Combining Dimension Reduction, Distance Measures and Covariance

Ravi Kashyap

SolBridge International School of Business / City University of Hong Kong

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1 Abstract

We develop a novel methodology based on the marriage between the Bhattacharyya distance, a measure of similarity across distributions of random variables, and the Johnson-Lindenstrauss Lemma, a technique for dimension reduction. The resulting technique is a simple yet powerful tool that allows comparisons between data-sets representing any two distributions. The degree to which different entities, (markets, universities, hospitals, cities, groups of securities, etc.), have different distance measures of their corresponding distributions tells us the extent to which they are different, aiding participants looking for diversification or looking for more of the same thing. We demonstrate a relationship between covariance and distance measures based on a generic extension of Stein's Lemma. We consider an asset pricing application and then briefly discuss how this methodology lends itself to numerous market-structure studies and even applications outside the realm of finance / social sciences by illustrating a biological application. We provide numerical illustrations using security prices, volumes and volatilities of both these variables from six different countries.

2 Introduction

The varying behavior of participants in a social system, which can also be viewed as unpredictable actions, will give rise to unintended consequences and as long as participants are free to observe the results and modify their actions, this effect will persist (Kashyap 2017; appendix 13 has a discussion of uncertainty and unintended consequences). Unintended consequences and its siamese twin, uncertainty, are fostering a trend of collecting data to improve decision making, which is perhaps leading to more analysis of the data and more actions, leading to a cycle of increasing data collection and actions, giving rise to information explosion (Dordick & Wang 1993; Korth & Silberschatz 1997; Sweeney 2001; Fuller 2010; Major & Savin-Baden 2010; Beath, Becerra-Fernandez, Ross & Short 2012). (Kashyap 2015) consider ways to reduce the complexity of social systems, which could be one way to mitigate the effect of unintended outcomes. While attempts at designing less complex systems are worthy endeavors, reduced complexity might be hard to accomplish in certain instances and despite successfully reducing complexity, alternate techniques at dealing with uncertainty are commendable complementary pursuits (Kashyap 2016).

While it might be possible to observe historical trends (or other attributes) and make comparisons across fewer number of entities; in large systems where there are numerous components or contributing elements, this can be a daunting task. In this present paper, we present quantitative measures across aggregations of smaller elements that can aid decision makers by providing simple yet powerful metrics to compare large groups of entities.

We consider a measure of similarity, the Bhattacharyya distance, across distributions of variables. We develop a novel methodology based on the marriage between the Bhattacharyya distance and the Johnson-Lindenstrauss Lemma (JL-Lemma), a technique for dimension re-

duction; providing us with a simple yet powerful tool that allows comparisons between data-sets representing any two distributions. The degree to which different entities, (markets, universities, hospitals, countries, cities, farms, forests, groups of securities, etc.), have different distance measures of their corresponding distributions tells us the extent to which they are different, aiding participants looking for diversification or looking for more of the same thing. To calculate the distance measure between any two random variables (both of which could be multi-variate with either different number of component random variables or observations), we utilize the JL-Lemma to reduce the dimensions of one of the random variables (either the number of component random variables or the number of observations) such that the two random variables have the same number of dimensions allowing us to apply distance measures.

We demonstrate a relationship between covariance and distance measures based on a generic extension of Stein's Lemma. We consider an asset pricing application and also briefly discuss how this methodology lends itself to numerous market-structure and asset pricing studies and even applications outside the realm of finance / social sciences by illustrating a biological application (sections 6, 6.1, 6.2). We illustrate some ways in which our techniques can be used in practice by comparing stock prices, trading volumes, price volatilities and volume volatilities across six markets (sections 7.1; 7.2).

An unintended consequence of our efforts, has become a review of the vast literature on distance measures and related statistical techniques, which can be useful for anyone that attempts to apply the corresponding techniques to the problems mentioned here and also to the many unmentioned but possibly related ones. The results and the discussion draw upon sources from statistics, probability, economics / finance, communication systems, pattern recognition and information theory; becoming one example of how elements of different fields can be combined to provide answers to the questions raised by a particular field. All the propositions are new results and they depend on existing results which are given as lemmas without proof. Such an approach ensures that the results are instructive and immediately applicable to a wider audience.

3 Literature Review of Methodological Fundamentals

3.1 Bhattacharyya Distance

We use the Bhattacharyya distance (Bhattacharyya 1943; 1946) as a measure of similarity or dissimilarity between the probability distributions of the two entities we are looking to compare. These entities could be two securities, groups of securities, markets or any statistical populations that we are interested in studying. The Bhattacharyya distance is defined as the negative logarithm of the Bhattacharyya coefficient.

$$D_{BC}\left(p_i, p_i'\right) = -\ln\left[\rho\left(p_i, p_i'\right)\right] \tag{1}$$

Here, $D_{BC}(p_i, p'_i)$, is the Bhattacharyya Distance between two multinomial populations each consisting of k categories or classes with associated probabilities $p_1, p_2, ..., p_k$ and $p'_1, p'_2, ..., p'_k$ respectively. The Bhattacharyya coefficient, $\rho(p_i, p'_i)$, is calculated as shown below for discrete and continuous probability distributions.

$$\rho(p_i, p'_i) = \sum_i^k \sqrt{p_i p'_i} \tag{2}$$

$$\rho(p_i, p'_i) = \int \sqrt{p_i(x)\, p'_i(x)} dx \tag{3}$$

Bhattacharyya's original interpretation of the measure was geometric (Derpanis 2008). He considered two multinomial populations each consisting of k categories or classes with associated probabilities $p_1, p_2, ..., p_k$ and $p'_1, p'_2, ..., p'_k$ respectively. Then, as $\sum_i^k p_i = 1$ and $\sum_i^k p'_i = 1$, he noted that $(\sqrt{p_1}, ..., \sqrt{p_k})$ and $(\sqrt{p'_1}, ..., \sqrt{p'_k})$ could be considered as the direction cosines of two vectors in $k-$dimensional space referred to a system of orthogonal co-ordinate axes. As a measure of divergence between the two populations Bhattacharyya used the square of the angle between the two position vectors. If θ is the angle between the vectors then:

$$\rho(p_i, p'_i) = cos\theta = \sum_i^k \sqrt{p_i p'_i} \tag{4}$$

Thus if the two populations are identical: $cos\theta = 1$ corresponding to $\theta = 0$, hence we see the intuitive motivation behind the definition as the vectors are co-linear. Bhattacharyya further showed that by passing to the limiting case a measure of divergence could be obtained between two populations defined in any way given that the two populations have the same number of variates. The value of coefficient then lies between 0 and 1.

$$0 \leq \rho(p_i, p'_i) = \sum_i^k \sqrt{p_i p'_i} = \sum_i^k p_i \sqrt{\frac{p'_i}{p_i}} \leq \sqrt{\sum_i^k p'_i} = 1 \tag{5}$$

$$0 \leq D_{BC}(p_i, p'_i) \leq \infty \tag{6}$$

Here, the last inequality follows from Jensen's inequality. (Comaniciu, Ramesh & Meer 2003) modify this as shown below and prove that this alternate measure, $d(p_i, p'_i)$ termed the modified Bhattacharyya Metric, follows all the metric axioms: positive, symmetric, is zero for the same two elements and satisfies the triangle inequality.

$$d(p_i, p'_i) = \sqrt{1 - \rho(p_i, p'_i)} \tag{7}$$

We get the following formulae (Lee and Bretschneider 2012) for the Bhattacharyya distance when applied to the case of two univariate normal distributions.

$$D_{BC-N}(p, q) = \frac{1}{4} \ln \left(\frac{1}{4} \left(\frac{\sigma_p^2}{\sigma_q^2} + \frac{\sigma_q^2}{\sigma_p^2} + 2 \right) \right) + \frac{1}{4} \left(\frac{(\mu_p - \mu_q)^2}{\sigma_p^2 + \sigma_q^2} \right) \tag{8}$$

Here, $D_{BC-N}(p,q)$ is the Bhattacharyya distance between p and q normal distributions or classes.σ_p^2 is the variance of the $p-$th distribution, μ_p is the mean of the $p-$th distribution, and p,q are two different distributions.

The original paper on the Bhattacharyya distance (Bhattacharyya 1943) mentions a natural extension to the case of more than two populations. For an M population system, each with k random variates, the definition of the coefficient becomes,

$$\rho\left(p_1,p_2,...,p_M\right)=\int\cdots\int\left[p_1\left(x\right)p_2\left(x\right)...p_M\left(x\right)\right]^{\frac{1}{M}}dx_1\cdots dx_k \tag{9}$$

For two multivariate normal distributions, $D_{BC-MN}\left(p_1,p_2\right)$ is the Bhattacharyya distance between two multivariate normal distributions, $\boldsymbol{p_1},\boldsymbol{p_2}$ where $\boldsymbol{p_i}\sim\mathcal{N}(\boldsymbol{\mu}_i,\,\boldsymbol{\Sigma}_i)$.

$$D_{BC-MN}\left(p_1,p_2\right)=\frac{1}{8}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}_1-\boldsymbol{\mu}_2)+\frac{1}{2}\ln\left(\frac{\det\boldsymbol{\Sigma}}{\sqrt{\det\boldsymbol{\Sigma}_1\,\det\boldsymbol{\Sigma}_2}}\right) \tag{10}$$

$\boldsymbol{\mu}_i$ and $\boldsymbol{\Sigma}_i$ are the means and covariances of the distributions, and $\boldsymbol{\Sigma}=\frac{\boldsymbol{\Sigma}_1+\boldsymbol{\Sigma}_2}{2}$. We need to keep in mind that a discrete sample could be stored in matrices of the form A and B, where, n is the number of observations and m denotes the number of variables captured by the two matrices.

$$\boldsymbol{A_{m\times n}}\sim\mathcal{N}\left(\boldsymbol{\mu_1},\boldsymbol{\Sigma_1}\right) \tag{11}$$

$$\boldsymbol{B_{m\times n}}\sim\mathcal{N}\left(\boldsymbol{\mu_2},\boldsymbol{\Sigma_2}\right) \tag{12}$$

The Bhattacharyya measure finds numerous applications in communications, pattern recognition and information theory (Mak and Barnard 1996; Guorong, Peiqi & Minhui 1996). Other measures of divergence include the Mahalanobis distance (Mahalanobis 1936) which is a measure of the distance between a point and a distribution, Kullback-Leibler (KL) divergence (Kullback & Leibler 1951) and the Hellinger or Matusita measure, $D_{H-M}\left(p_i,p_i'\right)$, (Hellinger 1909; Matusita 1955) which is related to the Bhattacharyya distance since minimizing the Matusita distance is equivalent to maximizing the Bhattacharyya distance. The KL measure is not symmetric and is discussed in (Duchi 2007; Contreras-Reyes & Arellano-Valle 2012) when applied to normal or skew normal distributions. In (Aherne, Thacker & Rockett 1998) it is shown that the Bhattacharyya coefficient approximates the χ^2-measure (Chi-Squared measure), while avoiding the singularity problem that occurs when comparing instances of the distributions that are both zero.

$$D_{H-M}\left(p_i,p_i'\right)=\sum_i^k\left(\sqrt{p_i}-\sqrt{p_i'}\right)^2=2-2\rho\left(p_i,p_i'\right) \tag{13}$$

$$\chi^2\left(p_i, p_i'\right) = \frac{1}{2}\sum_i^k \frac{\left(p_i - p_i'\right)^2}{\left(p_i + p_i'\right)} \tag{14}$$

For discrete probability distributions P and Q, the Kullback–Leibler divergence, $D_{\mathrm{KL}}(P\|Q)$, of Q from P is defined to be,

$$D_{\mathrm{KL}}(P\|Q) = \sum_i P(i)\,\log\frac{P(i)}{Q(i)} \tag{15}$$

For distributions P and Q, of continuous random variables, the Kullback–Leibler divergence is defined (Bishop 2006) to be the integral below, where p and q denote the densities of P and Q. (Huzurbazar 1955) proves a remarkable property that for all distributions admitting sufficient statistics the exact forms of the KL divergence come out as explicit functions of the parameters of the distribution.

$$D_{\mathrm{KL}}(P\|Q) = \int_{-\infty}^{\infty} p(x)\,\log\frac{p(x)}{q(x)}\,\mathrm{d}x \tag{16}$$

(Schweppe 1967b) develops new expressions for the KL divergence and the Bhattacharyya distance, in terms of the effects of various conditional expectation filters (physically realizable linear systems) acting on Gaussian processes. In particular, the distances are given by time integrals of the variances and mean values of the outputs of filters designed to generate the conditional expectations of certain processes. Defining a matched filter to be one whose output is the conditional expectation of the signal contained in the input, then the performance of a matched filter is much easier to analyze than the performance of a mismatched filter. The divergence involves mismatched filters while the Bhattacharyya distance uses only matched filters. Hence the Bhattacharyya distance is easier to analyze. In (Kailath 1967) it is shown that the two measures give similar results for Gaussian processes with unequal mean value functions and that the Bhattacharyya distance yields better results when the processes have unequal covariance functions, where in fact the divergence measure fails. In (Schweppe 1967a) the Bhattacharyya distance is specialized to Markov-Gaussian processes. (Cha 2007) is a comprehensive survey on distance/similarity measures between probability density functions.

For a discrete sample in a two class scenario, (Jain 1976) shows that any estimate of the Bhattacharyya coefficient is biased and consistent using a Taylor series expansion around the neighborhood of the points. We can write the Bhattacharyya coefficient as,

$$\rho\left(\alpha, \beta\right) = \sum_{i=1}^{N} \sqrt{\alpha_i \beta_i} \tag{17}$$

Here, α and β denote the sets of parameters $\{\alpha_1, ..., \alpha_N\}$ and $\{\beta_1, ..., \beta_N\}$ respectively. The class conditional densities are given by,

$$p_i = \sum_{i=1}^{N} \alpha_i \delta\left(x - i\right) \tag{18}$$

$$p_i' = \sum_{i=1}^{N} \beta_i \delta\left(x - i\right) \tag{19}$$

$$\text{such that} \qquad \sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{N} \beta_i = 1$$

$$\delta\left(i - j\right) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{20}$$

Let $\hat{\alpha_i}$ and $\hat{\beta}_i$ denote the maximum likelihood estimates of $\alpha$and β, respectively, $i = 1, ..., N$, based on m samples available from each of the two classes:

$$\hat{\alpha_i} = \frac{m_{i1}}{m} \qquad \hat{\beta}_i = \frac{m_{i2}}{m} \tag{21}$$

where m_{i1} and m_{i2} are the numbers of samples for which x takes the value i from class c_1 and c_2, respectively. We define a sample estimate of the Bhattacharyya coefficient as,

$$\hat{\rho} = \rho\left(\hat{\alpha}, \hat{\beta}\right) = \sum_{i=1}^{N} \sqrt{\hat{\alpha}_i \hat{\beta}_i} \tag{22}$$

where, $\hat{\alpha} = \{\hat{\alpha}_1, ..., \hat{\alpha}_N\}$ and $\hat{\beta} = \left\{\hat{\beta}_1, ..., \hat{\beta}_N\right\}$ respectively. (Djouadi, Snorrason & Garber 1990) derive closed-form expressions for the bias and variance of Bhattacharyya coefficient estimates based on a certain number of training samples from two classes, described by multivariate Gaussian distributions. Numerical examples are used to show the relationship between the true parameters of the densities, the number of training samples, the class variances, and the dimensionality of the observation space.

3.2 Dimension Reduction

A key requirement to apply the Bhattacharyya distance in practice is to have data-sets with the same number of dimensions. (Fodor 2002; Burges 2009; Sorzano, Vargas & Montano 2014) are comprehensive collections of methodologies aimed at reducing the dimensions of a data-set using Principal Component Analysis or Singular Value Decomposition and related techniques.

(Johnson & Lindenstrauss 1984) proved a fundamental result (JL Lemma) that says that any n point subset of Euclidean space can be embedded in $k = O(log\frac{n}{\epsilon^2})$ dimensions without distorting the distances between any pair of points by more than a factor of $(1 \pm \epsilon)$, for any $0 < \epsilon < 1$. Whereas principal component analysis is only useful when the original data points are inherently low dimensional, the JL Lemma requires absolutely no assumption on the original data. Also, note that the final data points have no dependence on d, the dimensions of the original data which could live in an arbitrarily high dimension.

Simplified versions of the original proof are in (Frankl and Maehara 1988; 1990). We use the version of

the bounds for the dimensions of the transformed subspace given in (Frankl & Maehara 1990; Dasgupta & Gupta 1999). (Nelson 2010) gives a proof using the Hanson and Wright inequality (Hanson & Wright 1971). (Achlioptas 2003) gives a proof of the JL Lemma using randomized algorithms. (Venkatasubramanian & Wang 2011) present a study of the empirical behavior of algorithms for dimensionality reduction based on the JL Lemma. We point out the wonderful references for the theory of matrix algebra and numerical applications (Gentle 2007; 2012).

Lemma 1. *For any $0 < \epsilon < 1$ and any integer n, let $k < d$ be a positive integer such that*

$$k \geq 4 \left(\frac{\epsilon^2}{2} - \frac{\epsilon^3}{3} \right)^{-1} \ln n$$

Then for any set V of n points in $\boldsymbol{R}^d$, there is a map $f : \boldsymbol{R}^d \to \boldsymbol{R}^k$ such that for all $u, v \in V$,

$$(1 - \epsilon) \|u - v\|^2 \leq \|f(u) - f(v)\|^2 \leq (1 + \epsilon) \|u - v\|^2$$

Furthermore, this map can be found in randomized polynomial time and one such map is $f(x) = \frac{1}{\sqrt{k}} Ax$ where, $x \in \boldsymbol{R}^d$ and A is a $k \times d$ matrix in which each entry is sampled i.i.d from a Gaussian $N(0, 1)$ distribution.

4 Intuition for Dimension Reduction

The above discussions of distance measures and dimension reduction are extensively used in many areas, but the combination of the two is bound to create confusions in the minds of the uninitiated. Hence, we provide examples from daily life, (both ours and from creatures in higher dimensions), to provide a convincing argument as to why the two used together can be a powerful tool for the study of complex systems governed by uncertainty.

4.1 Game of Darts

If we consider a cloud of points in multi-dimensional space. It would be reasonable to expect that the distance between the points, or how the points are distributed, gives a measure of the randomness inherent in the process generating them. When dimension reduction moves the points to lower dimensions, and the change in the distance between them stays bounded, the randomness properties of the original process are retained, to the extent as dictated, by the bounds established by the procedure performing the dimension transformation, which in our case is given by the JL Lemma.

Taking an example, from our own real lives, the marks on a dartboard in a game of darts are representative of the skills of the people throwing the darts. For simplicity, we could assume that there are three types of

dart throwers: novice, intermediate and advanced. Identifying the category of the person making the marks, would be similar to identifying the type of distribution of a stochastic process. If we map the marks, on the board, to a line using a transformation, that keeps the distances between the marks bounded, a flavor of the skill level would be retained and we would be able to identify the category of the person making the marks. Dimension reduction, using a transformation that keeps the distances bounded, is in essence the same undertaking.

4.2 The Merits and Limits of Four Physical Dimensions

Another example, of dimension transformation, is from the physical world, we live in. We are four dimensional creatures: latitude, longitude, height and time are our dimensions, since we need to know these four co-ordinates to fully specify the position of any object in our universe. This is perhaps, made clear to lay audiences, (with regards to physics, such as many of us), by the movie Interstellar (Thorne 2014). Also, (Sagan 2006), has a mesmerizing account of many physical aspects, including how objects, or, beings can transform from higher, to lower dimensions, and change shapes; but they would need to be obey the laws of the lower dimension. The last dimension, time, is the one we cannot control, or, move around in. But, we can change, the other three co-ordinates and hence we have three degrees of freedom.

(Appendix 12) has a detailed example to build the intuition related to dimension transformation from our four dimensional physical world. This example should also make it clear to us that, it is better to work with the highest dimension that we can afford to work with, since each higher dimension, retains some flavors of the object, we are trying to understand, that might be not discernible in lower dimensions. This should also tell us that objects we observe in our universe might have many interesting properties, that we are unable to observe due to the restrictions of our physical dimensions.

5 Methodological Innovations

In this section, we collect the new results developed in this paper. These can be broadly categorized into two buckets. The first is the types of distributions that we would obtain when a particular distribution type is transformed to a different dimension (different number of random variables) using the JL-Lemma (sections 5.1, 5.2, 5.3, 5.4). The second group of results considers the relationships between covariance and distance measures (section 5.5).

5.1 Normal Log-Normal Mixture

The normality or otherwise of stock price changes is discussed extensively in the literature: (Osborne 1959; 1962; Fama 1965; 1995; Mandelbrot & Taylor 1967; Kon 1984; Richardson & Smith 1993). Starting with a geometric Brownian motion for the stock price process, it can be established that stock prices are distributed log-normally, (Hull 2006). If we are looking at variables, that are only positive, such as prices, quantities

traded, or volatilities, then it is a reasonable initial assumption that they are distributed log normally (we relax this assumption to incorporate more generic settings in later sections).

Transforming log-normal multi-variate variables into a lower dimension by multiplication with an independent normal distribution (see: Lemma 1) results in the sum of variables with a normal log-normal mixture, (Clark 1973; Tauchen & Pitts 1983; Yang 2008), evaluation of which requires numerical techniques (Miranda & Fackler 2002). A random variable, U, would be termed a normal log-normal mixture if it is of the form,

$$U = Xe^{Y} \tag{23}$$

where, X and Y are random variables with correlation coefficient, ρ satisfying the below,

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}\right) \tag{24}$$

We note that for $\sigma_Y = 0$ when Y degenerates to a constant, this is just the distribution of X and ρ is unidentified.

To transform a column vector with d observations of a random variable into a lower dimension of order, $k < d$, we can multiply the column vector with a matrix, $A \sim N(0; \frac{1}{k})$ of dimension $k \times d$.

Proposition 1. *A dimension transformation of d observations of a log-normal variable into a lower dimension, k, using Lemma 1, yields a probability density function which is the sum of random variables with a normal log-normal mixture, given by the convolution,*

$$f_S(s) = f_{U_1}(u_1) * f_{U_2}(u_2) * ... * f_{U_k}(u_k)$$

$$\text{Here, } f_{U_i}(u_i) = \frac{\sqrt{k}}{2\pi\sigma_{Y_i}} \int_{-\infty}^{\infty} e^{-y - \frac{ku_i^2}{2e^{2y}} - \frac{\left[y-\mu_{Y_i}\right]^2}{2\sigma_{Y_i}^2}} dy$$

$$U_i = X_i e^{Y_i}$$

$$\begin{bmatrix} X_i \\ Y_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ \mu_{Y_i} \end{bmatrix}, \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \sigma_{Y_i}^2 \end{bmatrix}\right)$$

The convolution of two probability densities arises when we have the sum of two independent random variables, $Z = X + Y$. The density of Z, $h_Z(z)$ is given by,

$$h_Z(z) = (f_X * f_Y)(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(z-y) * f_Y(y)\, dy = \int_{-\infty}^{\infty} f_X(x) * f_Y(z-x)\, dx$$

When the number of independent random variables being added is more than two, or the reduced dimension after the Lemma 1 transformation is more than two, $k > 2$, then we can take the convolution of the density resulting after the convolution of the first two random variables, with the density of the third variable and so on in a pair wise manner, till we have the final density.

Proof. Appendix 14.2 sketches a general proof and then tailors it to our case where the normal distribution has zero mean and the two variables are uncorrelated.

□

Methods of estimating parameters and comparing the resulting distributions when normal (log-normal) distributions are mixed, are studied in (Fowlkes 1979; Vernic, Teodorescu & Pelican 2009). As noted earlier, the normal log-normal mixture tends to the normal distribution when the log-normal distribution has low variance and this property can be helpful for deciding when this approach is suitable.

5.2 Normal Normal Product

For completeness, we illustrate how dimension reduction would work on a data-set containing random variables that have normal distributions. This can also serve as a useful benchmark given the wide usage of the normal distribution and can be an independently useful result.

(Craig 1936) was one of the earlier attempts to investigate the probable error of the product of the two quantities, each of known probable error; becoming the first work to determine the algebraic expression for the moment-generating function of the product, but without being able to determine the distribution of the product. It was found that the distribution of $Z = XY$ is a function of the coefficient of correlation of both variables and of two parameters that are proportional to the inverse of the coefficient of variation of each variable. When the product of the means of the two random variables is nonzero, the distribution is skewed as well as having excess kurtosis, although (Aroian 1947; Aroian, Taneja & Cornwell 1978) showed that the product approaches the normal distribution as one or both of the ratios of the means to standard errors (the inverse of the coefficients of variation) of each random variable get large in absolute value.

(Aroian 1947) showed that the gamma distribution (standardized Pearson type III) can provide an approximation in some situations. Instead, the analytical solution for this product distribution is a Bessel function of the second kind with a purely imaginary argument (Aroian 1947; Craig 1936). The four moments of the product of two correlated normal variables are given in (Craig 1936; Aroian, Taneja & Cornwell 1978).

