for β_i. The difference in these gross-output and consumption growth rates is mainly due to the change in the level of the intermediate good usage due to the substitution effects. These show that the difference between the sector level growth rates of households' consumption and gross-output can be different for an automation good-using sector and an (directly or indirectly) automation-good producing sector. Our next result provides the differences in growth rate between sectors, by showing how each relates to overall output change.

PROPOSITION **7** *The growth rate of household consumption at the sector level in response to the technological changes is given by:*

$$\left[d\vec{ln}C_i\right] = \left[d\vec{ln}C\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L d\vec{lnw}_L - \alpha_i^H dlnw_H\right]$$

Propositions 4 and 7 together show how sector level consumption by households responses to the technological changes in sectors.

Without any substitution effect, the growth rates of sectoral gross output would be given by

$$\left[d\vec{ln}Y_i\right] = \left[d\vec{ln}C_i\right] = (I-\Omega)^{-1}\left[d\vec{ln}A_i^P\right] \tag{8}$$

which holds in response to the technological changes if the economy is in the pre-automation or post-automation phases, whereas the growth rate in the automation-phase reflects the changes in the wages of high- and low-skilled labor. Correspondingly, if a sector relies heavily on the high-skilled labor in its production process, then such a sector might experience a decrease in household consumption and even a decrease in gross-output level.

6.2 Labor Productivity at Sector Level

The amount of low- and high-skilled labor in each sector i is given by

$$L_i = L\frac{\alpha_i^L m_i}{\sum_{i\in N}\alpha_i^L m_i} \quad \text{and} \quad H_i = H\frac{\alpha_i^H m_i}{\sum_{i\in N}\alpha_i^H m_i},$$

and the productivity of low- and high-skilled labor in each sector is given by

$$\frac{Y_i}{L_i} = \frac{Y_i}{L\frac{\alpha_i^L m_i}{\sum_{i\in N}\alpha_i^L m_i}} \quad \text{and} \quad \frac{Y_i}{H_i} = \frac{Y_i}{H\frac{\alpha_i^H m_i}{\sum_{i\in N}\alpha_i^H m_i}}.$$

Correspondingly, during the automation phase, as reallocation of both low- and high-skilled labor occurs and the Domar weights change, sector level labor productivity changes as well. In the next result, we describe sector level labor productivity changes for different types of labor as well as the overall change in the sector level labor productivity.

PROPOSITION 8 *Changes in sector level labor productivities in response to technological changes are:*

$$\left[dln\vec{\frac{Y_i}{L_i}}\right] = \left[dln\vec{w}_L\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dln\vec{w}_L - \alpha_i^H dlnw_H\right] - \left[dln\,\vec{(}\alpha_i^L)\right]$$

$$\left[dln\vec{\frac{Y_i}{H_i}}\right] = \left[dln\vec{w}_H\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dln\vec{w}_L - \alpha_i^H dlnw_H\right]$$

$$\left[dln\vec{\frac{Y_i}{L_i+H_i}}\right] = (I-\Omega)^{-1}\left[dlnA_i^P + (\alpha_i^L s_L \vec{+} \alpha_i^H s_H)dln\left(\frac{w_H}{w_L}\right)\right] + \left[dln\,(L\alpha_i^L s_H\vec{} + H\alpha_i^H s_L)\right]$$

Proposition 8 implies that the high-skilled labor productivity in each sector rises following technological advancements. However, the sector level change in the productivity of low-skilled labor and the sector level change in the overall labor productivity is ambigious and depends on the weights of tasks that are performed by low- and high-skilled labor in various sectors. Nonetheless, the productivity of low- and high-skilled labor can be written as a function of changes in the technology and the wages, and only being a user or not matters.

7 Conclusion

We have analyzed the impact of technological change on an economy in which both complement and substitute inputs are present, as a function of the production network. Our results show that when there exists different types of labor and intermediate goods that can substitute for labor, then both the input-output structure and the intensity (or weights) of high-skilled and low-skilled labor tasks in each sector play key roles in determining the macroeconomic impacts of microeconomic shocks since they all determine both the total labor demand and the Domar weights of the producers of the substitutable intermediate goods.

Our results shed light on productivity paradoxes and wage inequality, and suggest that understanding the impact of technological change must account for substitution in production processes.

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8 APPENDIX

8.1 Equilibrium Conditions in Three-Sector Model

i) Firm maximization

The cost minimization problem for firm $i \in \{a, n\}$ is

$$\min_{L_{i0}, H_i} w_L L_{i0} + w_H H_i \text{ subject to } 1 = A_i^P (H_i)^{\alpha_i^H} (L_{i0})^{\alpha_{i0}^L}$$

The Lagrangian function is:

$$\mathcal{L} = w_L L_{i0} + w_H H_i - \lambda_i \left(A_i^P (H_i)^{\alpha_i^H} (L_{i0})^{\alpha_{i0}^L} - 1 \right)$$

The first order conditions are:

- $\frac{\partial \mathcal{L}}{\partial H_i} = w_H - \frac{\lambda_i^* \alpha_i^H A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L}}{H_i^*} = 0$

  $H_i^* = \frac{\lambda_i^* \alpha_i^H A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L}}{w_H}$

- $\frac{\partial \mathcal{L}}{\partial L_{i0}} = w_L - \frac{\lambda_i^* \alpha_{i0}^L \alpha_i^H A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L}}{L_{i0}} = 0$

  $L_{i0}^* = \frac{\lambda_i^* \alpha_{i0}^L \alpha_i^H A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L}}{w_L}$

- $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 1 - A_i^P (H_i^*)^{\alpha_i^H} (L_{i0}^*)^{\alpha_{i0}^L} = 0$

The FOCs above implies:

$$\frac{1}{\lambda_i^*} = A_i^P (\frac{\alpha_i^H}{w_H})^{\alpha_i^H} \left(\frac{\alpha_{i0}^L}{w_L} \right)^{\alpha_{i0}^L}$$

Then, zero profit condition implies $\lambda_i^* = w_H H_i^* + w_L L_{i0}^* = C_i(\mathbf{p}, w_L, w_H, 1) = p_i$.

The factor demands and priCes can be written as follows:

$$L_{a0} = \frac{\alpha_a^H p_a Y_a}{w_L} \text{ and } H_a = \frac{\alpha_a^H p_a Y_a}{w_H}$$

$$L_{n0} = \frac{\alpha_i^H p_i Y_i}{w_L} \text{ and } H_n = \frac{\alpha_n^H p_n Y_n}{w_H}$$

$$p_a = \frac{1}{A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}} (w_L)^{\alpha_{a0}^L} (w_H)^{\alpha_a^H} \tag{9}$$

$$p_n = \frac{1}{A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}} (w_L)^{\alpha_{n0}^L} (w_H)^{\alpha_n^H} \tag{10}$$

Next, we solve for the cost minimization for firm f.

$$\min_{L_{f0},L_{fa},H_f,X_{fa},X_{fn}} w_L(L_{f0}+L_{fa}) + w_H H_f + p_a X_{fa} + p_n X_{fn} \text{ subject to}$$
$$1 = A_f^P (H_f)^{\alpha_f^H} (L_{f0})^{\alpha_{f0}^L} [L_{fa} + A_a^Q X_{fa}]^{\alpha_{fa}^L} (A_n^Q X_{fn})^{\alpha_{fn}^n}$$

The Lagrangian function is:

$$\begin{aligned}\mathcal{L} =& w_L(L_{f0}+L_{fa}) + w_H H_f + p_a X_{fa} + p_n X_{fn} \\ -\lambda &\left(A_f^P (H_f)^{\alpha_f^H} (L_{f0})^{\alpha_{f0}^L} [L_{fa} + A_a^Q X_{fa}]^{\alpha_{fa}^L} (A_n^Q X_{fn})^{\alpha_{fn}^n} - 1\right)\end{aligned}$$

The FOCs imply:

- $\frac{\partial \mathcal{L}}{\partial \lambda} = A_f^P (H_f^*)^{\alpha_f^H} (L_{f0}^*)^{\alpha_{f0}^L} [L_{fa}^* + A_a^Q X_{fa}^*]^{\alpha_{fa}^L} (A_n^Q X_{fn}^*)^{\alpha_{fn}^n} = 1$

By plugging the equation above into the other FOCs, we get:

- $\frac{\partial \mathcal{L}}{\partial H_f} = w_H - \lambda_f^* \frac{\alpha_f^H}{H_f^*} = 0$

  $H_f^* = \lambda_f^* \frac{\alpha_f^H}{w_H}$

- $\frac{\partial \mathcal{L}}{\partial L_{f0}} = w_L - \lambda_f^* \frac{\alpha_{f0}^L}{L_{f0}^*} = 0$

  $L_{f0}^* = \lambda_f^* \frac{\alpha_{f0}^L}{w_L}$

- $\frac{\partial \mathcal{L}}{\partial X_{fn}} = p_n - \lambda_f^* \frac{\alpha_{fn}^n}{X_{fn}^{n*}} = 0$

  $X_{fn}^* = \lambda_f^* \frac{\alpha_{fn}^n}{p_n}$

- $\frac{\partial \mathcal{L}}{\partial L_{fa}} = \left(w_L - \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}\right) \geq 0$ and $L_{fa}^* \left(w_L - \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}\right) = 0$

  $\frac{\partial \mathcal{L}}{\partial X_{fa}} = \left(p_a - A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}\right) \geq 0$ and $X_{fa}^* \left(p_a - A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}\right) = 0$

  Both L_{fa}^* and X_{fa}^* can not be zero, otherwise $Y_f = 0$. Then,

  Case 1. $w_L = \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $L_{fa}^* = \lambda_f^* \frac{\alpha_{fa}^L}{w_L}$; and $p_a > A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $X_{fa}^* = 0$. In this case, $\frac{w_L}{p_a} < \frac{1}{A_a^Q}$.

  Case 2. $w_L > \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $L_{fa} = 0$; and $p_a = A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $X_{fa}^* = \lambda_f^* \frac{\alpha_{fa}^L}{p_a}$. In this case, $\frac{w_L}{p_a} > \frac{1}{A_a^Q}$.

