A New Keynesian Model with Wealth in the Utility Function

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The New Keynesian model suffers from several anomalies at the zero lower bound: explosive output and inflation, forward-guidance puzzle, and explosive government-spending multiplier. To resolve these anomalies, we introduce relative wealth into households' utility function; the justification is that relative wealth is a marker of social status, and people value high social status. Since people save not only for future consumption but also to accrue social status, the Euler equation is modified. As a result, when the marginal utility of wealth is sufficiently large, the dynamical system representing the equilibrium at the zero lower bound becomes a source instead of a saddle, which resolves all the anomalies.

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1. Introduction

A current issue in monetary economics is that the textbook New Keynesian model suffers from several anomalies when the zero lower bound on nominal interest rates (ZLB) is binding. Indeed, the model predicts that at the ZLB, output and inflation spiral down to implausibly low levels (Eggertsson and Woodford 2004; Cochrane 2017); forward-guidance policies have implausibly large effects (Del Negro, Giannoni, and Patterson 2015; Carlstrom, Fuerst, and Paustian 2015; Cochrane 2017); and government-spending multipliers become implausibly large (Christiano, Eichenbaum, and Rebelo 2011; Woodford 2011; Cochrane 2017).

The anomalies of the textbook model can all be traced to the behavior of the dynamical system representing the equilibrium at the ZLB. This dynamical system is composed of two differential equations: the Phillips curve, describing how producers set prices, and the Euler equation, describing how consumers spend and save. While the system is a source in normal times, at the ZLB the system becomes a saddle . The saddle property shapes the model's dynamics at the ZLB and is solely responsible for all the anomalies.

This paper proposes to remedy these anomalies by introducing relative wealth into households' utility function. The justification for this assumption is that relative wealth is a marker of social status, and people value high social status. Under this assumption, people save not only for future consumption but also because they accrue social status from their wealth. Thus, the Euler equation is modified—all the more so when the marginal utility of wealth is large. With sufficient marginal utility of wealth, the Euler equation is sufficiently altered that the dynamical system representing the equilibrium becomes a source instead of a saddle at the ZLB. At that point all the anomalies at the ZLB disappear. This is because when the system is a source, the response to any temporary shock is a muted version of the response to a permanent shock, which is itself finite.

Our results require that the marginal utility of wealth is large enough relative to the slope of the Phillips curve. Hence, we develop a method to measure the marginal utility of wealth. In steady state the Euler equation imposes that the real interest rate equals the time discount rate minus the marginal rate of substitution between wealth and consumption; hence the marginal utility of wealth is proportional to the gap between the time discount rate and the real interest rate. Since the time discount rate is significantly above market interest rates, we infer that the marginal utility of wealth is quite high. On the other hand, the Phillips curve is estimated to be quite flat. Overall, the requirement for our results is satisfied for a large majority of available estimates.

Justification for the Assumption of Wealth in the Utility Function. While modern macroeconomic models assume that people save only to be able to consume in the future, it has long

been recognized that people save perhaps for other reasons. For instance, observing the behavior of the European upper class at the beginning of the 20th century, Keynes (1919, chap. 2) noticed that "The duty of saving became nine-tenths of virtue and the growth of the cake the object of true religion. . . . There grew round the non-consumption of the cake all those instincts of puritanism. . . . Saving was for old age or for your children; but this was only in theory—the virtue of the cake was that it was never to be consumed, neither by you nor by your children after you." A few years later, Irving Fisher observed that "A man may include in the benefits of his wealth . . . the social standing he thinks it gives him, or political power and influence, or the mere miserly sense of possession, or the satisfaction in the mere process of further accumulation" (Fisher 1930, p. 17). Fisher's perspective is particularly interesting, since the modern view of saving as a way to optimize one's consumption stream is often attributed to him.

These observations that people derive utility from accumulating wealth in a way that is not tied to future consumption seems supported by neuroscientific evidence. In a survey of the literature, Camerer, Loewenstein, and Prelec (2005, p. 32) note that "brain-scans conducted while people win or lose money suggest that money activates similar reward areas as do other "primary reinforcers" like food and drugs, which implies that money confers direct utility, rather than simply being valued only for what it can buy."

In modeling the effect of wealth on utility, we need to decide what exactly about wealth brings utility to people. We assume that people enjoy accumulating wealth because it brings them social status: it marks their position on the totem pole. We therefore postulate that what enters into the utility function is the household's relative wealth: the distance of their wealth from the average. This assumption is convenient: in equilibrium, without heterogeneity, any household's wealth equals the average wealth, so relative wealth is zero. The assumptions that people seek to achieve high social status by accumulating wealth also seems plausible. First, Adam Smith, Ricardo, John Rae, J.S. Mill, Marshall, Veblen, and Frank Knight all supported the idea that people accumulate wealth to attain high social status (Steedman 1981). More recently, a vast literature across the social sciences has documented that people seek to achieve high social status; and that while there are various ways to gain social status, accumulating wealth is a prevalent pathway to do so.[1]

Related Literature. Before us, Eggertsson and Mehrotra (2014), Gabaix (2016), Diba and Loisel (2017), and Cochrane (2018) have developed variants of the New Keynesian model that cure all its ZLB anomalies. These strategies, like ours, result in determinacy of the ZLB equilibrium, which eliminates the ZLB anomalies of the New Keynesian model. Eggertsson

[1]Weiss and Fershtman (1998) and Heffetz and Frank (2011) review the evidence from economics; Fiske (2010), Anderson, Hildreth, and Howland (2015), and Cheng and Tracy (2013) from social psychology; Ridgeway (2014) from sociology; and Mattan, Kubota, and Cloutier (2017) from social neuroscience.

and Mehrotra modify the Euler equation with overlapping generations and the Phillips curve with downward nominal wage rigidity. Gabaix modifies the Euler equation and Phillips curve with boundedly rational households and firms. Diba and Loisel modify the Euler equation with a spread between interest rates on bonds and on bank reserves, and the Phillips curve with a banking cost. Finally, Cochrane maintains Euler equation and Phillips curve but also takes into account a fiscal theory of the price level.

More recently, Bilbiie (2018) and Acharya and Dogra (2018) have showed that under certain conditions, introducing heterogeneous agents into a New Keynesian model also yields a determinate ZLB equilibrium and thus resolves the New Keynesian anomalies at the ZLB. Bilbiie develops a model in which some agents cannot save at all while others can save without constraint. In such model an interest rate rule that responds to the price level instead of inflation leads to a determinate ZLB equilibrium. Acharya and Dogra propose a different model in which agents face idiosyncratic and uninsurable shocks to their endowment of labor. These shocks generate income risk, and when such risk is procyclical, the ZLB equilibrium becomes determinate.

These solutions to the ZLB anomalies tend to be analytically complicated, however. Eggertsson and Mehrotra (2014) require an overlapping-generations structure, Gabaix (2016) requires bounded rationality, and Bilbiie (2018) and Acharya and Dogra (2018) require heterogeneity—all of which increase the complexity of the derivations of the New Keynesian model. Diba and Loisel (2017) and Cochrane (2018) take into consideration one additional equation than in the textbook model, thus increasing the dimensionality of the dynamical system describing the equilibrium—from two to three dimensions. It may therefore be useful to strip the logic to the bone. We will do so by proposing a tractable model that deviates only minimally from the textbook model.

First, to improve the tractability of the New Keynesian model on which we base the analysis, we follow Benhabib, Schmitt-Grohe, and Uribe (2001) and introduce three modifications to the textbook version of the model (for example, Gali 2008, chap. 3): continuous time instead of discrete time; self-employed households instead of separate firms and households; and Rotemberg (1982) pricing instead of Calvo (1983) pricing. The modifications allow us to clarify how the ZLB anomalies are related and how they can be cured. Our New Keynesian model is also inspired by the deterministic continuous-time formulation of the New Keynesian model proposed by Werning (2011) to study optimal fiscal and monetary policy the ZLB; and also adopted by Cochrane (2017) to describe the ZLB anomalies and explain how they are affected by equilibrium selection. This continuous-time formulation offers a transparent description of the ZLB episode and is amenable to graphical analysis using phase diagrams. We rely on phase diagrams to illustrate the anomalies of the New Keynesian model, explain their origin, and resolve them by introducing wealth in the utility function.

Second, to solve the ZLB anomalies, we extend the New Keynesian model by assuming that households also derive utility from their relative wealth. The resulting deviation from the textbook model is minimal: the derivations of the Euler equation and Phillips curve are unchanged, and the equilibrium system remains 2-dimensional. Our method is related to the work of Ono and Yamada (2018), Michaillat and Saez (2014), and Michau (2018) in non-New-Keynesian models. These papers show that introducing absolute or relative wealth in the utility function may generate well-behaved zero-lower-bound steady states. These results are not directly portable to the New Keynesian model, however, because they require strong forms of wage or price rigidity: Ono and Yamada assume that wages follow an exogenous time path; Michaillat and Saez assume that inflation is fixed; and Michau assumes that nominal wages are downward rigid.[2]

Our method also relates to the work of Fisher (2015), Campbell et al. (2017), Rannenberg (2019), Hagedorn (2018), Kaplan and Violante (2018), and Auclert, Rognlie, and Straub (2018), who build New Keynesian models with government bonds in the utility function.[3] Our assumptions are slightly different: their households derive utility from safe and liquid assets (government bonds), not from their entire wealth (which also includes risky or illiquid assets); ours derive utility from the gap between their entire wealth and the average wealth in the economy.[4] The applications of the assumptions are also different. They find that their models are able to generate risk-premium shocks (Fisher), to ensure determinacy of the price level (Hagedorn), to alleviate the forward-guidance puzzle (Campbell et al.; Rannenberg), and to mimic the behavior of rich heterogeneous-agent models (Kaplan and Violante; Auclert, Rognlie, and Straub). We find that ours is able to ensure determinacy of the ZLB equilibrium and eradicate all New Keynesian anomalies at the ZLB.

2. Model

We present a tractable New Keynesian model in which households derive utility not only from consumption and leisure but also from relative wealth.

[2]Besides these papers, an emerging literature has introduced absolute or relative wealth in the utility function to revisit topics such as long-run growth (Kurz 1968; Konrad 1992; Zou 1994; Corneo and Jeanne 1997; Futagami and Shibata 1998; Corneo and Jeanne 2001), attitudes to risk (Robson 1992; Clemens 2004), asset pricing (Bakshi and Chen 1996; Gong and Zou 2002; Kamihigashi 2008; Michau, Ono, and Schlegl 2018), life-cycle consumption and saving (Zou 1995; Carroll 2000; Francis 2009), social stratification (Long and Shimomura 2004), international economics (Fisher 2005; Fisher and Hof 2005), financial crises (Kumhof, Ranciere, and Winant 2015), and capital taxation (Saez and Stantcheva 2018). The work by Cole, Mailath, and Postlewaite (1992) is also related: they provide a microfoundation for relative wealth in the utility function.

[3]The bonds-in-the-utility-function assumption was initially developed by Fair and Malkiel (1971), Poterba and Rotemberg (1987), and Krishnamurthy and Vissing-Jorgensen (2012).

[4]The two assumptions would only overlap in the special case where government bonds are the sole vehicule of wealth and are in zero net supply.

2.1. Assumptions

The economy is composed of a measure 1 of self-employed households. Each household $j \in [0,1]$ produces $y_j(t)$ units of a differentiated good j, sold to other households at a price $p_j(t)$. Household j's production function is

$$y_j(t) = a h_j(t), \tag{1}$$

where the parameter $a > 0$ represents the level of technology, and the variable $h_j(t)$ is hours of work. Household j incurs a disutility $h_j(t)$ from working.

The goods produced by households are imperfect substitutes for one another, so each household exercises some monopoly power. Moreover, households face a quadratic cost when they change their price: if household j changes its price at a rate $\pi_j(t) = \dot{p}_j(t)/p_j(t)$, it incurs a disutility

$$\frac{\gamma}{2}\pi_j(t)^2.$$

The parameter $\gamma > 0$ governs the cost to change prices and thus price rigidity.

Each household consumes goods produced by other households. Household j buys amounts $c_{jk}(t)$ of the different goods $k \in [0,1]$, which are aggregated into a consumption index

$$c_j(t) = \left[\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk\right]^{\epsilon/(\epsilon-1)}, \tag{2}$$

where $\epsilon > 1$ is the elasticity of substitution between goods. The consumption index yields utility $\ln(c_j(t))$ to the household. The price index is

$$p(t) = \left[\int_0^1 p_j(t)^{1-\epsilon}\, di\right]^{1/(1-\epsilon)}; \tag{3}$$

when households optimally allocate their consumption expenditure across goods, $p(t)$ is the price of one unit of consumption index. The inflation rate is $\pi(t) = \dot{p}(t)/p(t)$.

Next, households hold government bonds earning a nominal interest rate $i(t)$, determined by the central bank. Bonds are used to smooth consumption. In addition, people derive utility from their relative real wealth, which is the difference between their real wealth and the average real wealth in the economy. The motivation for this assumption is that relative wealth signals social status, and people value a high social status. Since bonds are the only store of wealth in the model, people also hold bonds to signal social status. Formally, each household j holds a

nominal quantity of bonds $b_j(t)$, which provides utility

$$u\left(\frac{b_j(t)}{p(t)} - \frac{b(t)}{p(t)}\right),$$

where the function u is increasing and concave, $b(t) = \int_0^1 b_k(t)\,dk$ is average nominal wealth, and $[b_j(t) - b(t)]/p(t)$ is household j's relative real wealth. The law of motion of bond holdings is

$$\dot{b}_j(t) = i(t)b_j(t) + p_j(t)y_j(t) - \int_0^1 p_k(t)c_{jk}(t)\,dk - \tau(t). \tag{4}$$

The term $i(t)b_j(t)$ is interest income; $p_j(t)y_j(t)$ is labor income; $\int_0^1 p_k(t)c_{jk}(t)\,dk$ is consumption expenditure; and $\tau(t)$ is a lump-sum tax used to service the government debt $b(t)$: $\tau(t) = i(t)b(t)$.

Then, taking as given aggregate variables, initial wealth $b_j(0)$, and initial price $p_j(0)$, each household j chooses time paths for $y_j(t)$, $p_j(t)$, $h_j(t)$, $\pi_j(t)$, $c_{jk}(t)$ for all $k \in [0,1]$, and $b_j(t)$ to maximize the discounted sum of instantaneous utilities

$$\int_0^\infty e^{-\delta t}\left[\ln\big(c_j(t)\big) + u\left(\frac{b_j(t)}{p(t)} - \frac{b(t)}{p(t)}\right) - h_j(t) - \frac{\gamma}{2}\pi_j(t)^2\right]dt, \tag{5}$$

where $\delta > 0$ is the time discount rate. The household faces four constraints: the production function (1); the budget constraint (4); the law of motion $\dot{p}_j(t) = \pi_j(t)p_j(t)$; and the demand for good j coming from other households' maximization:

$$y_j(t) = \left[\frac{p_j(t)}{p(t)}\right]^{-\epsilon} c(t), \tag{6}$$

where $c(t) = \int_0^1 c_k(t)\,dk$ is aggregate consumption. The household also faces a borrowing constraint preventing Ponzi schemes. Finally, we assume that all households face the same initial conditions, so they all behave the same.

Finally, we describe monetary policy. We first introduce the natural level of output, denoted y^n, and the natural rate of interest, denoted r^n:

$$y^n = a\frac{\epsilon - 1}{\epsilon} \tag{7}$$

$$r^n = \delta - u'(0)y^n. \tag{8}$$

We will see once we derive the Phillips curve and Euler equation that y^n and r^n have simple interpretations. The natural level of output is the level of output given by the steady-state Phillips

curve when prices are flexible ($\gamma = 0$), so it is the level of output maximizing firms' profits in the absence of price-adjustment cost. The natural rate of interest is the real interest rate given by the steady-state Euler equation when output is at its natural level, so it is the real rate under which households are happy to consume y^n. The steady state with zero inflation and output at its natural level is called natural steady state. In our analysis the central bank always aims to maintain the economy at the natural steady state.

In normal times, the natural rate of interest is positive, and the central bank follows a simple interest-rate rule:

$$i(\pi) = r^n + \phi\pi, \tag{9}$$

which implies a real interest rate

$$r(\pi) = r^n + (\phi - 1)\,\pi. \tag{10}$$

The parameter $\phi \geq 0$ governs the response of monetary policy to inflation: monetary policy is active when $\phi > 1$ and passive when $\phi < 1$. We will see that this policy allows the central bank to position the economy at the natural steady state.

When the natural rate of interest is negative, however, the natural steady state cannot be achieved because it would imply a negative nominal interest rate, violating the ZLB constraint. In that case, the central bank moves away from (9) and sets the nominal interest rate to zero in order to stimulate the economy:[5]

$$i = 0 \qquad \text{and} \qquad r = -\pi. \tag{11}$$

2.2. Equilibrium

The equilibrium is described by two differential equations: a Phillips curve and an Euler equation. These two equations determine two variables: inflation π and aggregate output y. (The formal derivations are fairly standard and relegated to appendix A.)

The Phillips curve arises from households' optimal pricing decisions:

$$\dot{\pi}(t) = \delta\pi(t) - \frac{\epsilon}{\gamma a}\left[y(t) - y^n\right], \tag{12}$$

[5]Following most of the literature, we separately consider the monetary policy (9) away from the ZLB and the monetary policy $i = 0$ at the ZLB. Then, we check that the ZLB is indeed not binding at the equilibrium obtained under (9). A more rigorous approach would consider a monetary-policy rule that directly incorporates the ZLB: $i(\pi) = \max\{0, r^n + \phi\pi\}$; we revisit our results under this approach in appendix D.

where y^n is given by (7). The steady-state Phillips curve, obtained by setting $\dot{\pi} = 0$ in (12), describes inflation as a linearly increasing function of output:

$$\pi = \frac{\epsilon}{\delta\gamma a}(y - y^n). \tag{13}$$

The steady-state Phillips curve shows that inflation is zero when output is at its natural level, and also that output is at the natural level of output y^n when prevailing when prices are flexible ($\gamma = 0$).

