Trade Theory with International Production Networks

David Rezza Baqaee
UCLA

Emmanuel Farhi*
Harvard

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Abstract

We characterize comparative statics for open production network economies in general equilibrium. We generalize growth-accounting to environments with international trade, input-output networks, and arbitrary distortions. We characterize how income responds to trade shocks up to a second-order approximation, and show that our result generalizes Arkolakis et al. (2012) to environments with nonlinear input-output networks. We show how aggregate trade elasticities can be constructed from microeconomic estimates. Finally, we also show that the costs of tariffs (but not iceberg costs) can be calculated by an appropriate summing up of deadweight loss triangles, and provide non-parametric formulae for these losses in terms of sufficient statistics. We apply our results to find that reallocation of resources has been a drag on the incomes of industrialized countries, the presence of intermediate inputs doubles the costs of Brexit (and changes its distributional consequences), and network spill-overs mean that main beneficiary of Trump tariffs on China may be Mexico.

*Emails: baqaee@econ.ucla.edu, efarhi@harvard.edu. We thank Natalie Bau for insightful comments. We are grateful to Maria Voronina and Chang He for outstanding research assistance.

1 Introduction

In this paper, we study how aggregate statistics, like GDP and real income, respond to disturbances in open-economies. Our comparative static results are general and non-parametric, allowing us to bring together results from several different literatures, including the growth-accounting, production networks, and international trade literatures. At an abstract level, we characterize how every variable (every price and quantity) responds to a microeconomic disturbance anywhere. However, we place our focus narrowly on how real income and real GDP respond to shocks.

This paper expands upon and generalizes a fundamental theorem of closed economies. Consider a closed, perfectly competitive economy where households have identical homothetic preferences and primary factors are inelastically supplied. Then, we can write

$$\frac{\mathrm{d}\log U}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log W}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_i} = \frac{sales_i}{GDP}, \tag{1}$$

where U is social welfare (in consumption equivalent units), W is real income, Y is real output, and A_i is a Hicks-neutral shock to some producer i. This means that the sales share of producer i is a sufficient statistic for understanding, to a first-order, the impact of a shock on aggregate welfare, aggregate income, and aggregate output. The first equality follows from homotheticity of preferences. The second equality follows from the fact that in a closed economy, real income and real output (when measured via a consumption price index) are identical. The final, and most subtle, equality is a consequence of the first welfare theorem known as Hulten's theorem. In open, and potentially distorted, economies, every one of the equality signs above can fail. In this paper, we generalize and clarify the relationship between each component of equation (1) in these environments.

To describe our results, we proceed equality by equality starting with

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log A_i} = \frac{sales_i}{GDP},$$

showing that the elasticity of real GDP with respect to shocks are given by sales shares. We show that if perfect competition holds within a country, then changes in aggregate productivity are still described by Hulten (1978) even if the economy is open to trade. However, when marginal cost pricing does not hold in a country, aggregate productivity is no longer described by Hulten's theorem, rather it behaves as described in Baqaee and Farhi (2017c).

Whereas Hulten's theorem applies unchanged to perfectly competitive open economies for real GDP, the relationship between real income and real GDP

$$\frac{\mathrm{d}\log W}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_i}$$

breaks down even if the economy is perfectly competitive. Intuitively, the simplicity of Hulten's theorem comes from the fact that, at the efficient allocation, a reallocation of resources has no effect on output and aggregate productivity, and hence, can be ignored. In an open-economy, reallocations do matter for real income up to a first order, even if the allocation is efficient. The reason is that in an open economy, one consumer's gain is another consumer's loss — since at the efficient point, reallocations are zero-sum. Therefore, whereas reallocations have no first-order effect on aggregate productivity and real GDP in efficient environments, they do have first-order effects on real income and consumption. We derive growth-accounting style expressions that apply to real income, rather than real GDP, and show that these expressions can decompose changes in welfare into a productivity effect and a reallocation effect. Once aggregated over all households in the world, in an efficient economy, the reallocation effects sum to zero, leaving behind only the pure technology effects. Hence, for the world as a whole (or equivalently, a closed economy), our welfare-accounting formulas collapse to be the same as traditional growth-accounting formulas. Outside of those cases however, the two formulas are very different.

Finally, social welfare in open economies differs from real income, breaking down

$$\frac{\mathrm{d}\log U}{\mathrm{d}\log A_i} = \frac{\mathrm{d}\log W}{\mathrm{d}\log A_i}$$

due to non-aggregability of consumer preferences. Therefore, we provide an approximation of a social welfare function for the world economy when each consumer's preferences are weighted according to their income share in the competitive equilibrium.

We state our results in two ways: (i) using ex-post sufficient statistics, and (ii) using ex-ante sufficient statistics. When stated in terms of ex-post sufficient statistics, comparative statics can be tied to changes in factor shares. These expressions are convenient for growth-accounting applications where our goal is measuring and decomposing the sources of real income growth, but are less useful if we need to perform counterfactuals or predict future changes (since we do not know how factor shares might respond to a shock). To make counterfactual predictions, we solve for the changes in factor income

shares implied by the model. These changes can be written in terms of microeconomic elasticities of substitution and the input-output matrix, and they do not require us to fully specify the structural model. By plugging these expressions for the changes in factor shares into our ex-post formulas, we can state our results in terms of ex-ante sufficient statistics.

Since solving for welfare changes requires us to first solve for how factor incomes change, our analysis also delivers, for free, the distributional consequences of shocks on the different factors. In addition, simple corollaries of our results show how aggregate trade elasticities, which are key statistics of interest in the international trade literature, can be written as linear combinations of underlying microeconomic elasticities of substitution.

Our approach allows us to spot some interesting connections between the closed-economy macro literature and the open-economy trade literature. The former is interested in tracing out the impact of productivity shocks on real GDP, whereas the latter is interested in tracing out the impact of trade shocks on real income. We prove that, under some conditions, there exists a useful isomorphism between these two types of questions. In particular, the income effects of iceberg shocks in CES trade models are equivalent to the output effects of productivity shocks in a dual closed economy. This means that we can use results from the closed-economy literature, principally Hulten (1978) and Baqaee and Farhi (2017a), to characterize the effects of iceberg shocks on welfare up to the second-order. Our formulas provide a generalization of the influential insights of Arkolakis et al. (2012) to environments with disaggregated, and non-linear, input-output connections.

For most of the paper, we restrict ourselves to efficient economies. Towards the end of the paper, we extend our results to allow for arbitrary distorting wedges (e.g tariffs). We show that extending our analysis to incorporate distortions via wedges is very straightforward. Conceptually, in a distorted economy, reallocations not only affect welfare, but they can also affect productivity and real GDP, and our set up allows for a unified treatment of both welfare and productivity in situations where reallocations matter to a first order.

We also show that, in general, the losses to society as a whole, and to the output of each country, from the imposition of tariffs or other distortions can be computed by an appropriate summing up of Harberger triangles. We provide explicit formulas for what these Harberger triangles are equal to in terms of microeconomic sufficient statistics, statistics that depend on microeconomic elasticities of substitution and the input-output

network. In particular, we show how the existence of global value chains can dramatically increase the costs of protectionism by both inflating the size of deadweight loss triangles *and* the weight on the importance of each triangle.

We include some empirical and quantitative applications of our results. We use our growth-accounting formulas to decompose changes in real income for different countries into pure technology and reallocation effects. We find that reallocation effects have had significant impacts on welfare for many countries. In particular, industrialized manufacturing dependent countries like Japan and Germany have lost much ground over our sample, whereas developing countries like China and commodity exporters like Australia and Canada have gained. The US has not been as dramatically affected by reallocation effects, and the evolution of real income has mostly kept pace with that of technological progress. However, even for countries like the US, where real income on the whole has not been affected by reallocation, factors within the US have experienced dramatic reallocations, with high-skilled real income gaining and low and medium skilled incomes losing.

Next, we use a quantitative model to predict how trade policy can affect different countries. We consider three exercises, each of which highlights a conceptual point. First, we consider the welfare changes associated with a reversal of globalization, achieved either via import tariffs or non-tariff trade barriers. We show that the latter is far more costly than the former, and link this to the fact that deadweight loss triangles are much smaller than the loss trapezoids generated by non-trade barriers. We also show that for both tariffs and non-tariff barriers, accounting for the existence of global value chains is crucial. Quantitatively, the losses from protectionism are tripled or quadrupled once we account for the existence of input-output connections.

Second, we consider a simple Brexit scenario where the remaining members of the European Union place tariffs or non-tariff trade barriers on British imports. We find that the former is much better than the latter, since in the latter some EU member states are even worse affected than Britain, while in the former scenario, European income on the whole rises from the collection of tariff revenues. Interestingly, regardless of whether the trade barriers are tariffs or not, low and medium skilled workers in Britain are better off after the tariffs are imposed, but only if we ignore the input-output connections between producers. When we allow for network linkages, all British factors are made worse off by Brexit (though capital and high-skilled labor are still more negatively affected).

Finally, we compute the effects of US tariffs on Chinese imports, motivated by recent

events. We find that, once we take into account network connections, Mexico, rather than US workers, are the primary beneficiaries of US tariffs on China, while the primary losers are China and China's major trading partners (Taiwan, Korea, and Australia). This last exercise emphasizes the interconnected nature of global trade, and the importance of considering the trading system as a whole when evaluating policy counterfactuals.

The outline of the paper is as follows. In Section 2, we set up the basic framework and define the relevant objets of interest. In Section 3, we derive our ex-post growth and welfare accounting expressions for efficient economies and use it to decompose income growth. In Section 4, we derive our ex-ante sufficient statistic formulas for doing counterfactuals in efficient economies and show how they can be used to aggregate trade elasticities and study the distributional consequences of shocks. In Section 5, we establish a duality between open and closed economies that allows us to extend the insights of Arkolakis et al. (2012) to economies with non-linear disaggregated production. We find that the welfare gains from trade can be much higher once one accounts for complementarities in production and intermediate inputs. In Section 6, we show how our results can be generalized to inefficient environments. In Section 7, we perform some quantitative counterfactual exercises. Before concluding in Section 9, we briefly describe how the results in prior sections, which occasionally assume CES functional forms, can easily be generalized to be fully non-parametric in Section 8.

Related Literature

Our paper is related to three literatures: the literature on production networks, the literature on growth accounting, and the literature on the gains from trade. We discuss each literature in turn starting with the one on production networks.

The literature on production networks has primarily been concerned with the propagation of shocks in closed economies, typically assuming a representative agent. For instance, Long and Plosser (1983), Acemoglu et al. (2012), Atalay (2017), Baqaee (2018), Carvalho et al. (2016), among many others. For an overview of this literature, see Carvalho and Tahbaz-Salehi (2018) and the references therein. Our results builds on the results by Baqaee and Farhi (2018) who show how shocks propagate in input-output networks with heterogenous consumers, since heterogeneity of consumers is a defining property of trade models.

Our growth accounting results are related to closed-economy results like Solow (1957), Hulten (1978), as well as the literature extending growth-accounting to open economies,

including Kehoe and Ruhl (2008), and Burstein and Cravino (2015) who show that Hulten's theorem can be applied to measure productivity changes in open-economies. Perhaps closest to us are Diewert and Morrison (1985) and Kohli (2004) who introduce output indices which account for terms-of-trade changes. Our real income and welfare-accounting measures share their goal, though our decomposition into pure productivity changes and reallocation effects is different. In explicitly accounting for the existence of intermediate inputs, our approach also speaks to how one can circumvent the double-counting problem and spill-overs arising from differences in gross and value-added trade, issues studied by Johnson and Noguera (2012) and Koopman et al. (2014). Relative to these other papers, our approach has the added bonus of easily being able to handle inefficiencies and wedges.

Finally, our results build on and contribute to the literature on the gains (or losses) from trade. In particular, our results generalize some of the results in Arkolakis et al. (2012) and Costinot and Rodriguez-Clare (2014) to environments with non-linear input-output connections. Our framework generalizes the input-output models emphasized in Caliendo and Parro (2015), Caliendo et al. (2017), Morrow and Trefler (2017), Fally and Sayre (2018), and Bosker and Westbrock (2018). Our results about the effects of trade in distorted economies also relates to Arkolakis et al. (2015). Our results also relate to work with non-parametric or semi-parametric models of trade like Adao et al. (2017) and Lind and Ramondo (2018).

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Trefler and Zhu (2010), Vanek (1968), Davis and Weinstein (2008), Costinot and Rodriguez-Clare (2014), Johnson (2014), Caliendo et al. (2017), , Cuñat and Zymek (2017), Feenstra and Sasahara (2017), Burstein and Vogel (2017),Krugman et al. (1995), Elsby et al. (2013), Koopman et al. (2014), Hummels et al. (2001), Blanchard et al. (2017), Costinot and Rodríguez-Clare (2018), Barattieri et al. (2018), Artuç et al. (2010), Dix-Carneiro (2014), Galle et al. (2017), Yi (2003), Epifani and Gancia (2011), Berthou et al. (2018), Bernard et al. (2018).

2 Framework

In this section, we set up the general framework of the model and define the key statistics whose properties we study. To start with, we will assume marginal cost pricing, so that there are no distorting wedges. In Section 6, we extend our results to environments with

distortions (e.g. markups, tariffs, taxes, etc.)

2.1 Model

There is a set of households H, a set of goods N, and a set of factors F. Each household, good, and factor is assigned to a country. Denote the set of households, goods, and factors belonging to country c by H_c, N_c, and F_c. Finally, C is the set of all countries. Each households $h \in H_c$ owns some share of the factors in that country $f_c \in F_c$. We denote this share by Φ_{hf}. We take the quantity of each factor, as well as the ownership structure of factors Φ as being exogenous.

Households

Each household $h \in H_c$ maximizes a homothetic demand aggregator

$$C_h = \mathcal{W}_h(c_{h1}, \ldots c_{hN}), \tag{2}$$

subject to the budget constraint

$$\sum_i p_i c_{hi} = \sum_{f \in F_c} \Phi_{hf} w_f L_f + t_h, \tag{3}$$

where p_i is the price of good i, c_{hi} is the quantity of good i consumed by household h, w_f is the wage of factor f and L_f is the quantity of factor f in country c. Finally, t_h is an exogenously given net lump sum transfer from the rest of the agents to household h.[1,2]

Goods

Each good is produced using a constant returns production function with the associated cost function

$$\frac{y_k}{A_k} \mathcal{C}_k(p_1, \ldots, p_N, w_1, \ldots, w_F), \tag{4}$$

[1]These exogenous transfers can be used to allow for trade imbalances in steady-state, similar to Dekle et al. (2008).

