Optimal Monetary Policy in Production Networks*

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Abstract

We study optimal monetary policy in a multi-sector economy in which firms buy and sell intermediate goods via an input-output network. Firms are subject to sectoral productivity shocks and face nominal rigidities; we abstract from inefficient cost-push shocks. We first show that while flexible-price equilibria are first-best efficient, generically these allocations cannot be acheived as equilibria under sticky prices for any monetary policy. We then study the second-best optimal monetary policy problem. We find that the optimal monetary policy trades off three types of welfare loss: efficiency loss due to price errors across sectors, efficiency loss due to price dispersion within sectors, and volatility of the output gap. Importantly, an output gap arises endogenously despite the fact that all underlying shocks are efficient.

Keywords: Business Cycles, Nominal Rigidities, Monetary Policy, Price Stability, Production Networks, Input-Output, Informational Frictions, Sectoral Distortions, Misallocation.

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1 Introduction

How does the input-output structure of the economy affect the optimal conduct of monetary policy?

In the canonical New Keynesian one-sector model with nominal rigidities, optimal monetary policy is well understood. As long as there are no missing tax instruments, it is optimal to stabilize the price level. Price level stabilization, along with a subsidy that eliminates any monopolistic mark-up, implements flexible price allocations and attains the first best. In the language of the New Keynesian literature, the "divine coincidence" holds: price stability minimizes both inflation and the output gap.

In order to break the divine coincidence and rationalize an active role for monetary policy, the New-Keynesian literature typically introduces inefficient, exogenous cost-push shocks. These are often modeled as shocks to the elasticity of substitution across goods which translate into inefficient movements in markups. Without state-contingent taxes/subsidies to eliminate these markup movements, the flexible-price equilibrum becomes inefficient. As long as firms face nominal rigidities, monetary policy becomes an imperfect instrument to fight against these shocks. This justifies an active role for the monetary authority who now faces a tradeoff between minimizing inflation and the output gap. In other words, the divine coincidence no longer holds.

In this paper we revert to the previous framework in which there are only efficient productivity shocks and the divine coincidence holds. However, rather than introduce exogenously inefficient cost-push shocks, we instead enrich the classic New Keynesian environment by allowing for multiple sectors and an input-output network of intermediate good trade. We ask whether this seemingly benign extension alters the underlying principles guiding optimal monetary policy. We find that in fact, it does!

Framework. We consider a multi-sector, general equilibrium model of inter-sectoral trade à la Long and Plosser (1983), and more recently, Acemoglu et al. (2012). Each sector consists of a continuum of firms that produce output using labor and intermediate goods. Technologies are homogeneous within sectors but heterogeneous across sectors. The output of each sector may be either used as intermediate goods or consumed, and the matrix of cross-sector input requirements defines the input-output production network. A representative household consumes a basket of final goods and supplies labor elastically.

As mentioned above, we assume that the only payoff-relevant shocks in the economy are efficient shocks to sectoral productivities. In order to accommodate price-setting, we make the typical assumption that within a sector firms are monopolistically competitive.[1] As in the New-Keynesian tradition, firms are subject to nominal rigidity. We model nominal rigidity as a form of incomplete information: firms must make nominal pricing decisions without common knowledge of the aggregate state and nominal marginal costs.

Finally, we allow for a government with control over two types of policy instruments: a rich set non-state contingent sectoral taxes/subsidies (set by the fiscal authority) and control over state-contingent nominal aggregate demand (set by the monetary authority).

[1] Goods are aggregated into a sectoral commodity which may then either be used for consumption or as an intermediate good to production. Firms are thus price-setters within a sector, but price-takers across sectors.

Methodology and Results. Our methodology for understanding optimal policy builds on the primal approach of the Ramsey literature applied to New Keynesian models, see e.g. Correia et al. (2008); Angeletos and La'O (2018). By the primal approach, we first characterize the entire set of allocations that can be implemented as market-based equilibria with the help of the available policy instruments. We then identify the welfare-maximizing allocation within that set, and back out the prices and policies that implement said allocation.

Armed with this methodology, we first formally show that in this class of economies with efficient shocks, the divine coincidence is a non-generic special case. Specifically, as long as we abstract from inefficient shocks and allow for corrective subsidies to eliminate sectoral mark-ups, flexible-price allocations remain first-best efficient. However, unlike the canonical one-sector New Keynesian model, the flexible-price allocation is unattainable. As long as there are multiple sectors in which firms face nominal rigidities, the divine coincidence fails.

The intuition for this is relatively simple. The first best allocation in this economy calls for a uniform price across all firms within a sector in all states—this maximizes within-sector productive efficiency. But at the same time the first best allocation calls for relative prices *across* sectors to move in response to changes in relative productivity. Such an allocation is impossible to acheive when firms in multiple sectors face nominal rigidities. This result may not be surprising as it has already been shown to be the case in special two-sector models such as Aoki (2001); Huang and Liu (2005) and Woodford (2010), as well as in the presense of both price and wage stickiness as in Henderson and Levine.

However, this result has the following implication. Nominal frictions open the door for relative pricing errors. Sectoral pricing errors act as wedges between marginal costs and marginal revenue. Following the work of Bigio and La'O (2018), we show that sectoral wedges in a multi-sector, input-output network manifest as both a fall in aggregate TFP, i.e. aggregate productive efficiency, as well as movement in the aggregate labor wedge, i.e. the output gap. However, unlike this previous work, sectoral wedges in our framework are endogenous outcomes of nominal rigidity, intermediate good trade, and monetary policy. Thus, an output gap arises *endogenously* in this model, even when the fundamental shocks are all efficient.

We thus consider the second best problem of optimal monetary policy. We derive an expression for welfare losses that results from deviations of the equilibrium allocation from the first best. We show that these welfare losses are a function of three components: (i) the volatility of the output gap, i.e. volatility of the aggregate labor wedge, (ii) efficiency loss due to pricing errors across sectors, and (iii) efficiency loss due to price dispersion within sectors.

We characterize the monetary policy that optimally trades off these three components. We find that the optimal policy depends non-trivially on the centrality of sectors, their elasticities of substitution, and their relative price stickiness. We derive a price stability policy with optimal weights on sectors.

Related Literature. The underlying economy is an input-output network economy and is thus related to the growing literature on network economies. Our model builds on general multi-sector

network models of Acemoglu et al. (2012) and Long and Plosser (1983). Dating back to Long and Plosser (1983), there is an extensive macroeconomic research on how input-output linkages affect the transmission of sector-specific productivity shocks—see the survey in Carvalho (2014) and Carvalho and Tahbaz-Salehi (2019). [2]

Due to the presense of sectoral wedges, this paper is most closely related to the production networks in Jones (2011); Bigio and La'O (2018); Baqaee and Farhi (2017). In comparison to these papers, the wedges that arise our paper are endogenous: they are functions of the realized pricing errors made by the firms. When studying the optimal second best policy, we use the results of Bigio and La'O (2018) to aggregate these wedges to an aggregate TFP and aggregate labor wedge.

In th New Keynesian literature. monetary policy papers. We show that when there are more multiple sectors with nominal rigidities, monetary policy cannot achieve the first best. There are a few papers that have studied optimal policy with two sectors. Woodford (2010) shows that with two sectors the second best monetary policy stabilizes a specific price index calculated by putting a higher weight on the stickier sector. An important special case is Aoki (2001), which showed that the completely flexible sector should be given a zero-weight in the stabilization target. Huang and Liu (2005) studied a case with intermediate input and concluded that the second best monetary policy stabilizes a combination of final good price and intermediate good price. Basu (1995) shows how introducing intermediate goods.. As far as we know, no paper has studied the second best monetary policy in an economy with a general input output network.

Methodologically, we follow the primal approach. understand the problem at a more general level. We first follow the primal approach as in Atkinson Stiglitz, Diamond and Mirrlees. Our framework resembles those found in either the Ramsey literature on optimal fiscal policy (Lucas and Stokey, 1983; Chari et al., 1994). We allow for nominal rigidities as in Correia et al. (2008).

Layout. The rest of the paper is organized as follows. Section 2 sets up the environment. Section 2.1 defines the sticky-price and flexible-price equilibria in our context. Section 3 defines and characterizes the first best optimal allocation as well as the flexible price equilibrium. Section 4 characterizes equilibrium allocations when firms face sticky prices and presents our first main result that the first best is unattainable. Section 5 defines and characterizes the second best problem. Finally, Section 6 explores optimal monetary policy. Section 8 concludes. The Appendix contains all proofs.

2 The Environment

Consider a static economy that is populated by a representative household and n production industries indexed by $i \in \{1, \ldots, n\}$. Each industry i consists of a continuum of monopolistically-competitive firms, indexed by $k \in [0, 1]$. Firms within a sector produce differentiated goods, which are aggregated at the sectoral level by a producer. This producer serves the sole purpose of aggregating the industry's differentiated goods into a uniform sectoral good, or commodity, which can in turn either be consumed by the household or used as an intermediate input for production. Finally, there is a

[2] This agenda began with Long and Plosser (1983) and was followed by Horvath (1998, 2000), Dupor (1999) and Shea (2002).

government with the ability to levy sector-specific taxes and control aggregate nominal demand.

Technology. Firms within the same industry use a common constant returns to scale technology to transform labor and intermediate inputs into their differentiated products. The production function of firm k in sector i is given by

$$y_{ik} = z_i F_i \left(l_{ik}, x_{ik} \right), \tag{1}$$

where y_{ik} is its output, l_{ik} is labor input, x_{ik} is a composite of the firm's intermediate inputs, and z_i is a sector-specific productivity shock. We assume that F_i is homogenous of degree 1. The composite of intermediate goods is given by

$$x_{ik} = G_i \left(\mathbf{x}_{ik} \right) \tag{2}$$

where G_i is a constant returns to scale production function, $\mathbf{x}_{ik} \equiv (x_{i1,k}, \ldots, x_{iN,k})'$ denotes the vector of intermediate good purchases, and $x_{ij,k}$ is the amount of sectoral commodity j purchased by firm (i,k). The nominal profits of firm (i,k) are thus given by

$$\pi_{ik} = (1 - \tau_i) \, p_{ik} y_{ik} - w l_{ik} - \sum_{j=1}^{n} p_j x_{ij,k}.$$

where p_{ik} is nominal price charged by the firm, p_j is the price at which it purchases input $x_{ik,j}$, w is the nominal wage, and τ_i is a sector specific revenue tax (or subsidy).

In order for firms to be price-setters, we make the typical assumption that firms within a sector are monopolistically competitive. As mentioned above, we thus assume that in each sector i, there is a special producer which aggregates the sectoral goods according to the following CES production function

$$y_i = \left(\int_0^1 y_{ik}^{(\theta_i - 1)/\theta_i} dk \right)^{\theta_i/(\theta_i - 1)}$$

with elasticity of substitution $\theta_i > 1$. This aggregator firm maximizes profits given by $p_i y_i - \int_0^1 p_{ik} y_{ik} dk$ where p_i is the price of the aggregated good produced by sector i.[3]

Recall that the uniform sectoral good produced by sector j may be used as either an intermediate input for other sectors or as final consumption. Therefore, market clearing in the commodity produced by sector j is given by

$$y_j = c_j + \sum_{i=1}^{n} x_{ij},$$

where $x_{ij} = \int_0^1 x_{ij,k} dk$ is the total amount of good j purchased by all firms in sector i.

The Household. The preferences of the representative household are given by

$$U(C) - V(L).$$

[3]We include this firm — which has zero value added — only for expositional purposes. The existence of such a firm ensures that that a homogenous good is produced by each industry, yet at the same time allowing for monopolistic competition among firms within a sector.

where C is the household's final consumption basket and L is the household's total labor supply. We assume the typical regularity conditions: U and V are twice differentiable, with $U', V' > 0$, $U'' < 0$, $V'' > 0$, and satisfy the Inada conditions. The final consumption basket of the household is $C = \mathcal{C}(c_1, \ldots, c_n)$, where c_i is the household's consumption of good i and $\mathcal{C}$ is homogenous of degree 1. The household's budget constraint is given by

$$\sum_{j=1}^{n} p_j c_j \leq wL + \sum_{i=1}^{n} \int_0^1 \pi_{ik} dk + T,$$

where the left-hand side denotes the total nominal expenditure on goods while the right-hand side is total nominal income; the latter includes the household's wage income, dividends from owning firms (both the monopolistic producers and the sectoral aggregators), and lump sum transfers from the government. We assume that the government budget constraint satisfies

$$T = \sum_{i=1}^{n} \tau_i \int_0^1 p_{ik} y_{ik} dk$$

so that all taxes/subsidies are rebated to the household. Finally, market clearing in the labor market is given by

$$L = \sum_{i=1}^{n} \int_0^1 l_{ik} dk.$$

Payoff-Relevant Shocks. The only payoff-relevant shocks in this economy are shocks to productivities. We let $\mathbf{z} = (z_1, \ldots, z_n)'$ denote the vector of all realizations of all productivity shocks.

2.1 Sticky vs Flexible Prices: Definitions

We will consider two scenarios: one in which firms face nominal rigidities and one in which they do not. In this section we define the sets of allocations, prices, and policies that can be part of an equilibrium under each scenario.

Sticky Prices. We model sticky prices as follows. Firms in sector i make their nominal pricing decisions before knowing their demand. Formally, firms make their their nominal decision based on an imperfect signal of the aggregate state, which we denote by ω_{ik}. These signals may differ both across firms and across sectors; we let Ω_i denote the realized distribution of signals in sector i and Ω denote the realized distribution of these distributions.We let $s \in S$ denote the aggregate state; the aggregate state includes not only the vector of realized productivity shocks, but also the entire realization of all signals:

$$s = (\mathbf{z}, \Omega)$$

Thus s not only contains all payoff-relevant shocks, it may also contain shocks to the aggregate profile of beliefs. We denote the unconditional probability of the aggregate state as $\mu(s)$.

We denote the prices set by the sticky price firms in sector i as $p_{ik}(\omega_{ik})$ and thereby imbed their informational constraint as a measurability condition on their nominal price.

Because the choice of nominal price by each firm ultimately affects their realized demand, we let $\{y_{ik}(\omega,s), \ell_{ik}(\omega,s), \mathbf{x}_{ik}(\omega,s)\}$ denote the output, labor, and inputs of the firms in sector i. This implies that we may express the aggregate level of sectoral output as

$$y_i(s,\Omega_i) = \left[\int y_{ik}(\omega,s)^{\frac{\theta_i-1}{\theta_i}} \mu(\omega|s)\, d\omega\right]^{\frac{\theta_i}{\theta_i-1}}, \tag{3}$$

and the sectoral price level as

$$p_i(\Omega_i) = \left[\int p_{ik}(\omega)^{1-\theta_i} \mu(\omega|s)\, d\omega\right]^{\frac{1}{1-\theta_i}}. \tag{4}$$

We thereby let ξ denote an allocation in this economy,

$$\xi(s) \equiv \{(y_{ik}(\omega,s), \ell_{ik}(\omega,s), \mathbf{x}_{ik}(\omega,s); y_i(\omega,\Omega); c_i(s))_i, C(s), L(s)\},$$

and we let ϱ denote a price system,

$$\varrho(s) \equiv \{(p_{ik}(\omega), p_i(\Omega))_i, P(s), W(s)\}.$$

Finally, consider policy. In particular we assume that the fiscal authority cannot set taxes contingent on the aggregate state, however it can set a different constant tax or subsidy, τ_i, for each sector. As for monetary policy, we assume the monetary authority has a tool, e.g. the interest rate, which it may use to target the nominal wage. This nominal target may be contingent on the aggregate state:

$$w(s) = \phi(s)$$

where $\phi(s)$ is an arbitrary function chosen by the monetary authority. While the idea of the nominal wage being the tool of the monetary authority does not seem realistic, per se, we establish in Section 6 that it is equivalent in our model to the monetary authority controlling nominal aggregate demand. In a dynamic version of the model, we can think of the monetary authority choosing a nominal interest rate which targets some nominal wage. In summary, we let ϑ denote a policy:

$$\vartheta(s) \equiv \{(\tau_i)_i, \phi(s)\}.$$

Our analysis thus abstracts from frictions on the side of either the representative household or the government. In contrast, the nominal or informational friction of the firms is embedded in the measurability constraint imposed on sticky price firms. Finally, we impose market clearing. The market for each sectoral commodity clears if and only if

$$c_i(s) + \sum_{j=1}^{N}\left(\int x_{ji}(\omega,s)\,\mu(\omega|s)\,d\omega\right) = y_i(\omega,s) = \left[\int y_{ik}(\omega,s)^{\frac{\theta_i-1}{\theta_i}}\mu(\omega|s)\,d\omega\right]^{\frac{\theta_i}{\theta_i-1}} \quad \forall i,\omega,s \tag{5}$$

with

$$y_{ik}(\omega,s) = z_i(s) F_i(\ell_{ik}(\omega,s), G_i(\mathbf{x}_{ik}(\omega,s))) \quad \forall i,\omega,s \tag{6}$$

Similarly, the labor market clears if and only if

$$L(s) = \sum_{i=1}^{N}\left(\int \ell_{ik}(\omega,s)\,\mu(\omega|s)\,d\omega\right) \quad \forall s; \tag{7}$$

We can then define our notion of sticky-price equilibria as follows.

