Unequal Growth*

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PRELIMINARY AND INCOMPLETE

Abstract

Over the past 50 years US households have experienced changes in earnings dynamics that have generated a large increase in inequality. The objective of this paper is to measure the impact of these changes on aggregate growth and welfare. We first apply, using US household panel data for the period 1967-2016, a simple statistical decomposition showing that aggregate growth is the sum of two terms. The first is the covariance between the level and growth of household earnings, which only depends on micro dynamics. The second is average growth across households, which depends both on micro and macro factors. In order to identify the impact of micro dynamics we bring two simple models of micro-founded growth to our data. We find that changes in micro earnings dynamics that are consistent with data involve *unequal growth* across the earnings distribution. These changes have a positive effect on aggregate growth, and, with incomplete markets, a negative and large, ex-ante, welfare effect.

JEL Classification Numbers: D31, O4

Key Words: Income distribution, Inequality, Growth slowdown

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1 Introduction

Over the past 50 years US households have experienced changes in earnings dynamics that have generated a large increase in earnings inequality. The objective of this paper is to measure the impact of these changes on aggregate growth and welfare.

We begin by presenting a statistical decomposition showing that aggregate earnings growth can be written as the sum of two terms: the first is the cross sectional (across households) covariance between earnings growth and earning levels, the second is the (un-weighted) average of household/individual earnings growth. The covariance can in turn be written as the product of the correlation between income level and growth (a measure of rank mobility), of a measure of income inequality and of a measure of dispersion of income growth across the income distribution. This decomposition provides a natural link between household earnings dynamics and aggregate earnings growth and thus we document the evolution of all its terms, for the United States, using micro data from the Panel Study of Income Dynamics (PSID) over the period 1968-2016. The data shows that aggregate growth is declining, that the average household earnings growth is stable, and that the covariance between growth and levels is declining. We then argue that these patterns can help us identify the impact of changes in micro dynamics on aggregate growth. The key insight is that the cross sectional covariance term (and all its components) is connected to aggregate growth, but only depends on micro dynamics, so we can identify changes in micro dynamics from changes in the covariance and its components. Importantly, changes in micro dynamics can have also an effect on the second term, which is, however, also affected by a common (across households) growth component. Therefore, once we have identified changes in the micro dynamics, the evolution of the second component can help us identify changes in the common component.

In order to make the identification exercise precise we present two simple models of micro-founded growth. The first one is a standard incomplete markets structure à la Aiyagari-Bewley-Huggett. The second one is a model of random exponential growth, related to the recent analysis by Jones and Kim (2018). While the first model has advantages in terms

of mapping to the micro data and in that it is a standard model in macro, the second one provides a lot of analytic tractability and allows us to derive several results in a sharp way. In both models we introduce changes in parameters governing income dynamics, and discipline these changes using standard studies on income micro dynamics and the observed aggregate moments, as they appear in the statistical decomposition described above. The idea is closely linked to the analysis by Gabaix et al. (2016), who frame the evolution of income inequality as a transition, from one invariant distribution to a new one, triggered by a change in the fundamentals of the household's income process. Our key contribution relative to the previous literature is the focus on the impact of these changes on aggregate growth. Both models show that the changes in micro income dynamics that are consistent with the decomposition involve sizeable a decline in the common component plus an *unequal growth* across the income distribution, that is growth declines that are much smaller for high income households than for low income households.

Both models suggest that the underlying changes in the income dynamics have had a positive effect on aggregate growth. This result is not obvious ex ante since the actual growth of the economy results from the combination of a common growth component and another component due to the changes in the shape of the income distribution. We use our models to evaluate the ex ante welfare consequences of such changes, and conclude that, in an economy with incomplete markets, the ex-ante welfare effect of the increase in *unequal growth* is negative and large.

Literature Review To be Completed Aiyagari (1994); Atkinson, Piketty, and Saez (2011); Arkolakis (2016); Benabou (1996); Benhabib and Bisin (2016); Chetty et al. (2014); **?**); Gabaix (2011); Guvenen, Ozkan, and Song (2014); Guvenen et al. (2015); Gabaix et al. (2016); Huggett (1993); Jovanovic (2014); Jones and Kim (2018); Kopczuk, Saez, and Song (2010); Krueger and Perri (2004); Lucas (2000); Luttmer (2011); Olley and Pakes (1996)

2 A micro decomposition of aggregate growth

In this section we present a simple statistical decomposition that connects aggregate income growth to micro-level (household or individual) income growth, cross sectional income inequality, and the cross sectional correlation between income growth and income level. These types of decompositions have been widely used in industrial organization to connect sectoral productivity growth to productivity growth in individual firms (see, among others, Olley and Pakes 1996). We find it useful to apply this decomposition to household level data (as opposed to firms), because it connects aggregate growth with household income inequality, which has a more direct and relevant welfare content than firms income inequality.

Let y_{it} be level of income of household/individual i at time t. Let Γ_{t+T} be the economy's aggregate growth over an horizon T, which is

$$\Gamma_{t+T} = \frac{E(y_{it+T})}{E(y_{it})} = E\left(\frac{y_{it+T}}{y_{it}}\frac{y_{it}}{E(y_{it})}\right)$$

where $E(.)$ is the cross sectional average. Now define

$$g_{i,t+T} \equiv \frac{y_{it+T}}{y_{it}} \quad , \quad s_{i,t} \equiv \frac{y_{it}}{E(y_{it})}$$

so that $\Gamma_{t+T} = E(g_{i,t+T} \cdot s_{i,t})$ where $g_{i,t+T}$ is income growth of unit i and $s_{i,t}$ the ratio between income of unit i and average income. Then, using the definition of covariance and the fact that $E(s_{i,t}) = 1$ we get

$$\Gamma_{t+T} = cov(g_{i,t+T}, s_{i,t}) + E(g_{i,t+T}) \tag{1}$$

or equivalently

$$\Gamma_{t+T} = corr(g_{i,t+T}, s_{i,t})\sigma(s_{i,t})\sigma(g_{i,t+T}) + E(g_{i,t+T}) \tag{2}$$

Equation (1) suggests that what matters for aggregate growth is not only the (unweighted) average individual growth $E(g_{i,t+T})$ but the distribution of growth opportunities, as summarized by $cov(g_{i,t+T}, s_{i,t})$. The intuition for why this is the case is straightforward: the higher the covariance, the faster higher income individuals grow; since they are high income they contribute more to aggregate growth and aggregate growth is higher. Equation (2) also suggests that $cov(g_{i,t+T}, s_{i,t})$ is linked to three cross sectional moments that have an intuitive economic interpretation. The first, $corr(g_{i,t+T}, s_{i,t})$, is the correlation between level and growth at the individual level. This measure captures the degree of mean reversion (or economic rank mobility) in individual income dynamics. The second, $\sigma(s_{i,t})$ is the standard deviation of $s_{i,t}$, which is essentially a measure of cross sectional income inequality. The third, $\sigma(g_{i,t+T})$, is the standard deviation of the growth rate of individual income, which is a measure of cross sectional income volatility. The equation suggests that changes in any of these three quantities will be associated, *ceteris paribus*, with changes in aggregate growth. It is important to note that this decomposition is a statistical identity, so, by itself, it cannot be used to make causal inferences on growth and inequality. Nevertheless it provides a useful starting point for assessing the impact of changing individual income dynamics on growth. To see why this is the case, note that all the moments in the first term of equation (2) are independent from the presence of a common growth factor, call it $\bar{g}$, that affects equally the growth of all households. All the terms in the product only depend on heterogenous individual income dynamics. The second term in equation (2) is instead potentially affected both by the common factor $\bar{g}$ and by individual income dynamics. So the evolution of the statistics in equation (2) will help us, with the aid of a simple statistical model, to identify the impact on growth of the changes in income dynamics, that drive in income inequality, from the changes in growth that are common across all households. For this reason the next section uses a panel of micro data to document how the terms in the decomposition has changed over time.

3 A decomposition of US growth: 1967-2016

Both equation (1) and equation (2) involve cross-sectional moments as well as moments related to individual income growth, so in order to bring them to the data we need panel data on household/individual earnings. Since our main focus is aggregate growth in the United States we also want a panel which captures well aggregate US growth. For these reasons we work with the Panel Study of Income Dynamics (PSID), which is a panel of US households, selected to be representative of the whole population, collected from 1967 to 2016. Figure 1 reports aggregate growth in per capita labor income (earnings) both in the PSID and the National Income and Product Accounting (NIPA).[1] The solid lines report the actual annualized growth (computed over a 4 years horizon), while the dotted lines are polynomial trends. The figure shows that growth in PSID tracks growth in NIPA quite closely, suggesting that the PSID sample is a good laboratory to study the connections between individual income dynamics and aggregate growth.

Figure 2 also shows that the PSID captures well the patterns of US household income inequality, as documented in from a much larger cross sectional survey, i.e. the March Current Population Survey. The figure plots a measure of inequality which is relevant for our decomposition, that is the standard deviations of income (in ratio to mean income), for 10 deciles of the earnings distribution, derived from the two surveys.[2] The figure shows that both surveys capture the well known secular increase in income inequality in the United States.

Since Figure 1 and Figure 2 show that the data in PSID capture well the evolution of

[1]The income measure in PSID is total wage and salary income plus farm income plus 50% of business income for each household in the sample, divided by the total number of persons in the sample. The income measure in NIPA is compensation of employees, wages and salaries disbursement plus 50% of proprietors income, per capita. All measures are deflated using the PCE deflator. See the data appendix for more details on data construction and for similar figures for different (narrower and broader) income measures. The reason why we focus on labor income is that other categories of income are notoriously not well measured in the PSID and in other micro surveys.

[2]The income measure in both PSID and CPS is total wage and salary income plus 50% of household business and farm income. Inequality measures are computed for households with heads between age 25 and 60. The average sample size in the PSID is around 4000 household per year, the size in CPS is 10 times larger.