  Case 3. $w_L = \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $L_{fa}^* = \lambda_f^* \frac{(1-t_{fa})\alpha_{fa}^L}{w_L}$; $p_a = A_a^Q \lambda_f^* \frac{\alpha_{fa}^L}{L_{fa}^* + A_a^Q X_{fa}^*}$, $X_{fa}^* = \lambda_f^* \frac{t_{fa}\alpha_{fa}^L}{p_a}$. In this case, $\frac{w_L}{p_a} = \frac{1}{A_a^Q}$.

$$\{L_{fa}^*, X_{fa}^*\} =: \begin{cases} \{\lambda_f^* \frac{\alpha_{fa}^L}{w_L}, 0\} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \{\lambda_f^* \frac{(1-t_{fa})\alpha_{fa}^L}{w_L}, \lambda_f^* \frac{t_{fa}\alpha_{fa}^L}{p_a}\} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \{0, \lambda_f^* \frac{\alpha_{fa}^L}{p_a}\} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases}$$

The FOCs above together imply:

$$\lambda_f^* = \begin{cases} \frac{1}{A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{1}{A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{1}{A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[A_a^Q \frac{\alpha_{fa}^L}{p_a}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases}$$

Under the zero profit conditions and the normalization of $p_f = 1$, we get:

$$\begin{cases} A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{A_a^Q\alpha_{fa}^L A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}}{(w_L)^{\alpha_{a0}^L}(w_H)^{\alpha_a^H}}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases}$$

Lastly, by plugging p_a and p_n into the equation above, it follows that:

$$\begin{cases} A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L} w_L^{1-\alpha_{a0}^L} w_H^{-\alpha_a^H} < 1 \\ A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L} w_L^{1-\alpha_{a0}^L} w_H^{-\alpha_a^H} = 1 \\ A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{A_a^Q\alpha_{fa}^L A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}}{(w_L)^{\alpha_{a0}^L}(w_H)^{\alpha_a^H}}\right]^{\alpha_{fa}^L}\left[\frac{A_n^Q\alpha_{fn}^n A_n^P(\alpha_n^H)^{\alpha_n^H}(\alpha_{n0}^L)^{\alpha_{n0}^L}}{(w_L)^{\alpha_{n0}^L}(w_H)^{\alpha_n^H}}\right]^{\alpha_{fn}^n} = 1 & \text{if } A_a^Q A_a^P(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L} w_L^{1-\alpha_{a0}^L} w_H^{-\alpha_a^H} > 1 \end{cases} \tag{11}$$

Then, the conditional factor demands are:

$$H_f =: \begin{cases} \frac{\alpha_f^H Y_f}{w_H A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{\alpha_f^H Y_f}{w_H A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{\alpha_f^H Y_f}{w_H A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[A_a^Q\frac{\alpha_{fa}^L}{p_a}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{12}$$

$$L_{f0} =: \begin{cases} \frac{\alpha_{f0}^L Y_f}{w_L A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{\alpha_{f0}^L Y_f}{w_L A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{\alpha_{f0}^L Y_f}{w_L A_f^P(\frac{\alpha_f^H}{w_H})^{\alpha_f^H}\left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L}\left[A_a^Q\frac{\alpha_{fa}^L}{p_a}\right]^{\alpha_{fa}^L}\left[A_n^Q\frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{13}$$

$$X_{fn} =: \begin{cases} \frac{\alpha_{fn}^L Y_f}{p_n A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{\alpha_{fn}^L Y_f}{p_n A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{\alpha_{fn}^L Y_f}{p_n A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[A_a^Q \frac{\alpha_{fa}^L}{p_a}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{14}$$

$$\{L_{fa}, X_{fa}\} =: \begin{cases} \{\frac{\alpha_{fa}^L Y_f}{w_L A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}}, 0\} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \{\frac{(1-t_{fa})\alpha_{fa}^L Y_f}{w_L A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}}, \frac{t_{fa}\alpha_{fa}^L Y_f}{p_a A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[\frac{\alpha_{fa}^L}{w_L}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}}\} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \{0, \frac{\alpha_{fa}^L Y_f}{p_a A_f^P (\frac{\alpha_f^H}{w_H})^{\alpha_f^H} \left(\frac{\alpha_{f0}^L}{w_L}\right)^{\alpha_{f0}^L} \left[A_a^Q \frac{\alpha_{fa}^L}{p_a}\right]^{\alpha_{fa}^L} \left[A_n^Q \frac{\alpha_{fn}^n}{p_n}\right]^{\alpha_{fn}^n}}\} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{15}$$

8.2 Proofs of Proposition 1 and Proposition 2

By combining the budget constraint and FOCs of the utility maximization:

$$C^L = w_L L \tag{16}$$

$$C^H = w_H H \tag{17}$$

$$Y_f = C = w_L L + w_H H \tag{18}$$

By combining the market clearing conditions, factor demands for low-skilled and high-skilled labor, we get:

$$L =: \begin{cases} \frac{p_n Y_n \alpha_{n0}^L + p_f Y_f \left(\alpha_{f0}^L + \alpha_{fa}^L\right)}{w_L} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{p_a Y_a \alpha_{a0}^L + p_n Y_n \alpha_{n0}^L + p_f Y_f \left(\alpha_{f0}^L + (1-t_{fa})\alpha_{fa}^L\right)}{w_L} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{p_a Y_a \alpha_{a0}^L + p_n Y_n \alpha_{n0}^L + p_f Y_f \alpha_{f0}^L}{w_L} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{19}$$

$$H =: \begin{cases} \frac{p_n Y_n \alpha_n^H + p_f Y_f \alpha_f^H}{w_H} & \text{if } \frac{w_L}{p_a} < \frac{1}{A_a^Q} \\ \frac{p_a Y_a \alpha_a^H + p_n Y_n \alpha_n^H + p_f Y_f \alpha_f^H}{w_H} & \text{if } \frac{w_L}{p_a} = \frac{1}{A_a^Q} \\ \frac{p_a Y_a \alpha_a^H + p_n Y_n \alpha_n^H + p_f Y_f \alpha_f^H}{w_H} & \text{if } \frac{w_L}{p_a} > \frac{1}{A_a^Q} \end{cases} \tag{20}$$

Market clearing for goods, factor demands, and $p_f = 1$ together imply:

$$p_a Y_a = p_a X_{fa} = t_{fa} \alpha^L_{fa} Y_f \tag{21}$$

where $t_{fa} = 1$ for $\frac{w_L}{p_a} > \frac{1}{A^Q_a}$, $t_{fa} = 0$ for $\frac{w_L}{p_a} < \frac{1}{A^Q_a}$, and $0 \leq t_{fa} \leq 1$ for $\frac{w_L}{p_a} = \frac{1}{A^Q_a}$.

$$p_n Y_n = p_n X_{fn} = \alpha^n_{fn} Y_f \tag{22}$$

Next, we write the condition for t_{fa} for further simplification.
Then, by using Eq. (18), Eq. (21), and Eq. (22), we can rewrite the market cleaaring for labor as follows:

$$L =: \begin{cases} \frac{\left(\alpha^L_{n0}\alpha^n_{fn} + \alpha^L_{f0} + \alpha^L_{fa}\right)(w_L L + w_H H)}{w_L} & \text{if } \frac{w_L}{p_a} < \frac{1}{A^Q_a} \\ \frac{\left(t_{fa}\alpha^L_{fa}\alpha^L_{a0} + \alpha^n_{fn}\alpha^L_{n0} + \alpha^L_{f0} + (1-t_{fa})\alpha^L_{fa}\right)(w_L L + w_H H)}{w_L} & \text{if } \frac{w_L}{p_a} = \frac{1}{A^Q_a} \\ \frac{\left(\alpha^a_{fa}\alpha^L_{a0} + \alpha^n_{fn}\alpha^L_{n0} + \alpha^L_{f0}\right)(w_L L + w_H H)}{w_L} & \text{if } \frac{w_L}{p_a} > \frac{1}{A^Q_a} \end{cases} \tag{23}$$

$$H =: \begin{cases} \frac{\left(\alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)(w_L L + w_H H)}{w_H} & \text{if } \frac{w_L}{p_a} < \frac{1}{A^Q_a} \\ \frac{\left(t_{fa}\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)(w_L L + w_H H)}{w_H} & \text{if } \frac{w_L}{p_a} = \frac{1}{A^Q_a} \\ \frac{\left(\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)(w_L L + w_H H)}{w_H} & \text{if } \frac{w_L}{p_a} > \frac{1}{A^Q_a} \end{cases} \tag{24}$$

$$\frac{w_H}{w_L} =: \begin{cases} \frac{L}{H}\frac{\left(\alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}{1-\left(\alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)} & \text{if } \frac{w_L}{p_a} < \frac{1}{A^Q_a} \\ \frac{L}{H}\frac{\left(t_{fa}\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}{1-\left(t_{fa}\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)} & \text{if } \frac{w_L}{p_a} = \frac{1}{A^Q_a} \\ \frac{L}{H}\frac{\left(\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}{1-\left(\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)} & \text{if } \frac{w_L}{p_a} > \frac{1}{A^Q_a} \end{cases} \tag{25}$$