The Euler equation arises from households' optimal saving decisions:

$$\frac{\dot{y}(t)}{y(t)} = r(\pi(t)) + u'(0)y(t) - \delta. \tag{14}$$

The term $r(\pi)$ describes the real interest rate set by the central bank—it is a linear function of inflation given either by (10) or by (11). The marginal utility of wealth, $u'(0)$, is evaluated at zero because in equilibrium relative wealth is zero (since all households hold the same wealth). The steady-state Euler equation, obtained by setting $\dot{y} = 0$ in (14), describes inflation as a linear function of output:

$$r(\pi) = \delta - u'(0)y. \tag{15}$$

Through the steady-state Euler equation, output is a decreasing function of the real interest rate—as in the old-fashioned Keynesian IS curve—and output is at its natural level when the real interest rate equals the natural rate of interest, given by (8).

We now compare two submodels:

DEFINITION 1: *The New Keynesian (NK) model has zero marginal utility of wealth:* $u'(0) = 0$. *The wealth-in-the-utility New Keynesian (WUNK) model has sufficient marginal utility of wealth:*

$$u'(0) > \frac{\epsilon}{\delta\gamma a}. \tag{16}$$

The NK model is the textbook model; the WUNK model is the alternative proposed in this paper. When prices are completely fixed ($\gamma \to \infty$), condition (16) simply becomes $u'(0) > 0$; when prices are perfectly flexible ($\gamma = 0$), condition (16) becomes $u'(0) > \infty$, which cannot be satisfied; hence, the WUNK model is well defined at the fixed-price limit but not at the flexible-price limit. In the WUNK model, we also impose $\delta > \sqrt{(\epsilon - 1)/\gamma}$ so that the model accommodates positive natural rates of interest while respecting (16).

2.3. Heuristic Derivation of the Equilibrium Conditions

All our results arise from the Phillips curve and Euler equation, so before delving into the analysis we review their economic interpretations. The interpretation are not exactly standard because the Phillips curve is derived from Rotemberg (1982) pricing instead of the standard Calvo (1983) pricing, and the Euler equation includes utility from wealth. Moreover the continuous-time formulation is less familiar than the standard discrete-time formulation. For these reasons, it is perhaps useful to provide heuristic derivations of the equations, as in Blanchard and Fischer (1989, pp. 40–42).

Let's start with the Euler equation. The equation is unconventional because wealth enters the utility function. The equation says that the household saves in an optimal fashion. At the optimum, the household cannot improve its situation by shifting consumption a little bit across time. Now, consider a household delaying one unit of consumption from time t to time $t + dt$; the unit of consumption can be invested at a real rate $r(t)$ during the interval and becomes $1 + r(t)dt$ at time $t + dt$. With log utility, the marginal utility from consumption at consumption y is $1/y$. Hence, the foregone consumption utility at time t would be $e^{-\delta t}/y(t)$; and the gained consumption utility at time $t + dt$ would be $e^{-\delta(t+dt)}[1 + r(t)dt]/y(t + dt)$. The one unit of consumption that is saved between t and $t + dt$ provides not only financial returns but also hedonic returns, since people derive utility from holding wealth. Since relative wealth is zero in equilibrium, the marginal utility from real wealth is $u'(0)$. Hence the utility gained from holding additional wealth for a duration dt is $e^{-\delta t}u'(0)dt$. At the optimum, the consumption reallocation has no effect on utility, so

$$0 = e^{-\delta(t+dt)}\left[1 + r(t)dt\right]\frac{1}{y(t + dt)} - e^{-\delta t}\frac{1}{y(t)} + e^{-\delta t}u'(0)dt.$$

After dividing by the marginal consumption utility at t, $e^{-\delta t}/y(t)$, we obtain

$$1 = e^{-\delta dt}[1 + r(t)dt]\frac{y(t)}{y(t + dt)} + u'(0)y(t)dt.$$

To obtain the Euler equation, note that up to second-order terms, $e^{-\delta dt} = 1 - \delta dt$ and

$$\frac{y(t + dt)}{y(t)} = 1 + \frac{\dot{y}(t)}{y(t)}dt.$$

Hence, up to second-order terms, the optimality condition becomes

$$1 = (1 - \delta dt)\left[1 + r(t)dt\right]\left[1 - \frac{\dot{y}(t)}{y(t)}dt\right] + u'(0)y(t)dt = 1 - \delta dt + r(t)dt - \frac{\dot{y}(t)}{y(t)}dt + u'(0)y(t)dt$$

which simplifies to

$$\frac{\dot{y}(t)}{y(t)} = r(t) - \delta + u'(0)y(t).$$

so that we recover (14).

The difference with the textbook model is clear. In the textbook Euler equation, consumption is governed by the cost of delaying consumption, given by the time discount rate δ, and the return on saving, given by the real interest rate r. With wealth in the utility function, the financial return on saving is supplemented by an hedonic return on saving, given by the marginal rate of substitution between real wealth and consumption $u'(0)y$; thus the total return on saving is $r + u'(0)y$ instead of r. Moreover, because the optimal consumption path depends not only on interest rates but also on the marginal rate of substitution between wealth and consumption, future interest rates have less impact on today's consumption. In fact, in appendix B, we derive the discrete-time version of the model and show that the Euler equation is discounted exactly as in McKay, Nakamura, and Steinsson (2017), and that discounting is stronger with higher marginal utility of wealth.

The derivation also clarifies the intuition for the property that in the steady-state Euler equation, consumption is a decreasing function of the real interest rate. When the real rate is higher, people have a financial incentive to save now and postpone consumption. People keep consumption constant only if the hedonic return on saving, given by the marginal rate of substitution $u'(0)y$, falls. Thus consumption y must decline.

We turn next to the Phillips curve. While its shape is conventional, its interpretation is nontraditional because price rigidity in our model arises from a price-adjustment cost (as in Rotemberg 1982) instead of staggered price adjustments (as in Calvo 1983). The curve says that any household prices in an optimal fashion, so it cannot improve its situation by shifting its inflation rate a little bit across time. Now, consider a household delaying one percentage point of of inflation from time t to time $t + dt$. The marginal disutility from inflation at inflation π is $\gamma\pi$. Hence, the foregone inflation disutility at time t would be $e^{-\delta t}\gamma\pi(t)\times 1\%$; and the increased inflation disutility at time $t+dt$ would be $e^{-\delta(t+dt)}\gamma\pi(t+dt)\times 1\%$. The one percentage point of inflation that is delayed reduces the household's price level between t and $t+dt$ by $dp(t) = -1\% \times p(t)$, which then affects sales, profits, and hours worked. Since the price elasticity of the demand faced by households is $-\epsilon$, the household's output increases by $dy(t) = \epsilon \times 1\% \times y(t)$ during a period dt. This forces them to increase hours by $dh(t) = dy(t)/a = \epsilon \times 1\% \times y(t)/a$ during the same period, incurring additional disutility of the same amount. The increase in sales combined with the increase in price yields an increase in revenue by $d(p(t)y(t)) = p(t)dy(t) + y(t)dp(t) = (\epsilon - 1) \times 1\% \times y(t) \times p(t)$. Since in equilibrium relative prices are 1, the increase in revenue yields an increase in consumption

$dc(t) = (\epsilon - 1) \times 1\% \times y(t)$; and since in equilibrium output and consumption are equal, this increase in consumption yields an gain of consumption utility by $e^{-\delta t} dc(t)/y(t) = e^{-\delta t} \times (\epsilon - 1) \times 1\%$, enjoyed during dt. At the optimum, the inflation reallocation has no effect on utility, so

$$0 = e^{-\delta t} \gamma \pi(t) \times 1\% - e^{-\delta(t+dt)} \gamma \pi(t + dt) \times 1\% - \epsilon \frac{y(t)}{a} dt \times 1\% + e^{-\delta t} (\epsilon - 1) dt \times 1\%,$$

which implies, up to second-order terms,

$$0 = -\gamma \left[\pi(t + dt) - \pi(t) \right] + \delta \gamma \pi(t) dt - \epsilon dt \left[\frac{y(t)}{a} - \frac{\epsilon - 1}{\epsilon} \right].$$

Moreover, up to second-order terms,

$$\pi(t + dt) = \pi(t) + \dot{\pi}(t) dt.$$

Hence, up to second-order terms, the optimality condition becomes

$$\dot{\pi}(t) = \delta \pi(t) - \frac{\epsilon}{\gamma a} \left[y(t) - \frac{\epsilon - 1}{\epsilon} a \right].$$

so that we recover (12).

The Phillips curve implies that in the absence of price-adjustment cost ($\gamma = 0$), households would like to produce at the natural level of output. This result directly comes from the monopolistic nature of competition. Without price-adjustment costs, it would be optimal to charge a relative price that is a markup $\epsilon/(\epsilon - 1)$ over the real marginal cost—which is the marginal rate of substitution between labor and consumption divided by the marginal product of labor. In a symmetric equilibrium, the relative price charged by any firm is 1, the marginal rate of substitution between labor and consumption is $1/(1/y) = y$, and the marginal product of labor is a, so at the optimum $1 = [\epsilon/(\epsilon - 1)] \times (y/a)$. This yields that $y = y^n$, where y^n is defined by (7).

The derivation also provides intuition for why in the steady-state Phillips curve, inflation is positive whenever output is above its natural level. When inflation is positive, reducing inflation lowers the disutility from price adjustment Since pricing is optimal, there must also be a cost to reducing inflation and thus increasing production. Therefore, production must already be excessive—output must be above its natural level.

2.4. Dynamical Properties

We describe the dynamical properties of the NK and WUNK models. This analysis is a required preliminary to the study of the ZLB. In normal times, the models have the following dynamical properties:

PROPOSITION 1: *Assume that the natural rate of interest is positive ($r^n > 0$) and monetary policy follows* (9)*. Then there is a unique steady-state equilibrium, with zero inflation ($\pi = 0$) and output at its natural level ($y = y^n$). At this natural steady state the ZLB is not binding ($i = r^n > 0$). Around the natural steady state, equilibrium dynamics are governed by*

$$\begin{bmatrix} \dot{y} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} u'(0)y^n & (\phi - 1)y^n \\ -\epsilon/(\gamma a) & \delta \end{bmatrix} \begin{bmatrix} y - y^n \\ \pi \end{bmatrix}. \tag{17}$$

– *NK model: The linear system* (17) *is a source when monetary policy is active ($\phi > 1$) and a saddle when monetary policy is passive ($\phi < 1$).*

– *WUNK model: The linear system* (17) *is a source whether monetary policy is active or passive.*

Proof: A steady state must satisfy the steady-state Phillips curve (13), and the steady-state Euler equation (15) combined with the monetary-policy rule (10). These equations form a linear system:

$$\pi = \frac{\epsilon}{\delta\gamma a}(y - y^n) \tag{18}$$

$$(\phi - 1)\pi = -u'(0)(y - y^n). \tag{19}$$

As $[y, \pi] = [y^n, 0]$ satisfies both equations, it is a steady state. Furthermore the two equations are non-parallel. In the NK model this is obvious since $u'(0) = 0$. In the WUNK model the slope of the second equation is $-u'(0)/(\phi - 1)$: if $\phi > 1$, the slope is negative; if $\phi \in [0, 1)$, the slope is positive and greater than $u'(0)$ and thus than $\epsilon/(\delta\gamma a)$, as (16) holds; in both cases the two equations have different slope. We conclude that the two equations admit at most one intersection, so $[y^n, 0]$ is the unique steady state. The nominal interest rate at $[y^n, 0]$ is given by (9): $i = r^n + \phi \times 0 = r^n > 0$.

Equilibrium dynamics are governed by the nonlinear dynamical system generated by the Phillips curve (12) and the Euler equation (14) combined with the monetary-policy rule (10). The Phillips curve can be written as $\dot{\pi} = P(y, \pi)$ where $P(y, \pi) = \delta\pi - \epsilon(y - y^n)/(\gamma a)$; the Euler equation (14) combined with the monetary-policy rule (10) can be written $\dot{y} = E(y, \pi)$

where $E(y,\pi) = y\left[(\phi-1)\pi + u'(0)(y-y^n)\right]$. The dynamical system can be linearized around the natural steady state as follows:

$$\begin{bmatrix} \dot{y} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} \frac{\partial E}{\partial y} & \frac{\partial E}{\partial \pi} \\ \frac{\partial P}{\partial y} & \frac{\partial P}{\partial \pi} \end{bmatrix} \begin{bmatrix} y - y^n \\ \pi \end{bmatrix},$$

where the partial derivatives are evaluated at $[y,\pi] = [y^n, 0]$. This linearized system is just (17). Around the natural steady state, equilibrium dynamics are described by this linearized system.

Next, we classify the linear dynamical system (17) following standard methodology (for example, Hirsch, Smale, and Devaney 2013). We denote by $\boldsymbol{M}$ the matrix in (17), and by $\lambda_1 \in \mathbb{C}$ and $\lambda_2 \in \mathbb{C}$ the two eigenvalues of $\boldsymbol{M}$. Without loss of generality, we assume that if the eigenvalues are real, $\lambda_1 < \lambda_2$. If the eigenvalues are complex, they are conjugates. Generically the solution to (17) takes the form

$$\begin{bmatrix} y(t) - y^n \\ \pi(t) \end{bmatrix} = x_1 e^{\lambda_1 t}\boldsymbol{v}_1 + x_2 e^{\lambda_2 t}\boldsymbol{v}_2,$$

where $\boldsymbol{v}_1 \in \mathbb{R}^2$ and $\boldsymbol{v}_2 \in \mathbb{R}^2$ are the linearly independent eigenvectors associated with the eigenvalues λ_1 and λ_2, and $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ are constants determined by the terminal condition. (This result holds as long as the eigenvalues are nonzero and distinct.)

From this solution, we see that the system is a source around the steady state either if λ_1 and λ_2 are real and positive (nodal source), or if they are complex with a positive real part (spiral source). If λ_1 and λ_2 are real, the trajectories are tangent to $\boldsymbol{v}_1$ when $t \to -\infty$ and are parallel to $\boldsymbol{v}_2$ when $t \to +\infty$.

On the other hand, the system is a saddle if λ_1 and λ_2 are real and $\lambda_1 < 0$ and $\lambda_2 > 0$. If the system is a saddle, the vector $\boldsymbol{v}_1$ gives the direction of the stable line (saddle path) while the vector $\boldsymbol{v}_2$ gives the direction of the unstable line.

To determine the properties of the system, we compute the trace and determinant of $\boldsymbol{M}$:

$$\begin{aligned} \text{tr}(\boldsymbol{M}) &= \delta + u'(0)y^n \\ \det(\boldsymbol{M}) &= \delta u'(0)y^n + \frac{(\phi-1)y^n\epsilon}{\gamma a}. \end{aligned}$$

We use the trace and determinant to determine the properties of the system, using the well-known results that $\text{tr}(\boldsymbol{M}) = \lambda_1 + \lambda_2$ and $\det(\boldsymbol{M}) = \lambda_1\lambda_2$.

Let's start with the NK model. Since $u'(0) = 0$, $\det(\boldsymbol{M}) = (\phi-1)y^n\epsilon/(\gamma a)$. We consider first the case $\phi < 1$. Then $\det(\boldsymbol{M}) < 0$, indicating that λ_1 and λ_2 are real and of opposite sign. (If they were real and of the same sign, $\det(\boldsymbol{M}) = \lambda_1\lambda_2 > 0$; if they were complex conjugates,

$\det(\boldsymbol{M}) = \lambda_1\overline{\lambda_1} > 0$.) Thus, the system is a saddle if $\phi < 1$.

Second, we consider the case $\phi > 1$. Then $\det(\boldsymbol{M}) > 0$, indicating that λ_1 and λ_2 are either real and of the same sign or complex conjugates. Since $u'(0) = 0$ in the NK model, we have $\operatorname{tr}(\boldsymbol{M}) = \delta > 0$, implying that λ_1 and λ_2 are either real and positive or complex with positive real part. (If they were real and negative, $\operatorname{tr}(\boldsymbol{M}) = \lambda_1 + \lambda_2 < 0$; if they were complex conjugates with negative real part, $\operatorname{tr}(\boldsymbol{M}) = \lambda_1 + \overline{\lambda_1} = 2\operatorname{Re}(\lambda_1) < 0$.) Either way, the system is a source.

We have just shown that the linear system (17) is a source when $\det(\boldsymbol{M}) > 0$ and $\operatorname{tr}(\boldsymbol{M}) > 0$, and it is a saddle when $\det(\boldsymbol{M}) < 0$. This classification of a linear dynamical system based on the trace and determinant of its matrix is well-known (see for instance fig. 4.1 in Hirsch, Smale, and Devaney 2013). We will use this classification result repeatedly below.