Proposition 2. *A dimension transformation of d observations of a normal variable into a lower dimension, k, using Lemma 1, yields a probability density function which is the sum of random variables with a normal normal product distribution, given by the convolution,*

$$f_S(s) = f_{U_1}(u_1) * f_{U_2}(u_2) * ... * f_{U_k}(u_k)$$

$$\textit{Here, } f_{U_i}(u_i) = \int_{-\infty}^{\infty} \left(\frac{1}{|x|}\right) \frac{1}{\sigma_{Y_i}\sqrt{2\pi}}\, e^{-\frac{\left(x-\mu_{Y_i}\right)^2}{2\sigma_{Y_i}^2}} \sqrt{\frac{k}{2\pi}}\, e^{-\frac{k\left(\frac{u_i}{x}\right)^2}{2}} dx$$

$$U_i = X_i Y_i$$

$$\begin{bmatrix} X_i \\ Y_i \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ \mu_{Y_i} \end{bmatrix}, \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \sigma_{Y_i}^2 \end{bmatrix}\right)$$

Proof. Appendix 14.3 sketches a general proof and then tailors it to our case where the normal distribution has zero mean and the two variables are uncorrelated.

□

Remark. We note the following two useful results.

1. By writing the product as the difference of two squared variables, it is easy to see that the product distribution is a linear combination of two Chi-Square random variables,

$$U_i = X_i Y_i = \frac{1}{4}\left\{[X_i + Y_i]^2 - [X_i - Y_i]^2\right\} \tag{25}$$

If $X_i \sim N\left(0, \sigma_{X_i}^2\right); Y_i \sim N\left(0, \sigma_{Y_i}^2\right)$ with a coefficient of correlation, $\rho_{X_i Y_i}$ then,

$$U_i = X_i Y_i \sim \frac{1}{4}\left\{\left(\sigma_{X_i}^2 + \sigma_{Y_i}^2 + 2\sigma_{X_i}\sigma_{Y_i}\rho_{X_i Y_i}\right) P - \left(\sigma_{X_i}^2 + \sigma_{Y_i}^2 - 2\sigma_{X_i}\sigma_{Y_i}\rho_{X_i Y_i}\right) Q\right\} \tag{26}$$

Here, $P, Q \sim \chi_1^2$ or central chi-squared random variables with one degree of freedom. P, Q are independent only if $\sigma_{X_i}^2 = \sigma_{Y_i}^2$. Hence, in general, P, Q are dependent non central chi-squared variables.

2. The result we use in (appendix 14.3) to derive the above convolution, can also be arrived at, by writing the density for $W = XY$ using the Dirac Delta function, $\delta(x)$ as,

$$f_W(w) = f_W(x, y) = \int_{-\infty}^{\infty} f_X(x|y)\, f_Y(y)\, dy = \int_{-\infty}^{\infty} f_X(x)\, dx \int_{-\infty}^{\infty} f_Y(y)\, \delta(w - xy)\, dy \tag{27}$$

$$= \int_{-\infty}^{\infty} f_X(x)\,dx \int_{-\infty}^{\infty} f_Y(y)\,\frac{\delta\left(y-\frac{w}{x}\right)}{|x|}dy \tag{28}$$

$$= \int_{-\infty}^{\infty} f_X(x)\,\frac{1}{|x|} f_Y\left(\frac{w}{x}\right)dx \tag{29}$$

(Glen, Leemis & Drew 2004) present an algorithm for computing the probability density function of the product of two independent random variables. (Springer & Thompson 1966) use the Mellin integral transform (Epstein 1948; End-note 3) to develop fundamental methods for the derivation of the probability distributions and density functions of the products of n independent random variables; (Springer & Thompson 1970) use these methods to show that the products of independent beta, gamma and central Gaussian random variables are Meijer G-functions (Mathai & Saxena 1973; End-note 4).

(Ware & Lad 2003) has results very closely aligned to our requirements. They attempt to calculate the probability that the sum of the product of variables with a Normal distribution is negative. They first assess the distribution of the product of two independent normally distributed variables by comparing three different methods: 1) a numerical method approximation, which involves implementing a numerical integration procedure on MATLAB; 2) a Monte Carlo construction and; 3) an approximation to the analytic result using the Normal distribution under certain conditions, by calculating the first two moments of the product, and then finding a distribution whose parameters match the moments. Second, they consider the sum of the products of two Normally distributed variables by applying the Convolution Formula. Lastly, they combine the two steps to arrive at the main results while also showing that a direct Monte Carlo approximation approach could be used. (Seijas-Macías & Oliveira 2012) is a recent work that has several comparisons using Newton-Cotes numerical integration (Weisstein 2004; End-note 5).

In addition, while it useful to keep these results at the back of our mind, it is worth finding simpler distributions instead of having to numerically calculate the distance based on the normal product or the normal log-normal mixture sum. An alternate is discussed in the next section, where we set both distributions to be truncated normal.

5.3 Truncated Normal / Multivariate Normal

A truncated normal distribution is the probability distribution of a normally distributed random variable whose value is either bounded below, above or both (Horrace 2005; Burkardt 2014) and hence seems like a natural candidate to fit the distributions we are dealing with. We can estimate the parameters of the distributions (both the unchanged one and the one with the reduced dimensions) by setting them as truncated multivariate normals and calculate the distance based on these estimated distributions.

To assess the suitability of normal distributions to fit the observed sample, there are a variety of tests. The univariate sample measures of skewness S and kurtosis K may be used for testing univariate normality. Under normality, the theoretical values of S and K are 0 and 3, respectively. One of the most famous tests for normality of regression residuals is the test of (Jarque & Bera 1980; 1987; End-notes 7; 8). The test statistic JB (Jarque-Bera) is a function of the measures of skewness and kurtosis computed from the sample and is based on the Lagrange multiplier test or score test.

The simplest testing problem assumes that the data y are generated by a joint density function $f\left(y,\theta^0\right)$ under the null hypothesis and by $f\left(y,\theta\right)$ under the alternative, with $\theta^0,\theta\in\boldsymbol{R}^k$. The Lagrange Multiplier test is derived from a constrained maximization principle (Engle 1984; End-note 6). Maximizing the log-likelihood subject to the constraint that $\theta^0=\theta$ yields a set of Lagrange Multipliers which measure the shadow price of the constraint. If the price is high, the constraint should be rejected as inconsistent with the data.

Another test frequently used is the sum of squares of the standardized sample skewness and kurtosis which is asymptotically distributed as a χ^2 variate (Doornik and Hansen 2008) along with transformations of skewness and kurtosis to facilitate easier implementation and also to contend with small sample issues. (Malkovich & Afifi 1973) generalize these statistics to test a hypothesis of multivariate normality including a discussion of transformations to simplify the computational methods.

(Székely & Rizzo 2005; End-note 9) propose a new class of consistent tests for comparing multivariate distributions based on Euclidean distance between sample elements. Applications include one-sample goodness-of-fit tests for discrete or continuous multivariate distributions in arbitrary dimension $d\geq 1$. The new tests can be applied to assess distributional assumptions for many classical procedures in multivariate analysis. (Wald & Wolfowitz 1946) consider the problem of setting tolerance limits for normal distributions with unknown mean and variance. For a univariate distribution with a sample of N independent observations, two functions L_1 and L_2 of the sample need to constructed such that the probability that the limits L_1 and L_2 will include at-least a given proportion γ of the population is equal to a preassigned value β.

Suppose $X\sim N(\mu,\sigma^2)$ has a normal distribution and lies within the interval $X\in(a,b),\ -\infty\leq a<b\leq\infty$. Then X conditional on $a<X<b$ has a truncated normal distribution. Its probability density function, f_X, for $a\leq x\leq b$, is given by

$$f_X\left(x\mid\mu,\sigma^2,a,b\right)=\begin{cases}\frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{\Phi\left(\frac{b-\mu}{\sigma}\right)-\Phi\left(\frac{a-\mu}{\sigma}\right)} & ;a\leq x\leq b\\ 0 & ;\text{otherwise}\end{cases}\tag{30}$$

Here, $\phi(\xi)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\xi^2\right)$ is the probability density function of the standard normal distribution and $\Phi(\cdot)$ is its cumulative distribution function. There is an understanding that if $b=\infty$, then $\Phi(\frac{b-\mu}{\sigma})=1$, and similarly, if $a=-\infty$, then $\Phi(\frac{a-\mu}{\sigma})=0$.

Proposition 3. *The Bhattacharyya distance, when we have truncated normal distributions p, q that do not overlap, is zero and when they overlap, it is given by*

$$\begin{aligned} D_{BC-TN}(p,q) &= \frac{1}{4}\left(\frac{(\mu_p-\mu_q)^2}{\sigma_p^2+\sigma_q^2}\right)+\frac{1}{4}\ln\left(\frac{1}{4}\left(\frac{\sigma_p^2}{\sigma_q^2}+\frac{\sigma_q^2}{\sigma_p^2}+2\right)\right) \\ &+\frac{1}{2}\ln\left[\Phi\left(\frac{b-\mu_p}{\sigma_p}\right)-\Phi\left(\frac{a-\mu_p}{\sigma_p}\right)\right]+\frac{1}{2}\ln\left[\Phi\left(\frac{d-\mu_q}{\sigma_q}\right)-\Phi\left(\frac{c-\mu_q}{\sigma_q}\right)\right] \\ &-\ln\left\{\Phi\left[\frac{u-\nu}{\varsigma}\right]-\Phi\left[\frac{l-\nu}{\varsigma}\right]\right\} \end{aligned}$$

Here,

$$p \sim N\left(\mu_p, \sigma_p^2, a, b\right)\ ;\ q \sim N\left(\mu_q, \sigma_q^2, c, d\right)$$

$$l = \min\left(a, c\right)\ ;\ u = \min\left(b, d\right)$$

$$\nu = \frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\ ;\ \varsigma = \sqrt{\frac{2\sigma_p^2\sigma_q^2}{\left(\sigma_p^2+\sigma_q^2\right)}}$$

Proof. Appendix 14.4.

□

It is easily seen that

$$\lim_{(a,c)\to-\infty;(b,d)\to\infty} D_{BC-TN}(p,q) = D_{BC-N}(p,q) \tag{31}$$

Looking at conditions when $D_{BC-TN}(p,q) \geq D_{BC-N}(p,q)$ gives the below. This shows that the distance measure increases with greater truncation and then decreases as the extent of overlap between the distributions decreases.,

$$\sqrt{\left[\Phi\left(\frac{b-\mu_p}{\sigma_p}\right)-\Phi\left(\frac{a-\mu_p}{\sigma_p}\right)\right]\left[\Phi\left(\frac{d-\mu_q}{\sigma_q}\right)-\Phi\left(\frac{c-\mu_q}{\sigma_q}\right)\right]} \geq \left\{\Phi\left[\frac{u-\nu}{\varsigma}\right]-\Phi\left[\frac{l-\nu}{\varsigma}\right]\right\} \tag{32}$$

(Kiani, Panaretos, Psarakis & Saleem 2008**;** Zogheib & Hlynka 2009; Soranzo & Epure 2014) list some of the numerous techniques to calculate the normal cumulative distribution. Approximations to the error function are also feasible options (Cody 1969; Chiani, Dardari & Simon 2003).

Similarly, a truncated multivariate normal distribution $\boldsymbol{X}$ has the density function,

$$f_{\mathbf{X}}\left(x_1, \ldots, x_k \mid \boldsymbol{\mu_p}, \boldsymbol{\Sigma_p}, \boldsymbol{a}, \boldsymbol{b}\right) = \frac{\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right))}{\int_{\boldsymbol{a}}^{\boldsymbol{b}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right))\,d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}_{\boldsymbol{a}\leq\boldsymbol{x}\leq\boldsymbol{b}}^{k}} \tag{33}$$

Here, $\boldsymbol{\mu_p}$ is the mean vector and $\boldsymbol{\Sigma_p}$ is the symmetric positive definite covariance matrix of the $\boldsymbol{p}$ distribution and the integral is a k dimensional integral with lower and upper bounds given by the vectors $(\boldsymbol{a}, \boldsymbol{b})$ and

$\boldsymbol{x} \in \boldsymbol{R}^{\boldsymbol{k}}_{\boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b}}$.

Proposition 4. *The Bhattacharyya coefficient when we have truncated multivariate normal distributions* $\boldsymbol{p}, \boldsymbol{q}$ *and all the* k *dimensions have some overlap, is given by*

$$
\begin{aligned}
D_{BC-TMN}\left(\boldsymbol{p}, \boldsymbol{q}\right) \quad = \quad & \frac{1}{8}(\boldsymbol{\mu_p} - \boldsymbol{\mu_q})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu_p} - \boldsymbol{\mu_q}) + \frac{1}{2} \ln \left(\frac{\det \boldsymbol{\Sigma}}{\sqrt{\det \boldsymbol{\Sigma_p} \, \det \boldsymbol{\Sigma_q}}} \right) \\
& + \frac{1}{2} \ln \left[\frac{1}{\sqrt{(2\pi)^k \left(\left| \boldsymbol{\Sigma_p} \right| \right)}} \int_{\boldsymbol{a}}^{\boldsymbol{b}} \exp \left(-\frac{1}{2} \left(\mathbf{x} - \boldsymbol{\mu_p} \right)^{\mathrm{T}} \boldsymbol{\Sigma_p}^{-1} \left(\mathbf{x} - \boldsymbol{\mu_p} \right) \right) d\boldsymbol{x}; \; \boldsymbol{x} \in \boldsymbol{R}^{k}_{\boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b}} \right] \\
& + \frac{1}{2} \ln \left[\frac{1}{\sqrt{(2\pi)^k \left(\left| \boldsymbol{\Sigma_q} \right| \right)}} \int_{\boldsymbol{c}}^{\boldsymbol{d}} \exp \left(-\frac{1}{2} \left(\mathbf{x} - \boldsymbol{\mu_q} \right)^{\mathrm{T}} \boldsymbol{\Sigma_q}^{-1} \left(\mathbf{x} - \boldsymbol{\mu_q} \right) \right) d\boldsymbol{x}; \; \boldsymbol{x} \in \boldsymbol{R}^{k}_{\boldsymbol{c} \leq \boldsymbol{x} \leq \boldsymbol{d}} \right] \\
& - \ln \left[\frac{1}{\sqrt{(2\pi)^k \det \left(\boldsymbol{\Sigma_p} \boldsymbol{\Sigma}^{-1} \boldsymbol{\Sigma_q} \right)}} \right. \\
& \left. \int_{\boldsymbol{l}}^{\boldsymbol{u}} \exp \left(-\frac{1}{2} \left\{ (\mathbf{x} - \mathbf{m})^{\mathrm{T}} \left(\boldsymbol{\Sigma_q}^{-1} \left[\boldsymbol{\Sigma} \right] \boldsymbol{\Sigma_p}^{-1} \right) (\mathbf{x} - \mathbf{m}) \right\} \right) d\boldsymbol{x}; \; \boldsymbol{x} \in \boldsymbol{R}^{k}_{\boldsymbol{\min(a,c)} \leq \boldsymbol{x} \leq \boldsymbol{\min(b,d)}} \right]
\end{aligned}
$$

Here,

$$\boldsymbol{p} \sim N\left(\boldsymbol{\mu_p}, \, \boldsymbol{\Sigma_p}, \, \boldsymbol{a}, \, \boldsymbol{b}\right)$$

$$\boldsymbol{q} \sim N\left(\boldsymbol{\mu_q}, \, \boldsymbol{\Sigma_q}, \, \boldsymbol{c}, \, \boldsymbol{d}\right)$$

$$\boldsymbol{u} = \boldsymbol{\min}\left(\boldsymbol{b}, \boldsymbol{d}\right) \; ; \; \boldsymbol{l} = \boldsymbol{\min}\left(\boldsymbol{a}, \boldsymbol{c}\right)$$

$$\mathbf{m} = \left[\left(\boldsymbol{\mu_p}^{\mathrm{T}} \boldsymbol{\Sigma_p}^{-1} + \boldsymbol{\mu_q}^{\mathrm{T}} \boldsymbol{\Sigma_q}^{-1} \right) \left(\boldsymbol{\Sigma_p}^{-1} + \boldsymbol{\Sigma_q}^{-1} \right)^{-1} \right]^{\mathrm{T}}$$

$$\boldsymbol{\Sigma} = \frac{\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}}{2}$$

Proof. Appendix 14.5

□

Again we see that,

$$\lim_{(\boldsymbol{a},\boldsymbol{c}) \to -\infty; (\boldsymbol{b},\boldsymbol{d}) \to \infty} D_{BC-TMN}\left(\boldsymbol{p}, \boldsymbol{q}\right) = D_{BC-MN}\left(\boldsymbol{p}, \boldsymbol{q}\right) \tag{34}$$

Looking at conditions when $D_{BC-TMN}(\boldsymbol{p}, \boldsymbol{q}) \geq D_{BC-MN}(\boldsymbol{p}, \boldsymbol{q})$ gives a similar condition as the univariate

case,

$$\frac{1}{2}\ln\left[\frac{1}{\sqrt{(2\pi)^k\left(|\boldsymbol{\Sigma_p}|\right)}}\int_{\boldsymbol{a}}^{\boldsymbol{b}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)\right)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{a}\leq\boldsymbol{x}\leq\boldsymbol{b}}\right] \tag{35}$$

$$+\frac{1}{2}\ln\left[\frac{1}{\sqrt{(2\pi)^k\left(|\boldsymbol{\Sigma_q}|\right)}}\int_{\boldsymbol{c}}^{\boldsymbol{d}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)\right)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{c}\leq\boldsymbol{x}\leq\boldsymbol{d}}\right] \tag{36}$$

$$\geq-\ln\left[\frac{1}{\sqrt{(2\pi)^k\det\left(\boldsymbol{\Sigma_p}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma_q}\right)}}\right. \tag{37}$$

$$\left.\int_{\boldsymbol{l}}^{\boldsymbol{u}}\exp\left(-\frac{1}{2}\left\{\left(\mathbf{x}-\mathbf{m}\right)^{\mathrm{T}}\left(\boldsymbol{\Sigma_q}^{-1}\left[\boldsymbol{\Sigma}\right]\boldsymbol{\Sigma_p}^{-1}\right)\left(\mathbf{x}-\mathbf{m}\right)\right\}\right)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\mathbf{min}(\boldsymbol{a},\boldsymbol{c})\leq\boldsymbol{x}\leq\mathbf{min}(\boldsymbol{b},\boldsymbol{d})}\right] \tag{38}$$

5.4 Discrete Multivariate Distribution

A practical approach would be to use discrete approximations for the probability distributions. This is typically done by matching the moments of the original and approximate distributions. Discrete approximations of probability distributions typically are determined by dividing the range of possible values or the range of cumulative probabilities into a set of collectively exhaustive and mutually exclusive intervals. Each interval is approximated by a value equal to its mean or median and a probability equal to the chance that the true value will be in the interval.

A criterion for judging the accuracy of the discrete approximation is that it preserve as many of the moments of the original distribution as possible. (Keefer & Bodily 1983; Smith 1993) provide comparisons of commonly used discretization methods. (Miller & Rice 1983) look at numerical integration using gaussian quadrature as a means of improving the approximation. This approach approximates the integral of the product of a function $g(x)$ and a weighting function $w(x)$ over the internal (a,b) by evaluating $g(x)$ at several values x and computing a weighted sum of the results.

$$\int_a^b g(x)\,w(x)\,dx=\sum_{i=1}^{N}w_i g(x_i) \tag{39}$$

For the case of a discrete approximation of a probability distribution, the density function $f(x)$ with support (a,b) is associated with the weighting function and the probabilities p_i with the weights w_i. We approximate $g(x)$ by a polynomial, and choose x_i and p_i (or w_i) to provide an adequate approximation for each term of the polynomial. This translates to finding a set of values and probabilities such that,

$$\int_a^b x^k f(x)\,dx=\sum_{i=1}^{N}p_i x_i^k\qquad\text{for } k=0,1,2,\ldots \tag{40}$$

A discrete approximation with N probability-value pairs can match the first $2N-1$ moments exactly by finding p_i and x_i that satisfy the below equations, which are solved using well known techniques.

$$p_1 + p_2 + \ldots + p_N = \int_a^b x^0 f(x)\, dx = 1 \tag{41}$$

$$p_1 x_1 + p_2 x_2 + \ldots + p_N x_N = \int_a^b x f(x)\, dx \tag{42}$$

$$p_1 x_1^2 + p_2 x_2^2 + \ldots + p_N x_N^2 = \int_a^b x^2 f(x)\, dx \tag{43}$$

$$\vdots\!\vdots \tag{44}$$

$$p_1 x_1^{2N-1} + p_2 x_2^{2N-1} + \ldots + p_N x_N^{2N-1} = \int_a^b x^{2N-1} f(x)\, dx \tag{45}$$

Among several alternatives to distribution approximations, (Wallace 1958), restricts attention to finding asymptotic expansions in which the errors of approximation approach zero as some parameter, typically sample size, tends to infinity. Estimating the parameters of the distributions (both the unchanged one and the one with the reduced dimensions) by setting them as discrete multivariate distributions and calculating the distance based on these estimated distributions is a practical alternative to using the truncated normal distributions.

5.5 Covariance and Distance

We compare the covariance between two random variables and the Bhattacharyya coefficient. Assuming support over relevant portions of the real line and that the density function is differentiable. Consider the covariance and the distance of $Y = X$ and $Y = -X$. It is easily seen that the distance between the distributions is *zero* in both cases, but the covariance is *one* when they are exactly the same and *minus one* in the other case. This shows that despite knowing the distance we do not have full information about the co-movements of the two distributions. Hence distance is not a substitute for covariance but rather a very useful complement, since knowing the two will tell us the magnitudes of how one variable will change if we know the other and the likelihood of observing certain values from one distribution if have observed values from the other one.

An interesting result concerning the covariance is given by Stein's lemma (Stein 1973; 1981; Rubinstein 1973; 1976) which connects the covariance between two random variables that are joint normally distributed and the covariance between any function of one of the random variables and the other. If X and Y have a joint distribution that is bivariate normal and if $c(\cdot)$ is a continuously differentiable function of X then,

$$\operatorname{Cov}[c(X), Y] = E[c'(X)] \operatorname{Cov}[X, Y] \tag{46}$$

It easily follows that, if X is a normal random variable with mean μ and variance σ^2 and if c is a differentiable

function with derivative c' such that $E\left[c'\left(X\right)\right] < \infty$, we then have,

$$E\left[\left(X-\mu\right)c\left(X\right)\right] = \sigma^2 E[c'\left(X\right)] \tag{47}$$

(Losq & Chateau 1982) extend the result to the case when $g\left(\cdot\right)$ is a function of n random variables. (Wei & Lee 1988) extend this further to the case where both variables are functions of multivariate normal random variables. (Siegel 1993) derives a remarkable formula for the covariance of X_1 with the minimum of the multivariate normal vector $(X_1, \ldots, X_n)$.

$$\mathrm{Cov}[X_1, \min\left(X_1, \ldots, X_n\right)] = \sum_{i=1}^{n} \mathrm{Cov}\left(X_1, X_i\right) \Pr\left[X_i = \min\left(X_1, \ldots, X_n\right)\right] \tag{48}$$

(Liu 1994) uses a multivariate version of Stein's identity to devise the below more generalized formula, where $X_{(i)}$ is the $i-$th largest among $X_1, \ldots, X_n$,

$$\mathrm{Cov}[X_1, X_{(i)}] = \sum_{j=1}^{n} \mathrm{Cov}\left(X_1, X_j\right) \Pr\left[X_j = X_{(i)}\right] \tag{49}$$

(Kattumannil 2009) extends the Stein lemma by relaxing the requirement of normality. If the continuous random variable X has support over the interval (a, b), that is $-\infty \leq a < X < b \leq \infty$, with mean $E(X) = \mu$, variance $Var(X) = \sigma^2$, density function $f_X\left(t\right)$ and cumulative distribution $F_X\left(t\right)$. Let h be a real valued function such that $E\left[h\left(X\right)\right] = \mu$ and $E\left[h^2\left(X\right)\right] < \infty$. Suppose $f_X\left(t\right)$ is a differentiable function with derivative $f'_X\left(x\right)$ and there exists a non-vanishing (non-zero over the support) function $g\left(t\right)$ such that,

$$\frac{f'_X\left(t\right)}{f_X\left(t\right)} = -\frac{g'\left(t\right)}{g\left(t\right)} + \frac{\left[\mu - h\left(t\right)\right]}{g\left(t\right)} \quad , \quad t \in (a, b) \tag{50}$$

$$\Rightarrow \quad f'_X\left(t\right) g\left(t\right) + g'\left(t\right) f_X\left(t\right) = \left[\mu - h\left(t\right)\right] f_X\left(t\right) \tag{51}$$

$$\Rightarrow \quad \frac{\partial f_X\left(t\right) g\left(t\right)}{\partial t} = \left[\mu - h\left(t\right)\right] f_X\left(t\right) \tag{52}$$

Integrating with respect to t from r to b and assuming $\lim_{t \to b} g\left(t\right) f_X\left(t\right) = 0$ shows that for a given $h\left(t\right)$ the value of $g\left(t\right)$ uniquely determines the distribution of X.

$$\left| f_X\left(t\right) g\left(t\right)\right|_r^b = \int_r^b \left[\mu - h\left(t\right)\right] f_X\left(t\right) dt \tag{53}$$

$$\Rightarrow \quad f_X\left(r\right) = \frac{1}{g\left(r\right)} \left\{ \int_r^b \left[h\left(t\right) - \mu\right] f_X\left(t\right) dt \right\} \tag{54}$$

Similarly, integrating with respect to t from a to r and assuming $\lim_{r\to a} g(t) f_X(t) = 0$,

$$f_X(r) = \frac{1}{g(r)} \left\{ \int_r^a [\mu - h(t)] f_X(t)\, dt \right\} \tag{55}$$

For any absolutely continuous function, $c(t)$, with derivative $c'(t)$ satisfying $E[c(X) h(X)] < \infty, E\left[c^2(X)\right] < \infty, E[g(X) c'(X)] < \infty$, and provided $\lim_{t\to b} g(t) f_X(t) = 0$ we have the following identity,

$$E[c(X)\{h(X) - \mu\}] = E[c'(X) g(X)] \tag{56}$$

(Teerapabolarn 2013) is a further extension of this normality relaxation to discrete distributions. Another useful reference, (Kimeldorf & Sampson 1973), provides a class of inequalities between the covariance of two random variables and the variance of a function of the two random variables. Let $\boldsymbol{A}$ be an open convex subset of the plane $\boldsymbol{R}^2$ and $\boldsymbol{\Upsilon}$ be the class of all pairs (X, Y) of real random variables having finite variances and $P\{(X, Y) \in \boldsymbol{A}\} = 1$. Assume that g is a function with continuous partial first derivatives in the domain $\boldsymbol{A}$ characterized by the following functional inequality and an equivalent partial differential inequality,

$$\forall (X, Y) \in \boldsymbol{\Upsilon}; \quad (x_1, y_1), (x_2, y_2) \in \boldsymbol{A} \Longleftrightarrow (x_2 - x_1)(y_2 - y_1) \leq [g(x_2, y_2) - g(x_1, y_1)]^2 \tag{57}$$

$$\left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial g}{\partial y}\right) \geq \frac{1}{4} \tag{58}$$

If the above equivalent necessary and sufficient conditions are satisfied we have,

$$\operatorname{cov}(X, Y) \leq \operatorname{var} g(X, Y) \tag{59}$$

We now derive the following general extension to Stein's lemma that does not require normality, involving the covariance between a random variable and a function of another random variable.