Figure 1: Growth in labor income: NIPA and PSID

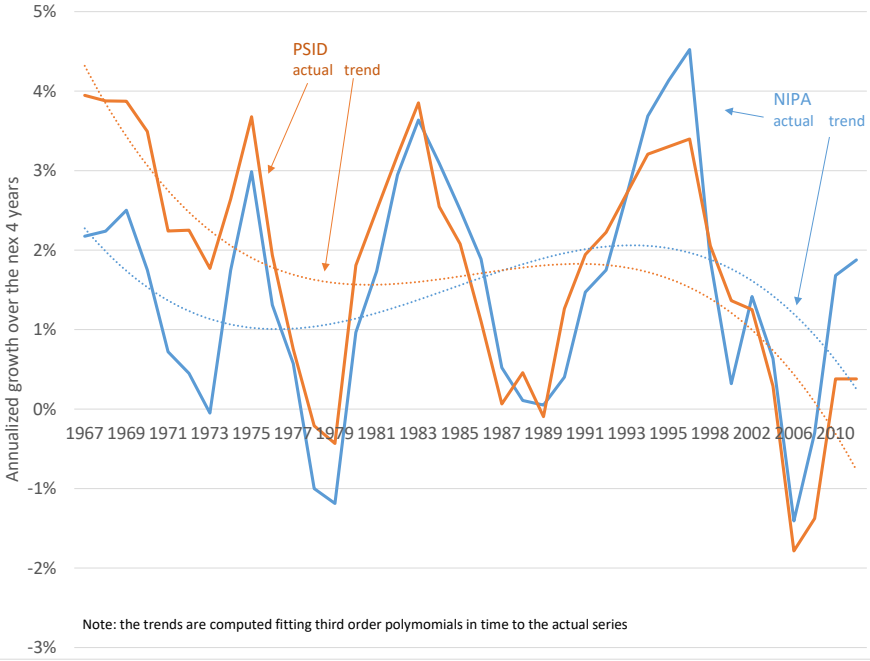


Figure 2: Inequality in labor income: PSID and CPS

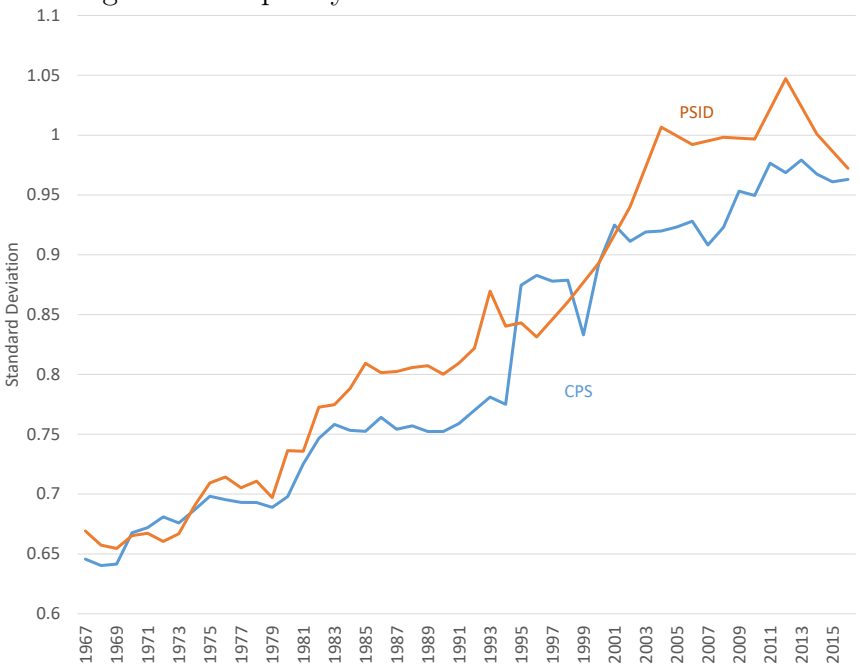
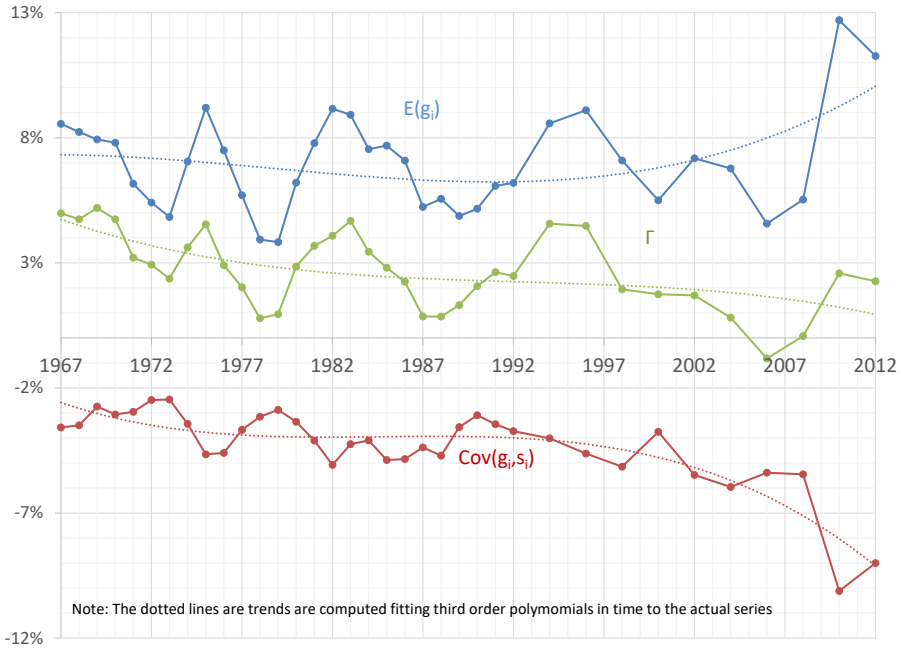


Figure 3: A decomposition of US aggregate growth

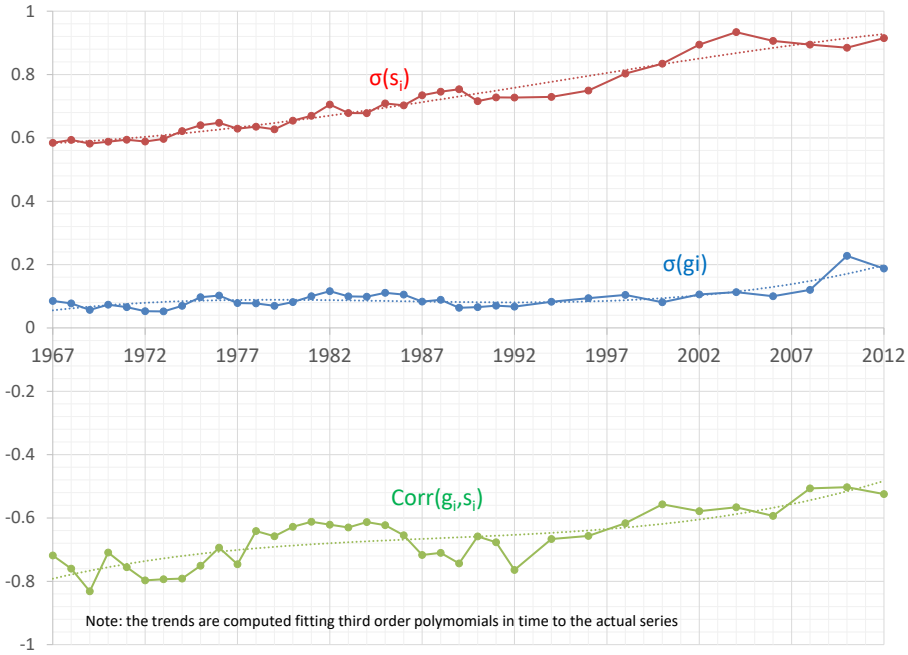


aggregate growth and inequality, we now proceed to compute the data equivalent in PSID of equation (1) and equation (2). Figure 3 shows the growth decomposition suggested by equation (1), where, in order to reduce noise due to measurement error in individual income, we aggregate households in 10 deciles.[3] The line labelled Γ reports aggregate growth rate (annualized) over the 4 years following the x-axis date, for our PSID sample. The line labelled $E(g_i)$ reports the (unweighted) average of the growth rate across deciles in our sample, and finally the line labelled $cov(g_{i,t+T}, s_{i,t})$ reports the covariance between the growth and the normalized level.

Figure 3 shows that decline in aggregate growth is associated to a even larger decline in the covariance between growth and level, $cov(g_{i,t+T}, s_{i,t})$, while the un-weighted average of growth rates in each decile, $E(g_{i,t+T})$, first declines and then slightly increases. Figure 4 further decomposes the trend in the covariance, using equation (2). The figure shows that

[3]Formally let I_t by the group of households who are in the i_{th} decile of the income distribution in period t. We define $g_{i,t+T} = \frac{\sum_{i \in I_t} y_{i,t+T}}{\sum_{i \in I_t} y_{i,t}}$, that is the growth rate of income in a given decile is computed using the same group of households in t and $t+T$. Our sample includes all households with head between age 25 and 60.

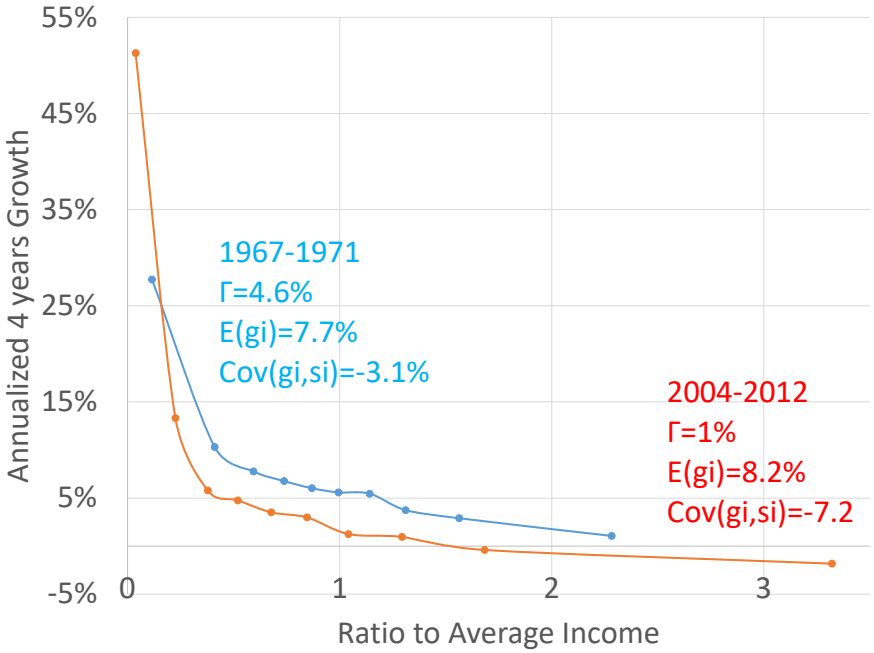
Figure 4: Decomposing the decline in covariance



the fall in the covariance is the result of two off-setting trends. On one hand the correlation between growth and levels ($corr(g_i, s_i)$) , which, in the beginning of the sample is around -0.8, becomes less negative. This would result, *ceteris paribus*, in an increase in the covariance. On the other hand the fact that income inequality ($\sigma(s_i)$) has increased, together with the fact that the correlation is negative, implies a decline in the covariance. Overall the increase in inequality dominates and thus a reduction of the covariance is observed. Nevertheless the increase (fall in absolute value) of the correlation is an important feature of the data. In particular it shows that, together with the increase in inequality, US households have experienced a substantial reduction in rank mobility, i.e. in recent years it is less likely for low income households to experience strong growth.