Then, we derive w_L by plugging Eq. (25) into the Eq. (11):

$$w_L =: \begin{cases} A^P_f \left(\alpha^H_f\right)^{\alpha^H_f} \left(\alpha^L_{f0}\right)^{\alpha^L_{f0}} \left[\alpha^L_{fa}\right]^{\alpha^a_{fa}} \left[A^Q_n \alpha^n_{fn} A^P_n (\alpha^H_n)^{\alpha^H_n} (\alpha^L_{n0})^{\alpha^L_{n0}}\right]^{\alpha^n_{fn}} \left(\frac{H}{L}\frac{1-\left(\alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}{\left(\alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}\right)^{\alpha^H_f + \alpha^H_n \alpha^n_{fn}} & \text{if } A^Q_a A^P_a < A^* \\ A^P_f \left(\alpha^H_f\right)^{\alpha^H_f} \left(\alpha^L_{f0}\right)^{\alpha^L_{f0}} \left[\alpha^L_{fa}\right]^{\alpha^a_{fa}} \left[A^Q_n \alpha^n_{fn} A^P_n (\alpha^H_n)^{\alpha^H_n} (\alpha^L_{n0})^{\alpha^L_{n0}}\right]^{\alpha^n_{fn}} \left(\frac{H}{L}\frac{1-\left(t_{fa}\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}{\left(t_{fa}\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}\right)^{\alpha^H_f + \alpha^H_n \alpha^n_{fn}} & \text{if } A^* \leq A^Q_a A^P_a \leq A^{**} \\ A^P_f \left(\alpha^H_f\right)^{\alpha^H_f} \left(\alpha^L_{f0}\right)^{\alpha^L_{f0}} \prod_{i=a,n} \left[A^Q_i \alpha^i_{fi} A^P_i (\alpha^H_i)^{\alpha^H_i} (\alpha^L_{i0})^{\alpha^L_{i0}}\right]^{\alpha^L_{fi}} \left(\frac{H}{L}\frac{1-\left(\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}{\left(\alpha^L_{fa}\alpha^H_a + \alpha^n_{fn}\alpha^H_n + \alpha^H_f\right)}\right)^{\alpha^H_f + \alpha^H_a \alpha^L_{fa} + \alpha^H_n \alpha^n_{fn}} & \text{if } A^Q_a A^P_a > A^{**} \end{cases} \tag{26}$$

where

$$\begin{cases} A^* = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}} \left(\frac{L}{H} \frac{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} \right)^{\alpha_a^H} \\ A^{**} = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}} \left(\frac{L}{H} \frac{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} \right)^{\alpha_a^H} \end{cases} \tag{27}$$

By plugging Eq. (26) into the Eq. (25), we get:

$$w_H =: \begin{cases} A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{L}{H} \frac{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} \right)^{1-\alpha_f^H - \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P < A^* \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{L}{H} \frac{\left(t_{fa} \alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(t_{fa} \alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} \right)^{1-\alpha_f^H - \alpha_n^H \alpha_{fn}^n} & \text{if } A^* \leq A_a^Q A_a^P \leq A^{**} \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \left(\frac{L}{H} \frac{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} \right)^{1-\alpha_f^H - \alpha_a^H \alpha_{fa}^L - \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P > A^{**} \end{cases} \tag{28}$$

Next, we derive the fraction t_{fa} from Eq. (28):

$$t_{fa} =: \begin{cases} 0 & \text{if } A_a^Q A_a^P < A^* \\ \frac{\frac{H}{L}\left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} \left(1-\alpha_{fn}^n \alpha_n^H - \alpha_f^H\right) - \left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\left(\alpha_{fa}^L \alpha_a^H\right)\left(1+\frac{H}{L}\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)} & \text{if } A^* \leq A_a^Q A_a^P \leq A^{**} \\ 1 & \text{if } A_a^Q A_a^P > A^{**} \end{cases} \tag{29}$$

Then, we plug Eq.(29) into Eq. (26) and Eq. (28), and get:

$$\frac{w_H}{w_L} =: \begin{cases} \frac{L}{H} \frac{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} & \text{if } A_a^Q A_a^P < A^* \\ \left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} & \text{if } A^* \leq A_a^Q A_a^P \leq A^{**} \\ \frac{L}{H} \frac{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)} & \text{if } A_a^Q A_a^P > A^{**} \end{cases} \tag{30}$$

Then, for constant weights and labor supply, by taking the logs and the total derivatives of each side of Eq. (30) (excluding the values for $A_a^Q A_a^P$ where $\frac{w_H}{w_L}$ is non-differentiable), we get:

$$dln\left(\frac{w_H}{w_L}\right) =: \begin{cases} 0 & \text{if } A_a^Q A_a^P < A^* \\ \frac{1}{\alpha_a^H} dln\left(A_a^Q A_a^P\right) & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ 0 & \text{if } A_a^Q A_a^P > A^{**} \end{cases} \tag{31}$$

Lastly, by plugging Eq. (29) into Eq. (26) and Eq. (28), we get:

$$w_L =: \begin{cases} A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{H}{L} \frac{1-\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}\right)^{\alpha_f^H + \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P \leq A^* \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H - \alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \left(\frac{H}{L} \frac{1-\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}\right)^{\alpha_f^H + \alpha_a^H \alpha_{fa}^L + \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P \geq A^{**} \end{cases} \tag{32}$$

$$w_H =: \begin{cases} A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(\frac{L}{H} \frac{\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}\right)^{1-\alpha_f^H - \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P < A^* \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^L\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(A_a^P A_a^Q (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H - \alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} & \text{if } A^* \leq A_a^Q A_a^P \leq A^{**} \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \left(\frac{L}{H} \frac{\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{1-\left(\alpha_{fa}^L \alpha_a^H + \alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}\right)^{1-\alpha_f^H - \alpha_a^H \alpha_{fa}^L - \alpha_n^H \alpha_{fn}^n} & \text{if } A_a^Q A_a^P > A^{**} \end{cases} \tag{33}$$

Then, by taking the logs and the total derivatives of the equation above (we exclude the values for $A_a^Q A_a^P$ where w_L and w_H are non-differentiable), we get:

$$dlnw_L =: \begin{cases} dlnA_f^P + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) & \text{if } A_a^P A_a^Q < A^* \\ dlnA_f^P + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) - \left(\frac{\alpha_f^H + \alpha_n^H \alpha_{fn}^n}{\alpha_a^H}\right) dln\left(A_a^P A_a^Q\right) & \text{if } A^* < A_a^P A_a^Q < A^{**} \\ dlnA_f^P + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) + \alpha_{fa}^L dln\left(A_a^P A_a^Q\right) & \text{if } A_a^P A_a^Q > A^{**} \end{cases}$$

$$dlnw_H =: \begin{cases} dlnA_f^P + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) & \text{if } A_a^P A_a^Q < A^* \\ dlnA_f^P + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) + \left(\frac{1-\alpha_f^H - \alpha_n^H \alpha_{fn}^n}{\alpha_a^H}\right) dln\left(A_a^P A_a^Q\right) & \text{if } A^* < A_a^P A_a^Q < A^{**} \\ dlnA_f^P + \alpha_{fn}^n dln\left(A_n^P A_n^Q\right) + \alpha_{fa}^L dln\left(A_a^P A_a^Q\right) & \text{if } A_a^P A_a^Q > A^{**} \end{cases}$$

This is the end of the proof of Proposition 1. Next, we derive $w_H H + w_L L = C$ by multiplying Eq. (26) by L and multiplying Eq. (28) by H and summing up these two, the consumption level in three phases are as follows with the ordering of pre-automation, automation, and post-automation phases:

$$C =: \begin{cases} A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^a\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \frac{L}{1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n} \left(\frac{H}{L} \frac{1-\left(\alpha_{fn}^n \alpha_n^H+\alpha_f^H\right)}{\left(\alpha_{fn}^n \alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_f^H+\alpha_n^H \alpha_{fn}^n} \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \left[\alpha_{fa}^a\right]^{\alpha_{fa}^a} \left[A_n^Q \alpha_{fn}^n A_n^P (\alpha_n^H)^{\alpha_n^H} (\alpha_{n0}^L)^{\alpha_{n0}^L}\right]^{\alpha_{fn}^n} \left(L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} + H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}}\right) \\ A_f^P \left(\alpha_f^H\right)^{\alpha_f^H} \left(\alpha_{f0}^L\right)^{\alpha_{f0}^L} \prod_{i=a,n} \left[A_i^Q \alpha_{fi}^i A_i^P (\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L}\right]^{\alpha_{fi}^L} \frac{L}{1-\left(\alpha_{fa}^a \alpha_a^H+\alpha_{fn}^n \alpha_n^H+\alpha_f^H\right)} \left(\frac{H}{L} \frac{1-\left(\alpha_{fa}^a \alpha_a^H+\alpha_{fn}^n \alpha_n^H+\alpha_f^H\right)}{\left(\alpha_{fa}^a \alpha_a^H+\alpha_{fn}^n \alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_f^H+\alpha_a^H \alpha_{fa}^a+\alpha_n^H \alpha_{fn}^n} \end{cases} \tag{34}$$

Then, by taking the logs of and totally differentiating both sides, we get:

$$dlnC = \begin{cases} dlnA_f^P + \alpha_{fn}^n dln(A_n^Q A_n^P) & \text{if } A_a^Q A_a^P < A^* \\ dlnA_f^P + \alpha_{fn}^n dln(A_n^Q A_n^P) + dln\left(L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} + H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}}\right) & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ dlnA_f^P + \alpha_{fn}^n dln(A_n^Q A_n^P) + \alpha_{fa}^L dln(A_a^P A_a^Q) & \text{if } A_a^Q A_a^P > A^{**} \end{cases}$$

Call $L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} + H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} = B$

Next, we derive $\frac{dlnB}{dln(A_a^Q A_a^P)} = \frac{(A_a^Q A_a^P)}{B} \frac{dB}{d(A_a^Q A_a^P)}$

$$\frac{dB}{d(A_a^Q A_a^P)} = \frac{\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} \left(H\left(1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n\right)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} + L\left(-\alpha_f^H-\alpha_n^H \alpha_{fn}^n\right)\right)}{\alpha_a^H A_a^Q A_a^P}$$

Then,

$$\frac{dlnB}{dln(A_a^Q A_a^P)} = \frac{\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} \left(H\left(1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n\right)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} + L\left(-\alpha_f^H-\alpha_n^H \alpha_{fn}^n\right)\right)}{\alpha_a^H \left(L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} + H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}}\right)}$$

$$= \frac{\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} \left(H\left(1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n\right)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} + L\left(-\alpha_f^H-\alpha_n^H \alpha_{fn}^n\right)\right)}{\alpha_a^H L\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{-\alpha_f^H-\alpha_n^H \alpha_{fn}^n}{\alpha_a^H}} \left(1+H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}$$

$$= \frac{\left(\frac{H}{L}\left(1-\alpha_f^H-\alpha_n^H \alpha_{fn}^n\right)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} - \left(\alpha_f^H+\alpha_n^H \alpha_{fn}^n\right)\right)}{\alpha_a^H \left(1+\frac{H}{L}\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}.$$

Eq. (29)implies that for $0 < t_{fa} < 1$:

$$t_{fa}\alpha_{fa}^{L} = \frac{\frac{H}{L}\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} \left(1-\alpha_{fn}^n \alpha_n^H - \alpha_f^H\right) - \left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)}{\alpha_a^H \left(1+\frac{H}{L}\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}\right)}$$

Thus, $dlnB = t_{fa}\alpha_{fa}^L dln(A_a^Q A_a^P)$ for $A^* < A_a^Q A_a^P < A^{**}$. Then, we get

$$dlnC = \begin{cases} dlnA_f^P + \alpha_{fn}^n dln(A_n^Q A_n^P) & \text{if } A_a^Q A_a^P < A^* \\ dlnA_f^P + \alpha_{fn}^n dln(A_n^Q A_n^P) + t_{fa}\alpha_{fa}^L dln(A_a^Q A_a^P) & \text{if } A^* < A_a^Q A_a^P < A^{**} \\ dlnA_f^P + \alpha_{fn}^n dln(A_n^Q A_n^P) + \alpha_{fa}^L dln(A_a^P A_a^Q) & \text{if } A_a^Q A_a^P > A^{**} \end{cases}$$

This completes the proof of Proposition 2.