Proposition 5. *If the continuous random variables X and Y have support over the interval (a, b), that is $-\infty \leq a < X, Y < b \leq \infty$, with means $E(X) = \mu_X; E(Y) = \mu_Y$, variances $E(X) = \mu_X; E(Y) = \mu_Y$, density functions $f_X(t); f_Y(u)$, cumulative distributions $F_X(t); F_Y(u)$ and joint density function, $f_{XY}(t, u)$. Let $h(u)$ be a real valued function such that $E[h(Y)] = \mu_Y$ and $E\left[h^2(Y)\right] < \infty$. Suppose $f_{XY}(t, u)$ is a differentiable function with derivative $f'_{XY}(t, u)$ and there exists a non-vanishing (non-zero over the support) function $g(t, u)$ such that,*

$$\frac{f'_{XY}(t, u)}{f_{XY}(t, u)} = -\frac{g'(t, u)}{g(t, u)} + \frac{[\mu_Y - h(u)]}{g(t, u)} \quad , \quad t, u \in (a, b)$$

Assuming $\lim_{t\to b} g(t,u) f_{XY}(t,u) = 0$ *shows that for a given* $h(u)$ *the value of* $g(t,u)$ *uniquely determines the joint distribution of* X *and* Y*.*

$$f_{XY}(r,u)\ g(r,u) = \int_r^b [h(u) - \mu_Y]\ f_{XY}(t,u)\ dt$$

$$\int_a^b f_{XY}(r,u)\ g(r,u)\ du = \int_a^b \int_r^b [h(u) - \mu_Y]\ f_{XY}(t,u)\ dt\, du$$

Similarly, assuming $\lim_{t\to a} g(t,u) f_{XY}(t,u) = 0$ gives,

$$f_{XY}(r,u)\ g(r,u) = \int_a^r [\mu_Y - h(u)]\ f_{XY}(t,u)\ dt$$

$$\int_a^b f_{XY}(r,u)\ g(r,u)\ du = \int_a^b \int_a^r [\mu_Y - h(u)]\ f_{XY}(t,u)\ dt\, du$$

For any absolutely continuous function, $c(t)$*, with derivative* $c'(t)$ *satisfying* $E[c(X) h(Y)] < \infty, E\left[c^2(X)\right] < \infty, E[g(X,Y) c'(X)] < \infty$*, and provided* $\lim_{t\to b} g(t,u) f_{XY}(t,u) = 0$*, we have the following identity,*

$$Cov[c(X), h(Y)] = E[c'(X)\ g(X,Y)]$$

Proof. Appendix 14.6.

□

Corollary 1. *It is easily seen that ,*

$$Cov[c(X), Y] = E[c'(X)\ g(X,Y)]$$

Proof. Substituting $h(u) = u$ in the proof of the proposition 5 and simplifying gives this result.

□

Building upon some of the above results, primarily the extension to Stein's lemma in corollary 1, we formulate the following connection between distance and covariance.

Proposition 6. *The following equations govern the relationship between the Bhattacharyya distance,* $\rho(f_X, f_Y)$*, and the covariance between any two distributions with joint density function,* $f_{XY}(t,u)$*, means,* μ_X *and* μ_Y

and density functions $f_X(t)$ *and* $f_Y(t)$*,*

$$Cov[c(X), Y] = Cov(X, Y) - E\left[\sqrt{\frac{f_Y(t)}{f_X(t)}}Y\right] + \mu_Y \rho(f_X, f_Y)$$

$$Cov(X, Y) + \mu_Y \rho(f_X, f_Y) = E[c'(X)\ g(X, Y)] + E\left[\sqrt{\frac{f_Y(t)}{f_X(t)}}Y\right]$$

Here,

$$c(t) = t - \sqrt{\frac{f_Y(t)}{f_X(t)}}$$

and $g(t, u)$ *is a non-vanishing function such that,*

$$\frac{f'_{XY}(t, u)}{f_{XY}(t, u)} = -\frac{g'(t, u)}{g(t, u)} + \frac{[\mu_Y - u]}{g(t, u)} \quad , \quad t, u \in (a, b)$$

Proof. Appendix 14.7.

□

Corollary 2. *It is easily seen that ,*

$$Cov[c(X), h(Y)] = Cov(X, h(Y)) - E\left[\sqrt{\frac{f_Y(t)}{f_X(t)}}h(Y)\right] + \mu_Y \rho(f_X, f_Y)$$

$$Cov(X, h(Y)) + \mu_Y \rho(f_X, f_Y) = E[c'(X), g(X, Y)] + E\left[\sqrt{\frac{f_Y(t)}{f_X(t)}}h(Y)\right]$$

Proof. Using $h(u)$ instead of u in the proof of the proposition 6 and simplifying gives this result.

□

6 Market-structure, Microstructure and Other Applications

Our methodologies can help in the comparison of economic systems that generate prices, quantities and aid in the analysis of shopping patterns and understanding consumer behavior. The systems could be goods transacted at different shopping malls or trade flows across entire countries. Study of traffic congestion, road accidents and other fatalities across two regions could be performed to get an idea of similarities and seek common solutions where such simplifications might be applicable. The diversity of applications is limited only by our imagination since these techniques can be used in any system where data is generated and is studied to obtain a better understanding of the system.

We point below one asset pricing and one biological application to show the limitless possibilities such a comparison affords. The empirical illustration below in section 7.1 is another example of how the techniques developed here can be immediately applicable for microstructure studies. We provide a definition of microstructure that justifies why information collection and dimension reduction can be helpful procedures in this space.

Definition 1. ***Market Microstructure is the investigation of the process and protocols that govern the exchange of assets with the objective of reducing frictions that can impede the transfer.***

In financial markets, where there is an abundance of recorded information, this translates to the study of the dynamic relationships between observed variables, such as price, volume and spread, and hidden constituents, such as transaction costs and volatility, that hold sway over the efficient functioning of the system. The degree to which different markets or sub groups of securities have different measures of their corresponding distributions tells us the extent to which they are different. This can aid investors looking for diversification or looking for more of the same thing.

6.1 Asset Pricing Application

The price p_t of an asset today that gives a payoff x_{t+1} in the future (next time period) is given by the fundamental asset pricing equation $p_t = E\left(m_{t+1}x_{t+1}\right)$ that makes use of a stochastic discount factor m_{t+1}. Multi-period versions are easily derived and hence we drop the time subscripts below. (Lucas 1978; Hansen & Richard 1987; Hansen & Jagannathan 1991; Cochrane 2009) are classic references. (Kashyap 2017) reviews the asset pricing literature with emphasis on the equity risk premium puzzle (Mehra & Prescott 1985; Mehra 1988), which is that the return on equities has far exceeded the average return on short-term risk-free debt and cannot be explained by conventional representative-agent consumption based equilibrium models. Some solution attempts and related debates are (Rietz 1988; Mehra & Prescott 1988; Weil 1989; Constantinides & Duffie 1996; Campbell & Cochrane 1999; Bansal & Yaron 2004; Barro 2006; Weitzman 2007). The puzzle is based on the assumption that consumption growth is log-normal. We can relax this assumption and derive the below relationships that can aid the many empirical attempts on asset pricing that seek to find proxies for the discount factor or to marginal utility growth, employing variables f of the form, $m = c\left(f\right)$.

Proposition 7. *The asset pricing equation $p = E\left(mx\right)$ can be written as,*

$$p = E\left(mx\right) \equiv E\left[c\left(f\right)x\right] \quad \text{Setting, } m = c\left(f\right)$$

$$p = E\left[c\left(f\right)\right] E\left(x\right) + \text{Cov}\left[c\left(f\right), x\right]$$

$$p = E\left[c\left(f\right)\right] E\left(x\right) + E\left[c'\left(f\right)\ g\left(f, x\right)\right]$$

Proof. Using proposition 5 and the definitions of $c\left(t\right)$ and $g\left(t, u\right)$ therein, gives this result.

□

Corollary 3. *With a further restriction on* $m \equiv c\left(f\right) = f - \sqrt{\frac{f_x(f)}{f_f(f)}}$*, where, expected payoff is* μ_x *with density function* $f_x\left(f\right)$ *and the density function of the factor* f *is* $f_f\left(f\right)$*, the asset pricing equation becomes,*

$$p = E\left[c\left(f\right)\right] E\left(x\right) + Cov(f, x) + \mu_x\, \rho\left(f_f, f_x\right) - E\left[\sqrt{\frac{f_x\left(f\right)}{f_f\left(f\right)}} x\right]$$

Proof. Follows immediately from proposition 6.

□

6.2 Biological Application

The field of medical sciences is not immune from the increasing collection of information (Berner & Moss 2005) and the side effects of growing research and literature (Huth 1989). The challenges of managing growing research output are significantly more complex than expanding amounts of data. This might require advances in artificial intelligence and a better understanding of the giant leap the human mind makes at times from sorting through information, summarizing it as knowledge and transcending to a state of condensed wisdom. We leave this problem aside for now; but tackle the slightly more surmountable problem of voluminous data sources.

Suppose, we have a series of observations of different properties (such as heart rate or blood sugar level and others as well) across different days from different people from different geographical regions. The goal would then be to identify people from which regions have a more similar constitution. The number of days on which the observations are made is the same for all regions; but there could be different number of participants in each region.

The data gathering could be the result of tests conducted at different testing facilities on a certain number of days and different number of people might show up at each facility on the different days the tests are done. The result of the data collection will be as follows: We have a matrix, $M_{i,j}$, for each region i and property j, with T observations across time (rows) and $N_{i,j}$ (columns) for number of participants from region i and

property j. Here, we consider all properties together as representing a region. The dimension of the data matrix is given by $\text{Dim}(M_{i,j}) = (T, N_{i,j})$. We could have two cases depending on the data collection results.

1. $N_{i,j} = N_{i,k}, \forall j, k$ That is the number of participants in a region are the same for all properties being measured. This would be the simple case.

2. $N_{i,j} \neq N_{i,k}, \forall j, k$ That is the number of participants in a region could be different for some of the properties being measured.

The simple scenario corresponding to case 1) of the data collection would be when we have a matrix, $M_{i,j}$, for each region i and property j, with T observations across time (rows) and N_i (columns) for number of participants from that region. If we consider each property separately, we can compute the Bhattacharyya Distance across the various matrices $M_{i,j}$ (in pairs) separately for each property j across the various regions or i's. The multinomial approach outlined earlier can be used to calculate one coefficient for each region by combining all the properties. For the second case, we can compute the Bhattacharyya Distance across the various matrices $M_{i,j}$ for all the properties combined (or separately for each property) for a region using the multinomial approach after suitable dimension transformations.

To illustrate this, suppose we have two matrices, A and B, representing two different probability distributions, with dimensions, $m \times n$ and $k \times n$, respectively. Here, m and k denote the number of variables captured by the two matrices A and B. In general, m and k are not equal. n is the number of observations, which is the same across the two distributions. We can calculate the Bhattacharyya distance or another measure of similarity or dissimilarity between A and B by reducing the dimensions of the larger matrix, (say $m > k$) to the smaller one so that we have $\underline{A}$ and B, with the same dimension, $k \times n$. Here, $\underline{A} = CA,\ \text{Dim}\,(C) = k \times m$. Each entry in C is sampled i.i.d from a Gaussian $N\left(0, \frac{1}{k}\right)$ distribution.

7 Empirical Illustrations

7.1 Comparison of Security Prices across Markets

We start with a simple example illustrating how this measure could be used to compare different countries based on the prices of all equity securities traded in that market. Our data sample contains closing prices for most of the securities from six different markets from Jan 01, 2014 to May 28, 2014 (Figure 1a). Singapore with 566 securities is the market with the least number of traded securities. Even if we reduce the dimension of all the other markets with more number of securities, for a proper comparison of these markets, we would

need more than two years worth of data. Hence as a simplification, we first reduce the dimension of the matrix holding the close prices for each market using PCA reduction, so that the number of tickers retained would be comparable to the number of days for which we have data. The results of such a comparison are shown in Figure 1b. Some implementation pointers using statistical packages such as R are given in section 7.7.

We report the full matrix and not just the upper or lower matrix since the PCA reduction we do takes the first country reduces the dimensions up-to a certain number of significant digits and then reduces the dimension of the second country to match the number of dimensions of the first country. For example, this would mean that comparing AUS and SGP is not exactly the same as comparing SGP and AUS. As a safety step before calculating the distance, which requires the same dimensions for the structures holding data for the two entities being compared, we could perform dimension reduction using JL Lemma if the dimensions of the two countries differs after the PCA reduction.

We repeat the calculations for different number of significant digits of the PCA reduction. ***This shows the fine granularity of the results that our distance comparison produces and highlights the issue that with PCA reduction there is loss of information, since with different number of significant digits employed in the PCA reduction, we get the result that different markets are similar.*** For example, in Figure 1b, AUS - SGP are the most similar markets when two significant digits are used and AUS - HKG are the most similar with six significant digits.

We illustrate another example, where we compare a randomly selected sub universe of securities in each market, so that the number of tickers retained would be comparable to the number of days for which we have data. The results are shown in Figure 2. The left table (Figure 2a) is for PCA reduction on a randomly chosen sub universe and the right table (Figure 2b) is for dimension reduction using JL Lemma for the same sub universe. We report the full matrix for the same reason as explained earlier and perform multiple iterations when reducing the dimension using the JL Lemma. A key observation is that the magnitude of the distances are very different when using PCA reduction and when using dimension reduction, due to the loss of information that comes with the PCA technique. It is apparent that using dimension reduction via the JL Lemma produces consistent results, since the same pairs of markets are seen to be similar in different iterations (in Figure 2b, AUS - IND are the most similar in iteration one and also in iteration five).

It is worth remembering that in each iteration of the JL Lemma dimension transformation we multiply by a different random matrix and hence the distance is slightly different in each iteration but within the bound established by JL Lemma. When the distance is somewhat close between two pairs of entities, we could observe an inconsistency due to the JL Lemma transformation in successive iterations. We discuss this as one area that would require further theoretical investigation in section 8.

Our approach could also be used when groups of securities are being compared within the same market, a

very common scenario when deciding on the group of securities to invest in a market as opposed to deciding which markets to invest in. Such an approach would be highly useful for index construction or comparison across sectors within a market. (Kashyap 2017) summarizes the theoretical results from this paper and expands the example illustrations to open, high, low prices, volumes and price volatilities to provide a complete market microstructure study; relevant bits of this study to illustrate how our techniques can be applied to different kinds of variables are reproduced in section 7.2.

Trading Country	Distinct Tickers
AUS	993
GBR	1497
HKG	698
IND	1562
JPN	3323
SGP	566

(a) Markets and Tickers Count

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	0.14	0.13	0.17	0.19	0.03
GBR	0.43	0	0.01	0.02	0.01	0.78
HKG	0.51	0.01	0	0.06	0.03	0.95
IND	1.58	0.1	0.1	0	0.03	2.32
JPN	1.16	0.04	0.03	0.03	0	1.79
SGP	0.03	0.23	0.23	0.25	0.27	0
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	1.24	1.34	2.02	1.72	0.18
GBR	3.83	0	0.03	0.51	0.2	10.15
HKG	3.64	0.03	0	0.41	0.15	9.01
IND	11.81	1.06	1.54	0	0.13	29.21
JPN	9.86	0.53	0.9	0.13	0	26.1
SGP	0.18	2.1	2.22	2.98	2.65	0
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	5.71	5.43	9.86	8.39	6.28
GBR	11.49	0	0.73	1.94	1.52	48.06
HKG	7.39	0.43	0	2.89	2.34	32.92
IND	20.54	2.05	4.45	0	0.13	60.42
JPN	19.12	1.64	4.02	0.13	0	58.73
SGP	1.68	7.67	7.84	10.21	9.19	0
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	11.49	8.43	19.89	18.47	24.16
GBR	11.81	0	0.75	2.05	1.64	50.11
HKG	8.63	0.75	0	4.45	4.02	45.78
IND	20.54	2.05	4.45	0	0.13	60.42
JPN	19.12	1.64	4.02	0.13	0	58.73
SGP	5.82	17.15	16.82	22.01	20.32	0
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	11.81	8.63	20.54	19.12	25.58
GBR	11.81	0	0.75	2.05	1.64	50.11
HKG	8.63	0.75	0	4.45	4.02	45.78
IND	20.54	2.05	4.45	0	0.13	60.42
JPN	19.12	1.64	4.02	0.13	0	58.73
SGP	13.42	31.43	29.24	38.88	37.01	0

(b) PCA Dimension Reduction

Figure 1: Distance Measures over Full Sample

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.46	1.24	3.68	0.33	11.53
GBR	0.76	-	0.61	2.50	0.15	15.07
HKG	1.19	0.52	-	0.61	0.60	10.98
IND	6.27	2.86	2.76	-	3.55	22.34
JPN	0.33	0.13	0.60	2.58	-	12.81
SGP	13.14	16.78	17.64	22.34	16.05	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.76	1.40	4.61	0.51	12.72
GBR	11.89	-	0.70	4.11	0.29	30.39
HKG	3.37	0.65	-	3.06	0.61	19.28
IND	18.04	3.97	4.27	-	5.23	37.14
JPN	8.29	0.26	0.62	4.96	-	25.63
SGP	14.95	36.87	37.18	47.71	35.12	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.74	2.25	6.27	1.38	13.14
GBR	24.90	-	0.72	6.45	0.32	46.38
HKG	16.95	0.70	-	5.24	0.64	37.18
IND	45.77	7.96	7.69	-	8.81	68.97
JPN	25.00	0.33	0.64	8.03	-	46.35
SGP	15.50	54.64	56.21	73.24	54.33	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	8.00	8.09	14.53	7.01	14.11
GBR	32.32	-	0.98	9.05	0.66	54.64
HKG	32.05	0.89	-	7.93	0.64	54.71
IND	82.09	35.77	45.20	-	12.44	94.43
JPN	40.99	34.98	3.47	10.81	-	62.56
SGP	15.57	69.76	61.37	89.41	65.97	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	18.32	18.30	28.66	16.92	15.02
GBR	33.36	-	1.26	9.91	0.97	55.66
HKG	35.16	1.26	-	8.57	0.66	57.79
IND	82.75	35.32	45.23	-	14.54	90.87
JPN	67.92	33.70	44.20	13.26	-	73.36
SGP	53.10	69.58	81.46	99.74	73.36	-

(a) PCA Dimension Reduction

Iteration=1	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	365,776	3,126,158	65,659	400,769	9,217,218
GBR	286,251	-	31,881	33,362	37,242	1,700,993
HKG	3,126,158	30,127	-	16,991	46,078	1,220,842
IND	43,505	22,088	35,519	-	9,189	99,572
JPN	589,164	26,210	55,591	10,220	-	41,351
SGP	5,729,568	1,926,989	576,043	107,999	18,700	-
Iteration=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	502,738	3,126,158	47,601	590,959	15,982,109
GBR	390,306	-	57,608	28,398	19,858	1,093,510
HKG	3,126,158	38,845	-	27,608	56,726	1,411,575
IND	15,478	31,908	22,963	-	18,192	108,002
JPN	389,431	33,110	41,238	27,142	-	20,152
SGP	19,803,314	1,566,940	566,872	110,645	66,599	-
Iteration=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	286,825	3,126,158	54,458	214,268	11,175,458
GBR	194,775	-	54,441	37,736	38,338	2,074,430
HKG	3,126,158	62,178	-	21,746	52,694	1,117,187
IND	67,113	20,674	12,701	-	15,499	125,484
JPN	710,536	32,260	67,444	15,402	-	14,429
SGP	13,822,251	1,294,142	1,563,725	87,795	21,591	-
Iteration=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	206,720	3,126,158	61,146	499,663	9,905,841
GBR	249,784	-	48,265	10,018	24,967	2,782,083
HKG	3,126,158	57,661	-	30,326	37,593	1,363,929
IND	41,889	28,625	44,232	-	13,091	111,664
JPN	619,108	50,946	41,268	12,422	-	23,960
SGP	15,744,248	2,078,534	1,487,785	102,756	19,925	-
Iteration=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	486,768	3,126,158	81,682	752,027	13,373,157
GBR	175,751	-	54,761	24,935	22,663	2,689,089
HKG	3,126,158	54,559	-	38,806	40,914	544,276
IND	51,493	44,493	46,420	-	17,287	92,262
JPN	215,560	18,928	56,973	11,849	-	45,916
SGP	10,990,677	1,697,948	929,998	98,233	38,624	-

(b) JL Lemma Dimension Reduction

Figure 2: Distance Measures over Randomly Chosen Sub Universe

7.2 Comparison of Security Trading Volumes, High-Low-Open-Close Prices and Volume-Price Volatilities

We provide several examples in (sections 7.3, 7.4, 7.5) complementary to (section 7.1) of how our techniques could be used to compare different countries based on the time series variables across all equity securities traded in that market. These illustrations are discussed in much greater detail as part of a complete study on market micro-structure in (Kashyap 2017). The data sample contains prices (open, close, high and low) and trading volumes for most of the securities from six different markets from Jan 01, 2014 to May 28, 2014 (Figure 3a). The time period under consideration, the number of securities and the bulk of the numerical analysis is similar to the comparisons done in section 7.1. To compare volatilities of prices and volumes, we calculate sixty day moving volatilities on the close price and trading volume and calculate the distance measure over the full sample and also across each of the randomly selected sub-samples.

7.3 Speaking Volumes Of: Comparison of Trading Volumes

The results of the volume comparison over the full sample are shown in Figure 3b. For example, in Figure 3b, AUS - GBR are the most similar markets when two significant digits are used and AUS - GBR are the most similar with six significant digits. In this case the PCA and JL Lemma dimension reduction give similar

results.

The random sample results are shown in Figure 4. The left table (Figure 4a) is for PCA reduction on a randomly chosen sub universe and the right table (Figure 4b) is for dimension reduction using JL Lemma for the same sub universe.