Let's turn to the WUNK model. Using $\phi - 1 \geq -1$ and (16), we have

$$\det(\boldsymbol{M}) \geq \delta u'(0) y^n - \frac{y^n \epsilon}{\gamma a} = \delta y^n \left[u'(0) - \frac{\epsilon}{\delta \gamma a} \right] > 0.$$

Furthermore, $\operatorname{tr}(\boldsymbol{M}) > \delta > 0$. Since both $\det(\boldsymbol{M}) > 0$ and $\operatorname{tr}(\boldsymbol{M}) > 0$, using the same logic as above, we conclude that the system is a source. □

When the natural rate of interest is negative, however, the natural steady state cannot be achieved because it would imply a negative nominal interest rate, violating the ZLB constraint. In the WUNK model, the natural rate becomes negative because the marginal utility of wealth increases sufficiently: $u'(0) > \delta / y^n$. In the NK model, we follow the literature and assume that the time discount rate in the Euler equation (now denoted $r^n < 0$) becomes negative, while the time discount rate in the Phillips curve (still denoted $\delta > 0$) does not change. This assumption can been justified in various ways: firm managers have constant discount rate while households' discount rate fluctuates (McKay, Nakamura, and Steinsson 2017, pp. 826–827); or financial intermediation creates a fluctuating spread between the central bank's interest rate and the interest rate used for intertemporal allocation of consumption expenditure (Woodford 2011, p. 16).[6] Once the natural rate of interest is negative, the dynamical properties of the models are modified as follows:

PROPOSITION 2: *Assume that the natural rate of interest is negative ($r^n < 0$) and monetary policy is at the ZLB: $i(\pi) = 0$ and $r(\pi) = -\pi$. Then there is a unique steady-state equilibrium,*

[6] Changing the discount rate only in the Euler equation does not affect the results: the properties of the NK model at the ZLB would remain the same if the discount rates in both Phillips curve and Euler equation became negative.

denoted $[y^z, \pi^z]$*, with*

$$y^z = y^n + \frac{r^n}{u'(0) - \epsilon/(\delta\gamma a)} \tag{20}$$

$$\pi^z = \frac{r^n}{u'(0)\delta\gamma a/\epsilon - 1}. \tag{21}$$

Around this ZLB steady state, equilibrium dynamics are governed by

$$\begin{bmatrix} \dot{y} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} u'(0)y^z & -y^z \\ -\epsilon/(\gamma a) & \delta \end{bmatrix} \begin{bmatrix} y - y^z \\ \pi - \pi^z \end{bmatrix}, \tag{22}$$

- *NK model: The ZLB steady state has positive inflation ($\pi^z = -r^n > 0$) and above-natural output ($y^z > y^n$). The linear system* (22) *is a saddle.*
- *WUNK model: The ZLB steady state has deflation ($\pi^z < 0$) and below-natural output ($y^z < y^n$). The linear system* (22) *is a source.*

Proof: The steady-state Phillips curve (13) and the steady-state Euler equation (15) combined with the monetary-policy rule $r(\pi) = -\pi$ form the following linear system:

$$\pi = \frac{\epsilon}{\delta\gamma a}(y - y^n) \tag{23}$$

$$\pi = -r^n + u'(0)(y - y^n). \tag{24}$$

A solution to this system with positive output is a steady state.

In the NK model, $u'(0) = 0$, so the system admits a unique solution:

$$y = y^n - \frac{\delta\gamma a r^n}{\epsilon} \tag{25}$$

$$\pi = -r^n. \tag{26}$$

Since $r^n < 0$, the solution satisfies $y > y^n > 0$: the solution has positive output so it is a steady state. Hence the NK model admits a unique steady state at the ZLB, where $\pi > 0$ (since $r^n < 0$) and $y > y^n$.[7] Note that the expressions (20) and (21) reduce to (25) and (26) when $u'(0) = 0$.

In the WUNK model, since (16) holds, the two equations in the linear system are non-parallel, so the system admits a unique solution. Inflation in that unique solution is given by (21). Condition (16) implies that $u'(0)\delta\gamma a/\epsilon > 1$, so π has the sign of r^n; since $r^n < 0$, we infer that $\pi < 0$.

[7] When $r^n > 0$ (for instance, during forward guidance) the steady state may not exist. This is because $y > 0$ only when $(\delta\gamma a r^n)/\epsilon < y^n$ or $r^n < (\epsilon - 1)/(\delta\gamma)$. When $r^n > (\epsilon - 1)/(\delta\gamma)$, then $y < 0$ and no steady state exists.

Since y and π in the solution are related by (23) and $\pi < 0$, we infer that $y < y^n$. In addition, as $\epsilon y^n/(\delta\gamma a) = (\epsilon - 1)/(\delta\gamma)$ and $r^n + u'(0)y^n = \delta$, output in the unique solution can be expressed as (20). In the WUNK model, we assume $u'(0) > \epsilon/(\delta\gamma a)$ and $\delta > \sqrt{(\epsilon - 1)/\gamma}$, which implies $\delta - (\epsilon - 1)/(\delta\gamma) > 0$. We infer that $y > 0$. Hence, the solution to the system has positive output, implying that it is indeed a steady state. In sum, the WUNK model admits a unique steady state at the ZLB, where $\pi < 0$ and $y < y^n$.

Next, as in the proof of proposition 1, equilibrium dynamics around the ZLB steady state are described by the system resulting from the linearization of the dynamical system generated by the Phillips curve (12), the Euler equation (14), and the monetary policy rule $i(\pi) = 0$. This linear system is (22); the matrix derives from the matrix in (17) by setting $\phi = 0$ and replacing y^n by y^z.

We denote by $\boldsymbol{M}$ the matrix in (22). As in the proof of proposition 1, we classify the linear system (22) by computing the trace and determinant of $\boldsymbol{M}$. We have $\text{tr}(\boldsymbol{M}) = \delta + u'(0)y^z > 0$ and

$$\det(\boldsymbol{M}) = \delta y^z \left[u'(0) - \frac{\epsilon}{\delta\gamma a} \right].$$

In the NK model $u'(0) = 0$ so $\det(\boldsymbol{M}) < 0$, implying that (22) is a saddle. In the WUNK model, (16) implies that $\det(\boldsymbol{M}) > 0$, implying that (22) is a source. In fact, the discriminant of the characteristic equation of the matrix $\boldsymbol{M}$ is strictly positive:

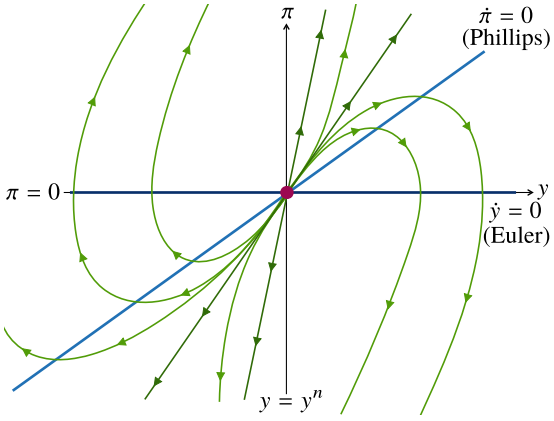
$$\text{tr}(\boldsymbol{M})^2 - 4\det(\boldsymbol{M}) = \delta^2 + [u'(0)y^n]^2 + 2\delta u'(0)y^n - 4\delta u'(0)y^n + 4\frac{y^n\epsilon}{\gamma a} = [\delta - u'(0)y^n]^2 + 4\frac{y^n\epsilon}{\gamma a} > 0.$$

Hence the eigenvalues of $\boldsymbol{M}$ are real, not complex: (22) is a nodal source, not a spiral source. □

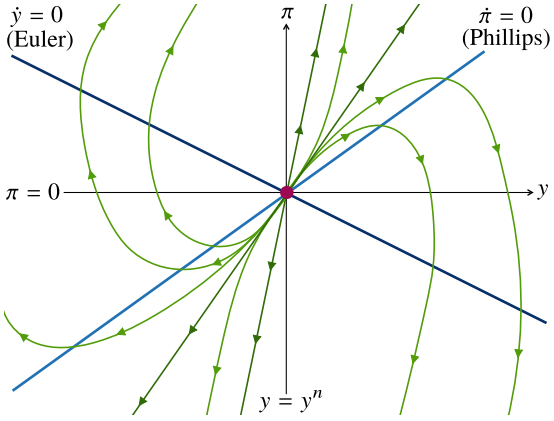
The results in the propositions are illustrated by the phase diagrams in figure 1. The phase diagrams represent the dynamical systems (17) and (22) in the NK and WUNK models. They are constructed using standard methods for planar linear systems (the proof of proposition 1 provides more details). Five points are worth noting.

First, the π-nullcline is the same as the steady-state Phillips curve, and the y-nullcline the same as the steady-state Euler equation.

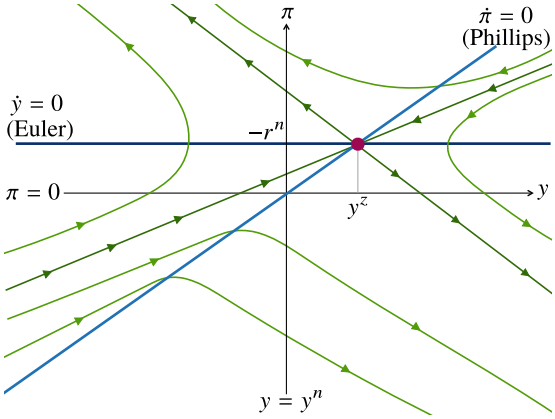
Second, the sole difference between NK and WUNK models is the Euler line (y-nullcline). In the NK model the Euler line is always horizontal because the steady-state Euler equation imposes that the real rate $r(\pi)$ equals the time discount rate, which determines inflation independently of output. In the WUNK model the Euler line is not horizontal because the steady-state Euler equation makes the real rate $r(\pi)$ a decreasing function of output. When monetary policy is active, $r(\pi)$ is increasing in π, so the Euler line slopes downward; when monetary policy is passive, including at the ZLB, $r(\pi)$ is decreasing in π, so the Euler line slopes upward.



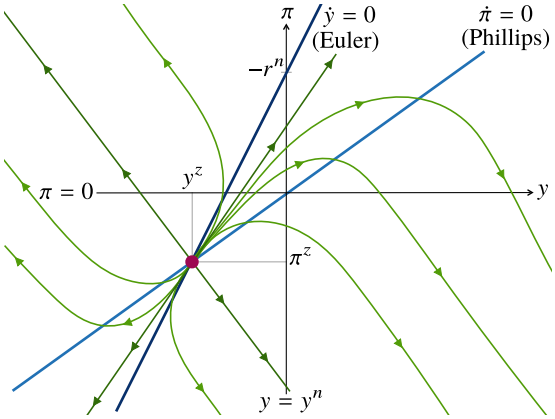
A. NK model with active monetary policy: source



B. WUNK model with active monetary policy: source



C. NK model at the ZLB: saddle



D. WUNK model at the ZLB: source

Figure 1. Equilibrium Phase Diagrams in Normal Times and at the ZLB

Notes: π is inflation, y is output, y^n is the natural level of output, r^n is the natural rate of interest, and $[y^z, \pi^z]$ is the ZLB steady state. The NK model has no marginal utility of wealth: $u'(0) = 0$. The WUNK model has sufficient marginal utility of wealth: $u'(0) > \epsilon/(\delta\gamma a)$. In normal times, $r^n > 0$ and the monetary policy is given by $i = r^n + \phi\pi$. Monetary policy is active when $\phi > 1$ and passive when $\phi < 1$. At the ZLB, $r^n < 0$ and the monetary policy is given by $i = 0$. The phase diagrams in panels A and B are constructed from (17) using standard methods from planar linear systems. The phase diagrams in panels C and D are constructed similarly from (22). The systems (17) and (22) are obtained by linearizing the Phillips curve and Euler equation around the natural and ZLB steady states; they describe local equilibrium dynamics. Panel A shows that in the NK model the equilibrium system is a source in normal times when monetary policy is active. Panels B and D show that in the WUNK model the equilibrium system is a source both in normal times and at the ZLB. In panels A and B, we have plotted a nodal source, but the system could also be a spiral source, depending on value of ϕ; in panel D the system is always a nodal source. In these three panels, the equilibrium is determinate: the only bounded equilibrium trajectory is to jump to the steady state and stay there. Finally, panel C shows that in the NK model the equilibrium system is a saddle at the ZLB.

Third, only the Euler line changes between normal times and ZLB, because the natural rate of interest and monetary policy change. In the NK model, the Euler line shifts up from $\pi = 0$ to $\pi = -r^n > 0$. In the WUNK model, the Euler line is upward sloping (as monetary policy is passive) and shifts inward to cross the point $[y^n, -r^n]$ instead of the point $[y^n, 0]$.

Fourth, we omit the phase diagrams when monetary policy is passive in normal times, but they can easily be visualized. These phase diagrams are like the ZLB phase diagrams, except that the Euler line goes through the point $[y^n, 0]$.

Fifth, panel D illustrate the origin of condition (16). The dynamical system representing the WUNK equilibrium remains a source at the ZLB as long as the Euler line is steeper than the Phillips line (π-nullcline). The slope of the Euler line at the ZLB is the marginal utility of wealth, $u'(0)$, so a certain marginal utility of wealth is required—this amount is given by (16).

The propositions have several implications. First, they have implications for equilibrium determinacy. When the system is a source, the equilibrium is determinate: the only bounded equilibrium trajectory is to jump to the steady state and stay there; if the economy jumped somewhere else, output or inflation would become infinite following a trajectory similar to those plotted in panels A, B, or D. When the system is a saddle, in contrast, the equilibrium is indeterminate: any trajectory jumping somewhere on the saddle path and converging to the steady state is an equilibrium, as illustrated in panel C. Hence, in the NK model, the ZLB equilibrium is indeterminate, which has forced researchers to focus on temporary ZLB episodes, following Krugman (1998) and Eggertsson and Woodford (2003). In the WUNK model, on the other hand, the ZLB equilibrium is determinate: the economy jumps to the ZLB steady state and remains there. The ZLB could last forever: absent fundamental shocks, no market forces or change in sentiment could pull the economy away from it.

Second, the propositions imply that away from the ZLB, monetary policy plays quite a different role in the NK and WUNK models. In the NK model, the equilibrium is determinate only when monetary policy is active—the Taylor principle. Hence, the central bank must always adhere to an active policy to avoid indeterminacy. The Taylor principle does not apply in the WUNK model, however: the equilibrium is determinate whether monetary policy is active or passive. Hence, the central bank does not need to worry about how strongly it responds to inflation; it only needs to worry about the rate of interest.

Third, the propositions are the basis for the analysis in the rest of the paper. In particular, as we will see in the next section, it is because the dynamical system representing the ZLB equilibrium is a source in the WUNK model instead of a saddle that the anomalies of the NK model at the ZLB disappear in the WUNK model.

The results of propositions 1 and 2 on the NK model are well-known (for example, Woodford

2001). The results of the propositions on the WUNK model share many similarities with the results obtained by Gabaix (2016) in his proposition 3.1. In his model with bounded rationality he finds that when bounded rationality is strong enough, the dynamical system representing the equilibrium is indeed a source even at the ZLB (although he presents his model in discrete time and does not use our phase-diagram language). He also find that when prices are more flexible, more bounded rationality is required to maintain the source property. The same is true in our model: higher marginal utility of wealth is required for condition (16) to hold when the price-adjustment cost γ is lower. The logic is illustrated in panel D of figure 1. The figure shows that the system remains a source at the ZLB only when the Euler line is steeper that the Phillips line. As prices are more flexible, the Phillips line becomes steeper, and the required steepness for the Euler line is higher. The slope of the Euler line is directly determined by bounded rationality in the model of Gabaix (see his equation (19)) and by marginal utility of wealth in our model (see our equation (15)), so these have to be larger when prices are more flexible.

3. Empirical Support for the WUNK Assumption

In the WUNK model the marginal utility of wealth is above the threshold specified in (16). Before analyzing the model, we assess this assumption using empirical evidence.

As a first step, we rewrite (16) to express it with estimable statistics. Multiplying the inequality (16) by δy^n, we obtain $\delta u'(0) y^n > y^n \epsilon/(\gamma a)$. The definition of the natural rate of interest, given by (8), implies that $u'(0) y^n = \delta - r^n$. This shows how the marginal rate of substitution between wealth and consumption can be measured: by estimating the gap between the time discount rate and the natural rate of interest. The expression for natural output, given by (7), implies that $y^n \epsilon/(\gamma a) = (\epsilon - 1)/\gamma$. The statistic $(\epsilon - 1)/\gamma$ has been estimated before because, as shown by equation (A16) in appendix B, it is the coefficient on the output gap in the discrete-time Phillips curve. Hence, we rewrite (16) as

$$\delta \times (\delta - r^n) > \frac{\epsilon - 1}{\gamma}. \tag{27}$$

We need estimates of three statistics to assess the WUNK assumption: the natural rate of interest r^n, the time discount rate δ, and the output-gap coefficient in the Phillips curve, $(\epsilon - 1)/\gamma$.

Estimates of the natural rate of interest are uncontroversial.Del Negro et al. (2017, fig. 1) and Holston, Laubach, and Williams (2017, fig. 1) estimate the US natural rate of interest around $r^n = 2\%$. It is much more complicated to estimate δ and $(\epsilon - 1)/\gamma$. We now survey the literatures estimating these two statistics.

3.1. Time Discount Rate

A vast literature has attempted to estimate time discount rates. Frederick, Loewenstein, and O'Donoghue (2002, table 1) survey the estimates obtained from real-world behavior and elicitation in field or laboratory experiments. There is a lot of variation in the estimates, but the majority of them points to high time discounting, much higher than prevailing market interest rates. We take the average estimate in each of the 43 studies covered by the survey, and then take the median of these averages. We obtain an annual discount rate of $\delta = 0.35$.

Most studies discussed by Frederick, Loewenstein, and O'Donoghue assume that people use a single rate to exponentially discount future utility. This exponential-discounting model is subject to many anomalies, however. Recent work allows for more general discounting, such as hyperbolic or quasi-hyperbolic (β-δ) discounting. They also separate between time discounting and risk aversion, which were sometimes mingled in previous studies. Andersen et al. (2014, table 3) survey 16 such studies. Taking again the average estimate in each study, and then the median of these averages, we obtain $\delta = 0.43$.

Overall, then, a time discount rate of $\delta = 0.4$ seems to be a midpoint estimate. Using the natural rate of interest $r^n = 2\%$, we obtain an estimate of the marginal rate of substitution between wealth and consumption: $u'(0)y^n = \delta - r^n = 0.4 - 0.02 = 0.38$. Although the financial return on wealth (r^n) is much lower than the time discount rate (δ), people are willing to hold wealth because they derive direct utility from it.

3.2. Slope of Phillips Curve

The slope of the Phillips curve has been estimated by a large literature. Mavroeidis, Plagborg-Moller, and Stock (2014, fig. 3) review the literature. For a specification similar to (A16), estimates of $(\epsilon - 1)/\gamma$ vary between 0.005 and 0.08, with a median estimate across 16 studies of 0.03. Mavroeidis, Plagborg-Moller, and Stock (2014, table 3) also propose their own estimate of the Phillips curve using two vintages of data. Their estimates are slightly lower than the median: 0.026 and 0.018.