8.3 Derivations for Examples

8.3.1 Example 1

In this part, we derive the derivation for Example 1. The derivations for Example 2 will follow from the equilibrium conditions, similarly.

$Y_a = A_a^P L_{a0}^{0.5} H_a^{0.5}$,

$Y_f = H_f^{0.5}(L_{fa} + A_a^Q X_{fa})^{0.5}$.

We plug the given paramters into the Eq.(27) and Eq. (**??**), and get:

$$t_{fa} =: \begin{cases} 0 & \text{if} \quad A_1^P A_{21}^Q < 2 \\ \left(\frac{2(A_a^P A_a^Q)^2 - 8}{(A_a^P A_a^Q)^2 + 4}\right) & \text{if} \quad 2 \leq A_1^P A_{21}^Q \leq 2\sqrt{3} \\ 1 & \text{if} \quad A_1^P A_{21}^Q > 2\sqrt{3} \end{cases} \tag{35}$$

which can be rewritten as:

$$t_{fa} = min\left\{max\left\{0, \left(\frac{2(A_a^P A_a^Q)^2 - 8}{(A_a^P A_a^Q)^2 + 4}\right)\right\}, 1\right\}$$

By using the equation for t_{fa}, we derive the following equations:

$$p_f = 1$$

$$p_a = \begin{cases} \frac{1}{A_a^P} & \text{if} \quad A_a^P A_a^Q < 2 \\ \frac{1}{A_a^P} & \text{if} \quad 2 \leq A_a^P A_a^Q \leq 2\sqrt{3} \\ \sqrt{\frac{A_a^Q}{2\sqrt{3}A_a^P}} & \text{if} \quad A_a^P A_a^Q \geq 2\sqrt{3} \end{cases} \tag{36}$$

$$w_L =: \begin{cases} \frac{1}{2} & \text{if } A_a^P A_a^Q < 2 \\ \frac{1}{A_a^P A_a^Q} & \text{if } 2 \leq A_a^P A_a^Q \leq 2\sqrt{3} \\ \sqrt{\frac{A_a^P A_a^Q}{2\sqrt{3}}} \left(\frac{1}{2\sqrt{3}}\right) & \text{if } A_a^P A_a^Q > 2\sqrt{3} \end{cases} \tag{37}$$

$$w_H =: \begin{cases} \frac{1}{2} & \text{if } A_a^P A_a^Q < 2 \\ \frac{A_a^P A_a^Q}{4} & \text{if } 2 \leq A_a^P A_a^Q \leq 2\sqrt{3} \\ \sqrt{\frac{A_a^P A_a^Q}{2\sqrt{3}}} \left(\frac{2\sqrt{3}}{4}\right) & \text{if } A_a^P A_a^Q > 2\sqrt{3} \end{cases} \tag{38}$$

$$\frac{w_H}{w_L} =: \begin{cases} 1 & \text{if } A_a^P A_a^Q < 2 \\ \frac{\left(A_a^P A_a^Q\right)^2}{4} & \text{if } 2 \leq A_a^P A_a^Q \leq 2\sqrt{3} \\ 3 & \text{if } A_a^P A_a^Q > 2\sqrt{3} \end{cases} \tag{39}$$

Phases	t_{fa}	p_a	p_f	w_L	w_H	$\frac{w_H}{w_L}$	Y_a	$Y_f = C$
$A_a^P A_a^Q \leq 2$ (pre-automation phase)	0	$\frac{1}{A_a^P}$	1	$\frac{1}{2}$	$\frac{1}{2}$	1	0	1
$2 < A_a^P A_a^Q < 2\sqrt{3}$ (automation phase)	$2\left(\frac{\left(A_a^P A_a^Q\right)^2 - 4}{\left(A_a^P A_a^Q\right)^2 + 4}\right)$	$\frac{1}{A_a^P}$	1	$\frac{1}{A_a^P A_a^Q}$	$\frac{A_a^P A_a^Q}{4}$	$\frac{\left(A_a^P A_a^Q\right)^2}{4}$	$A_a^P \left(\frac{\left(A_a^P A_a^Q\right)^2 - 4}{4\left(A_a^P A_a^Q\right)}\right)$	$\frac{4 + \left(A_a^P A_a^Q\right)^2}{4\left(A_a^P A_a^Q\right)}$
$A_a^P A_a^Q \geq 2\sqrt{3}$ (post-automation phase)	1	$\sqrt{\frac{A_a^Q}{A_a^P 2\sqrt{3}}}$	1	1	3	3	$\frac{A_a^P}{\sqrt{3}}$	$\sqrt{\frac{2A_a^P A_a^Q}{3\sqrt{3}}}$

8.3.2 Example 2

The derivations for Example 2 will follow from the equilibrium conditions, similarly. Thus, we omit the derivations for Example 2 here, and provide only the result for parameter t_{fa}. The equilibrium level of $Y_f = C$ in Example 2 are depicted in Table 4 below.

Phases	$Y_f = C$
$A_a^P A_a^Q \leq \frac{2}{\sqrt{3}}$ (pre-automation phase)	$\left(\frac{4}{27}\right)^{\frac{1}{4}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}}$
$\frac{2}{\sqrt{3}} < A_a^P A_a^Q < 2$ (automation phase)	$\frac{1}{2\sqrt{2}} \left(A_n^Q A_n^P\right)^{\frac{1}{2}} \left(\left(\frac{A_a^P A_a^Q}{2}\right)^{-\frac{1}{2}} + \left(\frac{A_a^P A_a^Q}{2}\right)^{\frac{3}{2}}\right)$
$A_a^P A_a^Q \geq 2$ (post-automation phase)	$\frac{1}{2} \left(A_n^P A_n^Q\right)^{\frac{1}{2}} \left(A_a^Q A_a^P\right)^{\frac{1}{2}}$

Table 4. Total consumption (or net-output) in Example 2

$$t_{fa} \begin{cases} 0 & \text{if } A_a^P A_a^Q \leq \frac{2}{\sqrt{3}} \left(\frac{L}{H}\right)^{0.5} \\ \frac{\frac{H}{L}\left(\frac{A_a^Q A_a^P}{2}\right)^2 \left(\frac{3}{4}\right) - \left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)\left(1 + \frac{H}{L}\left(\frac{A_a^Q A_a^P}{2}\right)^2\right)} & \frac{2}{\sqrt{3}} \left(\frac{L}{H}\right)^{0.5} < A_a^P A_a^Q < 2 \left(\frac{L}{H}\right)^{0.5} \\ 1 & A_a^P A_a^Q \geq 2 \left(\frac{L}{H}\right)^{0.5} \end{cases}$$

8.4 Derivations for Section 4.1

The Domar weight of firm a is given by $\alpha_{fa}^L t_{fa}$. Then,

$$\begin{aligned} \frac{d^2 lnC}{d^2 ln(A_a^P A_a^Q)} &= \frac{d(\alpha_{fa}^L t_{fa})}{dln(A_a^P A_a^Q)} \\ &= \frac{HL\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}}}{\alpha_a^H \left(H\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} + L\right)\left(H(\alpha_{fn}^n \alpha_n^H + \alpha_f^H - 1)\left(A_a^Q A_a^P (\alpha_a^H)^{\alpha_a^H} (\alpha_{a0}^L)^{\alpha_{a0}^L}\right)^{\frac{1}{\alpha_a^H}} + L\left(\alpha_{fn}^n \alpha_n^H + \alpha_f^H\right)\right)} \end{aligned} \tag{40}$$

As one can see from the equation above, the second order term depends on the high-skilled and low-skilled labor supply, weights of tasks in each production process and the initial level of productivity in automation sector.

Proof of Proposition 4.

Proof of part i :

The growth of consumption in response to sectoral productivity growth can be rewritten as follows:

$\frac{dlnC}{dlnA_i^P} = \frac{A_i^P}{C} \frac{dC}{dA_i^P}$

If we show $\frac{dC}{dA_i^P} = \frac{p_i Y_i}{A_i^P}$, then we are done. Next, we derive $\frac{dC}{dA_i^P}$.

The social planners problem is as follows:

$$\max_{\{C_i\},\{X_{ij}\},\{L_{ij}\},\{L_{i0}\},\{H_i\}} \phi C^L + (1-\phi) C^H + \sum_i \lambda_i \left(A_i^P F_i - \sum_j X_{ji} - C_i\right) + \eta \left(L - \sum_i \sum_{j \in a_i \cup 0} L_{ij}\right) + \mu \left(H - \sum_i H_i\right)$$

For $\phi = \frac{1}{2}$, social planners problem is equivalent to

$$\max_{\{C_i\},\{X_{ij}\},\{L_{ij}\},\{L_{i0}\},\{H_i\}} \left(C^L + C^H\right) + \sum_i \lambda_i \left(A_i^P F_i - \sum_j X_{ji} - C_i\right) + \eta \left(L - \sum_i \sum_{j \in a_i \cup 0} L_{ij}\right) + \mu \left(H - \sum_i H_i\right)$$

Then, the envelope theorem implies:

$\frac{dC}{dA_i^P} = -\lambda_i \frac{Y_i}{A_i^P}$.

If $-\lambda_i = p_i$, then we are done.

The social planners problem also implies that:

$\frac{dC}{dC_i} = -\lambda_i$

Moreover, FOCs of the profit maximization of the represenative firm in the final good sector implies that:

$p_i C_i = \beta_i p_f C$, which further implies

$\sum_i p_i C_i = C$.