Definition 1. A *sticky price equilibrium* is a triplet $(\xi, \varrho, \vartheta)$ of allocations, prices, and policies such that

(i) firms maximize their expected real value of profits given their information set;

(ii) households optimize;

(iii) the government budget constraint is satisfied; and

(iv) goods and labor markets clear.

We next define our notion of flexible-price equilibria by dropping the measurability constraint on prices. Flexible-price firms, on the other hand, make their nominal decision based on complete information of the aggregate state. Formally, we simply consider the special case in which all firms prices are measurable in the aggregate state, $p_i(s)$. Accordingly, we adjust the equation for output in (3) and the sectoral price level in condition (4): so that $y_{ik}(\omega, s) = y_i(s)$. We then state the relevant definition as follows.

Definition 2. A *flexible price equilibrium* is a triplet (ξ, δ, ϑ) of allocations, prices, and policies that satisfy the same conditions as those stated in Definition 1 with the special case that $\omega = s$

We conclude this section with one additional definition, whose meaning is self-evident.

Definition 3. An allocation ξ is *feasible* if and only if it satisfies resource constraints (5), (6), and (7). We denote the set of all feasible allocations by $\mathcal{X}$.

Our information structure can be generalized to allow for any profile of information sets ω_{ik} across firms and across sectors.

We may nest Correia Nicolini Teles.. For now, as mentioned above we consider the special case in which $\omega_{ik} = \omega_i$for all sticky price firms. This simplification is not central to our results. We relax this assumption in Section 5 when we consider the second best policy.

This is similar to dynamic models of "sticky information" as in Mankiw and Reis (2002) and Ball et al. (2005). If we were to make our model dynamic, these settings would be directly nested in our framework by letting φ assign probability λ to $\omega_i^t = (\omega_i^{t-1}, s^t)$ and probability $1-\lambda$ to $\omega_i^t = \omega_i^{t-1}$, where $\lambda \in (0,1)$ is the probability with which a firm updates its information set with the perfect observation of the underlying state and $1-\lambda$ is the probability with which the firm is stuck with her old information set.

3 The First Best and Flexible Price Allocations

We first abstract from sticky prices. In this section we consider the first-best allocation as well as the set of allocations that may be implemented under flexible prices.

3.1 The First-Best Allocation

Consider the first-best planner who chooses the welfare-maximizing allocation among all feasible allocations. Note first that by symmetry of firms within each sector, the planner dictates that all firms

within a sector choose the same input, labor, and output quantities.[4] Then the first- best planner's problem thus reduces to

$$\max_{\xi} \int_{s\in S} \left[U\left(\mathcal{C}\left(c_1\left(s\right),\ldots,c_N\left(s\right)\right)\right) - V\left(L\left(s\right)\right)\right]\mu\left(s\right)ds \tag{8}$$

subject to resource constraints

$$c_i\left(s\right) + \sum_j x_{ji}\left(s\right) = y_i\left(s\right) = z_i\left(s\right)F_i\left(\ell_i\left(s\right), G_i\left(\boldsymbol{x}_i\left(s\right)\right)\right) \text{ for all } i, s \tag{9}$$

$$L\left(s\right) = \sum_{i=1}^{N} \ell_i\left(s\right) \text{ for all } s \tag{10}$$

The equations that characterize the planner's optimum are straight forward. We characterize these in the following lemma.

Proposition 1. *The first-best optimal allocation, $\xi^* \in \mathcal{X}$, is characterized by the following set of equations*

$$V'\left(L(s)\right) = U'\left(C(s)\right)\frac{d\mathcal{C}(s)}{dc_i(s)}z_i\left(s\right)\frac{dF_i\left(s\right)}{d\ell_i\left(s\right)} \quad \textit{for all } i, s \tag{11}$$

and

$$\frac{d\mathcal{C}(s)/dc_j(s)}{d\mathcal{C}(s)/dc_i(s)} = z_i\left(s\right)\frac{dF_i\left(s\right)}{dx_i\left(s\right)}\frac{dG_i\left(s\right)}{dx_{ij}\left(s\right)} \quad \textit{for all } i, j, s \tag{12}$$

along with resource constraints (9) and (10).

Conditions (11) and (12) along with the resource constraints hold state-by-state in the first best allocation. The first condition (11) indicates that for any good, it is optimal to equate the marginal rate of substitution between consumption of that good and labor with the marginal rate of transformation. To see this more clearly, note that left hand side of condition (11) is the marginal social cost of labor: simply the marginal disutility of labor of the household. The planner equates this marginal cost with the marginal social benefit. The marginal social benefit consists of three multiplicative components: the marginal product of labor in the production of commodity i, the marginal product of good i in the production of the final good, and the marginal utility of consumption of the final good.

The second condition (12) similarly indicates that the planner finds it optimal to equate the marginal rate of substitution between two goods to their marginal rate of transformation. The marginal rate of substitution, the left hand side of equation (12), is simply the ratio of marginal utilities from consumption of either good. The marginal rate of transformation is simply the marginal product of good i in the production of good j, as shown in the right hand side of this condition.

3.2 Flexible-Price Equilibrium

We now consider the set of allocations that may be implementable as a flexible price equilibrium in this economy.

[4] Specifically, the planner chooses, $x_{ij} = x_{ij,k}$, $\ell_i = \ell_{i,k}$, and $y_i = y_{i,k}$ for all firms k within sector i.

Consider the household. The household simply maximizes its utility subject to its budget constraint. This leads to the following household optimality conditions:

$$U'(C)\frac{d\mathcal{C}}{dc_i}\frac{W}{p_i} = V'(L) \text{ for all } i \tag{13}$$

$$\frac{d\mathcal{C}/dc_j}{d\mathcal{C}/dc_i} = \frac{p_j}{p_i} \text{ for all } i,j \tag{14}$$

These equations are straightforward. The first condition (13) imposes that the household sets it's marginal rate of substiution between consumption and labor equal to their price ratio, w/p_i. The second condition (14) simply states that the household's marginal rate of substitution between two goods is equal to their price ratio.

Next consider the problem of the sectoral aggregator firm. Recall that this firm is simply a Dixit-Stiglitz aggregator of the goods produced by all firms within that sector. The sectoral aggregator firm's optimality conditions lead to the following familiar CES demand function,

$$y_{ik}(s) = \left(\frac{p_{ik}(s)}{p_i(s)}\right)^{-\theta_i} y_i(s) \tag{15}$$

The Individual Firm. We now consider the problem of the typical monopolistically competitive firm. Consider a firm within sector i. The firm's problem can be easily split into a dual problem. The firm solves an outer problem of profit maximization given by

$$\max (1-\tau_i) p_{ik}(s) y_{ik}(s) - mc_{ik}(s) y_{ik}(s) \tag{16}$$

where $mc_i(s)$ is the marginal cost of producing goods $y_i(s)$, subject to the firm's demand function (15).

$$y_{ik}(s) = \left(\frac{p_{ik}(s)}{p_i(s)}\right)^{-\theta_i} y_i(s)$$

The inner problem determines the firm's marginal cost function. In particular, the firm's inner problem is a cost minimization problem given by

$$mc_i(s) y_{ik}(s) = \min w(s) \ell_i(s) + \sum_j p_j(s) x_{ij,k}(s) \tag{17}$$

subject to the firm's production function in ().

Inner problem. From the firm's inner problem we have that

$$mc_i(s) = w(s)\left(z_i(s)\frac{dF_i(s)}{d\ell_i(s)}\right)^{-1} \text{ and } mc_i(s) = p_j(s)\left(z_i(s)\frac{dF_i(s)}{dx_i(s)}\frac{dG_i(s)}{dx_{ij}(s)}\right)^{-1} \tag{18}$$

From now on we drop the subscript k on the firm's marginal cost. As long as firms have constant returns to scale, their marginal costs are sector (and not firm) specific.

Outer problem. Now consider the firm's outer problem. Substituting the demand function (15) into the firm's objective in (16) we have that the firm solves

$$\max_{y_{i,k}} (1-\tau_i)\left(\frac{y_{ik}(s)}{y_i(s)}\right)^{-1/\theta_i} p_i(s) y_{ik}(s) - mc_{ik}(s) y_{ik}(s)$$

This yields the typical monopolistically-competitive firm's choice for the price as a markup over marginal cost

$$p_i(s) = \left[(1-\tau_i)\left(\frac{\theta_i - 1}{\theta_i}\right)\right]^{-1} mc_i(s) \tag{19}$$

Combining this expression with the firm's marginal cost conditions in (18), using the fact that $p_{ik}(s) = p_i(s)$ for every sector i, and finally combining this with the household's optimality conditions (13)-(14), we come to the following characterization of the set of flexible price allocations.

Proposition 2. *A feasible allocation, $\xi \in \mathcal{X}$, is part of a flexible-price equilibrium if and only if there exists a set of scalars $(\chi_1, \ldots, \chi_N) \in \mathbb{R}^N_+$ such that the following two sets of conditions hold:*

$$V'(L(s)) = \chi_i U'(C(s)) \frac{d\mathcal{C}(s)}{dc_i(s)} z_i(s) \frac{dF_i(s)}{d\ell_i(s)} \quad \textit{for all } i \in I, s \in S \tag{20}$$

and

$$\frac{d\mathcal{C}(s)}{dc_j(s)} = \chi_i \frac{d\mathcal{C}(s)}{dc_i(s)} z_i(s) \frac{dF_i(s)}{dx_i(s)} \frac{dG_i(s)}{dx_{ij}(s)} \quad \textit{for all } i, j, s \in S. \tag{21}$$

The conditions in Proposition 2 are almost identical to those characterizing the planner's optimum in Lemma 1, aside from the set of scalars $(\chi_1, \ldots, \chi_N)$. The first condition (20) indicates that for any good, the marginal rate of substitution between consumption and labor is equal to the marginal rate of transformation, modulo a non-state contingent wedge χ_i. Similary, the second condition equates the marginal rate of substitution between two goods to their marginal rate of transformation, again subject to the wedge χ_i. In both conditions, a non-stochastic wedge arises given by

$$\chi_i = (1-\tau_i)\frac{\theta_i - 1}{\theta_i}$$

Each sectoral wedge consist of two terms: the tax or subsidy levied by the government, and the monopolistic markup that arises due to monopolistic-competition among firms within each sector. As a result, the scalars $(\chi_1, \ldots, \chi_N) \in \mathbb{R}^N_+$ parameterize the power of the fiscal authority. In particular, with sectoral taxes/or subsidies, the fiscal authority can move allocations by inducing wedges in the conditions (20) and (21).

Let $\mathcal{X}^f$ denote the entire set of flexible-price equilibria. Comparing the conditions in Propositions 2 to their counterparts in 1 it is immediately clear that the first best can be acheived under flexible prices when $\chi_i = 1$ for all i. we see that the optimal allocation is always contained in the set of flexible price equilibrium allocations. This leads to the following result.

Proposition 3. *The first best allocation belongs to the set of flexible price allocations: $\xi^* \in \mathcal{X}^f$. The first best may be implemented as a flexible-price equilibrium with sector-specific subsidies that undo all monopolistic markups:*

$$1 - \tau_i = \frac{\theta_i}{\theta_i - 1} > 1$$

This is the multi-sector version of the typical result that in a flexible-price economy without nominal rigidities, the only distortion in this economy comes from the monopolistic competition. Thus under flexible prices it is optimal for the government to set sector-specific subsidies which are invariant with the aggregate state.

4 Sticky-Price Equilibrium

We now consider the economy under sticky prices. The household's problem remains the same. This leads to the same optimality conditions as in (13) and (14). Similarly, the sectoral aggregator firm's problem and optimality conditions remain the same. Therefore the Dixit-Stiglitz demand function faced by firms is given by

$$y_{ik}(\omega, s) = \left(\frac{p_{ik}(\omega)}{p_i(\omega)}\right)^{-\theta_i} y_i(\omega, s) \tag{22}$$

The Individual Sticky Price Firm. Sticky-price individual firms profit maximization is given by

$$\max \mathbb{E}\left[\mathcal{M}(s)\left\{(1-\tau_i)\, p_{ik}(\omega)\, y_{ik}(\omega, s) - mc_{ik}(s)\, y_{ik}(\omega, s)\right\} \middle| \omega\right]$$

where $\mathcal{M}(s)$ is the household's nominal stochastic discount factor, subject to the demand function (15). Plugging in the demand function (15), this becomes

$$\max_{p_{ik}} \mathbb{E}\left[\mathcal{M}(s)\left((1-\tau_i)\, p_{ik}(\omega)\left(\frac{p_{ik}(\omega)}{p_i(\omega, s)}\right)^{-\theta_i} y_i(\omega, s) - mc_{ik}(\omega, s)\left(\frac{p_{ik}(\omega)}{p_i(\omega, s)}\right)^{-\theta_i} y_i(\omega, s)\right) \middle| \omega\right]$$

As a result, sticky price firms set their price as follows

$$p_{ik}(\omega) = \left[(1-\tau_i)\left(\frac{\theta_i - 1}{\theta_i}\right)\right]^{-1} \frac{\mathbb{E}\left[v_i(\omega, s)\, mc_{ik}(\omega, s) \middle| \omega\right]}{\mathbb{E}\left[v_i(\omega, s) \middle| \omega\right]} \tag{23}$$

where

$$v_i(\omega, \Omega, s) \equiv U'(C(s)) \frac{d\mathcal{C}(s)}{dc_i(s)} \left(\frac{y_{ik}(\omega, s)}{y_i(\Omega, s)}\right)^{\frac{\theta_i - 1}{\theta_i}} y_i(\Omega, s) \tag{24}$$

That is, the firm sets its price as a weighted expectation of marginal cost. Equation (23) is similar to a condition in Correia Nicolini Teles (but with the expression for v_i being slightly different).

The flexible price firm faces the same exact problem as before; its price is thereby given by (19). That is, the flexible price firm continues to set its price as a constant markup over marginal cost. This implies that the nominal price of the sticky price firm's may be alternatively written as:

$$p_{ik}(\omega) = \frac{\mathbb{E}\left[v_i(\omega, s)\, p_{ik}(s) \middle| \omega\right]}{\mathbb{E}\left[v_i(\omega, s) \middle| \omega\right]}$$

where $p_i(s)$ is the price it would set if it were flexible. Instead, the sticky price firm sets a price that is equal to it's expectation of its optimal price. Therefore we may write the price chosen by the sticky price firm in terms of its pricing error relative to what it would have set had it known the aggregate state:

$$p_{ik}(\omega) = \left[(1-\tau_i)\left(\frac{\theta_i - 1}{\theta_i}\right)\right]^{-1} \varepsilon_{ik}(\omega, s)^{-1}\, mc_i(s) \tag{25}$$

where $mc_i(s)$ is the realized marginal cost and $\varepsilon_{ik}(\omega, s)$ as the sticky-price firm's pricing error given by,

$$\varepsilon_{ik}(\omega, s) \equiv \frac{\mathbb{E}\left[v_i(\omega, s) \middle| \omega\right] mc_i(s)}{\mathbb{E}\left[v_i(\omega, s)\, mc_i(s) \middle| \omega\right]} \tag{26}$$

Combining this expression with the firm's marginal cost conditions in (18) and with the household's optimality conditions (13)-(14), we come to the following characterization of the set of sticky-price equilibria.