As an alternative, we can measure $(\epsilon - 1)/\gamma$ from microestimates of ϵ and γ. Using firm-level data, De Loecker and Eeckhout (2017, fig. 1) attempt to measure the goods-market markup in the United States. They find that the average markup $\epsilon/(\epsilon - 1)$ hovers between 1.2 and 1.3 between 1950 and 1980 before continuously rising to 1.7 between 1980 and 2014. Since 1990 the average markup is around 1.5, implying an average elasticity $\epsilon = 3$, which we use as our midrange estimate. Next, following Michaillat (2014, p. 206), we calibrate the price-adjustment cost to $\gamma = 61$. This cost is obtained from the microevidence provided by Zbaracki et al.

(2004), who describe the pricing process of a large industrial US firm and measure the physical, managerial, and customer costs of changing prices. Combining the estimates for ϵ and γ yields $(\epsilon - 1)/\gamma = (3 - 1)/61 = 0.033$. This number is close to the estimate from the Phillips-curve literature, so we use $(\epsilon - 1)/\gamma = 0.03$ as a midpoint estimate.

3.3. Assessment of the WUNK Assumption

We combine our preferred estimates of δ, r^n, and $(\epsilon-1)/g$ to assess the WUNK assumption, which can be written as (27). The estimates imply that (27) easily holds: $\delta\times(\delta-r^n) = 0.4\times(0.4-0.02) = 0.15 > 0.03 = (\epsilon - 1)/\gamma$. Hence the assumption seems to hold comfortably in US data.

Of course, there remains significant uncertainty about the exact estimates of the statistics, especially δ, and $(\epsilon - 1)/\gamma$. How large is the range of estimates under which the WUNK assumption holds? First, if we settle on a median estimate of the time discount rate, (27) holds for any of the Phillips-curve estimates reported by Mavroeidis, Plagborg-Moller, and Stock (2014). Indeed, with $\delta = 0.4$, we have $\delta \times (\delta - r^n) = 0.4 \times (0.4 - 0.02) = 0.15 > 0.08$, where 0.08 is the highest estimate of $(\epsilon - 1)/\gamma$ in Mavroeidis, Plagborg-Moller, and Stock (2014, fig. 3). In fact, even if we used a lower time discount rate of $\delta = 0.3$, we would have $0.3 \times (0.3 - 0.02) = 0.084 > 0.08$, so (27) would hold for any of the Phillips-curve estimates reported by Mavroeidis, Plagborg-Moller, and Stock (2014). Second, if we settle on a median estimate of output-gap coefficient in the Phillips curve, (27) holds for any time discount rate above $\delta = 0.19$ (since $0.19 \times (0.19 - 0.02) = 0.032 > 0.03$). If we use the output-gap coefficient in the Phillips curve estimated by Mavroeidis, Plagborg-Moller, and Stock (2014) from a modern vintage of data, (27) holds for any time discount rate above $\delta = 0.15$ (since $0.15 \times (0.15 - 0.02) = 0.019 > 0.018$). Such values of the time discount rate are at the low end of available estimates: combining the estimates surveyed by Frederick, Loewenstein, and O'Donoghue (2002) and Andersen et al. (2014), we find that more than 70% of estimates of the time discount rate are above $\delta = 0.19$.

4. New Keynesian Anomalies at the ZLB

This section illustrates the anomalies of the NK model at the ZLB: the explosive paths of output and inflation, the forward-guidance puzzle, and the explosive government-spending multiplier. It then shows that these anomalies do not exist in the WUNK model.

4.1. Output and Inflation

We study the paths of output and inflation during a temporary ZLB episode. We consider a two-stage scenario, as in Werning (2011) and Cochrane (2017). First, between times 0 and $T > 0$, the natural rate of interest becomes negative: $r^n = r^z < 0$. In response, the central bank maintains the nominal interest rate at the ZLB: $i = 0$. Second, after time T, the natural rate becomes positive again, and the central bank returns to the normal monetary-policy rule, given by (9). (In the NK model, monetary policy must be active to ensure equilibrium determinacy.)

We analyze the ZLB episode by going backward in time. In the second stage, $t > T$, monetary policy maintains the economy at the natural steady state. We then move back to the ZLB episode, $t \in (0, T)$. The corresponding phase diagrams are shown in figure 2. The phase diagrams for the NK model are the same as in panel C of figure 1. At the end of the ZLB, at time T, the economy must be at the natural steady state. Because the equilibrium system is a saddle, at time 0 inflation and output jump down to $\pi(0) < 0$ and $y(0) < y^n$. After that initial contraction, they recover following the unique trajectory leading to $y(T) = y^n$ and $\pi(T) = 0$. Hence the ZLB creates a slump, with below-natural output and deflation. Furthermore output and inflation become unboundedly low as the ZLB lasts longer (panel C).

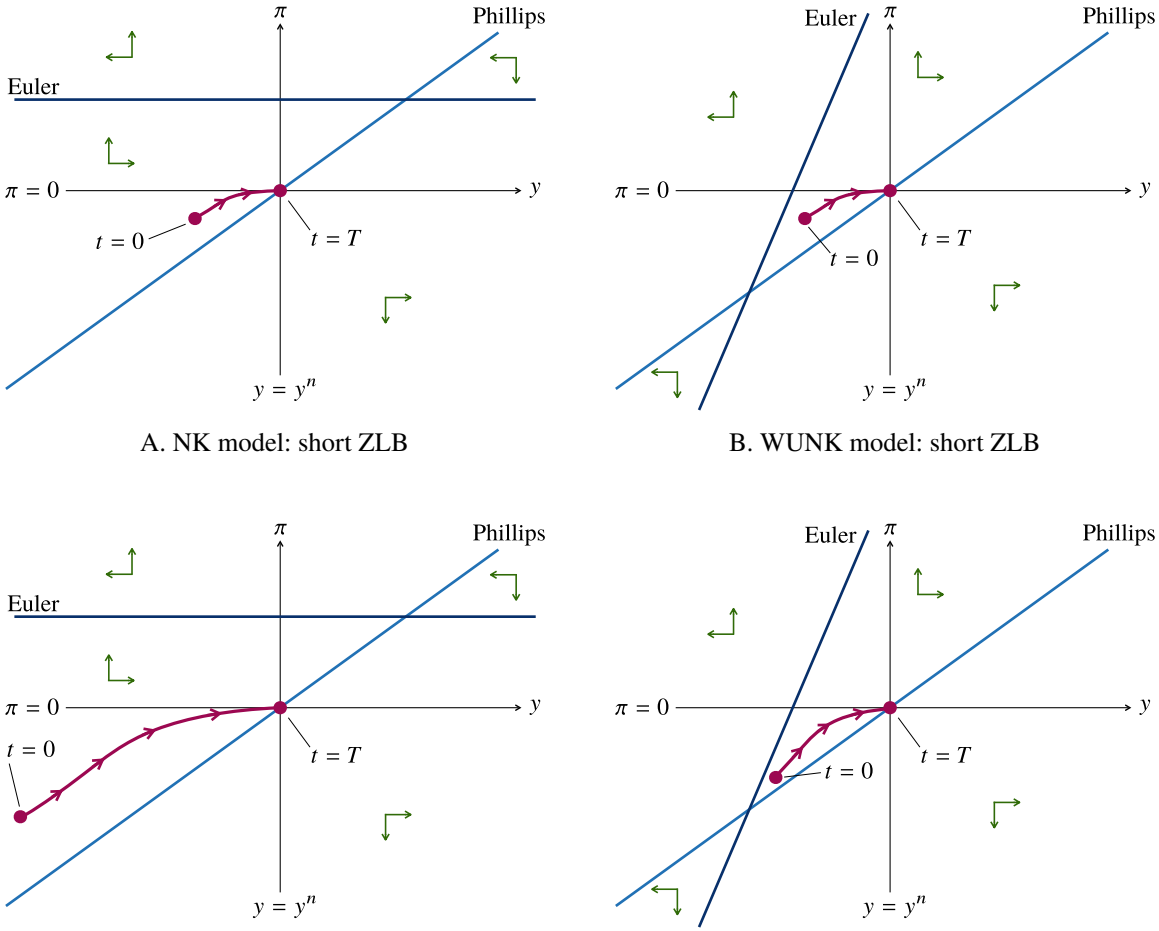
In the WUNK model, output and inflation do not take explosive paths during the ZLB. The phase diagrams for the ZLB episode (panels B and D) are the same as in panel D of figure 1. Because the equilibrium system remains a source at the ZLB, at time 0 inflation and output jump down toward the ZLB steady state, but not all the way: $\pi^z < \pi(0) < 0$ and $y^z < y(0) < y^n$. Then they recover over time, following the unique trajectory leading to $\pi(T) = 0$ and $y(T) = y^n$. Thus, as in the NK model, the ZLB episode creates a slump. Furthermore, the initial contraction is deeper when the ZLB lasts longer (panel D). But unlike in the NK model, the paths of output and inflation are bounded below: irrespective of the length of the ZLB, inflation and output always fall less than if the ZLB were permanent. In fact, if the natural rate of interest is negative but close to 0 at the ZLB, such that π^z is close to 0 and y^z to y^n, output and inflation barely deviate from the natural steady state during the ZLB, even if the ZLB lasts an arbitrarily long time.

The following proposition summarizes these results:[8]

PROPOSITION 3: *A ZLB episode between times* 0 *and* T *has the following properties:*

– *NK model: The economy enters a slump:* $y(t) < y^n$ *and* $\pi(t) < 0$ *for all* $t \in (0, T)$. *The slump*

[8]The result in proposition 3 that output becomes infinitely negative when the ZLB becomes infinitely long should not be interpreted literally—because the result is obtained using a local approximation of the dynamical system (12)–(14). The interpretation is that output falls much below its natural level (so much so that the local approximation stops being valid). The nonapproximated dynamical system would guarantee that output remains positive.



A. NK model: short ZLB

B. WUNK model: short ZLB

C. NK model: long ZLB

D. WUNK model: long ZLB

Figure 2. Output and Inflation during ZLB Episodes of Various Durations

Notes: The figure displays the equilibrium trajectory in the NK and WUNK models during a ZLB episode. A ZLB episode lasts between times 0 and T. During the episode, the natural rate of interest is negative and the central bank responds by setting the nominal interest rate to zero. At the end of the ZLB, at time T, the central bank brings the economy to the natural steady state, where $y = y^n$ and $\pi = 0$. The phase diagrams in panels A and C come from figure 1, panel C. The phase diagrams in panels B and D come from figure 1, panel D. (The Phillips line is the π-nullcline; the Euler line is the y-nullcline.) The figure contrasts a short ZLB episode (small T) in panels A and B with a long ZLB episode (large T) in panels C and D. In both models, at time 0, inflation is negative and output is below its natural level: the ZLB creates a slump. Then the economy recovers along the unique trajectory reaching the natural steady state at time T. Critically, the WUNK model eliminates the explosive paths of output and inflation during long ZLB episodes: in the NK model, the initial slump becomes unboundedly large as the ZLB becomes longer; in the WUNK model, in contrast, the initial slump is bounded below by the ZLB steady state.

becomes infinitely severe as the ZLB becomes infinitely long:

$$\lim_{T\to\infty} y(0) = -\infty$$
$$\lim_{T\to\infty} \pi(0) = -\infty.$$

– *WUNK model: The economy enters a slump:* $y(t) < y^n$ *and* $\pi(t) < 0$ *for all* $t \in (0,T)$*. The slump is bounded below by the ZLB steady state:* $y(t) > y^z$ *and* $\pi(t) > \pi^z$ *for all* $t \in (0,T)$*. In fact, the slump approaches the ZLB steady state as the ZLB becomes infinitely long:*

$$\lim_{T\to\infty} y(0) = y^z = y^n + \frac{1}{u'(0) - \epsilon/(\delta\gamma a)} \cdot r^z \tag{28}$$

$$\lim_{T\to\infty} \pi(0) = \pi^z = \frac{\epsilon/(\delta\gamma a)}{u'(0) - \epsilon/(\delta\gamma a)} \cdot r^z. \tag{29}$$

The explosive paths of output and inflation at the ZLB are a well-known anomaly of the NK model (Eggertsson and Woodford 2004, fig. 1; Cochrane 2017, fig. 1). Such pattern is difficult to reconcile with real-world observations. The Japanese ZLB episode that started in 1995 has been lasting more than twenty years without sustained deflation. The ZLB episode that started in 2009 in the euro area has been lasting for more than 10 years, and it did not yield sustained deflation either.

In the WUNK model, in contrast, the ZLB slump is bounded below by the ZLB steady state. So irrespective of the length of the ZLB episode, inflation and output do not spiral down. Instead, as the length of the ZLB increases, both variables converge to their values at the ZLB steady state. And these steady-state values may not be far from the steady state in normal times. For instance, if the natural interest rate is only slightly negative during the ZLB, steady-state inflation is only slightly below zero (see (29)); then, in any temporary ZLB, inflation always remains above that steady-state level—so there is no reason to expect sharp deflation during a ZLB, even if it lasts a very long time.

The result obtained here is similar to proposition 3.2 in Gabaix (2016). In his model with bounded rationality, Gabaix shows that output and inflation also converge to the ZLB steady state as the ZLB becomes arbitrarily long; he also obtains closed-form expressions for the steady-state values of the two variables.

4.2. Forward Guidance

We study the paths of output and inflation when the central bank conducts forward guidance during the ZLB. We consider a three-stage scenario, as in Cochrane (2017). First, between time

0 and time $T > 0$, there is a ZLB episode. To alleviate the ZLB, the central bank makes a forward-guidance promise at time 0: that it will maintain the nominal interest rate at zero for a duration Δ once the ZLB is over.[9] Second, between time T and time $T + \Delta$, the natural rate of interest becomes positive: $r^n = r^f > 0$. The central bank abides by its forward-guidance promise and keeps the nominal interest rate at zero. Third, after time $T + \Delta$, the natural rate of interest remains positive, and the central bank returns to normal monetary policy, given by (9).

Let's start with the NK model. The analysis of forward guidance is in figure 3. We construct the paths of output and inflation by going backward in time. In the third stage, $t > T + \Delta$, monetary policy maintains the economy at the natural steady state. The second stage, $t \in (T, T + \Delta)$, is the forward-guidance episode, depicted in panel A. The phase diagram is similar to that in panel C of figure 1, except that $r^n > 0$ instead of $r^n < 0$. Since the Euler line goes through the point $[y^n, -r^n]$, we infer that the Euler line is lower; however, the equilibrium system remains a saddle.[10] Following the logic of figure 2, we find that at time T, inflation and output must be at $\pi(T) > 0$ and $y(T) > y^n$. They then decrease over time, following the unique trajectory leading to $\pi(T + \Delta) = 0$ and $y(T + \Delta) = y^n$. Accordingly, the economy booms during forward guidance: inflation is always positive and output above its natural level. Furthermore, as the duration of forward guidance increases, inflation and output at the beginning of forward guidance, $[y(T), \pi(T)]$, becomes unboundedly high.

Next, we turn to the ZLB episode, $t \in (0, T)$, depicted in panels B, C, and D. The three panels differ by the duration of the forward-guidance episode following the ZLB episode. The phase diagrams in these panels are the same as that in panel C of figure 1. Because of the boom engineered through forward guidance, the situation is improved at the ZLB: instead of reaching the natural steady state at time T, the economy must reach $\pi(T) > 0$ and $y(T) > y^n$, so at any time before T, the economy tends to have higher inflation and output than without forward guidance. Forward guidance can actually have tremendously strong effects. For small durations of forward guidance, the position at the beginning of forward guidance, $[y(T), \pi(T)]$, is on the left-hand side of the unstable line of the dynamical system representing the ZLB equilibrium. Hence, it is connected to trajectories coming from the southwest quadrant of the phase diagram (panel B). As the ZLB lasts longer, initial output and inflation become lower and lower. When the duration of forward guidance is such that the position at the beginning of forward guidance, $[y(T), \pi(T)]$, is exactly on the unstable line of the dynamical system, then the position at the beginning of the

[9] Eggertsson and Woodford (2003), Jung, Teranishi, and Watanabe (2005), and Werning (2011) show that such path of nominal interest rates is indeed optimal in this kind of ZLB situation.

[10] Proposition 2 was derived for a ZLB situation: $i = 0$ and $r^n < 0$. But almost all the results of the proposition also hold for a forward-guidance situation: $i = 0$ and $r^n > 0$. The only different results concern the steady state. In the NK model, the steady state has deflation and below-natural output (in addition the steady state only exists if $r^n < (\epsilon - 1)/(\delta\gamma)$). In the WUNK model the steady state has positive inflation and above-natural output.