Thus, $\frac{dC}{dC_i} = p_i$. By combining this result with the FOCs of the social planner's problem, we get $-\lambda_i = p_i$, which completes the proof.

Proof of part ii and iii :

m_i is the Domar weight of sector i, which implies:

$p_i Y_i = m_i(w_L L + w_H H)$

We multiply both sides by α_i^H and sum across sectors:

$\alpha_i^H p_i Y_i = \alpha_i^H m_i(w_L L + w_H H)$

$\sum_{i \in N} \alpha_i^H p_i Y_i = \sum_{i \in N} \alpha_i^H m_i(w_L L + w_H H)$,

FOCs of firm maximization imply that $\sum_{i \in N} \alpha_i^H p_i Y_i = w_H H$. Then.

$w_H H = \sum_{i \in N} \left[\alpha_i^H m_i w_L L + \alpha_i^H m_i w_H H\right]$

$w_H H(1 - \sum_{i \in N} \alpha_i^H m_i) = \sum_{i \in N} \left[\alpha_i^H m_i w_L L\right]$

Thus;

$$w_L L = (w_H H) \frac{1 - \sum_{i \in N} \alpha_i^H m_i}{\sum_{i \in N} \alpha_i^H m_i}$$

For $C = w_L L + w_H H$, we have:

$$C = \frac{w_H H}{\sum_{i \in N} \alpha_i^H m_i} \quad \text{and} \quad C = \frac{w_L L}{1 - \sum_{i \in N} \alpha_i^H m_i}$$

By taking logs of both sides and totally differentiating both sides, we have:

$$dln w_H = dln C + dln(\sum_{i \in N} \alpha_i^H m_i)$$

$$dlnw_L = dlnC - dln(1 - \sum_{i \in N} \alpha_i^H m_i)$$

Proof of Proposition 5

Proof of part i)

LEMMA **1** *Consider the $n \times n$ matrix $D = (I - \Omega')^{-1}$, where Ω' is the transpose of the given matrix Ω. Then, the following conditions hold:*

$D_{ij} = \sum_k D_{ik}\Omega_{jk}$ *for all pairs* $[i, j\}$ *s.t.* $i \neq j$

$D_{ii} = 1 + \sum_k D_{ik}\Omega_{ik}$

Proof of Lemma 1.

For $D = (I - \Omega')^{-1}$, we claim that $D = I + D\Omega'$ holds. Supposr it holds. Then, by plugging $D = (I - \Omega')^{-1}$ into $D = I + D\Omega'$, we get $(I - \Omega')^{-1} = I + (I - \Omega')^{-1}\Omega'$, which can be rewritten as $(I - \Omega')^{-1}(I - \Omega') = I$. Thus, our claim holds. Then, $D = I + D\Omega'$ implies that:

$[D_{ij}]_{i \neq j} = \sum_k D_{ik}\Omega'_{kj}$

$D_{ii} = 1 + \sum_k D_{ik}\Omega'_{ki}$

Then, by using $\Omega'_{kj} = \Omega_{jk}$ and $\Omega'_{ki} = \Omega_{ik}$, we can write:

$$[D_{ij}]_{i \neq j} = \sum_k D_{ik}\Omega_{jk}$$

$$D_{ii} = 1 + \sum_k D_{ik}\Omega_{ik}$$

This completes the proof of Lemma 1.

By Lemma 1,

- $D_{ii} = 1 + \sum_{k \in N} D_{ik}\Omega_{ik}$ for all i,
- $D_{ij} = D_{ii}\Omega_{ji} + \sum_{k \neq i} D_{ik}\Omega_{jk}$ for all pairs $\{i, j\}$ such that $i \neq j$ and $\Omega_{ji} > 0$ (i is a direct supplier of j), and
- $D_{ij} = \sum_{k \neq i} D_{ik}\Omega_{jk}$ for all pairs $\{i, j\}$ such that $i \neq j$ and $\Omega_{ji} = 0$ (i is not a direct supplier of j).

Given that $\Omega_{ij} \geq 0$ for all pairs $\{i, j\}$, the set of equations above imply that each element of matrix D is

i) non-negative,

ii) non-decreasing in each element of matrix Ω.

Proof of part ii):

Denote the initial economy given in part ii) by E^0, and denote the equilibrium input-output matrix of economy E^0 by Ω.

Then, Ω has the following properties:

i) $[\Omega]_{ij} = t_{ij}\alpha_{ij}^L \in [0, \alpha_{ij}^L]$ for all $i \in N$, $j \in a_i$.

ii) $[\Omega]_{ij} = \alpha_{ij}^n$ for all $i \in N$, $j \in n_i$.

iii) $[\Omega]_{ij} = 0$ for all $i \in N$, $j \in N \setminus \{a_i \cup n_i\}$.

In economy E^0, consider the set $\left[S_j^0\right]$ such that there exists a directed upstream supply path from each $s \in S_j^0 \setminus j$ to j and $j \in S_j^0$ as well.

Thus, for any given j, S_j^0 is the set of sectors including sector j and its all direct and indirect suppliers in Economy E^0. In addition, denote the set of sectors that has no upstream supply path to sector j by $\left\{S_j^0\right\}^C$. For any given economy, we can find these sets for each $j \in N$.

Step 1) First, we show that if there exists any $j \in N$ such that $\left\{S_j^0\right\}^C \neq \emptyset$, then $D_{kl}^0 = [(I - \Omega')^{-1}]_{kl} = 0$ holds for all pairs $\{k, l\}$ such that $k \in \left\{S_j^0\right\}^C$ and $l \in S_j^0$.

In order to show this, first, we show the condition below holds:

- $D_{kl}^0 > 0$ iff there exists a directed path from k to l.

In order to show this, we use the result in Lemma 1.

Take any $k \in \left\{S_j^0\right\}^C$. For any such k, there exists no directed upstream path from k to any $l \in S_j^0$ holds. Otherwise, if there exists a directed path from k to at least one $l \in S_j^0$, then $k \in S_j^0$ must hold as well.

Then, by Lemma 1, we have:

$D_{kl} = \sum_{i \neq k} D_{ki}\Omega_{li}$ for all pairs $\{k, l\}$ such that $k \in \left\{S_j^0\right\}^C$ and $l \in S_j^0$.

For any $\Omega_{li} > 0$, $i \in S_l^0$. If $i \in S_l^0$, then $i \in S_j^0$ also holds since there exists a directed path from i to l and from l to j.

Then, for each ordered pair $\{k, l\}$ such that $k \in \left\{S_j^0\right\}^C$ and $l \in S_j^0$, we have a set of equations such that:

$[D_{kl}]_{l \in S_j^0} = \sum_{i \in S_j^0} D_{ki}\Omega_{li}$

The same argument applies for each $k \in \left\{S_j^0\right\}^C$. Then, we have a system of equations, which has a unique solution. Otherwise, for given Ω, the matrix D wouldn't be unique as well, because Lemma 1 implies that the set of equations above is the full set of equations that consists any $[D_{kl}]_{k \in \left\{S_j^0\right\}^C, l \in S_j^0}$.

$D_{kl} = 0$ for each ordered pair $\{k, l\}$ such that $k \in \left\{S_j^0\right\}^C$ and $l \in S_j^0$ is a solution, which gives us the unique values for each such D_{kl}.

Next, we show that $D_{kl}^0 > 0$ if there exists a directed path from k to l.

Suppose that there exists at least one directed upstream path from k to l. Consider any of these directed paths from k to l, and order the firms in a selected directed path as follows: $S_{kl}^0 = \{i_0, i_1, i_2, .., i_n\}$ where $i_0 = k$ and $i_n = l$, and $\Omega_{i_{t+1}, i_t} > 0$ for all $0 \leq t \leq n - 1$.

Then, by using Lemma 1:

$D_{i_0 i_1} = D_{i_0 i_0}\Omega_{i_1 i_0} + \sum_{j \neq i_0} D_{i_0 j}\Omega_{i_1 j}$

Moreover, Lemma 1 implies that $D_{ii} \geq 1$ for all i. Thus, for $\Omega_{i_1 i_0} > 0$, $D_{i_0 i_1} > 0$ holds.

Lastly, the property $D_{ij} = \sum_{k \neq i} D_{ik}\Omega_{jk}$ for all $i, j : \Omega_{ji} = 0$ (i is not a direct supplier of j) implies that $D_{i_j i_{j+t}} > 0$ holds for all $t \leq n - j$, which further implies $D_{kl} > 0$.

Step 2) Next, consider the economy E^* that is more automated than E. Then, the following conditions hold for the equilibrium input-output network at the economy E^*:

i) $[\Omega^*]_{ij} \geq [\Omega]_{ij} = t_{ij}\alpha^L_{ij} \in [0, \alpha^L_{ij}]$ for all $i \in N$, $j \in a_i$.

ii) $[\Omega^*]_{ij} = t^*_{ij}\alpha^L_{ij} > [\Omega]_{ij} = t_{ij}\alpha^L_{ij}$ for some $i \in N$ for some $j \in a_i$.

iii) $[\Omega^*]_{ij} = [\Omega]_{ij} = \alpha^n_{ij}$ for all $i \in N$, $j \in n_i$.

iv) $[\Omega^*]_{ij} = [\Omega]_{ij} = 0$ for all $i \in N$, $j \in N \setminus \{a_i \cup n_i\}$.

Take one increase in automation at a time. In order to do that, take any one of the ordered pairs $\{i, j\}$ such that $[\Omega^*]_{ij} > [\Omega]_{ij}$.

Consider the matrix Ω^1 such that Ω^1 differs from Ω only in its $(ij)^{th}$ element, all else equal, where $[\Omega^1]_{ij} = [\Omega^*]_{ij}$. Call it economy E^1.

Similarly, in economy E^1, consider the set $\left[S^1_j\right]$ such that there exists a directed upstream supply path from each $l \in S^1_j \setminus j$ to the given more automated task j. In addition, denote the set of sectors that has no direct upstream supply path to sector j by $\left\{S^1_j\right\}^C$.

If there is no change in the set of existing paths from Economy E^0 to the economy E^1, but only the weight of an existence link increase, then the result in Step 1 above still holds.