Proposition 4. *A feasible allocation, $\xi \in \mathcal{X}$, is part of a sticky-price equilibrium if and only if there exists a set of scalars $\{\chi_1, \ldots, \chi_N\} \in \mathbb{R}_+^N$ and a function $\phi : \mathcal{S} \to \mathbb{R}_+$, such that the following conditions hold:*

(i) The allocation, along with the set of scalars $(\chi_1, \ldots, \chi_N)$ and the set of wedge functions $(\varepsilon_1, \ldots, \varepsilon_N)$, jointly satisfy

$$V'(L(s)) = \chi_i \varepsilon_{ik}(\omega, s) U'(\mathcal{C}(s)) \frac{d\mathcal{C}(s)}{dc_i(s)} \left(\frac{y_{ik}(\omega, s)}{y_i(s)} \right)^{-1/\theta_i} z_i(s) \frac{dF_i}{d\ell_{ik}}(\omega, s) \quad \forall i, \omega, s \tag{27}$$

and

$$\frac{d\mathcal{C}(s)}{dc_j(s)} = \chi_i \varepsilon_{ik}(\omega, s) \frac{d\mathcal{C}(s)}{dc_i(s)} \left(\frac{y_{ik}(\omega, s)}{y_i(s)} \right)^{-1/\theta_i} z_i(s) \frac{dF_i}{dx_{ik}} \frac{dG_i}{dx_{ij,k}}(\omega, s) \quad \forall i, j, \omega, s. \tag{28}$$

(ii) The function $\phi : \mathcal{S} \to \mathbb{R}_+$ induces a set of error wedge functions $\varepsilon_1, \ldots, \varepsilon_N : \Omega \times \mathcal{S} \to \mathbb{R}_+$ given by

$$\varepsilon_{ik}(\omega, s)^{-1} = \frac{\mathbb{E}\left[v_i(\omega, s) \phi(s) \left(z_i(s) \frac{dF_{ik}}{d\ell_{ik}}(\omega, s) \right)^{-1} \middle| \omega \right]}{\mathbb{E}\left[v_i(\omega, s) | \omega \right] \phi(s) \left(z_i(s) \frac{dF_{ik}}{d\ell_{ik}}(\omega, s) \right)^{-1}} \quad \forall i, \omega, s \tag{29}$$

with $v_i(\omega, s)$ defined in (24).

Proposition 4 is the sticky-price counterpart of Proposition 2 for flexible prices. To understand this, first consider part (ii) of Proposition 4. Conditions (27) and (28) are exactly the same as conditions (20) and (21), except for a new wedge, given by $\varepsilon_{ik}(\omega, s)$. The new wedge $\varepsilon_{ik}(\omega, s)$ is a sector-specific and represents an *additional* control variable for the government, one that encapsulates the power of monetary policy over real allocations when prices are sticky.

As with taxes, the monetary authority can move allocations by moving the wedges in conditions (27) and (28). This power is non-trivial, but it is also restrained by condition (29). This is because the wedges that arise are only due to mistakes by the sticky-price firms in setting their prices. Therefore, the condition (29) imposes the following constraint on these errors:

$$\mathbb{E}\left[v_i(\omega, s) (\varepsilon_i(\omega, s) - 1) | \omega \right] = 0$$

error must be equal 1 in expectation! This is the pricing error made by the sticky price firm. But it must be the case that this pricing error is equal to 1 in expectation.

Flexible-Price versus Sticky-Price allocations. Let $\mathcal{X}^s$ denote the entire set of sticky-price equilibria. We now consider how this set relates to the set of flexible-price equilibria.

Theorem 1. *The sets $\mathcal{X}^f$ and $\mathcal{X}^s$ are disjoint:*

$$\mathcal{X}^f \cap \mathcal{X}^s = \varnothing$$

That is, any allocation implementable as an equilibrium under flexible prices cannot be implemented as an equilibrium under sticky prices.

The following result is then immediate.

Corollary 1. *The first best allocation cannot be implemented as an equilibrium under sticky prices:*

$$\xi^* \notin \mathcal{X}^s.$$

The intuition for Theorem 1 and Corollary 1 is straightforward. Consider the planner's optimal allocation which can itself be implemented as a flexible-price equilibrium with appropriate sectoral taxes (subsidies). The planner would like relative quantities across sectors to move efficiently with productivity shocks, but the planner would like all firms within each sector to produce the same quantity. In order to implement this under flexible prices, relative prices across sectors should similarly move with productivity shocks, while prices should be uniform across firms within each sector. This partition of price movements is necessary both for flexible-price allocations and in particular for first best allocation.

However, this partitition of price movements is impossible under sticky prices when there are multiple sectors. In order to ensure that prices are uniform within a particular sector, the monetary authority must target price stability for that sector. This is the typical first-best policy in one-sector New Keynesian models and it implements zero pricing errors within that particular sector so that all sticky and flexible-price firms set the same nominal price. In this case, with only one sector, price stability implements first best allocations: the nominal wage simply moves freely with the productivity shock.

But when there are multiple sectors, and monetary policy is used to acheive price stability within one particular sector, it cannot also used to target price stability in any other sector. That is, monetary policy cannot stabilize prices in all sectors at once. And even if it could–suppose there were two sectors and there were somehow a policy that could stabilize nominal prices across all firms in both sectors–this still would not acheive first best: the planner would have wanted the relative price between the two sectors to move with productivity shocks. As a result, in a generic multi-sector economy, the first best cannot be achieved as a sticky-price equilibrium.

In the appendix we consider some special, non-generic cases in which monetary policy can implement the first best. The first case is the familiar New Keynesian one-sector model. In this case, the first best may be achieved with price stability and a subsidy which eliminates the monopolistic markup.

A second case in which the first best is implementable is the Aoki two-sector model. Suppose there are two sectors, $i = 1, 2$, but sector 1 has sticky prices while sector 2 is fully flexible. In this case the monetary policy can implement flexible price allocations by stabilizing prices in sector 1. This, along with the optimal subsidy, implements first best.

5 Welfare Losses

In a generic multi-sector economy with nominal rigidities the first best cannot be attained. We now consider the Ramsey's planner's problem. The Ramsey planner maximizes welfare over the set of sticky-price implementable allocations. To simplify this problem, we make the following set of assumptions in order to calculate welfare.

5.1 The Ramsey Problem with Cobb-Douglas technology

First, we assume that Cobb-Douglas assumptions on utility and technology. The production firm in sector i produces output according to the following Cobb-Douglas technology

$$F_i(\ell_i, x_i) = \ell_i^{\alpha_i} x_i^{1-\alpha_i} \tag{30}$$

The parameter α_i is the labor share and the vector of the labor shares is denoted by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)'$. The composite of sectoral inputs is given by the Cobb-Douglas function

$$x_i = G_i(\mathbf{x}_i) = \prod_{j \in I} x_{ij}^{g_{ij}}. \tag{31}$$

Each parameter $g_{ij} \in [0,1]$ denotes the share of good j in the total intermediate input use of the firm in sector i. Finally, to maintain constant returns to scale, we assume that $\sum_{j=1}^{N} g_{ij} = 1, \forall i \in \{1, ..., N\}$.

Finally, we assume the final consumption basket is a Cobb-Douglas composite of sectoral goods given by

$$\mathcal{C}(c_1, \ldots, c_N) = \prod_{i \in I} c_i^{v_i}. \tag{32}$$

The coefficient $v_i \in [0,1]$ is the household's expenditure share on good i and, $\sum_{i=1}^{N} v_i = 1$. The vector of household expenditure shares is $\mathbf{v} = (v_1, \ldots, v_N)'$.

Log-Normal dispersion within sectors. Second we assume that output within a sector is log-normally distributed.

$$\log y_{ik}(\omega_{ik}, s) \sim N\left(\mathbb{E} \log y_{ik}(\omega_{ik}, s), Var_\omega\left[\log y(\omega, s) | \Omega, s\right]\right)$$

We can either assume a signal structure that delivers this strucure in equilibrium or one may think of this as a second order approximation. Given this, we have the following lemma.

Lemma 1. *Suppose output within a sector is log-normally distributed. Then total output of sector i may be written as*

$$y_i(\Omega_i, s) = \vartheta_i(s)\, y_i(\omega_i, s)$$

where

$$y_i(\omega, s) \equiv \mathbb{E} y_{ik}(\omega_{ik}, s)$$

is mean output within a sector and

$$\vartheta_i \equiv \exp\left\{-\frac{1}{2}\frac{1}{\theta_i} Var_\omega\left[\log y_{ik}(\omega_{ik}, s) - \mathbb{E} \log y_{ik}(\omega_{ik}, s) | \Omega, s\right]\right\}$$

is a decreasing function of within-sector cross-sectional dispersion.

Therefore, dispersion within a sector is isomorphic to a negative productivity shock to this sector.

Sticky-Price Equilibrium With these two assumptions we may rewrite Proposition 4 as follows.

Proposition 5. *A feasible allocation, $\xi \in \mathcal{X}$, is part of a sticky-price equilibrium if and only if there exists a set of scalars $\{\chi_1, \ldots, \chi_N\} \in \mathbb{R}_+^N$ and a function $\phi : \mathcal{S} \to \mathbb{R}_+$, such that the following conditions hold:*

$$V'(L(s)) = \chi_i \varepsilon_i(\omega, s) U'(\mathcal{C}(s)) v_i \frac{\mathcal{C}(s)}{c_i(s)} \alpha_i \frac{\vartheta_i y_i(\omega, s)}{\ell_i(s)} \quad \forall i, \omega, s \tag{33}$$

and

$$v_j \frac{\mathcal{C}(s)}{c_j(s)} = \chi_i \varepsilon_i(\omega, s) v_i \frac{\mathcal{C}(s)}{c_i(s)} (1 - \alpha_i) g_{ij} \frac{\vartheta_i y_i(\omega, s)}{x_{ij}(\omega, s)} \quad \forall i, j, \omega, s. \tag{34}$$

where the function $\phi : \mathcal{S} \to \mathbb{R}_+$ induces a set of error wedge functions $\varepsilon_1, \ldots, \varepsilon_N : \Omega \times \mathcal{S} \to \mathbb{R}_+$ given by

$$\varepsilon_i(\omega, s)^{-1} = \frac{\mathbb{E}\left[u_i(\omega, s) \phi(s) \left(\frac{y_i(\omega,s)}{\ell_i(\omega,s)} \right)^{-1} \middle| \omega \right]}{\mathbb{E}\left[u_i(\omega, s) | \omega \right] \phi(s) \left(\frac{y_i(\omega,s)}{\ell_i(\omega,s)} \right)^{-1}} \quad \forall i, \omega, s \tag{35}$$

with $u_i(\omega, s)$ given by

$$u_i(\omega, s) = U'(C(s)) v_i \frac{\mathcal{C}(s)}{c_i(s)} y_i(\omega, s)$$

We may use the aggregation results derived in Bigio and La'O (2018). They show that in a Cobb-Douglas economy with exogenous wedges at the sectoral level, the wedges aggregate to an aggregate TFP shock and an aggregate labor wedge. The difference here is that the wedges at the sectoral level are now endogenous and depend on the monetary policy. For now, let us set

$$\chi_i = 1 \quad \forall i$$

We thus obtain the following aggregation result.

Theorem 2. *Let $\boldsymbol{\epsilon}(s)$ denote the realized vector of sectoral errors, $\boldsymbol{\epsilon}(s) = (\varepsilon_1(\omega, s), \ldots, \varepsilon_N(\omega, s))$. Then there exists functions $TFP : S \to \mathbb{R}_+$ and $\Lambda : \mathbb{R}^N \to \mathbb{R}_+$ such that the equilibrium quantities aggregate to a representative firm, representative household economy with a constant returns-to-scale aggregate production function given by:*

$$C(s) = A(\mathbf{z}(s)) A(\vartheta) TFP(\boldsymbol{\epsilon}(s)) L(s), \tag{36}$$

where $TFP(\boldsymbol{\epsilon}(s))$ is total factor productivity due to errors, and an equilibrium intratemporal condition given by

$$\frac{V'(L(s))}{U'(C(s))} = \Lambda(\boldsymbol{\epsilon}(s)) \frac{C(s)}{L(s)}, \tag{37}$$

where $\Lambda(\boldsymbol{\epsilon}(s))$ is the aggregate labor wedge.

(i) The aggregate labor wedge is given by

$$\Lambda(\boldsymbol{\epsilon}(s)) \equiv \boldsymbol{\alpha}' \left\{ diag(\boldsymbol{\epsilon}(s))^{-1} - \mathbf{G}' diag(\mathbf{1} - \boldsymbol{\alpha}) \right\}^{-1} v, \tag{38}$$

and satisfies $\Lambda(\mathbf{1}) = 1$.

(ii) Total factor productivity due to the error wedges are given by

$$TFP\left(\boldsymbol{\epsilon}\left(s\right)\right) \equiv \frac{A\left(\boldsymbol{\epsilon}\left(s\right)\right)}{\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}, \tag{39}$$

where for any vector $\boldsymbol{w}$ of length N, the function $A:\mathbb{R}^{N}\rightarrow\mathbb{R}_{+}$ is a log-linear operator defined as

$$\log A\left(\mathbf{w}\right) \equiv \boldsymbol{v}'\mathbf{L}\log\mathbf{w}. \tag{40}$$

where $\mathbf{L}$ is the $N\times N$ Leontief inverse matrix,

$$\mathbf{L} \equiv \left[\mathbb{I}_{N} - diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right]^{-1}, \tag{41}$$

and satisfies $A\left(\mathbf{1}\right)=1$.

Using this result, we may then state the Ramsey planner's problem as follows.

Planner's Problem The Ramsey planner chooses consumption, labor, and a monetary policy $\{C\left(s\right),L\left(s\right),\phi\left(s\right)\}$ in order to maximize ex-ante utility

$$\max_{C(s),L(s),\phi(s)} \mathbb{E}\left[U\left(C\left(s\right)\right)-V\left(L\left(s\right)\right)\right]$$

subject to

$$\begin{aligned} C\left(s\right) &= A\left(\mathbf{z}\left(s\right)\right)A\left(\vartheta\right)TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)L\left(s\right), \\ V'\left(L(s)\right)L(s) &= \Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)U'\left(C\left(s\right)\right)C\left(s\right) \end{aligned}$$

where the errors depend on monetary policy

$$\varepsilon_{i}\left(\omega,s\right) = \varepsilon_{i}\left(\phi\left(s\right),\omega,s\right) \;\; \forall i,\omega,s$$

Therefore errors show up in three different ways. The average pricing error of a sector manifests in TFP and in the labor wedge. Furthermore, cross-sectional dispersion within a sector manifests as TFP.

5.2 Welfare Losses with Homothetic Utility

Theorem 2 holds for any specification of the utility functions U,V. We may solve for equilibrium consumption (GDP) and labor in this economy under a particular form of preferences. We make the standard assumption that preferences be homothetic; in particular, we let:

$$U\left(C\right) \equiv \frac{C^{1-\gamma}}{1-\gamma} \text{ and } V\left(L\right) \equiv \frac{L^{1+1/\eta}}{1+1/\eta}. \tag{42}$$

for parameters $\gamma\geq 0$ and $\epsilon>0$. Given the static nature of our model, the parameter γ captures both risk aversion and the income elasticity of labor supply and η corresponds to the Frisch elasticity of labor supply. With the homothetic preference specification and we may explicitly solve for equilibrium labor supply and equilibrium aggregate output, which are simply determined by the two equations in (36) and (37). We solve explicitly for these functions in the appendix.

The objective of the Ramsey planner is to maximize welfare, given by the unconditional expectation of the utility of the representive household:

$$\mathcal{W} \equiv \mathbb{E}\left[\frac{C(s)^{1-\gamma}}{1-\gamma} - \frac{L(s)^{1+1/\eta}}{1+1/\eta}\right]$$

With homothetic utility, we may write welfare as follows.