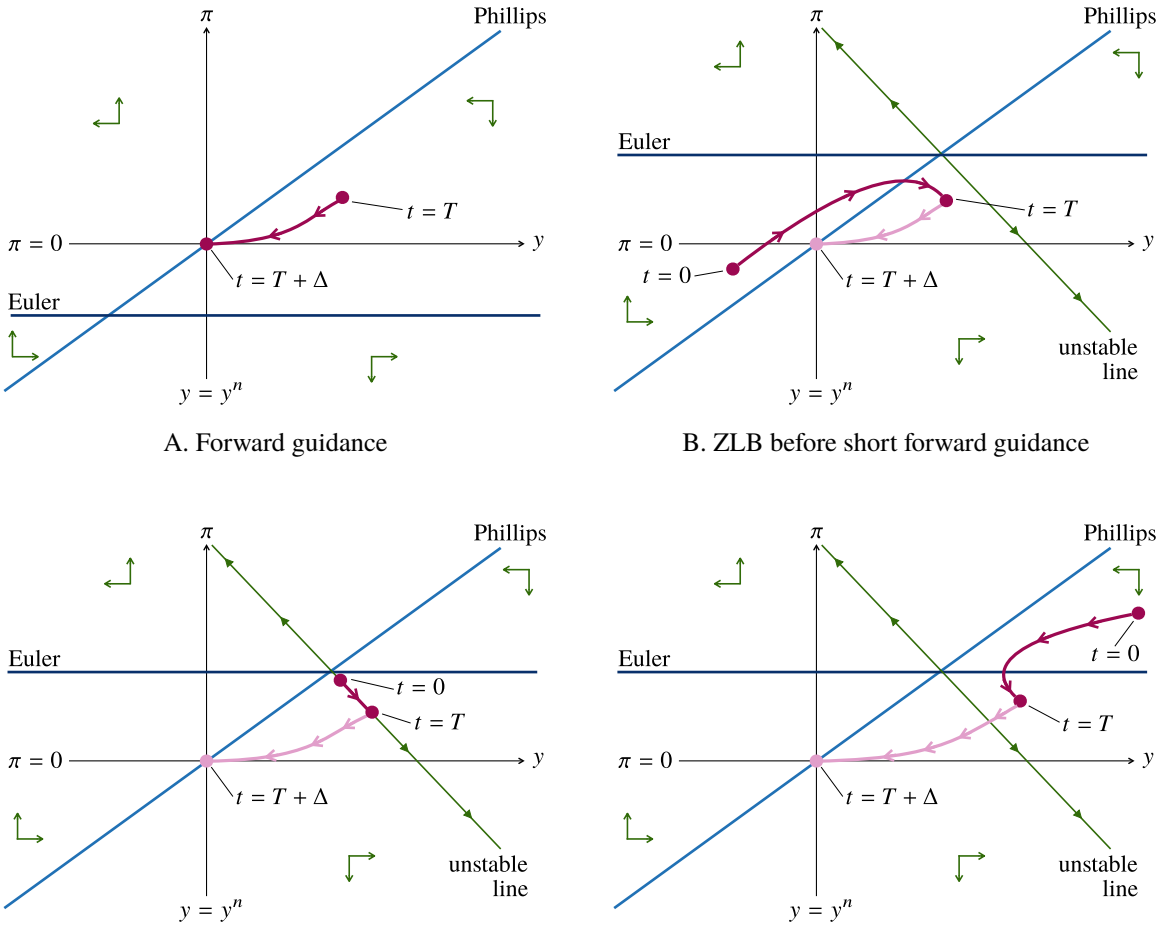


Figure 3. NK Model: ZLB Episode Followed by Forward-Guidance Episodes of Various Durations

Notes: The figure displays equilibrium trajectories in the NK model during a ZLB episode followed by a forward-guidance episode.The ZLB episode between times 0 and T is as in figure 2. It is followed by a forward-guidance episode between times T and $T + \Delta$: the natural rate of interest becomes positive but the central bank maintains the nominal interest rate at zero. At the end of forward guidance, at time $T + \Delta$, the central bank brings the economy to the natural steady state. The phase diagrams in panel A (similar to panel C in figure 1 but with $r^n > 0$) depict the forward-guidance episode. The phase diagrams in panels B, C, and D (same as panel C in figure 1) depict the ZLB episode with different durations for the following forward guidance (different Δ). In panels B, C, and D, the unstable line represents the eigenvector associated with the positive eigenvalue of (22). The NK model suffers from two problems, which are two facets of the forward-guidance puzzle: the forward-guidance boom becomes arbitrarily large as forward guidance becomes longer (panel A); and there is a reversal as the duration of forward guidance crosses a threshold, whereby the ZLB slump becomes a boom (panel C versus panel E).

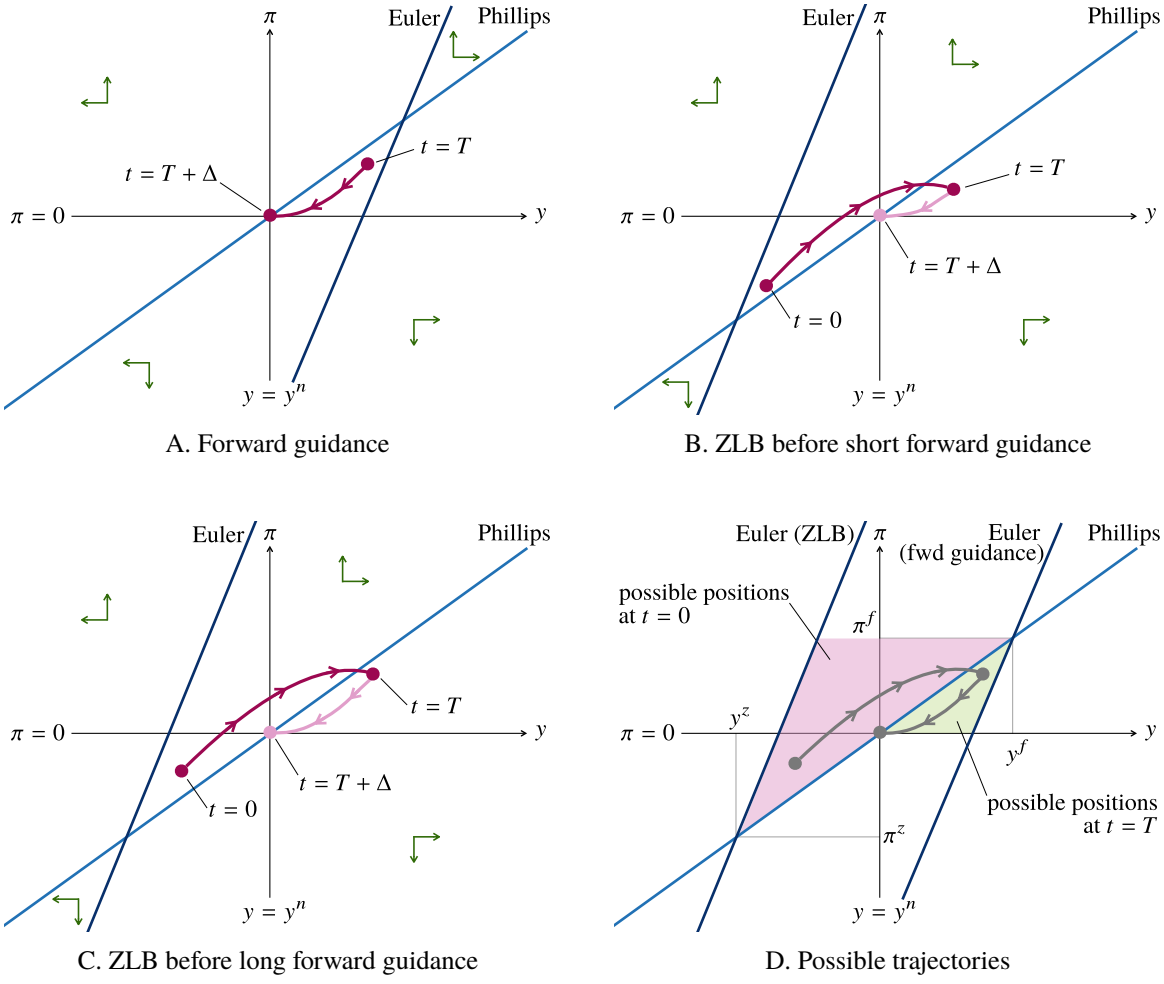


Figure 4. WUNK Model: ZLB Episode Followed by Forward-Guidance Episodes of Various Durations

Notes: The figure displays equilibrium trajectories in the NK and WUNK models during a ZLB episode followed by a forward-guidance episode.The ZLB episode between times 0 and T is as in figure 2. It is followed by a forward-guidance episode between times T and $T + \Delta$: the natural rate of interest becomes positive but the central bank maintains the nominal interest rate at zero. At the end of forward guidance, at time $T + \Delta$, the central bank brings the economy to the natural steady state. The phase diagram in panel A (similar to panel D in figure 1 but with $r^n > 0$) depict the forward-guidance episode. The phase diagrams in panels B and C (same as panels D in figure 1) depict the ZLB episode. Panel C is like panel B, but with longer forward guidance (longer Δ). Panel D is a generic version of panels B and C, describing any duration of ZLB and forward guidance. The forward-guidance puzzle disappears in the WUNK model as equilibrium trajectories are bounded in the green and red areas (panel F).

ZLB, $[y(0), \pi(0)]$, must be on the unstable line, between the ZLB steady state and $[y(T), \pi(T)]$ (panel C). As the ZLB lasts longer, the initial position inches closer to the steady state. Last, for longer forward guidance, the position at the beginning of forward guidance, $[y(T), \pi(T)]$, is on the right-hand side of the unstable line, so it is connected to trajectories coming from the northeast quadrant of the phase diagram (panel D). As a result, initial output and inflation are much higher than when $[y(T), \pi(T)]$ was on the other side of the unstable line. Then, as the ZLB lasts longer, initial output and inflation become higher and higher.

The power of forward guidance is much more subdued in the WUNK model, as illustrated in figure 4. In the third stage, $t > T + \Delta$, the economy is at the natural steady state. We then move to the forward-guidance episode, $t \in (T, T + \Delta)$, depicted in panel A. The phase diagram is similar to that in panel D of figure 1, except that $r^n > 0$ instead of $r^n < 0$. Since the Euler line goes through the point $[y^n, -r^n]$, we infer that the Euler line is shifted outward, and that the steady state, denoted $[y^f, \pi^f]$, shifts accordingly. Nevertheless, the equilibrium system remains a source. Hence, at time T, inflation and output must be at $\pi(T) > 0$ and $y(T) > y^n$; then they decrease over time, following the unique trajectory leading to $\pi(T + \Delta) = 0$ and $y(T + \Delta) = y^n$. As in the NK model, the economy booms during forward guidance; but unlike in the NK model, the paths of output and inflation are bounded above by the forward-guidance steady state, $[y^f, \pi^f]$.

Before forward guidance comes the ZLB episode, $t \in (0, T)$, depicted in panels B and C. The phase diagrams are the same as those in panel D of figure 1. As in the NK model, because of the boom engineered through forward guidance, the situation is improved at the ZLB: at any time before T, the economy tends to have higher inflation and output than without forward guidance. Unlike in the NK model, however, output during the ZLB episode is always below the level it reaches at the beginning of forward guidance, $y(T)$. This is because $[y(T), \pi(T)]$ is only connected to trajectories coming from the northeast quadrant of the ZLB phase diagram. So forward guidance cannot generate unbounded booms in the WUNK model. The ZLB cannot generate unbounded slumps either: the paths of output and inflation are bounded below by the ZLB steady state, $[y^z, \pi^z]$. These properties are showed in panel F: $[y(T), \pi(T)]$ necessarily falls in the green area so $[y(0), \pi(0)]$ is necessarily in the red area (or green area if T is tiny).

Based on these dynamics, we isolate three anomalies in the NK model, which are resolved in the WUNK model:

PROPOSITION 4: *Consider ZLB episode during* $(0, T)$*, with* $i = 0$ *and* $r^n = r^z < 0$*, followed by a forward-guidance episode during* $(T, T + \Delta)$*, with* $i = 0$ *and* $r^n = r^f > 0$*. Let* $y(t; T, \Delta)$ *and* $\pi(t; T, \Delta)$ *be output and inflation at time* t*.*

– *NK model: There exists a forward-guidance duration* $\Delta^* > 0$ *with the following properties.*

- *For any forward guidance shorter than Δ^*, a long-enough ZLB is an arbitrarily severe slump: for any $\Delta < \Delta^*$,*

$$\lim_{T\to\infty} y(0;T,\Delta) = -\infty$$
$$\lim_{T\to\infty} \pi(0;T,\Delta) = -\infty.$$

- *For any forward guidance longer than Δ^*, a long-enough ZLB is an arbitrarily large boom: for any $\Delta > \Delta^*$,*

$$\lim_{T\to\infty} y(0;T,\Delta) = +\infty$$
$$\lim_{T\to\infty} \pi(0;T,\Delta) = +\infty.$$

  In addition, a long-enough forward guidance generates an arbitrarily large boom: for any T,

$$\lim_{\Delta\to\infty} y(0;T,\Delta) = +\infty$$
$$\lim_{\Delta\to\infty} \pi(0;T,\Delta) = +\infty.$$

- *As the ZLB becomes arbitrarily long, the forward-guidance multiplier evaluated at the duration Δ^* becomes infinite. This implies that as the ZLB is long enough, an arbitrarily small increase or decrease in the duration of forward guidance has an arbitrarily large effect on output. Formally, we define the forward-guidance multiplier at Δ by*

$$\frac{dy}{d\Delta}(T,\Delta) = \lim_{x\to 0} \frac{y(0;T,\Delta+x) - y(0;T,\Delta)}{x};$$

  then we have

$$\lim_{T\to\infty} \frac{dy}{d\Delta}(T,\Delta^*) = +\infty.$$

– *WUNK model: For any durations T and Δ, the following properties arise.*

- *At all times the economy remains bounded above by the forward-guidance steady state and bounded below by the ZLB steady state. Formally, for any $t \in (0, T+\Delta)$,*

$$y(t;T,\Delta) \in [y^z, y^f]$$
$$\pi(t;T,\Delta) \in [\pi^z, \pi^f].$$

- *The effect of forward-guidance vanishes as the ZLB becomes infinitely long, such that the forward-guidance multiplier evaluated at any duration goes to zero. For any* Δ*,*

$$\lim_{T\to\infty} \frac{dy}{d\Delta}(T,\Delta) = 0.$$

The proposition describes several anomalies associated with forward guidance in the NK model. First, if the duration of forward guidance is short (as in panel B of figure 3), an arbitrarily long ZLB leads to an arbitrarily severe slump. Hence, the result from proposition 3, which was derived without forward guidance, extends to situations with short forward guidance.

Second, if the duration of forward guidance is above some threshold Δ^* (as in panel D of figure 3), an arbitrarily long ZLB leads to an arbitrarily large boom.

Hence, forward guidance cannot explain the stability of the Japanese economy after nearly thirty years at the ZLB. Even with forward guidance, except in the knife-edge case in which forward guidance lasts exactly Δ^*, output and inflation follow explosive paths as the ZLB is arbitrarily long-lasting.

Third, a large boom also occurs if instead of the ZLB episode, the forward-guidance episode is arbitrarily long-lasting.

Fourth, the multiplier of forward-guidance policy at duration Δ^* is infinite. This is because an arbitrary small increase in the duration of forward guidance move the economy at the beginning of forward guidance from one side of the ZLB unstable line to the other side, which can generate an arbitrarily large increase in output.

The anomalies identified in the proposition correspond to the instances of forward-guidance puzzle described by Carlstrom, Fuerst, and Paustian (2015, fig. 1), and Cochrane (2017, fig. 6). These papers also show that above a threshold duration, forward guidance transforms the ZLB into a boom whose amplitude increases with the duration of forward guidance.

In the WUNK model, such anomalous patterns vanish. Irrespective of forward guidance, an arbitrarily long ZLB always leads to the ZLB steady state, with mild deflation and below-natural output. An arbitrarily long forward-guidance episode brings the economy at the beginning of forward guidance at the forward-guidance steady state, with mild inflation and above-natural steady state. The forward-guidance multiplier is always finite, irrespective of forward-guidance duration. And irrespective of the the durations of ZLB and forward guidance, the economy always remains bounded between the ZLB and forward-guidance steady states.

Like our paper, the papers mentioned previously that obtain a well-behaved ZLB steady state address the forward-guidance puzzle. Beside these papers, a growing literature focuses solely on addressing the forward-guidance puzzle. Within this literature, our paper follows the same path

as those developing discounted Euler equations—in which future interest rates have less effect on today's consumption than in the standard version. For example, Del Negro, Giannoni, and Patterson (2015) generate discounting through overlapping generations. McKay, Nakamura, and Steinsson (2016) and Bilbiie (2017) generate discounting by introducing heterogeneous agents facing borrowing constraints. Angeletos and Lian (2018) generate discounting by introducing incomplete information. More related to our approach, Rannenberg (2019) generates discounting by introducing two types of households (one type is credit constrained and simply consumes its disposable income; one type is unconstrained and borrows and saves using safe assets) and by assuming that safe assets enter unconstrained households' utility function. Last, also closely related to our approach, Campbell et al. (2017) generate discounting by introducing bonds in the utility function.[11] By introducing relative wealth in the utility function we also generate a discounted Euler equation; but going one step further, we find that sufficient discounting transforms the dynamical system representing the ZLB equilibrium from a saddle to a source, resolving at once all anomalies at the ZLB. Furthermore, most of the literature base their analysis on numerical results, while we are able to establish our results using phase diagrams.

4.3. Government Spending

The government purchases quantities $g_k(t)$ of the differentiated goods $k \in [0,1]$. These quantities are aggregated into an index of public consumption that takes the same form as the index of private consumption:

$$g(t) \equiv \left[\int_0^1 g_k(t)^{(\epsilon-1)/\epsilon} dk\right]^{\epsilon/(\epsilon-1)}. \tag{30}$$

The public-consumption index $g(t)$ enters separately into households' utility functions. Government expenditure is financed with lump-sum taxation.

In addition, we assume that the disutility of labor is convex (finite Frisch elasticity). We alter the shape of the utility function to generate richer policy effects, as in Werning (2011), Cochrane (2017), and Wieland (2018). Indeed, the convexity allows us to have nonneutral government spending in steady state. With linear disutility of labor instead, real marginal costs are constant, so government spending is neutral in steady state: it does not affect private consumption or inflation.

[11] Other approaches have been used to address the forward-guidance puzzle. These include modifying the Phillips curve (for example, Carlstrom, Fuerst, and Paustian 2015), considering the economy in a temporary equilibrium with agents having "reflective" expectations Garcia-Schmidt and Woodford (2019), combining bounded rationality and incomplete markets (Farhi and Werning 2017), or introducing a spread between the federal funds rate and the interest rate on government bonds (for example, Bredemeier, Kaufmann, and Schabert 2018).

Specifically, household j incurs disutility

$$\frac{1}{1+\eta}h_j(t)^{1+\eta}$$

from producing goods and services, where the parameter $\eta > 0$ is the inverse of the Frisch elasticity of labor supply.