[2]In mapping our model to data, we interpret domestic "households" as any agent which consumes resources without producing resources to be used by other agents. Specifically, this means that we include domestic investment and government expenditures in our definition of "households" when we map this model to the data.

where y_k is the total quantity of good k produced and C_k / A_k is the unit cost function. The parameter A_k is an exogenous Hicks-neutral productivity shifter. As pointed out by McKenzie (1959), neoclassical production functions are constant returns to scale without loss of generality, since any decreasing returns production function can always be written as a constant returns production function by adding quasi-fixed factors.

The assumption that productivity shifters are Hicks-neutral is also made without loss of generality. This is because biased technical change, say for k's ability to use an input j, can be captured by adding a new (constant-returns) producer who buys from j and sells to k. A Hicks-neutral productivity shifter for this new producer is isomorphic to an input-biased technology shock. Similarly, iceberg trade costs can be represented as productivity shifters for specialized exporters.

Equilibrium

Given productivities A_i and a feasible vector of transfers (a vector satisfying $\sum_i t_i = 0$), a general equilibrium is a set of prices p_i, intermediate input choices x_{ij}, factor input choices l_{if}, outputs y_i, and consumption choices c_{ci}, such that: (i) each producer chooses inputs to minimize costs taking prices as given; (ii) each household maximizes utility subject to its budget constraint taking prices as given; (iii) the markets for all goods and factors clear.

2.2 Definitions

To characterize comparative statics, we introduce some input-output notation and definitions.

Input-Output Definitions

Let $b_{(h)}$ be the $N \times 1$ vector whose ith element is the share of good i in consumer h's budget

$$b_{hi} = \frac{p_i c_{hi}}{\sum_{i \in N} p_i c_{hi}}. \tag{5}$$

Let χ_h be consumer h's share of total income

$$\chi_h = \frac{\sum_{i \in N} p_i c_{hi}}{\sum_{j \in N} \sum_{d \in H} p_j c_{dj}}. \tag{6}$$

Following Baqaee and Farhi (2018), define the heterogenous-agent input-output matrix to be the $(H+N+F)\times(H+N+F)$ matrix Ω whose ijth element is equal to i's expenditures on inputs from j as a share of its total revenues/income

$$\Omega_{ij} \equiv \frac{p_j x_{ij}}{p_i y_i}. \tag{7}$$

The HAIO matrix Ω includes the factors of production and the households, where factors consume no resources (zero rows), while households produce no resources (zero columns). The Leontief inverse matrix is

$$\Psi \equiv (I-\Omega)^{-1} = I + \Omega + \Omega^2 + \dots. \tag{8}$$

While the input-output matrix Ω records the *direct* exposures of one agent to another, the Leontief inverse matrix Ψ records instead the *direct and indirect* exposures through the production network.

The *global* Domar weight λ_i of producer i is its sales as a fraction of world GDP

$$\lambda_i \equiv \frac{p_i y_i}{GDP} = \sum_{h\in H} \chi_h \Psi_{hi}, \tag{9}$$

where the second equality is an accounting identity linking the Leontief inverse to the Domar weight.

The *local* Domar weight λ_{ci} of producer i in country c is its sales as a fraction the GDP of country c

$$\lambda_{ci} = \frac{p_i y_i}{GDP_c}.$$

Note that $\sum_{i=1}^{N} \lambda_i > 1$ in general since some sales are not final sales but intermediate sales. For expositional convenience, for a factor f, we use Λ_f and Λ_{cf} instead of λ_f and λ_{cf}.

Nominal Output and Nominal Income

Gross domestic product (GDP) or nominal output for country c is the total earnings of the factors located in that country

$$GDP_c = \sum_{f\in F_c} w_f L_f = \sum_{h\in H_c}\sum_{i\in N_c} p_i c_{hi} + \sum_{i\in N_c}\sum_{j\notin N_c} p_j x_{ji} - \sum_{i\in N_c}\sum_{j\notin N_c} p_j x_{ij} = \sum_{i\in N} p_i q_{ci}^{GDP}. \tag{10}$$

This is, in turn, equal to the revenue of all domestic firms minus the intermediate input costs of all domestic firms (including both imported and foreign intermediates).[3] The scalar q_{ci}^{GDP} is the net quantity of good i in the GDP of country c (which means that it can be positive or negative).

Gross domestic income (GDI) or nominal income for country c is the total income earned by domestic households in that country

$$GDI_c = \sum_{f \in F_c} w_f L_f + t_c = \sum_{h \in H_c} \sum_{i \in N} p_i c_{hi}. \tag{11}$$

Nominal GDP and GDI can differ from one another due to trade imbalances.

Define world GDP and GDI by summing across all households in the world

$$GDP = \sum_{c \in C} \sum_{h \in H_c} \sum_{i \in N} p_i c_{hi} = \sum_{c \in C} \sum_{f \in F_c} \left(w_f L_f + t_c \right) = GDI.$$

Of course, at the global level, GDP and GDI are equal since the world has no net exports. When we write GDP without a subscript, this refers to world GDP.

Real Output and Real Income

Since the economy has heterogeneous households, the level of real GDP or GDI is ambiguous to define. Hence, we do not offer a definition for the *level* of real GDP, defining instead only the *changes* in real output and income using the Divisia index. The change in real GDP of country c is

$$\mathrm{d} \log Y_c = \sum_{i \in N} F_{ci} \, \mathrm{d} \log q_{ci}^{GDP}, \tag{12}$$

where F_{ci} is good i's net share in GDP $p_i q_{ci}^{GDP} / GDP_c$. Similarly, we can define changes in the GDP deflator as

$$\mathrm{d} \log P_c = \sum_{c} F_{ci} \, \mathrm{d} \log p_i. \tag{13}$$

We can also measure changes in household h's welfare

$$\mathrm{d} \log W_h = \sum_{i \in N} \Omega_{hi} \, \mathrm{d} \log c_{hi}, \tag{14}$$

[3] Equivalently, GDP is the receipts of domestic producers from sales to households and foreign firms minus foreign purchases of intermediate inputs. Since we interpret household demand as consisting of consumption, investment, and government purchases, this is the familiar $GDP = C + I + G + X - M$.

where Ω_{hi} is good i's share in household h's expenditures.[4] With some abuse of notation, we can also define changes in real income (real GDI) for country c by aggregating across households in that country

$$\mathrm{d}\log W_c = \sum_{i\in N}\sum_{h\in H_c}\left(\frac{\chi_h}{\sum_{k\in H_c}\chi_h}\Omega_{hi}\right)\mathrm{d}\log c_{ci}, \tag{15}$$

where $c_{ci} = \sum_{h\in H_c} c_{hi}$. If there is a representative household in country c (H_c is a singleton), then changes in real GDI are a first-order approximation to changes in the representative agent's welfare, and (14) is the same as (15).[5]

Recall that whenever a country has balanced trade, nominal income and nominal output are the same. However, even when trade is balanced, real output and real income are *not* the same. This is because real GDP and real GDI use different price deflators, the former uses the GDP deflator (that accounts for imports and exports directly), whereas the latter uses a consumption basket price index (which only accounts for prices faced directly by consumers).

For the entire world, real output $\mathrm{d}\log Y$ and real income $\mathrm{d}\log W$ are the same, since the world is in autarky. If the world has a representative consumer (H is a singleton) then $\mathrm{d}\log Y$ also measures changes in the welfare of that consumer. In this extreme case, the model in this paper simplifies to the one studied by Baqaee and Farhi (2017b).[6]

3 Comparative Statics: Ex-Post Sufficient Statistics

We now characterize how real output and real income respond to shocks. Our results in this section will be in terms of expenditure shares and changes in expenditure shares. Hence, the results will only be useful with knowledge of expenditure shares, and in the case of welfare, knowledge of changes in expenditure shares. This makes the results ideal for applications where there is information on the expenditure shares before and after

[4] Shephard's lemma implies that the right-hand side of (14) is equal to the variation in household h's welfare (the left-hand side).

[5] Our notion of real GDI growth is related to, but distinct from, the command-basis real GDP growth computed by the BEA and discussed by, for example Kehoe and Ruhl (2008). The primary difference is in the treatment of imported intermediate inputs — whereas the command-basis notion of GDP uses (final and intermediate) import prices to deflate net exports, our price deflator would only use prices of imported final goods.

[6] See Hulten (1973) for more details on the relationship between Divisia indices and cardinal measures of welfare and output.

the shock (i.e. growth-accounting). In the next section, we characterize how expenditure shares change in response to shocks conditional on knowledge of elasticities of substitution. This can then be combined with the results in this section to produce counterfactual predictions about how income and welfare would respond to shocks.

3.1 Output and Productivity-Accounting

In response to a productivity shock to some producer i in country c,

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log A_i} = \lambda_{ci}\mathbf{1}(i \in N_c). \tag{16}$$

Hence, to a first-order, a shock to producer i moves real output in country c by an amount equal to that producer's local Domar weight. Perhaps counterintuitively, (16) implies that shocks to foreign producers have *no* effect on domestic real GDP (to any order of approximation)![7]

Since changes in iceberg trade costs, and changes in the quantity of factors, can be represented as productivity shocks, equation (16) leads to the following characterization of changes in real GDP.

Theorem 1 (Productivity-Accounting)**.** *If domestic producers price goods at marginal cost then*

$$\mathrm{d}\log Y_c = \sum_{i\in N_c} \lambda_{cf}\,\mathrm{d}\log A_i + \sum_{f\in F_c} \Lambda_{cf}\,\mathrm{d}\log L_c. \tag{17}$$

Hence, as shown by Burstein and Cravino (2015), productivity-accounting, a la Hulten (1978), is the same in an open economy as it is in a closed economy, namely, the Solow (1957) residual is equal to the Domar-weighted sum of productivity shocks of domestic producers. Of course, what does break down in the open economy is the connection between real output growth, and changes in real income.

[7] There are two important caveats to this statement: (i) shocks to foreign producers may change domestic local Domar weights, and thereby change the way local shocks affect the domestic economy (a nonlinear interaction), (ii) Equation (16) is false if the economy is not efficient. We shall discuss both of these issues at length in Sections 5 and 6.

3.2 Income and Welfare-Accounting

In a closed economy, real income and real output are the same, and so Hulten's theorem applies not just to output, but also to income and welfare. In order to characterize changes in real income, we need to define network-adjusted exposures. For each household h, define

$$\lambda_k^h = \Omega_{hk} + \sum_{i \in N} \Omega_{hi}\Omega_{ik} + \sum_{i,j \in N} \Omega_{hi}\Omega_{ij}\Omega_{jk} + \cdots = \Psi_{hk} \tag{18}$$

as the *exposure* of consumer h to producer k. We can use a similar formula to define the network-adjusted exposure of each consumer h to each factor f, which we denote with capital letters Λ_f^h. The Domar weight is related to the exposures via the accounting identity

$$\lambda_k = \sum_{h \in H} \chi_h \lambda_k^h; \tag{19}$$

in other words, the Domar weight of each good k can be decomposed into an average of household exposures to that good, weighted by household income.

In response to a productivity shock to some producer i ,

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_i} = \lambda_k^h + \sum_{f \in F_c} \Phi_{hf} \frac{\Lambda_f}{\chi_c} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_i} - \sum_{f \in F} \Lambda_f^c \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_i}. \tag{20}$$

Hence, unlike real GDP, welfare and real income move in more complex ways in an open economy than in a closed one.

Equation (20) shows that in response to a productivity shock to k, holding fixed the distribution of income Λ, consumer h is made better off due to their network-adjusted exposure λ_k^h to producer k. However, in equilibrium, the distribution of income can shift in response to the shock, and these "terms-of-trade" effects have welfare consequences. These consequences are captured by the last two summands on the right hand side of (20): the first is the gains to income from an appreciation in the price of factors h owns, while the second is the losses to income from an appreciation in the price of factors h consumes. In a closed economy, the last two summands exactly offset one another and so can be neglected. Furthermore, in a closed economy, the network-adjusted exposure of the representative consumer λ_k^h is exactly equal to the Domar weight λ_k.[8]

[8]Equation (20) collapses to Hulten's theorem for a closed economy even in the absence of a structural representative agent. To see this, construct changes in world real income using the weighted average of individual real incomes: $\mathrm{d}\log W = \sum_{h \in H} \chi_h \, \mathrm{d}\log W_h$. Then applying equation (20) results in

To complete our characterization of welfare, we also show how welfare responds to a change in transfers. Let $\tau_h = t_h / \sum_{i \in N, h \in H} p_i c_{hi}$ be the transfers to household h as a share of global GDP. Then, in response to a transfer shock $\mathrm{d}\,\tau$:

$$\mathrm{d} \log W_h = \frac{1}{\chi_h} \mathrm{d}\, \tau_h + \sum_{f \in F_c} \Phi_{hf} \frac{\Lambda_f}{\chi_c} \mathrm{d} \log \Lambda_f - \sum_{f \in F} \Lambda_f^h \mathrm{d} \log \Lambda_f, \tag{21}$$

where χ_h is household h's share of aggregate income. In response to a transfer shock, there is the direct effect of the transfer, as well as the accompanying effect of changes in relative factor rewards. Equations (20) and (21) allow the following (first-order) characterization of changes in welfare (or real income).

Theorem 2 (Welfare-Accounting)**.**

$$\mathrm{d} \log W_h - \sum_{f \in F} \Lambda_f^c \mathrm{d} \log L_f = \sum_{i \in N} \lambda_i^c \mathrm{d} \log A_i + \sum_{f \in F} \left(\Phi_{hf} \Lambda_{cf} - \Lambda_f^c \right) \mathrm{d} \log \Lambda_f + \frac{1}{\chi_h} \mathrm{d}\, \tau. \tag{22}$$

3.3 Interpretation of Growth-Accounting Formulas

Theorem 2 shows how changes in consumer welfare can be related to changes in factor income shares, so that conditional on knowledge of how factor income shares will respond to a shock, we can deduce how welfare responds to that shock. In the next section, we show how factor income shocks will move in equilibrium, and hence, how welfare will move in response to shocks.