Trading Country	Distinct Tickers
AUS	993
GBR	1497
HKG	698
IND	1562
JPN	3323
SGP	566

(a) Markets and Tickers Count

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.71	14.80	8.48	2.46	7.34
GBR	0.59	-	4.04	3.63	1.36	2.00
HKG	9.57	8.10	-	18.22	13.30	1.43
IND	4.34	5.66	18.22	-	1.57	13.53
JPN	0.64	1.47	8.33	1.00	-	5.65
SGP	5.05	3.70	1.32	12.71	8.24	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.31	26.12	20.02	7.42	9.82
GBR	0.74	-	15.76	9.70	2.74	7.47
HKG	19.41	20.04	-	43.24	30.94	3.83
IND	16.78	14.86	54.82	-	4.46	37.77
JPN	3.62	3.18	29.50	3.63	-	18.18
SGP	8.63	8.83	3.59	29.52	18.18	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.31	26.12	20.02	7.42	9.82
GBR	1.31	-	29.50	16.84	4.94	11.74
HKG	26.12	29.50	-	63.13	46.07	7.06
IND	20.02	16.84	63.13	-	4.87	42.32
JPN	7.42	4.94	46.07	4.87	-	26.05
SGP	9.80	11.66	6.87	41.85	25.78	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.31	26.12	20.02	7.42	9.82
GBR	1.31	-	29.50	16.84	4.94	11.74
HKG	26.12	29.50	-	63.13	46.07	7.06
IND	20.02	16.84	63.13	-	4.87	42.32
JPN	7.42	4.94	46.07	4.87	-	26.05
SGP	9.82	11.74	7.06	42.32	26.05	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.31	26.12	20.02	7.42	9.82
GBR	1.31	-	29.50	16.84	4.94	11.74
HKG	26.12	29.50	-	63.13	46.07	7.06
IND	20.02	16.84	63.13	-	4.87	42.32
JPN	7.42	4.94	46.07	4.87	-	26.05
SGP	9.82	11.74	7.06	42.32	26.05	-

(b) Volume PCA Dimension Reduction

Figure 3: Security Count by Market / Volume Distance Measures over Full Sample

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.18	1.41	5.17	8.89	5.13
GBR	0.18	-	2.20	3.97	7.57	6.32
HKG	1.22	1.72	-	5.14	7.44	1.63
IND	3.35	2.47	6.38	-	1.02	10.36
JPN	5.39	4.52	8.41	0.90	-	12.07
SGP	3.29	3.72	1.49	5.71	6.90	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	4.19	1.85	15.37	24.12	7.40
GBR	0.44	-	3.07	6.24	11.08	8.35
HKG	1.54	2.48	-	10.62	14.99	2.18
IND	7.80	5.90	12.35	-	1.56	18.23
JPN	10.66	9.07	14.99	1.43	-	20.35
SGP	4.25	5.07	1.71	9.42	12.07	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	14.76	2.28	30.50	43.97	8.88
GBR	2.58	-	5.97	7.21	14.00	12.79
HKG	1.88	8.80	-	22.46	31.90	2.93
IND	15.37	7.44	21.42	-	3.09	29.57
JPN	18.98	13.52	24.62	2.29	-	32.37
SGP	6.47	9.00	2.56	20.57	26.21	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	55.70	5.45	44.78	55.94	40.58
GBR	6.77	-	11.21	7.71	17.10	18.42
HKG	1.96	15.74	-	31.47	45.27	2.95
IND	21.11	7.78	27.82	-	4.60	36.03
JPN	28.28	16.64	35.05	3.73	-	43.38
SGP	7.40	15.18	2.93	29.57	38.60	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	55.72	46.92	45.68	66.87	40.66
GBR	11.28	-	15.74	8.06	19.60	22.69
HKG	1.96	17.74	-	34.97	48.87	3.54
IND	24.80	8.06	31.47	-	5.33	39.44
JPN	36.55	19.02	43.51	5.07	-	51.61
SGP	7.53	20.22	2.93	36.03	48.31	-

(a) Volume PCA Dimension Reduction

Iteration=1	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	84.81	76.82	85.01	88.46	65.64
GBR	93.91	-	94.34	82.85	82.30	91.94
HKG	74.70	87.92	-	82.37	93.98	70.89
IND	91.10	81.72	82.37	-	62.41	89.73
JPN	90.30	87.39	98.24	61.81	-	97.81
SGP	69.21	89.96	69.95	85.73	98.17	-
Iteration=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	81.86	78.23	96.07	90.57	70.70
GBR	90.42	-	92.18	78.68	87.65	90.18
HKG	67.18	92.73	-	82.37	95.56	72.50
IND	85.54	81.43	82.37	-	62.35	86.19
JPN	82.29	85.22	92.64	61.67	-	97.58
SGP	71.25	89.07	76.35	88.51	96.66	-
Iteration=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	91.33	67.99	87.39	86.02	64.20
GBR	91.75	-	93.97	80.84	84.74	92.59
HKG	73.46	95.92	-	82.37	96.44	72.98
IND	86.44	80.28	82.37	-	62.12	85.65
JPN	92.56	84.56	95.61	63.02	-	102.12
SGP	63.40	89.66	69.56	86.12	96.98	-
Iteration=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	78.91	72.97	83.66	90.74	64.95
GBR	81.57	-	92.61	80.52	84.93	93.73
HKG	69.17	91.17	-	82.37	96.88	67.78
IND	86.65	82.47	82.37	-	62.77	91.47
JPN	83.34	85.57	96.62	63.14	-	97.85
SGP	66.94	92.89	74.65	84.17	96.33	-
Iteration=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	84.48	71.55	83.71	90.53	72.69
GBR	80.39	-	89.52	78.57	86.37	91.60
HKG	79.32	92.58	-	82.37	92.38	71.09
IND	91.27	83.53	82.37	-	63.25	85.50
JPN	91.05	86.06	98.11	62.77	-	95.25
SGP	65.91	91.14	68.00	89.46	95.75	-

(b) Volume JL Lemma Dimension Reduction

Figure 4: Volume Distance Measures over Randomly Chosen Sub Universe

7.4 A Pricey Prescription: Comparison of Prices (Open, Close, High and Low)

7.4.1 Open Close

The results of a comparison between open and close prices over the full sample are shown in Figures 5, 5a, 5b. For example, in Figure 5b, AUS - SGP are the most similar markets when two significant digits are used and AUS - HKG are the most similar with six significant digits. The similarities between open and close prices, in terms of the distance measure, are also easily observed.

The random sample results are shown in Figures 6, 7. The left table (Figures 6a, 7a) is for PCA reduction on a randomly chosen sub universe and the right table (Figures 6b, 7b) is for dimension reduction using JL Lemma for the same sub universe. In Figure 7b, AUS - IND are the most similar in iteration one and also in iteration five.

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.14	0.13	0.18	0.20	0.03
GBR	0.44	-	0.01	0.02	0.01	0.80
HKG	0.49	0.02	-	0.07	0.04	0.99
IND	1.28	0.07	0.07	-	0.02	2.01
JPN	1.20	0.05	0.04	0.02	-	1.92
SGP	0.03	0.23	0.23	0.27	0.29	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.50	1.53	2.17	2.02	0.26
GBR	6.74	-	0.21	0.39	0.10	19.10
HKG	3.91	0.06	-	0.42	0.14	10.50
IND	14.44	0.59	2.26	-	0.24	37.46
JPN	10.35	0.12	0.82	0.22	-	28.88
SGP	0.22	2.33	2.38	3.03	2.87	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	7.63	5.93	10.76	8.85	7.63
GBR	15.77	-	1.90	0.97	0.26	57.92
HKG	8.16	1.29	-	3.46	2.14	37.90
IND	21.98	0.97	4.89	-	0.33	65.00
JPN	18.58	0.26	3.15	0.33	-	61.14
SGP	1.49	7.44	7.21	9.13	8.30	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	15.77	9.28	21.98	18.58	28.21
GBR	15.77	-	1.90	0.97	0.26	57.92
HKG	9.28	1.90	-	4.89	3.15	49.67
IND	21.98	0.97	4.89	-	0.33	65.00
JPN	18.58	0.26	3.15	0.33	-	61.14
SGP	6.55	20.11	18.26	23.56	21.47	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	15.77	9.28	21.98	18.58	28.21
GBR	15.77	-	1.90	0.97	0.26	57.92
HKG	9.28	1.90	-	4.89	3.15	49.67
IND	21.98	0.97	4.89	-	0.33	65.00
JPN	18.58	0.26	3.15	0.33	-	61.14
SGP	13.59	34.11	29.74	39.03	36.17	-

(a) Open PCA Dimension Reduction

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	0.14	0.13	0.17	0.19	0.03
GBR	0.43	0	0.01	0.02	0.01	0.78
HKG	0.51	0.01	0	0.06	0.03	0.95
IND	1.58	0.1	0.1	0	0.03	2.32
JPN	1.16	0.04	0.03	0.03	0	1.79
SGP	0.03	0.23	0.23	0.25	0.27	0
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	1.24	1.34	2.02	1.72	0.18
GBR	3.83	0	0.03	0.51	0.2	10.15
HKG	3.64	0.03	0	0.41	0.15	9.01
IND	11.81	1.06	1.54	0	0.13	29.21
JPN	9.86	0.53	0.9	0.13	0	26.1
SGP	0.18	2.1	2.22	2.98	2.65	0
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	5.71	5.43	9.86	8.39	6.28
GBR	11.49	0	0.73	1.94	1.52	48.06
HKG	7.39	0.43	0	2.89	2.34	32.92
IND	20.54	2.05	4.45	0	0.13	60.42
JPN	19.12	1.64	4.02	0.13	0	58.73
SGP	1.68	7.67	7.84	10.21	9.19	0
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	11.49	8.43	19.89	18.47	24.16
GBR	11.81	0	0.75	2.05	1.64	50.11
HKG	8.63	0.75	0	4.45	4.02	45.78
IND	20.54	2.05	4.45	0	0.13	60.42
JPN	19.12	1.64	4.02	0.13	0	58.73
SGP	5.82	17.15	16.82	22.01	20.32	0
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	0	11.81	8.63	20.54	19.12	25.58
GBR	11.81	0	0.75	2.05	1.64	50.11
HKG	8.63	0.75	0	4.45	4.02	45.78
IND	20.54	2.05	4.45	0	0.13	60.42
JPN	19.12	1.64	4.02	0.13	0	58.73
SGP	13.42	31.43	29.24	38.88	37.01	0

(b) Close PCA Dimension Reduction

Figure 5: Open / Close Distance Measures over Full Sample

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.61	1.47	2.48	0.07	9.13
GBR	3.62	-	0.44	1.55	1.03	18.37
HKG	1.89	0.42	-	0.78	1.69	14.29
IND	5.57	1.52	1.18	-	3.83	20.31
JPN	1.05	0.89	1.82	3.83	-	13.64
SGP	10.36	12.91	14.29	15.86	10.65	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.38	2.31	4.24	0.33	11.41
GBR	12.34	-	0.47	1.95	2.04	30.83
HKG	7.95	0.46	-	1.39	2.34	24.91
IND	20.45	2.23	1.80	-	7.04	40.52
JPN	8.79	2.10	2.98	6.64	-	26.21
SGP	13.90	30.83	31.93	36.40	24.81	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	2.40	3.24	5.57	1.05	11.87
GBR	26.29	-	0.64	3.70	2.91	48.79
HKG	22.40	0.52	-	2.36	3.71	44.14
IND	40.97	4.38	2.83	-	12.01	64.91
JPN	17.78	2.81	4.19	10.11	-	38.40
SGP	15.42	51.81	54.53	62.91	41.97	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	9.90	10.44	14.16	6.04	13.48
GBR	33.29	-	1.30	5.43	3.30	57.35
HKG	40.62	2.09	-	3.11	6.94	65.40
IND	52.11	7.27	3.20	-	15.37	77.25
JPN	25.20	3.31	6.54	14.27	-	47.78
SGP	15.76	61.71	68.54	77.95	51.10	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	21.25	22.40	28.96	14.72	14.95
GBR	36.23	-	2.50	7.27	3.34	60.73
HKG	42.22	2.50	-	3.20	7.24	67.15
IND	52.11	7.27	3.20	-	15.37	77.25
JPN	28.09	3.34	7.41	15.43	-	51.10
SGP	15.78	75.73	73.67	88.79	63.77	-

(a) Open PCA Dimension Reduction

Iteration=1	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	147,332	357,105	72,252	447,338	11,760,081
GBR	64,024	-	23,900	6,985	29,057	733,083
HKG	263,487	49,759	-	15,888	20,577	563,200
IND	82,491	14,386	9,684	-	5,029	61,990
JPN	426,948	13,373	13,078	7,474	-	11,441
SGP	7,879,885	854,575	671,440	69,954	8,488	-
Iteration=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	79,232	437,898	34,660	723,681	10,352,861
GBR	83,584	-	22,291	14,406	16,543	505,788
HKG	261,184	25,308	-	18,538	35,131	362,887
IND	44,316	13,045	25,246	-	7,986	65,117
JPN	120,124	16,647	26,602	7,153	-	9,187
SGP	9,594,568	561,372	664,757	71,920	17,134	-
Iteration=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	159,101	78,753	63,290	438,398	7,680,674
GBR	173,521	-	25,823	7,215	17,145	907,660
HKG	290,054	28,928	-	11,158	26,866	546,871
IND	97,118	7,770	10,263	-	9,279	63,926
JPN	726,018	19,079	13,403	6,391	-	13,355
SGP	5,556,248	516,818	742,894	68,860	22,389	-
Iteration=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	137,104	212,410	39,547	603,107	8,457,391
GBR	142,636	-	35,139	12,768	17,116	669,073
HKG	205,720	19,393	-	11,048	20,371	478,788
IND	50,186	12,734	16,334	-	7,315	58,361
JPN	233,014	21,248	19,775	12,809	-	14,787
SGP	18,167,921	781,053	618,734	61,358	34,356	-
Iteration=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	152,307	60,820	64,520	242,028	17,000,095
GBR	332,989	-	31,501	7,566	26,028	925,237
HKG	221,826	38,972	-	19,559	31,486	432,616
IND	99,696	9,554	9,488	-	6,775	65,333
JPN	258,864	14,837	34,881	5,165	-	14,320
SGP	8,304,606	510,917	720,135	62,932	10,476	-

(b) Open JL Lemma Dimension Reduction

Figure 6: Open Distance Measures over Randomly Chosen Sub Universe

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.46	1.24	3.68	0.33	11.53
GBR	0.76	-	0.61	2.50	0.15	15.07
HKG	1.19	0.52	-	0.61	0.60	10.98
IND	6.27	2.86	2.76	-	3.55	22.34
JPN	0.33	0.13	0.60	2.58	-	12.81
SGP	13.14	16.78	17.64	22.34	16.05	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.76	1.40	4.61	0.51	12.72
GBR	11.89	-	0.70	4.11	0.29	30.39
HKG	3.37	0.65	-	3.06	0.61	19.28
IND	18.04	3.97	4.27	-	5.23	37.14
JPN	8.29	0.26	0.62	4.96	-	25.63
SGP	14.95	36.87	37.18	47.71	35.12	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.74	2.25	6.27	1.38	13.14
GBR	24.90	-	0.72	6.45	0.32	46.38
HKG	16.95	0.70	-	5.24	0.64	37.18
IND	45.77	7.96	7.69	-	8.81	68.97
JPN	25.00	0.33	0.64	8.03	-	46.35
SGP	15.50	54.64	56.21	73.24	54.33	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	8.00	8.09	14.53	7.01	14.11
GBR	32.32	-	0.98	9.05	0.66	54.64
HKG	32.05	0.89	-	7.93	0.64	54.71
IND	82.09	35.77	45.20	-	12.44	94.43
JPN	40.99	34.98	3.47	10.81	-	62.56
SGP	15.57	69.76	61.37	89.41	65.97	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	18.32	18.30	28.66	16.92	15.02
GBR	33.36	-	1.26	9.91	0.97	55.66
HKG	35.16	1.26	-	8.57	0.66	57.79
IND	82.75	35.32	45.23	-	14.54	90.87
JPN	67.92	33.70	44.20	13.26	-	73.36
SGP	53.10	69.58	81.46	99.74	73.36	-

(a) Close PCA Dimension Reduction

Iteration=1	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	365,776	3,126,158	65,659	400,769	9,217,218
GBR	286,251	-	31,881	33,362	37,242	1,700,993
HKG	3,126,158	30,127	-	16,991	46,078	1,220,842
IND	43,505	22,088	35,519	-	9,189	99,572
JPN	589,164	26,210	55,591	10,220	-	41,351
SGP	5,729,568	1,926,989	576,043	107,999	18,700	-
Iteration=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	502,738	3,126,158	47,601	590,959	15,982,109
GBR	390,306	-	57,608	28,398	19,858	1,093,510
HKG	3,126,158	38,845	-	27,608	56,726	1,411,575
IND	15,478	31,908	22,963	-	18,192	108,002
JPN	389,431	33,110	41,238	27,142	-	20,152
SGP	19,803,314	1,566,940	566,872	110,645	66,599	-
Iteration=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	286,825	3,126,158	54,458	214,268	11,175,458
GBR	194,775	-	54,441	37,736	38,338	2,074,430
HKG	3,126,158	62,178	-	21,746	52,694	1,117,187
IND	67,113	20,674	12,701	-	15,499	125,484
JPN	710,536	32,260	67,444	15,402	-	14,429
SGP	13,822,251	1,294,142	1,563,725	87,795	21,591	-
Iteration=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	206,720	3,126,158	61,146	499,663	9,905,841
GBR	249,784	-	48,265	10,018	24,967	2,782,083
HKG	3,126,158	57,661	-	30,326	37,593	1,363,929
IND	41,889	28,625	44,232	-	13,091	111,664
JPN	619,108	50,946	41,268	12,422	-	23,960
SGP	15,744,248	2,078,534	1,487,785	102,756	19,925	-
Iteration=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	486,768	3,126,158	81,682	752,027	13,373,157
GBR	175,751	-	54,761	24,935	22,663	2,689,089
HKG	3,126,158	54,559	-	38,806	40,914	544,276
IND	51,493	44,493	46,420	-	17,287	92,262
JPN	215,560	18,928	56,973	11,849	-	45,916
SGP	10,990,677	1,697,948	929,998	98,233	38,624	-

(b) Close JL Lemma Dimension Reduction

Figure 7: Close Distance Measures over Randomly Chosen Sub Universe

7.4.2 High Low

The results of a comparison between high and low prices over the full sample are shown in Figures 8, 8a, 8b. For example, in Figure 8a, AUS - SGP are the most similar markets when two significant digits are used and AUS - HKG are the most similar with six significant digits. The similarities between high and low prices are also easily observed.

The random sample results are shown in Figures 9, 10. The left table (Figures 9a, 10a) is for PCA reduction on a randomly chosen sub universe and the right table (Figures 9b, 10b) is for dimension reduction using JL Lemma for the same sub universe. In Figures 9b and 10b, AUS - IND are the most similar in iteration one and also in iteration five.

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.13	0.13	0.19	0.20	0.04
GBR	0.52	-	0.02	0.04	0.02	0.95
HKG	0.76	0.02	-	0.07	0.03	1.21
IND	1.31	0.06	0.07	-	0.02	1.79
JPN	1.19	0.04	0.03	0.02	-	1.66
SGP	0.04	0.23	0.24	0.29	0.29	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.40	1.51	2.07	1.88	0.11
GBR	5.81	-	0.23	0.27	0.08	8.41
HKG	3.33	0.06	-	0.28	0.10	3.74
IND	9.68	0.30	1.14	-	0.12	14.17
JPN	8.55	0.09	0.69	0.12	-	13.29
SGP	0.13	3.01	3.04	4.22	3.69	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	6.85	5.18	9.32	7.86	1.31
GBR	14.87	-	2.04	0.46	0.34	35.25
HKG	7.08	1.35	-	2.47	2.06	17.89
IND	19.42	0.51	3.84	-	0.14	41.57
JPN	18.45	0.43	3.69	0.14	-	40.51
SGP	0.58	7.34	6.30	9.54	8.23	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	14.87	8.37	18.87	17.82	11.51
GBR	15.21	-	2.07	0.51	0.43	36.99
HKG	8.56	2.07	-	3.84	3.69	28.67
IND	19.42	0.51	3.84	-	0.14	41.57
JPN	18.45	0.43	3.69	0.14	-	40.51
SGP	2.83	16.28	12.51	19.36	17.73	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	15.21	8.56	19.42	18.45	12.60
GBR	15.21	-	2.07	0.51	0.43	36.99
HKG	8.56	2.07	-	3.84	3.69	28.67
IND	19.42	0.51	3.84	-	0.14	41.57
JPN	18.45	0.43	3.69	0.14	-	40.51
SGP	6.68	26.12	19.77	29.74	28.38	-

(a) High PCA Dimension Reduction

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.12	0.13	0.17	0.19	0.03
GBR	0.54	-	0.01	0.04	0.02	0.94
HKG	0.49	0.01	-	0.06	0.03	0.90
IND	1.20	0.06	0.07	-	0.02	1.73
JPN	1.08	0.04	0.03	0.02	-	1.59
SGP	0.03	0.20	0.23	0.25	0.26	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.43	1.40	2.07	1.90	0.13
GBR	6.81	-	0.38	0.11	0.06	13.22
HKG	3.24	0.07	-	0.21	0.12	4.97
IND	8.62	0.11	0.71	-	0.04	16.17
JPN	8.51	0.06	0.65	0.04	-	16.60
SGP	0.14	3.00	2.89	3.83	3.56	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	7.13	5.07	8.28	7.82	2.69
GBR	17.13	-	2.42	0.14	0.13	44.01
HKG	7.62	1.68	-	2.12	2.12	24.82
IND	18.91	0.14	3.05	-	0.05	45.98
JPN	18.86	0.13	3.21	0.05	-	45.90
SGP	0.92	7.60	6.59	8.73	8.26	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	15.94	8.75	17.62	17.54	13.54
GBR	17.13	-	2.42	0.14	0.13	44.01
HKG	9.34	2.42	-	3.05	3.21	34.56
IND	18.91	0.14	3.05	-	0.05	45.98
JPN	18.86	0.13	3.21	0.05	-	45.90
SGP	3.96	17.72	14.23	19.07	18.57	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	17.13	9.34	18.91	18.86	15.70
GBR	17.13	-	2.42	0.14	0.13	44.01
HKG	9.34	2.42	-	3.05	3.21	34.56
IND	18.91	0.14	3.05	-	0.05	45.98
JPN	18.86	0.13	3.21	0.05	-	45.90
SGP	9.04	30.49	23.73	32.12	31.80	-

(b) Low PCA Dimension Reduction

Figure 8: High / Low Distance Measures over Full Sample

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.71	2.09	2.52	0.11	3.25
GBR	6.91	-	0.71	1.08	0.85	13.47
HKG	2.67	0.67	-	0.31	1.57	7.93
IND	4.92	1.05	0.41	-	2.43	10.70
JPN	2.45	0.56	1.67	2.60	-	8.06
SGP	0.01	0.16	1.08	0.84	0.02	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.18	2.67	3.34	0.35	4.16
GBR	18.46	-	0.74	1.83	1.68	26.24
HKG	9.44	0.72	-	0.58	2.00	16.30
IND	25.51	1.98	1.63	-	6.06	33.66
JPN	11.20	1.49	2.43	4.96	-	18.32
SGP	0.01	0.16	1.08	0.84	0.02	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	2.50	3.93	4.92	1.35	4.32
GBR	34.58	-	0.75	4.22	2.45	44.40
HKG	23.62	0.74	-	1.97	2.72	31.96
IND	56.37	5.95	5.68	-	14.07	67.04
JPN	25.46	2.40	3.20	10.09	-	34.68
SGP	0.94	1.48	2.50	2.47	1.10	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	9.67	10.81	12.92	6.69	4.70
GBR	45.68	-	1.13	6.86	3.33	56.59
HKG	39.75	0.75	-	4.91	3.65	50.07
IND	80.11	39.50	46.59	-	38.57	80.51
JPN	33.96	3.23	4.07	14.80	-	44.20
SGP	4.51	11.67	12.92	14.47	9.42	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	24.33	25.05	31.19	18.27	4.92
GBR	47.25	-	1.51	7.44	3.39	58.36
HKG	44.82	0.77	-	6.05	4.07	55.52
IND	78.98	39.21	46.10	-	38.22	79.26
JPN	37.96	3.45	4.60	17.19	-	48.86
SGP	4.80	26.24	27.15	31.61	20.90	-

(a) High PCA Dimension Reduction

Iteration=1	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1,136,789	1,035,029	62,336	285,299	8,861,726
GBR	260,317	-	54,234	11,853	20,017	560,188
HKG	262,093	45,389	-	8,583	29,900	574,598
IND	21,578	11,494	15,910	-	14,096	49,364
JPN	210,287	20,528	22,761	18,787	-	20,877
SGP	6,290,821	233,946	1,059,991	121,144	15,969	-
Iteration=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	272,510	326,018	37,758	198,698	8,150,262
GBR	1,357,619	-	43,911	6,813	18,465	164,969
HKG	200,980	69,434	-	10,954	25,776	315,811
IND	121,515	16,390	12,922	-	14,461	145,572
JPN	363,923	29,124	32,262	14,606	-	7,711
SGP	13,814,434	304,504	305,510	83,479	31,267	-
Iteration=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	858,176	441,543	98,931	202,658	10,092,986
GBR	750,609	-	36,957	17,704	30,381	174,224
HKG	981,359	52,791	-	20,361	23,209	506,752
IND	43,493	28,691	25,489	-	23,532	84,349
JPN	384,847	20,335	29,712	10,567	-	19,023
SGP	12,652,910	456,558	844,319	86,701	6,604	-
Iteration=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	318,293	569,600	75,164	238,532	4,742,125
GBR	606,041	-	55,490	17,544	32,817	379,511
HKG	495,372	27,143	-	13,924	29,809	950,869
IND	64,320	13,531	23,475	-	18,880	35,207
JPN	330,397	15,052	22,570	8,377	-	25,638
SGP	14,465,992	245,619	241,314	34,732	19,457	-
Iteration=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	183,034	874,949	21,725	357,254	11,238,310
GBR	569,244	-	64,096	20,076	31,230	194,679
HKG	624,445	27,196	-	28,036	38,832	675,894
IND	54,740	19,631	11,161	-	21,869	72,540
JPN	334,172	11,263	28,547	20,940	-	10,348
SGP	6,452,800	291,636	259,187	103,512	15,874	-