Lemma 2. *Up to a second order approximation, welfare is given by*

$$\mathcal{W} \approx \mathcal{W}^* \exp\left\{\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\left(\mathbb{E}\log TFP(\boldsymbol{\epsilon}(s)) + \log A(\vartheta)\right) - \frac{1}{2}\frac{(1-\gamma)(1+1/\eta)}{(1/\eta+\gamma)^2}\mathbb{E}\left[\left(\log\Lambda(\boldsymbol{\epsilon}(s))\right)^2\right]\right\}$$

where $\mathcal{W}^*$ is the level of first-best welfare.

Let $C^*(s)$ denote first best GDP. Let $\hat{C}(s)$ denote GDP in the absence of a labor wedge and in the absense of within-sector dispersion. Thus

$$\log\hat{C}(s) - \log C^*(s) = \log TFP(\boldsymbol{\epsilon}(s))$$

Finally, let $C(s)$ denote equilibrium output. We call the difference between $C(s)$ and $\hat{C}(s)$ the "output gap" which is the consequence of the labor wedge.

$$\log C(s) - \log\hat{C}(s) = \frac{1}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))$$

We may then rewrite Lemma () as follows.

Theorem 3. *To a second-order approximation, welfare is given by*

$$\mathcal{W} \approx \mathcal{W}^* \exp\left(-\frac{1}{2}\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\Omega\right)$$

where Ω denotes welfare losses. Welfare losses may be written as function of three terms:

$$\Omega = \frac{1}{1/\eta+\gamma}\mathbb{V}_\Lambda + \mathbb{V}_{TFP} + \sum_{i\in I}\lambda_i\frac{1}{\theta_i}\sigma_i^y.$$

These terms are: (i) the volatility of the output gap due to sectoral wedges

$$\mathbb{V}_\Lambda \equiv \mathbb{E}\left[\left(\log\Lambda(\boldsymbol{\epsilon}(s))\right)^2\right] = (1/\eta+\gamma)^2 Var\left(\log C(s) - \log\hat{C}(s)\right),$$

(ii) the expected efficiency loss due to sectoral wedges

$$\mathbb{V}_{TFP} = -2\mathbb{E}\left(\log\hat{C}(s) - \log C^*(s)\right) = \sum_i\sum_j \left.\frac{d^2\log\Lambda(\boldsymbol{\epsilon}(s))}{(d\log\epsilon_j)(d\log\epsilon_i)}\right|_{\boldsymbol{\epsilon}=\mathbf{1}} \mathbb{E}\left[(\log\epsilon_i)(\log\epsilon_i)\right],$$

and finally, (iii) a weighted sum of within-sector output dispersion

$$\sigma_i^y = Var_\omega\left[\log y_{ik}(\omega_{ik}, s) - \log y_i(\omega, s)\,|\Omega, s\right].$$

First, note that the third term is written in terms of within-sector output dispersion. However, we may instead write it in terms of price dispersion as follows. First note that from the dixit-stiglitz demand function, we have

$$\log y_{ik}\left(\omega_{ik},s\right)-\log y_i\left(\bar{\omega}_i,s\right)=-\theta_i\left(\log p_{ik}\left(\omega_{ik}\right)-\log p_i\left(s\right)\right)$$

As a result, $\sigma_i^y=\theta_i^2\sigma_i^p$ where $\sigma_i^p\equiv Var_\omega\left[\log p\left(\omega\right)-\log P\left(s\right)|s\right]$. Therefore the third term in the welfare loss function may be written as

$$\sum_{i\in I}\lambda_i\theta_i\sigma_i^p.$$

Next, consider the first two terms in the welfare loss function (). We may write the volatility of the output gap,$\mathbb{V}_\Lambda$, as the following function

$$\mathbb{V}_\Lambda=\mathbb{E}\left[\left(\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)^2\right]=\sum_j\sum_i\left\{\left(\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\epsilon_j}\right)\left(\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\epsilon_i}\right)\Bigg|_{\boldsymbol{\epsilon}=\mathbf{1}}\right\}\mathbb{E}\left[\left(\log\epsilon_i\right)\left(\log\epsilon_j\right)\right]$$

Combining this with our expression for $\mathbb{V}_{TFP}$ gives us the following lemma.

Lemma 3. *Let*

$$\Lambda_i^1=\left.\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\epsilon_i}\right|_{\boldsymbol{\epsilon}=\mathbf{1}}\ \textit{and}\ \Lambda_{ij}^2=\left.\frac{d^2\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{\left(d\log\epsilon_j\right)\left(d\log\epsilon_i\right)}\right|_{\boldsymbol{\epsilon}=\mathbf{1}}$$

Then

$$\frac{1}{1/\eta+\gamma}\mathbb{V}_\Lambda+\mathbb{V}_{TFP}=\sum_i\sum_j\left(\frac{1}{1/\eta+\gamma}\Lambda_i^1\Lambda_j^1+\Lambda_{ij}^2\right)\mathbb{E}\left[\left(\log\epsilon_i\right)\left(\log\epsilon_j\right)\right]$$

5.3 The Ramsey Problem Restated

In light of our previous results, we may now restate the Ramsey problem as follows

Planner's Problem The Ramsey planner chooses policy instruments $(\phi_1,\ldots,\phi_N)$ so as to minimize welfare losses Ω

$$\min\sum_i\sum_j\left(\frac{1}{1/\eta+\gamma}\Lambda_i^1\Lambda_j^1+\Lambda_{ij}^2\right)\mathbb{E}\left[\left(\log\epsilon_i\right)\left(\log\epsilon_j\right)\right]+\sum_{i\in I}\lambda_i\theta_i\sigma_i^p\left(\phi\right)$$

subject to

$$\log\boldsymbol{\epsilon}\left(s\right)\sim N\left(0,\Sigma\left(\phi\right)\right)$$

That is, the planner minimizes welfare losses due to these three terms. With his/her policy instruments, the planner controls both the variance-covariance matrix of errors, as well as the price dispersion within sectors.

6 Optimal Monetary Policy

Our results in Theorem 1 and Corollary 1 establish that, in general, the first best allocation cannot be implemented as a sticky price equilibrium. In this section, we study optimal monetary policy, i.e., the policy that maximizes household welfare over the set of all possible sticky price implementable allocations. As our main result, we establish that the optimal monetary policy not only depends on the extent of price stickiness across the economy, but also on the economy's production network.

In order to obtain closed-form expressions for the optimal policy, we impose a number of simplifying assumptions on preferences, technologies, and the nature of price stickiness in the economy. More specifically, we assume that firms in each industry employ Cobb-Douglas technologies to transform labor and intermediate goods into their differentiated products, with the production technology of a firm k in industry i given by

$$x_{ik} = z_i \zeta_i l_{ik}^{\alpha_i} \prod_{j=1}^{n} x_{ij,k}^{a_{ij}}. \tag{43}$$

In the above expression, l_{ik} denotes the amount of labor hired by the firm, $x_{ij,k}$ is the quantity of good j used as an intermediate input, $\alpha_i \geq 0$ denotes the share of labor in industry i's production technology, z_i is a Hicks-neutral productivity shock that is common to all firms in industry i, and ζ_i is a normalization constant, the value of which only depends on model parameters and is independent of the shocks.[5] The exponent $a_{ij} \geq 0$ in (43) captures the fact that firms in industry i may rely on the goods produced by other industries as intermediate inputs for production. Note that, to ensure all technologies exhibit constant returns, we impose the restriction that $\alpha_i + \sum_{j=1}^{n} a_{ij} = 1$. We summarize the input-output linkages at the industry-level with a matrix $A = [a_{ij}]$, which, with some abuse of terminology, we refer to as the economy's input-output matrix.

As in Section 2, we assume a monopolistically competitive market structure within each industry, with the output of in firms industry i is aggregated into a sectoral good according to the following CES production function

$$y_i = \left(\int_0^1 y_{ik}^{(\theta_i - 1)/\theta_i} dk \right)^{\theta_i/(\theta_i - 1)}$$

with elasticity of substitution $\theta_i > 1$. We assume that sector-specific subsidies are set to $1 - \tau_i = \theta_i/(\theta_i - 1)$ in order to undo monopolistic markups. As established in Proposition 3, this guarantees that the flexible price equilibrium is efficient.

We also assume that the basket of consumption goods is a Cobb-Douglas aggregator of the goods produced by the various industries given by

$$C = \prod_{i=1}^{n} (c_i / v_i)^{v_i},$$

where c_i is the amount of good i consumed and the constants $v_i \geq 0$ measure various goods' shares in the household's consumption basket, normalized such that $\sum_{i=1}^{n} v_i = 1$. The representative

[5] In what follows, we set the value of this constant to $\zeta_i = \alpha_i^{-\alpha_i} \prod_{j=1}^{n} a_{ij}^{-a_{ij}}$. This choice has no bearing on the results, as the sole purpose of this constant is to simplify the analytical expressions.

household has homothetic preferences given by

$$W(C,L)=\frac{C^{\gamma-1}}{\gamma-1}-\frac{L^{1+1/\eta}}{1+1/\eta},$$

where C denotes the household's consumption and L denotes her labor supply. Given the static nature of our model, the parameter γ captures both risk aversion and the income elasticity of labor supply and η corresponds to the Frisch elasticity of labor supply.

Finally, we define an industry's Domar weight as that industry's sales as a fraction of GDP. More specifically, the Domar weight of industry i is defined as

$$\lambda_i=\frac{p_i y_i}{\text{GDP}}$$

where p_i is the price of good i and y_i is industry i's output.

6.1 Welfare Losses

Before specifying the nature of nominal rigidities and monetary policy, we first provide a characterization of the extent of resulting welfare losses in the economy. Let $(p_1,\ldots,p_n,w)$ and $(p_1^*,\ldots,p_n^*,w^*)$ denote the vectors of industry-level nominal prices and the wage in the the presence and absence of nominal rigidities, respectively. Given the indeterminacy of prices in the absence of nominal rigidities, we normalize the nominal wage — without any loss of generality — such that $w^*=w$.

In order to obtain closed-form expressions, we follow the rest of the literature by providing a second-order (log) approximation for welfare losses arising due to nominal rigidities. Formally, we assume that $e_i=\log p_i-\log p_i^*=O(\delta)$ for all industries i, i.e., the presence of nominal rigidities results in "pricing errors" — defined as departures of nominal prices from their flexible counterparts — of the same order of magnitude δ. As we will show in subsequent sections, $e_i=O(\delta)$ for all i whenever the deviation of (log) productivity shocks $\log z$ from their steady state is of order δ. We have the following result:

Theorem 4. *The welfare loss due to the presence of nominal rigidities is given by*

$$\log W-\log W^*=-\frac{1}{2}\frac{(1+\eta)(1-\gamma)}{1+\eta\gamma}\left(\sum_{i=1}^{n}\lambda_i\theta_i\sigma_i^2+\frac{1+\eta\gamma}{\eta}\Sigma+\sum_{i=0}^{n}\lambda_i\text{var}_i(e_1,\ldots,e_n)\right)+o(\delta^2) \tag{44}$$

up to a second-order approximation in pricing errors, where λ_i denotes the Domar weight of industry i, $\sigma_i^2=\int_0^1\log p_{ik}^2dk-(\int_0^1\log p_{ik}dk)^2$ is price dispersion in industry i, $\Sigma=(\log y-\log y^*)^2$ is the volatility of output gap,

$$\text{var}_i(e_1,\ldots,e_n)=\sum_{j=1}^{n}a_{ij}e_j^2-\left(\sum_{j=1}^{n}a_{ij}e_j\right)^2 \tag{45}$$

is the across-industry dispersion of pricing errors from the perspective of industry i, with the convention that $i=0$ corresponds to the industry producing the consumption good bundle.

The above theorem generalizes the well-known results for welfare losses in single-sector New Keynesian models (e.g., Woodford (2003) and Galí (2008)) to our multi-sector framework. It illustrates that the loss in welfare due to the presence of nominal rigidities manifests itself via three separate terms. The first term, $\sum_{i=1}^{n} \lambda_i \theta_i \sigma_i^2$ — which is present in the baseline single-sector New Keynesian models — measures welfare losses due to price dispersion *within* each industry. The second term, on the other hand, captures the loss due to inefficient supply of labor by the household, i.e., the aggregate labor wedge. This term is also present in baseline single-sector models. In contrast to the first two terms, the third term on the right-hand side of (44) only arises in multi-sector economies and corresponds to welfare losses that arise as a consequence of misallocation of resources *across* industries. More specifically, the expression $\text{var}_i(e_1, \ldots, e_n)$ in equation (45) measures the dispersion in pricing errors of i's supplier industries, with higher weights assigned to industries that are more important input-suppliers to industry i. This latter term therefore captures that more dispersion in pricing errors in i's supplier industries translates to more misallocation of resources and hence, larger welfare losses.[6]

A few remarks are in order. First, note that Theorem 4 provides a second-order approximation to welfare losses regardless of the source of pricing errors. Therefore, to be able to use this result in our New Keynesian framework, all we need to do is to determine how any specific form of nominal rigidity translates into the pricing errors $(e_1, \ldots, e_n)$ and price dispersions $(\sigma_1, \ldots, \sigma_n)$. Second, we note that Theorem 4 has important implications for the conduct of monetary policy. Recall from Corollary 1 that the monetary authority can never implement the first-best allocation. As a result, when setting the policy, the monetary authority faces an inherent trade-off between minimizing the various losses captured by equation (44). In subsequent sections, we therefore use Theorem 4 to provide a characterization of optimal monetary policy as a function of the economy's production network and the extent of price rigidities across various industries.

6.2 Information Structure and Nominal Rigidities

We now turn to specifying the information structure of firms in each industry and the resulting nominal rigidities. We assume that (log) productivity shock $\log z_i$ is drawn independently from some distribution $\mathcal{F}_i$, commonly known to all firms in the economy. In addition, we assume that the distributions $(\mathcal{F}_1, \ldots, \mathcal{F}_n)$ — and hence, the firms' common prior beliefs — are such that, absent any informative signals, all firms in the economy find it optimal to set their nominal prices equal to 1, i.e., $\log p_{ik} = 0$ for all firms (i, k). Note that this assumption is without much loss of generality, as it is equivalent to normalizing nominal prices in the economy's steady-state. More importantly, we parameterize the distributions $\mathcal{F}_i$ by a parameter δ such that $\text{stdev}(\log(z_i)) = O(\delta)$. This ensures that firms' ex ante uncertainty about the realizations of each of the (log) productivity shocks is of order δ.

As in our framework in Section 2, each firm receives a private signal about the realization of productivity shocks $(z_1, \ldots, z_n)$ prior to setting its price. More specifically, we assume that a fraction $\rho_i \in [0, 1]$ of firms in industry i receive no informative signals about the realization of the shocks,

[6] When the economy of consists of only two industries that sell to the household but with no input-output linkages, this term reduces to the expression for quadratic welfare losses in Woodford (2010).

whereas the remainder fraction of $1-\rho_i$ can set their prices after observing the realization of all shocks (equivalently, these firms have access to perfectly informative signals). While extreme, focusing on this information structure has two advantages. First, it allows us to parametrize the degree of price stickiness in each industry i by a single dimensional parameter ρ_i. Second, the collection of parameters $(\rho_1,\ldots,\rho_n)$ resemble the "sticky information" formulation of Mankiw and Reis (2002) and Ball, Mankiw, and Reis (2005).