To get a better understanding of the individual income dynamics generating these changes in Figure 5 we plot the average 4-year growth in each decile (g_i) v/s the ratio of the income of the decile to average income (s_i). To average out the effects of the cycle, in the figure we average this statistic over the first 5 years of the PSID sample and over the last 5 years of

Figure 5: Growth and ratios by deciles of the income distribution



the sample (for which we can compute growth the 4-year growth rate).[4] There are several features we would like to point out in the graph. The first is that the relation is negative in both periods. Low income households tend to grow faster. This feature explains the negative correlation (and covariance) between g_i and s_i. The second feature is that over time there has been a substantial change in this relation. The support of the (s_i) has expanded both to the right and to the left, reflecting the increase in income inequality. Also growth (g_i) in recent years is slower for the top and mid deciles of the income distribution, while is faster for the bottom deciles. This twisting of the curve explains, in an accounting sense, why in recent years we observe a lower aggregate growth rate (4.5% v/s 1%), despite a higher average growth of each decile ($E(g_i)$ has increased from 7.7% to 8.2%).

So far we have documented a series of facts relating growth and inequality in the United States over the past 50 years. Aggregate growth has declined and inequality has increased. The decline in growth has not been uniform across the income distribution, which results in

[4]The first 5 years are 1967, 1968, 1969, 1970 and 1971, while the last 5 years for which we can compute growth are 2004,2006,2008, 2010 and 2012. We average the 4-year growth over five periods to smooth out cyclical components in the growth in each decile.

a lower covariance between level and growth. In the next sections we consider two simple models of household income, to propose a theory of changes in income dynamics. Using the facts with some simple theory will allow us to identify separately the changes that drive inequality and growth (unequal growth) separately from changes that only affect growth (common growth factors).

4 A Bewley-Aiyagari-Huggett model

We consider a standard Bewley-Aiyagari-Hugget small open economy, with few simple modifications to the household income process, introduced to capture the features and the changes in the income distribution documented above.[5] We then explore the effect of these changes on aggregate growth and on welfare. The economy is inhabited by a continuum of infinitely lived households with standard preferences over consumption flows, denoted by

$$E_t \sum_{j=0}^{\infty} \beta^t u(c_{t+j}).$$

where $\beta > 0$ is the discount factor and $u(.)$ is a standard utility function, which will assume to be $CRRA$, *i.e.* $u(c) = \frac{c^{1-\theta}}{1-\theta}$.

4.1 Earning Potential

Each household in each period receives an idiosyncratic realization of its earning potential Y_{it}. We model earning potential as

$$\log Y_{it} \equiv y_{it} = \alpha_i + e_{it} + f_{it}.$$

[5]The assumption of small open economy is made for computational convenience. For completeness we will also solve a closed economy version of the model, where the interest rate is endogenous.

The first component, α_i, is a standard fixed effect, meant to capture permanent differences in earning potential across households. We assume

$$\alpha_i \sim N(0, \sigma_\alpha)$$

The second component e_{it} is a standard autoregressive process, which we model as

$$\begin{aligned} e_{it} &= \rho e_{it-1} + \varepsilon_{it} \quad , \quad \varepsilon_{it} \sim N(\mu_\varepsilon(s_{it}), \sigma^2_\varepsilon(s_{it})) \\ s_{it} &\equiv \frac{Y_{it}}{E_i(Y_{it})}, \qquad \sigma^2_\varepsilon(s_{it}) = \sigma^2_\varepsilon + \chi \frac{(1 - s_{it})}{(1 + s_{it})} \end{aligned}$$

Note that the parameter χ links the volatility of shocks of the income process, $\sigma^2_\varepsilon(s_{it})$, to s_{it}, the position of household i in the income distribution.[6] This is motivated by a large body of research which has documented that households at the bottom of the income distribution face higher volatility in their earnings shocks (see, among others, Meghir and Pistaferri (2004)). The third component of the income process, f_{it}, which we name the growth factor, is going to be the driver of the increase in income inequality and it evolves according to

$$f_{it} = f_{it-1} + \bar{g}_t \ + \delta_t \frac{(s_{it} - 1)}{(1 + s_{it})} \tag{3}$$

The important element in equation (3) is that earnings growth of household i can depend on s_{it}. First consider the case in which $\delta_t = 0$. In this case each household experiences a common income growth rate $\bar{g}_t$. In our experiments this is going to be the relevant case in the initial and final steady state. During finite time transitions, however, we will allow the parameter δ_t to be different from 0, and in particular to be positive, so that households with income above the mean ($s_{it} > 1$), will have faster growth than households with income below

[6] Since $\exp(e_{it})$ is distributed log normally changing the volatility of e_{it} also mechanically change its mean. To eliminate this effect we also allow also the mean of the shocks $\mu_\varepsilon(s_{it})$ to depend on s_{it} and we set it so that $E(\exp(\varepsilon_{it}))$ does not vary across the income distribution. This is done to separate heterogeneity in variance (captured in the autoregressive component of income) from heterogeneity in means, which in our specification is captured by the fixed effects and by the growth factor.

the mean. As we will show below, when $\delta_t > 0$, income inequality is increasing, so this will be our modelling device to obtain the observed trends in income inequality.

4.2 Work choices and earnings

In each period each household with earning potential Y_{it} has the option to work on the market, and earn its potential minus taxes, or work at home and earn a transfer income $\exp(\phi_t)$, which grows at the common growth rate of the economy

$$\phi_t = \phi_{t-1} + \bar{g}_t$$

When households work on the market they pay a flat tax that the government uses to finance the transfer income. The process for earnings (before transfer and taxes) of household i, which we denote by $h(Y_{it})$ is thus given by

$$h(Y_{it}) = \begin{cases} Y_{it} & \text{if } Y_{it}(1-\tau) \geqslant \exp(\phi_t) \\ 0 & \text{if } Y_{it}(1-\tau) < \exp(\phi_t) \end{cases}$$

This feature of the model will generate household earnings that feature positive as well as 0 values.

4.3 The household problem

The household consumption saving problem is standard. In the baseline case we assume incomplete markets so that each household can borrow and save using an uncontingent bond, which pays an exogenously given interest rate r. Bond holdings have to be above a borrowing

constraint $\bar{b} \leq 0$. The problem can then be written as

$$\max_{c_{t+j}, b_{t+j}} E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \tag{4}$$

$$s.t.$$

$$c_{t+j} = b_{t+j-1}(1+r) + \max(h(Y_{it+j}), \exp(\phi_{t+j})) - b_{t+j},\ b_{t+j} \geq \bar{b} \text{ for every } j$$

$$b_{t+j} \geq \bar{b} \qquad b_{t-1} \text{ given}$$

4.4 Equal growth stationary equilibria

We first analyze stationary equilibria in which there is no unequal growth ($\delta = 0$) and in which all parameters, including the long run growth rate of the economy α are constant. An *equal growth* equilibrium is a distribution of households over earning potential and asset of $\mu(Y, b)$, plus household decision rule $b'(b, Y)$ satisfying the following conditions

1. The decision rules solve the household decision problem 4
2. Given the decision rules of the households the distribution is time invariant
3. The government budget constraint is satisfied

$$\int \tau h(Y) d\mu = \int \phi I(h(Y) = 0) d\mu$$

where $I(.)$ is the indicator function.

Note that in an *equal growth* equilibrium, all individual and aggregate variables grow at the constant rate of α, hence when we solve for it, we solve for equilibrium in an economy where all variables are detrended by the growth factor f_t and where the discount factor β and the interest rate on bonds $1 + r$ are suitably rescaled. [7]

[7] In particular the interest rate in the detrended economy is equal to $\frac{1+r}{1+\alpha}$ and the discount factor, in the case where utility is CRRA with risk aversion parameter equal to θ, is equal to $\beta * (1+\alpha)^{(1-\theta)}$

4.5 Unequal growth equilibria

We label *unequal growth* equilibria, the equilibria that arise during a transition from one stationary distirbution to another. We assume the economy start in an stationary equilibrium and at time t_0, then experiences a change in parameters for $N < \infty$ periods. In particular we will consider the case in which $\delta_t > 0$ and in which α_t is not constant for $t \in [t_0, t_0 + N]$. After period $t_0 + N + 1$, we assume that the economy settles to a constant growth rate $\bar{\alpha}$ and that $\delta_t = 0$. An unequal growth equilibrium is a sequence of distributions $\mu_t(Y, b)$, and a sequence of decision rules $b'_t(b, Y)$, for $t \in [t_0, \infty]$, satisfying the following conditions:

1. Given perfect foresight on the path of parameters changes, the decision rules solve the household decision problem 4
2. The sequence of distributions are consistent with the decision rules
3. The government budget constraint is satisfied in every period

$$\int \tau_t h(Y) d\mu_t = \int \phi_t I(h(Y) = 0) d\mu_t$$

Note that the assumption of perfect foresight might sound a bit extreme, as it implies that high income households in 1975 (the date at which we will start our transition), learn that they have faster growth for the next N years (which in the baseline calibration we set to 40). For this reason we will also present results for *unequal growth* equilibria where agents do not expect the change in parameters and are "surprised" every period.