(b) High JL Lemma Dimension Reduction

Figure 9: High Distance Measures over Randomly Chosen Sub Universe

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.73	2.04	2.97	0.25	5.19
GBR	7.03	-	0.71	2.18	1.00	17.55
HKG	2.46	0.68	-	0.45	1.46	10.44
IND	6.40	2.02	0.83	-	3.67	16.43
JPN	0.32	0.35	1.46	2.73	-	7.32
SGP	0.02	0.08	0.63	0.68	0.08	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.34	2.90	4.67	0.45	7.47
GBR	18.73	-	0.76	2.71	2.23	30.83
HKG	9.68	0.73	-	1.17	2.12	20.82
IND	19.97	2.44	1.40	-	6.40	32.05
JPN	6.58	1.48	2.27	5.36	-	16.90
SGP	0.02	0.08	0.63	0.68	0.08	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.34	2.90	4.67	0.45	7.47
GBR	35.43	-	0.77	3.64	3.39	48.98
HKG	24.41	0.76	-	1.94	3.35	37.05
IND	37.95	3.32	2.25	-	9.62	51.13
JPN	20.28	2.80	3.70	9.35	-	32.12
SGP	2.56	3.30	4.48	4.79	2.93	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	8.53	9.68	13.13	5.57	8.08
GBR	49.31	-	0.91	4.53	4.29	64.34
HKG	43.43	0.78	-	2.97	4.85	58.58
IND	57.60	4.38	3.19	-	13.30	73.44
JPN	32.85	3.92	4.95	12.92	-	47.20
SGP	7.83	13.96	15.79	18.52	11.69	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	20.22	21.44	26.99	14.27	8.33
GBR	73.79	-	1.25	40.39	4.79	63.99
HKG	48.68	0.80	-	3.36	5.11	64.60
IND	61.78	4.53	3.64	-	13.96	74.15
JPN	38.79	4.35	5.14	14.13	-	54.35
SGP	8.34	35.57	37.05	43.42	28.28	-

(a) Low PCA Dimension Reduction

Iteration=1	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	730,968	260,160	42,949	119,615	3,141,558
GBR	727,754	-	21,673	22,205	43,511	255,735
HKG	529,013	55,050	-	11,719	26,304	531,520
IND	37,621	17,579	11,282	-	12,847	61,039
JPN	121,720	40,337	31,205	7,666	-	32,840
SGP	4,559,242	189,211	434,402	61,039	12,514	-
Iteration=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	282,531	153,498	43,470	178,108	5,414,456
GBR	1,278,439	-	45,722	22,952	41,980	410,496
HKG	231,228	32,926	-	11,255	32,263	846,343
IND	63,492	10,607	17,175	-	7,040	61,039
JPN	151,883	36,875	28,303	5,930	-	16,353
SGP	5,045,445	374,646	284,311	61,039	26,401	-
Iteration=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	707,672	464,398	38,825	409,387	9,099,068
GBR	1,080,809	-	41,095	14,243	13,147	138,746
HKG	79,074	37,726	-	24,387	46,036	1,123,944
IND	31,139	34,849	21,067	-	9,882	61,039
JPN	160,496	15,568	53,609	9,250	-	13,068
SGP	8,406,786	305,797	716,506	61,039	13,597	-
Iteration=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	977,857	374,215	53,744	230,691	7,193,157
GBR	573,856	-	31,660	24,332	49,753	406,460
HKG	124,577	29,814	-	24,784	23,501	1,073,209
IND	107,403	19,574	18,487	-	15,592	61,039
JPN	144,123	31,802	46,399	10,054	-	7,446
SGP	8,732,192	213,074	508,333	61,039	24,479	-
Iteration=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	397,266	262,786	60,288	137,397	9,900,417
GBR	631,324	-	18,766	6,964	22,831	530,999
HKG	231,667	39,690	-	23,492	46,673	1,203,763
IND	53,796	26,806	8,340	-	10,841	61,039
JPN	76,818	29,587	25,712	15,652	-	19,338
SGP	6,575,871	419,610	146,474	61,039	18,501	-

(b) Low JL Lemma Dimension Reduction

Figure 10: Low Distance Measures over Randomly Chosen Sub Universe

7.5 Taming the (Volatility) Skew: Comparison of Close Price / Volume Volatilities

The results of a comparison between close price volatilities and volume volatilities over the full sample are shown in Figures 11, 11a, 11b. For example, in Figure 11a, AUS - GBR are the most similar markets when two significant digits are used and AUS - HKG are the most similar with six significant digits. In Figure 11b, AUS - GBR - IND are equally similar markets when two significant digits are used and AUS - GBR are the most similar with six significant digits. The difference in magnitudes of the distance measures for prices, volumes and volatilities is also easily observed. What this indicates is that, prices are from the most dissimilar or distant distributions, volatilities are less similar and volumes are from the most similar or overlapping distributions. As also observed in the volume comparisons, volume volatility comparisons give seemingly similar results when PCA or JL Lemma dimension reductions are used. By considering the price volatilities, and creating portfolios of instruments that have dissimilar volatility distributions, we could reduce the overall risk or variance of the portfolio returns, becoming one potential way of mitigating the effects of wild volatility swings.

The random sample results are shown in Figures 12, 13. The left table (Figures 12a, 13a) is for PCA reduction on a randomly chosen sub universe and the right table (Figures 12b, 13b) is for dimension reduction

using JL Lemma for the same sub universe. In Figure 12b AUS - SGP are the most similar in iteration one and also in iteration five. In Figure 13b, AUS - SGP are the most similar in iteration one and AUS- GBR in iteration five.

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.02	0.10	0.28	0.12	0.38
GBR	0.02	-	0.04	0.16	0.05	0.40
HKG	0.09	0.04	-	0.06	0.02	0.62
IND	0.24	0.13	0.06	-	0.04	0.78
JPN	0.10	0.04	0.02	0.04	-	0.51
SGP	0.22	0.32	0.48	0.78	0.51	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.04	0.14	0.47	0.22	1.28
GBR	0.04	-	0.06	0.28	0.10	1.42
HKG	0.13	0.05	-	0.13	0.05	1.64
IND	0.41	0.24	0.12	-	0.05	2.17
JPN	0.21	0.10	0.05	0.06	-	1.91
SGP	0.88	1.11	1.45	2.17	1.68	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.32	0.51	1.07	0.60	3.46
GBR	0.38	-	0.07	0.37	0.12	5.83
HKG	0.55	0.07	-	0.18	0.06	6.23
IND	0.97	0.34	0.17	-	0.10	6.62
JPN	0.57	0.11	0.06	0.10	-	5.85
SGP	2.38	3.27	3.86	5.02	4.06	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.23	1.20	2.66	1.56	5.44
GBR	2.74	-	0.27	0.60	0.16	12.93
HKG	1.67	0.21	-	0.64	0.20	11.37
IND	4.87	0.60	0.90	-	0.25	17.05
JPN	3.47	0.18	0.39	0.26	-	14.73
SGP	5.24	9.11	9.65	12.25	10.27	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	3.06	2.25	5.64	3.88	6.04
GBR	3.06	-	0.28	0.72	0.22	13.76
HKG	2.25	0.28	-	1.06	0.43	13.06
IND	5.64	0.72	1.06	-	0.27	18.42
JPN	3.88	0.22	0.43	0.27	-	15.50
SGP	6.04	13.76	13.06	18.42	15.50	-

(a) Close Volatility PCA Dimension Reduction

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.01	0.43	0.01	0.62	0.11
GBR	0.01	-	0.42	0.02	0.61	0.07
HKG	0.43	0.52	-	0.33	0.03	0.88
IND	0.01	0.03	0.33	-	0.50	0.17
JPN	0.62	0.72	0.03	0.50	-	1.11
SGP	0.11	0.08	0.88	0.17	1.11	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.02	1.09	0.04	1.37	0.28
GBR	0.01	-	1.08	0.06	1.37	0.20
HKG	1.09	1.21	-	0.76	0.03	2.24
IND	0.04	0.07	0.72	-	0.96	0.46
JPN	1.46	1.60	0.03	1.06	-	2.79
SGP	0.31	0.27	2.57	0.58	2.97	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.02	2.29	0.10	2.65	0.65
GBR	0.02	-	2.20	0.13	2.54	0.51
HKG	2.22	2.40	-	1.46	0.04	4.79
IND	0.10	0.14	1.41	-	1.72	1.14
JPN	2.80	3.01	0.04	1.93	-	5.72
SGP	0.67	0.59	5.10	1.24	5.57	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.03	3.74	0.12	3.80	1.02
GBR	0.03	-	3.84	0.17	3.98	0.86
HKG	3.80	4.05	-	2.74	0.08	8.16
IND	0.12	0.17	2.69	-	2.73	1.74
JPN	4.01	4.26	0.10	2.85	-	8.59
SGP	1.09	0.97	8.48	1.85	8.40	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.03	4.20	0.14	4.15	1.12
GBR	0.03	-	4.40	0.18	4.36	1.02
HKG	4.20	4.40	-	3.00	0.10	8.90
IND	0.14	0.18	3.00	-	2.94	1.92
JPN	4.15	4.36	0.10	2.94	-	8.78
SGP	1.12	1.02	8.90	1.92	8.78	-

(b) Volume Volatility PCA Dimension Reduction

Figure 11: Close / Volume Volatility Distance Measures over Full Sample

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.88	0.17	0.67	1.42	0.89
GBR	1.04	-	1.24	4.10	0.31	0.09
HKG	0.18	1.21	-	0.80	1.97	1.27
IND	2.12	4.10	1.93	-	5.34	3.98
JPN	1.89	0.31	2.13	5.34	-	0.34
SGP	0.97	0.21	1.35	4.67	0.68	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.14	0.27	2.53	2.11	0.97
GBR	1.32	-	1.30	6.28	0.75	0.30
HKG	0.24	1.24	-	1.93	2.13	1.30
IND	2.90	5.55	3.19	-	7.39	4.87
JPN	2.88	0.61	2.44	8.61	-	1.20
SGP	0.98	0.33	1.56	5.39	1.81	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.35	0.57	3.32	3.37	0.98
GBR	1.58	-	1.35	7.06	2.06	0.47
HKG	0.38	1.27	-	3.51	2.40	1.39
IND	3.32	6.43	4.30	-	9.44	5.37
JPN	4.52	1.47	3.19	11.17	-	2.71
SGP	0.99	0.48	1.91	5.56	3.82	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.58	0.98	3.44	5.55	0.99
GBR	1.59	-	1.47	7.49	3.66	0.52
HKG	0.79	1.31	-	4.85	3.35	1.72
IND	3.43	6.98	4.99	-	11.98	5.51
JPN	6.41	2.77	3.92	13.80	-	4.53
SGP	1.02	0.52	2.08	5.85	5.47	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1.59	1.14	3.67	6.79	1.02
GBR	3.99	-	1.80	8.27	43.65	0.70
HKG	1.14	1.45	-	5.84	4.16	2.07
IND	3.67	7.34	5.84	-	14.58	5.72
JPN	7.74	4.18	4.95	16.59	-	5.85
SGP	1.08	0.64	2.09	6.68	6.61	-

(a) Close Volatility PCA Dimension Reduction

Iteration=1	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	3,564	4,294	7,437	2,510	1,166
GBR	3,225	-	1,259	1,500	701	772
HKG	3,828	1,320	-	2,357	778	670
IND	3,625	2,544	2,588	-	2,384	1,156
JPN	7,939	827	1,248	3,291	-	1,532
SGP	5,071	772	889	1,189	874	-
Iteration=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	2,690	3,084	3,402	3,394	2,123
GBR	2,025	-	719	2,965	1,474	772
HKG	2,851	985	-	3,385	1,308	541
IND	2,859	2,899	2,801	-	3,723	2,009
JPN	3,711	1,303	934	3,189	-	828
SGP	2,694	772	715	1,099	1,077	-
Iteration=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	1,727	3,196	4,768	3,158	3,195
GBR	3,720	-	508	2,738	1,777	772
HKG	3,762	604	-	4,297	875	1,431
IND	5,539	996	3,746	-	2,703	1,611
JPN	5,393	1,764	948	2,676	-	1,310
SGP	2,435	772	675	2,354	1,210	-
Iteration=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	2,397	3,638	3,509	6,393	1,604
GBR	1,999	-	1,083	2,855	1,003	772
HKG	3,896	627	-	2,847	1,081	579
IND	6,069	1,698	3,401	-	2,502	1,901
JPN	4,050	1,332	1,910	4,238	-	1,137
SGP	2,106	772	580	923	837	-
Iteration=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	3,372	2,665	5,358	4,090	1,359
GBR	2,142	-	669	1,630	1,246	772
HKG	2,437	1,909	-	2,837	662	666
IND	5,617	2,073	2,886	-	3,338	1,485
JPN	4,786	1,074	540	4,378	-	704
SGP	2,583	772	776	1,217	1,006	-

(b) Close Volatility JL Lemma Dimension Reduction

Figure 12: Close Volatility Distance Measures over Randomly Chosen Sub Universe

Sig Level=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.05	0.58	0.15	0.70	0.14
GBR	0.05	-	0.79	0.24	0.96	0.07
HKG	0.49	0.72	-	0.29	0.04	0.88
IND	0.15	0.25	0.38	-	0.45	0.48
JPN	0.70	0.96	0.04	0.37	-	1.20
SGP	0.14	0.07	1.01	0.40	1.20	-
Sig Level=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.18	0.69	0.16	0.91	0.48
GBR	0.21	-	1.25	0.32	1.81	0.18
HKG	0.70	1.30	-	0.51	0.16	2.05
IND	0.24	0.34	0.65	-	1.14	0.87
JPN	0.96	1.69	0.11	0.74	-	2.40
SGP	0.71	0.20	2.05	0.78	2.84	-
Sig Level=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.53	0.78	0.39	1.31	1.54
GBR	0.55	-	1.97	0.35	3.05	0.41
HKG	0.80	2.02	-	1.18	0.25	3.75
IND	0.72	0.43	1.66	-	2.97	1.15
JPN	1.36	3.05	0.24	1.89	-	4.94
SGP	1.91	0.54	3.93	1.10	5.48	-
Sig Level=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.68	0.84	0.91	1.86	2.56
GBR	0.81	-	2.38	0.56	31.99	0.88
HKG	36.89	2.43	-	2.41	29.89	5.26
IND	34.23	0.87	2.71	-	28.22	32.62
JPN	1.72	3.84	0.39	3.23	-	6.84
SGP	2.96	0.88	5.17	1.15	31.38	-
Sig Level=6	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	0.89	0.86	1.46	32.29	3.08
GBR	36.75	-	2.44	0.87	31.07	36.10
HKG	35.62	43.30	-	2.95	30.49	38.19
IND	33.06	40.24	2.95	-	27.90	34.79
JPN	1.99	4.37	0.62	4.04	-	7.68
SGP	36.06	0.89	5.26	1.19	30.86	-

(a) Volume Volatility PCA Dimension Reduction

Iteration=1	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	7,763	6,535	8,149	5,664	4,812
GBR	6,875	-	3,043	5,474	5,797	7,534
HKG	5,941	3,043	-	4,935	8,760	3,560
IND	4,104	10,331	5,829	-	4,648	8,989
JPN	4,064	10,568	8,188	5,113	-	6,743
SGP	4,184	5,164	6,178	4,507	4,400	-
Iteration=2	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	3,702	7,378	4,910	7,247	10,018
GBR	7,525	-	3,043	5,821	7,454	9,053
HKG	5,573	3,043	-	4,302	6,469	7,147
IND	5,167	5,413	4,981	-	6,836	4,961
JPN	6,885	9,429	9,128	4,418	-	4,394
SGP	5,325	8,479	2,586	6,773	7,978	-
Iteration=3	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	5,185	7,671	5,267	7,528	5,057
GBR	6,172	-	3,043	8,065	9,304	4,471
HKG	5,240	3,043	-	10,982	14,587	5,191
IND	3,287	9,208	5,014	-	5,642	8,628
JPN	7,061	8,964	5,860	4,587	-	3,065
SGP	5,492	5,957	5,827	7,769	8,161	-
Iteration=4	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	4,914	4,807	5,403	6,821	3,615
GBR	9,158	-	3,043	4,686	3,566	4,035
HKG	5,655	3,043	-	4,997	7,552	4,085
IND	4,559	11,613	6,126	-	4,865	8,521
JPN	5,258	8,402	5,535	5,182	-	8,778
SGP	7,837	6,986	8,885	6,297	4,594	-
Iteration=5	AUS	GBR	HKG	IND	JPN	SGP
AUS	-	3,099	4,559	6,843	5,945	5,381
GBR	5,035	-	3,043	8,025	5,299	6,914
HKG	12,538	3,043	-	5,717	4,808	7,028
IND	8,141	12,006	4,641	-	5,296	5,811
JPN	5,536	7,473	3,719	8,451	-	4,109
SGP	5,770	4,704	3,833	2,778	6,316	-

(b) Volume Volatility JL Lemma Dimension Reduction

Figure 13: Volume Volatility Distance Measures over Randomly Chosen Sub Universe

7.6 Other Complementary Case Studies

1. **Please note, this text in RED is only to solicit feedback from the reviewers and editors. It will be DELETED.**

2. **If this section is found to be too preliminary, this can be entirely deleted; but it has been included mostly to show the wide range of applications the techniques in this article can facilitate.**

3. **If it is felt that having a more detailed example from these projects would help this paper, we are happy to include it here. Though, it would take more time to develop these case studies from these other projects to act as preliminary findings.**

7.6.1 United Kingdom Crime Analysis

We are developing a separate application (Kashyap 2017) which applies the techniques in this article and also from network theory to a crime data set from the United Kingdom. The goal of this project is to identify regions with similar levels and types of crimes so that policy and enforcement decisions can be made based on lessons learnt from earlier episodes or from regions where certain legal measures had a decent level of success. The Crime Data-set is publicly available at: UK Police Data.

7.6.2 Hong Kong and Shanghai Securities Comparison across Industrial Sectors

We are developing another numerical example (Kashyap 2017) on real data which applies the techniques in this article to find out which sectors in China and Hong Kong are more similar based on the stock prices and trading volumes of all securities listed on the Shanghai and Hong Kong stock exchange. We are using a publicly available data set downloadable from many market data provider websites ranging from Jan-02-2014 to Nov-04-2014.

7.7 Implementation Pointers

(Chaussé 2010) is a good reference for estimating the parameters of a normal distribution through the Generalized Method of Moments, GMM, (Cochrane 2009) using R package **gmm**. The numerical computation of a multivariate normal is often a difficult problem. (Genz 1992; Genz & Bretz 2009) describe a transformation that simplifies the problem and places it into a form that allows efficient calculation using standard numerical multiple integration algorithms. (Hothorn, Bretz & Genz 2001) give pointers for the numerical computation

of multivariate normal probabilities based on the R package **mvtnorm**. (Manjunath & Wilhelm 2012) derive an explicit expression for the truncated mean and variance for the multivariate normal distribution with arbitrary rectangular double truncation. (Wilhelm & Manjunath 2010; Wilhelm 2015) have details on the R package **tmvtnorm**, for the truncated multivariate normal distribution including routines for the GMM Estimation for the Truncated Multivariate Normal Distribution. In (appendix 15), we list some R code snippets, that includes the Johnson-Lindenstrauss matrix transformation and a modification to the routine to calculate the Bhattacharyya distance, available currently in the R package **fps**. This modification allows much larger numbers and dimensions to be handled, by utilizing the properties of logarithms and the Eigen values of a matrix.

8 Possibilities for Future Research

There is a lot of research being done in developing distance measures, dimension reduction techniques and understanding the properties of distributions whose dimensions have been transformed. Such studies are aimed at developed better theoretical foundations as well faster algorithms for implementing newer techniques. We point out alternative approaches that might hold potential starting with some not so recent methods and then some newer techniques. (Chow & Liu 1968) present a method to approximate an n dimensional discrete distribution by a product of several of its component distributions of lower order in such a way that the product is a probability extension of these distributions of lower order. A product approximation is a valid probability distribution. Only the class of second order distributions are used. There are $n\left(n-1\right)/2$ second order approximations of which at most $n-1$ can be used in the approximation.

The uncertainty associated with a state estimate can be represented most generally by a probability distribution incorporating all knowledge about the state. The Kalman filter (Kalman 1960; End-note 10) exploits the fact that 1) given only the mean and variance (or covariance in multiple dimensions) of a distribution, the most conservative assumption that can be made about the distribution is that it is a Gaussian having the given mean and variance and 2) the fact that the application of a linear operator to a Gaussian distribution always yields another Gaussian distribution. Given the assumptions of 1) and 2) it is straightforward to show that the Kalman filter will yield the best possible estimate of the mean and variance of the state. The requirement that the mean and variance of the state is measurable represents little practical difficulty but the requirement that all observation and process models be linear is rarely satisfied in nontrivial applications. (Julier & Uhlmann 1996) examine an alternative generalization of the Kalman filter that accommodates nonlinear transformations of probability distributions through the use of a new representation of the mean and variance information about the current state.

(Székely, Rizzo & Bakirov 2007; Székely & Rizzo 2009; Lyons 2013) develop a new measure of dependence between random vectors called the Distance correlation. Distance covariance and distance correlation are analogous to product-moment covariance and correlation, but unlike the classical definition of correlation, distance correlation is zero only if the random vectors are independent. This is applicable to random vectors of arbitrary and not necessarily equal dimension and to any distributions with finite first moments. (Székely & Rizzo 2013) come up with the Energy distance, which is a statistical distance between the distributions of random vectors characterizing equality of distributions. The name energy derives from Newton's gravitational potential energy (which is a function of the distance between two bodies), and there is an elegant relation to the notion of potential energy between statistical observations. Energy statistics (E-statistics) are functions of distances between statistical observations. The idea of energy statistics is to consider statistical observations as heavenly bodies governed by a statistical potential energy, which is zero if and only if an underlying statistical null hypothesis is true.

We chronicle some key simplifications we have used in our illustrative examples:

1. A key limitation of our study is that we have reduced dimensions using PCA or randomly sampled a sub matrix from the overall data-set so that the length of time series available is in the range of the number of securities that could be compared. Using a longer time series for the variables would be a useful extension and a real application would benefit immensely from more history.

2. We have used the simple formula for the Bhattacharyya distance applicable to multivariate normal distributions. The formulae we have developed over a truncated multivariate normal distribution or using a Normal Log-Normal Mixture could give more accurate results. Again, later works should look into tests that can establish which of the distributions would apply depending on the data-set under consideration.

3. For the examples in section 7.2, for each market we have looked at seven variables, open, close, low, high, trading volume, close volatility and volume volatility. These variables can be combined using the expression for the multinomial distance to get a complete representation of which markets are more similar than others. We aim to develop this methodology and illustrate these techniques further in later works.

Once we have the similarity measures across groups of entities, a separate set of analysis can be performed to prudently select which policies or procedures can be transferred across similar entities. For example, across groups of securities, portfolios could be constructed to see how sensitive they are to different explanatory factors and then performance benchmarks could be used to gauge the risk return relationship.

When the distance is somewhat close between two pairs of entities, we could observe an inconsistency due to the JL Lemma transformation in successive iterations. It is worth remembering that if we perform

multiple iterations of the JL Lemma dimension transformation (multiplying by a different random matrix); the distance could be slightly different in each iteration but within the bound established by JL Lemma. Though, in one iteration one pair could seem to be more similar than the other and the result could be reversed in the other iteration. When compared to other dimension reduction techniques such as PCA, due to loss of data, it is hard to know if the results are accurate at all when the distance between entities is too small. Theoretical bounds could be established with regards to what are the limits on the distance measure where such an inconsistency could result. Further to this, a huge branch of theoretical investigations can be undertaken regarding the interval over which the distance measure will fall when one of the random variables is multiplied by different random matrices governed by the JL-Lemma to effect the same dimension transformation.

A key point to remember is that the findings with respect to the similarity of different entities are based on the data available for the particular time period under consideration. There could be fundamental changes in the true data generating processes, which are generally unknown and the same entities for a different time period could give entirely different results. Hence, any real application that continues to collect the relevant data that is generated, should calculate the metrics on a moving time period basis (rolling) and use the latest results that are computed. Needless to say, this brings up the question of how much of the recent history we should use as we make our decisions. While our present econometric tools fall short of providing concrete answers, many practical aids and suggestions are available in any econometric guide (Hamilton 1994). This is of course true for most (all?) empirical data investigations.