We follow the same steps as in the baseline model to determine the Euler equation and Phillips curve. (The derivation is relegated to appendix C.) Because output is no longer equal to private consumption, the Euler equation must remain expressed in terms of private consumption:

$$\frac{\dot{c}(t)}{c(t)} = r(\pi(t)) + u'(0)c(t) - \delta. \tag{31}$$

Furthermore, since output now includes private consumption plus public consumption, and the disutility of labor is convex, the Phillips curve is modified as follows:

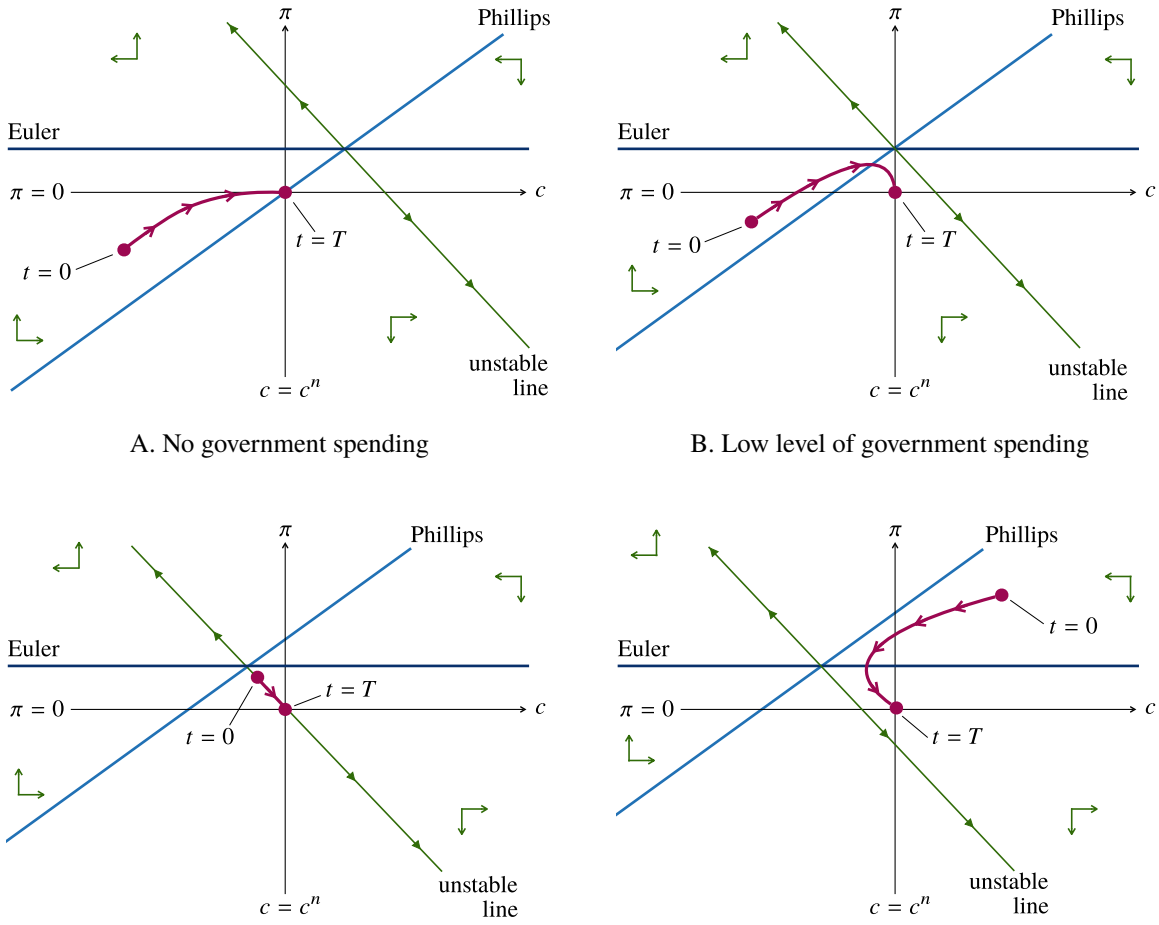
$$\dot{\pi}(t) = \delta\pi(t) + \frac{(\epsilon-1)\left[c(t)+g(t)\right]}{\gamma c(t)}\left\{1 - \frac{\epsilon}{\epsilon-1}\cdot\frac{\left[c(t)+g(t)\right]^{\eta}c(t)}{a^{1+\eta}}\right\}. \tag{32}$$

The interpretation of the Phillips curve remains the same, except that the real marginal cost takes a more complicated form. As before, the real marginal cost is the marginal rate of substitution between labor and consumption divided by the marginal product of labor. The marginal product of labor remains a. The marginal rate of substitution is the marginal disutility of labor divided by the marginal utility of consumption. The former remains $1/c(t)$, but the latter has become $h(t)^{\eta} = (y(t)/a)^{\eta} = \left[c(t)+g(t)\right]^{\eta}/a^{\eta}$. So the term after $\epsilon/(\epsilon-1)$ in the curly brackets is the real marginal cost.

We study the paths of consumption and inflation when the government uses government spending during the ZLB. We consider a two-stage scenario, as in Cochrane (2017). First, between time 0 and time $T > 0$, there is a ZLB episode. The natural rate of interest is negative: $r^n = r^z < 0$. In response, the central bank maintains the nominal interest rate at the ZLB: $i = 0$. To alleviate the ZLB, the government provides an amount $g > 0$ of public consumption. Second, after time T, the natural rate becomes positive again: $r^n > 0$. The central bank returns to the normal monetary-policy rule, given by (9). The government stops providing public consumption: $g = 0$.

The following proposition summarizes these results:

PROPOSITION 5: *Consider ZLB episode during $(0,T)$, with $i = 0$ and $r^n = r^z < 0$, accompanied by government spending $g > 0$. Let $c(t;T,g)$ and $\pi(t;T,g)$ be output and inflation at time t.*



A. No government spending

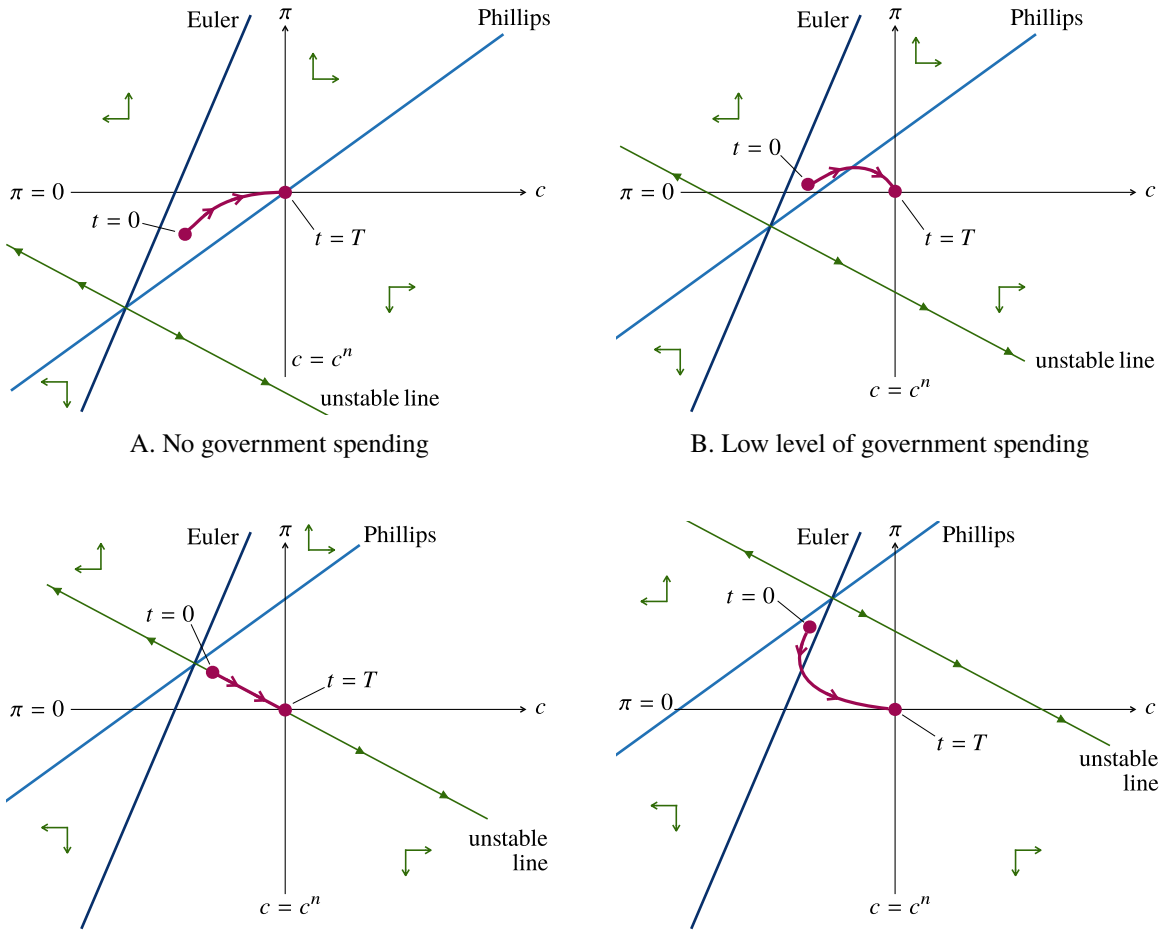
B. Low level of government spending

C. Medium level of government spending

D. High level of government spending

Figure 5. NK Model: ZLB Episode with Various Levels of Government Spending

Notes: The figure displays equilibrium trajectories in the NK model during ZLB episodes accompanied by various levels of government spending. A ZLB episode lasts between times 0 and T. During the episode, the natural rate of interest is negative and the central bank responds by setting the nominal interest rate to zero. At the same time, the government spends a positive amount. At the end of the ZLB, at time T, the central bank brings the economy to the natural steady state, where $y = y^n$ and $\pi = 0$. The phase diagrams are the same as in figure 1, panel C, except that the Phillips line is shifted inward by government spending.



A. No government spending

B. Low level of government spending

C. Medium level of government spending

D. High level of government spending

Figure 6. WUNK Model: ZLB Episode with Various Levels of Government Spending

Notes: The figure displays equilibrium trajectories in the WUNK model during ZLB episodes accompanied by various levels of government spending. A ZLB episode lasts between times 0 and T. During the episode, the natural rate of interest is negative and the central bank responds by setting the nominal interest rate to zero. At the same time, the government spends a positive amount. At the end of the ZLB, at time T, the central bank brings the economy to the natural steady state, where $y = y^n$ and $\pi = 0$. The phase diagrams are the same as in figure 1, panel D, except that the Phillips line is shifted inward by government spending.

– *NK model: There exists a government-spending level $g^* > 0$ with the following properties.*

  - *For any government spending below g^*, a long-enough ZLB is an arbitrarily severe slump: for any $g < g^*$,*

$$\lim_{T\to\infty} y(0;T,g) = -\infty$$
$$\lim_{T\to\infty} \pi(0;T,g) = -\infty.$$

  - *For any forward guidance longer than g^*, a long-enough ZLB is an arbitrarily large boom: for any $g > g^*$,*

$$\lim_{T\to\infty} y(0;T,g) = +\infty$$
$$\lim_{T\to\infty} \pi(0;T,g) = +\infty.$$

  - *As the ZLB becomes arbitrarily long, the government-spending multiplier evaluated at the government-spending level g^* becomes infinite. This implies that as the ZLB is long enough, an arbitrarily small increase or decrease in the government spending has an arbitrarily large effect on output. Formally, we define the government-spending multiplier at g by*

$$\frac{dy}{dg}(g) = \lim_{x\to 0} \frac{y(0;T,g+x) - y(0;T,g)}{x};$$

  then we have

$$\lim_{T\to\infty} \frac{dy}{dg}(T,g^*) = +\infty.$$

– *WUNK model: For any durations T, the following properties arise.*

  - *At all times the economy remains bounded. The bounds depend on the level of government spending g.*
  - *As the ZLB becomes infinitely long, the government-spending multiplier evaluated at any duration goes to the following finite value*

$$\lim_{T\to\infty} \frac{dy}{dg}(T,g) = 1 + \frac{\eta}{\frac{u'(0)\delta\gamma a}{\epsilon^{1/(1+\eta)}(\epsilon-1)^{\eta/(1+\eta)}} - (1+\eta)}.$$

5. New Keynesian Properties at the ZLB

Beside the anomalous properties described in the previous section, the New Keynesian model has several other, intriguing properties at the ZLB. Some of these properties are labeled "paradoxes": not because they are anomalous but because they defy usual economic logic. Here we show that the WUNK model shares these properties: paradox of thrift, paradox of toil, paradox of flexibility, and above-one government-spending multiplier.

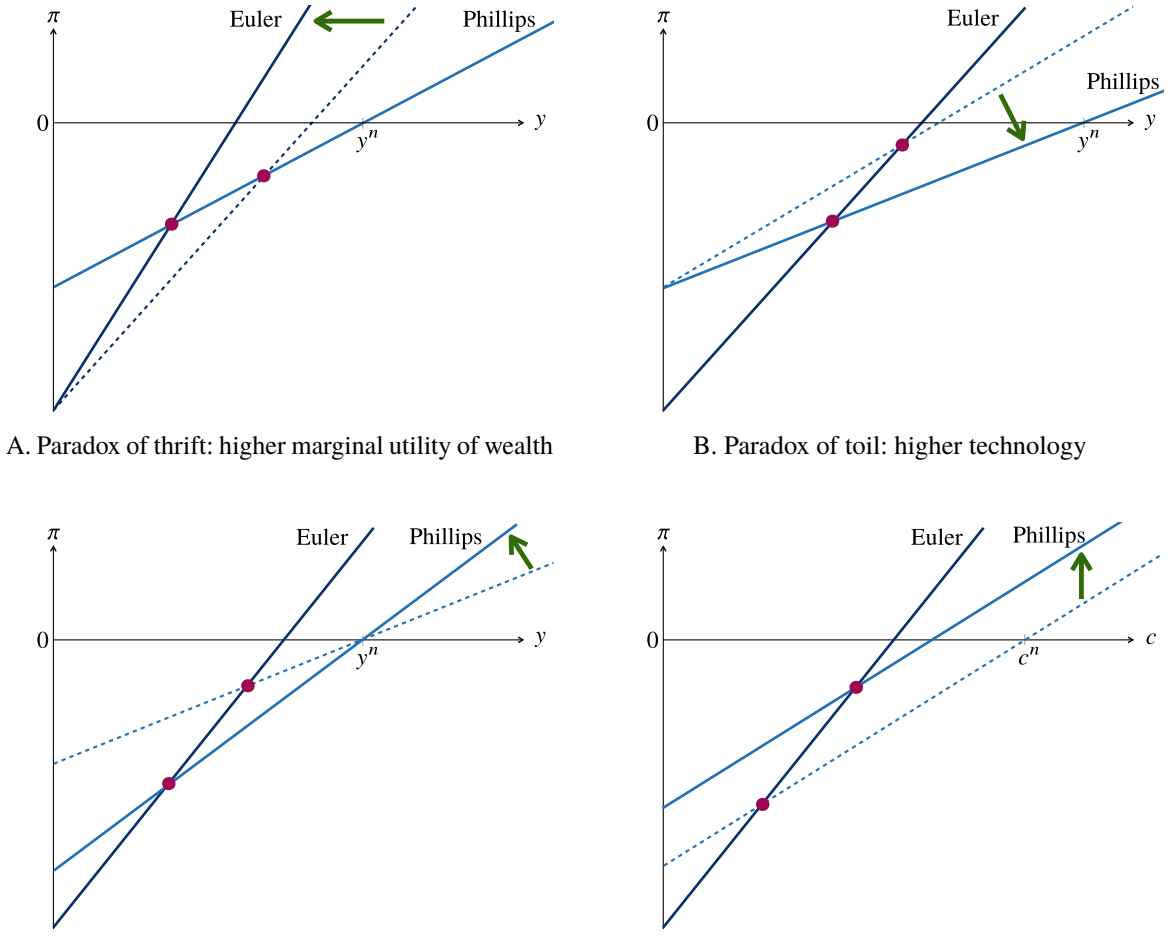
Since the ZLB equilibrium is determinate in the WUNK model, we are not forced to introduce temporary ZLB episodes: we simply analyze these paradoxes at a permanent ZLB. We assume that the natural rate of interest is negative and the central bank maintains the nominal interest rate at zero. The only equilibrium is to jump to and stay at the ZLB steady state, where the economy is in a slump: inflation is negative and output is below its natural level. Then an unexpected permanent shock occurs, immediately bringing the economy to a new ZLB steady state. We aim to compare the new equilibrium to the old one. Since each equilibrium is the intersection of the steady-state Phillips curve and steady-state Euler equation (15), we determine the effects of the shocks graphically, as shown in figure 7.

5.1. Paradox of Thrift

We first study the effect of an increase in the marginal utility of wealth ($u'(0)$). At the ZLB the steady-state Euler equation is $\pi = -\delta + u'(0)y$. Hence the increase in $u'(0)$ makes the Euler line steeper and moves the economy inward along the Phillips line: both output and inflation decrease (figure 7, panel A). The following proposition summarizes the result:

PROPOSITION 6: *Paradox of thrift: At the ZLB in the WUNK model, an unexpected and permanent increase in the marginal utility of wealth reduces output and inflation.*

The increase in the marginal utility of wealth operates exactly like the Keynesian paradox of thrift, which also appears in the New Keynesian model (Eggertsson 2010, p. 16). After the increase in the marginal utility of wealth, households want to increase their wealth holdings relative to their peers, so they favor saving over consumption. But in equilibrium relative wealth is fixed since all households are the same; hence the only way to save more relative to consumption is to reduce consumption. In normal times the central bank would offset this reduction in aggregate demand by reducing the nominal interest rate. As this is not an option at the ZLB, output falls.