4.6 Calibration

Table 1 summarizes our parameter values for the equal growth equilibrium, which we calibrate to match feature of the US economy in the late 1960s, before the increase in inequality started. We now briefly describe how we set those. Starting first with the persistence of the autoregressive part, ρ, we set it equal to 0.6, following previous quantitative studies that have

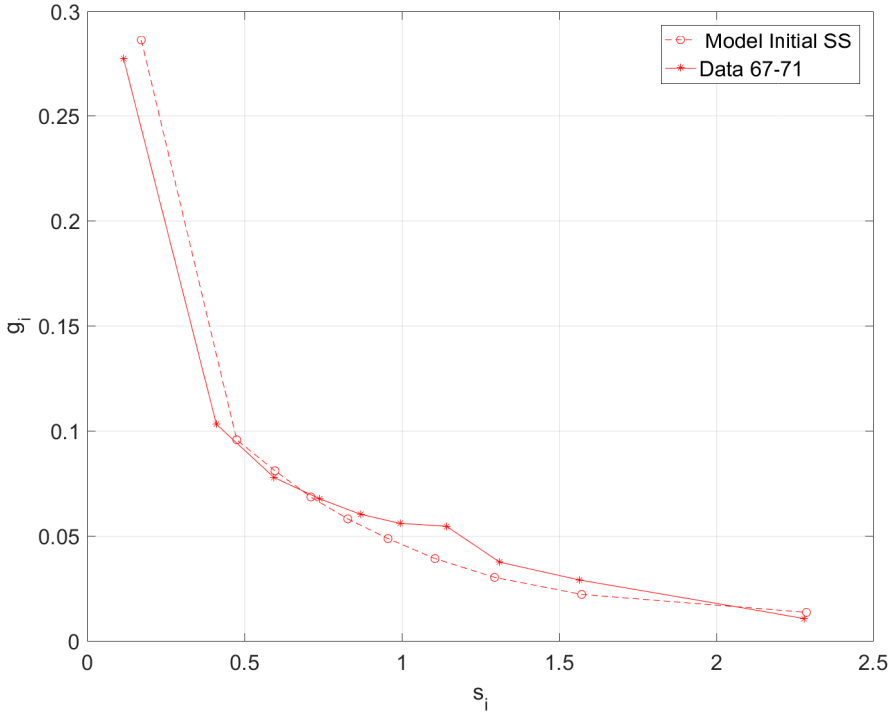
Table 1: Parameters in the initial stationary equilibrium

Income Process Parameters		
Name	Symbol	Value
Variance of fixed effects	σ_α	0.45
Persistence of shocks	ρ	0.6
Baseline sd of shocks	σ_ε	0.21
Standard deviation gradient	χ	$0.75\sigma_\varepsilon$
Common growth	$\bar{g}$	4.5%
Transfer income (% of average Y)	ϕ	0.3
Tax rate	τ	1.5%
Unequal growth	δ	0
Preference Parameters		
Discount Factor	β	0.97
Risk Aversion	θ	2
Other Parameters		
Borrowing Constraint	$\bar{b}$	0
Risk free rate	r	2.5%

estimated AR(1) processes on household income process (see Aiyagari (1994), Heaton and Lucas (1996)). We then set the parameter ϕ, which determines the threshold under which an household does not work in the initial steady state to match a percentage of non working households with head of age 25-60, in the years 1967-1968 of 5.3%. We define non working those households for which annual hours worked by the head and spouse are less than 500. We then set the parameter χ which determines how much the volatility of the shocks decline with income level to be equal to $3/4\sigma_\varepsilon$ which implies that a household at the low end of the income distribution ($s = 1/4$) has roughly twice the variance of the shocks of a household in the high end of the distribution ($s = 2$). This is broadly consistent with the figures reported by Meghir and Pistaferri (2004).

To conclude we set the three remaining parameters, $\bar{g}, \sigma_\alpha$ and σ_ε, to match the shape of the relation between income growth and income levels in the first years of our sample (the line labelled 1968-71 in Figure 5). Figure 6 shows the relation between growth and level in the data and in the model. Our stylized income process capture this relation reasonably well. The mean reverting component of the process is responsible for the overall negative

Figure 6: Initial steady state: data and model



slope of the curve. The presence of fixed effects is necessary to generate the flat part of the curve on the right, and the extensive margin decision plus the heteroscedasticity in the volatility of income shocks is important to generate the spike in the growth of the first decile of the distribution. The spike is generated by households in the bottom decile which have low income and experience frequent transitions in and out of employment, and those two features of the model capture those households. Finally we set the interest rate on bonds to 2.5% and the discount factor to 0.97, so to generate, in the initial steady state, a wealth to income ratio of around 3.0.

4.7 Results

Once we have calibrated the model to the initial steady state, we consider a transition period. In particular we assume that starting in 1975 the parameter δ increases from 0, its steady state value, to 0.03, during a period of 40 years. This implies, for example, that during that

period a household with earnings that are twice the mean ($s_i = 2$) grows 1% more per year than a household with earnings at the mean ($s_i = 1$). After 40 years the parameter δ reverts to 0. This parameter change is chosen so that the model exactly replicates the increase in earnings inequality (the increase in standard deviation of the s_i) documented in figure 4. If that was the only change in the transition the model would imply a share of non participant households that would rise "too much" relative to the data. Low s_i households experience negative growth in their potential income which induce them not participate, so the share of non working households would rise to 13% (relative to 8.7% in our PSID sample). For this reason in our baseline calibration we also change the time path for the transfer income ϕ_t so that along the transition the model matches the increase in non-working households that we observe in our sample (which goes from 5.3% in 1967-68, to 8.7% in 2014-16).[8]

4.7.1 Growth impact

Figure 7 shows the time paths implied by the model for all the terms of the decomposition in equation (2) that do not depend on the aggregate growth factor $\bar{g}_t$. The figure suggests that the increase in unequal growth captures well the type of income dynamics in the data. Note that unequal growth is able to generate an increasing path for correlation between level and growth, together with a declining pattern for covariance between the two variables. Initially, as unequal growth takes place, it increases both the covariance and the correlation between income levels and growth.

As time goes by, more unequal growth results in poor agents experiencing larger shocks, because the variance of earning shocks increases when income falls, and because they move more between working and not working. These larger shocks at the bottom result in higher $\sigma(s_i)$ and higher $\sigma(g_i)$ which result in falling covariance. Next we can asses the impact on

[8] In order to match the increase in non working households in the data the model calls for a *decline* in the transfer income (from about 30% of mean income to about 18% of mean labor income). The reason is that unequal growth would imply too much non participation, and we need a reduction in transfer income to induce households to participate in the model as they do in the data. We view this decline in transfer income as a reduced form way to capture an increasing incentive for labor force participation, which is particular relevant for women.

Figure 7: Micro moments during the transition

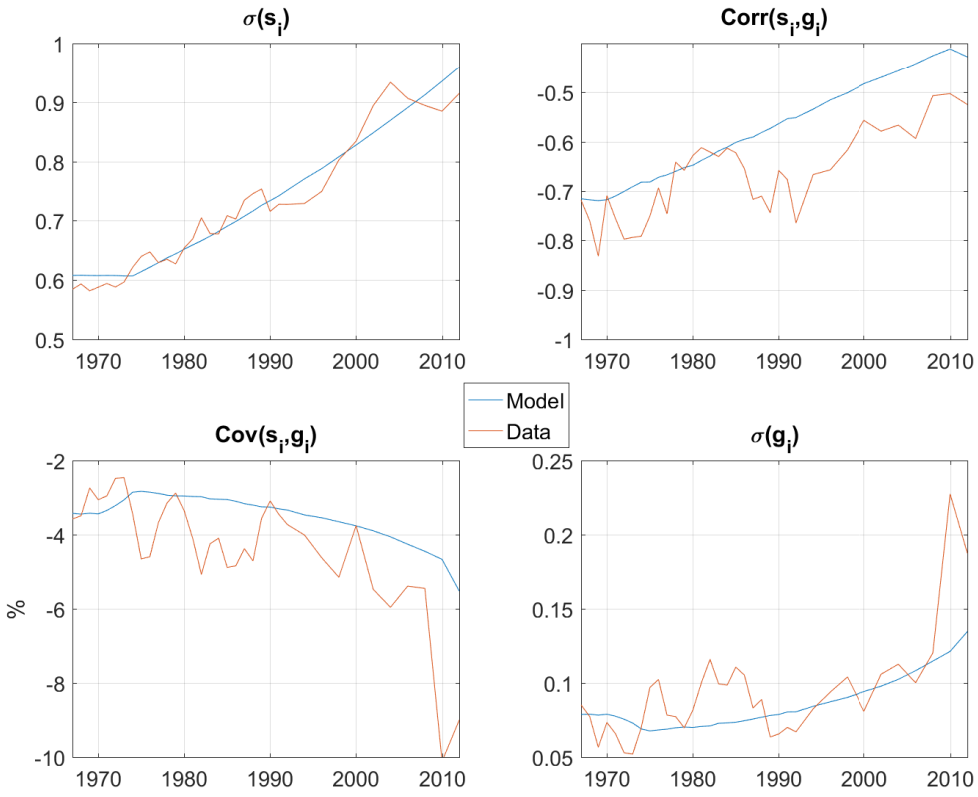
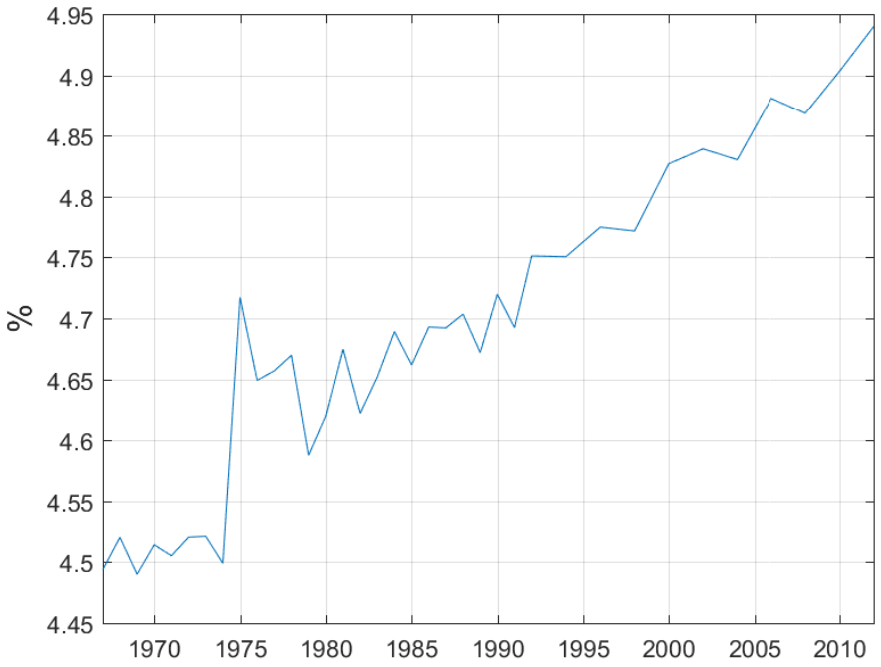