Another improvement, could be to normalize, or, standardize distance measures. This is perhaps, applicable not just to the Bhattacharyya distance, but to other types of distance metrics, as well. For example, the covariance is a useful number, but its use is magnified and the intuitive lessons compounded, by the use of correlations derived from covariance measures. Again, we need to be wary of this path, since anything that can be measured is magical; that is, there is always something bigger or something smaller, which can be expressed as a need to contend with the concepts of the infinite, or, the infinitesimal. So by standardizing anything, we might reduce the range of the values, but it might be necessary, to retain the number of significant digits, that give us meaningful comparisons. Also, since we wish to compare more than two entities, it might be helpful, to come up with metrics, that can combine, multiple distance measures, at the same time. Though, it would suffice to be able to combine two of the distance measures at a time, since that is how we would compare any two probabilistic entities; just as in the covariance case other possibilities such as combining more than two at a time need not be ruled out.

9 Conclusions

We have discussed various measures of similarity between probability distributions. We have shown how the combination of the Bhattacharyya distance and the Johnson-Lindenstrauss Lemma provides us with a novel and practical methodology that allows comparisons between any two probability distributions. We demonstrated a relationship between covariance and distance measures based on a generic extension of Stein's Lemma. We have provided numerical examples based on security prices across six countries where we have compared trading volumes, open-close-high-low prices, price and volume volatilities; all of which act as how to guides of the techniques developed here for real life applications. We have also discussed how this methodology lends itself to numerous applications outside the realm of finance and economics.

10 Acknowledgements and End-notes

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2. Numerous seminar participants, particularly at a few meetings of the econometric society and various finance organizations, suggested ways to improve the core results in the manuscript and prompted us to find ways to express the intuition for why we need the main results. The views and opinions expressed in this article, along with any mistakes, are mine alone and do not necessarily reflect the official policy or position of either of my affiliations or any other agency.

3. The Mellin transform is an integral transform that may be regarded as the multiplicative version of the two-sided Laplace transform (End-note 12). The Mellin transform of a function f is

$$\{\mathcal{M}f\}(s) = \varphi(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

The inverse transform is

$$\left\{\mathcal{M}^{-1}\varphi\right\}(x) = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\varphi(s)\,ds$$

This integral transform is closely connected to the theory of Dirichlet series (End-note 11), and is often used in number theory, mathematical statistics, and the theory of asymptotic expansions; it is closely related to the Laplace transform (End-note 13) and the Fourier transform (End-note 14), and the theory of the gamma function and allied special functions. To avoid extended discussions, we only provide some details here. The following link has more details: Mellin Transform, Wikipedia Link

4. The G-function was introduced by (Cornelis Simon Meijer 1936) as a very general function intended to include most of the known special functions as particular cases. This was not the only attempt of its kind: the generalized hypergeometric function and the MacRobert E-function had the same aim, but Meijer's G-function was able to include those as particular cases as well. The first definition was made by Meijer using a series; nowadays the accepted and more general definition is via a line integral in the complex plane, introduced in its full generality by Arthur Erdelyi in 1953. Meijer G-Function, Wikipedia Link

5. The Newton-Cotes formulas, also called the Newton-Cotes quadrature rules or simply Newton-Cotes rules, are a group of formulas for numerical integration (also called quadrature) based on evaluating the integrand at equally spaced points. Newton-Cotes formulas, Wikipedia Link

6. Rao's score test, also known as the score test or the Lagrange multiplier test (LM test) in econometrics, is a statistical test of a simple null hypothesis that a parameter of interest θ is equal to some particular value θ_0. It is the most powerful test when the true value of θ is close to θ_0. The main advantage of the score test is that it does not require an estimate of the information under the alternative hypothesis or unconstrained maximum likelihood. Score or Lagrange Multiplier Test, Wikipedia Link

7. In probability theory and statistics, skewness is a measure of the asymmetry of the probability distribution of a real-valued random variable about its mean. The skewness value can be positive or negative, or undefined. Skewness, Wikipedia Link

8. In probability theory and statistics, kurtosis (from Greek: kyrtos or kurtos, meaning "curved, arching") is a measure of the "tailedness" of the probability distribution of a real-valued random variable. In a similar way to the concept of skewness, kurtosis is a descriptor of the shape of a probability distribution and, just as for skewness, there are different ways of quantifying it for a theoretical distribution and corresponding ways of estimating it from a sample from a population. Depending on the particular

measure of kurtosis that is used, there are various interpretations of kurtosis, and of how particular measures should be interpreted. Kurtosis, Wikipedia Link

9. The Euclidean distance or Euclidean metric is the "ordinary" straight-line distance between two points in Euclidean space. With this distance, Euclidean space becomes a metric space. The associated norm is called the Euclidean norm. Older literature refers to the metric as the Pythagorean metric. A generalized term for the Euclidean norm is the L^2 norm or L^2 distance. Euclidean Distance, Wikipedia Link

10. In statistics and control theory, Kalman filtering, also known as linear quadratic estimation (LQE), is an algorithm that uses a series of measurements observed over time, containing statistical noise and other inaccuracies, and produces estimates of unknown variables that tend to be more accurate than those based on a single measurement alone, by estimating a joint probability distribution over the variables for each timeframe. The Kalman filter has numerous applications for guidance, navigation, and control of vehicles, particularly aircraft and spacecraft. Furthermore, the Kalman filter is a widely applied concept in time series analysis used in fields such as signal processing and econometrics. Kalman filters also are one of the main topics in the field of robotic motion planning and control, and they are sometimes included in trajectory optimization. Kalman Filter, Wikipedia Link

11. A Dirichlet series is any series of the form
$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$
where s is complex, and a_n is a complex sequence. Dirichlet Series, Wikipedia Link

12. The two-sided Laplace transform or bilateral Laplace transform is an integral transform equivalent to probability's moment generating function. Two-sided Laplace transforms are closely related to the Fourier transform, the Mellin transform, and the ordinary or one-sided Laplace transform. If $f(t)$ is a real or complex valued function of the real variable t defined for all real numbers, then the two-sided Laplace transform is defined by the integral,
$$\mathcal{B}\{f\}(s) = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$
Two-Sided Laplace Transform, Wikipedia Link

13. The Laplace transform of a function $f(t)$, defined for all real numbers $t \geq 0$, is the function $F(s)$, which

is a unilateral transform defined by

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

where s is a complex number frequency parameter. $s = \sigma + i\omega$, with real numbers σ and ω. An alternate notation for the Laplace transform is $\mathcal{L}\{f\}$ instead of F. Laplace Transform, Wikipedia Link

14. The Fourier transform (FT) decomposes a function of time (a signal) into the frequencies that make it up, in a way similar to how a musical chord can be expressed as the frequencies (or pitches) of its constituent notes. The Fourier transform of a function of time is itself a complex-valued function of frequency, whose absolute value represents the amount of that frequency present in the original function, and whose complex argument is the phase offset of the basic sinusoid in that frequency. The Fourier transform is called the frequency domain representation of the original signal. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time. The Fourier transform is not limited to functions of time, but in order to have a unified language, the domain of the original function is commonly referred to as the time domain. The Fourier transform of the function f is traditionally denoted by adding a circumflex: $\hat{f}$. There are several common conventions for defining the Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{C}$. One definition that is commonly used for any real number ξ is,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\ e^{-2\pi i x \xi}\,dx$$

When the independent variable x represents time, the transform variable ξ represents frequency (e.g. if time is measured in seconds, then the frequency is in hertz). Under suitable conditions, f is determined by $\hat{f}$ for any real number x, via the inverse transform:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\ e^{2\pi i x \xi}\,d\xi$$

Fourier Transform, Wikipedia Link

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12 Appendix: Example for the Merits and Limits of Four Physical Dimensions

Now, imagine an object in our world, say a soda can, or, a laptop. Observing these articles, in three dimensions, can tell us a great deal about these items, and help us understand their nature and properties. But now, suppose that we put light, on these objects, from different directions, and observe the shadow in a lower dimension, which is a flat surface. Here, we are restricted to two dimensions and perhaps we are also losing the color of the objects. But say, we could view the shadow, with spectacles having different colored glasses (we could, of course, drop this additional constraint and deal only with a grey shadow, without any loss of intuition). We could capture, a great deal of the properties of the three dimensional object, but we are restricted by the characteristics, that could be captured in a lower dimension. Another point to bear in mind, is that the shadow only projects certain points onto the flat surface, as dictated by the properties of physics, more specifically, perhaps, the laws of light; whereas with the JL Lemma transformation, every point in the higher dimension is mapped to a point in the lower dimension.

When we do the JL Lemma transformation, by multiplying the original set of observations we have, (a probability distribution) by another probability distribution; or, a conversion from a higher dimensional object, to a lower dimension, can be compared to putting light, on the object in the three dimensional world, from different angles and capturing the different shadows. The shadow changes in the two dimensional world depending on the angle from which the light is coming (or, which particular probability distribution, is being used for the transformation), but the original object, lives on, in the higher world, with no change whatsoever. There is definitely some loss, but that loss is the reality or the limitation of the dimension we are forced to work with. If we have one object in a lower dimension and that is the best information, we have about it; we cannot take it to the higher dimension, to compare it with an object, in the higher dimension. The best we can do, is bring the object from the higher dimension, to the lower dimension and compare them, in the lower dimension.

From our example, this is essentially, comparing a shadow that we have, to the shadows we can produce, by bringing objects from the higher dimension to the lower dimension. To illustrate this further, imagine that we have just observed the shadow of a thief, and that is the best information we have captured (say from a camera footage). The best way to go about tracking this thief, is trying to understand ,what sort of people can produce such shadows, and this might reduce the range of possibilities, as we try to look for our

burglar. For someone, that would say that it is not too much information, surely that is more data points, than having nothing and we should be thankful and look into ways in which, we can use, the only lead we have (in this case), in tracking down the shadowy suspect.

13 Appendix: Uncertainty and Unintended Consequences

(Lawson 1985) argues that the Keynesian view on uncertainty (that it is generally impossible, even in probabilistic terms, to evaluate the future outcomes of all possible current actions; Keynes 1937; 1971; 1973), far from being innocuous or destructive of economic analysis in general, can give rise to research programs incorporating, amongst other things, a view of rational behavior under uncertainty, which could be potentially fruitful. (McManus and Hastings 2005) clarify the wide range of uncertainties that affect complex engineering systems and present a framework to understand the risks (and opportunities) they create and the strategies system designers can use to mitigate or take advantage of them. (Simon 1962) points out that any attempt to seek properties common to many sorts of complex systems (physical, biological or social), would lead to a theory of hierarchy since a large proportion of complex systems observed in nature exhibit hierarchic structure; that a complex system is composed of subsystems that, in turn, have their own subsystems, and so on. These viewpoints hold many lessons for policy designers in the social sciences and could be instructive for researchers looking to create metrics to compare complex systems, keeping in mind the caveats of dynamic social systems.

A hall mark of the social sciences is the lack of objectivity. Here we assert that objectivity is with respect to comparisons done by different participants and that a comparison is a precursor to a decision.

Conjecture 1. ***Despite the several advances in the social sciences, we have yet to discover an objective measuring stick for comparison, a so called, True Comparison Theory, which can be an aid for arriving at objective decisions.***

The search for such a theory could again (Kashyap 2017) be compared, to the medieval alchemists' obsession with turning everything into gold. For our present purposes, the lack of such an objective measure means that the difference in comparisons, as assessed by different participants, can effect different decisions under the same set of circumstances. Hence, despite all the uncertainty in the social sciences, the one thing we can be almost certain about is the subjectivity in all decision making. Restricted to the particular sub-universe of economic and financial theory, this translates to the lack of an objective measuring stick of value, a so called, True Value Theory. This lack of an objective measure of value, (hereafter, value will be

synonymously referred to as the price of an instrument), leads to dissimilar decisions and actions by different participants, making prices react at varying degrees and at varying speeds to the pull of diverse macro and micro factors.

14 Appendix of Proofs

14.1 Notation and Terminology for Key Results

- $D_{BC}\left(p_i, p_i'\right)$, the Bhattacharyya Distance between two multinomial populations each consisting of k categories classes with associated probabilities $p_1, p_2, ..., p_k$ and $p_1', p_2', ..., p_k'$ respectively.
- $\rho\left(p_i, p_i'\right)$, the Bhattacharyya Coefficient.
- $d\left(p_i, p_i'\right)$, the modified Bhattacharyya Metric.
- $\chi^2\left(p_i, p_i'\right)$ Chi-Squared measure.
- $D_{H-M}\left(p_i, p_i'\right)$ is the Hellinger or Matusita Distance.
- $D_{BC-N}(p, q)$ is the Bhattacharyya distance between p and q normal distributions or classes.
- $D_{BC-MN}\left(p_1, p_2\right)$ is the Bhattacharyya distance between two multivariate normal distributions, $\boldsymbol{p_1}, \boldsymbol{p_2}$ where $\boldsymbol{p_i} \sim \mathcal{N}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$.
- $D_{BC-TN}(p, q)$ is the Bhattacharyya distance between p and q truncated normal distributions or classes.
- $D_{BC-TMN}\left(p_1, p_2\right)$ is the Bhattacharyya distance between two truncated multivariate normal distributions, $\boldsymbol{p_1}, \boldsymbol{p_2}$ where $\boldsymbol{p_i} \sim \mathcal{N}(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i, \boldsymbol{a}_i, \boldsymbol{b}_i)$.
- $D_{\mathrm{KL}}(P \| Q)$ is the Kullback-Leibler divergence of Q from P.

14.2 Proof of Proposition 1

Proof. From the properties of a bivariate distribution (Bertsekas 2002) of two normal random variables, V, W, the conditional expectation is given by

$$E\left[V \mid W\right] = E\left[V\right] + \frac{cov\left(V, W\right)}{\sigma_W^2}\left(W - E\left[W\right]\right)$$

This is a linear function of W and is a normal distribution. The conditional distribution of V given W is normal with mean $E[V \mid W]$ and variance, $\sigma^2_{V|W}$ using the correlation coefficient ρ_{VW} between them is,

$$Var\left(V \mid W\right) = \sigma^2_{V|W} = (1 - \rho^2_{VW})\sigma^2_V$$

This is easily established by setting,

$$\hat{V} = \rho_{VW}\frac{\sigma_V}{\sigma_W}\left(W - \mu_W\right)$$

$$\tilde{V} = V - \hat{V}$$

$$E\left[\tilde{V}\right] = E\left[V\right] - E\left[\rho_{VW}\frac{\sigma_V}{\sigma_W}\left(W - \mu_W\right)\right] = E\left[V\right]$$

W and $\tilde{V}$ are independent as are $\hat{V}$ and $\tilde{V}$, since $\hat{V}$ is a scalar multiple of W and $E\left[W\tilde{V}\right] = \mu_W\mu_V$ implying that $Cov\left(W, \tilde{V}\right) = E\left[W\tilde{V}\right] - \mu_W\mu_V = 0$ as shown below,

$$E\left[W\tilde{V}\right] = E\left[WV\right] - \rho_{VW}\frac{\sigma_V}{\sigma_W}E\left[W^2\right] + \rho_{VW}\frac{\sigma_V}{\sigma_W}E\left[W\right]\mu_W$$

$$= \rho_{VW}\sigma_V\sigma_W + \mu_W\mu_V - \rho_{VW}\frac{\sigma_V}{\sigma_W}\left\{\sigma^2_W + \mu^2_W\right\} + \rho_{VW}\frac{\sigma_V}{\sigma_W}\mu^2_W = \mu_W\mu_V$$

$$E\left[V \mid W\right] = E\left[\tilde{V} + \hat{V} \mid W\right] = E\left[\tilde{V} \mid W\right] + E\left[\hat{V} \mid W\right]$$

$$= \mu_V + \rho_{VW}\frac{\sigma_V}{\sigma_W}\left(W - \mu_W\right)$$

$$Var\left(V \mid W\right) = Var\left(\hat{V} + \tilde{V} \mid W\right)$$

$$= Var\left(\tilde{V} \mid W\right) = Var\left(\tilde{V}\right) = Var\left(V - \rho_{VW}\frac{\sigma_V}{\sigma_W}W\right)$$

$$= Var\left(V\right) + \rho^2_{VW}\frac{\sigma^2_V}{\sigma^2_W}Var\left(W\right) - 2\rho_{VW}\frac{\sigma_V}{\sigma_W}Cov\left(V, W\right)$$

$$= \sigma^2_V + \rho^2_{VW}\sigma^2_V - 2\rho^2_{VW}\sigma^2_V$$

$$= \sigma^2_V\left(1 - \rho^2_{VW}\right)$$

Now consider the random variable, U, given by,

$$U = Xe^Y$$

Here, X and Y are random variables with correlation coefficient, ρ satisfying,

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{bmatrix}\right)$$

The conditional distribution of U given Y is normal with mean, variance and density given by,

$$E\left[U \mid Y\right] = E\left[Xe^Y \mid Y\right] = e^Y E\left[X \mid Y\right]$$

$$= e^Y\left\{\mu_X + \frac{cov\left(X,Y\right)}{\sigma_Y^2}\left(Y-\mu_Y\right)\right\}$$

$$= e^Y\left\{\mu_X + \frac{\rho\sigma_X\sigma_Y}{\sigma_Y^2}\left(Y-\mu_Y\right)\right\}$$

$$= e^Y\left\{\mu_X + \frac{\rho\sigma_X}{\sigma_Y}\left(Y-\mu_Y\right)\right\}$$

$$\sigma_{U|Y}^2 = Var\left(U \mid Y\right) = Var\left(Xe^Y \mid Y\right) = e^{2Y}Var\left(X \mid Y\right)$$

$$= e^{2Y}(1-\rho^2)\sigma_X^2$$

$$f_{U|Y}\left(u \mid Y\right) \sim N\left(e^Y\left\{\mu_X + \frac{\rho\sigma_X}{\sigma_Y}\left(Y-\mu_Y\right)\right\}, e^{2Y}(1-\rho^2)\sigma_X^2\right)$$

We observe that this conditional distribution is undefined for $\sigma_Y = 0$ when Y degenerates to a constant. We then get the joint density of U, Y as,

$$f_{UY}\left(u,y\right) = f_{U|Y}\left(u \mid Y\right) \times f_Y\left(y\right)$$

$$= \left\{2\pi e^{2y}(1-\rho^2)\sigma_X^2\right\}^{-\frac{1}{2}} e^{-\frac{\left[u-e^y\left\{\mu_X+\frac{\rho\sigma_X}{\sigma_Y}(y-\mu_Y)\right\}\right]^2}{2e^{2y}(1-\rho^2)\sigma_X^2}} \times \left\{2\pi\sigma_Y^2\right\}^{-\frac{1}{2}} e^{-\frac{[y-\mu_Y]^2}{2\sigma_Y^2}}$$

The marginal density of U is given by,

$$f_U\left(u\right) = \int_{-\infty}^{\infty}\left\{2\pi e^{2y}(1-\rho^2)\sigma_X^2\right\}^{-\frac{1}{2}} e^{-\frac{\left[u-e^y\left\{\mu_X+\frac{\rho\sigma_X}{\sigma_Y}(y-\mu_Y)\right\}\right]^2}{2e^{2y}(1-\rho^2)\sigma_X^2}} \times \left\{2\pi\sigma_Y^2\right\}^{-\frac{1}{2}} e^{-\frac{[y-\mu_Y]^2}{2\sigma_Y^2}}\, dy$$

In our case, we have, $\mu_X = 0, \sigma_X^2 = 1/k$ and $\rho = 0$ giving,

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}\right)$$

$$f_U\left(u\right) = \int_{-\infty}^{\infty}\left\{\frac{2\pi e^{2y}}{k}\right\}^{-\frac{1}{2}} e^{-\frac{ku^2}{2e^{2y}}} \times \left\{2\pi\sigma_Y^2\right\}^{-\frac{1}{2}} e^{-\frac{[y-\mu_Y]^2}{2\sigma_Y^2}}\, dy$$

$$f_U(u) = \frac{\sqrt{k}}{2\pi\sigma_Y} \int_{-\infty}^{\infty} e^{-y - \frac{ku^2}{2e^{2y}} - \frac{[y-\mu_Y]^2}{2\sigma_Y^2}} dy$$

□

14.3 Proof of Proposition 2

Proof. First we establish the density of the product of two independent random variables. Let $W = XY$, a continuous random variable, product of two independent continuous random variables X and Y. The distribution function of W is,

$$F_W(w) = \int_{\{(x,y)|xy \le w\}} f_{XY}(x,y)\, dxdy$$

Here, $\{(x,y)\,|xy \le w\} = \left\{-\infty < x \le 0, \frac{w}{x} \le y < \infty\right\} \cup \left\{0 \le x \le \infty, -\infty < y \le \frac{w}{x}\right\}$. We can write the above as,

$$F_W(w) = \int_{-\infty}^{0} \int_{\frac{w}{x}}^{\infty} f_{XY}(x,y)\, dydx + \int_{0}^{\infty} \int_{-\infty}^{\frac{w}{x}} f_{XY}(x,y)\, dydx$$

Differentiating with respect to, w, using the Leibniz integral rule gives the required density,

$$f_W(w) = \int_{-\infty}^{0} \left(-\frac{1}{x}\right) f_{XY}\left(x, \frac{w}{x}\right) dx + \int_{0}^{\infty} \left(\frac{1}{x}\right) f_{XY}\left(x, \frac{w}{x}\right) dx$$

$$f_W(w) = \int_{-\infty}^{\infty} \left(\frac{1}{|x|}\right) f_{XY}\left(x, \frac{w}{x}\right) dx = \int_{-\infty}^{\infty} \left(\frac{1}{|x|}\right) f_X(x)\, f_Y\left(\frac{w}{x}\right) dx$$

In our case, the density of the two independent normal variables, where, μ_X is the mean and σ_X^2 is the variance of the X distribution, is given by,

$$f_X\left(x \mid \mu_X, \sigma_X^2\right) = \frac{1}{\sigma_X\sqrt{2\pi}}\, e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \quad ; \quad f_Y\left(y \mid 0, \frac{1}{k}\right) = \sqrt{\frac{k}{2\pi}}\, e^{-\frac{k(y)^2}{2}}$$

$$f_W(w) = \int_{-\infty}^{\infty} \left(\frac{1}{|x|}\right) \frac{1}{\sigma_X\sqrt{2\pi}}\, e^{-\frac{(x-\mu_X)^2}{2\sigma_X^2}} \sqrt{\frac{k}{2\pi}}\, e^{-\frac{k\left(\frac{w}{x}\right)^2}{2}} dx$$

Leibniz Integral Rule: Let $f(x,\theta)$ be a function such that $f_\theta(x,\theta)$ exists, and is continuous. Then,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\int_{a(\theta)}^{b(\theta)} f(x,\theta)\,\mathrm{d}x\right) = \int_{a(\theta)}^{b(\theta)} \partial_\theta f(x,\theta)\,\mathrm{d}x \,+\, f\big(b(\theta),\theta\big)\cdot b'(\theta) \,-\, f\big(a(\theta),\theta\big)\cdot a'(\theta)$$

where the partial derivative of f indicates that inside the integral only the variation of $f(x,)$ with θ is considered in taking the derivative. Extending the above to the case of double integrals gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{c(t)}^{d(t)}\int_{a(t)}^{b(t)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\right) = \int_{c(t)}^{d(t)}\left\{ f\big(b(t),y\big)\cdot b'(t) - f\big(a(t),y\big)\cdot a'(t)\right\}\,\mathrm{d}y + \int_{a(t)}^{b(t)}\left\{ f(d(t),x)\cdot d'(t) - f(c(t),x)\cdot c'(t)\right\}\,\mathrm{d}x$$

□

14.4 Proof of Proposition 3

Proof. Suppose we have two truncated normal distributions p, q with density functions.

$$f_p\left(x \mid \mu_p, \sigma_p^2, a, b\right) = \begin{cases} \frac{\frac{1}{\sigma_p}\phi\left(\frac{x-\mu_p}{\sigma_p}\right)}{\Phi\left(\frac{b-\mu_p}{\sigma_p}\right)-\Phi\left(\frac{a-\mu_p}{\sigma_p}\right)} & ; a \le x \le b \\ 0 & ; \text{otherwise} \end{cases}$$

$$f_q\left(x \mid \mu_q, \sigma_q^2, c, d\right) = \begin{cases} \frac{\frac{1}{\sigma_q}\phi\left(\frac{x-\mu_q}{\sigma_q}\right)}{\Phi\left(\frac{d-\mu_q}{\sigma_q}\right)-\Phi\left(\frac{c-\mu_q}{\sigma_q}\right)} & ; c \le x \le d \\ 0 & ; \text{otherwise} \end{cases}$$