A. Paradox of thrift: higher marginal utility of wealth

B. Paradox of toil: higher technology

C. Paradox of flexibility: lower price-adjustment cost

D. Above-one government-spending multiplier

Figure 7. WUNK Model: Comparative Statics at the ZLB

Notes: This figure displays the equilibrium during a permanent ZLB in the WUNK model, and its response to three unexpected permanent shocks. π is inflation, y is output, and y^n is the natural output. The Phillips line represents the steady-state Phillips curve, given by (13). The Euler line represents the steady-state Euler equation at the ZLB, given by (15) with $r(\pi) = -\pi$ and $r^n < 0$. The ZLB equilibrium is given by the intersection of the two lines: output is below its natural level and inflation is negative. Panel A illustrates the paradox of thrift: an increase in the marginal utility for wealth moves the Euler line inward, which depresses output and inflation. Panel B illustrates the paradox of toil: an increase in technology moves the Phillips line outward, which depresses output and inflation. Panel C illustrates the paradox of flexibility: a decrease in the price-adjustment cost rotates the Phillips line counterclockwise around the point $[y^n, 0]$, which depresses output and inflation.

5.2. Paradox of Toil

Next we examine the effect of an increase in technology (a). Using (7), the steady-state Phillips curve (13) can be rewritten as $\pi = \epsilon y/(\delta\gamma a) - (\epsilon - 1)/(\delta\gamma)$. Hence the increase in a makes the Phillips line flatter and moves the economy inward along the Euler line: both output and inflation decrease (figure 7, panel B). The following proposition summarizes the result:

PROPOSITION 7: *Paradox of toil: At the ZLB in the WUNK model, an unexpected and permanent increase in technology reduces output and inflation.*

This reduction in output is the paradox of toil, discovered by Eggertsson (2010). It operates as follows. With higher technology, real marginal costs are lower, and the natural level of output is higher: firms would like to produce and sell more. Hence, firms tend to reduce their prices, reducing inflation. Away from the ZLB, the central bank would offset this reduction by lowering the nominal interest rate; but this cannot happen at the ZLB. Then the reduction in inflation raises the real interest rate—as the nominal interest rate is at zero—which makes households more prone to save and in equilibrium leads to lower output.

5.3. Paradox of Flexibility

We finally turn to the effect of a decrease in the price-adjustment cost (γ). The steady-state Phillips curve (13) shows that the increase in γ leads to a counterclockwise rotation of the Phillips line around the point $[y^n, 0]$, which moves the economy inward along the Euler line: both output and inflation decrease (figure 7, panel C). The following proposition summarizes the results:

PROPOSITION 8: *Paradox of flexibility: At the ZLB in the WUNK model, an unexpected and permanent decrease in price-adjustment cost reduces output and inflation.*

This reduction in output is the paradox of flexibility, introduced in the New Keynesian model by Eggertsson and Krugman (2012, pp. 1487–1488). With a lower price-adjustment cost, firms are keener to adjust their prices to bring production closer to the natural level of output, which accentuates the existing deflation. Hence, as in the paradox of toil, the real interest rate rises, which results in lower output.

5.4. Above-One Government-Spending Multiplier

We finally turn to the effect of an increase in government spending (g). The steady-state Phillips curve (32) shows that the increase in γ leads to an inward shift the Phillips line, which moves the economy upward along the Euler line: both private consumption and inflation increase

(figure 7, panel D). Since private consumption increases when public consumption does, the government-spending multiplier, which satisfies $dy/dg = 1 + dc/dg$, is necessarily greater than one:

PROPOSITION 9: *At the ZLB in the WUNK model, an unexpected and permanent increase in government spending raises private consumption and inflation. Hence the government-spending multiplier is above one:*

$$\frac{dy}{dg} = 1 + \frac{\eta}{\frac{[u'(0)+r^z/c^n]\delta\gamma a}{\epsilon^{1/(1+\eta)}(\epsilon-1)^{\eta/(1+\eta)}} - (1+\eta)}. \tag{33}$$

The result that the government-spending multiplier is necessarily above one at the ZLB was established by Eggertsson (2011) and Christiano, Eichenbaum, and Rebelo (2011), among other. The intuition is the following. With higher government spending, real marginal costs for a given level of private consumption c are higher, so firms would like to reduce their sales to households. Hence, firms tend to increase their prices, raising inflation. At the ZLB, the increase in inflation lowers the real interest rate—as the nominal interest rate is at zero—which makes households more prone to consume. This, in equilibrium, leads to higher private consumption and a multiplier above one, as $dy/dg = 1 + dc/dg$.

Note that the multiplier remains finite although we consider permanent ZLB episodes and policy, consistent with the proposition 5, but different from the results in the existing ZLB literature.

6. Conclusion

This paper extends the textbook New Keynesian model by introducing relative wealth into the utility function. The marginal utility of wealth is assumed to be above a threshold that depends on price rigidity—an assumption that seems to hold in the data. Although our model deviates only minimally from the textbook model, it resolves all its anomalies at the ZLB: there is no explosive slump if the ZLB is long-lasting, no explosive boom if forward guidance is long-lasting, and no infinite forward-guidance and government-spending multipliers. Hence in our model, the ZLB might last forever as a steady state, and forward-guidance and government-spending policies have plausible, limited effects. At the same time, our model retains other properties of the New Keynesian model at the ZLB: paradox of thrift, paradox of toil, paradox of flexibility, and above-one government-spending multiplier.

Beyond the New Keynesian model, the wealth-in-the-utility-function assumption might be a simple way to model people's saving behavior better. First, it reconciles the single-digit interest

rates observed on many markets with the double-digit time discount rates measured in most studies. Second, it leads to a negative relationship between consumption and real interest rate, as in the old-fashioned IS curve. Third and related, it accommodates a broad range of steady-state real interest rates—including negative ones. Fourth, it reduces the effect of future interest rates on today's consumption. These properties contrast with the standard model, in which the steady-state real interest rate equals the time discount rate, and future interest rates have an implausibly large effect on today's consumption.

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Appendix A. Derivation of Phillips Curve and Euler Equation

We derive the two differential equations that govern the equilibrium of our New Keynesian model: the Phillips curve, given by (12), and the Euler equation, given by (14).

Household Saving and Pricing. We begin by characterizing households' optimal saving and pricing. To solve household j's optimization problem, we write the current-value Hamiltonian:

$$\mathcal{H}_j = \frac{\epsilon}{\epsilon-1}\ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\,dk\right) + u\left(\frac{b_j(t)-b(t)}{p(t)}\right) - \frac{y_j^d(p_j(t),t)}{a} - \frac{\gamma}{2}\pi_j(t)^2$$
$$+ \mathcal{A}_j(t)\left[i(t)b_j(t) + p_j(t)y_j^d(p_j(t),t) - \int_0^1 p_k(t)c_{jk}(t)\,dk - \tau(t)\right] + \mathcal{B}_j(t)\pi_j(t)p_j(t),$$

with control variables $c_{jk}(t)$ for all $k \in [0,1]$ and $\pi_j(t)$, state variables $b_j(t)$ and $p_j(t)$, and costate variables $\mathcal{A}_j(t)$ and $\mathcal{B}_j(t)$. To simplify we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Hamiltonian. To ease notation, we now drop the time index t.

The first optimality conditions are $\partial\mathcal{H}_j/\partial c_{jk} = 0$ for all $k \in [0,1]$. The conditions yield

$$\frac{1}{c_j}\left(\frac{c_{jk}}{c_j}\right)^{-1/\epsilon} = \mathcal{A}_j p_k. \tag{A1}$$

Appropriately integrating (A1) over all $k \in [0,1]$, and using (2) and (3), we find

$$\mathcal{A}_j = \frac{1}{pc_j}. \tag{A2}$$

Combining (A1) and (A2), we then obtain

$$c_{jk} = \left(\frac{p_k}{p}\right)^{-\epsilon} c_j. \tag{A3}$$

Integrating (A3) over all $j \in [0,1]$, we get the usual demand for good k:

$$y_k^d(p_k) = \int_0^1 c_{jk}\,dj = \left(\frac{p_k}{p}\right)^{-\epsilon} c, \tag{A4}$$

where $c = \int_0^1 c_j\,dj$ measures aggregate consumption. We use this expression for $y_k^d(p_k)$ in household k's Hamiltonian. We also obtain $\int_0^1 p_k c_{jk}\,dk = pc_j$: the price of one unit of consumption index is indeed p.

The second optimality condition is $\partial \mathcal{H}_j / \partial b_j = \delta \mathcal{A}_j - \dot{\mathcal{A}}_j$, which can be expressed

$$-\frac{\dot{\mathcal{A}}_j}{\mathcal{A}_j} = i + \frac{1}{p\mathcal{A}_j} \cdot u'\left(\frac{b_j - b}{p}\right) - \delta.$$

Using (A2), we obtain the household's Euler equation:

$$\frac{\dot{c}_j}{c_j} = i - \pi + c_j u'\left(\frac{b_j - b}{p}\right) - \delta. \tag{A5}$$

This Euler equation describes the optimal path of household j's consumption.

The third optimality condition is $\partial \mathcal{H}_j / \partial \pi_j = 0$, which yields

$$\mathcal{B}_j p_j = \gamma \pi_j. \tag{A6}$$

Differentiating (A6) with respect to time, we obtain

$$\frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j} = \frac{\dot{\pi}_j}{\pi_j} - \pi_j. \tag{A7}$$

The fourth optimality condition is $\partial \mathcal{H}_j / \partial p_j = \delta \mathcal{B}_j - \dot{\mathcal{B}}_j$, which implies

$$\epsilon \frac{y_j}{a p_j} - (\epsilon - 1)\mathcal{A}_j y_j + \mathcal{B}_j \pi_j = \delta \mathcal{B}_j - \dot{\mathcal{B}}_j.$$

Reshuffling the terms, we obtain

$$\pi_j - \frac{(\epsilon - 1) y_j \mathcal{A}_j}{\mathcal{B}_j p_j}\left(p_j - \frac{\epsilon}{\epsilon - 1} \cdot \frac{1}{a\mathcal{A}_j}\right) = \delta - \frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j}.$$

Then, using (A2), (A6), and (A7), we obtain the household's Phillips curve:

$$\frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon - 1) y_j}{\gamma c_j \pi_j}\left(\frac{p_j}{p} - \frac{\epsilon}{\epsilon - 1} \cdot \frac{c_j}{a}\right). \tag{A8}$$

This equation describes the optimal path of the price set by household j.

The previous four conditions, together with the transversality conditions $\lim_{t \to +\infty} e^{-\delta t} \mathcal{B}_j(t) p_j(t) = 0$ and $\lim_{t \to +\infty} e^{-\delta t} \mathcal{A}_j(t) b_j(t) = 0$, are necessary and sufficient for a maximum to the household's problem (Benhabib, Schmitt-Grohe, and Uribe 2001, p. 48).

Equilibrium Saving and Pricing. We now turn to saving and pricing in equilibrium. All households have the same initial wealth and initial price, so they all behave the same. We therefore omit the subscripts j and k.

Then, we simplify the household's Euler equation, given by (A5), and the household's Phillips curve, given by (A8), using two equilibrium conditions: relative wealth is zero; and production and consumption are equal ($y = c$). Accordingly, we simplify the household's Euler equation to

$$\frac{\dot{y}}{y} = r - \delta + u'(0)y, \tag{A9}$$

where $r = i - \pi$. This is just the Euler equation given by (14). Since $y^n = (\epsilon - 1)a/\epsilon$, we also simplify the household's Phillips curve to

$$\dot{\pi} = \delta\pi - \frac{\epsilon}{\gamma a}\left(y - y^n\right). \tag{A10}$$

This is the Phillips curve given by (12).

Appendix B. Phillips Curve and Euler Equation in Discrete Time

We recast the Phillips curve (12) and Euler equation (14) in discrete time. The discrete-time formulation is helpful to compare our model with the textbook New Keynesian model, which is usually presented in discrete time. This formulation also shows that introducing wealth in the utility function yields a discounted Euler equation.

In discrete time, households trade one-period government bonds. Bonds purchased in period t have a price $q(t)$ and pay one unit of money at maturity, in period $t + 1$. The nominal interest rate between t and $t + 1$ is defined as $i(t) = -\ln(q(t))$.

Household Saving and Pricing. We begin by characterizing saving and pricing by households. Household j chooses sequences $\left\{y_j(t), p_j(t), h_j(t), \left[c_{jk}(t)\right]_{k=0}^{1}, b_j(t)\right\}_{t=0}^{\infty}$ to maximize the discounted sum of instantaneous utilities

$$\sum_{t=0}^{\infty} \beta^t \left\{ \frac{\epsilon}{\epsilon - 1} \ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\, dk \right) + u\left(\frac{b_j(t) - b(t)}{p(t)} \right) - h_j(t) - \frac{\gamma}{2}\left[\frac{p_j(t)}{p_j(t-1)} - 1 \right]^2 \right\} dt,$$

where $\beta < 1$ is the time discount factor. The maximization is subject to three constraints. First, there is the production function (1). Second, there is the demand for good j, given by (6). The demand for good j remains the same as in continuous time because the allocation of consumption expenditure across goods is a static decision, so it is unaffected by the representation of time.

And third, there is a budget constraint:

$$\int_0^1 p_k(t)c_{jk}(t)\,dk + q(t)b_j(t) + \tau(t) = p_j(t)y_j(t) + b_j(t-1).$$

Household j is also subject to a solvency constraint preventing Ponzi schemes: $\lim_{T\to\infty} b_j(T) \geq 0$. Finally, household j takes as given the initial conditions $b_j(-1)$ and $p_j(-1)$, as well as the sequences of aggregate variables $\{p(t), q(t), c(t)\}_{t=0}^{\infty}$. We assume that all households face the same initial conditions, so they all behave the same.

To solve household j's problem, we set up the Lagrangian:

$$\mathcal{L}_j = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\epsilon}{\epsilon-1} \ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon}\,dk \right) + u\left(\frac{b_j(t) - b(t)}{p(t)} \right) - \frac{y_j^d(p_j(t),t)}{a} - \frac{\gamma}{2}\left[\frac{p_j(t)}{p_j(t-1)} - 1 \right]^2 \right.$$
$$\left. + \mathcal{A}_j(t)\left[p_j(t)y_j^d(p_j(t),t) + b_j(t-1) - \int_0^1 p_k(t)c_{jk}(t)\,dk - q(t)b_j(t) - \tau(t) \right] \right\}$$

where $\mathcal{A}_j(t)$ is the Lagrange multiplier on the budget constraint in period t, and

$$y_j^d(p_j(t),t) = \left[\frac{p_j(t)}{p(t)} \right]^{-\epsilon} c(t) \tag{A11}$$

is the demand for good j in period t. To simplify we have used the production and demand constraints to substitute $y_j(t)$ and $h_j(t)$ out of the Lagrangian.

We begin by computing the first-order conditions with respect to $c_{jk}(t)$ for all $k \in [0,1]$. As in the continuous-time case, we obtain

$$\mathcal{A}_j(t) = \frac{1}{p(t)c_j(t)}. \tag{A12}$$

We then turn to the first-order condition with respect to $b_j(t)$. The condition is

$$q(t)\mathcal{A}_j(t) = \frac{1}{p(t)} u'\left(\frac{b_j(t) - b(t)}{p(t)} \right) + \beta \mathcal{A}_j(t+1).$$

Using (A12), we obtain the household's Euler equation for consumption:

$$q(t) = c_j(t)u'\left(\frac{b_j(t) - b(t)}{p(t)} \right) + \beta \frac{p(t)c_j(t)}{p(t+1)c_j(t+1)}. \tag{A13}$$

Finally, the first-order condition with respect to $p_j(t)$ is

$$\epsilon\frac{y_j(t)}{a}+\beta\gamma\frac{p_j(t+1)}{p_j(t)}\left[\frac{p_j(t+1)}{p_j(t)}-1\right]=\gamma\frac{p_j(t)}{p_j(t-1)}\left[\frac{p_j(t)}{p_j(t-1)}-1\right]+(\epsilon-1)\mathcal{A}_j(t)p_j(t)y_j(t).$$

Using (A12) again, we obtain the household's Phillips curve:

$$\epsilon\frac{y_j(t)}{a}+\beta\gamma\frac{p_j(t+1)}{p_j(t)}\left[\frac{p_j(t+1)}{p_j(t)}-1\right]=\gamma\frac{p_j(t)}{p_j(t-1)}\left[\frac{p_j(t)}{p_j(t-1)}-1\right]+(\epsilon-1)\frac{p_j(t)y_j(t)}{p(t)c_j(t)}. \tag{A14}$$

Equilibrium Saving and Pricing. We now turn to saving and pricing in equilibrium. Since all households behave the same in equilibrium, we drop the subscripts j and k. The fact that all households hold the same wealth implies that relative wealth is zero. Next, production and consumption are equal, so we set $y(t)=c(t)$. Then, from (A13), we obtain the Euler equation:

$$q(t)=u'(0)y(t)+\beta\frac{p(t)y(t)}{p(t+1)y(t+1)}.$$

Similarly, from (A14), we obtain the Phillips curve:

$$\frac{p(t)}{p(t-1)}\left[\frac{p(t)}{p(t-1)}-1\right]=\beta\frac{p(t+1)}{p(t)}\left[\frac{p(t+1)}{p(t)}-1\right]-\frac{\epsilon}{\gamma a}\left[y(t)-y^n\right].$$

The next step is to log-linearize the Euler equation and Phillips curve around the natural steady state, where $y=y^n$, $\pi=0$, and $i=r^n$. We introduce the log-deviation of output from its steady-state level: $\hat{y}(t)=\ln(y(t))-\ln(y^n)$. We also introduce the inflation rate between periods t and $t+1$: $\pi(t+1)=\ln(p(t+1))-\ln(p(t))$.