Consider that $\Omega_{ij} = 0$ and an increase in Ω_{ij} results in a new upstream link from j to i in Economy E^1. However, in such a case, the set of upstream suppliers of sector j does not change and the following conditions hold:

$S^0_j = S^1_j$

$\left\{S^0_j\right\}^C = \left\{S^1_j\right\}^C$

Thus for any k such that $k \in \left\{S^1_j\right\}^C$ and $l \in S^1_j$, $[D^0]_{kl} = [D^1]_{kl} = 0$ holds.

Next, we show that for each ordered pair $\{k, l\}$ such that $k \in \left\{S^1_j\right\}^C$ and $l \in \left\{S^1_j\right\}^C$, the following condition holds.

$D^1_{kl} = D^0_{kl}$

In order to show this, first, by Lemma 1:

$D^0_{kl} = D^0_{kk}\Omega_{lk} + \sum_{s \neq k} D^0_{ks}\Omega_{ls}$

$D^1_{kl} = D^1_{kk}\Omega^1_{lk} + \sum_{s \neq k} D^1_{ks}\Omega^1_{ls}$

For any $l \in \left\{S^0_j\right\}^C$ and $k \in \left\{S^0_j\right\}^C$, $\Omega_{lk} = \Omega^1_{lk}$ and $\Omega_{li} = \Omega^1_{li}$ holds if $l \neq i$, where i is the sector that uses more of automation good j. On the other hand, if $l = i$, then since Ω_{ij} rises, $D^0_{kj}\Omega_{ij}$ enters into the equation above. However since $D^0_{kj} = D^1_{kj} = 0$ holds, $D^0_{kj}\Omega_{ij} = D^1_{kj}\Omega^1_{ij} = 0$ holds. Moreover, the set $\left\{S^0_j\right\}^C$ remains same. Thus, the system of equations above remain same in both Economy E^0 and economy E^1. Similar to the previous part, there must exists a unique solution for the system of equations above. Therefore, the unique solution in Economy E^0 is exactly the same as in Economy E_1.

Lastly, the vector of Domar weights (so the centralities) is equal to $\mathbf{m} = (I - \Omega')^{-1}\overrightarrow{\beta}$.

Thus, in Economy E^0, $m_i = \sum_j D^0_{ij}\beta_j$, and in economy E^1, we have $m^1_i = \sum_j D^1_{ij}\beta_j$.

For constant β, for any $k \in \left\{S^0_j\right\}^C \left(= \left\{S^1_j\right\}^C\right)$, $\sum_i D^1_{ki}\beta_i = \sum_i D^0_{ki}\beta_i$ holds, which implies that:

$m^1_k = m_k$ holds for all $k \in \left\{S^1_j\right\}^C$.

Step 3) Next, we show that $D^1_{li} > D^0_{li}$ for all $l \in S^0_j$, where i is the sector that increases its automation in task j.

In order to show that,

$[D^0_{li}]_{l\in S^0_j} = D^0_{ll}\Omega_{il} + \sum_{k\neq l} D^0_{lk}\Omega_{ik}$

$[D^1_{li}]_{l\in S^1_j} = D^1_{ll}\Omega^1_{il} + \sum_{k\neq l} D^1_{lk}\Omega^1_{ik}$, which can be rewritten as:

$[D^1_{li}]_{l\in S^0_j} = D^1_{ll}\Omega^1_{il} + \sum_{k\neq l} D^1_{lk}\Omega^1_{ik}$,

For $l = j$, we have

$D^0_{ji} = D_{jj}\Omega_{ij} + \sum_{k\neq l} D_{jk}\Omega_{ik}$

$D^1_{ji} = D^1_{jj}\Omega^1_{ij} + \sum_{k\neq l} D^1_{jk}\Omega^1_{ik}$

Since, each $[D_{ij}]_{i,j\in N}$ is non-decreasing in any element $[\Omega_{kl}]_{k,l\in N}$ and since $\Omega^1_{ij} > \Omega_{ij}$, we conclude from above that $D^1_{ji} > D^0_{ji}$.

Next, consider any $l \in S^1_j$ and $l \neq j$:

$[D^0_{li}]_{l\in S^0_j\setminus j} = D^0_{ll}\Omega_{il} + D^0_{lj}\Omega_{ij} + \sum_{k\neq j} D^0_{lk}\Omega_{ik}$

$[D^1_{li}]_{l\in S^1_j\setminus j} = D^1_{ll}\Omega^1_{il} + D^1_{lj}\Omega^1_{ij} + \sum_{k\neq j} D^1_{lk}\Omega^1_{ik}$, which can be rewritten as

$[D^1_{li}]_{l\in S^0_j\setminus j} = D^1_{ll}\Omega^1_{il} + D^1_{lj}\Omega^1_{ij} + \sum_{k\neq j} D^1_{lk}\Omega^1_{ik}$

Next, by combining:

i) for each $l \in S^0_j \setminus j$, $D_{lj} > 0$,

ii) each $[D_{ij}]_{i,j\in N}$ is non-decreasing in any element $[\Omega_{kl}]_{k,l\in N}$, and

iii) $\Omega^1_{ij} > \Omega_{ij}$,

we conclude that $\{D^1_{li}\} > \{D^0_{li}\}$ for each $l \in S_j$.

Thus, any sector k that is direct or indirect supplier of sector j ($k \in S^0_j = S^1_j$) has a higher dependency to sector i.

Lastly, for constant β, for any $l \in S^0_j = S^1_j$, $\sum_i D^1_{li}\beta_i > \sum_i D^0_{li}\beta_i$ holds, which implies that:

$m^1_i > m_i$ holds for all $l \in S_j$.

Step 4) We do this iteration one by one for each increase in automation for an ordered pair $\{i, j\}$ such that $[\Omega^{t+1}]_{ij} > [\Omega^t]_{ij} > 0$ until we reach the equilibrium input-output network Ω^*, where the same results above hold at each step.

end of proof.

8.5 Equilibrium Analysis of N-Sector Economy

The cost minimization problem for each firm $i \in N$ is:

$$\min_{\{L_{ij}\}_{j\in 0\cup K}, H_i, \{X_{ij}\}_{j\in N}} \sum_{j\in 0\cup K} w_L L_{ij} + w_H H_i + \sum_{j\in N} p_j X_{ij}$$
$$\text{subject to } 1 = A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j\in a_i} (L_{ij} + X_{ij})^{\alpha_{ij}^L}\right] \left[\prod_{j\in n_i} (X_{ij})^{\alpha_{ij}^n}\right]$$

The Lagrangian function is:

$$\mathcal{L} = \sum_{j\in 0\cup K} w_L L_{ij} + w_H H_i + \sum_{j\in N} p_j X_{ij} - \lambda_i \left(A_i^P (L_{i0})^{\alpha_{i0}^L} (H_i)^{\alpha_i^H} \left[\prod_{j\in a_i} (L_{ij} + X_{ij})^{\alpha_{ij}^L}\right] \left[\prod_{j\in n_i} (X_{ij})^{\alpha_{ij}^n}\right] - 1 \right)$$

The first order conditions are:

- $\frac{\partial \mathcal{L}}{\partial \lambda_i} = 1 - A_i^P (L_{i0}^*)^{\alpha_{i0}^L} (H_i^*)^{\alpha_i^H} \left[\prod_{j\in a_i} \left(L_{ij}^* + X_{ij}^*\right)^{\alpha_{ij}^L}\right] \left[\prod_{j\in n_i} (X_{ij}^*)^{\alpha_{ij}^n}\right] = 0$

  $1 = A_i^P (L_{i0}^*)^{\alpha_{i0}^L} (H_i^*)^{\alpha_i^H} \left[\prod_{j\in a_i} \left(L_{ij}^* + X_{ij}^*\right)^{\alpha_{ij}^L}\right] \left[\prod_{j\in n_i} (X_{ij}^*)^{\alpha_{ij}^n}\right]$

  $\frac{\partial \mathcal{L}}{\partial H_i} = w_H - \frac{\lambda_i^* \alpha_i^H}{H_i^*} = 0$

  $H_i^* = \frac{\lambda_i^* \alpha_i^H}{w_H}$

- $\frac{\partial \mathcal{L}}{\partial L_{i0}} = w_L - \frac{\lambda_i^* \alpha_{i0}^L}{L_{i0}^*} = 0$

  $L_{i0}^* = \frac{\lambda_i^* \alpha_{i0}^L}{w_L}$

- $\frac{\partial \mathcal{L}}{\partial \{X_{ij}\}_{j\in n_i}} = p_j - \frac{\lambda_i^* \alpha_{ij}^n}{\{X_{ij}^*\}_{j\in n_i}} = 0$

  $\{X_{ij}^*\}_{j\in n_i} = \frac{\lambda_i^* \alpha_{ij}^n}{p_j}$

- $\{L_{ij}^*, X_{ij}^*\}_{j\in a_i} =: \begin{cases} \{\lambda_i^* \frac{\alpha_{ij}^L}{w_L}, 0\} & \text{if } \frac{w_L}{p_j} < 1 \\ \{\lambda_i^* \frac{(1-t_{ij}^*)\alpha_{ij}^L}{w_L}, \lambda_i^* \frac{t_{ij}^* \alpha_{ij}^L}{p_j}\} & \text{if } \frac{w_L}{p_j} = 1 \\ \{0, \lambda_i^* \frac{\alpha_{ij}^L}{p_j}\} & \text{if } \frac{w_L}{p_j} > 1 \end{cases}$

- $\{\mathbf{L_{ij}^*}, \mathbf{X_{ij}^*}\}_{\mathbf{j\notin a_i\cup n_i}} = \mathbf{0}$

The FOCs above together imply:

$1 = A_i^P (\frac{\lambda_i^* \alpha_{i0}^L}{w_L})^{\alpha_{i0}^L} (\frac{\lambda_i^* \alpha_i^H}{w_H})^{\alpha_i^H} \left[\prod_{j\in a_i} \left(L_{ij}^* + X_{ij}^*\right)^{\alpha_{ij}^L}\right] \left[\prod_{j\in n_i} (\frac{\lambda_i^* \alpha_{ij}^n}{p_j})^{\alpha_{ij}^n}\right]$