Here, μ_p is the mean and σ_p^2 is the variance of the p distribution with bounds a, b. The Bhattacharyya coefficient is given by the below when the distributions overlap and it is zero otherwise,

$$\rho\left(p,q\right) = \int_{l=\min(a,c)}^{u=\min(b,d)} \sqrt{f_p\left(x \mid \mu_p, \sigma_p^2, a, b\right) f_q\left(x \mid \mu_q, \sigma_q^2, c, d\right)}dx$$

$$= \int_l^u \sqrt{\frac{\frac{1}{\sigma_p}\phi\left(\frac{x-\mu_p}{\sigma_p}\right)}{\left[\Phi\left(\frac{b-\mu_p}{\sigma_p}\right)-\Phi\left(\frac{a-\mu_p}{\sigma_p}\right)\right]}\frac{\frac{1}{\sigma_q}\phi\left(\frac{x-\mu_q}{\sigma_q}\right)}{\left[\Phi\left(\frac{d-\mu_q}{\sigma_q}\right)-\Phi\left(\frac{c-\mu_q}{\sigma_q}\right)\right]}}dx$$

$$\int_l^u \sqrt{\phi\left(\frac{x-\mu_p}{\sigma_p}\right)\phi\left(\frac{x-\mu_q}{\sigma_q}\right)}dx = \frac{1}{\sqrt{2\pi}}\int_l^u \sqrt{\exp\left[-\frac{(x-\mu_p)^2}{2\sigma_p^2}\right]\exp\left[-\frac{(x-\mu_q)^2}{2\sigma_q^2}\right]}$$

$$= \frac{1}{\sqrt{2\pi}}\int_l^u \exp\left[-\frac{\left(\sigma_p^2+\sigma_q^2\right)}{4\sigma_p^2\sigma_q^2}\left\{x^2 - 2x\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)} + \frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)^2}{\left(\sigma_p^2+\sigma_q^2\right)^2} - \frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)^2}{\left(\sigma_p^2+\sigma_q^2\right)^2} + \frac{\left(\mu_p^2\sigma_q^2+\mu_q^2\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_l^u \exp\left[-\frac{1}{2}\frac{\left(\sigma_p^2+\sigma_q^2\right)}{2\sigma_p^2\sigma_q^2}\left\{x - \frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\right\}^2\right]\exp\left[-\frac{1}{4}\left\{\frac{\left(\mu_p-\mu_q\right)^2}{\sigma_p^2+\sigma_q^2}\right\}\right]$$

$$= \sqrt{\frac{2\sigma_p^2\sigma_q^2}{\left(\sigma_p^2+\sigma_q^2\right)}} \exp\left[-\frac{1}{4}\left\{\frac{\left(\mu_p-\mu_q\right)^2}{\sigma_p^2+\sigma_q^2}\right\}\right] \left\{\Phi\left[\frac{u-\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}}{\sqrt{\frac{2\sigma_p^2\sigma_q^2}{\left(\sigma_p^2+\sigma_q^2\right)}}}\right] - \Phi\left[\frac{l-\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}}{\sqrt{\frac{2\sigma_p^2\sigma_q^2}{\left(\sigma_p^2+\sigma_q^2\right)}}}\right]\right\}$$

$$D_{BC-TN}(p,q) = -\ln\left[\rho\left(p,q\right)\right]$$

$$\begin{aligned} D_{BC-TN}(p,q) &= \frac{1}{4}\left(\frac{\left(\mu_p-\mu_q\right)^2}{\sigma_p^2+\sigma_q^2}\right) + \frac{1}{4}\ln\left(\frac{1}{4}\left(\frac{\sigma_p^2}{\sigma_q^2}+\frac{\sigma_q^2}{\sigma_p^2}+2\right)\right) \\ &\quad + \frac{1}{2}\ln\left[\Phi\left(\frac{b-\mu_p}{\sigma_p}\right)-\Phi\left(\frac{a-\mu_p}{\sigma_p}\right)\right] + \frac{1}{2}\ln\left[\Phi\left(\frac{d-\mu_q}{\sigma_q}\right)-\Phi\left(\frac{c-\mu_q}{\sigma_q}\right)\right] \\ &\quad - \ln\left\{\Phi\left[\frac{u-\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}}{\sqrt{\frac{2\sigma_p^2\sigma_q^2}{\left(\sigma_p^2+\sigma_q^2\right)}}}\right] - \Phi\left[\frac{l-\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}}{\sqrt{\frac{2\sigma_p^2\sigma_q^2}{\left(\sigma_p^2+\sigma_q^2\right)}}}\right]\right\} \end{aligned}$$

Let, $\nu = \frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}$ and $\varsigma = \sqrt{\frac{2\sigma_p^2\sigma_q^2}{\left(\sigma_p^2+\sigma_q^2\right)}}$. Looking at conditions when $D_{BC-TN}(p,q) \geq D_{BC-N}(p,q)$ gives,

$$\frac{1}{2}\ln\left[\Phi\left(\frac{b-\mu_p}{\sigma_p}\right)-\Phi\left(\frac{a-\mu_p}{\sigma_p}\right)\right] + \frac{1}{2}\ln\left[\Phi\left(\frac{d-\mu_q}{\sigma_q}\right)-\Phi\left(\frac{c-\mu_q}{\sigma_q}\right)\right] \geq \ln\left\{\Phi\left[\frac{u-\nu}{\varsigma}\right]-\Phi\left[\frac{l-\nu}{\varsigma}\right]\right\}$$

$$\sqrt{\left[\Phi\left(\frac{b-\mu_p}{\sigma_p}\right)-\Phi\left(\frac{a-\mu_p}{\sigma_p}\right)\right]\left[\Phi\left(\frac{d-\mu_q}{\sigma_q}\right)-\Phi\left(\frac{c-\mu_q}{\sigma_q}\right)\right]} \geq \left\{\Phi\left[\frac{u-\nu}{\varsigma}\right]-\Phi\left[\frac{l-\nu}{\varsigma}\right]\right\}$$

For completeness, let us also consider two univariate normal distributions, p, q with density function,

$$f_p\left(x \mid \mu_p, \sigma_p^2\right) = \frac{1}{\sigma_p\sqrt{2\pi}}\, e^{-\frac{(x-\mu_p)^2}{2\sigma_p^2}}$$

Here, μ_p is the mean and σ_p^2 is the variance of the p distribution. The Bhattacharyya coefficient is given by,

$$\rho\left(p,q\right) = \int \sqrt{f_p\left(x \mid \mu_p, \sigma_p^2\right) f_q\left(x \mid \mu_q, \sigma_q^2\right)}dx$$

$$= \int \sqrt{\frac{1}{\sigma_p\sqrt{2\pi}}\, e^{-\frac{\left(x^2+\mu_p^2-2x\mu_p\right)}{2\sigma_p^2}} \frac{1}{\sigma_q\sqrt{2\pi}}\, e^{-\frac{\left(x^2+\mu_q^2-2x\mu_q\right)}{2\sigma_q^2}}}\, dx$$

$$= \int \sqrt{\frac{1}{\sigma_p\sqrt{2\pi}}\,\frac{1}{\sigma_q\sqrt{2\pi}}\, e^{-\left\{\frac{x^2\left(\sigma_p^2+\sigma_q^2\right)-2x\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)+\left(\mu_p^2\sigma_q^2+\mu_q^2\sigma_p^2\right)}{2\sigma_p^2\sigma_q^2}\right\}}}\, dx$$

$$= \int \sqrt{\frac{1}{\sigma_p\sqrt{2\pi}}\,\frac{1}{\sigma_q\sqrt{2\pi}}\, e^{-\frac{\left(\sigma_p^2+\sigma_q^2\right)}{2\sigma_p^2\sigma_q^2}\left\{x^2-2x\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}+\frac{\left(\mu_p^2\sigma_q^2+\mu_q^2\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\right\}}}\, dx$$

$$= \frac{1}{\sqrt{\sigma_p\sigma_q}}\,\frac{1}{\sqrt{2\pi}} \int e^{-\frac{\left(\sigma_p^2+\sigma_q^2\right)}{4\sigma_p^2\sigma_q^2}\left\{x^2-2x\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}+\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)^2}{\left(\sigma_p^2+\sigma_q^2\right)^2}-\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)^2}{\left(\sigma_p^2+\sigma_q^2\right)^2}+\frac{\left(\mu_p^2\sigma_q^2+\mu_q^2\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\right\}} dx$$

$$= \frac{1}{\sqrt{\sigma_p \sigma_q}} \sqrt{\frac{2\sigma_p^2\sigma_q^2}{\left(\sigma_p^2+\sigma_q^2\right)}} \sqrt{\frac{\left(\sigma_p^2+\sigma_q^2\right)}{2\sigma_p^2\sigma_q^2}} \frac{1}{\sqrt{2\pi}} \int e^{-\frac{1}{2}\frac{\left(\sigma_p^2+\sigma_q^2\right)}{2\sigma_p^2\sigma_q^2}\left\{x-\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\right\}^2 - \frac{\left(\sigma_p^2+\sigma_q^2\right)}{4\sigma_p^2\sigma_q^2}\left\{-\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)^2}{\left(\sigma_p^2+\sigma_q^2\right)^2}+\frac{\left(\mu_p^2\sigma_q^2+\mu_q^2\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\right\}} dx$$

$$= \sqrt{\frac{2\sigma_p\sigma_q}{\left(\sigma_p^2+\sigma_q^2\right)}}\, e^{-\frac{1}{4}\frac{(\mu_p-\mu_q)^2}{\sigma_p^2+\sigma_q^2}}$$

$$\left[\because -\frac{\left(\sigma_p^2+\sigma_q^2\right)}{4\sigma_p^2\sigma_q^2}\left\{-\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)^2}{\left(\sigma_p^2+\sigma_q^2\right)^2}+\frac{\left(\mu_p^2\sigma_q^2+\mu_q^2\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\right\} = \frac{1}{4\sigma_p^2\sigma_q^2}\left\{\frac{\left(\mu_p\sigma_q^2+\mu_q\sigma_p^2\right)^2-\left(\sigma_p^2+\sigma_q^2\right)\left(\mu_p^2\sigma_q^2+\mu_q^2\sigma_p^2\right)}{\left(\sigma_p^2+\sigma_q^2\right)}\right\}\right.$$

$$= \frac{1}{4\sigma_p^2\sigma_q^2}\left\{\frac{\mu_p^2\sigma_q^4+\mu_q^2\sigma_p^4+2\mu_p\sigma_q^2\mu_q\sigma_p^2-\sigma_p^2\mu_p^2\sigma_q^2-\sigma_q^4\mu_p^2-\sigma_p^4\mu_q^2-\sigma_q^2\mu_q^2\sigma_p^2}{\left(\sigma_p^2+\sigma_q^2\right)}\right\}$$

$$\left.= \frac{1}{4\sigma_p^2\sigma_q^2}\left\{\frac{2\mu_p\sigma_q^2\mu_q\sigma_p^2-\sigma_p^2\mu_p^2\sigma_q^2-\sigma_q^2\mu_q^2\sigma_p^2}{\left(\sigma_p^2+\sigma_q^2\right)}\right\} = -\frac{1}{4}\left\{\frac{(\mu_p-\mu_q)^2}{\sigma_p^2+\sigma_q^2}\right\}\right]$$

The Bhattacharyya distance then becomes,

$$D_{BC-N}(p,q) = -\ln\left[\rho\left(p,q\right)\right] = -\ln\sqrt{\left\{\frac{\left(\sigma_p^2+\sigma_q^2\right)^2}{4\sigma_p^2\sigma_q^2}\right\}^{-\frac{1}{2}}} - \ln\left\{e^{-\frac{1}{4}\frac{(\mu_p-\mu_q)^2}{\sigma_p^2+\sigma_q^2}}\right\}$$

$$= \frac{1}{4}\ln\left(\frac{1}{4}\left(\frac{\sigma_p^2}{\sigma_q^2}+\frac{\sigma_q^2}{\sigma_p^2}+2\right)\right)+\frac{1}{4}\left(\frac{(\mu_p-\mu_q)^2}{\sigma_p^2+\sigma_q^2}\right)$$

□

14.5 Proof of Proposition 4

Proof. Suppose we have two truncated normal distributions $\boldsymbol{p}, \boldsymbol{q}$ with density functions,

$$f_{\mathbf{p}}\left(x_1,\ldots,x_k \mid \boldsymbol{\mu_p},\, \boldsymbol{\Sigma_p},\, \boldsymbol{a},\, \boldsymbol{b}\right) = \frac{\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right))}{\int_{\boldsymbol{a}}^{\boldsymbol{b}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right))\,d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{a}\leq\boldsymbol{x}\leq\boldsymbol{b}}}$$

$$f_{\boldsymbol{q}}\left(x_1,\ldots,x_k \mid \boldsymbol{\mu_q},\, \boldsymbol{\Sigma_q},\, \boldsymbol{c},\, \boldsymbol{d}\right) = \frac{\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_q}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_q}\right))}{\int_{\boldsymbol{c}}^{\boldsymbol{d}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_q}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_q}\right))\,d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{c}\leq\boldsymbol{x}\leq\boldsymbol{d}}}$$

Here, $\boldsymbol{\mu_p}$ is the mean vector and $\boldsymbol{\Sigma_p}$ is the symmetric positive definite covariance matrix of the $\boldsymbol{p}$ distribution and the integrals are k dimensional integrals with lower and upper bounds given by the vectors $(\boldsymbol{a}, \boldsymbol{b})$ and $\boldsymbol{x}\in\boldsymbol{R}^{\boldsymbol{k}}_{\boldsymbol{a}\leq\boldsymbol{x}\leq\boldsymbol{b}}$. The Bhattacharyya coefficient is given by,

$$\rho\left(\boldsymbol{p},\boldsymbol{q}\right) = \int_{\boldsymbol{l}=\mathbf{min}(\boldsymbol{a},\boldsymbol{c})}^{\boldsymbol{u}=\mathbf{min}(\boldsymbol{b},\boldsymbol{d})} \sqrt{f_{\mathbf{p}}\left(x_1,\ldots,x_k \mid \boldsymbol{\mu_p},\, \boldsymbol{\Sigma_p},\, \boldsymbol{a},\, \boldsymbol{b}\right) f_{\boldsymbol{q}}\left(x_1,\ldots,x_k \mid \boldsymbol{\mu_q},\, \boldsymbol{\Sigma_q},\, \boldsymbol{c},\, \boldsymbol{d}\right)}d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\mathbf{min}(\boldsymbol{a},\boldsymbol{c})\leq\boldsymbol{x}\leq\mathbf{min}(\boldsymbol{b},\boldsymbol{d})}$$

$$\begin{aligned}\rho\left(\boldsymbol{p},\boldsymbol{q}\right) &= \left[\int_{\boldsymbol{a}}^{\boldsymbol{b}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)\Big)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^{k}_{\boldsymbol{a}\leq\boldsymbol{x}\leq\boldsymbol{b}}\right]^{-\frac{1}{2}}\\ &\left[\int_{\boldsymbol{c}}^{\boldsymbol{d}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_q}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)\Big)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^{k}_{\boldsymbol{c}\leq\boldsymbol{x}\leq\boldsymbol{d}}\right]^{-\frac{1}{2}}\\ &\int_{\boldsymbol{l}}^{\boldsymbol{u}}\exp\left(-\frac{1}{4}\left\{\left(\mathbf{x}-\mathbf{m}\right)^{\mathrm{T}}\left(\boldsymbol{S}^{-1}\right)\left(\mathbf{x}-\mathbf{m}\right)+\boldsymbol{M}\right\}\right)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^{k}_{\boldsymbol{\min(a,c)}\leq\boldsymbol{x}\leq\boldsymbol{\min(b,d)}}\end{aligned}$$

Here,

$$\boldsymbol{S}=\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}=\boldsymbol{\Sigma_p}\left[\boldsymbol{\Sigma_q}+\boldsymbol{\Sigma_p}\right]^{-1}\boldsymbol{\Sigma_q}$$

$$\mathbf{m}=\left[\left(\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\right)\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}\right]^{\mathrm{T}}$$

$$\boldsymbol{M} = \left(\boldsymbol{\mu_p}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\left[\boldsymbol{\Sigma_p}+\boldsymbol{\Sigma_q}\right]^{-1}\left(\boldsymbol{\mu_p}-\boldsymbol{\mu_q}\right)$$

$$\begin{aligned}\rho\left(\boldsymbol{p},\boldsymbol{q}\right) &= \left[\frac{1}{\sqrt{(2\pi)^{k}\left(\left|\boldsymbol{\Sigma_p}\right|\right)}}\int_{\boldsymbol{a}}^{\boldsymbol{b}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)\Big)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^{k}_{\boldsymbol{a}\leq\boldsymbol{x}\leq\boldsymbol{b}}\right]^{-\frac{1}{2}}\\ &\left[\frac{1}{\sqrt{(2\pi)^{k}\left(\left|\boldsymbol{\Sigma_q}\right|\right)}}\int_{\boldsymbol{c}}^{\boldsymbol{d}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_q}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)\Big)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^{k}_{\boldsymbol{c}\leq\boldsymbol{x}\leq\boldsymbol{d}}\right]^{-\frac{1}{2}}\\ &\left[\exp\left(-\frac{1}{4}\boldsymbol{M}\right)\frac{\sqrt{\det\left(\boldsymbol{\Sigma_p}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma_q}\right)}}{\left(\left|\boldsymbol{\Sigma_p}\right|\left|\boldsymbol{\Sigma_q}\right|\right)^{\frac{1}{4}}}\right]\left[\frac{1}{\sqrt{(2\pi)^{k}\det\left(\boldsymbol{\Sigma_p}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma_q}\right)}}\right.\\ &\left.\int_{\boldsymbol{l}}^{\boldsymbol{u}}\exp\left(-\frac{1}{2}\left\{\left(\mathbf{x}-\mathbf{m}\right)^{\mathrm{T}}\left(\boldsymbol{\Sigma_q}^{-1}\left[\boldsymbol{\Sigma}\right]\boldsymbol{\Sigma_p}^{-1}\right)\left(\mathbf{x}-\mathbf{m}\right)\right\}\right)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^{k}_{\boldsymbol{\min(a,c)}\leq\boldsymbol{x}\leq\boldsymbol{\min(b,d)}}\right]\end{aligned}$$

Here,

$$\boldsymbol{\Sigma}=\frac{\boldsymbol{\Sigma_p}+\boldsymbol{\Sigma_q}}{2}$$

$$D_{BC-TMN}\left(\boldsymbol{p},\boldsymbol{q}\right)=-\ln\left[\rho\left(\boldsymbol{p},\boldsymbol{q}\right)\right]$$

$$
\begin{aligned}
D_{BC-TMN}\left(\boldsymbol{p},\boldsymbol{q}\right) &= \frac{1}{8}(\boldsymbol{\mu_p}-\boldsymbol{\mu_q})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu_p}-\boldsymbol{\mu_q})+\frac{1}{2}\ln\left(\frac{\det\boldsymbol{\Sigma}}{\sqrt{\det\boldsymbol{\Sigma_p}\,\det\boldsymbol{\Sigma_q}}}\right)\\
&\quad+\frac{1}{2}\ln\left[\frac{1}{\sqrt{(2\pi)^k\left(|\boldsymbol{\Sigma_p}|\right)}}\int_{\boldsymbol{a}}^{\boldsymbol{b}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)\Big)\,d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{a}\le\boldsymbol{x}\le\boldsymbol{b}}\right]\\
&\quad+\frac{1}{2}\ln\left[\frac{1}{\sqrt{(2\pi)^k\left(|\boldsymbol{\Sigma_q}|\right)}}\int_{\boldsymbol{c}}^{\boldsymbol{d}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_q}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)\Big)\,d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{c}\le\boldsymbol{x}\le\boldsymbol{d}}\right]\\
&\quad-\ln\left[\frac{1}{\sqrt{(2\pi)^k\det\left(\boldsymbol{\Sigma_p}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma_q}\right)}}\right.\\
&\quad\left.\int_{\boldsymbol{l}}^{\boldsymbol{u}}\exp\left(-\frac{1}{2}\left\{\left(\mathbf{x}-\mathbf{m}\right)^{\mathrm{T}}\left(\boldsymbol{\Sigma_q}^{-1}\left[\boldsymbol{\Sigma}\right]\boldsymbol{\Sigma_p}^{-1}\right)\left(\mathbf{x}-\mathbf{m}\right)\right\}\right)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{\min(a,c)}\le\boldsymbol{x}\le\boldsymbol{\min(b,d)}}\right]
\end{aligned}
$$

Looking at conditions when $D_{BC-TMN}(\boldsymbol{p},\boldsymbol{q})\ge D_{BC-MN}(\boldsymbol{p},\boldsymbol{q})$ gives,

$$
\begin{aligned}
&\frac{1}{2}\ln\left[\frac{1}{\sqrt{(2\pi)^k\left(|\boldsymbol{\Sigma_p}|\right)}}\int_{\boldsymbol{a}}^{\boldsymbol{b}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)\Big)\,d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{a}\le\boldsymbol{x}\le\boldsymbol{b}}\right]\\
&+\frac{1}{2}\ln\left[\frac{1}{\sqrt{(2\pi)^k\left(|\boldsymbol{\Sigma_q}|\right)}}\int_{\boldsymbol{c}}^{\boldsymbol{d}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_q}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)\Big)\,d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{c}\le\boldsymbol{x}\le\boldsymbol{d}}\right]\\
\ge\ &-\ln\left[\frac{1}{\sqrt{(2\pi)^k\det\left(\boldsymbol{\Sigma_p}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma_q}\right)}}\right.\\
&\left.\int_{\boldsymbol{l}}^{\boldsymbol{u}}\exp\left(-\frac{1}{2}\left\{\left(\mathbf{x}-\mathbf{m}\right)^{\mathrm{T}}\left(\boldsymbol{\Sigma_q}^{-1}\left[\boldsymbol{\Sigma}\right]\boldsymbol{\Sigma_p}^{-1}\right)\left(\mathbf{x}-\mathbf{m}\right)\right\}\right)d\boldsymbol{x};\ \boldsymbol{x}\in\boldsymbol{R}^k_{\boldsymbol{\min(a,c)}\le\boldsymbol{x}\le\boldsymbol{\min(b,d)}}\right]
\end{aligned}
$$

For completeness, let us also consider two multivariate normal distributions, $\boldsymbol{p},\boldsymbol{q}$ (k dimensional random vectors in our case) where, $\boldsymbol{p}\sim\mathcal{N}(\boldsymbol{\mu_p},\,\boldsymbol{\Sigma_p})$, $\boldsymbol{q}\sim\mathcal{N}(\boldsymbol{\mu_q},\,\boldsymbol{\Sigma_q})$ with density function,

$$
f_{\mathbf{p}}\left(x_1,\ldots,x_k\mid\boldsymbol{\mu_p},\,\boldsymbol{\Sigma_p}\right)=\frac{1}{\sqrt{(2\pi)^k|\boldsymbol{\Sigma_p}|}}\exp\left(-\frac{1}{2}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\right)\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)\Big)
$$

Here, $\boldsymbol{\mu_p}$ is the mean vector and $\boldsymbol{\Sigma_p}$ is the symmetric positive definite covariance matrix of the $\boldsymbol{p}$ distribution. The Bhattacharyya coefficient is given by,

$$
\rho\left(\boldsymbol{p},\boldsymbol{q}\right)=\int\cdots\int\sqrt{f_{\mathbf{p}}\left(x_1,\ldots,x_k\mid\boldsymbol{\mu_p},\,\boldsymbol{\Sigma_p}\right)f_{\mathbf{q}}\left(x_1,\ldots,x_k\mid\boldsymbol{\mu_q},\,\boldsymbol{\Sigma_q}\right)}dx_1\cdots dx_k
$$

$$
\rho\left(\boldsymbol{p},\boldsymbol{q}\right)=\int\cdots\int\sqrt{\frac{1}{(2\pi)^k\sqrt{|\boldsymbol{\Sigma_p}||\boldsymbol{\Sigma_q}|}}\exp\left(-\frac{1}{2}\left\{\left(\mathbf{x}-\boldsymbol{\mu_p}\right)^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_p}\right)+\left(\mathbf{x}-\boldsymbol{\mu_q}\right)^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\left(\mathbf{x}-\boldsymbol{\mu_q}\right)\right\}\right)}dx_1\cdots dx_k
$$

$$
\rho\left(\boldsymbol{p},\boldsymbol{q}\right)=\int\cdots\int\frac{1}{(2\pi)^{\frac{k}{2}}\left(|\boldsymbol{\Sigma_p}||\boldsymbol{\Sigma_q}|\right)^{\frac{1}{4}}}\exp\left(-\frac{1}{4}\left\{\left(\mathbf{x}-\mathbf{m}\right)^{\mathrm{T}}\left(\boldsymbol{S}^{-1}\right)\left(\mathbf{x}-\mathbf{m}\right)+\boldsymbol{M}\right\}\right)dx_1\cdots dx_k
$$

$$(\mathbf{x}-\boldsymbol{\mu_p})^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}(\mathbf{x}-\boldsymbol{\mu_p})+(\mathbf{x}-\boldsymbol{\mu_q})^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}(\mathbf{x}-\boldsymbol{\mu_q}) \quad =$$

$$\mathbf{x}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\mathbf{x}-\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\mathbf{x}-\mathbf{x}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}+\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}+\mathbf{x}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\mathbf{x}-\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\mathbf{x}-\mathbf{x}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}$$

$$=\mathbf{x}^{\mathrm{T}}\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)\mathbf{x}-2\left(\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\right)\mathbf{x}+\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}$$

We want this to be of the form,

$$(\mathbf{x}-\mathbf{m})^{\mathrm{T}}\left(\boldsymbol{S}^{-1}\right)(\mathbf{x}-\mathbf{m})+\boldsymbol{M}=\mathbf{x}^{\mathrm{T}}\boldsymbol{S}^{-1}\mathbf{x}-\mathbf{m}^{\mathrm{T}}\boldsymbol{S}^{-1}\mathbf{x}-\mathbf{x}^{\mathrm{T}}\boldsymbol{S}^{-1}\mathbf{m}+\mathbf{m}^{\mathrm{T}}\boldsymbol{S}^{-1}\mathbf{m}+\boldsymbol{M}$$