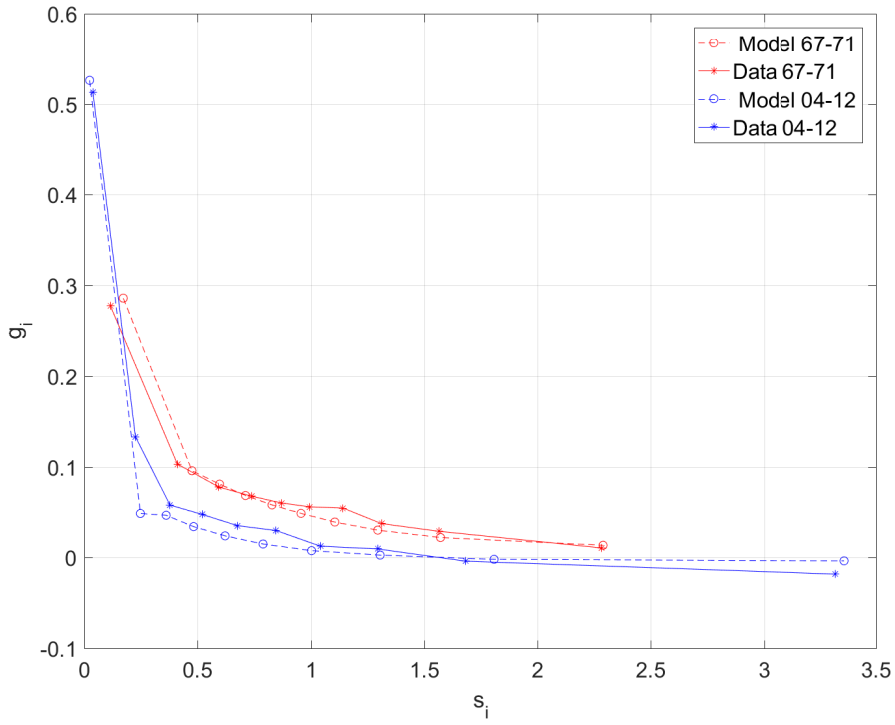
Figure 8: Impact of micro changes on aggregate growth during transition



aggregate growth (separate from changes in the growth factor), which is reported in Figure 8. The Figure shows that during the transition aggregate growth increases, but only to a modest extent (from 4.5% to 4.9%).

Our final result involves backing out the changes in the common component of aggregate growth. To do so, we allow the parameter $\bar{g}_t$ to change over time so that in the last decade of the transition, namely over the period 2004-2012, the average of growth rates by decile in the model, $E(g_i)$, matches the $E(g_i)$ in the data. Figure 9 shows the result of this exercise: a slow down of the common growth factor (from 4.5% to 1%), matched with the micro changes of $\sigma(s_i)$ and of $cov(s_i, g_i)$, provide a fairly accurate match of the evolution of earnings growth across the entire wealth distribution. Our estimate of the reduction in the common factor of earnings growth is large, suggesting that changes such as technological slowdown (see, for example Gordon 2012), or the decline in labor share (see, for example, Elsby, Hobijn, and Sahin 2013) have had an important effect on the evolution of the growth in labor earnings in the United States. Our analysis suggests that this effect has been partly muted by the

Figure 9: Distribution of growth and shares by decile: 1967-1971 v/s 2004-2012



unequal growth in earning dynamics.

4.7.2 Welfare

We conclude this section with an analysis of the welfare impact of "unequal growth", that is of the changes that are triggered by the new income dynamics. As is intuitive, the impact crucially depends on two factors: the curvature in utility, which in this class of models captures the social cost of consumption inequality, and the degree of market incompleteness. In Table 2 we measure the welfare cost (in lifetime consumption equivalent units) of moving from a steady state with equal growth, to an unequal growth equilibrium. In other words, the number in the table measure the percentage of lifetime consumption a household, under the veil of ignorance, is willing to give up to avoid the period of unequal growth. We consider two values of the risk aversion (2 and 4) and three market structures, complete markets (CM), bond economy (BE, the economy described above) and autarky (A), the economy in which

Table 2: Welfare costs of unequal growth

	Market Structure		
Risk aversion (θ)	CM	BE	A
$\theta = 2$	-6.1%	+2%	+10.3%
$\theta = 4$	-17.2%	+28.5%	+50.6%

household simply consume their (after transfer) earnings. In the bond and the complete markets economy the welfare numbers are computed assuming that households are surprised by changes in the growth factor every period (but expect them to be permanent).[9]

The table shows first that in complete markets unequal growth produce welfare gains. The logic is obviously that the benefits of the higher aggregate growth are shared among all households. When markets are incomplete, however, high earnings household benefit and low earnings lose, and this results in ex-ante welfare losses, that with curvature equal to 4 can be very substantial. It is useful to think of the losses in incomplete markets as arising from two features. The first is that poor agents experience negative growth and thus are stuck with permanently lower component of their income. The other is that with lower income, they also experience more volatile shocks. In financial autarky both these features affect welfare negatively, hence the large welfare losses. In the bond economy agents can (partly) insure against the more volatile shocks, but still suffer the adverse consequences of the permanently lower component of income and that explain why the welfare losses in the bond economy are also quite high. Another way to understand the large welfare losses in the bond economy is that the process of unequal growth causes increase dispersion in "permanent income" (see Bowlus and Robin (2004), Abbott and Gallipoli (2019) and Straub (2019)) which translates in dispersion in consumption and in welfare losses. Note also that with when curvature is high ($\theta = 4$) the gap in the welfare impact of unequal growth between complete markets and incomplete markets gets very large. This is not surprising, but it highlights that a

[9]In the autarky economy the welfare impact is independent on whether or not the changes in the income process are anticipated.

period of unequal growth increases the social value of better risk sharing or social insurance mechanisms.

5 A Pareto-model with random growth opportunities

This section presents a simple aggregative model of the evolution of the income distribution. The primitive of the analysis is the individual income process, which is exogenous. The thought experiment is to analyse a transition, starting from one invariant distribution, then assuming a change of the fundamental process, and studying the convergence towards the new invariant distribution. The spirit of this exercise is analogue to the one developed by Gabaix et al. (2016). The model is deliberately simple to be analytically tractable. We will use this feature to match the model to the observed cross sectional moments before the shock, which we assume to occur in the early 70s, and abut 35 years in the process, in the early 2000s.

Setup. Assume a cross section of agents, each of which can matched with a project producing an income. Agents without a project have income y_0 and are are matched with a project at rate φ. New projects start at income y_1, which grows at rate γ until the project is destroyed, which happens at rate δ. Thus a project surviving t periods yields the income $y(t) = y_1 e^{\gamma t}$. Let $p = \{0, 1\}$ denote the agent's state (with or without a project).

In steady state the economy will have a fraction $\omega \equiv \delta/(\varphi + \delta) \in (0, 1)$ of agents with no projects, i.e. income y_0. Noting that the distribution of project durations is exponential, i.e. it has density $f(t) = \delta e^{-\delta t}$, we compute the density of y by change of variables using $y(t)$ gives

$$h(y) = \frac{\alpha\, y_1^{\alpha}}{y^{(\alpha+1)}} \quad \text{where} \quad \alpha \equiv \frac{\delta}{\gamma} \tag{5}$$

which is a Pareto distribution with CDF $H(y) = 1 - \left(\frac{y_1}{y}\right)^{\alpha}$. For the distribution to have a finite mean (income) it must be that $\alpha > 1$ i.e. $\delta > \gamma$, in which case we have that the mean

income (conditional on $p = 1$) is

$$\mathbb{E}\left(y\middle|p=1\right) = y_1 \frac{\alpha}{\alpha - 1} \quad , \quad \text{median: } \; y_1\, 2^{(1/\alpha)}$$

Thus the mean income in the population is

$$\mathbb{E}\,(y) = y_0\omega + y_1(1-\omega)\frac{\alpha}{\alpha-1} \tag{6}$$

which is decreasing in α and decreasing in ω.

Steady state moments Now we compute $\mathbb{E}(g_i|y,T)$ the expected income growth for an agent with income y over a time period of length T. Let $M(y,T) \equiv \mathbb{E}(y(T)|y(0)=y)$ denote the expected value of income in T periods for an agent with current income y. This is

$$M(y,T) = ye^{(\gamma-\delta)T} + \int_0^T \delta e^{-\delta s} m(T-s)ds \tag{7}$$

where $m(T) \equiv \mathbb{E}(y(T)|y(0)=y_0)$ is the expected value of income T periods from now for an agent whose current state is $p = 0$ (i.e. in a no growth state, hence with income level $y = y_0$). Some analysis reveals that

$$m(T) = y_1 \int_0^T \varphi\, \theta_0(s) e^{(\gamma-\delta)(T-s)} ds + y_0\theta_0(T) \;\; \text{where} \;\; \theta_0(s) = \frac{\delta + \varphi e^{-(\varphi+\delta)s}}{\varphi+\delta} \tag{8}$$

where $\theta_0(s)$ is a type of "survival-statistic" that, conditional on an agent being in state $p = 0$ at time zero, gives the probability that the agent is still in that state after s periods (this takes into account the possibility of leaving the state and coming back to it).[10] Notice that the steady state fraction of agents at $p = 0$ defined above, namely $\omega = \delta/(\delta+\varphi)$, obtains in

[10] This is easily derived as the limit of the discrete time flow accounting. Let $\Delta > 0$ be the time period length, then $\theta_0(s+\Delta) = (1-\Delta\varphi)\theta_0(s) + (1-\theta_0(s))\delta\Delta$. Using a first order expansion an taking the limit for $\Delta \to 0$ gives the ode $\theta_0'(s) = \delta - (\varphi+\delta)\theta_0(s)$ with boundary condition $\theta_0(0) = 1$, which gives the equation in the text.

the limit as $\lim_{s\to\infty}\theta_0(s)=\omega$. We get

$$m(T)=y_1\left(\frac{\alpha-\omega}{\alpha-1}+\frac{\varphi\gamma}{\varphi+\gamma}\left(\frac{e^{-(\varphi+\delta)T}}{\varphi+\delta}-\frac{e^{-(\delta-\gamma)T}}{\delta-\gamma}\right)\right)+(y_0-y_1)\frac{\delta+\varphi e^{-(\varphi+\delta)T}}{\varphi+\delta} \tag{9}$$

where we used $\alpha\equiv\delta/\gamma$. Notice again that as $T\to\infty$ the expected income level converges to the average cross sectional income computed above.