However, Theorem 2 is also interesting because it can be used to *measure* the reason why welfare changed in equilibrium. Conceptually, disturbances affect welfare by changing the size of the pie and each agent's share of the pie. Theorem 2 provides precisely this decomposition of welfare. In a world with heterogenous consumption baskets, the notion of "the pie" is ambiguous, since different agents value goods differently. Hence, our decomposition breaks changes in welfare into (i) the change in welfare from increased production of each good holding fixed every agent's share of each good, and (ii) the change in welfare from changes in the shares of each good consumed by each agent. Formally, borrowing language from Baqaee and Farhi (2017c), let $\mathcal{X}$ be an $(H + N + F) \times (H + N + F)$ admissible allocation matrix, where $\mathcal{X}_{ij} = x_{ij}/y_j$ is the share of the physical output y_j

$\mathrm{d} \log W / \mathrm{d} \log A_k = \mathrm{d} \log Y / \mathrm{d} \log A_k = \lambda_k$, even though there is no structural representative agent.

used by agent i (including factors and households). Specify the vector of productivities A and denote by $\mathcal{W}_h(A, \mathcal{X})$ the welfare W_h of household h achieved by this allocation.[9,10]

Now consider how welfare changes in response to shocks in equilibrium:

$$\begin{aligned} \mathrm{d}\log W_h &= \underbrace{\frac{\partial \log \mathcal{W}_h}{\partial \log A}\,\mathrm{d}\log A}_{\Delta\text{Technology}} + \underbrace{\frac{\partial \log \mathcal{W}_h}{\partial \mathcal{X}}\,\mathrm{d}\,\mathcal{X}}_{\Delta\text{Reallocation}}, \\ &= \frac{\partial \log \mathcal{W}_h}{\partial \log A}\,\mathrm{d}\log A + \frac{\partial \log \mathcal{W}_h}{\partial \mathcal{X}}\left(\frac{\partial \mathcal{X}}{\partial \log A}\,\mathrm{d}\log A + \frac{\partial \mathcal{X}}{\partial \tau}\,\mathrm{d}\,\tau\right). \end{aligned}$$

The change in welfare can be broken down into two components: the direct or "pure" effect of changes in technology $\mathrm{d}\log A$, holding the distribution of resources $\mathcal{X}$ constant; and the indirect effects arising from the equilibrium changes in the distribution of resources $\mathrm{d}\,\mathcal{X}$. Reallocation effects arise for two reasons: (i) a change in the trade deficit, and (ii) due to changes in the relative prices. Specifically, changes in terms of trade do not necessarily offset changes in the quantity of exports, hence, the share of income claimed by each household can rise or fall in response to movements in relative prices.[11]

We can interpret Theorem 2 as giving us components of this decomposition

$$\frac{\partial \log \mathcal{W}_h}{\partial \log A_i}\,\mathrm{d}\log A_i = \lambda_i^h\,\mathrm{d}\log A_i, \tag{23}$$

[9]The allocation matrix is admissible if the following conditions are verified: $0 \leq \mathcal{X}_{ij} \leq 1$ for all i and j; $\mathcal{X}_{ij} = 0$ for all j and for $N+1 \leq i \leq N+F$; $\sum_{i=1}^{N+F} \mathcal{X}_{ij} \leq 1$ for all $1 \leq j \leq N$; $\sum_{i=1}^{N+F} \mathcal{X}_{ij} = 1$ for all $N+1 \leq j \leq N+F$; and there exists a unique resource-feasible allocation such that the share x_{ij}/y_j of the output y_j of producer i which is used by producer j is equal to $\mathcal{X}_{ij}$, so that $\mathcal{X}_{ij} = \frac{x_{ij}}{y_j}$.

[10]To see how to construct this allocation, consider the production functions F_i defined as the duals of the cost functions C_i in the usual way. Then the vector of outputs y_i solves the system of equations $y_i = F_i(\mathcal{X}_{1i}y_1, \ldots, \mathcal{X}_{(N+F)i}, y_{N+F})$ for $1 \leq i \leq N$ and $y_{N+f} = L_f$ for $1 \leq f \leq F$.

[11]The reallocation effect measured here relates to the gains from international risk-sharing identified by Cole and Obstfeld (1991). They showed that in a Cobb-Douglas model, there may be no gains from allowing countries to insure each other against productivity shocks. The reason for this intuition is that in a Cobb-Douglas world, where all price-elasticities are unity, the terms of trade move exactly to offset the change in net exports, hence providing perfect insurance without the need for financial contracts. In our context, assuming that all production and cost functions are Cobb-Douglas would imply that factor shares are constant, which would then eliminate the terms associated with reallocation arising from productivity shocks. Viewed in this way, it is also intuitive why these terms disappear in an economy with a representative agent: there are no reallocation effects in this case.

and

$$\frac{\partial \log \mathcal{W}_h}{\partial \mathcal{X}} \mathrm{d}\,\mathcal{X} = \sum_{f \in c} \frac{\Lambda_f}{\chi_c} \mathrm{d} \log \Lambda_f - \sum_{f=1}^{F} \Lambda_f^c \,\mathrm{d} \log \Lambda_f + \frac{1}{\chi_h} \mathrm{d}\,\tau. \tag{24}$$

An alternative decomposition that's common in the literature is to break down changes in welfare into the change in income and the change in prices:

$$\mathrm{d} \log W_h = \mathrm{d} \log \chi_h - \mathrm{d} \log P_h,$$

where χ_h is the share of income and P_h is the price index of consumer h. One might be tempted to interpret changes in income shares as changes in the share of the pie, and changes in prices as the change in the size of the pie. To see why this interpretation is flawed, consider the example in Figure 1. In this example, each household produces a good using their domestic factor and sells the entire output to the other household. The sales of the domestic good finance the purchases of the foreign good. Now consider a perturbation of transfers $\mathrm{d}\,\tau$ which moves income from home to foreign. This transfer will shift both income and prices, however, since this economy is capable of only a single feasible allocation, it must be that

$$\frac{\mathrm{d} \log W_h}{\mathrm{d}\,\tau} = 0$$

and hence

$$\frac{\mathrm{d} \log \chi_h}{\mathrm{d}\,\tau} = -\frac{\mathrm{d} \log P_h}{\mathrm{d}\,\tau}.$$

Hence, with the income vs. expenditures decomposition, we are led to the conclusion that the share of the pie changed (income shares), and there were offsetting changes to the size of the pie (prices). However, the allocation did not change — these transfers were simply a veil, and any reasonable observer agrees that neither the size nor the share of the pie changed. This type of logic can also be applied to productivity accounting, see Appendix R for more details.

3.4 Application to Welfare Accounting

In this section, we use Theorem 2 to decompose changes in income for the representative consumer in different countries over time. We decompose income into changes due to changes in the physical environment (holding fixed the allocation of resources), changes in the allocation of resources due to changes in relative factor prices (terms of

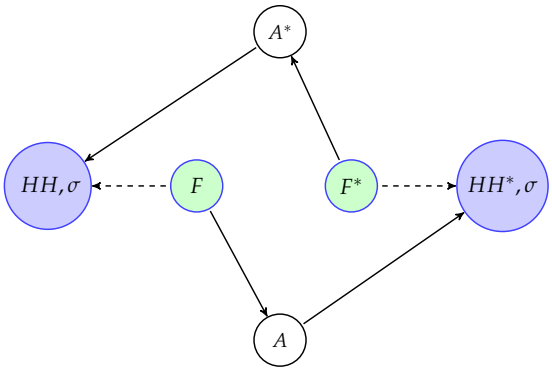


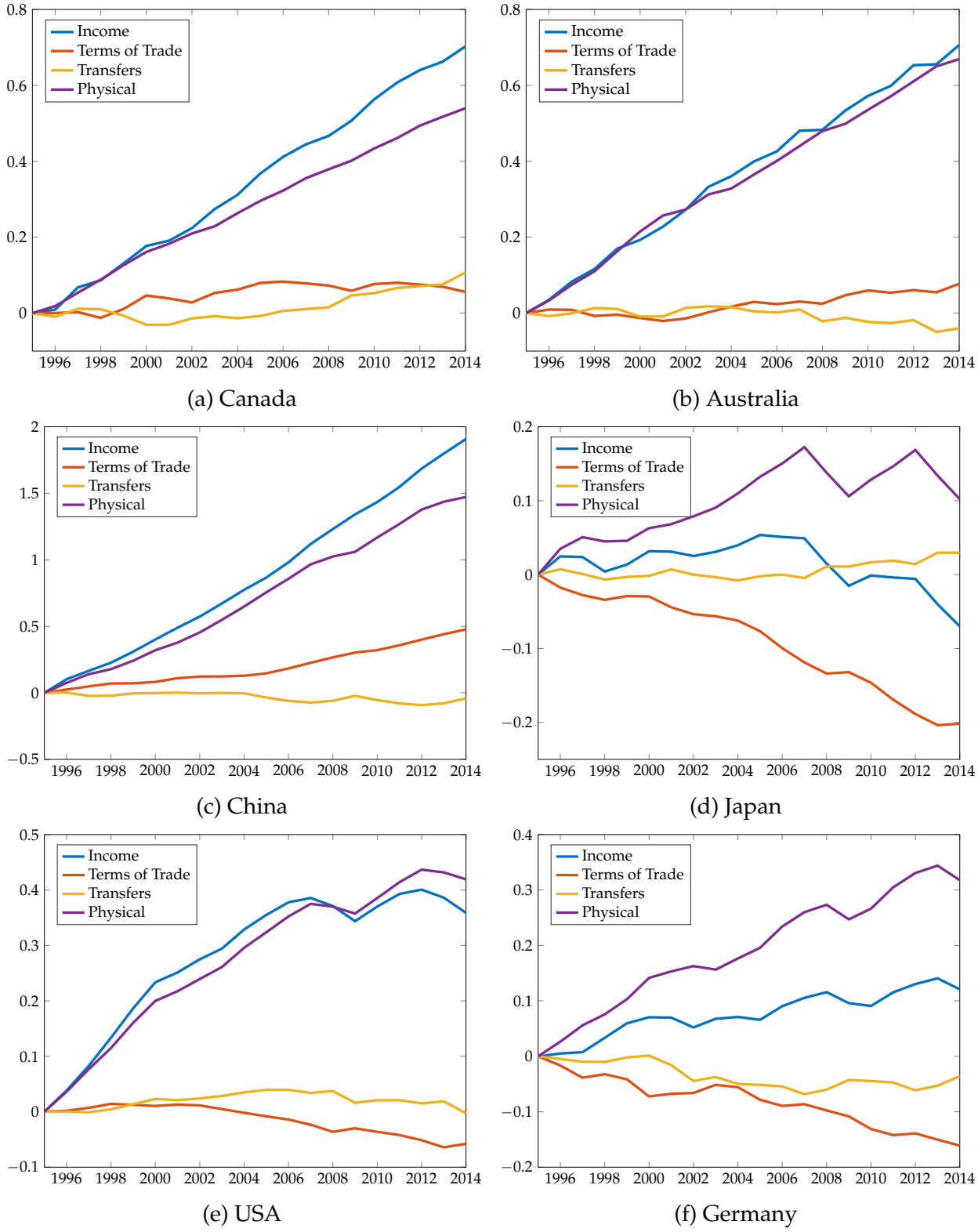
Figure 1: The solid lines show the flow of goods while the dashed lines show the flow of wage payments. Each node label gives the name of the node.

trade effects), and changes in the allocation of resources due to changes in net transfers. Of course, if we were to aggregate all countries, then the effects of changes in the allocation of resources would be exactly zero, and we would only be left to changes due to the physical environment (changes to the production functions and factor quantities).

We use data from the World Input-Output Database (Timmer et al., 2015). The dataset contains input-output tables as well as price and quantity indices for 41 countries and 35 sectors from 1995-2014. The database measures four primary factors for each country: low-skill, medium-skill, and high-skill labor, as well as the capital stock. In Figure 3 we display the cumulative change in each component over time for a sample of countries. The full list of countries can be found in the Appendix E. Focusing on the terms of trade movements, by and large, we find that manufacturing-reliant developed countries in Western Europe (Britain, France, and in particular Germany) and developed countries in East Asia (Japan, South Korea) have lost out over time, while Eastern Europe and China have gained massively. Developed nations that are more reliant on commodity exports, like Canada, Norway, and Australia, have not lost out. Developed countries reliant on financial services, like Ireland and Luxembourg, have also fared better.

We can also decompose real income earned by factors inside a specific country per unit of factor supply. For instance, zooming into the US, we find that high-skilled workers in the US have benefitted from reallocation over time, while medium and low-skilled have lost out to the rest of the world. So despite the fact that the US has been slightly hurt by reallocation in aggregate, we the impact of this reallocation has been very heterogonously felt by owners of different factors.

While Theorem 2 allows us to decompose real income growth after the fact, it does not

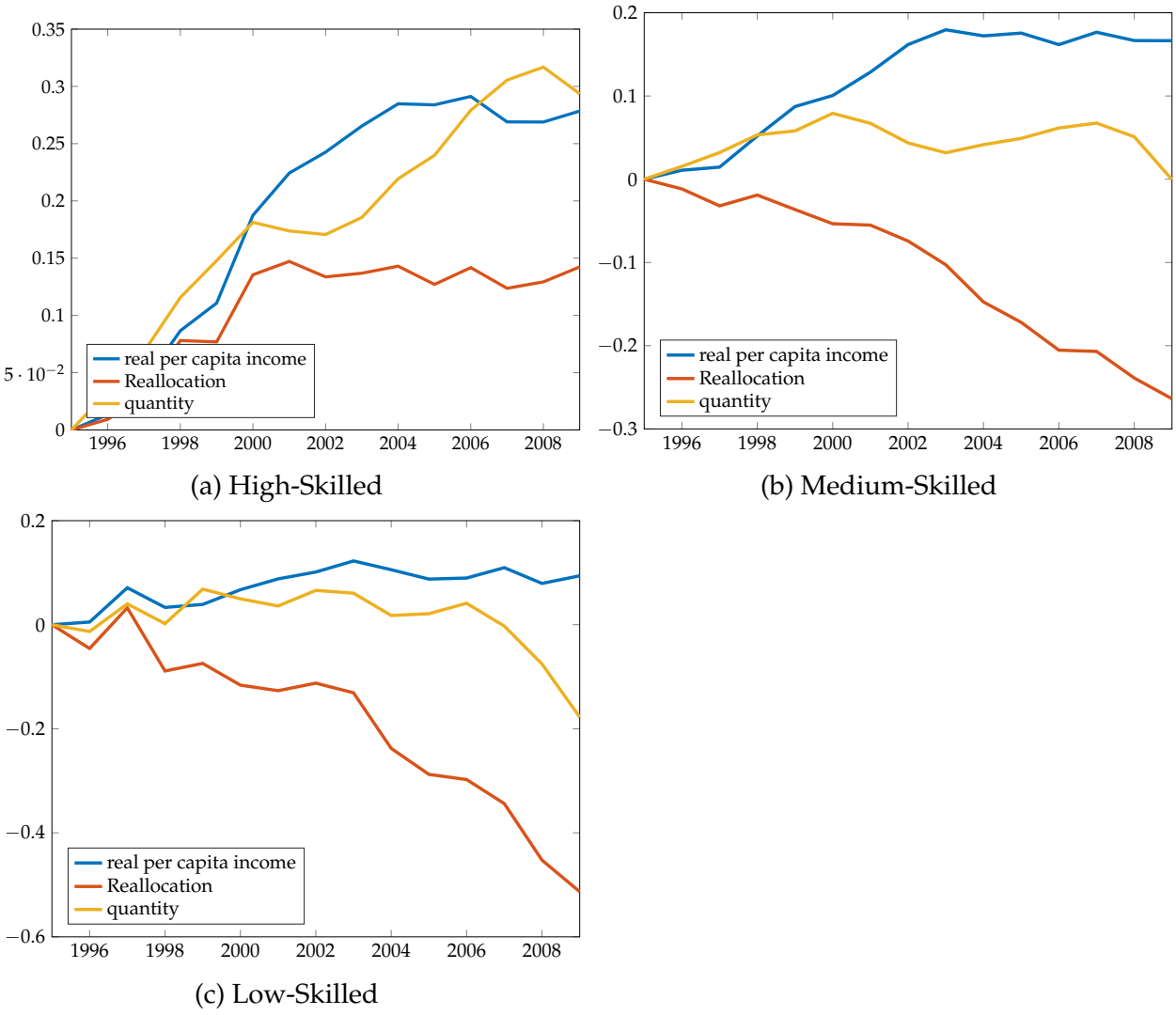


(a) Canada (b) Australia

(c) China (d) Japan

(e) USA (f) Germany

Figure 2: Welfare accounting results for a sample of countries using the WIOD data.