6.3 Monetary Policy Instruments

In Section 2, we assumed that the monetary authority can set the nominal wage $w(z)$ as a function of the realized productivity shocks z. Assuming a log-linear approximation, this means that the monetary authority can freely choose the coefficients $\phi=(\phi_1,\ldots,\phi_n)$ such that

$$\log w=\sum_{i=1}^{n}\phi_i\log z_i+o(\delta). \tag{46}$$

We now establish that any such policy can always be implemented using the money supply as the monetary instrument. To establish this result formally, we assume that the representative household is subject to a cash-in-advance constraint in the form of

$$\sum_{i=1}^{n}p_ic_i\leq m,$$

where the left-hand side denotes nominal expenditure and m denotes the money supply. We assume that the monetary authority has the ability to commit to a specific monetary policy, in the form of setting $m(z)$ as a pre-specified function of the realized productivities z. Under a log-linear approximation, this means that the monetary authority chooses the coefficients $\varphi=(\varphi_1,\ldots,\varphi_n)$ such that

$$\log m=\sum_{i=1}^{n}\varphi_i\log z_i+o(\delta). \tag{47}$$

Therefore, even though some firms may not know the realized productivity shocks when they set their nominal prices (due to incomplete information), they take into account how these shocks impact the money supply. We have the following result:

Theorem 5. *Suppose no industry is perfectly sticky, i.e., $\rho_i<1$ for all i. Then, up to a first-order approximation, the monetary authority can*

(a) implement any nominal wage, or alternatively

(b) stabilize an arbitrary nominal price index

using the money supply as an instrument;

The above result therefore establishes that, as long as some firms in any given industry receive informative signals about realized productivities, there is an isomorphism between three different

representations of monetary policy: (i) setting the money supply; (ii) setting the nominal wage; or (iii) stabilizing a price index. To be more specific, statement (a) of Theorem 5 shows that for any given collection of coefficients $\phi = (\phi_1, \ldots, \phi_n)$ in equation (46), there exists another collection of coefficients φ in (47) such that all nominal and real variables would coincide. Furthermore, statement (b) of Theorem 5 provides an alternative representation for any policy by establishing a one-to-one correspondence between setting the money supply and a price stabilization policy of the form

$$\sum_{i=1}^{n} \chi_i \log p_i = o(\delta). \tag{48}$$

More specifically, it illustrates that for any $\chi = (\chi_1, \ldots, \chi_n)$ satisfying $\sum_{i=1}^{n} \chi_i = 1$ such that (48), there exists a ϕ that satisfies (46).[7]

In view of Theorem 5, in what follows we will use the three different representations of monetary policy interchangeably, with the understanding that each representation can be implemented by controlling the money supply in the economy.

We conclude this discussion by defining a few concepts that will play a key role in our characterization of optimal policy. We define the *pass-through of monetary policy to industry* i as the sensitivity of i's marginal cost to the nominal wage,[8] i.e.,

$$\kappa_i = \frac{d \log \mathrm{mc}_i}{d \log w}. \tag{49}$$

It is easy to verify that that κ_i only depends on the extent of nominal rigidities to industries that are upstream to industry i. In particular, $\kappa_i = 1$ if no upstream industry is subject to nominal rigidities, whereas κ_i becomes smaller the more upstream industries are subject to nominal rigidities. Similarly, we define *monetary non-neutrality of industry* i as

$$\nu_i = 1 - \frac{d \log p_i}{d \log w}. \tag{50}$$

To see why ν_i can be a measure of monetary non-neutrality at industry i, note that $\nu_i = 0$ if industry i's nominal price moves with the nominal wage one-for-one. This would be the case as long as neither i nor any of its upstream industries are subject to any nominal rigidity. In contrast, $\nu_i > 0$, whenever the nominal price of industry i responds to changes in nominal wage sluggishly. Such sluggishness can arise if either i or any of its upstream industries are subject to nominal rigidities.[9]

As already mentioned, both κ_i and ν_i depend on nominal rigidities in not only industry i but also all its direct and indirect upstream suppliers. This is a consequence of the strategic complementarities in firms' price-setting behavior. In particular, all else equal, more upstream rigidities make a firm's marginal cost less responsive to productivity shocks as well as the monetary policy. Hence, even if the firm can observe the full realization of the shocks, it has less incentives to move its nominal price in accordance with the realized shocks.

[7] For the relationship between ϕ, φ, and χ, see the proof of Theorem 5.

[8] Recall that, in view of Theorem 5, we can assume the monetary authority sets the nominal wage.

[9] One can also define a measure of monetary non-neutrality in terms of the monetary policy's effect on real variables (such as industry-level output). While one can transform each representation to another, we find it more convenient to work with the definition in (50).

6.4 Optimal Policy

With Theorems 4 and 5 in hand, we are now ready to present our characterization of optimal monetary policy as a function of the economy's production network and the extent of nominal rigidities throughout the economy.

Recall from our discussion in Subsection 6.1 that the monetary authority faces a trade-off between minimizing the various forms of welfare losses in equation (44). Given the complexity of this expression, and in order to present our results in the most transparent manner, in what follows, we focus on three separate scenarios: in each case the monetary authority is concerned with minimizing the losses in one of the terms on the right-hand side of (44). The optimal policy can then be characterized as a trade-off between the three terms.[10]

Theorem 6. *Suppose the monetary authority can stabilize any nominal price index, by setting* $(\chi_1, \ldots, \chi_n)$ *such that* $\sum_{i=1}^{n} \chi_i \log p_i = 0$.

(a) The policy that minimizes output gap volatility is given by

$$\chi_i^{\text{og}} \propto \lambda_i \rho_i / (1 - \rho_i). \tag{51}$$

(b) The policy that minimizes within industry misallocation is given by

$$\chi_i^{\text{within}} \propto \lambda_i \theta_i \rho_i \kappa_i, \tag{52}$$

where κ_i *is the pass-through of monetary policy to industry* i.

(c) The policy that minimizes across industry misallocation is given by

$$\chi_i^{\text{across}} \propto \frac{\rho_i}{1 - \rho_i} \sum_{j=0}^{n} \lambda_j \left[\sum_{r=1}^{n} a_{jr} \ell_{ri} \nu_r - \left(\sum_{r=1}^{n} a_{jr} \ell_{ri} \right) \left(\sum_{r=1}^{n} a_{jr} \nu_r \right) \right], \tag{53}$$

where ν_j *is the degree of monetary non-neutrality of industry* j *and* $L = [\ell_{ij}] = (I - A)^{-1}$ *denotes the economy's Leontief inverse.*

Theorem 6 provides a full characterization of the optimal monetary policy in terms of model primitives. As a first observation, note that the policies that minimizes the three kinds of welfare losses — characterized in parts (a)–(c) of the theorem — are in general not identical. As a result, in line with Corollary 1, the monetary authority cannot implement the first best by setting all losses equal to zero, and instead, faces a trade-off. Next, note that Theorem 6 illustrates that the optimal policy depends not only on the extent of price rigidities (as captured by the stickiness parameters ρ) but also on the economy's production network. As a result, it underscores that the nature of optimal policy in our multi-sector framework can be quite distinct from that of baseline single-sector New Keynesian models. To clarify the dependence of optimal policy on model primitives, we next discuss each part of Theorem 6.

[10] See the appendix for a complete characterization of the optimal policy.

Statement (a) of Theorem 6 illustrates that the price stabilization policy of a monetary authority who is purely interested in minimizing output gap volatility targets industries that are (i) larger (higher Domar weight λ_i) and (ii) stickier (higher ρ_i).

Statement (b) of the theorem then provides a characterization of the policy that minimizes price dispersion within each industry. Not surprisingly, it establishes that the monetary authority targets industries that are larger, stickier, and have higher elasticities of substitution, as a higher elasticity translates the same level of price dispersion into larger misallocation. More interestingly, however, equation (52) illustrates that the weight assigned to industry i in the optimal policy is also increasing in the pass-through parameter κ_i defined in (49).

Finally, statement (c) of Theorem 6 characterizes the stabilization policy that minimizes welfare losses arising from price dispersion across various industries.

We conclude by characterizing the optimal policy for a simple special case:

Corollary 2. *Suppose $\rho_i = 0$ for all $i \neq j$. Then, the optimal price stabilization policy is given by $\chi_j = 1$ and $\chi_i = 0$ for all $i \neq j$. This policy implements the first-best allocation.*

The above corollary therefore extends the result of Aoki (2001) to a general multi-sector economy with input-output linkages: as long as there is only a single industry subject to nominal rigidities, the monetary authority can implement the first-best allocation by stabilizing the price of that industry.

7 Quantitative Illustration

We use a back of the envelope calculation to derive the optimal price level. Need to add.

8 Conclusion

We study optimal monetary policy when firms are arranged in a production network.

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Appendix: Proofs

In this Appendix, we first state and solve for the household and firms' problems.We then proceed with the proofs of the results that appear in the main text.
The Household. Consider first the household.

Household's Problem In any state s, the household chooses consumption and labor

$$\max u\left(\mathcal{C}\left(c_1\left(s\right),\ldots,c_N\left(s\right)\right)\right)-v\left(L\left(s\right)\right)$$

subject to its budget constraint

$$\sum_{i=1}^{N}p_i\left(s\right)c_i\left(s\right)\leq w\left(s\right)L\left(s\right)+\sum_{i=1}^{N}\left(\int\pi_{ik}dk+\pi_i\right)+T\left(s\right)$$

FOCs

$$\begin{aligned}u'\left(C\left(s\right)\right)\frac{d\mathcal{C}\left(s\right)}{dc_i\left(s\right)}-\lambda\left(s\right)p_i\left(s\right) &= 0\\ -v'\left(L\left(s\right)\right)+\lambda\left(s\right)w\left(s\right) &= 0\end{aligned}$$

The Sectoral Aggregator Firm. The sectoral aggregator firm solves

$$p_iy_i=\min\int p_{i,k}y_{i,k}dk$$

subject to

$$y_i=\left[\int y_{i,k}^{\frac{\theta_i-1}{\theta_i}}dk\right]^{\frac{\theta_i}{\theta_i-1}}$$

This implies demand functions

$$y_{ik}=\left(\frac{p_{ik}}{p_i}\right)^{-\theta_i}y_i$$

where the price index for sector i is given by

$$p_i=\left[\int p_{ik}^{1-\theta_i}dk\right]^{\frac{1}{1-\theta_i}}$$

Proof of Proposition 1. FOCs of the planner's problem are given by

$$\begin{aligned}u'\left(C\left(s\right)\right)\frac{d\mathcal{C}\left(s\right)}{dc_i\left(s\right)}-\mu_i\left(s\right) &= 0 \text{ for all } i\\ -v'\left(L\left(s\right)\right)+\mu_i\left(s\right)z_i\left(s\right)\frac{dF_i\left(s\right)}{d\ell_i\left(s\right)} &= 0 \text{ for all } i, \text{ and}\\ \mu_i\left(s\right)z_i\left(s\right)\frac{dF_i\left(s\right)}{dx_i\left(s\right)}\frac{dG_i\left(s\right)}{dx_{ij}\left(s\right)}-\mu_j\left(s\right) &= 0 \text{ for all } i,j\end{aligned}$$

Thus for any i

$$v'\left(L\left(s\right)\right)=u'\left(C\left(s\right)\right)\frac{d\mathcal{C}\left(s\right)}{dc_i\left(s\right)}z_i\left(s\right)\frac{dF_i\left(s\right)}{d\ell_i\left(s\right)}$$

And for any two goods i,j

$$\frac{d\mathcal{C}\left(s\right)}{dc_i\left(s\right)}z_i\left(s\right)\frac{dF_i\left(s\right)}{dx_i\left(s\right)}\frac{dG_i\left(s\right)}{dx_{ij}\left(s\right)}=\frac{d\mathcal{C}\left(s\right)}{dc_j\left(s\right)}$$

Proof of Proposition 2. *The Individual Monopolisticly-Competitive Firm.* The firm's problem can be split into a dual problem. The outer problem is the individual firm's profit maximization

$$\max\left(1-\tau_i\right)p_{ik}y_{i,k}-mc_{i,k}y_{i,k}$$

where $mc_{i,k}$ is the marginal cost of producing goods $y_{i,k}$, subject to the demand function (15).

The inner problem is a cost minimization problem

$$mc_{i,k}y_{i,k}=\min w\ell_{i,k}+\sum_{j=1}^{N}p_jx_{ij,k}$$

subject to the firm's production function

$$y_{i,k}=z_iF_i\left(\ell_{i,k},G_i\left(\{x_{ij,k}\}\right)\right)$$

We first solve the individual firm's cost minimization problem stated above. Due to fact that F and G are CRS (homogenous of degree 1) we can write this problem as follows

$$mc_{ik}=\min w\tilde{\ell}_{ik}+\sum_{j=1}^{N}p_j\tilde{x}_{ij,k}$$

subject to

$$1=z_iF_i\left(\tilde{\ell}_{ik},G_i\left(\{\tilde{x}_{ij,k}\}\right)\right)$$

where

$$\tilde{\ell}_{ik}=\ell_{ik}/y_{ik}\text{ and }\tilde{x}_{ij,k}=x_{ij,k}/y_{ik}$$

Next, given that this problem is identical for all firms within sector k, we may thereby drop the k subscript as follows.

$$mc_i=\min w\tilde{\ell}_i+\sum_{j=1}^{N}p_j\tilde{x}_{ij}$$

subject to

$$1=z_iF_i\left(\tilde{\ell}_i,G_i\left(\{\tilde{x}_{ij}\}\right)\right)$$

The FOCs are given by

$$\begin{aligned}w-\lambda_iz_i\frac{dF_i}{d\ell_i}&=0\\p_j-\lambda_iz_i\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}}&=0\quad\text{for all }j\end{aligned}$$

We thus have that

$$\frac{w}{p_j}=\frac{\frac{dF_i}{d\ell_i}}{\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}}}$$

and

$$\frac{p_j}{p_{j'}}=\frac{\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}}}{\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij'}}}$$

Plugging this back into the firm's cost function, we have

$$\begin{aligned} mc_i \frac{dF_i}{d\ell} &= w\frac{dF_i}{d\ell}\tilde{\ell}_i + \sum_{j=1}^{N} w\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}}\tilde{x}_{ij} \\ mc_i \frac{dF_i}{d\ell} &= w\left(\frac{dF_i}{d\ell}\tilde{\ell}_i + \sum_{j=1}^{N} \frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}}\tilde{x}_{ij}\right) \end{aligned}$$

Using the fact that F and G are homothetic, we then get

$$mc_i \frac{dF_i}{d\ell} = wF_i\left(\tilde{\ell}_i, G_i\left(\{\tilde{x}_{ij}\}\right)\right)$$

therefore

$$mc_i = \frac{w\left(s\right)}{z_i\left(s\right)\frac{dF_i}{d\ell_i}\left(s\right)}$$

Similarly, we can also get that

$$mc_i = \frac{p_j}{z_i\left(s\right)\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}}}$$

From now on we drop the subscript k on the firm's marginal cost.

Outer problem. Plugging in the demand function into (), we have that the firm solves

$$\max_{y_{i,k}} \left(1-\tau_i\right)\left(\frac{y_{i,k}}{y_i}\right)^{-1/\theta_i} p_i y_{i,k} - mc_i y_{i,k}$$

FOC with respect to output is given by

$$\begin{aligned} \left(1-\tau_i\right)\frac{\theta_i - 1}{\theta_i}\left(y_{ik}\right)^{-\frac{1}{\theta_i}} p_i \left(\frac{1}{y_i}\right)^{-1/\theta_i} &= mc_i \\ \left(1-\tau_i\right)\frac{\theta_i - 1}{\theta_i}\left(\frac{y_{ik}}{y_i}\right)^{-\frac{1}{\theta_i}} &= \frac{mc_i}{p_i} \end{aligned}$$

By symmetry, $y_{ik} = y_i$ for all k in sector i. Therefore

$$\frac{1}{p_i} = \left(1-\tau_i\right)\frac{\theta_i - 1}{\theta_i}\frac{1}{mc_i}$$

We now combine this with the household's optimality equations in (), we have

$$\begin{aligned} u'\left(C\right)\frac{d\mathcal{C}}{dc_i}\left(1-\tau_i\right)\frac{\theta_i - 1}{\theta_i}\frac{w}{mc_i} &= v'\left(L\right) \text{ for all } i \\ \frac{d\mathcal{C}/dc_i}{d\mathcal{C}/dc_j}\left(1-\tau_i\right)\frac{\theta_i - 1}{\theta_i}\frac{1}{mc_i} &= \frac{1}{p_j} \text{ for all } i,j \end{aligned}$$

Let

$$\chi_i = \left(1-\tau_i\right)\frac{\theta_i - 1}{\theta_i}$$

Then

$$\begin{aligned} u'\left(C\right)\frac{d\mathcal{C}}{dc_i}\chi_i\frac{w}{mc_i} &= v'\left(L\right) \text{ for all } i \\ \frac{d\mathcal{C}/dc_i}{d\mathcal{C}/dc_j}\chi_i\frac{1}{mc_i} &= \frac{1}{p_j} \text{ for all } i,j \end{aligned}$$

We combine these with equations

$$\begin{aligned} u'(C)\frac{d\mathcal{C}}{dc_i}\chi_i\frac{w}{mc_i} &= v'(L) \quad \text{for all } i \\ \frac{d\mathcal{C}/dc_i}{d\mathcal{C}/dc_j}\chi_i\frac{1}{mc_i} &= \frac{1}{p_j} \quad \text{for all } i,j \end{aligned}$$

to get

$$v'(L) = \chi_i u'(C)\frac{d\mathcal{C}}{dc_i}z_i\frac{dF_i}{d\ell} \quad \text{for all } i$$

and

$$\frac{d\mathcal{C}/dc_i}{d\mathcal{C}/dc_j}\chi_i z_i\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}} = 1 \quad \text{for all } i,j$$

or

$$\frac{d\mathcal{C}}{dc_i}\chi_i z_i\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}} = \frac{d\mathcal{C}}{dc_j} \quad \text{for all } i,j$$

Proof of Proposition 3.