Using Equation (9) into equation (7) we thus have an analytic expression to compute the expected income growth in the cross section over a period of length T, which is given by

$$\mathbb{E}(g_i,T)\equiv\omega\frac{m(T)}{y_0}+(1-\omega)\int_{y_1}^{\infty}h(y)\frac{M(y,T)}{y}dy \tag{10}$$

5.1 A permanent shock and the distribution of incomes during the transition

Suppose at time $t=0$ the fundamental parameters φ, δ and γ change to new values $\tilde{\varphi}$, $\tilde{\delta}$ and $\tilde{\gamma}$. In particular assume that the new success rate $\tilde{\varphi}$, as well as failure rates $\tilde{\delta}$ applies for all $t>0$. The new growth rate $\tilde{\gamma}$ will only apply to successful project initiated after $t=0$.

In this section we solve in closed form the PDE for the Kolmogorov Forward equation to compute the density of income levels during the transition. In particular, we assume that right after the shock the new projects are created at rate $\tilde{\varphi}$ and grow at rate $\tilde{\gamma}$, die at rate $\tilde{\delta}$, so that the new steady state fraction of agents with no project will be $\tilde{\omega}\equiv\tilde{\delta}/(\tilde{\varphi}+\tilde{\delta})$. We also assume that existing active projects will fade out at the old rate δ, so that t periods after the shock there will be a mass of survivors equal to $(1-\omega)e^{-\delta t}$.

The fraction of agents with $p=0$ after t periods in the transition is given by the sum of the new and old survivors, given by the function

$$\tilde{\omega}(t)=\tilde{\omega}+(\omega-\tilde{\omega}+\chi)\,e^{-(\tilde{\delta}+\tilde{\varphi})t}-\chi e^{-\delta t}\quad\text{where}\quad\chi\equiv\frac{(\tilde{\delta}-\delta)(1-\omega)}{\tilde{\varphi}+\tilde{\delta}-\delta} \tag{11}$$

Notice that after the shock the agents with the project (i.e. with $p = 1$) come in 2 types. Agents with a new project and agents with the old project, and that $\tilde{\omega}(0) = \omega$ while as $t \to \infty$ then $\tilde{\omega}(t) \to \tilde{\omega}$.

The domain for y for the new type is $y \in (y_1, y_M(t))$ with $y_M(t) = y_1 e^{\tilde{\gamma} t}$ where t is the time elapsed since the shock. Let $\tilde{f}(y,t)$ denote the density of y at time t conditional on $p = 1$ *and the project being a new variety.* We want to characterize the density $\tilde{f}(y,t)$ during a transition towards the new invariant Pareto distribution. Note that the support of this distribution is $(y_1, y_1 e^{\tilde{\gamma} t})$ where y_1 is the injection point where mass flows in at a rate $\varphi\tilde{\omega}(t)$.

The density obeys $\tilde{f}(y,t)$ the KFE

$$\frac{\partial}{\partial t}\tilde{f}(y,t) = -\frac{\partial}{\partial y}\left(\tilde{f}(y,t)\tilde{\gamma} y\right) - \tilde{\delta}\tilde{f}(y,t) \tag{12}$$

After t periods the mass of agents with an active project of the new type, with $y \in (y_1, y_M(t))$ with $y_M(t) = y_1 e^{\tilde{\gamma} t}$, is given by $\eta(t)$ which is[11]

$$\eta(t) = 1 - \tilde{\omega}(t) - (1 - \omega)e^{-\delta t} \tag{13}$$

Next we use an eigenvalue-eigenfunction decomposition to solve the above PDE by separating its variables. Conjecture that the solution is separable

$$\tilde{f}(y,t) = \sum_{j=1}^{\infty} e^{\lambda_j t} f_j(y)$$

then the KFE gives

$$\lambda_j f_j(y) = -f_j'(y)\tilde{\gamma} y - (\tilde{\delta} + \tilde{\gamma}) f_j(y) \quad \text{for} \quad j = 0, 1, 2, ...$$

So that $f_j(y) = A_j y^{-\left(1 + \frac{\tilde{\delta} + \lambda_j}{\tilde{\gamma}}\right)}$. We solve for λ_j, A_j by ensuring the density satisfies mass

[11] This is immediate since the total mass of active projects at time t is given by $1 - \tilde{\omega}(t)$, from which we substract the mass of old projects equal to $(1 - \omega)e^{-\delta t}$.

preservation as given by

$$\int_{y_1}^{y_1 e^{\tilde{\gamma}t}} \tilde{f}(y,t)dy = \eta(t) \tag{14}$$

This gives

$$\sum_{j=0}^{\infty} \frac{A_j \tilde{\gamma}}{\tilde{\delta}+\lambda_j} y_1^{-\frac{\tilde{\delta}+\lambda_j}{\tilde{\gamma}}} \left(e^{\lambda_j t} - e^{-\tilde{\delta}t}\right) = 1-\tilde{\omega} - (\omega - \tilde{\omega} + \chi)e^{-(\tilde{\delta}+\tilde{\varphi})t} - (1-\omega-\chi)e^{-\delta t} \tag{15}$$

and matching coefficients gives the eigenvalue-eigenfunction pair associated to the invariant distribution $\lambda_0 = 0, A_0 = (1-\tilde{\omega})\tilde{\alpha}y_1^{\tilde{\alpha}}$, and two more eigenvalue-eigenfunction pair associated to the transition, $\lambda_1 = -\delta, A_1 = -(1-\omega-\chi)\frac{(\tilde{\delta}-\delta)}{\tilde{\gamma}} y_1^{\frac{(\tilde{\delta}-\delta)}{\tilde{\gamma}}}$ and $\lambda_2 = -(\tilde{\delta}+\tilde{\varphi}), A_2 = (\omega - \tilde{\omega} + \chi)\frac{\tilde{\varphi}}{\tilde{\gamma}} y_1^{-\frac{\tilde{\varphi}}{\tilde{\gamma}}}$ while all other pairs $\{\lambda_j = A_j = 0\}$ for $j = 3, 4...$, so that we have

$$\tilde{f}(y,t) = (1-\tilde{\omega})\frac{\tilde{\alpha}y_1^{\tilde{\alpha}}}{y^{1+\tilde{\alpha}}} - e^{-\delta t}(1-\omega-\chi)\frac{\frac{(\tilde{\delta}-\delta)}{\tilde{\gamma}} y_1^{\frac{(\tilde{\delta}-\delta)}{\tilde{\gamma}}}}{y^{1+\frac{(\tilde{\delta}-\delta)}{\tilde{\gamma}}}} + e^{-(\tilde{\delta}+\tilde{\varphi})t}(\omega-\tilde{\omega}+\chi)\frac{\frac{\tilde{\varphi}}{\tilde{\gamma}} y_1^{-\frac{\tilde{\varphi}}{\tilde{\gamma}}}}{y^{1-\frac{\tilde{\varphi}}{\tilde{\gamma}}}} \tag{16}$$

The fading-out of old active projects. As mentioned we assume that after the shock the mass of survivors with the old project continues to grow at the old rate γ and gradually fades out at the old rate δ. At time t this mass is distributed over the domain $y \in (y_m(t), \infty)$ where $y_m(t) = y_1 e^{\gamma t}$.

The transition density of the active ($p = 1$) old agents, given by $h(y,t)$, obeys the kolmogorov forward equation

$$\frac{\partial}{\partial t}h(y,t) = -\frac{\partial}{\partial y}\left(h(y,t)\gamma y\right) - \delta h(y,t) \tag{17}$$

which is analogue to equation (12), with mass-preserving constraint

$$\int_{y_m(t)}^{\infty} h(y,t)dy = (1-\omega)e^{-\delta t} \tag{18}$$

It is easy to verify that the density function $(1-\omega)h(y)$, where $h(y)$ is the invariant density function given in equation (5), associated to the zero eigenvalue, satisfies the above equations.

Density of income for active ($p = 1$) projects t periods after the shock. Hence t periods after the shock the density over the interval (y_1, ∞) is

$$f(y,t) = \tilde{f}(y,t)\mathcal{I}_n(y) + (1-\omega)h(y)\mathcal{I}_o(y) \quad \text{for} \quad y \in (y_1, \infty) \tag{19}$$

where

$$\begin{cases} \mathcal{I}_n(y) = 1 \ \text{ if } \ y \in (y_1, y_M(t)), \quad \mathcal{I}_n(y) = 0 \ \text{ otherwise} \\ \\ \mathcal{I}_o(y) = 1 \ \text{ if } \ y \in (y_m(t), \infty), \quad \mathcal{I}_o(y) = 0 \ \text{ otherwise} \end{cases}$$

Some algebra shows that $\int_{y_1}^{\infty} f(y,t)dy = 1 - \tilde{\omega}(t)$ for all t.

5.2 Moments during transition

The mass of agents without a project in the cross section is equal to $\omega = \delta/(\delta+\varphi)$ before the shock, while t periods after the shock the mass of agents at y_0 is given by $\tilde{\omega}(t)$ in equation (11), where $\tilde{\omega} = \tilde{\delta}/(\tilde{\delta}+\tilde{\varphi})$ is the new steady state value as $t \to \infty$.