(a) High-Skilled

(b) Medium-Skilled

(c) Low-Skilled

Figure 3: Welfare accounting for different types of labor in the US using the WIOD data.

allow us to ask counterfactual questions. In the next section, we characterize how Domar weights, including factor shares, respond to shocks. This can then be used to predict welfare changes to a first-order using Theorem 2 and output changes to a second-order using Theorem 1.

4 Comparative Statics: Ex-Ante Sufficient Statistics

In the previous section, we related changes in real output $\mathrm{d}\log Y$ and real income $\mathrm{d}\log W$ to expenditure shares λ and changes in expenditure shares $\mathrm{d}\log\Lambda$. Baqaee and Farhi (2018) characterize how every price and quantity changes in response to shocks as a function of the HAIO table and elasticities of substitution. In this section, we use those results, in conjunction with Section 3, to derive comparative statics.

Throughout the rest of the paper, we restrict our attention to the class of models that belong to the nested-CES class (this is the case where every production and consumption function can be written as a nested-CES function). The reason for this is pedagogical clarity, and all of our results, as well as the intuitions we provide, can be extended to non-CES economies (unless otherwise stated). See Appendix B for more information about how to generalize the results to non-CES functions.

For convenience, we write the nested-CES economy in a standard form where each producer in the matrix Ω corresponds to a CES nest.

4.1 Comparative Statics

To state our results, following Baqaee and Farhi (2017b), we introduce the input-output covariance operator

$$Cov_{\Omega^{(j)}}(\Psi_{(k)},\Psi_{(l)}) = \sum_i \Omega_{ji}\Psi_{ik}\Psi_{il} - \left(\sum_i \Omega_{ji}\Psi_{ik}\right)\left(\sum_i \Omega_{ji}\Psi_{il}\right). \tag{25}$$

Response of Sales Shares

Proposition 3. *[Productivity/Iceberg Shocks] Consider a nested-CES model in standard form. Then,*

$$\lambda_l \frac{\mathrm{d}\log\lambda_l}{\mathrm{d}\log A_k} = \sum_{i\in\{H,N\}} \lambda_j(1-\theta_j) Cov_{\Omega^{(j)}}\left(\Psi_{(k)} + \sum_f \Psi_{(f)}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log A_k}, \Psi_{(l)}\right)$$

$$+ \sum_{h \in H} (\lambda_l^h - \lambda_l) \left(\sum_{f \in F_c} \Phi_{hf} \Lambda_f \frac{\mathrm{d} \log \Lambda_f}{\mathrm{d} \log A_k} \right).$$

Letting l denote a factor in Proposition 3 allows us to write linear system whose fixed point gives the equilibrium change in factor shares in response to a shock $\mathrm{d} \log A_i$:

$$\frac{\mathrm{d} \log \Lambda}{\mathrm{d} \log A_k} = \Gamma \frac{\mathrm{d} \log \Lambda}{\mathrm{d} \log A_k} + \Theta \frac{\mathrm{d} \log \Lambda}{\mathrm{d} \log A_k} + \delta_{(k)}, \tag{26}$$

where

$$\Gamma_{fg} = -\sum_j (\theta_j - 1) \frac{\lambda_j}{\Lambda_f} Cov_{\Omega^{(j)}} \left(\Psi_{(g)}, \Psi_{(f)} \right), \tag{27}$$

$$\delta_{fk} = \sum_j (\theta_j - 1) \frac{\lambda_j}{\Lambda_f} Cov_{\Omega^{(j)}} \left(\Psi_{(k)}, \Psi_{(f)} \right), \tag{28}$$

and

$$\Theta_{fg} = \frac{1}{\Lambda_f} \sum_{c \in C} (\Lambda_f^c - \Lambda_f) \Phi_{cg} \Lambda_g. \tag{29}$$

Basically, (127) can be interpreted term-by-term. The vector δ is the *impulse* — its fth element in (28) encodes how demand for each factor changes in response to substitution by all nodes in the network *holding fixed* the relative price of factors. Of course, in equilibrium, the relative price of factors changes, and the change in factor prices sets off additional rounds of substitution in the economy. The matrix Γ accounts for how the change in factor prices changes demand for factors.

If the economy had a representative consumer, Γ and δ would be all we need. However, heterogeneity in household preferences means that demand for factors depends not only on factor prices, but also on the distribution of income across consumers, which itself depends on factor prices. The term Θ captures how changes in the relative factor shares affect the distribution of income across households, and how that affects demands for different factors. Note that if all households are symmetrically exposed to a factor then $\lambda_l^c = \lambda_l$, and the changes in the distribution of income will have no effect on demand for that factor.

Response of Real GDI and GDP

We can combine Proposition 3 with Theorem 2 to trace out the welfare consequences of a shock $\mathrm{d}\log A_i$,

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_i} = \lambda_k^h + \sum_{f\in F_c} \Phi_{hf}\frac{\Lambda_f}{\chi_c}\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_i} - \sum_{f\in F} \Lambda_f^c \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_i}. \tag{30}$$

This expression takes into account changes in the allocation matrix, or equivalently, price effects which are not offset by volume effects. Similarly, we can combine Proposition 1 to compute effects on output to a second-order by noting that

$$\frac{\partial^2 \log Y_c}{\partial \log A_k \partial \log A_j} = \frac{\mathrm{d}\,\lambda_{ck}}{\mathrm{d}\log A_j}.$$

Note that since real output does not correspond to a physical good, the Domar weights are not a conservative vector field, meaning that the order of differentiation in the expression above is not arbitrary.

Response of Prices and Quantities

We now characterize the elasticities of prices and quantities to these shocks

Proposition 4. *(Prices) The elasticities of the prices of the different producers to the different productivities are given by*

$$\frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}, \tag{31}$$

$$\frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k} = -\Psi_{ik} + \sum_g \Psi_{ig}\frac{\mathrm{d}\log w_g}{\mathrm{d}\log A_k}, \tag{32}$$

where $\mathrm{d}\log Y/\mathrm{d}\log A_k$ *and* $\mathrm{d}\log \Lambda_f/\mathrm{d}\log A_k$ *are given in Proposition 2.*

Armed with Propositions 2 and 4, we can finish the characterization by showing how sales and output quantities of the different producers respond to shocks.

Corollary 1. *(Sales and Quantities) The elasticities of the sales and output quantities of the dif-*

ferent producers to the different productivities are given by

$$\frac{\mathrm{d}\log p_i y_i}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log \lambda_i}{\mathrm{d}\log A_k} + \frac{\mathrm{d}\log Y}{\mathrm{d}\log A_k}, \tag{33}$$

$$\frac{\mathrm{d}\log y_i}{\mathrm{d}\log A_k} = \frac{\mathrm{d}\log p_i y_i}{\mathrm{d}\log A_k} - \frac{\mathrm{d}\log p_i}{\mathrm{d}\log A_k}, \tag{34}$$

where $\mathrm{d}\log \lambda_i / \mathrm{d}\log A_k$, $\mathrm{d}\log Y / \mathrm{d}\log A_k$, *and* $\mathrm{d}\log p_i / \mathrm{d}\log A_k$ *are given in Propositions 2 and 4. These formulas can be applied to factors by treating them as producers of non-reproducible goods using* $i = f$ *and replacing* p_i *by* w_f, y_i *by* L_f, *and* λ_i *by* Λ_f.

An Example: Cost-Disease vs. Specialization

To build some intuition for these results, we work through a simple example showing how trade can result in export-led growth. As discussed by Baqaee and Farhi (2017b),

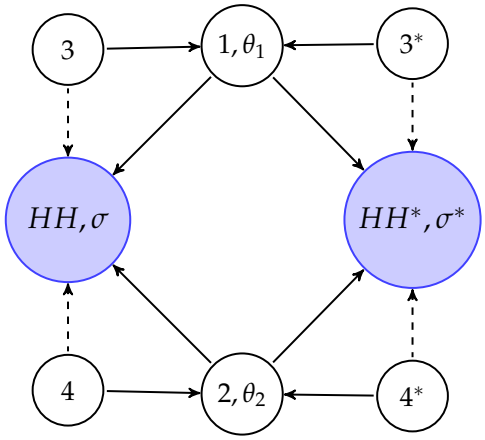


Figure 4: The solid lines show the flow of goods while the dashed lines show the flow of wage payments. Each node label gives the name of the node, as well as its elasticity of substitution.

Baumol's cost disease, whereby sectors with faster productivity growth shrink over time, is a manifestation of non-log-linearity. In particular, Baumol's cost disease occurs when the general equilibrium elasticity of substitution between sectors is less than 1. In this example, we show how the general equilibrium elasticity of substitution of a country can depend on its openness to trade.

Consider the economy depicted in Figure 4. Goods 1 and 2 are final goods, and goods $3, 3^*, 4$, and 4^* are intermediate goods. The goods 3 and 4 are produced using domestic

labor and 3^* and 4^* are produced using foreign labor. The elasticity of substitution between 3 and 3^* is given by θ_1 and between 4 and 4^* is θ_2. The elasticity of substitution for home consumers is σ and for foreign consumers is σ^*. To simplify the algebra, suppose that both household consume the same basket of goods, and this demand is split evenly across goods 3 and 4. We also assume $\Omega_{13} = \Omega_{24} = \Omega$, and $\theta_2 = \sigma^* = 1$. Then applying Propositions 1 and 3, we have

$$\frac{\partial^2 \log Y}{\partial \log A_3^2} = \frac{\mathrm{d}\,\lambda_3}{\mathrm{d}\log A_3} = \frac{\Omega^3(\sigma-1)}{4} + \frac{\Omega(1-\Omega)(\theta-1)}{(\theta+1)}. \tag{35}$$

Letting $\mathrm{d}\log Y_h$ denote the GDP of the home country, we can also characterize[12]

$$\mathrm{d}^2 \log Y_h = \frac{\mathrm{d}\,\lambda_{h3}}{\mathrm{d}\log A_3} = \Omega^2 \frac{(\sigma-1)}{4} + (1-\Omega)\frac{(\theta-1)}{(\theta+1)}. \tag{36}$$

The parameter Ω measures the degree of openness: when $\Omega = 1$, then the home country is effectively closed since the foreign country contributes nothing to the value-added of the world; when $\Omega = 0$, then the home country becomes a small open economy. It is instructive to consider these two extreme cases

$$\mathrm{d}^2 \log Y_h = \frac{1}{4}(\sigma-1), \quad \Omega = 1,$$
$$\mathrm{d}^2 \log Y_h = \frac{1}{2}\frac{(\theta-1)}{\theta+1}, \quad \Omega = 0.$$

Typical estimates of elasticities of substitution across sectors suggest that $\sigma \ll 1$ whereas trade elasticities within sectors $\theta \gg 1$. Hence, in the closed economy limit, we find $\mathrm{d}^2 \log Y_h / \mathrm{d}\log A_3^2 < 0$ whereas in the small open economy $\mathrm{d}^2 \log Y_h / \mathrm{d}\log A_3^2 > 0$ — hence, opening to international trade can help the home country overcome Baumol's cost-disease by relying on export-led growth. Of course, conditional on matching the size of the two sectors in the home country's GDP, the first-order effect of the productivity shock is given by Hulten's theorem, and does not depend on the degree of openness

$$\frac{\mathrm{d}\log Y_h}{\mathrm{d}\log A_3} = \lambda_{h3}. \tag{37}$$

[12]To characterize $\mathrm{d}^2\,\mathrm{d}\log Y_h$ we use $\mathrm{d}^2 \log Y_h = \mathrm{d}\,\lambda_{h3} = \mathrm{d}(\lambda_3/\Lambda_h) = \lambda_3/\Lambda_h\,(1/\lambda_3\,\mathrm{d}\,\lambda_3 - \mathrm{d}\log\Lambda_h)$, where Λ_h is the global Domar weight of the home factor.

4.2 Aggregating Trade Elasticities

Proposition 3 can also be used to delineate the mapping from microeconomic elasticities of substitution up to aggregate trade elasticities. The *trade elasticity* is a key object of interest in international economics. Costinot and Rodriguez-Clare (2014) define this as the elasticity of imports relative to domestic demand with respect to bilateral iceberg costs, holding fixed income levels. In our context, the trade elasticity of imports of j into i is

$$\varepsilon_{ij,kl} = \frac{\partial \log \lambda_{ij}/\lambda_{ii}}{\partial \log \tau_{kl}},$$

where λ_{ij} is the value of imports from country j by country i and τ_{kl} is the iceberg cost of imports from k to l. The partial derivative is holding fixed income levels (the wages of the primary factors and transfers). Other than being an object of independent interest, the trade elasticity is one of the key sufficient statistics that pin down gravity equations, and inform welfare counterfactuals.[13] Typically, the trade elasticity between i and j is defined only relative to the trade costs between i and j — this is because in most models, cross-elasticities are all equal to zero. In general however, the trade cross-elasticities are not equal to zero, hence our more general definition.