Proof of Proposition 4. *The Individual Sticky Price Firm.* Sticky-price individual firms profit maximization

$$\max \mathbb{E}\left[\mathcal{M}(s)\left((1-\tau_i)p_{ik}(\omega)y_{ik}(\omega,s) - mc_i(\omega,s)y_{ik}(\omega,s)\right)\right]$$

where $\mathcal{M}$ is the household's nominal stochastic discount factor, subject to demand. Plugging in the demand function (15), this becomes

$$\max \mathbb{E}\left[\mathcal{M}(s)\left((1-\tau_i)p_{ik}(\omega)^{1-\theta_i}p_i(s)^{\theta_i}y_i(s) - mc_i(\omega,s)p_{ik}(\omega)^{-\theta_i}p_i(s)^{\theta_i}y_i(s)\right)\right]$$

The FOC with respect to the price is given by

$$\mathbb{E}\left[\mathcal{M}(s)y_i(s)\left((1-\tau_i)\left(\frac{\theta_i-1}{\theta_i}\right)p_{ik}(\omega)^{-\theta_i}p_i(s)^{\theta_i} - mc_i(\omega,s)p_{ik}(\omega)^{-\theta_i-1}p_i(s)^{\theta_i}\right)\right] = 0$$

The household's marginal value of wealth satisfies

$$U'(C(s))\frac{d\mathcal{C}(s)}{dc_i(s)} - \mathcal{M}(s)p_i(s) = 0$$

The household's nominal marginal value of wealth is given by

$$\mathcal{M}(s) = \frac{U'(\mathcal{C}(s))}{p_i(s)}\frac{d\mathcal{C}(s)}{dc_i(s)}$$

Plugging this in, we get

$$\mathbb{E}\left[U'(\mathcal{C}(s))\frac{d\mathcal{C}(s)}{dc_i(s)}y_i(s)\left((1-\tau_i)\left(\frac{\theta_i-1}{\theta_i}\right)p_{ik}(\omega)^{-\theta_i}p_i(s)^{\theta_i-1} - mc_i(\omega,s)p_{ik}(\omega)^{-\theta_i-1}p_i(s)^{\theta_i-1}\right)\middle|\,\omega\right] = 0$$

rewriting

$$\mathbb{E}\left[U'(\mathcal{C}(s))\frac{d\mathcal{C}(s)}{dc_i(s)}y_i(s)\left(\frac{p_{ik}(\omega)}{p_i(s)}\right)^{1-\theta_i}\left((1-\tau_i)\left(\frac{\theta_i-1}{\theta_i}\right) - \frac{mc_{ik}(\omega,s)}{p_{ik}(\omega)}\right)\middle|\,\omega\right] = 0$$

Using the demand function

$$\frac{y_{ik}}{y_i} = \left(\frac{p_{ik}}{p_i}\right)^{-\theta_i}$$

we get

$$\mathbb{E}\left[U'\left(\mathcal{C}\left(s\right)\right)\frac{d\mathcal{C}\left(s\right)}{dc_i\left(s\right)}y_i\left(s\right)\left(\frac{y_{ik}\left(\omega,s\right)}{y_i\left(s\right)}\right)^{\frac{\theta_i-1}{\theta_i}}\left(\left(1-\tau_i\right)\left(\frac{\theta_i-1}{\theta_i}\right)-\frac{mc_i\left(\omega,s\right)}{p_{ik}\left(\omega\right)}\right)\middle|\,\omega\right]=0$$

Rewriting,

$$\mathbb{E}\left[U'\left(\mathcal{C}\left(s\right)\right)\frac{d\mathcal{C}\left(s\right)}{dc_i\left(s\right)}y_i\left(s\right)\left(\frac{y_{ik}\left(\omega,s\right)}{y_i\left(s\right)}\right)^{\frac{\theta_i-1}{\theta_i}}\left(p_{ik}\left(\omega\right)-\left[\left(1-\tau_i\right)\left(\frac{\theta_i-1}{\theta_i}\right)\right]^{-1}mc_i\left(\omega,s\right)\right)\middle|\,\omega\right]=0$$

As a result, sticky price firms sets its price as follows

$$p_{ik}\left(\omega\right)=\left[\left(1-\tau_i\right)\left(\frac{\theta_i-1}{\theta_i}\right)\right]^{-1}\frac{\mathbb{E}\left[v_i\left(\omega,s\right)mc_{ik}\left(\omega,s\right)|\,\omega\right]}{\mathbb{E}\left[v_i\left(\omega,s\right)|\,\omega\right]}$$

where the stochastic discount factor of the firm is given by

$$v_i\left(\omega,s\right)\equiv U'\left(\mathcal{C}\left(s\right)\right)\frac{d\mathcal{C}\left(s\right)}{dc_i\left(s\right)}y_i\left(s\right)\left(\frac{y_{ik}\left(\omega,s\right)}{y_i\left(s\right)}\right)^{\frac{\theta_i-1}{\theta_i}}$$

Sticky price firm

$$p_{ik}\left(\omega\right)=\left[\left(1-\tau_i\right)\left(\frac{\theta_i-1}{\theta_i}\right)\right]^{-1}\frac{\mathbb{E}\left[v_i\left(\omega,s\right)mc_{ik}\left(\omega,s\right)|\,\omega\right]}{mc_{ik}\left(\omega,s\right)\mathbb{E}\left[v_i\left(\omega,s\right)|\,\omega\right]}mc_{ik}\left(\omega,s\right)$$

where $mc_{ik}\left(\omega,s\right)$ is the realized nominal marginal cost of the firm. Therefore we may write the sticky price firm in terms of its realized pricing error

$$p_{ik}\left(\omega\right)=\left[\left(1-\tau_i\right)\left(\frac{\theta_i-1}{\theta_i}\right)\right]^{-1}\varepsilon_{ik}\left(\omega,s\right)^{-1}mc_{ik}\left(\omega,s\right)$$

where $\varepsilon_i\left(\omega,s\right)$ is the firm's pricing error is given by

$$\varepsilon_{ik}\left(\omega,s\right)^{-1}\equiv\frac{\mathbb{E}\left[v_i\left(\omega,s\right)mc_{ik}\left(\omega,s\right)|\,\omega\right]}{mc_{ik}\left(\omega,s\right)\mathbb{E}\left[v_i\left(\omega,s\right)|\,\omega\right]}$$

is the pricing error made by the sticky price firm.

$$\varepsilon_{ik}\left(\omega,s\right)=\left[\left(1-\tau_i\right)\left(\frac{\theta_i-1}{\theta_i}\right)\right]^{-1}\frac{mc_{ik}\left(\omega,s\right)}{p_{ik}\left(\omega\right)}$$

Taking again equation (), this implies

$$\mathbb{E}\left[v_i\left(\omega,s\right)\left(\varepsilon_{ik}\left(\omega,s\right)-1\right)|\,\omega\right]=0$$

error must be equal 1 in expectation.

Recall that

$$mc_{ik}\left(\omega,s\right)=\frac{W\left(s\right)}{z_i\left(s\right)\frac{dF_i}{d\ell_i}\left(\omega,s\right)}$$

and

$$mc_{ik}(\omega, s) = \frac{p_j(s)}{z_i(s)\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}}(\omega, s)}$$

Thus

$$p_{ik}(\omega) = \chi_i^{-1}\varepsilon_{ik}(\omega, s)^{-1}\frac{W(s)}{z_i(s)\frac{dF_i}{d\ell_i}(\omega, s)}$$

$$p_{ik}(\omega) = \chi_i^{-1}\varepsilon_{ik}(\omega, s)^{-1}\frac{p_j(s)}{z_i(s)\frac{dF_i}{dx_i}\frac{dG_i}{dx_{ij}}(\omega, s)}$$

Work with the first equation in (). From the household' problem, the wage satisfies

$$W(s) = \frac{V'(L(s))}{U'(\mathcal{C}(s))\frac{d\mathcal{C}(s)}{dc_i(s)}}p_i(s)$$

Thus

$$p_{ik}(\omega) = \chi_i^{-1}\varepsilon_{ik}(\omega, s)^{-1}\frac{V'(L(s))}{U'(\mathcal{C}(s))\frac{d\mathcal{C}(s)}{dc_i(s)}z_i(s)\frac{dF_i}{d\ell_i}(\omega, s)}p_i(s)$$

$$V'(L(s)) = \chi_i\varepsilon_{ik}(\omega, s)U'(\mathcal{C}(s))\frac{d\mathcal{C}(s)}{dc_i(s)}\frac{p_{ik}(\omega)}{p_i(s)}z_i(s)\frac{dF_i}{d\ell_i}(\omega, s)$$

$$\left(\frac{y_{ik}}{y_i}\right)^{-1/\theta_i} = \left(\frac{p_{ik}}{p_i}\right)$$

$$V'(L(s)) = \chi_i\varepsilon_{ik}(\omega, s)U'(\mathcal{C}(s))\frac{d\mathcal{C}(s)}{dc_i(s)}\left(\frac{y_{ik}(\omega, s)}{y_i(s)}\right)^{-1/\theta_i}z_i(s)\frac{dF_i}{d\ell_i}(\omega, s)$$

Work with the second equation in (). Frow the household' problem, the price ratio satisfies

$$\frac{d\mathcal{C}/dc_i}{d\mathcal{C}/dc_j} = \frac{p_i(s)}{p_j(s)} \quad \text{for all } i, j$$

Thus

$$p_{ik}(\omega) = \chi_i^{-1}\varepsilon_{ik}(\omega, s)^{-1}\frac{p_i(s)}{\frac{d\mathcal{C}/dc_i}{d\mathcal{C}/dc_j}z_i(s)\frac{dF_{ik}}{dx_{ik}}\frac{dG_{ik}}{dx_{ij,k}}(\omega, s)}$$

Therefore

$$\frac{d\mathcal{C}(s)}{dc_j(s)} = \chi_i\varepsilon_{ik}(\omega, s)\frac{d\mathcal{C}(s)}{dc_i(s)}\frac{p_{ik}(\omega)}{p_i(s)}z_i(s)\frac{dF_{ik}}{dx_{ik}}\frac{dG_{ik}}{dx_{ij,k}}(\omega, s)$$

which gives us

$$\frac{d\mathcal{C}(s)}{dc_j(s)} = \chi_i\varepsilon_{ik}(\omega, s)\frac{d\mathcal{C}(s)}{dc_i(s)}\left(\frac{y_{ik}(\omega, s)}{y_i(s)}\right)^{-1/\theta_i}z_i(s)\frac{dF_{ik}}{dx_{ik}}\frac{dG_{ik}}{dx_{ij,k}}(\omega, s)$$

QED.

Proof of Theorem 1. *Proof.* We prove Theorem 1 by contradiction. Suppose there exists some allocation, $\hat{\xi}$ such that

$$\hat{\xi} \in \mathcal{X}^f \cap \mathcal{X}^s.$$

Because this allocation is implementable as an equilibrium under both flexible and sticky prices, it must simultaneously satisfy equations (20)-(21) and (27)-(28). In order to satisfy both sets of equations, it is necessary that $\eta_i(\omega, s)\chi_i^s = \chi_i^f$ for all i, ω, s. Because the fiscal policy is assumed to be invariant to the state (ω, s), it must be the case that $\eta_i(\omega, s)$ be equal to some constant for all i, ω, s.

This implies $\eta_i(\omega, s) = 1$ for all i, ω, s. This is without loss of generality as one can always change the constant level with χ_i^s. From condition (**??**), it is thus necessary that $\varepsilon_i(\omega, s) = 1$ for all i, ω, s.

This implies that there exists some function $\phi : \mathcal{S} \to \mathbb{R}_+$ such that it induces a set of error functions $\varepsilon_1, \ldots, \varepsilon_N : \Omega \times \mathcal{S} \to \mathbb{R}_+$ according to (29) such that

$$\varepsilon_i(\omega, s) = 1 \text{ for all } i, \omega, s$$

That is, there must exist some monetary policy ϕ such that no pricing errors are made.

We now show that this cannot be possible in a generic multi-sector economy. Consider the possible pricing errors made by sticky price firms in sector 1. Equation (29) implies that

$$\varepsilon_1(\omega, s) = 1 \text{ for all } \omega, s \quad \text{if and only if} \quad \phi(s)\left(z_1(s)\frac{dF_1(s)}{d\ell_1(s)}\right)^{-1} = A, \forall s$$

where A is a constant. Therefore, let us set monetary policy as follows

$$\phi(s) = Az_1(s)\frac{dF_1(s)}{d\ell_1(s)}$$

to ensure $\varepsilon_1(\omega, s) = 1$ for all realizations ω, s.

But now that monetary policy is set to eliminate all pricing errors for firms in sector 1, consider the possible pricing errors made by sticky price firms in any other sector $j \neq 1$. For any sticky-price firm in sector j, we have that

$$\phi(s)\left(z_j(s)\frac{dF_j(s)}{d\ell_j(s)}\right)^{-1} = Az_1(s)\frac{dF_1(s)}{d\ell_1(s)}\left(z_j(s)\frac{dF_j(s)}{d\ell_j(s)}\right)^{-1}$$

But note that this quantity is no more a constant–it varies with the aggregate state s. As a result, it cannot be the case that in sector $j \neq 1$, $\varepsilon_j(\omega, s) = 1$ for all ω, s. That is, sticky-price firms in other sectors will make pricing errors. Therefore, in a multi-sector economy, no such function $\phi(s)$ exists such that it induces $\varepsilon_i(\omega, s) = 1$ for all i, ω, s according to (29). Therefore, the allocation$\hat{\xi}$ does not exist and$\mathcal{X}^f \cap \mathcal{X}^s = \varnothing$ **QED.**

Special Case: The Single Sector Model with all sticky price firms

Suppose there is only one sector, so that we may drop the i subscript. .

Proposition 6. *A feasible allocation, $\xi \in \mathcal{X}$, is part of a sticky-price equilibrium if and only if there exists a scalar $\chi^s \in \mathbb{R}_+$ and a function $\phi : \mathcal{S} \to \mathbb{R}_+$, such that the following conditions hold:*

(i) The function $\phi : \mathcal{S} \to \mathbb{R}_+$ induces a wedge function $\eta : \Omega \times \mathcal{S} \to \mathbb{R}_+$ given by

$$\epsilon(\omega, s) = \frac{\mathbb{E}\left[v(\omega, s)\, \phi(s) \left(z(s) \frac{dF(s)}{d\ell(s)} \right)^{-1} \middle| \omega \right]}{\mathbb{E}\left[v(\omega, s) | \omega \right] \phi(s) \left(z(s) \frac{dF(s)}{d\ell(s)} \right)^{-1}} \tag{54}$$

with $v(\omega, s)$ defined in (24).