Assume also that the income for an agent without a project evolves exogenously, t periods after the shock, as $y_0(t)$, with $y_0(0) = y_0$. We will still assume that whenever the agent exits the $p = 0$ state, she will start from income y_1. Thus y_1 is the (time invariant) "initial income" agents get when they start a new project. Thus the aggregate income of the economy, t periods after the shock is

$$\mathbb{E}(y,t) = \tilde{\omega}(t)y_0(t) + \int_{y_1}^{\infty} y\, f(y,t)\, dy \tag{20}$$

where $f(y,t)$ was given in equation (19).

Now we compute $\mathbb{E}(g_i|y,T,t)$ the expected income growth over a time-period of length T, computed t periods after the shock, for an agent who has income y. We begin by computing the expected income levels conditional on the current y in a horizon of T periods.

If the agent is in state $p = 0$ then the expected value of income over a time period T is computed following the logic used for the steady state in equation (8), with the difference that the formula will now use the new parameters and that the income of the poor ($p = 0$) at time t is $y_0(t)$:

$$\tilde{m}(t,T) = y_1 \int_0^T \tilde{\varphi}\, \theta_0(s) e^{(\tilde{\gamma}-\tilde{\delta})(T-s)} ds + \theta_0(T) y_0(t+T) \quad \text{where} \quad \theta_0(s) = \frac{\tilde{\delta} + \tilde{\varphi} e^{-(\tilde{\varphi}+\tilde{\delta})s}}{\tilde{\varphi}+\tilde{\delta}} \tag{21}$$

where $\theta_0(s)$ denotes the probability that an agent who does not grow ($p = 0$) at time $s = 0$ is in the same state after s periods.

We get the closed form expression:

$$\tilde{m}(t,T) = y_1 \left(\frac{\tilde{\alpha}-\tilde{\omega}}{\tilde{\alpha}-1} + \frac{\tilde{\varphi}\tilde{\gamma}}{\tilde{\varphi}+\tilde{\gamma}} \left(\frac{e^{-(\tilde{\varphi}+\tilde{\delta})T}}{\tilde{\varphi}+\tilde{\delta}} - \frac{e^{-(\tilde{\delta}-\tilde{\gamma})T}}{\tilde{\delta}-\tilde{\gamma}} \right) \right) + (y_0(t+T) - y_1) \frac{\tilde{\delta} + \tilde{\varphi} e^{-(\tilde{\varphi}+\tilde{\delta})T}}{\tilde{\varphi}+\tilde{\delta}} \tag{22}$$

where we used $\tilde{\alpha} \equiv \tilde{\delta}/\tilde{\gamma}$ and $\tilde{\omega} \equiv \tilde{\delta}/(\tilde{\delta}+\tilde{\varphi})$.

Note that t periods after the shock there is a mass of agents in state $p = 1$ which keeps growing at rate γ and terminate operating at rate δ. The expected value of income over an horizon T for these "old" agents with current income y is

$$M^o(y,t,T) = y e^{(\gamma-\delta)T} + \int_0^T \delta e^{-\delta s} \tilde{m}(t+s, T-s) ds \tag{23}$$

Notice that this statistic depends on t, the time elapsed since the shock, only through the income of the poor $y_0(t)$ which depends on "calendar" time t and is one component of the function $\tilde{m}(t,T)$.

The new projects initiated after the shock grow at rate $\tilde{\gamma}$. After t periods since the shock occurred the expected value of income for an agent with a new project and current income y, over an horizon T, is

$$M^n(y,t,T) = y e^{(\tilde{\gamma}-\tilde{\delta})T} + \int_0^T \tilde{\delta} e^{-\tilde{\delta} s} \tilde{m}(t+s, T-s) ds \tag{24}$$

The expected cross-sectional income growth varies with t, the time elapsed since the shock, because of the time varying composition of the agents (e.g. the agents with the old projects growing at rate γ will gradually disappear) and because the income of the poor $y_0(t)$ (may) change.

After t periods the expected growth rate of income over an horizon T is

$$\mathbb{E}(g_i, t, T) \equiv \tilde{\omega}(t)\frac{\tilde{m}(t,T)}{y_0(t)} + \int_{y_1}^{y_1 e^{\tilde{\gamma}t}} \tilde{f}(y,t)\frac{M^n(y,t,T)}{y}dy + \int_{y_1 e^{\gamma t}}^{\infty} h(y,t)\frac{M^o(y,t,T)}{y}dy \tag{25}$$

Notice that $\mathbb{E}(g_i, t, T)$ is a forward looking variable (expectation over future horizons). It jumps the moment the shock hits since agents know the new parameters will apply from that moment onwards. An identical logic is used to compute higher moments such as the variance of growth rates.

Using the equation (20) for the aggregate output dynamics and equation (25) it is also straightforward to compute the expected value for the covariance between g_i and s_i after t periods since the shock occurred.

5.3 A quantitative exercise

Next we parametrize the model using the data form the early 70s and the recent data. The thought experiment is that the economy was initially in a steady state, when a shock to the fundamental parameters of the income process occurred. We first parametrize the model to fit the steady state of the 1970s. We then choose the post-shock parameters assuming they changed in the early 70s and that, after $t = 35$ years (in the early 2000s), they generate the moments we see in the data.

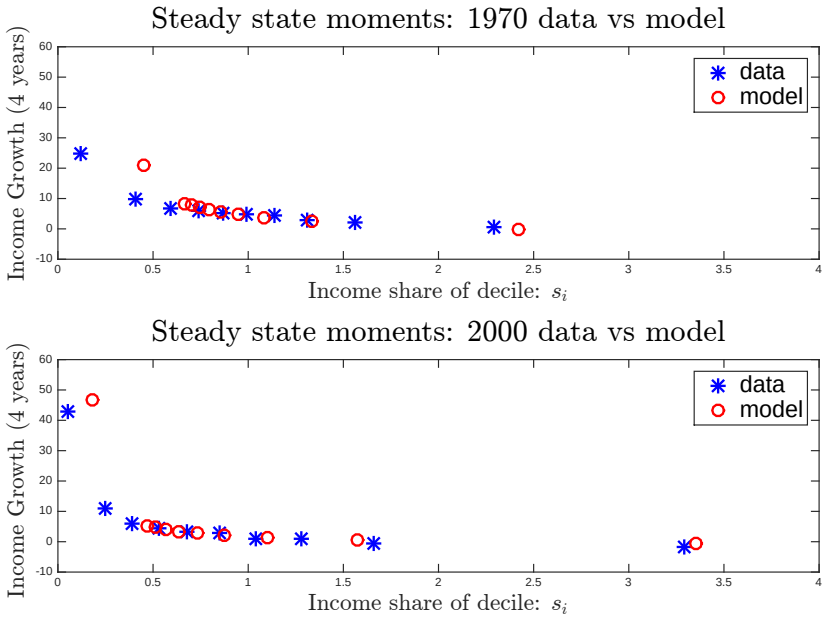
To bring the model to the data we assume that the individual growth rates are made of a common trend component, $\bar{g}_t$, and of an individual specific component g_i. The growth decomposition of Section 2 implies that $\Gamma_t = \bar{g}_t + \mathbb{E}(g_i) + cov(s_i, g_i)$. In a steady state, where the shape of the income distribution is constant, then $\Gamma_t = \bar{g}_t$ and $\mathbb{E}(g_i) = -cov(s_i, g_i)$.

Without loss of generality we normalize $y_1 = 1$ and use three set of moments to calibrate

Table 3: Target moments and model parameters

	1967-1973		2000-2010	
	data	model	data	model
$\text{std}(s_i)$	0.62	0.53	0.94	0.87
$\text{cov}(g_i, s_i)$	-0.03	-0.02	-0.06	-0.05
$\text{std}(g_i)$	0.07	0.06	0.13	0.15
Model calibrated parameters				
δ		0.16		0.09
γ		0.064		0.067
φ		3.9		1.1
y_0/y_1		0.25		0.25

Figure 10: Cross sectional patterns in data vs model



the remaining model parameters $\{\delta, \gamma, \varphi, y_0\}$. Since our model is mute about the common growth factor we only use moments that do not depend on it. We interpret ω as the fraction of households with a near zero income, a fraction which is about 3% of the population in the early 70s and almost doubles in the early 2000. The value of y_0 is chosen to match the growth rate of incomes observed in the first decile, a value which is high since it involves

some households jump from y_0 to y_1. Second, we choose δ to match the probability that a household in the top decile remains in the same decile after 4 years. Since the probability of remaining in the top decile moves from about 0.52 to 0.68 from the early 1970s to the early 2000s, we use $\delta \approx 16\%$ and $\tilde{\delta} \approx 9\%$ for each of these periods, respectively. The targets for the fraction of agents at y_0, respective 3% and 6% in each period, allow us to retrieve information on φ using $\omega = \delta/(\varphi + \delta)$ and using equation (11) for the early 2000s assuming $t = 35$. Finally we choose a value for γ in the 1970s to match the inequality, as measured by $std(s_i)$ and to match the observed $cov(s_i, g_i)$, where both moments are measured at the level of incomes deciles, indexed by $i = 1, ..., 10$.

Table 3 reports the targeted moments, in the data and as fitted by the model, and model parametrization. To fit the moments in the early 2000 we assumed that the shock (change in the fundamental parameters) occurred in 1970, so that the time elapsed since the shock in the early 2000 is $t = 35$ years. We then used the transition moments discussed in Section 5.2 to choose the parameter $\tilde{\gamma}$ to best match the targeted model moments.