In many, if not most, workhorse trade models the trade elasticity is equal to some constant primitive so that $\varepsilon_{ij,kl} = \varepsilon > 0$ whenever $ij = kl$ and zero otherwise. However, much like the macro-elasticity of substitution between labor and capital, the aggregate trade elasticity is a reduced-form object that emerges endogenously from the interactions of heterogenous producers and consumers.

Proposition 5. *(Aggregating Trade Elasticities) The trade elasticity of imports from i into j relative to the trade cost between k and l is*

$$\varepsilon_{ij,kl} = \sum_{m \in C,N} (1-\theta_m)\lambda_m Cov_{\Omega^{(m)}}(\Psi_{(kl)}, \Psi_{(ij)}/\lambda_{ij} - \Psi_{(ii)}/\lambda_{ii}). \tag{38}$$

Proposition 5 shows that the aggregate trade elasticity is a linear combination of microeconomic elasticities of substitution, where the weights depend on the input-output structure. These weights capture the change in demand expenditure on ij vs. ii as a result of substitution by each producer/household m in response to a change in the iceberg cost to kl. This means that the aggregate trade elasticity is not necessarily constant across different counterfactuals.

[13] See Head and Mayer (2014) and Costinot and Rodriguez-Clare (2014).

Variable Trade Elasticities and Intermediates

In many trade models, the trade elasticity is constant.[14] A notable exception is provided by Yi (2003). Yi observes that the response of trade volumes to reductions in trade costs has varied greatly over time — large reductions in trade barriers resulted in relatively small movements in the volume of trade during the middle of the 20th century, whereas small reductions caused large changes in the 80s and 90s. Yi's explanation for these facts is the existence of vertical specialization, or in our language, the presence of intermediate inputs, which can cause the trade elasticity to become variable even if all the microeconomic elasticities are constant and uniform. The example, which uses Proposition 5 gives a flavor of Yi's basic insight.

Consider the two-country, two-good economy depicted in Figure 5. Each country's good is produced using domestic factors and imports. Suppose that the micro elasticity of substitution is equal to θ. Then the trade elasticity is

$$\varepsilon = \frac{(1-\theta)}{1-\omega\omega^*},$$

where ω and ω^* are the steady-state intermediate input shares of the domestic and foreign producer. We see that as the intermediate input shares increase, the trade elasticity becomes larger. Of course, since the intermediate input shares ω and ω^* are themselves endogenous (depending on the iceberg shock), this means that ε varies as a function of the iceberg shocks. In particular, if $\theta > 1$, then the trade elasticity increases as the iceberg costs fall.

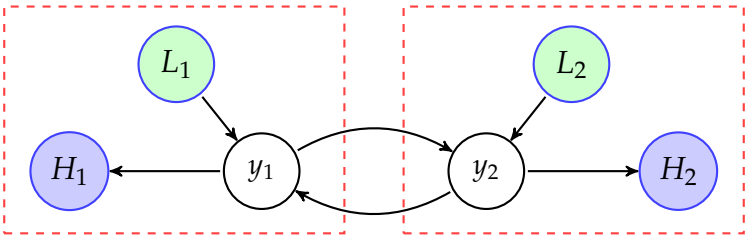


Figure 5: The solid lines show the flow of goods. Green nodes are factors, purple nodes are households, and white nodes are goods. The boundaries of each country are denoted by dashed box.

[14] See Appendix TBA for conditions under which the trade elasticity is constant.

Capital Reswitching and the Trade Elasticity

Yi (2003) shows that the trade elasticity can be nonlinear due to vertical specialization. Imagine there are two ways of producing a good, one uses a domestic supply chain and the other uses a global value chain. Whenever the good is domestically produced, the iceberg costs of transporting the good are, at most, incurred once. However, when the good is made via a global value chain, the iceberg costs are incurred as many times as the good is shipped across borders. As a function of the iceberg cost parameter τ, the difference in the price of these two goods (holding factor prices fixed) is a polynomial of the form

$$B_n \tau^n - B_1 \tau,$$

where B_n and B_1 are some coefficients and n is the number of times the border is crossed. The nonlinearity in τ, whereby the iceberg cost's effects are compounded by crossing the border, drives the excess sensitivity of trade to trade barriers in Yi (2003).

An interesting connection can be made between the behavior of the trade elasticity, along the lines of Yi (2003), and the (closed-economy) reswitching debates of the 1960s. Specifically, the Yi set up is one special case. In general, the cost difference between producing goods using supply chains of different lengths is a polynomial in τ – and this polynomial can, in principle, have more than one root. This means that the trade elasticity can be non-monotonic as a function of the trade costs, in fact, it can even have the "wrong" sign, where the volume of trade decreases as the iceberg costs fall. This mirrors the apparent paradoxes in capital theory where the relationship between the capital stock and the return on capital can be non-monotonic, and an increase in the interest rate can cause the capital stock to increase.[15]

Such examples are extreme, but they illustrate the point that in the presence of input-output networks, the trade elasticity even in partial equilibrium (holding factor prices constant) can behave quite unlike any microeconomic demand elasticity.

[15]To see this in the trade context, imagine two perfectly substitutable goods, one of which is produced by using 10 units of foreign labor, the other is produced by shipping 1 unit of foreign labor to the home country, back to the foreign country, and then back to the home country and combining it with 10 units of domestic labor. If we normalize both foreign and domestic wages to be unity, then the he costs of producing the first good is $10(1+\tau)$, whereas the cost of producing the second good is $(1+\tau)^3 + 10$, where τ is the iceberg trade cost. When $\tau = 0$, the first good dominates and goods are only shipped once across borders. When τ is sufficiently high, the cost of crossing the border is high enough that the first good again dominates. However, when τ has an intermediate value, then it can become worthwhile to produce the second good, which causes goods to be shipped across borders many times, thereby inflating the volume of trade.

Non-Symmetry and Non-Triviality of Trade Elasticities

Another interesting subtlety of the Proposition 5 is that the aggregate trade elasticities are non-symmetric. That is, in general $\varepsilon_{ij,kl} \neq \varepsilon_{ji,kl}$. Furthermore, unlike the standard gravity equation, Proposition 5 shows that the cross-trade elasticities are, in general, nonzero. Hence, changes in trade costs between k and l can affect the volume of trade between i and j *holding fixed* relative factor prices and incomes. This is due to the presence of global value chains, which transmit shocks in one part of the economy to another independently of the usual general equilibrium effects (which work through the price of factors).

5 Duality Between Open and Closed Economies

Folk wisdom among trade economists suggests that an increase in trade costs are, in some sense, isomorphic to negative productivity shocks. Formally, this intuition is incorrect. For instance, as Kehoe and Ruhl (2008) point out, in an efficient model, trade shocks have no effect on TFP whereas domestic productivity shocks clearly do. Their observation is implied by Theorem 2, which shows that Hulten's theorem holds in an international context as long as the economy is efficient.

However, there is also a sense in which the folk intuition is correct. In this section, we establish a useful isomorphism between the effects of trade shocks in an open model and the effects of productivity shocks in a closed one. Besides bridging the open and closed economy literatures, the isomorphism allows us to use results from closed models to study the behavior of open ones.

Specifically, we prove that under some conditions, the effects of iceberg trade shocks on real income/welfare in an open economy can be analyzed by studying the effects of productivity shocks on real output/GDP in a dual closed economy. In other words, there exists a useful mapping between closed and open economies. Since Hulten (1978) and Baqaee and Farhi (2017a) have characterized the behavior of real GDP up to a second-order in closed-economies, we can use the isomorphism to study the welfare effects of trade shocks non-linearly. This dual relationship is also valuable since it can be seen as a direct generalization of the insights of Arkolakis et al. (2012) (ACR).

5.1 Environment

Unlike the rest of the results in the paper, this isomorphism relies on having CES aggregators in traded goods.[16] For any open economy c, there are some nodes D_c whose production functions do not rely directly on foreign goods, while the rest of the nodes M_c have production functions that do rely on foreign goods. We assume that any node $i \in M_c$ is a CES aggregator with elasticity of substitution ζ_i. In other words, for any $i \in M_c$, the cost of producing i is given by

$$p_i = \frac{1}{A_i}\left(\sum_{j \in N_c} \Omega_{ij} p_j^{1-\zeta_i} + \sum_{j \notin N_c} \Omega_{ij} p_j^{1-\zeta_i}\right)^{\frac{1}{1-\zeta_i}}.$$

For the open economy c, we construct a fictitious closed economy which shares the nodes D_c with its open economy counterpart, but for any node $i \in N_c - D_c$, it instead has

$$\check{p}_i = \frac{1}{\check{A}_i}\left(\sum_{j \in N_c} \frac{\Omega_{ij}}{\sum_{j \in N_c} \Omega_{ij}} p_j^{1-\zeta_i}\right)^{\frac{1}{1-\zeta_i}} = \frac{1}{\check{A}_i}\left(\sum_{j \in N_c} \check{\Omega}_{ij} p_j^{1-\zeta_i}\right)^{\frac{1}{1-\zeta_i}},$$

where $\frac{\Omega_{ij}}{\sum_{j \in N_c} \Omega_{ij}} = \check{\Omega}_{ij}$. Intuitively, the closed economy replaces any good which relies on foreign goods with a good that only relies on domestic goods but has a different productivity shifter.

In the body of the paper, we consider one-factor models, like ACR, and in Appendix J, we extend the result to multi-factor models. The latter case is useful in studying the distributional consequences of trade.

5.2 Duality for One-Factor Economies

Assume that the domestic economy has one primary factor of production (say, labor).[17] Let $W_c(z)$ denote real income of a country c as a function of iceberg trade costs z. Let $\check{W}_c(\check{A})$ denote the real income of the closed-economy counterpart of c as a function of fictitious productivity parameters. Since the "caron" economy is closed, real income is equal to real output $\check{W}_c(\check{A}) = \check{Y}_c(\check{A})$.[18]

[16] This is because the result relies on the Feenstra (1994) formula.

[17] The assumption that the domestic economy have one factor of production will be relaxed shortly.

[18] Caron is the name of the inverted hat symbol $\check{}$.

Proposition 6 (Isomorphism)**.** *Consider a change in iceberg trade costs from $\overline{z}$ to z, then*

$$\Delta \log W = \log\left(\frac{W_c(z)}{W_c(\overline{z})}\right) = \log\left(\frac{\check{Y}_c(\check{A})}{\check{Y}_c(\overline{\check{A}})}\right) = \Delta \log \check{Y}.$$

where

$$\Delta \log \check{A}_i = \log\left(\frac{\check{A}_i}{\overline{\check{A}}_i}\right) = \frac{1}{\zeta_i - 1}\log\left(\frac{\sum_{j\in N_c}\Omega_{ij}}{\sum_{j\in N_c}\overline{\Omega}_{ij}}\right).$$

In words, the effect of iceberg trade shocks on *income* in an *open economy* are isomorphic to the effects of productivity shocks on *output* in a fictitious *closed economy*. Given this isomorphism to a closed-economy, we can use results from the literature on closed-economy models to characterize the effects of trade shocks.

We begin by looking at the first-order effects:

Corollary 2 (First-Order)**.** *When Proposition 6 holds, a first-order approximation to the change in welfare from iceberg shocks $\Delta \log z$ is*

$$\Delta \log W \approx \sum_{i\in N_c} \check{\lambda}_i \Delta \log \check{A}_i. \tag{39}$$

where $\mathrm{d}\log \check{W}_c / \mathrm{d}\log \check{A}_i = \check{\lambda}_i$ by Hulten (1978).

Hence, conditional on the size of the implied productivity shocks, the presence of intermediate inputs amplifies the effects of trade shocks much in the same way that the effect of intermediate inputs amplifies the effects of productivity shocks in closed economies. This is due to the intermediate input multiplier effect discussed by, among others, Jones (2011).

Next, we consider the second-order effect of these shocks.

Corollary 3 (Second-Order)**.** *When Proposition 6 holds, a second-order approximation to the change in welfare from iceberg shocks $\Delta \log z$ is*

$$\Delta \log W \approx \sum_{i\in N_c} \check{\lambda}_i \Delta \log \check{A}_i + \frac{1}{2}\Delta \log \check{A}\frac{d^2 \log \check{Y}_c}{\mathrm{d}\log \check{A}^2}\Delta \log \check{A}. \tag{40}$$

From Baqaee and Farhi (2017b),

$$\frac{d^2 \log \check{W}_c}{\mathrm{d}\log \check{A}_j \,\mathrm{d}\log \check{A}_i} = \frac{\mathrm{d}\,\check{\lambda}_i}{\mathrm{d}\log \check{A}_j} = \sum_{j\in N_c}(1-\theta_j)\check{\lambda}_j Cov_{\check{\Omega}^{(j)}}\left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)}\right),$$

whenever the model has a nested CES form.[19]

Hence, nonlinearities in consumption and production can act to amplify or mitigate the effects of the shock relative to the first-order approximation. Cobb-Douglas ($\theta_j = 1$ for all j) is the case that keeps the isomorphic closed economy log-linear.

Connection to ACR

Proposition 6 provides a way to generalize the insight of Arkolakis et al. (2012) to models with nonlinear input-output linkages. We can also recover the basic results of ACR in the one-sector and multi-sector case. In the one-sector case, $\check{\lambda}_i = 1$, and hence

$$\mathrm{d} \log W_h = \frac{1}{1 - \zeta_i} \mathrm{d} \log \Omega_{hh}, \tag{41}$$

where Ω_{hh} is the domestic input share. This formula can be integrated in closed form to yield the classic ACR result. For the multi-sector version, ACR assume that the elasticity of substitution across nests is 1, in which case,

$$\mathrm{d} \log W_h = \sum_i \check{\lambda}_i \frac{\mathrm{d} \log \Omega_{ii}}{1 - \zeta_i}, \tag{42}$$

but Corollary 3 shows that the second-order effects $\mathrm{d}\, \check{\lambda}_i = 0$ in this case. Hence, this can also be integrated in closed-form to yield their multi-sector formula.