(ii) The allocation, the scalar χ^s, and the wedge function η, jointly satisfy

$$V'(L(s)) = \epsilon(\omega, s)\, \chi^s U'(C(s))\, z(s) \frac{dF(s)}{d\ell(s)} \text{ for all } \omega, s \tag{55}$$

Proof of Proposition 6. Need to add. **QED.**

Proofs for Ramsey Problem.

Proof of Lemma 1. Let output within a sector be log-normally distributed. Then

$$\left[\int y_{ik}(\omega_{ik}, s)^{\frac{\theta-1}{\theta}} dk \right] = \exp\left\{ \left(\frac{\theta - 1}{\theta} \right) \mathbb{E} \log y_{ik}(\omega_{ik}, s) + \frac{1}{2} \left(\frac{\theta - 1}{\theta} \right)^2 Var_\omega \left[\log y_{ik}(\omega_{ik}, s) | \Omega, s \right] \right\}$$

$$\int y_{ik}(\omega_{ik}, s)^{\frac{\theta-1}{\theta}} dk = \left(\mathbb{E} y_{ik}(\omega_{ik}, s) \right)^{\frac{\theta-1}{\theta}} \exp\left(\frac{1}{2} \frac{\theta - 1}{\theta} \left(-\frac{1}{\theta} \right) Var_\omega \left[\log y(\omega, s) | \Omega, s \right] \right)$$

Therefore

$$y_i(\Omega, s) = \left[\int y_{ik}(\omega_{ik}, s)^{\frac{\theta-1}{\theta}} dk \right]^{\frac{\theta}{\theta-1}} = \mathbb{E} y_{ik}(\omega_{ik}, s) \exp\left(-\frac{1}{2} \frac{1}{\theta} Var_\omega \left[\log y(\omega, s) | \Omega, s \right] \right)$$

Let

$$y_i(\omega, s) \equiv \mathbb{E} y_{ik}(\omega_{ik}, s)$$

And as a result

$$y_i(\Omega, s) = \vartheta_i y_i(\omega, s)$$

where

$$\vartheta_i \equiv \exp\left(-\frac{1}{2} \frac{1}{\theta_i} Var_\omega \left[\log y(\omega, s) | \Omega, s \right] \right)$$

$$y_{ik}(\omega, s) = \left(\frac{p_{ik}(\omega_{ik})}{p_i(\omega)} \right)^{-\theta_i} y_i(\omega, s)$$

Thus, we may write

$$\log p_{ik}(\omega_{ik}) = \log p_i(\omega_i) + \sigma_i \zeta_{ik}$$

where

$$y_{ik}(\omega, s) = \left(\frac{p_{ik}(\omega_{ik})}{p_i(\omega)} \right)^{-\theta_i} y_i(\omega, s)$$

Thus

$$\log y(\omega, s) - \log y_i(\omega, s) = -\theta_i \left(\log p_{ik}(\omega_{ik}) - \log p_i(\omega) \right)$$

Therefore, cross-sectional variance in output is given by

$$Var_{\omega}\left[\log y\left(\omega,s\right)-\log Y\left(s\right)|s\right]=\theta_{i}^{2}Var\left[\log p\left(\omega\right)-\log P\left(s\right)|s\right]$$

Thus

$$\vartheta_{i}=\exp\left(-\frac{1}{2}\frac{1}{\theta}Var_{\omega}\left[\log y\left(\omega,s\right)|\Omega,s\right]\right)$$

as in (). **QED.**

Proof of Proposition 5. Need to add.

Proof of Theorem 2. Need to add.

Lemma 4. *Suppose utility of consumption and labor is homothetic and given by (42). Then, equilibrium aggregate labor and consumption (GDP) are given by the following functions, respectively:*

$$\log L\left(s\right) = \frac{1-\gamma}{1/\eta+\gamma}\left[\log A\left(\mathbf{z}\left(s\right)\right)+\log A\left(\vartheta\right)+\log TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)\right]+\frac{1}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right) \tag{56}$$

$$\log C\left(s\right) = \frac{1+1/\eta}{1/\eta+\gamma}\left[\log A\left(\mathbf{z}\left(s\right)\right)+\log A\left(\vartheta\right)+\log TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)\right]+\frac{1}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right). \tag{57}$$

Lemma 4 thus provides explicit formulas for equilibrium output and labor in this economy.

Proof of Lemma 4. With the homothetic preferences stated in (42), we have the following system of two equations and two unknowns:

$$L\left(s\right)^{1/\eta}=\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)C\left(s\right)^{-\gamma}\frac{C\left(s\right)}{L\left(s\right)} \tag{58}$$

$$C\left(s\right)=A\left(\mathbf{z}\left(s\right)\right)TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)L\left(s\right) \tag{59}$$

Substituting expression (59) for consumption into (58), we get

$$L\left(s\right)^{1+1/\eta}=\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\left(A\left(\mathbf{z}\left(s\right)\right)TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)L\left(s\right)\right)^{1-\gamma}$$

Solving this expression for $L\left(s\right)$ we get

$$L\left(s\right)^{1/\eta+\gamma}=\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\left(A\left(\mathbf{z}\left(s\right)\right)TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)^{1-\gamma},$$

and taking logs of both sides gives us equilibrium aggregate labor

$$\log L\left(s\right)=\frac{1-\gamma}{1/\eta+\gamma}\log\left(A\left(\mathbf{z}\left(s\right)\right)TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)+\frac{1}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)$$

as in (56). Finally, substituting this back into the aggregate production function (59) yields the following expression for equilibrium aggregate consumption

$$\log C\left(s\right)=\frac{1+1/\eta}{1/\eta+\gamma}\log\left(A\left(\mathbf{z}\left(s\right)\right)TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)+\frac{1}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)$$

as in (57). **QED.**

Next we plug this in to obtain equilibrium welfare.

Proposition 7. *Suppose utility of consumption and labor is homothetic and given by (42). Given any realization of the state, realized welfare is given by*

$$\mathcal{W}(s) = \mathcal{W}^*(s) \frac{(1-\gamma)(1+\eta)}{1+\eta\gamma} \left(A(\vartheta) TFP(\boldsymbol{\epsilon}(s))\right)^{\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}} \left[\frac{1}{1-\gamma}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1-\gamma}{1/\eta+\gamma}} - \frac{1}{1+1/\eta}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1+1/\eta}{1/\eta+\gamma}}\right] \tag{60}$$

where

$$\mathcal{W}^*(s) = \frac{1+\eta\gamma}{(1-\gamma)(1+\eta)} A(\mathbf{z}(s))^{\frac{(1-\gamma)(1+\eta)}{1+\eta\gamma}}$$

is first best welfare.

Consider the special case in which $\eta \to 0$ so that labor is infinitely inelastic. In this case,

$$\mathcal{W}(s) = \mathcal{W}^*(s) TFP(\boldsymbol{\epsilon}(s))^{(1-\gamma)}$$

so that there is no Labor Wedge.

Proof of Proposition 7. Plugging our expressions for consumption and labor, () and (), into welfare

$$\begin{aligned}\mathcal{W}(s) =& \frac{1}{1-\gamma}\left(A(\mathbf{z}(s))A(\vartheta)TFP(\boldsymbol{\epsilon}(s))\right)^{\left(\frac{1+1/\eta}{1/\eta+\gamma}\right)(1-\gamma)}\Lambda(\boldsymbol{\epsilon}(s))^{\left(\frac{1}{1/\eta+\gamma}\right)(1-\gamma)} \\ & - \frac{1}{1+1/\eta}\left(A(\mathbf{z}(s))A(\vartheta)TFP(\boldsymbol{\epsilon}(s))\right)^{\left(\frac{1-\gamma}{1/\eta+\gamma}\right)(1+1/\eta)}\Lambda(\boldsymbol{\epsilon}(s))^{\left(\frac{1}{1/\eta+\gamma}\right)(1+1/\eta)}\end{aligned}$$

First best welfare

$$\mathcal{W}^*(s) = \frac{1}{1-\gamma}A(\mathbf{z}(s))^{\frac{(1+1/\eta)(1-\gamma)}{1/\eta+\gamma}} - \frac{1}{1+1/\eta}A(\mathbf{z}(s))^{\frac{(1+1/\eta)(1-\gamma)}{1/\eta+\gamma}}$$

Thus

$$\mathcal{W}^*(s) = \frac{1+\eta\gamma}{(1-\gamma)(1+\eta)}A(\mathbf{z}(s))^{\frac{(1+1/\eta)(1-\gamma)}{1/\eta+\gamma}}$$

Plugging this in, we have that

$$\mathcal{W}(s) = \mathcal{W}^*(s) \frac{(1-\gamma)(1+\eta)}{1+\eta\gamma} \left(A(\vartheta) TFP(\boldsymbol{\epsilon}(s))\right)^{\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}} \left[\frac{1}{1-\gamma}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1-\gamma}{1/\eta+\gamma}} - \frac{1}{1+1/\eta}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1+1/\eta}{1/\eta+\gamma}}\right]$$

as in (). **QED.**

Note that welfare can equivalently be written as

$$\mathcal{W}(s) = \mathcal{W}^*(s)\left(A(\vartheta)TFP(\boldsymbol{\epsilon}(s))\right)^{\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}}\left[\frac{1+\eta}{1+\eta\gamma}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{\eta(1-\gamma)}{1+\eta\gamma}} - \frac{\eta(1-\gamma)}{1+\eta\gamma}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1+\eta}{1+\eta\gamma}}\right]$$

or

$$\mathcal{W}(s) = \mathcal{W}^*(s)\left(A(\vartheta)TFP(\boldsymbol{\epsilon}(s))\right)^{\frac{(1-\gamma)(1+\eta)}{1+\eta\gamma}}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{\eta(1-\gamma)}{1+\eta\gamma}}\left[\frac{1+\eta}{1+\eta\gamma} - \frac{\eta(1-\gamma)}{1+\eta\gamma}\Lambda(\boldsymbol{\epsilon}(s))\right]$$

or

$$\mathcal{W}(s) = \mathcal{W}^*(s)\frac{1+\eta}{1+\eta\gamma}TFP(\boldsymbol{\epsilon}(s))^{\frac{(1-\gamma)(1+\eta)}{1+\eta\gamma}}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{\eta(1-\gamma)}{1+\eta\gamma}}\left[1 - \frac{\eta(1-\gamma)}{1+\eta}\Lambda(\boldsymbol{\epsilon}(s))\right]$$

Note: This seems correct so far. Translation to Alireza's notes:

$$\begin{aligned}TFP(\boldsymbol{\epsilon}(s)) &= y/y^* \\ A(\boldsymbol{\epsilon}(s)) &= (p/p^*)^{-1} \\ \Lambda(\boldsymbol{\epsilon}(s)) &= 1 - \sum \lambda_i (1-Q_i)\end{aligned}$$

We now take a second order approximation of welfare in () and get the following result.

Lemma 5. *Suppose utility of consumption and labor is homothetic. Then, to a second order approximation around the efficient benchmark, realized welfare satisfies*

$$\log \mathcal{W}(s) \approx \log \mathcal{W}^*(s) + \frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}(\log TFP(\boldsymbol{\epsilon}(s)) + \log A(\vartheta)) - \frac{1}{2}\frac{(1-\gamma)(1+1/\eta)}{(1/\eta+\gamma)^2}(\log \Lambda(\boldsymbol{\epsilon}(s)))^2$$

where

$$\mathcal{W}^*(s) = \frac{1+\eta\gamma}{(1-\gamma)(1+\eta)} A(\mathbf{z}(s))^{\frac{(1-\gamma)(1+\eta)}{1+\eta\gamma}}$$

is relaized first best welfare.

Consider the special case in which $\eta \to 0$ so that labor is infinitely inelastic. In this case,

$$\log \mathcal{W}(s) \approx \log \mathcal{W}^*(s) + (1-\gamma)(\log TFP(\boldsymbol{\epsilon}(s)) + \log A(\vartheta))$$

so that there is no Labor Wedge.

Proof of Lemma 5. We have that

$$\mathcal{W}(s) = \mathcal{W}^*(s)\frac{(1-\gamma)(1+\eta)}{1+\eta\gamma}(A(\vartheta)TFP(\boldsymbol{\epsilon}(s)))^{\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}}\left[\frac{1}{1-\gamma}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1-\gamma}{1/\eta+\gamma}} - \frac{1}{1+1/\eta}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1+1/\eta}{1/\eta+\gamma}}\right]$$

Let

$$H(\boldsymbol{\epsilon}(s)) = \frac{1}{1-\gamma}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1-\gamma}{1/\eta+\gamma}} - \frac{1}{1+1/\eta}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1+1/\eta}{1/\eta+\gamma}}$$

$$\mathcal{W}(s) = \mathcal{W}^*(s)\frac{(1-\gamma)(1+\eta)}{1+\eta\gamma}TFP(\boldsymbol{\epsilon}(s))^{\frac{(1+1/\eta)(1-\gamma)}{1/\eta+\gamma}}H(\boldsymbol{\epsilon}(s))$$

Then

$$\log \mathcal{W}(s) = \log \mathcal{W}^*(s) + \frac{(1-\gamma)(1+\eta)}{1+\eta\gamma}\log TFP(\boldsymbol{\epsilon}(s)) + \log H(\boldsymbol{\epsilon}(s)) + \log\left(\frac{(1-\gamma)(1+\eta)}{1+\eta\gamma}\right)$$

where

$$\log H(\boldsymbol{\epsilon}(s)) = \log\left[\frac{1}{1-\gamma}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1-\gamma}{1/\eta+\gamma}} - \frac{1}{1+1/\eta}\Lambda(\boldsymbol{\epsilon}(s))^{\frac{1+1/\eta}{1/\eta+\gamma}}\right] = \log\left[\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right) - \frac{1}{1+1/\eta}\epsilon\right.$$

Log approximation

$$\log H(\boldsymbol{\epsilon}(s)) \approx \log H(\Lambda(1)) + \left.\frac{d\log H(\boldsymbol{\epsilon}(s))}{d\log\Lambda(\boldsymbol{\epsilon}(s))}\right|_{\Lambda(1)}(\log\Lambda(\boldsymbol{\epsilon}(s)) - \log\Lambda(1)) + \frac{1}{2}\left.\frac{d^2\log H(\boldsymbol{\epsilon}(s))}{d\log\Lambda(\boldsymbol{\epsilon}(s))^2}\right|_{\Lambda(1)}(\log\Lambda(\boldsymbol{\epsilon}(s)) - \log\Lambda$$

where $\log\Lambda(1) = 0$ and

$$\log H(1) = \log\left(\frac{1}{1-\gamma} - \frac{1}{1+1/\eta}\right) = \log\left(\frac{1+\eta\gamma}{(1-\gamma)(1+\eta)}\right)$$

$$\begin{aligned}
\frac{d\log H(\boldsymbol{\epsilon}(s))}{d\log\Lambda(\boldsymbol{\epsilon}(s))} &= \frac{d\log\left(\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right) - \frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right)\right)}{d\log\Lambda(\boldsymbol{\epsilon}(s))} \\
&= \frac{1}{\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right) - \frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right)}\left[\frac{1}{1-\gamma}\frac{1-\gamma}{1/\eta+\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}\right.\right. \\
&= \frac{\frac{1}{1/\eta+\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right) - \frac{1}{1/\eta+\gamma}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right)}{\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right) - \frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda(\boldsymbol{\epsilon}(s))\right)}
\end{aligned}$$

$$\left.\frac{d\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}\right|_{\epsilon=1}=\frac{\frac{1}{1/\eta+\gamma}-\frac{1}{1/\eta+\gamma}}{\frac{1}{1-\gamma}-\frac{1}{1+1/\eta}}=0$$