Figure 11: Aggregate dynamics during the transition

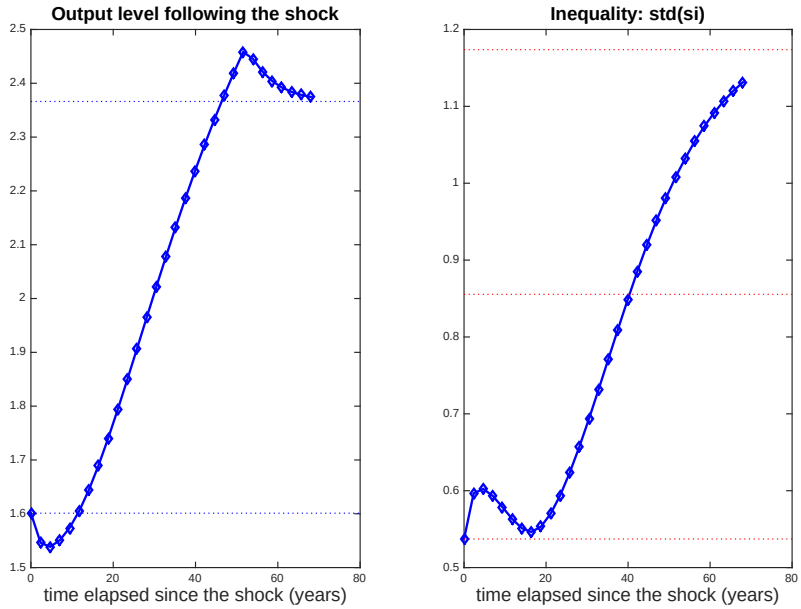


Figure 10 plots the cross sectional patterns implied by the model over these two target periods. It is apparent that the model requires a longer Pareto tail to fit the more recent data, something which is achieved by picking a lower value for $\tilde{\alpha} = \tilde{\delta}/\tilde{\gamma}$. As shown in Table 3, the reduction occurs because the duration of projects increases (lower $\tilde{\delta} < \delta$) and the growth of the active project increases relative to the old ones ($\tilde{\gamma} > \gamma$).

Figure 11 displays the Aggregate dynamics during the transition implied by this model calibration. The transition unfolds slowly and takes about 60 years. The aggregate time series displays an initial oscillation, due to the changing mass of agents at y_0, and is then followed by a long period in which both output and inequality increase. Over the first 40 years inequality increases by an amount that is comparable to the one seen in the data, and the average output effect of these changes is an average 0.8% growth per year over this period.

5.4 Welfare assessment under incomplete markets

As a preliminary (and worst case) assessment we consider the case of autarky (next we'll consider a bond economy) for which a simple analytic solution exists. We thus assume that there are no financial markets and that consumption equals income. The value of the consumption stream of an agent without project $p = 0$ is

$$v_0(\delta, \gamma, \varphi) = \mathbb{E}\left(\int_0^\infty e^{-\rho t} u(y(t)) dt \,|y(0) = y_0, p = 0\right) \tag{26}$$

where ρ is the intertemporal discount rate, and the arguments δ, γ, φ stress that the value depends on this triplet of fundamental parameters. Likewise, let $v_1(y; \delta, \gamma, \varphi)$ denote the value for agent with an active project, so that $p = 1$, income level y and parameters δ, γ, φ, which is

$$v_1(y; \delta, \gamma, \varphi) = \mathbb{E}\left(\int_0^\infty e^{-\rho t} u(y(t)) dt \,|y(0) = y, p = 1\right) \tag{27}$$

Notice that v_0 and v_1 solve the following HJB equations

$$\rho v_0 = u(y_0) + \varphi(v_1(y_1) - v_0) \quad , \quad \rho v_1(y) = u(y) + v_1'(y) y \gamma + \delta(v_0 - v_1(y))$$

Assume $u(y) = \frac{y^{1-\theta}}{1-\theta}$, as done above. Guess the solution is $v_1(y) = A + By^{\beta}$, substitute and match coefficients to get

$$v_0(\delta, \gamma, \varphi) = \frac{\rho + \delta}{(\rho + \delta + \varphi)\rho(1-\theta)} \left(y_0^{1-\theta} + \frac{\varphi y_1^{1-\theta}}{\rho + \delta - (1-\theta)\gamma} \right) \tag{28}$$

and for $v_1(y)$

$$v_1(y; \delta, \gamma, \varphi) = \frac{1}{\rho(1-\theta)} \left(\frac{\delta y_0^{1-\theta}}{\rho + \delta + \varphi} + \frac{\delta \varphi y_1^{1-\theta}}{(\rho + \delta + \varphi)(\rho + \delta - (1-\theta)\gamma)} + \frac{\rho y^{1-\theta}}{\rho + \delta - (1-\theta)\gamma} \right) \tag{29}$$

We use the parametrization of Table 3 to develop a simple welfare assessment under autarky. We construct the ex ante welfare measure for the 1970s using the parameters of the left side of the table and compute the ex ante welfare using the steady state distribution of incomes for that period, which we denote by $F_{70}(y)$, so that the ex ante welfare measure is $W_{70} = \omega_{70} v_0(\delta, \gamma, \varphi) + (1 - \omega_{70}) \int_y v_1(y; \delta, \gamma, \varphi) dF_{70}(y)$.

We compare this welfare measure with two alternatives. One is the ex-ante welfare in the steady state that will be achieved once the distribution of income settles on the new invariant implied by the change of parameters, $F_{00}(y)$. The ex-ante welfare under this measure is given by $W_{00} = \omega_{00}(\tilde{\gamma}, \tilde{\varphi}, \tilde{\delta}) + (1 - \omega_{00}) \int_y v_1(y; \tilde{\gamma}, \tilde{\varphi}, \tilde{\delta}) dF_{00}(y)$, where the notation stresses that the value functions v_0 and v_1 depend on the new parameters (indexed by a tilde). Obviously reaching the new state-state takes a long time, about 60 years in the simulations, so this welfare measure ignores the transition costs. An alternative welfare measure is the value function for an agent right after the change in parameter occurs. For an agent with $p = 0$ this is simply v_0 evaluated at the new parameters. The value for an agent with $p = 1$ and income y solves $(\rho + \delta)\tilde{v}_1(y) = u(y) + \tilde{v}_1'(y) y \gamma + \delta v_0(\tilde{\gamma}, \tilde{\varphi}, \tilde{\delta})$ where the notation stresses that

Table 4: Welfare-cost under autarky

	Cost on impact	Cost in 2000
$\theta = 2$	+8%	-9%
$\theta = 4$	+24%	+19%

Ex-ante Welfare is expressed in consumption equivalent units, relative to consumption in the 1970. As for the Bewley model the computation assumes a time discount $\rho = 0.03$.

the value v_0 depends on the new parameters (indexed by a tilde) while the function $\tilde{v}_1$ (for the active projects at the time of the shocks) continue to obey the old law of motion given by δ, γ. This gives

$$\tilde{v}_1(y; \delta, \gamma) = \frac{\delta v_0(\tilde{\delta}, \tilde{\gamma}, \tilde{\varphi})}{\rho + \delta} + \frac{y^{1-\theta}}{(1-\theta)\left(\rho + \delta - (1-\theta)\gamma\right)} \tag{30}$$

The ex-ante welfare one instant after the shock is thus given by $\tilde{W}_{70} = \omega_{70} v_0(\tilde{\gamma}, \tilde{\varphi}, \tilde{\delta}) + (1 - \omega_{70}) \int_y \tilde{v}_1(y; \delta, \gamma) dF_{70}(y)$.

Table 4 reports some preliminary welfare results under alternative assumptions about the agent's risk aversion, a crucial parameter as the ex ante welfare crucially hinges on the concavity of the utility function. With a risk aversion of about $\theta = 2$ the change in the fundamental parameters leads to a welfare loss on impact in the order of about 8% of permanent consumption (second column of the table). This welfare loss originates since under the parameters it becomes harder for agents to come up with a successful project ($\tilde{\varphi} < \varphi$), so that the prospects of being stuck at $p = 0$ makes agents ex-ante unhappy. Eventually, though, once the new invariant distribution settles in, there are enough agents with high income for whom the new regime yields substantial welfare gains. This is why, in the long run, the new regime leads to a welfare improvement (in the order of 9% of permanent consumption).

Increasing the risk aversion of agents to $\theta = 4$ further decreases the ex-ante desirability of the new regime. The table shows that the impact effect now leads to a larger welfare reduction (-24% in consumption equivalent terms), and that the welfare loss persists even

Table 5: Welfare Cost under a Bond-economy

	Cost on impact	Cost in 2000
$\theta = 2$	-1.4%	-26%
$\theta = 4$	-1%	-15%
$\theta = 6$	+0.2%	-14%

Ex-ante Welfare is expressed in consumption equivalent units, relative to consumption in the 1970. As for the Bewley model the computation assumes a time discount $\rho = 0.03$, and a risk free bond $R = 1.025$.

once the new steady state is eventually reached.

The welfare effects under autarky are obviously an extreme example of incomplete markets. Next we present a welfare analysis in an economy that allows agents to self-insure using a a riskless bond with gross return R and a borrowing constraint at zero, so that each agent assets is $a \geq 0$. This economy has a 2 dimensional state space, given by $\{a, y\}$. We solve numerically for the value function $v_0(a, y_0)$ and $v_1(a, y)$ where $y \in (y_1, \infty)$, where the subscript indicates whether the agent income grows ($p = 0$ vs $p = 1$). We compute the invariant distribution over income and assets numerically, and develop simple welfare analysis as done above.

Table 5 reports the preliminary results of the analysis. It is evident that the bond allows the agent to self-insure against adverse income shocks, so that the overall welfare outcomes are superior to autarky. in spite of the fact that under the new income process the welfare of agents without a job ($p = 0$) is much smaller than before, the ex-ante welfare effect of the shock is to increase welfare for $\theta = 2$ and $\theta = 4$. A large risk aversion equal to $\theta = 6$ is necessary to yield a negative ex ante welfare effect on impact. In the long run, the welfare consequences of the shock are positive and large for the parameters considered in the table.

6 Conclusions

We have shown that a statistical process for household earnings that involve more "unequal growth" , i.e. high earnings households growing (over time) faster and low earnings growing

(over time) slower can account well for the evolution of the US earnings distribution over the past 50 years. We have also shown that more unequal growth has a mild (between 0.5 and 1% per year) positive effect on aggregate growth, and a potentially very large (as high as 50% of lifetime consumption) negative welfare effect, when markets are incomplete. The natural next question is what is the driver of this increase in unequal growth? For some times there has been a lot of very exciting work that thinks about sources of unequal growth (see, for two recent examples of such work, Fogli and Guerrieri 2019, Moll, Rachel, and Restrepo 2019), and we believe that integrating our framework to these works can help us understand better the aggregate consequences of changes in the formation of individual earnings. We also find that, with the increase in unequal growth the social value of better (private or public) insurance mechanisms increase tremendously, and thus another relevant research direction is how to improve such mechanisms.

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