Corollary 2 shows that by ignoring intermediate inputs, we are likely to significantly mitigate the impact of trade shocks due to the intermediate multiplier effect. Furthermore, Corollary 3 shows that by restricting ourselves to a Cobb-Douglas specification in the fictitious closed-economy, we are assuming away nonlinearities in production which can act to amplify or mitigate trade shocks.[20]

[19] Corollary can easily be extended to models that do not have a nested CES form in domestic consumption and production by replacing the input-output covariance operator with its generalization, the substitution operator, so that

$$\frac{\mathrm{d}^2 \log \check{W}_c}{\mathrm{d} \log \check{A}_j \, \mathrm{d} \log \check{A}_i} = \sum_{j \in N_c} \check{\lambda}_j \Phi_{\check{\Omega}^{(j)}} \left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)} \right),$$

where $\Phi_{\check{\Omega}^{(j)}} \left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)} \right)$ is the substitution operator defined by Baqaee and Farhi (2017b). See Baqaee and Farhi (2017b) for more details. For convenience, we restrict our attention to the nested-CES class of models.

[20] This observation accords with the findings of Costinot and Rodriguez-Clare (2014) who find that allowing for intermediate inputs significantly increases the predicted gains from trade, although they restrict themselves to a Cobb-Douglas model.

Example with a Critical Input

Consider the economy depicted in Figure 6, where the only traded good is E. We interpret E as energy. Consumers in country H consume domestic goods 1 through to N with some elasticity of substitution θ_0. Goods 1 through to M are made via labor L and a composite energy good E, which is a CES aggregate of domestic and foreign energy with elasticity of substitution ζ_E. Domestic energy E, as well as goods $M+1$ through to N are made using domestic labor only.

Let Ω_{EH} denote the expenditure share of domestic varieties in the composite good E, denoting its steady-state value by $\overline{\Omega}_{EH}$. Now, consider an increase in iceberg trade costs z_E that reduces the share of foreign varieties in composite good E (or equivalently, increases Ω_{EH}). We can analyze the impact of this shock on the welfare W_H of consumer H by making use of Proposition 6. Namely, we can analyze the welfare effect of the shock by considering a negative productivity shock of size

$$\Delta \log \check{A}_E = \frac{1}{1-\zeta_E} \left(\log \Omega_{EH} - \log \overline{\Omega}_{Eh} \right)$$

to the isomorphic closed-economy.

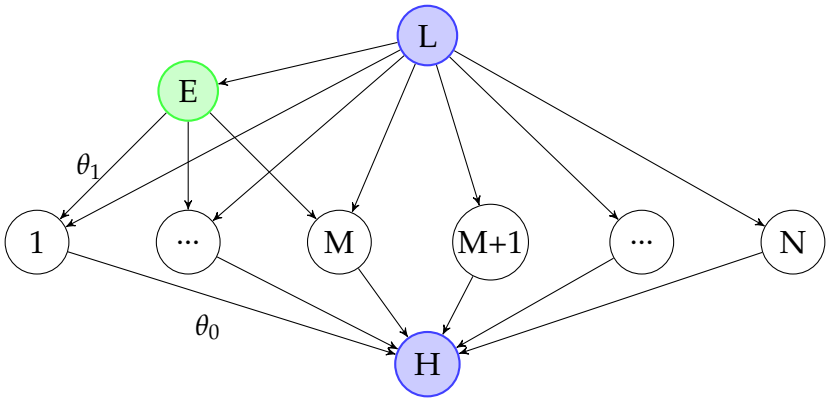


Figure 6: An illustration of the economy with a key input E. Each industry has different shares of labor and energy and substitutes across labor and energy with elasticity $\theta_1 < 1$. The household can substitute across goods with elasticity of substitution $\theta_0 > \theta_1$. Energy is a traded good, which can either be produced domestically or sourced from the rest of the world.

In this case, using Hulten (1978), we can write

$$\frac{\mathrm{d}\log W_H}{\mathrm{d}\log z_E} = -\frac{\mathrm{d}\log \check{W}_H}{\mathrm{d}\log \check{A}_E} = -\check{\lambda}_E \tag{43}$$

where $\check{\lambda}_E$ is the domestic sales of E as a share of domestic consumption (the Domar weight of E in the isomorphic closed economy). Hence, the larger are the domestic sales of E, the larger is the reduction to welfare. As usual, the existence of intermediate inputs amplifies the effect of the shock

With the aid of Proposition 6, and the generalization of Hulten's theorem from Baqaee and Farhi (2017b), we can also analyze the nonlinear effect on welfare of the trade shock

$$\frac{\mathrm{d}^2\log W_H}{\mathrm{d}\log z_E} = -\frac{\mathrm{d}\,\check{\lambda}_E}{\mathrm{d}\log \check{A}_E} = \sum_{j\in N_h}(1-\theta_j)\check{\lambda}_j Var_{\check{\Omega}^{(j)}}\left(\check{\Psi}_E\right), \tag{44}$$

$$= \check{\lambda}_E^2\frac{N-M}{N}(1-\theta_0) + \check{\lambda}_E\left(1-\frac{N}{M}\check{\lambda}_E\right)(1-\theta_1), \tag{45}$$

$$= \check{\lambda}_E(1-\check{\lambda}_E)(1-\theta_0) + \check{\lambda}_E\left(1-\frac{N}{M}\check{\lambda}_E\right)(\theta_0-\theta_1). \tag{46}$$

Equation (44) is the general form of the second-order term, as shown by Baqaee and Farhi (2017b). Specialized to this example, gives (45), which can also be written as (46). Hence, the elasticities of substitution in domestic production θ_1 and consumption θ_0 affect the way a given trade shock is felt by domestic consumers. If θ_0 and θ_1 are very low, then the impact of a negative trade shock is amplified. In the case when $M = N$, so that E becomes a universal input in the economy, then the elasticity of substitution in consumption θ_0 drops out of the expression. This is because when $M = N$, the household is uniformly exposed to the trade shock $Var_{\Omega^{(0)}}(\check{\Psi}_{(E)}) = 0$, so that substitution by the household is irrelevant.

So, intuitively, relative to the formula in ACR, we see that complementarities in production in the domestic economy can amplify the effects of negative trade shocks. These effects are further amplified if flexible producers (those with high elasticities of substitution in production) are uniformly exposed to the shock, while less flexible producers (those with low elasticities of substitution in production) are heterogonously exposed to the shock.

Gains from Trade Using the Isomorphism

Using the World Input-Output Database (WIOD) (see Timmer et al., 2015), we can apply our isomorphism to study the gains from trade by moving each country to Autarky and using Proposition 6. We assume that production takes a nested CES form, where σ is the elasticity of substitution across industries in consumption, ϵ is the elasticity of substitution between value-added and intermediate inputs, and θ is the elasticity of substitution across industries in intermediate input use. Finally, we assume that the output of each industry is a CES aggregate of domestic and foreign varieties, with the relevant elasticity of substitution set to match the estimates of Caliendo and Parro (2015).

$(\sigma, \epsilon, \theta)$	ACR	$(1, 1, 1)$	$(1, 0.5, 0.6)$	$(0.8, 0.3, 0.2)$
MEX	11.5%	18.4%	23.7%	69.5%
FRA	9.8%	17.2%	24.3%	99.9%
USA	4.5%	8.3%	9.6%	13.1%
JPN	2.4%	4.6%	4.9%	5.5%

Table 1: Percentage change in consumer welfare for a selection of countries associated with going to Autarky. The ACR column assumes that there are no intermediate inputs, and that the elasticities of substitution across industries in final demand is one. In other words, the ACR column implements (42). The trade elasticities are the same across all columns, so the size of the trade shock to each industry is the same across all columns.

The results of this exercise are in Table 1 for different values of the elasticities of substitution $(\sigma, \epsilon, \theta)$. As expected, moving from the multi-sector ACR column to the Cobb-Douglas model with intermediate inputs increases the numbers. This is a "first-order" effect associated with the fact that intermediate inputs amplify TFP shocks. Moving across columns towards more complementarity increases the gains from trade (but by different amounts in different countries), and this is a nonlinear effect.

6 Tariffs and Other Distortions

So far, we have maintained the assumption of perfect competition. In this section, we extend our results to allow for tariffs, or more generally distortions (modelled via implicit or explicit taxes). We prove a sufficient statistics result showing that the way output changes when tariffs change is *only* a function three statistics: the sales shares of goods with wedges, the size of each wedge, and the quantity of goods with wedges. All other

variables (elasticities of substitution, returns to scale, input-output linkages, non-taxed goods production etc.) matter only in so far as they play a role in determining the equilibrium value of these sufficient statistics.[21]

We use this sufficient statistic formula to characterize, up to a second order, the gains in output and welfare from eliminating tariffs. This second-order approximation, which is exactly a deadweight loss triangle, depends on an unobservable and endogenous sufficient statistic: the change in the quantities of taxed goods. Therefore, we use a generalized version of our propagation equations to derive counterfactual formulas for the gains from eliminating tariffs. These formulas show how the nature of input-output connections, and elasticities of substitution, affect the size of welfare losses.

In Appendix M, we also show how all the results in the previous sections, including the growth-accounting results in Theorem 1 and 2 and the propagation equations in Proposition 20, can be generalized straightforwardly to a context with distortions.

6.1 Allowing for Distortions

Let μ represent the $N \times 1$ matrix of tax wedges, where the ith element is an ad valorem tax on the output of i. To model trade tariffs, we simply need to add specialized importers and exporters to the model who buy finished domestic goods and sell them across borders. An import or export tax would then correspond to output taxes on these specialized importers/exporters.

To state our results, we assume that the revenue generated by the wedge μ_i are *included* in the revenue of i—that is, i collects the tax revenue as part of its revenue, and then pays the government. This is merely an accounting convention, and it is straightforward to convert our results for situations where the revenue generated by the wedge are not included in the revenue of i.

6.2 Comparative Statics

The following result, taken from Baqaee and Farhi (2019) and building on Harberger (1964), shows the way tariffs affect output depends on the sales shares of taxed goods, and how their quantities change; nothing else matters.

[21] This accords with Paul Krugman's view that trade in intermediate inputs does not alter the logic of why trade barriers are harmful, see, *Does Trade in Intermediate Goods Alter the Logic of Costs From Protectionism?* by Paul Krugman (2018).

Proposition 7.

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log \mu_i} = \frac{\mathrm{d}\log W}{\mathrm{d}\log \mu_i} = \sum_{l\in N} \lambda_l \left(1 - \frac{1}{\mu_l}\right) \frac{\mathrm{d}\log y_l}{\mathrm{d}\log \mu_i}. \tag{47}$$

For any country c,

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log \mu_i} = \sum_{l\in N_c} \lambda_{cl} \left(1 - \frac{1}{\mu_l}\right) \frac{\mathrm{d}\log y_l}{\mathrm{d}\log \mu_i}. \tag{48}$$

Proposition 7 shows that production networks, or global value chains, matter only in so far as they affect the equilibrium Domar weights λ or the elasticity of taxed quantities $\mathrm{d}\log y$ with respect to taxes. Of course, production networks will affect both the size of the Domar weights as well as the output elasticity of taxed goods, but at least in terms of Proposition 7, these are quantitative and not qualitative changes.[22]

An immediate consequence of Proposition 7 is that at the efficient equilibrium, the elasticity of output with respect to taxes is zero. This holds at the world level, and at the country level as long as there are no wedges inside the country.

6.3 Costs of Tariffs: Ex-Post Sufficient Statistics

Using Proposition 7, we can calculate the losses from protectionism in world GDP, country-level GDP, and social welfare (defined via a social welfare function), up to a second-order approximation.

Proposition 8. *At the efficient (no wedge) equilibrium, up to a second-order approximation*

1. *World real GDP and real GDI:*

$$\Delta\log Y = \Delta\log W \approx \frac{1}{2}\sum_{i\in N} \lambda_i \Delta\log y_i \Delta\log \mu_i. \tag{49}$$

2. *Country c real GDP:*

$$\Delta\log Y_c \approx \frac{1}{2}\sum_{i\in N_c} \lambda_{ci} \Delta\log y_i \Delta\log \mu_i. \tag{50}$$

[22] This runs counter to Melitz and Redding (2014) and their argument that intermediate inputs introduce a new channel for why trade is beneficial, an effect they dub "trade-induced productivity changes." As shown by Theorem 1, in a competitive model, changes in import costs have no effect on domestic productivity regardless of whether or not there are intermediate inputs. In inefficient models, changes in trade can affect domestic productivity but the existence of this effect is not contingent on the existence of intermediate inputs.

Hence, for both the world and for each country, the reduction in output is given by the sum of all the deadweight-loss triangles $1/2\Delta \log y_i \Delta \log \mu_i$ weighted by their Domar weights. [23] These expressions rely on ex-post sufficient statistics in the sense that they require data from after the shock. In the next subsection, we write these in terms of ex-ante sufficient statistics instead.

We can compare these costs to the cost of an increase in transport costs, rather than tariffs, whose effect on output, up to a second order, is given by the sum of trapezoids rather than triangles

$$\Delta \log Y = \Delta \log W \approx \sum_{i \in N} \lambda_i \left(1 + \frac{1}{2} \Delta \log \lambda_i\right) \Delta \log A_i. \tag{51}$$

In the expression above, $\Delta \log A_i$ are changes in iceberg costs (or equivalently, Hicks-Neutral shocks to exporters).

Propositions 7 and 8 are predictions of the model about how GDP and GDI respond to taxes. They do not have any normative content since real GDP and GDI are problematic as measures of welfare in situations where household preferences are not aggregable.

To study welfare, we define a Bergson-Samuelson social welfare function

$$W^{BS} = \sum_h \overline{\chi}_h \log W_h,$$

where $\overline{\chi}_h$ is the income share of consumer h at the equilibrium with no distortionary taxes. This is the social welfare function which, at the undistorted equilibrium, has no incentives to redistribute resources from one agent to another. Now, we compute the cost of tariffs in terms of the proportional reduction in every agent's consumption which would make social welfare at the undistorted equilibrium equal social welfare after the imposition of tariffs. Formally, we measure the social cost of tariffs as the number δ that solves

$$W^{BS}(\delta \overline{W}_1, \ldots, \delta \overline{W}_H) = W^{BS}(W_1, \ldots, W_H),$$

where $\overline{W}_h$ and W_h is the consumption of agent h in the undistorted and distorted equilibrium.

[23] These results are simple consequences of differentiating Proposition (48) and evaluating this derivative at $\mu = 1$.

Proposition 9. *At the efficient (no wedge) equilibrium, up to a second-order approximation*

$$\delta \approx \Delta \log Y + Cov_{\overline{\chi}} \left(\Delta \log \chi, \Delta \log P_{cpi} \right). \tag{52}$$

Hence, world welfare is lowered by an amount equal to the reduction in world GDP plus a redistributive effect. This effect is positive whenever the covariance between the change in income shares and the change in consumer price indices is positive. This is because the social planner would prefer that those consumers whose income share has increased also be the consumers whose prices have increased.