Thus

$$\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)\approx\log H\left(\Lambda\left(1\right)\right)+\frac{1}{2}\left.\frac{d^{2}\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{2}}\right|_{\Lambda(1)}\left(\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)^{2}$$

$$\frac{d^{2}\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{2}}=\frac{d}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}\left\{\frac{d\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}\right\}=\frac{d}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}\left\{\frac{\frac{1}{1/\eta+\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1/\eta+\gamma}\exp\left(\frac{1+1/}{1/\eta+}\right.\right.}{\left.\left.\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+}\right.\right.}$$

$$=\frac{\left(\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)\right)\frac{d}{d\log\Lambda(\boldsymbol{\epsilon}(s))}\left[\frac{1}{1/\eta+\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\right.\right.}{\left.\left.\left(\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right.\right.\right.}$$

$$-\frac{\left(\frac{1}{1/\eta+\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1/\eta+\gamma}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)\right)\frac{d}{d\log\Lambda(\boldsymbol{\epsilon}(s))}\left[\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\right.\right.}{\left.\left.\left(\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\right.\right.\right.\right.}$$

$$=\frac{\left(\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)\right)\left[\frac{1}{1/\eta+\gamma}\frac{1-\gamma}{1/\eta+\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1/\eta+\gamma}\frac{1+1/}{1/\eta+}\right.}{\left.\left(\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)\right)^{2}\right.}$$

$$-\frac{\left(\frac{1}{1/\eta+\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1/\eta+\gamma}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)\right)\left[\frac{1}{1-\gamma}\frac{1-\gamma}{1/\eta+\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1+1/\eta}\frac{1+1/}{1/\eta+}\right.}{\left.\left(\frac{1}{1-\gamma}\exp\left(\frac{1-\gamma}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)-\frac{1}{1+1/\eta}\exp\left(\frac{1+1/\eta}{1/\eta+\gamma}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)\right)^{2}\right.}$$

$$\left.\frac{d^{2}\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{2}}\right|_{\Lambda(1)}=\frac{\left(\frac{1}{1-\gamma}-\frac{1}{1+1/\eta}\right)\left[\frac{1}{1/\eta+\gamma}\frac{1-\gamma}{1/\eta+\gamma}-\frac{1}{1/\eta+\gamma}\frac{1+1/\eta}{1/\eta+\gamma}\right]-\left(\frac{1}{1/\eta+\gamma}-\frac{1}{1/\eta+\gamma}\right)\left[\frac{1}{1-\gamma}\frac{1-\gamma}{1/\eta+\gamma}-\frac{1}{1+1/\eta}\frac{1+1/\eta}{1/\eta+\gamma}\right]}{\left(\frac{1}{1-\gamma}-\frac{1}{1+1/\eta}\right)^{2}}$$

$$\left.\frac{d^{2}\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{2}}\right|_{\Lambda(1)}=\frac{\left[\frac{1}{1/\eta+\gamma}\frac{1-\gamma}{1/\eta+\gamma}-\frac{1}{1/\eta+\gamma}\frac{1+1/\eta}{1/\eta+\gamma}\right]}{\left(\frac{1}{1-\gamma}-\frac{1}{1+1/\eta}\right)}=\frac{1}{1/\eta+\gamma}\frac{\left[\frac{1-\gamma-(1+1/\eta)}{1/\eta+\gamma}\right]}{\left(\frac{1+1/\eta-(1-\gamma)}{(1-\gamma)(1+1/\eta)}\right)}$$

$$\left.\frac{d^{2}\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{2}}\right|_{\Lambda(1)}=\frac{1}{1/\eta+\gamma}\frac{\left[\frac{-\gamma-1/\eta}{1/\eta+\gamma}\right]}{\left(\frac{1/\eta+\gamma}{(1-\gamma)(1+1/\eta)}\right)}=-\frac{1}{1/\eta+\gamma}\frac{\left[\frac{1/\eta+\gamma}{1/\eta+\gamma}\right]}{\left(\frac{1/\eta+\gamma}{(1-\gamma)(1+1/\eta)}\right)}=-\frac{1}{1/\eta+\gamma}\frac{1}{\left(\frac{1/\eta+\gamma}{(1-\gamma)(1+1/\eta)}\right)}$$

$$\left.\frac{d^{2}\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{2}}\right|_{\Lambda(1)}=-\frac{\left(1-\gamma\right)\left(1+1/\eta\right)}{\left(1/\eta+\gamma\right)^{2}}$$

$$\log H\left(\boldsymbol{\epsilon}\left(s\right)\right)\approx\log H\left(\Lambda\left(1\right)\right)-\frac{1}{2}\frac{\left(1-\gamma\right)\left(1+1/\eta\right)}{\left(1/\eta+\gamma\right)^{2}}\left(\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)^{2}$$

$$\log\mathcal{W}\left(s\right)\approx\log\mathcal{W}^{*}\left(s\right)+\frac{\left(1-\gamma\right)\left(1+1/\eta\right)}{1/\eta+\gamma}\log TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)-\frac{1}{2}\frac{\left(1-\gamma\right)\left(1+1/\eta\right)}{\left(1/\eta+\gamma\right)^{2}}\left(\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)^{2}$$

as in (). **QED.**

Proof of Lemma 2. We have

$$\log \mathcal{W}(s) \approx \log \mathcal{W}^*(s) + \frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma} \log TFP(\boldsymbol{\epsilon}(s)) - \frac{1}{2}\frac{(1-\gamma)(1+1/\eta)}{(1/\eta+\gamma)^2} (\log \Lambda(\boldsymbol{\epsilon}(s)))^2$$

We want

$$\mathcal{W} = \mathbb{E}\mathcal{W}(s) = \exp\left(\mathbb{E}\log \mathcal{W}(s) + \frac{1}{2} Var \log \mathcal{W}(s)\right)$$

where

$$\mathbb{E}\log \mathcal{W}(s) = \mathbb{E}\log \mathcal{W}^*(s) + \frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\mathbb{E}\log TFP(\boldsymbol{\epsilon}(s)) - \frac{1}{2}\frac{(1-\gamma)(1+1/\eta)}{(1/\eta+\gamma)^2}\mathbb{E}(\log \Lambda(\boldsymbol{\epsilon}(s)))^2$$

and

$$Var \log \mathcal{W}(s) = Var \log \mathcal{W}^*(s) + \left(\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\right)^2 Var \log TFP(\boldsymbol{\epsilon}(s)) + 2\left(\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\right) Cov(\log \mathcal{W}^*(s), l$$

$$\mathcal{W} = \exp\left(\mathbb{E}\log \mathcal{W}^*(s) + \frac{1}{2} Var \log \mathcal{W}^*(s)\right) \exp\left(\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\mathbb{E}\log TFP(\boldsymbol{\epsilon}(s))\right)$$
$$\exp\left(-\frac{1}{2}\frac{(1-\gamma)(1+1/\eta)}{(1/\eta+\gamma)^2}\mathbb{E}(\log \Lambda(\boldsymbol{\epsilon}(s)))^2 + \frac{1}{2}\left(\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\right)^2 Var \log TFP(\boldsymbol{\epsilon}(s)) + \left(\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\right.\right.$$

Let

$$\mathcal{W}^*(s) = \mathbb{E}\mathcal{W}^*(s) = \exp\left(\mathbb{E}\log \mathcal{W}^*(s) + \frac{1}{2} Var \log \mathcal{W}^*(s)\right)$$

Welfare is thus given by

$$\mathcal{W} \approx \mathcal{W}^* \exp\left(\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\mathbb{E}\log TFP(\boldsymbol{\epsilon}(s)) + \frac{1}{2}\left(\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\right)^2 Var \log TFP(\boldsymbol{\epsilon}(s))\right)$$
$$\exp\left(-\frac{1}{2}\frac{(1-\gamma)(1+1/\eta)}{(1/\eta+\gamma)^2}\mathbb{E}\left[(\log \Lambda(\boldsymbol{\epsilon}(s)))^2\right] + \left(\frac{(1-\gamma)(1+1/\eta)}{1/\eta+\gamma}\right) Cov(\log \mathcal{W}^*(s), \log TFP(\boldsymbol{\epsilon}(s)))\right)$$

where $\mathcal{W}^* = \mathbb{E}\mathcal{W}^*(s)$. as in (). **QED.**

Proof of proposition.

It will turn out that in fact the TFP term only has higher order terms. Thus the Var and Cov terms disappear. As a result, we get the following

Proof of Theorem 3. First, we have that log tfp term is given by

$$\log TFP(\boldsymbol{\epsilon}(s)) = \log A(\boldsymbol{\epsilon}(s)) - \log \Lambda(\boldsymbol{\epsilon}(s))$$

where, to a second order approximation

$$\log \Lambda(\boldsymbol{\epsilon}(s)) \approx \log \Lambda(\mathbf{1}) + \sum_i \left.\frac{d\log \Lambda(\boldsymbol{\epsilon}(s))}{d\log \epsilon_i}\right|_{\boldsymbol{\epsilon}=\mathbf{1}} \log \epsilon_i + \frac{1}{2}\sum_j\sum_i \left.\frac{d^2 \log \Lambda(\boldsymbol{\epsilon}(s))}{(d\log \epsilon_j)(d\log \epsilon_i)}\right|_{\boldsymbol{\epsilon}=\mathbf{1}} (\log \epsilon_i)(\log \epsilon_i)$$

where $\log \Lambda(\mathbf{1}) = 0$. Thus

$$\log \Lambda(\boldsymbol{\epsilon}(s)) \approx \sum_i \left.\frac{d\log \Lambda(\boldsymbol{\epsilon}(s))}{d\log \epsilon_i}\right|_{\boldsymbol{\epsilon}=\mathbf{1}} \log \epsilon_i + \frac{1}{2}\sum_j\sum_i \left.\frac{d^2 \log \Lambda(\boldsymbol{\epsilon}(s))}{(d\log \epsilon_j)(d\log \epsilon_i)}\right|_{\boldsymbol{\epsilon}=\mathbf{1}} (\log \epsilon_i)(\log \epsilon_i)$$

We have that

$$\log A\left(\boldsymbol{\epsilon}\left(s\right)\right)=\sum_{i}\left.\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\epsilon_{i}}\right|_{\boldsymbol{\epsilon}=\mathbf{1}}\log\epsilon_{i}$$

Thus

$$\log TFP\left(\boldsymbol{\epsilon}\left(s\right)\right)\approx-\frac{1}{2}\sum_{j}\sum_{i}\left.\frac{d^{2}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{\left(d\log\epsilon_{j}\right)\left(d\log\epsilon_{i}\right)}\right|\left(\log\epsilon_{i}\right)\left(\log\epsilon_{i}\right)$$

As a result, welfare is given by

$$\mathcal{W}=\mathcal{W}^{*}\exp\left(-\frac{1}{2}\frac{\left(1-\gamma\right)\left(1+1/\eta\right)}{1/\eta+\gamma}\mathbb{E}\left[\sum_{j}\sum_{i}\left.\frac{d^{2}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{\left(d\log\epsilon_{j}\right)\left(d\log\epsilon_{i}\right)}\right|_{\boldsymbol{\epsilon}=\mathbf{1}}\left(\log\epsilon_{i}\right)\left(\log\epsilon_{i}\right)\right]-\frac{1}{2}\frac{\left(1-\gamma\right)\left(1+1/\eta\right)}{\left(1/\eta+\gamma\right)^{2}}\mathbb{E}\left[\left(\log\Lambda\left(\right.\right.\right.$$

Thus

$$\mathcal{W}=\mathcal{W}^{*}\exp\left(-\frac{1}{2}\frac{\left(1-\gamma\right)\left(1+1/\eta\right)}{1/\eta+\gamma}\Omega\right)$$

where

$$\Omega=\mathbb{E}\left[\sum_{j}\sum_{i}\left.\frac{d^{2}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{\left(d\log\epsilon_{j}\right)\left(d\log\epsilon_{i}\right)}\right|_{\boldsymbol{\epsilon}=\mathbf{1}}\left(\log\epsilon_{i}\right)\left(\log\epsilon_{i}\right)\right]+\frac{1}{\left(1/\eta+\gamma\right)}\mathbb{E}\left[\left(\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)\right)^{2}\right]$$

$$\Omega=\sum_{j}\sum_{i}\left.\frac{d^{2}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{\left(d\log\epsilon_{j}\right)\left(d\log\epsilon_{i}\right)}\right|_{\boldsymbol{\epsilon}=\mathbf{1}}\mathbb{E}\left[\left(\log\epsilon_{i}\right)\left(\log\epsilon_{j}\right)\right]+\frac{1}{\left(1/\eta+\gamma\right)}\sum_{j}\sum_{i}\left.\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\epsilon_{j}}\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\epsilon_{i}}\right|_{\boldsymbol{\epsilon}=\mathbf{1}}\mathbb{E}\left[\left(\log\epsilon_{i}\right)\left(\log\right.\right.$$

$$\Omega=\sum_{j}\sum_{i}\left\{\frac{d^{2}\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{\left(d\log\epsilon_{j}\right)\left(d\log\epsilon_{i}\right)}+\frac{1}{\left(1/\eta+\gamma\right)}\left.\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\epsilon_{j}}\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\epsilon_{i}}\right|_{\boldsymbol{\epsilon}=\mathbf{1}}\right\}\mathbb{E}\left[\left(\log\epsilon_{i}\right)\left(\log\epsilon_{j}\right)\right]$$

as in (). **QED.**

Derivatives

One of these terms is pretty simple.

$$\frac{d\log A\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\varepsilon_{j}}=\boldsymbol{v}'\mathbf{L}e_{i}=\boldsymbol{v}'\left[\mathbb{I}_{N}-diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right]^{-1}e_{i}\tag{61}$$

This term is log-linear and hence holds globally, i.e. for any $\boldsymbol{\epsilon}\left(s\right)$.

The other term. We may write it as:

$$\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\varepsilon_{j}}=-\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}\frac{d\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\varepsilon_{j}}.$$

where

$$\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)=\boldsymbol{\alpha}'\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}^{-1}\boldsymbol{v}$$

Thus

$$\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\varepsilon_{j}}=-\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}\boldsymbol{\alpha}'\frac{d\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}^{-1}}{d\log\varepsilon_{j}}\boldsymbol{v}.\tag{62}$$

Next, we use the fact that for any invertible matrix A, the following is true: $\frac{dA^{-1}}{dx} = -A^{-1}\frac{dA}{dx}A^{-1}$. Applying this, we have that

$$\begin{aligned}\frac{d\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}^{-1}}{d\log\varepsilon_j} =& -\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}^{-1}\\ &\times\frac{d\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}}{d\log\phi_i}\\ &\times\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}^{-1}\end{aligned}$$

where:

$$\begin{aligned}\frac{d\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}}{d\log\varepsilon_j} &= \frac{d\left\{diag\left(\exp\left(\log\boldsymbol{\epsilon}\left(s\right)\right)\right)^{-1}\right\}}{d\log\varepsilon_j} = diag\left(e_j\right)diag\left(\exp\left(\log\boldsymbol{\epsilon}\left(s\right)\right)\right)^{-1}\\ &= diag\left(e_j\right)diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}\end{aligned}$$

Plugging this into (62) gives us,

$$\begin{aligned}\frac{d\log\Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)}{d\log\varepsilon_j} =& \Lambda\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}\boldsymbol{\alpha}'\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}^{-1}\\ &\times diag\left(e_j\right)diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}\left\{diag\left(\boldsymbol{\epsilon}\left(s\right)\right)^{-1}-\left(diag\left(\mathbf{1}-\boldsymbol{\alpha}\right)\mathbf{G}\right)'\right\}^{-1}\boldsymbol{v}.\end{aligned}$$

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First. Take any variable X that is log-normal with

$$\log X \sim N\left(\mathbb{E}\log X, Var\log X\right)$$

Then for any scalar δ,

$$\begin{aligned}\mathbb{E}\left[X^{\delta}\right] &= \exp\left(\delta\mathbb{E}\log X+\frac{1}{2}\delta^2 Var\log X\right)\\ &= \exp\left(\delta\mathbb{E}\log X+\frac{1}{2}\delta Var\log X+\frac{1}{2}\delta\left(\delta-1\right)Var\log X\right)\\ &= \exp\left(\mathbb{E}\log X+\frac{1}{2}Var\log X\right)^{\delta}\exp\left(\frac{1}{2}\delta\left(\delta-1\right)Var\log X\right)\end{aligned}$$

Therefore

$$\mathbb{E}\left[X^{\delta}\right] = \left(\mathbb{E}X\right)^{\delta}\exp\left(\frac{1}{2}\delta\left(\delta-1\right)Var\log X\right)$$

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