$1 = A_i^P (\frac{\lambda_i^* \alpha_{i0}^L}{w_L})^{\alpha_{i0}^L} (\frac{\lambda_i^* \alpha_i^H}{w_H})^{\alpha_i^H} \left[\prod_{j\in a_i} \left(\frac{\lambda_i^* \alpha_{ij}^L}{w_L}\right)^{(1-t_{ij}^*)\alpha_{ij}^L}\right] \left[\prod_{j\in a_i} \left(\frac{\lambda_i^* \alpha_{ij}^L}{p_j}\right)^{t_{ij}^* \alpha_{ij}^L}\right] \left[\prod_{j\in n_i} (\frac{\lambda_i^* \alpha_{ij}^n}{p_j})^{\alpha_{ij}^n}\right]$

Lastly, profit maximization implies that the unit cost for each i is equal to p_i. Thus $\lambda_i^* = p_i$ can be rewritten as:

$$p_i = B_i \left[\prod_{j \in N} (p_j)^{t_{ij}^* \alpha_{ij}} \right] (w_L)^{\alpha_{i0}^L + \sum_{j \in N} (1 - t_{ij}^*) \alpha_{ij}^L} (w_H)^{\alpha_i^H} \tag{41}$$

where

$$B_i = \frac{1}{(A_i^P)(\alpha_i^H)^{\alpha_i^H} (\alpha_{i0}^L)^{\alpha_{i0}^L} \left[\prod_{j \in N} (\alpha_{ij}^L)^{\alpha_{ij}^L} \right] \left[\prod_{j \in N} (\alpha_{ij}^n)^{\alpha_{ij}^n} \right]} \tag{42}$$

and T^* is the equilibrium expenditure-weight matrix satsfying the following conditions:
$t_{ij}^* = 1$ for all $j \in n_i$
$t_{ij}^* \in [0, 1]$ for all $j \in a_i$.
Then, the equilibrium input-output matrix is Ω is the Hadamard product of two matrices:

$$\Omega = T^* \circ \Omega^S$$

where Ω^S is the structural input-output matrix satfying the conditions below:
$[\Omega^S]_{ij} = \alpha_{ij}^n > 0$ is the task-weight of non-automatable task $j \in n_i$ in sector i.
$[\Omega^S]_{ij} = \alpha_{ij}^L > 0$ is the task-weight of automatable task $j \in a_i$ in sector i.
$[\Omega^S]_{ij} = 0$ for any $j \in N$ such that $j \notin a_i \cup n_i$.
Then, we can write the equation for prices as follows:

$$p_i = B_i \left[\prod_{j \in N} (p_j)^{\omega_{ij}} \right] (w_L)^{\alpha_i^L} (w_H)^{\alpha_i^H} \tag{43}$$

where

- $\alpha_i^L = \alpha_{i0}^L + \sum_{j \in N} (1 - t_{ij}^*) \alpha_{ij}^L$
- $\omega_{ij} = t_{ij} \alpha_{ij}^L$ for all pairs (i, j) such that $j \in a_i$
- $\omega_{ij} = \alpha_{ij}^n$ for all pairs (i, j) such that $j \in n_i$.

i) Low- and High-skilled labor wages
$C_i = \frac{\beta_i C}{p_i}$
$Y_i = (\sum_{j \in N} \frac{\omega_{ji} p_j Y_j}{p_i}) + \frac{\beta_i (w_L L + w_H H)}{p_i}$
$p_i Y_i = (\sum_{j \in N} \omega_{ji} p_j Y_j) + \beta_i (w_L L + w_H H)$
In matrix notation:
$\overrightarrow{p_i Y_i} = (I - \Omega')^{-1} \left[\overrightarrow{\beta_i (w_L L + w_H H)} \right]$
Then, we can write $p_i Y_i$ as follows:

$p_i Y_i = m_i(w_L L + w_H H)$
where $m_i = \sum_{j \in N} [(I - \Omega')^{-1}]_{ij} \beta_j$.
$\alpha_i^H p_i Y_i = \alpha_i^H [m_i(w_L L + w_H H)]$
$\sum_{i \in N} \alpha_i^H p_i Y_i = \sum_{i \in N} \left[\alpha_i^H [m_i(w_L L + w_H H)]\right]$
$w_H H = \sum_{i \in N} \left[\alpha_i^H [m_i(w_L L + w_H H)]\right]$
$w_H H = \sum_{i \in N} \left[\alpha_i^H m_i w_L L + \alpha_i^H m_i w_H H\right]$
$w_H H(1 - \sum_{i \in N} \alpha_i^H m_i) = \sum_{i \in N} \left[\alpha_i^H m_i w_L L\right]$
Thus:

$$\frac{w_L}{w_H} = \left(\frac{H}{L}\right) \frac{1 - \sum_{i \in N} \alpha_i^H m_i}{\sum_{i \in N} \alpha_i^H m_i} \tag{44}$$

where m_i is the Domar weight of firm i.

Proof of Proposition 6:

Part i) The proof immediately follows from Eq. (44). Since the input-output network, so the m_i remains fixed for each sector, $\frac{w_H}{w_L}$ also remains constant.

Part ii) By using $(I - \Omega)^{-1} \left[\alpha_i^L + \vec{\alpha_i^H}\right] = \mathbf{1}$, we can rewrite Eq. (49) as follows

$$dln\left(\frac{p_i}{w_L}\right) = (I - \Omega)^{-1} \left[-dlnA_i^P + \vec{\alpha_i^H} dln\left(\frac{w_H}{w_L}\right)\right]$$

Consider an automation good sector j that is in transition, which implies that $p_j = w_L$. Then, following a shock to productivities, $p_j = w_L$ still holds for any such sector j and, thus, $dln\left(\frac{p_j}{w_L}\right) = 0$ holds. Then, for $K = (I - \Omega)^{-1}$, we have $dln\left(\frac{w_H}{w_L}\right) = \frac{\sum_{i \in N} K_{ji} \frac{dlnA_i^P}{\alpha_i^H}}{\sum_{i \in N} K_{ji}}$.

Proof of Proposition 7:

By taking logs of both sides of Equation (43), in matrix notation we have:

$$\left[ln\vec{p}_i\right] = (I - \Omega)^{-1} \left[lnB_i + \alpha_i^L ln\vec{w}_L + \alpha_i^H lnw_H\right] \tag{45}$$

By taking the total derivative of both sides of Eq. (45):

$$\left[dln\vec{p}_i\right] = (I - \Omega)^{-1} \left[dlnB_i + d\left(\alpha_i^L ln\vec{w}_L\right) + d\left(\alpha_i^H lnw_H\right)\right] + d\left((I - \Omega)^{-1}\right) \left[lnB_i + \alpha_i^L ln\vec{w}_L + \alpha_i^H lnw_H\right] \tag{46}$$

Next, we write the total derivative of $(I - \Omega)^{-1}$

$$(I - \Omega)^{-1}(I - \Omega) = I$$

$$(I-\Omega)^{-1}d\left[(I-\Omega)\right]+d\left((I-\Omega)^{-1}\right)(I-\Omega)=0$$

$$(I-\Omega)^{-1}d\left[(I-\Omega)\right]=-d\left((I-\Omega)^{-1}\right)(I-\Omega)$$

Lastly, by multiplying both sides by $-(I-\Omega)^{-1}$ from right and plugging $d\left((I-\Omega)\right)=-d\Omega$, we get

$$d\left((I-\Omega)^{-1}\right)=(I-\Omega)^{-1}(d\Omega)(I-\Omega)^{-1} \tag{47}$$

Then, by plugging Eq. (47) into Eq. (46)

$$\left[d\vec{ln}p_i\right]=(I-\Omega)^{-1}\left[\left[dlnB_i+\alpha_i^L dlnw_L+\left(\vec{d}\alpha_i^L\right)lnw_L+\alpha_i^H dlnw_H\right]+(d\Omega)(I-\Omega)^{-1}\left[lnB_i+\alpha_i^L\vec{lnw}_L+\alpha_i^H lnw_H\right]\right] \tag{48}$$

By taking log of Eq. (43), we get :

$$lnp_i-\sum_j\omega_{ij}lnp_j=lnB_i+\alpha_i^L lnw_L+\alpha_i^H lnw_H$$

In matrix notation:

$$(I-\Omega)\left[\vec{lnp}_i\right]=\left[lnB_i+\alpha_i^L\vec{lnw}_L+\alpha_i^H lnw_H\right]$$

Then, we can rewrite Eq. (48) as follows:

$$\left[d\vec{ln}p_i\right]=(I-\Omega)^{-1}\left[\left[dlnB_i+\alpha_i^L dlnw_L+(\vec{d}\alpha_i^L)lnw_L+\alpha_i^H dlnw_H\right]+(d\Omega)\left[\vec{lnp}_i\right]\right]$$

$[d\Omega]_{ij}=0$ if automation of task j is not in the transition path in sector i. For any j such that j is in the transition path in sector i, $[d\Omega]_{ij}=-d\alpha_{ij}^L$.