6.4 Costs of Tariffs: Ex-Ante Sufficient Statistics

Propositions 8 and 9 express the costs of tariffs in terms of individual output elasticities. In this subsection, we provide explicit formulas for these costs in terms of micro-elasticities of substitution and the HAIO network.

Proposition 10. *At the efficient (no wedge) equilibrium, up to a second-order approximation, the change in world GDP due to a change in wedges is*

$$-\Delta \log Y \approx \mathcal{L}_I + \mathcal{L}_X + \mathcal{L}_H, \tag{53}$$

where

$$\mathcal{L}_{\mathcal{I}} = \frac{1}{2} \sum_l \sum_k (\mathrm{d} \log \mu_l)(\mathrm{d} \log \mu_k) \sum_j \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}), \tag{54}$$

$$\mathcal{L}_X = -\frac{1}{2} \sum_l \sum_g \mathrm{d} \log \mu_l \, \mathrm{d} \log \Lambda_g \sum_j \lambda_j \theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}), \tag{55}$$

$$\mathcal{L}_{\mathcal{H}} = -\frac{1}{2} \sum_l \sum_{h \in H} (\mathrm{d} \log \mu_l)(\mathrm{d} \log \chi_h) \chi_h (\lambda_l^h - \lambda_l). \tag{56}$$

Changes in country-level GDP and social welfare W^{BS} are treated along similar lines in Appendix M. The formula for $\mathrm{d} \log \Lambda$ *and* $\mathrm{d} \log \chi$ *are in the same Appendix.*

Although production networks do not matter conditional on the sufficient statistics in Proposition 7, Proposition 10 shows that these sufficient statistics are functions of the details of the production network. Furthermore, even though goods which are not taxed do not matter from the perspective of Proposition 7, they *do* matter in the sense that they shape the sufficient statistics.

Equation (53) shows that losses to world output are the sum of three components. The first component $\mathcal{L}_I$ shows that when wedge μ_k changes, this changes j's demand for goods produced by l. The extent to which j's demand for l is affected depends on the elasticity of substitution θ_j and the covariance between j's exposure to k and l. If θ_j is very low, then j substitutes very little. If the covariance $Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)})$ is very low, then changes in j's input demands, caused by μ_k, do not affect demand for l. This change in the demand for l, multiplied by the size of the wedge $\mathrm{d}\log\mu_l$ is the size of the lth Harberger triangle.

A helpful term to consider for intuition in $\mathcal{L}_I$ is the $k = l$ term:

$$(\mathrm{d}\log\mu_k)^2 \sum_j \lambda_j \theta_j Var_{\Omega^{(j)}}(\Psi_{(k)}).$$

If j is uniformly exposed to k, then $Var(\Omega^{(j)}(\Psi_{(k)}) = 0$, in which case substitution by j is irrelevant, since j cannot change the demand for k by varying its input choices.

The term $\mathcal{L}_I$ would be all we would have *if* the economy contained a single household and a single factor. The other two terms, $\mathcal{L}_X$ and $\mathcal{L}_H$ account for the fact that relative factor prices (given by $\mathrm{d}\log\Lambda$) and relative income shares (given by $\mathrm{d}\log\chi$) also change in response to the change in wedges.

The term associated with changing factor prices, $\mathcal{L}_X$, behaves much the same way as $\mathcal{L}_I$. The only difference is that it traces out how a change in the price $\mathrm{d}\log\Lambda_g$ of some factor g causes demand for l to change. This is compared to $\mathcal{L}_I$ which instead tracked how a change in the wedge to k changed demand for l. s

The final term $\mathcal{L}_H$, which is associated with the fact that there are heterogeneous consumers, accounts for the fact that changes in the income shares of households $\mathrm{d}\log\chi$ reallocates household demand for different goods. An increase in household h's income $\mathrm{d}\log\chi_h$ increases demand for l by $\chi_h(\lambda_l^h - \lambda_l)$. If household h is exposed to good l in the same way as the economy in general, then $\lambda_l^h = \lambda_l$, and this change in income shares in neutral. Of course, the assumption of a representative agent will always cause this final term to be zero.

Proposition 10 is not fully explicit in the sense that it depends on some endogenous elasticities — namely the elasticity of factor shares $\mathrm{d}\log\Lambda$, income shares $\mathrm{d}\log\chi$ and consumer price indices $\mathrm{d}\log P_{cpi}$. In Appendix M, we provide the structural formulas for these objects as well, but we omit them from the body of the paper since we have studied those propagation equations in other papers, and because they are quite similar to the

propagation equations for the efficient case studied in Section 4.

Round-about Example

To see how the details of the production network can matter, consider again the example in Figure 5. Suppose that both the home country 1 and the foreign country 2 introduce a tax on their imports. If these taxes are identical, then by symmetry, there will be no terms of trade effects. This means that changes in country-level GDP, country-level GDP, world GDP, world GDI, and world welfare are all the same. Hence, using Proposition 8, the losses are given by

$$\sum_{i \in N} \lambda_i \, \mathrm{d} \log y_i = \frac{2\theta\omega^2}{(\omega-1)^2(\omega+1)^2} \, \mathrm{d} \log \mu,$$

where $\mathrm{d} \log \mu$ is the change in import tariffs. The increase as the intermediate input share ω tends to 1. This is because as $\omega \to 1$, both the Domar weight λ_i and the output elasticity $\mathrm{d} \log y_i$ diverge. For this example,

$$\lambda_i = \frac{1}{2(1-\omega)}, \quad \text{and} \quad \mathrm{d} \log y_i = -\frac{2\theta\omega^2}{(1-\omega)(1+\omega)^2} \, \mathrm{d} \log \mu.$$

The Domar weight diverges since a higher intermediate input share expands sales relative to value-added. The output elasticity diverges since a higher intermediate input share means that goods cross the border more times, and each time the border is crossed, the tariff has to be paid.[24]

7 Quantitative Examples

In this section, we use a multi-factor production network model, calibrated to fit world input-output data, to quantify the way trade costs (tariffs or iceberg) affect output, welfare, and factor rewards. We also compare our benchmark model to alternative calibrations which trivialize the input-output connections or global value chains as a way to gauge the quantitative significance of allowing for these effects.

Our benchmark model consists of X countries, each with four factors of production: high-skilled, medium-skilled, low-skilled labor, and capital. Due to lack of data, we assume that each country has a single consumer price index. Hence, different consumers in

[24]This simple example recreates the spirit of the argument in Melitz and Redding (2014).

the same country may draw their income from different factors, but they all consume the same basket of goods as their compatriots. Each country has 30 industries each of which produces a single output good.

The model has a nested-CES structure. Each industry produces output by combining its value-added (consisting of the four domestic factors) with intermediate goods (consisting of the 30 industries). The elasticity of substitution across primary factors is γ, and across intermediates is ε. The elasticity of substitution between primary factors and intermediate inputs is ζ. When industry i in country c purchases intermediate inputs of some industry j, it consumes a CES aggregate of goods from this industry sourced from various countries with some elasticity of substitution θ_j. The final consumer in each country consumes goods with an elasticity of substitution σ across industries. When the consumer consumes goods from some industry j, it does so via CES aggregate of goods from industry j sourced from various countries with elasticity of substitution θ_j. Finally, there are both import tariffs and iceberg trade shocks.

We use data from the world input-output database (see Timmer et al., 2015) to calibrate the model's expenditure shares. The world input-output database (WIOD) contains the expenditures of each industry in each country on intermediate input purchases from every other industry in every other country, as well as on the four primary factors. It also contains data on final consumption demand. We use the data from the year 2008 to calibrate the CES share parameters in the model. We also assume there are transfers between countries so that we can match trade imbalances.

To calibrate the elasticities of substitution, we use the estimates from Caliendo and Parro (2015) for the elasticity of substitution θ_j across traded goods. We set the elasticity of substitution across primary factors $\gamma = 0.5$, the one across industries $\varepsilon = 0.2$, and the one between value-added and intermediates $\zeta = 0.5$. These elasticities are broadly consistent with the estimates of Atalay (2017) and Boehm et al. (2015). Finally, we set the elasticity of substitution in consumption $\sigma = 0.9$, following Atalay (2017), Herrendorf et al. (2013), and Oberfield and Raval (2014), who use an elasticity of substitution in consumption (across industries) of slightly less than one.

Of course, there is substantial uncertainty about these parameter values, since the relevant elasticity value is context dependent (depending on the time-horizon and the level of aggregation in the data among other things). Therefore, we perform some robustness checks on our results to find out how sensitive the numbers are to the precise values of the elasticities of substitution.

7.1 Reversing Globalization: The Role of Intermediates

In Table 2, we consider the impact on the real income of each country, as well as the effect on world welfare, if the extent of globalization is rolled back. In particular, we compute the model's response to a 10% increase in either the iceberg costs of trade or a 10% increase in import tariffs. We compare the response of our benchmark model to the response of a version of the model which trivializes input-output connections and assumes value-added production functions.

Across the board, as expected, we see that an increase in iceberg trade costs (or non-tariff barriers to trade) are significantly more costly than tariffs. As an example, the losses to US income are slightly positive for a rollback of globalization due to tariffs, but are ten times larger and negative for an increase in iceberg cost. For the world, the losses in welfare due to an increase in iceberg costs are almost six times larger at 2.3% relative to 0.4% for tariffs.

Secondly, when we compare the model to the one with value-added production functions, again we find that intermediate inputs act to magnify the losses greatly. For the world as a hole, the losses are more than twice as big for both tariffs and iceberg costs. The presence of intermediates also makes tariffs more costly, both on aggregate, and individually. For almost every country in the world, accounting for intermediate inputs makes tariffs more costly. Hence, although Proposition 22 shows that the losses from tariffs are not qualitatively different in the presence of global value chains, and are still estimable from an appropriate summing up of Harberger triangles, quantitatively, accounting for global value chains makes a significant difference.

7.2 Brexit: Distributional Consequences

We use the quantitative model to consider how Britain's exit from the European Union might affect both the real income, as well as the distribution of income, across different countries. Although the terms of Britain's divorce from the EU are still not settled, Britain's departure from the EU is likely to set off an increase in both tariff and non-tariff barriers to trade with the rest of Europe. We start with the tariff scenario, where we consider the effects of a 10% import tariff imposed on all British imports into EU countries. The effects of this tariff on world GDP is modest at around -0.01%. The change in real GDI for each country in the database is shown in Table 3. As one might expect, Britain is most negatively affected by the tariffs, experiencing a reduction in real income of around

	AUS	AUT	BEL	BGR	BRA	CAN	CHN	CYP	CZE
IO Iceberg	-2.04%	-4.45%	-6.69%	-5.89%	-0.80%	-3.27%	-1.30%	-3.96%	-5.10%
VA Iceberg	-0.75%	-1.67%	-2.30%	-1.94%	-0.32%	-1.36%	-0.52%	-1.43%	-1.71%
IO Tariff	-0.72%	-1.16%	-2.26%	-0.67%	-0.09%	-1.23%	-0.16%	-0.23%	-0.55%
VA Tariff	-0.35%	-0.50%	-0.35%	0.29%	-0.04%	-0.73%	0.10%	-0.05%	0.10%

	DEU	DNK	ESP	EST	FIN	FRA	GBR	GRC	HUN
IO Iceberg	-3.04%	-3.55%	-2.09%	-5.54%	-3.32%	-1.98%	-2.52%	-2.37%	-5.81%
VA Iceberg	-1.24%	-1.59%	-0.76%	-1.86%	-1.27%	-0.67%	-1.00%	-0.84%	-2.34%
IO Tariff	-0.91%	-0.39%	0.12%	-1.38%	-0.82%	-0.18%	-0.65%	0.34%	-0.53%
VA Tariff	-0.45%	-0.30%	0.12%	-0.36%	-0.13%	0.04%	-0.38%	0.17%	-0.31%

	IDN	IND	IRL	ITA	JPN	KOR	LTU	LUX	LVA
IO Iceberg	-2.08%	-1.58%	-8.35%	-1.82%	-1.02%	-3.31%	-5.16%	-16.89%	-4.27%
VA Iceberg	-0.90%	-0.58%	-3.65%	-0.68%	-0.47%	-1.34%	-1.73%	-5.05%	-1.38%
IO Tariff	-0.62%	-0.06%	-3.71%	0.07%	-0.01%	-0.30%	-1.17%	-5.38%	-0.52%
VA Tariff	-0.46%	-0.03%	-1.32%	0.12%	0.02%	0.16%	0.03%	-0.73%	-0.09%

	MEX	MLT	NLD	POL	PRT	ROU	RUS	SVK	SVN
IO Iceberg	-1.71%	-7.36%	-5.50%	-3.11%	-2.97%	-3.16%	-2.97%	-6.14%	-5.17%
IO Tariff	-0.08%	-1.84%	-2.59%	-0.14%	0.07%	-0.07%	-1.59%	-0.66%	-0.89%
VA Iceberg	-0.83%	-2.91%	-2.02%	-1.05%	-1.04%	-1.12%	-1.40%	-2.17%	-1.87%
VA Tariff	-0.37%	-0.84%	-0.66%	0.06%	0.10%	-0.01%	-1.10%	-0.16%	-0.46%

	SWE	TUR	TWN	USA	ROW	World
IO Iceberg	-4.22%	-1.91%	-5.42%	-1.06%	-3.99%	-2.26%
VA Iceberg	-1.61%	-0.62%	-1.71%	-0.35%	-1.52%	-0.86%
IO Tariff	-1.48%	0.12%	-1.78%	0.09%	-1.20%	-0.43%
VA Tariff	-0.50%	0.06%	-0.03%	0.07%	-0.58%	-0.17%

Table 2: Percentage change in real income by country in response to a universal 10% change in iceberg trade costs or import tariffs. We compare the results from the Benchmark model (IO) with a model that assumes only value-added production functions (VA).

-0.3%. As one might expect, most European countries, with the exception of some Balkan countries, gain from the imposition of tariffs, with much of the gain coming from the collection of tariff revenues. The countries gaining the most are Luxembourg and Ireland, both of whom are engaged in significant financial services trade. Next, we consider the effects of iceberg trade costs. Now, most European countries lose, since no tariff revenues are being collected. Furthermore, Luxumbourg and Ireland now lead the losses (whereas previously they lead the gains), and these losses are in fact larger than those borne by Britain. This underscores the same lesson as in the previous section: non-trade barriers are much more costly than tariffs.