We start by log-linearizing the Euler equation:

$$-i(t)=-r^n+(1-\alpha)\hat{y}(t)+\alpha\left[\hat{y}(t)-\hat{y}(t+1)-\pi(t+1)\right]$$

where

$$\alpha=\frac{\beta}{\beta+u'(0)y^n}.$$

Reshuffling the terms yields

$$\hat{y}(t)=\alpha\hat{y}(t+1)-\left[i(t)-r^n-\alpha\pi(t+1)\right]. \tag{A15}$$

Because the marginal utility of wealth is positive ($u'(0)>0$), we have $\alpha<1$. The Euler equation is therefore discounted: future output, $\hat{y}(t+1)$, appears discounted by a coefficient $\alpha<1$. Diba

and Loisel (2017, sec. 6) show that such discounting occurs in a variety of models. For example, in McKay, Nakamura, and Steinsson (2017) it occurs because of income risk and borrowing constraints. As another example, in Gabaix (2016) it appears because of consumers' bounded rationality. To make discounting more apparent, we can solve the Euler equation forward:

$$\hat{y}(t) = -\sum_{k=0}^{+\infty} \alpha^k \left[i(t+k) - r^n - \alpha\pi(t+k+1) \right].$$

As noted by McKay, Nakamura, and Steinsson (2017, p. 821), in this type of Euler equations, the effect of interest rates k periods in the future on current output is discounted by $\alpha^k < 1$. Hence, discounting is stronger for interest rates further in the future.

Finally we log-linearize the Phillips curve:

$$\pi(t) = \beta\pi(t+1) + \kappa\hat{y}(t) \tag{A16}$$

where

$$\kappa = \frac{\epsilon}{\gamma} \cdot \frac{y^n}{a} = \frac{\epsilon - 1}{\gamma}.$$

Despite wealth in the utility function, the Phillips curve remains the same as in a typical New Keynesian model: for instance, (A16) is the same as equation (21) in Gali (2008, p. 49).

Appendix C. Phillips Curve and Euler Equation with Government Spending

We rederive the Phillips curve and Euler equation in presence of government spending and convex disutility of labor. We obtain the Phillips curve given by (32) and the Euler equation given by (31).

Government Consumption. We begin by computing the government's demand for each good. At any point in time the government chooses the amounts g_j of each good $j \in [0,1]$ to minimize the expenditure

$$\int_0^1 p_j g_j \, dj$$

subject to the constraint of providing an amount of public consumption g:

$$\left[\int_0^1 g_j^{(\epsilon-1)/\epsilon} \, dj \right]^{\epsilon/(\epsilon-1)} = g.$$

To solve the government's problem we set up a Lagrangian:

$$\mathcal{L} = \int_0^1 p_j g_j \, dj + C \cdot \left\{ g - \left[\int_0^1 g_j^{(\epsilon-1)/\epsilon} \, dj \right]^{\epsilon/(\epsilon-1)} \right\},$$

where C is the Lagrange multiplier on the public-consumption constraint.

We broadly follow the same steps as in the derivation of (A4). The first-order conditions with respect to g_j for all $j \in [0,1]$ are $\partial \mathcal{L}/\partial g_j = 0$. These conditions imply

$$p_j = C \cdot \left(\frac{g_j}{g} \right)^{-1/\epsilon}. \tag{A17}$$

Appropriately integrating (A17) over all $j \in [0,1]$, and using (3) and (30), we find

$$C = p. \tag{A18}$$

Combining (A17) and (A18), we then obtain the government's demand for good j:

$$g_j = \left(\frac{p_j}{p} \right)^{-\epsilon} g. \tag{A19}$$

Household Saving and Pricing. We begin by characterizing households' optimal saving and pricing. To solve household j's optimization problem, we write the current-value Hamiltonian:

$$\mathcal{H}_j = \frac{\epsilon}{\epsilon - 1} \ln\left(\int_0^1 c_{jk}(t)^{(\epsilon-1)/\epsilon} \, dk \right) + u\left(\frac{b_j(t) - b(t)}{p(t)} \right) - \frac{1}{1+\eta} \left[\frac{y_j^d(p_j(t),t)}{a} \right]^{1+\eta} - \frac{\gamma}{2} \pi_j(t)^2$$
$$+ \mathcal{A}_j(t) \left[i(t) b_j(t) + p_j(t) y_j^d(p_j(t),t) - \int_0^1 p_k(t) c_{jk}(t) \, dk - \tau(t) \right] + \mathcal{B}_j(t) \pi_j(t) p_j(t).$$

Compared to the baseline case of appendix A, the Hamiltonian's terms including the consumption levels $c_{jk}(t)$ have not changed. Thus the optimality conditions $\partial \mathcal{H}_j / \partial c_{jk} = 0$ remain the same, which implies that (A1), (A2), and (A3) remain valid. Adding the government's demand, given by (A19), to households' demand, given by (A3), we obtain the total demand for good j at time t:

$$y_j^d(p_j(t),t) = g_j(t) + \int_0^1 c_{jk}(t) \, dk = \left[\frac{p_j(t)}{p(t)} \right]^{-\epsilon} y(t),$$

where $y(t) \equiv g(t) + \int_0^1 c_j(t) \, dj$ measures total consumption. The expression for $y_j^d(p_j(t),t)$ enters into the Hamiltonian $\mathcal{H}_j$.

The Hamiltonian's terms including bond holdings $b_j(t)$ have not changed either compared to the case of appendix A. Hence the optimality condition $\partial \mathcal{H}_j / \partial b_j = \delta \mathcal{A}_j - \dot{\mathcal{A}}_j$ remains the same, which implies that the Euler equation (A5) remains valid.

Next, since the Hamiltonian's terms including inflation $\pi_j(t)$ have not changed compared to the case of appendix A, the optimality condition $\partial \mathcal{H}_j / \partial \pi_j = 0$ is unchanged. Equations (A6) and (A7) therefore remain valid.

Last, because the disutility from labor is convex, the optimality condition $\partial \mathcal{H}_j / \partial p_j = \delta \mathcal{B}_j - \dot{\mathcal{B}}_j$ is modified. The condition now implies

$$\frac{\epsilon}{p_j} \left(\frac{y_j}{a} \right)^{1+\eta} - (\epsilon - 1) \mathcal{A}_j y_j + \mathcal{B}_j \pi_j = \delta \mathcal{B}_j - \dot{\mathcal{B}}_j,$$

which can be rewritten as

$$\pi_j - \frac{(\epsilon - 1) y_j \mathcal{A}_j}{\mathcal{B}_j p_j} \left(p_j - \frac{\epsilon}{\epsilon - 1} \cdot \frac{(y_j / a)^\eta}{a \mathcal{A}_j} \right) = \delta - \frac{\dot{\mathcal{B}}_j}{\mathcal{B}_j}.$$

Then, using (A2), (A6), and (A7), we obtain the household's Phillips curve:

$$\frac{\dot{\pi}_j}{\pi_j} = \delta + \frac{(\epsilon - 1) y_j}{\gamma c_j \pi_j} \left[\frac{p_j}{p} - \frac{\epsilon}{\epsilon - 1} \cdot \frac{(y_j / a)^\eta c_j}{a} \right]. \tag{A20}$$

Equilibrium Saving and Pricing. We simplify the household's Euler equation, given by (A5), using the facts that in equilibrium, all households behave the same (so $c_j = c$), and relative wealth is zero (so $b_j - b = 0$). Accordingly, we obtain the following Euler equation:

$$\frac{\dot{c}}{c} = r - \delta + u'(0) c.$$

This is just the Euler equation given by (31). Unlike in appendix A, production and consumption are not equal (since $y = c + g$); therefore, we can no longer use y instead of c.

Next, we turn to the household's Phillips curve, given by (A20). In equilibrium, all households behave the same, so $p_j = p$, $\pi_j = \pi$, $y_j = y$, and $c_j = c$. Moreover, output equals private plus public consumption: $y = c + g$. Hence, we obtain the following Phillips curve:

$$\dot{\pi} = \delta \pi + \frac{(\epsilon - 1)(c + g)}{\gamma c} \left[1 - \frac{\epsilon}{\epsilon - 1} \cdot \frac{(c + g)^\eta c}{a^{1+\eta}} \right].$$

This is just the Phillips curve given by (32).

Appendix D. Global Monetary-Policy Rule

In the main text, following most of the literature, we separately consider two local monetary-policy rules: the policy rule (9) away from the ZLB, and the monetary policy $i = 0$ at the ZLB. Then, we check that the ZLB is indeed not binding at the equilibrium obtained under (9). In this appendix we discuss a more rigorous approach—we consider a global monetary-policy rule, which directly incorporates the ZLB:

$$i(\pi) = \max\{0, r^n + \phi\pi\}. \tag{A21}$$

The policy rule implies a real interest rate

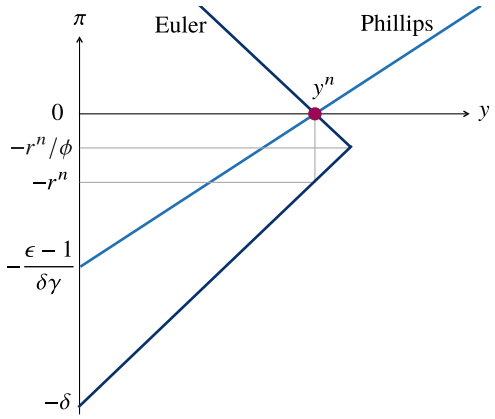
$$r(\pi) = \max\{-\pi, r^n + (\phi - 1)\pi\}. \tag{A22}$$

In the NK model, it is well-known that things are more complicated under the global monetary-policy rule. As showed by Benhabib, Schmitt-Grohe, and Uribe (2001), when $r^n > 0$, the natural steady state may coexist with another, ZLB steady state. As a result, Specifically, we show that even when monetary policy is active, there usually exists many equilibrium trajectories that start in the neighborhood of the natural steady state and converge to the second, ZLB steady state.

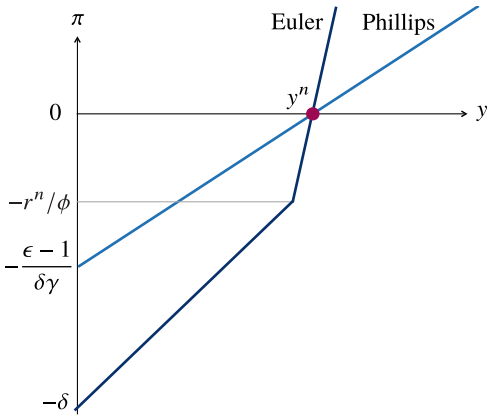
In the WUNK model, on the other hand, considering a global monetary-policy rule would not make much difference—it would only be more cumbersome. This is because, unlike in the NK model, the WUNK model continues to feature a unique steady state under the global policy rule. If $r^n > 0$, the natural steady state is the unique steady state, and local dynamics follow (17); if $r^n < 0$, the ZLB steady state is the unique steady state, and local dynamics follow (22).

Let's establish the result that the WUNK model admits a unique steady state under the global monetary-policy rule (A21). A steady-state equilibrium must satisfy the steady-state Phillips curve, given by (13), and the steady-state Euler equation, given by (15), in which the real interest rate comes from (A22). The monetary-policy rule (A22) implies that $r(\pi) = -\pi$ if $\pi < -r^n/\phi$ and $r(\pi) = r^n + (\phi - 1)\pi$ if $\pi \geq -r^n/\phi$. Hence, we first search for potential steady states with $\pi \geq -r^n/\phi$, before turning to potential steady states with $\pi < -r^n/\phi$.

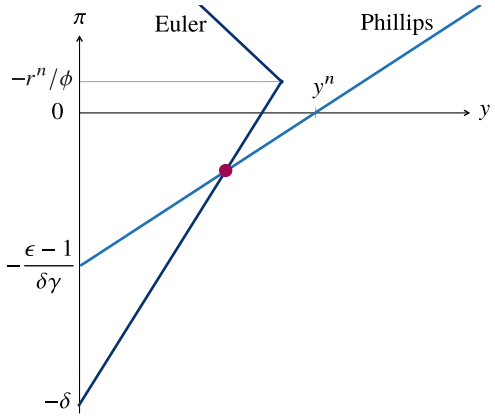
We start with the range $\pi \geq -r^n/\phi$. In that range, $r(\pi) = r^n + (\phi - 1)\pi$, so the steady-state Phillips curve and Euler equation form the linear system (18)–(19). As argued in the proof of proposition 1, $[y, \pi] = [y^n, 0]$ is the unique solution of the system. But that solution is a steady state only when $\pi \geq -r^n/\phi$. Since $\pi = 0$ and $\phi > 0$, that solution is a steady state only when $r^n \geq 0$. When $r^n < 0$, there is no steady state in the range $\pi \geq -r^n/\phi$.



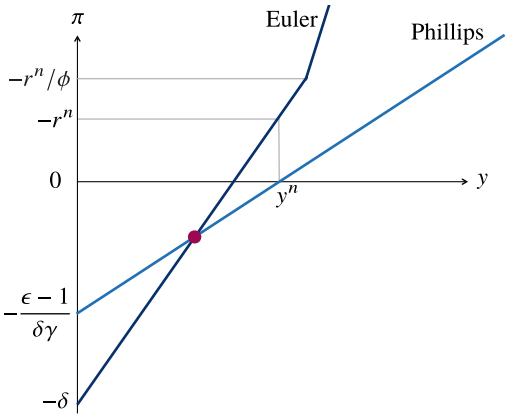
A. Positive natural rate, active monetary policy



B. Positive natural rate, passive monetary policy



C. Negative natural rate, active monetary policy



D. Negative natural rate, passive monetary policy

Figure A1. WUNK Model: Steady State in Various Situations

Notes: π is inflation, y is output, y^n is the natural level of output, r^n is the natural rate of interest, and ϕ indicates the response of monetary policy to inflation ($\phi > 1$: active monetary policy; $\phi < 1$: passive monetary policy). The Phillips line represents the steady-state Phillips curve, given by (13). The Euler curve represents the steady-state Euler equation for the WUNK model, given by (15) and (A22) with the marginal utility of wealth satisfying (16). The intersection of the Phillips line and Euler curve gives the steady state of the WUNK model. The four panels describe the steady state in four different situations: positive versus negative natural rate of interest, and active versus passive monetary policy. The main result illustrated in the figure is that in the WUNK model the steady state is unique. If the natural rate of interest is positive, the ZLB is not binding and the steady state has zero inflation and output at its natural level. If the natural rate of interest is negative, the ZLB is binding and the steady state has deflation and output below its natural level.

We then turn to the range $\pi < -r^n/\phi$. In that range, $r(\pi) = -\pi$, so the steady-state Phillips curve and Euler equation form the linear system (23)–(24). As argued in the proof of proposition 2, the system admits a unique solution $[y, \pi] \in \mathbb{R}^2$. Inflation in that unique solution is given by (21). Condition (16) implies that $u'(0)\delta\gamma a/\epsilon > 1$, so π has the sign of r^n. For that solution to be a steady state, however, we need $\pi < -r^n/\phi$. Hence, as $\phi > 0$, the inequality $\pi < -r^n/\phi$ holds only when $r^n < 0$. This means that there is no steady state on the range $\pi < -r^n/\phi$ when $r^n \geq 0$, but there may be one when $r^n < 0$: the requirement is that $y \geq 0$. We have seen in the proof of proposition 2 that this requirement is always satisfied; therefore, the solution to the system (23)–(24) is a steady state when $r^n < 0$. This steady state has $\pi < 0$ (from (21) and $r^n < 0$) and $y < y^n$ (from (23) and $\pi < 0$).

These results are summarized in the following proposition:

PROPOSITION A1: *Under the global monetary-policy rule* (A21)*, the WUNK model admits a unique steady-state equilibrium. If the natural rate of interest is positive, the unique steady state is the natural steady state, with zero inflation, output at its natural level, and nominal interest rate at the natural rate of interest. If the natural rate of interest is negative, the unique steady state is a ZLB steady state, with deflation, output below its natural level, and zero nominal interest rate.*

These results are illustrated in figure A1. The Phillips line represents the steady-state Phillips curve (13): it is linear, upward sloping, and goes through the point $[y, \pi] = [y^n, 0]$.

The Euler curve represents the steady-state Euler equation (15) combined with the interest-rate rule (A22). The Euler curve is composed of two linear branches because (A22) defines the real interest rate as a piecewise linear function of inflation. The non-ZLB branch is defined for $\pi > -r^n/\phi$ by (15) and $r = r^n + (\phi - 1)\pi$ (corresponding to $i = r^n + \phi\pi$). The ZLB branch is defined for $\pi < -r^n/\phi$ by (15) and $r = -\pi$ (corresponding to $i = 0$). The non-ZLB branch is downward sloping when monetary policy is active, and upward sloping when monetary is passive. Because of (16), when it is upward sloping, the non-ZLB branch is always steeper than the Phillips line. The non-ZLB branch (or its continuation) goes through the point $[y, \pi] = [y^n, 0]$. The ZLB branch is upward sloping and steeper than the Phillips line due to (16). The condition $\delta > \sqrt{(\epsilon - 1)/\gamma}$ ensures that the intercept of the ZLB branch with the y-axis is below the intercept of the Phillips line. Finally, the ZLB branch (or its continuation) goes through the point $[y, \pi] = [y^n, -r^n]$.

Consequently, if $r^n > 0$, then $[y^n, 0]$ is the unique steady state, whether monetary policy is active (panel A) or passive (panel B). This is the natural steady state, at the intersection of the Phillips line and non-ZLB Euler branch. If $r^n < 0$, on the other hand, $[y^n, 0]$ is not a steady state. Instead, the unique steady state is a ZLB steady state, at the intersection of the Phillips line and the ZLB Euler branch. This steady state features deflation and below-natural output. Such a ZLB

steady state may prevails with active monetary policy away from the ZLB (panel C) or passive monetary policy away from the ZLB (panel D).

To conclude, we note that the results in this appendix are less robust than those in the main text. The uniqueness of the steady state relies on the functional-form assumptions of log utility over consumption and linear disutility of labor. These assumptions lead to the linearity of the steady-state Phillips curve and Euler equation, which ensures the uniqueness of the steady state. With general isoelastic utility functions instead, the steady-state Phillips curve and Euler equation would be nonlinear, and several new steady states may appear. The results in the text, on the other hand, would continue to hold with general isoelastic utility functions, as long as the analysis concentrates on the neighborhood of the natural steady state (when the natural rate of interest is positive) or the closest ZLB steady state (when the natural rate of interest is negative).

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