Moreover, for any such j, $\frac{p_j}{w_L}=1$, which implies that $lnp_j=lnw_L$. Then, $\left(d\alpha_i^L\right)lnw_L$ and $(d\Omega)\left[\vec{lnp}_i\right]$ cancel out and we get:

$$\left[d\vec{ln}p_i\right]=(I-\Omega)^{-1}\left[dlnB_i+\alpha_i^L d\vec{lnw}_L+\alpha_i^H dlnw_H\right]$$

Eq. (42) implies that $dlnB_i=-dlnA_i^P$, and so that

$$\left[d\vec{ln}p_i\right]=(I-\Omega)^{-1}\left[-dlnA_i^P+\alpha_i^L d\vec{lnw}_L+\alpha_i^H dlnw_H\right] \tag{49}$$

Lastly, the FOCs of the profit maximization of the final good producer implies that $\beta_i p_f C=p_i C_i$. By taking the log and differentiating both sides, we have:

$$\left[d\vec{ln}p_i\right]=\left[d\vec{ln}C\right]-\left[d\vec{ln}C_i\right]$$

By plugging this into Eq. (49) , we get:

$$\left[dl\vec{n}C_i\right] = \left[dl\vec{n}C\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{n}w_L - \alpha_i^H dlnw_H\right] \tag{50}$$

$$\left[dl\vec{n}C_i\right] = \left[dl\vec{n}C\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{n}w_L - \alpha_i^H dlnw_H\right] \tag{51}$$

Proof of Proposition 8:

ii) High-skilled labor (H_i)

$$H_i = \frac{\alpha_i^H p_i Y_i}{w_H} = \frac{\left[\alpha_i^H m_i (L + w_H H)\right] H\left(1 - \sum\limits_{i\in N}\alpha_i^H m_i\right)}{L\left(\sum\limits_{i\in N}\alpha_i^H m_i\right)}$$

$$\frac{H_i}{H} = \frac{\left[\alpha_i^H m_i\left(L + \frac{L\left(\sum\limits_{i\in N}\alpha_i^H m_i\right)}{\left(1-\sum\limits_{i\in N}\alpha_i^H m_i\right)}\right)\right]\left(1-\sum\limits_{i\in N}\alpha_i^H m_i\right)}{L\left(\sum\limits_{i\in N}\alpha_i^H m_i\right)}$$

Thus, the unit of high-skilled worker in sector i, H_i, is equal to:

$$H_i = H\frac{\alpha_i^H m_i}{\sum\limits_{i\in N}\alpha_i^H m_i} \tag{52}$$

iii) Low-skilled labor (L_i)

$$w_L L = \sum_{i\in N}\left[\sum_{j\in T_i^L}\alpha_{ij}^L p_i Y_i\right]$$

where T_i^L is the set of tasks that are performed by labor in sector i. For $w_L = 1$,

$$L = \sum_{i\in N}\left[\sum_{j\in T_i^L}\alpha_{ij}^L p_i Y_i\right]$$

$$L = \sum_{i\in N}\alpha_i^L p_i Y_i$$

where $\alpha_i^L = \alpha_{i0}^L + \sum\limits_{j\in T_i^L}\alpha_{ij}^L$.

$$\frac{L_i}{L} = \frac{\alpha_i^L p_i Y_i}{\sum\limits_{i\in N}\alpha_i^L p_i Y_i}$$

$$\frac{L_i}{L} = \frac{\alpha_i^L m_i\left(L + \frac{L\left(\sum\limits_{i\in N}\alpha_i^H m_i\right)}{\left(1-\sum\limits_{i\in N}\alpha_i^H m_i\right)}\right)}{\sum\limits_{i\in N}\left(\alpha_i^L m_i\left(L + \frac{L\left(\sum\limits_{i\in N}\alpha_i^H m_i\right)}{\left(1-\sum\limits_{i\in N}\alpha_i^H m_i\right)}\right)\right)}$$

Thus, the unit of low-skilled worker in sector i, L_i, is:

$$L_i = L\frac{\alpha_i^L m_i}{\sum\limits_{i\in N}\alpha_i^L m_i} \tag{53}$$

$$\frac{Y_i}{L_i} = \frac{Y_i}{L\frac{\alpha_i^L m_i}{\sum\limits_{i\in N}\alpha_i^L m_i}} \tag{54}$$

By taking log and total derivative of both sides:

$$dln\left(\frac{Y_i}{L_i}\right) = dlnY_i + dln\left(\sum_{i\in N}\alpha_i^L m_i\right) - dln\alpha_i^L - dlnm_i$$

The profit maximization of representative final good producer ($p_iC_i = \beta_i p_f C$) and the equation for Domar weight $\left(m_i = \frac{p_iY_i}{p_fC}\right)$ together imply:

$$\frac{Y_i}{C_i} = \frac{m_i}{\beta_i}$$

By taking logs and totally differentiating both sides:

$$dlnY_i = dlnC_i + dlnm_i$$

Then,

$$\left[dl\vec{n}Y_i\right] = \left[dl\vec{n}C\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{n}w_L - \alpha_i^H dlnw_H\right] + \left[dl\vec{n}m_i\right]$$

$$\left[dln\vec{\frac{Y_i}{L_i}}\right] = \left[dl\vec{n}C\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{n}w_L - \alpha_i^H dlnw_H\right] + \left[dln\left(\sum_{i\in N}\vec{\alpha_i^L m_i}\right)\right] - \left[dl\vec{n}\alpha_i^L\right]$$

By plugging $dlnC + dln\left(\sum\limits_{i\in N}\alpha_i^L m_i\right) = dlnw_L$ into above:

$$\left[dln\vec{\frac{Y_i}{L_i}}\right] = \left[dl\vec{n}w_L\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{n}w_L - \alpha_i^H dlnw_H\right] - \left[dl\vec{n}\alpha_i^L\right]$$

Next, we compare $\left[dln\vec{\frac{Y_i}{L_i}}\right]$ and $\left[dl\vec{n}Y_i\right]$:

$$\left[dln\vec{\frac{Y_i}{L_i}}\right] = \left[dl\vec{n}C\right] + \left[dln\left(\sum_{i\in N}\vec{\alpha_i^L m_i}\right)\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{n}w_L - \alpha_i^H dlnw_H\right] - \left[dl\vec{n}\alpha_i^L\right]$$

$$\left[dl\vec{n}Y_i\right] = \left[dl\vec{n}C\right] + \left[dl\vec{n}m_i\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{n}w_L - \alpha_i^H dlnw_H\right]$$

$$\left[dl\vec{n}L_i\right] = \left[dl\vec{n}Y_i\right] - \left[dln\vec{\frac{Y_i}{L_i}}\right] = \left[dl\vec{n}m_i\right] + \left[dl\vec{n}\alpha_i^L\right] - \left[dln\left(\sum_{i\in N}\vec{\alpha_i^L m_i}\right)\right]$$

Next, we derive $dln\left(\frac{Y_i}{H_i}\right)$:

$$\frac{Y_i}{H_i} = \frac{Y_i}{H\frac{\alpha_i^H m_i}{\sum\limits_{i\in N}\alpha_i^H m_i}} \tag{55}$$

$$dln\frac{Y_i}{H_i} = dlnY_i + dln\left(\sum_{i\in N}\alpha_i^H m_i\right) - dln\alpha_i^H - dlnm_i$$

$$\left[dln\vec{\frac{Y_i}{H_i}}\right] = \left[dl\vec{n}C\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{nw}_L - \alpha_i^H dlnw_H\right] + \left[dln\left(\sum_{i\in N}\vec{\alpha_i^H m_i}\right)\right]$$

By plugging $dlnC + dln\left(\sum_{i\in N}\alpha_i^H m_i\right) = dlnw_H$ into above:

$$\left[dln\vec{\frac{Y_i}{H_i}}\right] = \left[dl\vec{nw}_H\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{nw}_L - \alpha_i^H dlnw_H\right]$$

Next, we compare $\left[dl\vec{n\frac{Y_i}{H_i}}\right]$ and $\left[dl\vec{n}Y_i\right]$:

$$\left[dl\vec{n}Y_i\right] = \left[dl\vec{n}C\right] + \left[dl\vec{n}m_i\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{nw}_L - \alpha_i^H dlnw_H\right]$$

$$\left[dln\vec{\frac{Y_i}{H_i}}\right] = \left[dl\vec{n}C\right] + \left[dln\left(\sum_{i\in N}\vec{\alpha_i^H m_i}\right)\right] + (I-\Omega)^{-1}\left[dlnA_i^P - \alpha_i^L dl\vec{nw}_L - \alpha_i^H dlnw_H\right]$$

$$\left[dl\vec{n}H_i\right] = \left[dl\vec{n}Y_i\right] - \left[dln\vec{\frac{Y_i}{H_i}}\right] = \left[dl\vec{n}m_i\right] - \left[dln\left(\sum_{i\in N}\vec{\alpha_i^H m_i}\right)\right]$$

Proof of Proposition 3:

$$\begin{cases} A^* = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}}\left(\frac{L}{H}\frac{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H} \\ A^{**} = \frac{1}{(\alpha_a^H)^{\alpha_a^H}(\alpha_{a0}^L)^{\alpha_{a0}^L}}\left(\frac{L}{H}\frac{\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H} \end{cases}$$

i) A^* and A^{**} are increasing in $\frac{L}{H}$

$$\frac{\partial A^*}{\partial\left(\frac{L}{H}\right)} = \left(\frac{L}{H}\right)^{\alpha_a^H-1}\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{1-\alpha_a^H}\left(\frac{\alpha_{fn}^n\alpha_n^H+\alpha_f^H}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H} > 0$$

$$\frac{\partial A^{**}}{\partial\left(\frac{L}{H}\right)} = \left(\frac{L}{H}\right)^{\alpha_a^H-1}\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{1-\alpha_a^H}\left(\frac{\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H} > 0$$

ii) A^* and A^{**} are increasing in α_n^H

$$\frac{\partial A^*}{\partial(\alpha_n^H)} = \frac{\left(\frac{a_a^H}{\alpha_{a0}^L}\right)^{\alpha_{a0}^L}\alpha_{fn}^n\left(\frac{L}{H}\frac{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H}}{\left(1-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} > 0$$

$$\frac{\partial A^{**}}{\partial(\alpha_n^H)} = \frac{(\frac{a_a^H}{\alpha_{a0}^L})^{\alpha_{a0}^L}\alpha_{fn}^n\left(\frac{L}{H}\frac{\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H}}{\left(1-\alpha_{fa}^L\alpha_a^H-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} > 0$$

iii) For constant α_{fn}^n and α_{fa}^L, A^* and A^{**} are increasing in α_f^H (decreasing in α_{f0}^L). (done)

$$\frac{\partial A^{*}}{\partial(\alpha_f^H)} = \frac{(\frac{a_a^H}{\alpha_{a0}^L})^{\alpha_{a0}^L}\left(\frac{L}{H}\frac{\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H}}{\left(1-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)\left(\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} > 0$$

$$\frac{\partial A^{**}}{\partial(\alpha_f^H)} = \frac{(\frac{a_a^H}{\alpha_{a0}^L})^{\alpha_{a0}^L}\left(\frac{L}{H}\frac{\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}{1-\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)}\right)^{\alpha_a^H}}{\left(1-\alpha_{fa}^L\alpha_a^H-\alpha_{fn}^n\alpha_n^H-\alpha_f^H\right)\left(\alpha_{fa}^L\alpha_a^H+\alpha_{fn}^n\alpha_n^H+\alpha_f^H\right)} > 0$$