Comparing the coefficients of $\mathbf{x}$,

$$\boldsymbol{S}^{-1}=\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)$$

$$\mathbf{m}^{\mathrm{T}}\boldsymbol{S}^{-1}=\left(\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\right)\Rightarrow\mathbf{m}=\left[\left(\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\right)\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}\right]^{\mathrm{T}}$$

$$\boldsymbol{M}=\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}-\mathbf{m}^{\mathrm{T}}\boldsymbol{S}^{-1}\mathbf{m}$$

$$=\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}-\left(\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\right)[\boldsymbol{S}]^{\mathrm{T}}\left(\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\right)^{\mathrm{T}}$$

$$=\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}-\left(\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\right)[\boldsymbol{S}]\left(\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}+\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}\right)$$

$$\begin{aligned}=\quad&\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}+\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}-\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{S}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}\\&-\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{S}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}-\boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{S}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}-\boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{S}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q}\end{aligned}$$

Using the result in (Henderson and Searle 1981, eq-22),

$$\boldsymbol{S}=\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}=\boldsymbol{\Sigma_p}-\boldsymbol{\Sigma_p}\left(\boldsymbol{I}+\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_p}\right)^{-1}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_p}$$

$$=\boldsymbol{\Sigma_p}-\boldsymbol{\Sigma_p}\left(\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_p}+\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_p}\right)^{-1}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_p}$$

$$=\boldsymbol{\Sigma_p}-\boldsymbol{\Sigma_p}\boldsymbol{\Sigma_p}^{-1}\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_p}$$

$$\Rightarrow\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}\left[\boldsymbol{I}+\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_p}\right]=\boldsymbol{\Sigma_p}$$

$$\Rightarrow\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}=\boldsymbol{\Sigma_p}\left[\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_q}+\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_p}\right]^{-1}$$

$$\Rightarrow\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}=\boldsymbol{\Sigma_p}\left[\boldsymbol{\Sigma_q}+\boldsymbol{\Sigma_p}\right]^{-1}\boldsymbol{\Sigma_q}$$

From symmetry we can write,

$$\boldsymbol{S}=\left(\boldsymbol{\Sigma_p}^{-1}+\boldsymbol{\Sigma_q}^{-1}\right)^{-1}=\boldsymbol{\Sigma_q}-\boldsymbol{\Sigma_q}\left(\boldsymbol{I}+\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_q}\right)^{-1}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_q}$$

$$= \boldsymbol{\Sigma_q} - \boldsymbol{\Sigma_q}\left(\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_q} + \boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_q}\right)^{-1}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_q}$$

$$= \boldsymbol{\Sigma_q} - \boldsymbol{\Sigma_q}\boldsymbol{\Sigma_q}^{-1}\left(\boldsymbol{\Sigma_q}^{-1} + \boldsymbol{\Sigma_p}^{-1}\right)^{-1}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_q}$$

$$\Rightarrow \left(\boldsymbol{\Sigma_p}^{-1} + \boldsymbol{\Sigma_q}^{-1}\right)^{-1}\left[\boldsymbol{I} + \boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_q}\right] = \boldsymbol{\Sigma_q}$$

$$\Rightarrow \left(\boldsymbol{\Sigma_p}^{-1} + \boldsymbol{\Sigma_q}^{-1}\right)^{-1} = \boldsymbol{\Sigma_q}\left[\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_q}\right]^{-1}$$

$$\Rightarrow \left(\boldsymbol{\Sigma_p}^{-1} + \boldsymbol{\Sigma_q}^{-1}\right)^{-1} = \boldsymbol{\Sigma_q}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\Sigma_p}$$

Using this in the result for $\boldsymbol{M}$,

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p} - \boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{S}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p} - \boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\Sigma_p}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\Sigma_q}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q} \\ &\quad + \boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q} - \boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{S}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\mu_q} - \boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\boldsymbol{\Sigma_q}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\Sigma_p}\boldsymbol{\Sigma_p}^{-1}\boldsymbol{\mu_p}\end{aligned}$$

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\left[\boldsymbol{S}\boldsymbol{S}^{-1} - \boldsymbol{S}\boldsymbol{\Sigma_p}^{-1}\right]\boldsymbol{\mu_p} - \boldsymbol{\mu_p}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_q} \\ &\quad + \boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\left[\boldsymbol{S}\boldsymbol{S}^{-1} - \boldsymbol{S}\boldsymbol{\Sigma_q}^{-1}\right]\boldsymbol{\mu_q} - \boldsymbol{\mu_q}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_p}\end{aligned}$$

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\left[\boldsymbol{S}\left\{\boldsymbol{S}^{-1} - \boldsymbol{\Sigma_p}^{-1}\right\}\right]\boldsymbol{\mu_p} - \boldsymbol{\mu_p}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_q} \\ &\quad + \boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\left[\boldsymbol{S}\left\{\boldsymbol{S}^{-1} - \boldsymbol{\Sigma_q}^{-1}\right\}\right]\boldsymbol{\mu_q} - \boldsymbol{\mu_q}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_p}\end{aligned}$$

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{\mu_p}^{\mathrm{T}}\boldsymbol{\Sigma_p}^{-1}\left[\boldsymbol{\Sigma_p}\left[\boldsymbol{\Sigma_q} + \boldsymbol{\Sigma_p}\right]^{-1}\boldsymbol{\Sigma_q}\boldsymbol{\Sigma_q}^{-1}\right]\boldsymbol{\mu_p} - \boldsymbol{\mu_p}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_q} \\ &\quad + \boldsymbol{\mu_q}^{\mathrm{T}}\boldsymbol{\Sigma_q}^{-1}\left[\boldsymbol{\Sigma_q}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\Sigma_p}\boldsymbol{\Sigma_p}^{-1}\right]\boldsymbol{\mu_q} - \boldsymbol{\mu_q}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_p}\end{aligned}$$

$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{\mu_p}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_p} - \boldsymbol{\mu_p}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_q} \\ &\quad + \boldsymbol{\mu_q}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_q} - \boldsymbol{\mu_q}^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\boldsymbol{\mu_p}\end{aligned}$$

$$\boldsymbol{M} = \left(\boldsymbol{\mu_p} - \boldsymbol{\mu_q}\right)^{\mathrm{T}}\left[\boldsymbol{\Sigma_p} + \boldsymbol{\Sigma_q}\right]^{-1}\left(\boldsymbol{\mu_p} - \boldsymbol{\mu_q}\right)$$

Let $\boldsymbol{\Sigma} = \frac{\boldsymbol{\Sigma_p}+\boldsymbol{\Sigma_q}}{2}$,

$$\begin{aligned}\rho(\boldsymbol{p},\boldsymbol{q}) &= \int\cdots\int\left[\frac{\sqrt{\det\left(\boldsymbol{\Sigma_p}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma_q}\right)}}{\left(|\boldsymbol{\Sigma_p}||\boldsymbol{\Sigma_q}|\right)^{\frac{1}{4}}\sqrt{(2\pi)^k\det\left(\boldsymbol{\Sigma_p}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma_q}\right)}}\right.\\ &\left.\exp\left(-\frac{1}{4}\boldsymbol{M}\right)\exp\left(-\frac{1}{2}\left\{(\mathbf{x}-\mathbf{m})^{\mathrm{T}}\left(\boldsymbol{\Sigma_q}^{-1}[\boldsymbol{\Sigma}]\,\boldsymbol{\Sigma_p}^{-1}\right)(\mathbf{x}-\mathbf{m})\right\}\right)dx_1\cdots dx_k\right]\end{aligned}$$

$$\rho(\boldsymbol{p},\boldsymbol{q}) = \frac{\left(|\boldsymbol{\Sigma_p}||\boldsymbol{\Sigma_q}|\right)^{\frac{1}{4}}}{\left(|\boldsymbol{\Sigma}|\right)^{\frac{1}{2}}}\exp\left(-\frac{1}{4}\boldsymbol{M}\right)$$

$$D_{BC-MN}(\boldsymbol{p},\boldsymbol{q}) = -\ln\left[\rho(\boldsymbol{p},\boldsymbol{q})\right] = \frac{1}{8}(\boldsymbol{\mu_p}-\boldsymbol{\mu_q})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu_p}-\boldsymbol{\mu_q}) + \frac{1}{2}\ln\left(\frac{\det\boldsymbol{\Sigma}}{\sqrt{\det\boldsymbol{\Sigma_p}\,\det\boldsymbol{\Sigma_q}}}\right)$$

□

14.6 Proof of Proposition 5

Proof. Since the following differential equation is satisfied,

$$\frac{f'_{XY}(t,u)}{f_{XY}(t,u)} = -\frac{g'(t,u)}{g(t,u)} + \frac{[\mu_Y - h(u)]}{g(t,u)} \quad , \quad t,u\in(a,b)$$

$$\Rightarrow \quad f'_{XY}(t,u)\,g(t,u) + g'(t,u)\,f_{XY}(t,u) = [\mu_Y - h(u)]\,f_{XY}(t,u)$$

$$\Rightarrow \quad \frac{\partial f_{XY}(t,u)\,g(t,u)}{\partial t} = [\mu_Y - h(u)]\,f_{XY}(t,u)$$

Integrating with respect to t from r to b and assuming $\lim_{t\to b} g(t,u)\,f_{XY}(t,u) = 0$ shows that for a given $h(u)$ the value of $g(t,u)$ uniquely determines the joint distribution of X and Y.

$$|f_{XY}(t,u)\,g(t,u)|_r^b = \int_r^b [\mu_Y - h(u)]\ f_{XY}(t,u)\ dt$$

$$f_{XY}(r,u)\ g(r,u) = \int_r^b [h(u)-\mu_Y]\ f_{XY}(t,u)\ dt$$

$$\int_a^b f_{XY}(r,u)\ g(r,u)\ du = \int_a^b\int_r^b [h(u)-\mu_Y]\ f_{XY}(t,u)\ dt\,du$$

Similarly, integrating with respect to t from a to r and assuming $\lim_{t\to a} g(t,u) f_{XY}(t,u) = 0$ gives,

$$f_{XY}(r,u)\ g(r,u) = \int_a^r [\mu_Y - h(u)]\ f_{XY}(t,u)\ dt$$

$$\int_a^b f_{XY}(r,u)\ g(r,u)\ du = \int_a^b \int_a^r [\mu_Y - h(u)]\ f_{XY}(t,u)\ dt\, du$$

Now consider,

$$\mathrm{Cov}[c(X), h(Y)] = \int_a^b \int_a^b [c(t) - E\{c(X)\}]\ [h(u) - E\{h(Y)\}]\ f_{XY}(t,u)\ dt\, du$$

$$= \int_a^b \int_a^b c(t)\ [h(u) - \mu_Y]\ f_{XY}(t,u)\ dt\, du - \int_a^b \int_a^b E\{c(X)\}\ (h(u) - \mu_Y)\ f_{XY}(t,u)\ dt\, du$$

$$= \int_a^b \int_a^b c(t)\ [h(u) - \mu_Y]\ f_{XY}(t,u)\ dt\, du \quad \left\{ \because \int_a^b \int_a^b E\{c(X)\}\ (h(u) - \mu_Y) f_{XY}(t,u)\ dt\, du = 0 \right\}$$

$$= \int_a^b \int_a^b [c(t) - c(a)]\ [h(u) - \mu_Y]\ f_{XY}(t,u)\ dt\, du$$

$$= \int_a^b \int_a^b \left[\int_a^t c'(r)\ dr \right] [h(u) - \mu_Y]\ f_{XY}(t,u)\ dt\, du$$

Using Fubini's theorem and interchanging the order of integration,

$$\mathrm{Cov}[c(X), h(Y)] = \int_a^b \int_a^b c'(r) \left\{ \int_r^b [h(u) - \mu_Y]\ f_{XY}(t,u)\ dt \right\} dr\, du$$

$$= \int_a^b \int_a^b c'(r)\ f_{XY}(r,u)\ g(r,u)\ dr\, du$$

$$= E[c'(r)\ g(r,u)] = E[c'(X)\ g(X,Y)]$$

□

14.7 Proof of Proposition 6

Proof.

$$\mathrm{Cov}[c(X), Y] = \int\int [c(t) - E\{c(X)\}](u - \mu_Y) f_{XY}(t,u)\ dt\, du$$

$$= \int\int [c(t)\, u - E\{c(X)\}\, u - c(t)\, \mu_Y + E\{c(X)\}\, \mu_Y] f_{XY}(t,u)\ dt\, du$$

$$= \int\int c(t)\, u\, f_{XY}(t,u)\ dt\, du - E\{c(X)\}\, \mu_Y$$

$$= \int\int c(t)\ u\, f_{(Y|X)}(u \mid t)\ f_X(t)\ dt\, du - E\{c(X)\}\mu_Y$$

$$= \int\int c(t)\ u\, f_{(X|Y)}(t \mid u)\ f_Y(u)\ dt\, du - E\{c(X)\}\mu_Y$$

Set,

$$c(t) = t - \sqrt{\frac{f_Y(t)}{f_X(t)}}$$

$$\mathrm{Cov}[c(X), Y] = \int\int \left\{ t - \sqrt{\frac{f_Y(t)}{f_X(t)}} \right\} u\, f_{XY}(t,u)\ dt\, du - E\left[\left\{ t - \sqrt{\frac{f_Y(t)}{f_X(t)}} \right\} \right] \mu_Y$$

$$= \mathrm{Cov}(X,Y) - \int\int \sqrt{\frac{f_Y(t)}{f_X(t)}}\, u\, f_{(Y|X)}(u \mid t)\ f_X(t)\ dt\, du + \mu_Y \rho(f_X, f_Y)$$

$$= \mathrm{Cov}(X,Y) - \int \sqrt{\frac{f_Y(t)}{f_X(t)}}\, E[Y \mid X = t]\ f_X(t)\ dt + \mu_Y \rho(f_X, f_Y)$$

$$= \mathrm{Cov}(X,Y) - E\left[\sqrt{\frac{f_Y(t)}{f_X(t)}} Y \right] + \mu_Y \rho(f_X, f_Y)$$

$$= \mathrm{Cov}(X,Y) - \int\int \sqrt{\frac{f_Y(t)}{f_X(t)}}\, u\, f_{(X|Y)}(t \mid u)\ f_Y(u)\ dt\, du + \mu_Y \rho(f_X, f_Y)$$

□

14.8 Proof of Proposition 7

Proof. Proved within the main body of the article.

□

15 R Code Snippets

```
#Johnson Linderstrauss Lemma
minimumDimension = function(numberOfPoints, errorTolerance){
        result = (4 * log(numberOfPoints)) /
                                (((errorTolerance^2)/2) - ((errorTolerance^3)/3));
        result = ceiling(result);
        return(result);
};

#Johnson Linderstrauss Lemma
```

```
reduceDimension = function(inputTable, errorTolerance,minDimension){
        nRows = dim(inputTable)[1];
        nColumns = dim(inputTable)[2];
        randomMatrix = matrix(rnorm(nColumns*minDimension,
                mean=0,sd=1/sqrt(minDimension)), nColumns, minDimension);
        cat(paste("\t\tRandom Dim Reduction Matrix : ",
                dim(randomMatrix)[1],"*",dim(randomMatrix)[2],"\n"));
        inputTable = (inputTable%*%randomMatrix);
        return(inputTable);
};

#Overflow Bug Fix to the bhattacharya.dist() function available in library("fps")
bhattacharyyaDistanceEigenValues = function (mu1, mu2, Sigma1, Sigma2) {
        aggregatesigma <- (Sigma1 + Sigma2)/2;
        d1 <- mahalanobis(mu1, mu2, aggregatesigma)/8;
        #d2 <- log(det(as.matrix(aggregatesigma))/sqrt(det(as.matrix(Sigma1)) *
        #det(as.matrix(Sigma2))))/2;
        eigenAggregate=log(eigen(as.matrix(aggregatesigma))$values);
        eigenOne=(log(eigen(as.matrix(Sigma1))$values)/2);
        eigenTwo=(log(eigen(as.matrix(Sigma2))$values)/2);
        d2 <- sum(eigenAggregate)-sum(eigenOne)-sum(eigenTwo);
        out <- d1 + (d2/2);
        return(out);
};

#Perform PCA Rotation with different number of components
pcaRotation = function(inputTable,significantDigits=3,numberOfCOmponents=-1,
        returnTruncated=FALSE,transposeIfNeeded=TRUE){
        numRows = dim(inputTable)[1];
        numColumns = dim(inputTable)[2];
        transposeDone = FALSE;
        if(transposeIfNeeded){
                if(numRows<numColumns){
                inputTable = (t(inputTable));
```

```
        numRows = dim(inputTable)[1];
        numColumns = dim(inputTable)[2];
        transposeDone = TRUE;
        }
}
pcaResult = prcomp(inputTable);
#Plot is extremely useful. This graphic is highly insightful
plot(cumsum(pcaResult$sdev^2/sum(pcaResult$sdev^2)));

if(numberOfCOmponents==-1){
        cummulativeContribution = cumsum(pcaResult$sdev^2/sum(pcaResult$sdev^2));
        cummulativeContribution = round(diff(cummulativeContribution),
                significantDigits);
        pcaSignifcantDimension = sum(cummulativeContribution>0)+1;
        cat(paste("\t\tCalculating Contributions and Retaining: ",
        pcaSignifcantDimension,"out of: ",numColumns," total Dimensions\n"));
        if(!returnTruncated) {
                inputTable = (pcaResult$x[,1:pcaSignifcantDimension]) %*%
                        (t(pcaResult$rotation[,1:pcaSignifcantDimension]));
                #and add the center (and re-scale) back to data
        if(pcaResult$scale != FALSE){
                inputTable <- scale(inputTable, center = FALSE ,
                                        scale=1/pcaResult$scale)
        }
        if(pcaResult$center != FALSE){
                inputTable <- scale(inputTable, center = -1 * pcaResult$center,
                                        scale=FALSE)
        }
        }else {
                inputTable = (pcaResult$x[,1:pcaSignifcantDimension]);
        }
        } else {
                cat(paste("\t\tComponent Count Received and Retaining: ",
                        numberOfCOmponents,"out of: ",numColumns,
```

```
                numberOfCOmponents = min(numberOfCOmponents,numColumns);
                cat(paste("\t\tMinimum dimension: ",
                        numberOfCOmponents,"out of: ",numColumns,

                inputTable = (pcaResult$x[,1:numberOfCOmponents]);
            }
        if(transposeDone){
            inputTable = (t(inputTable));
        }
        return (inputTable);
};
```

DELETION: Is there a Linear Transformation to Send Both Datasets to the same dimension? This is instead of the JL Lemma

THIS SECTION AND PART BELOW ARE FOR DELETION AND VERIFICATION

15.1 Proof of Lemma

$$Cov(X,Y) = \int\int (t-\mu_X)(u-\mu_Y) f_{XY}(t,u)\ dt\ du$$

$$= \int\int (t\,u) f_{XY}(t,u)\ dt\ du - \mu_X\,\mu_Y$$

Taking derivatives and First Order Conditions,

$$\frac{\partial Cov(X,Y)}{\partial \mu_X} = \int\int ut\left[\frac{\partial\{f_{XY}(t,u)\}}{\partial \mu_X}\right] dt\ du - \mu_Y$$

$$\Rightarrow \mu_Y = \int\int ut\left[\frac{\partial\{f_{XY}(t,u)\}}{\partial \mu_X}\right] dt\ du$$

$$\frac{\partial Cov(X,Y)}{\partial \mu_Y} = \int\int ut\left[\frac{\partial\{f_{XY}(t,u)\}}{\partial \mu_Y}\right] dt\ du - \mu_X$$

$$\Rightarrow \mu_X = \int\int ut\left[\frac{\partial\{f_{XY}(t,u)\}}{\partial \mu_Y}\right] dt\ du$$

We write the Bhattacharyya distance with the above notation for the density function,

$$\rho(f_X, f_Y) = \int \sqrt{f_X(t)\, f_Y(t)} dt$$

$$\frac{\partial \rho\left(f_X, f_Y\right)}{\partial \mu_X} = \int \frac{\partial \sqrt{f_X\left(t\right) f_Y\left(t\right)}}{\partial \mu_X} dt$$

$$\frac{\partial \rho\left(f_X, f_Y\right)}{\partial \mu_X} = \int \frac{1}{2\sqrt{f_X\left(t\right) f_Y\left(t\right)}} \left\{ f_Y\left(t\right) \frac{\partial f_X\left(t\right)}{\partial \mu_X} \right\} dt$$

15.2 Ratio of Exponentials and Limit of Density Functions

Limit of ratio of exponentials,

$$\lim_{x \to \infty} \left[\frac{e^x}{e^{x^2}} \right]$$

Limit of Multivariate Probability Density Function as one or more or all variables approach positive or negative infinity,

$$\lim_{x_1 \to \pm\infty} f\left(x_1, x_2, \ldots, x_n\right) = 0$$

$$\lim_{x_1, x_2 \to \pm\infty} f\left(x_1, x_2, \ldots, x_n\right) = 0$$

$$\lim_{x_1, x_2, \ldots, x_n \to \pm\infty} f\left(x_1, x_2, \ldots, x_n\right) = 0$$

15.3 Proof of Lemma

THIS PART BELOW IS INCORRECT

$$Cov\left(X, Y\right) = \int \int \left(t - \mu_X\right)\left(u - \mu_Y\right) f_{XY}\left(t, u\right)\ dt\ du$$

$$= \int \int \left(t\ u\right) f_{XY}\left(t, u\right)\ dt\ du - \mu_X\ \mu_Y$$

Taking derivatives,

$$\frac{\partial Cov\left(X, Y\right)}{\partial t} = \int u \left[\int \frac{\partial \left\{ t f_{XY}\left(t, u\right) \right\}}{\partial t}\ dt \right]\ du$$

$$= \int u \int t\ \frac{\partial f_{XY}\left(t, u\right)}{\partial t}\ dt\ du + \int \int u\ f_{XY}\left(t, u\right)\ dt\ du$$

$$= \int u \left\{ t\ f_{XY}\left(t, u\right) - \int f_{XY}\left(t, u\right)\ dt \right\}\ du + \int \int u\ f_{XY}\left(t, u\right)\ dt\ du$$

$$\frac{\partial Cov\left(X, Y\right)}{\partial t} = t \int u\ f_{XY}\left(t, u\right)\ du$$

$$\frac{\partial^2 Cov\left(X, Y\right)}{\partial t^2} = \int u\ f_{XY}\left(t, u\right)\ du$$

$$\frac{\partial^2 Cov\left(X, Y\right)}{\partial u \partial t} = t\ u\ f_{XY}\left(t, u\right)$$

By Symmetry,

$$\frac{\partial Cov\left(X,Y\right)}{\partial u}=u\int t\,f_{XY}\left(t,u\right)\,dt$$

$$\frac{\partial^{2}Cov\left(X,Y\right)}{\partial u^{2}}=\int t\,f_{XY}\left(t,u\right)\,dt$$

We write the Bhattacharya distance with the above notation for the density function,

$$\rho\left(f_{X},f_{Y}\right)=\int\sqrt{f_{X}\left(t\right)f_{Y}\left(t\right)}dt$$

$$\frac{\partial\rho\left(f_{X},f_{Y}\right)}{\partial t}=\int\frac{\partial\sqrt{f_{X}\left(t\right)f_{Y}\left(t\right)}}{\partial t}dt$$

$$\frac{\partial\rho\left(f_{X},f_{Y}\right)}{\partial t}=\int\frac{1}{2\sqrt{f_{X}\left(t\right)f_{Y}\left(t\right)}}\left\{f_{X}\left(t\right)\frac{\partial f_{Y}\left(t\right)}{\partial t}+f_{Y}\left(t\right)\frac{\partial f_{X}\left(t\right)}{\partial t}\right\}dt$$

15.4 Miscellaneous

$$E\left(X\right)=\int_{-\infty}^{\infty}tf_{X}\left(t\right)dt$$

Using Leibniz Integral Rule and assuming the density is differentiable,

$$\frac{dE\left(X\right)}{dt}=\int_{-\infty}^{\infty}\left\{f_{X}\left(t\right)+t\frac{\partial f_{X}\left(t\right)}{\partial t}\right\}dx+f_{X}\left(\infty\right)\frac{\partial\infty}{\partial t}-f_{X}\left(-\infty\right)\frac{\partial\left(-\infty\right)}{\partial t}$$

Using integration by parts,

$$\frac{dE\left(X\right)}{dt}=1+\left|tf_{X}\left(t\right)\right|_{-\infty}^{\infty}-1+f_{X}\left(\infty\right)\frac{\partial\infty}{\partial t}-f_{X}\left(-\infty\right)\frac{\partial\left(-\infty\right)}{\partial t}$$

Or using the definition of Integral as the Anti derivative,

$$\frac{dE\left(X\right)}{dt}=\left|tf_{X}\left(t\right)\right|_{-\infty}^{\infty}$$