In Table 4 we consider how the real wages of primary factors change in various European countries in response to the imposition of the tariffs. In almost every European country, capital and low-skilled labor loses and high-skilled workers gain. This suggests that Britain was primarily engaged in selling high-skilled labor services to the rest of Europe. Interestingly, in Britain, the losers are capital and high-skilled workers, whereas low-skilled and medium-skilled worker wages actually rise in response to Brexit, suggesting that Britain was primarily exporting capital and high-skilled labor services to the rest of Europe in exchange for low and medium-skill competing workers.

Of course, this exercise abstracts from the fact that Brexit is likely to also affect the mobility of labor across borders within Europe. In particular, one might expect the diverging distributional consequences of Brexit on British primary factors to become even starker if Brexit results in reduced migration of low-skilled workers from Eastern Europe into Britain.

In Table 4, we tabulate the change in the price of each factor of production in units of domestic consumption. For brevity, we suppress non-European countries who are not greatly affected by the imposition of the tariffs. In general, we see that capital income declines most significantly in response to the tariffs, while high-skilled income mostly rises. Low-skilled and Medium-skilled labor tend to also lose, although not by as much as capital. For Britain, all factors lose, the losses being led by capital, followed by high-skilled, medium-skilled, and low-skilled workers. Table 7 in Appendix Q shows the breakdown between overall changes in real income into changes due to factor prices and changes due to tariffs and transfers. Table 7 shows that on the whole, every single EU country's factor real factor prices fall due to the tariffs — the reason EU members gain from imposing tariffs is due to the increased tariff revenue. Of course, the overall reduction in factor rewards masks the heterogeneity shown in Table 4.

Tariffs					
AUT	0.02%	GRC	0.00%	ROU	0.00%
BEL	0.01%	HUN	0.01%	SVK	0.01%
BGR	-0.01%	IRL	0.23%	SVN	0.02%
CYP	0.02%	ITA	0.00%	SWE	0.01%
CZE	0.01%	LTU	-0.01%	GBR	-0.30%
DEU	0.02%	LUX	0.33%		
DNK	0.02%	LVA	0.00%		
ESP	0.00%	MLT	0.07%		
EST	0.02%	NLD	0.04%		
FIN	0.03%	POL	0.02%		
FRA	0.00%	PRT	0.01%		
Non-Tariff Barrier					
AUT	-0.03%	GRC	-0.03%	ROU	-0.03%
BEL	-0.11%	HUN	-0.06%	SVK	-0.03%
BGR	-0.08%	IRL	-0.66%	SVN	-0.08%
CYP	-0.12%	ITA	-0.02%	SWE	0.00%
CZE	-0.06%	LTU	-0.03%	GBR	-0.27%
DEU	-0.05%	LUX	-1.04%		
DNK	-0.07%	LVA	-0.04%		
ESP	-0.03%	MLT	-0.36%		
EST	-0.06%	NLD	-0.10%		
FIN	-0.07%	POL	-0.04%		
FRA	-0.04%	PRT	-0.04%		

Table 3: The change in real income for each country when EU countries impose a 10% import tariff or iceberg cost on all British trade.

In general, the increases in tariffs reduce the volume of trade, and since traded goods are more capital intensive, this negatively affects capital earnings across the board. On the flip-side, factors which are less trade intensive, typically but not always high-skilled labor, tend to gain. Most of the gains to Ireland and Luxembourg (the countries which gain the most from the tariffs) are actually coming tariff revenues collected from financial intermediation. Interestingly, for Britain itself, the tariffs have very little effect on the real income of low-skilled workers, with most of the costs being borne by capital and high-skilled labor.

	AUT	BEL	BGR	CYP	CZE	DEU	DNK	ESP	EST
Capital	-0.09%	-0.16%	-0.09%	-0.18%	-0.07%	-0.11%	-0.13%	-0.02%	-0.17%
Low	-0.06%	-0.26%	-0.13%	-0.47%	-0.12%	-0.16%	-0.22%	-0.04%	-0.09%
Med	-0.02%	-0.10%	0.06%	-0.05%	-0.07%	-0.02%	-0.09%	-0.01%	-0.04%
High	0.07%	0.17%	0.15%	0.14%	0.12%	0.00%	0.12%	0.00%	0.12%
	FIN	FRA	GRC	HUN	IRL	ITA	LTU	LUX	LVA
Capital	-0.20%	-0.05%	-0.07%	-0.14%	-1.26%	-0.04%	-0.08%	-0.71%	-0.04%
Low	-0.06%	-0.04%	0.04%	-0.12%	-0.36%	-0.06%	-0.02%	-1.06%	-0.09%
Med	-0.07%	-0.04%	0.04%	-0.08%	-0.37%	0.00%	0.00%	-0.97%	-0.07%
High	0.12%	0.02%	-0.01%	0.15%	0.05%	0.05%	0.05%	-0.78%	0.04%
	MLT	NLD	POL	PRT	ROU	SVK	SVN	SWE	GBR
Capital	-0.53%	-0.13%	0.02%	-0.02%	0.02%	-0.08%	-0.16%	0.01%	-0.87%
Low	-0.53%	-0.31%	-0.15%	-0.12%	-0.09%	-0.11%	-0.12%	-0.01%	0.40%
Med	-0.03%	-0.16%	-0.17%	0.04%	-0.06%	-0.03%	-0.07%	-0.01%	0.14%
High	0.25%	0.11%	0.08%	0.09%	0.01%	0.06%	0.06%	0.01%	-0.19%

Table 4: The change in factor rewards, in units of domestic consumption, for each primary factor in response to a 10% increase in European taxes on British imports.

7.3 Trump Tariffs: Spillovers in Trade Policy

Next, we consider the way tariffs between two countries can affect a third country. The Trump administration has recently imposed a 10% tariff on a broad range of Chinese imports, with the possibility that these tariffs will be increased further to 25%. Meanwhile, the administration has also renegotiated the North American Free Trade Agreement with

Mexico and Canada, replacing with the USMCA. In this subsection, we analyze an impressionistic version of this policy, computing the effects of the imposition of a tariff on China while maintaining free trade with Mexico.

In Table 5, we show the effects on the real income of each country due to a 10% tax on Chinese imports. As one might expect, China is affected most negatively by this policy, followed by some of China's close trading partners, like Taiwan, Korea, and Australia. The United States gains slightly from the imposition of tariffs, and these gains are roughly divided between increases in real factor prices and increases in tariff revenues. Interestingly, the country that gains the most from the tariffs is Mexico due to the increase in demand for Mexican goods. Table 6 in Appendix Q breaks down how the factor prices (in units of domestic consumption) change in response to the tariffs. This table shows that the overall trend, across all countries including the USA and Mexico, is that capital loses at the expense of labor. This is similar to the Brexit scenario, and indicates that typically tariffs seem to tilt the distribution of income in favor of labor due to the fact that trade is more capital intensive. For most countries, except Mexico, most of labor's gains accrue to high-skilled workers. For Mexico, most of the gains go to low and medium skilled labor instead.

AUS	-0.01%	ESP	0.01%	JPN	0.00%	RUS	0.00%
AUT	0.00%	EST	0.02%	KOR	-0.02%	SVK	0.01%
BEL	0.00%	FIN	0.01%	LTU	0.02%	SVN	0.01%
BGR	0.02%	FRA	0.01%	LUX	-0.01%	SWE	0.00%
BRA	0.00%	GBR	0.01%	LVA	0.01%	TUR	0.01%
CAN	0.02%	GRC	0.01%	MEX	0.03%	TWN	-0.06%
CHN	-0.24%	HUN	0.02%	NLD	0.02%	USA	0.01%
CYP	0.04%	IDN	0.01%	NOR	0.01%	ROW	0.01%
CZE	0.01%	IND	0.02%	POL	0.01%		
DEU	0.00%	IRL	0.00%	PRT	0.01%		
DNK	0.00%	ITA	0.01%	ROU	0.01%		

Table 5: The change in real income for each country when the USA imposes a 10% import tariff on all Chinese trade.

8 Extensions

8.1 Appendix on Beyond CES

9 Conclusion

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A Proofs

Proof of Theorem 1. Let H be the set of domestically produced goods and let F be the set of foreign-produced goods. Let Ω^h be the matrix whose elements are

$$\Omega^h_{ij} = \frac{M_{ij}}{R_i} \quad (i, j \in H), \tag{57}$$

where M_{ij} is the value of i's purchases from j and R_i is the revenues of i. Similarly, let Ω^f be given by

$$\Omega^f_{ij} = \frac{M_{ij}}{R_i} \quad (i \in H, j \in F). \tag{58}$$

Finally, let

$$\alpha_{ij} = \frac{w_j L_{ij}}{R_i} \quad (i \in H), \tag{59}$$

where $w_j L_{ij}$ are factor payments by i to factor j.

Denoting the vector of domestic prices by p^h and foreign prices by p^f, Shephard's lemma implies that

$$\mathrm{d}\log p^h = (I - \Omega^h)^{-1}\left(\Omega^f \,\mathrm{d}\log p^f + \alpha\,\mathrm{d}\log w - \mathrm{d}\log A\right), \tag{60}$$

where $\mathrm{d}\log A$ are perturbations to the productivity functions of domestic producers.

Now, let γ be each good's share in GDP, so for domestically produced goods

$$\gamma_i = \frac{F_i + X_i}{GDP}, \tag{61}$$

where F_i is final consumption and X_i is exports. For imported goods

$$\gamma_i = -\sum_{j \in H} \Omega_{ji}\lambda_j. \tag{62}$$

Hence,

$$\mathrm{d}\log RGDP_c = \Lambda'(\mathrm{d}\log w + \mathrm{d}\log L) - \mathrm{d}\log P_{GDP_c}, \tag{63}$$

can be written as

$$\mathrm{d}\log RGDP_c = \Lambda'(\mathrm{d}\log w + \mathrm{d}\log L) - (\gamma^h)'\,\mathrm{d}\log p^h + (\gamma^f)'\,\mathrm{d}\log p^f. \tag{64}$$

Substituting the expression for $\mathrm{d}\log p^h$ gives

$$\begin{aligned}\mathrm{d}\log RGDP_c = \Lambda'(\mathrm{d}\log w + \mathrm{d}\log L) - (\gamma^h)'(I-\Omega^h)^{-1}\left(\Omega^f\,\mathrm{d}\log p^f + \alpha\,\mathrm{d}\log w - \mathrm{d}\log A\right)\\ + (\gamma^f)'\,\mathrm{d}\log p^f.\end{aligned}$$

To complete the proof, note that $(\gamma^h)'(I-\Omega^h)^{-1}\alpha = \Lambda$, and $(\gamma^h)'(I-\Omega^h)^{-1}\Omega^f = \gamma^f$. These facts follow from market clearing conditions:

$$\lambda_i^h = b_i^h + \sum_j \Omega_{ji}^h \lambda_j^h + \epsilon_i, \tag{65}$$

where b_i^h is final domestic demand as a share of GDP and ϵ_i is exports as a share of GDP. This implies that

$$(\lambda^h)' = (b^h + \epsilon)'(I-\Omega^h)^{-1}. \tag{66}$$

Similarly,

$$\gamma_i^f = \sum_j \Omega_{ji}^f \lambda_j^h, \tag{67}$$

and

$$\Lambda_i = \sum_j \alpha_{ji}\lambda_j^h. \tag{68}$$

Putting these market clearing conditions together gives us our desired result.

■

Proof of Theorem 2. TBA. ■

Proof of Proposition 3. This follows as a corollary of Theorem 19. ■

Proof of Proposition **??**. *Part I:*

In the household CES nest, we have

$$\mathrm{d}\log b_{cj} = (1-\theta_c)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_{y_c}\right). \tag{69}$$

In particular, this implies that

$$\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci} = (1-\theta_c)\left(\mathrm{d}\log p_j - \mathrm{d}\log p_i\right). \tag{70}$$

Substituting the equilibrium prices gives

$$\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci} = (1-\theta_c)\left(\Psi_{jk} + \sum^{F}\Psi_{jf}\,\mathrm{d}\log\Lambda_f - \Psi_{ik} - \sum^{F}\Psi_{if}\,\mathrm{d}\log\Lambda_f\right). \tag{71}$$

Hence, we have

$$\Psi_{jk} + \sum^{F}\Psi_{jf}\,\mathrm{d}\log\Lambda_f = \frac{\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci}}{1-\theta_c} + \Psi_{ik} + \sum^{F}\Psi_{if}\,\mathrm{d}\log\Lambda_f. \tag{72}$$

From Theorem 2, we know that

$$\mathrm{d}\log Y_c = -\lambda_k^c + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f - \sum_{f=1}^{F}\Lambda_f^c\,\mathrm{d}\log\Lambda_f. \tag{73}$$

In other words,

$$\mathrm{d}\log Y_c = -\sum_{j=1}^{N} b_{cj}\left(\Psi_{jk} + \sum^{F}\Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f. \tag{74}$$

Substituting in (72) into this expression above we get

$$\mathrm{d}\log Y_c = -\sum_{j=1}^{N} b_{cj}\left(\frac{\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci}}{1-\theta_c} + \Psi_{ik} + \sum^{F}\Psi_{if}\,\mathrm{d}\log\Lambda_f\right) + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f. \tag{75}$$

Since, by assumption i is entirely domestically produced $\Psi_{ik} = 0$. Furthermore, since there is only one factor in country c, $\Psi_{iL} = 1$ and $\Psi_{if} = 0$ for all factors $f \neq L$. So, this becomes

$$\mathrm{d}\log Y_c = -\sum_{j=1}^{N} b_{cj}\left(\frac{\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci}}{1-\theta_c}\right) - \mathrm{d}\log\Lambda_L + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f. \tag{76}$$

Since there is only one factor in country c, this means that $\Lambda_L = \chi_c$ and so

$$\mathrm{d}\log Y_c = -\sum_{j=1}^{N} b_{cj}\left(\frac{\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci}}{1-\theta_c}\right). \tag{77}$$

Since $\sum b_{cj}\,\mathrm{d}\log b_{cj} = 0$, this leaves us with

$$\mathrm{d}\log Y_c = \sum_{j=1}^{N} b_{cj}\left(\frac{\mathrm{d}\log b_{ci}}{1-\theta_c}\right). \tag{78}$$

Integrate this expression to get the desired result.

Part II:

If c has more than one factor, then we have

$$\mathrm{d}\log Y_c = -\sum_{j=1}^{N} b_{cj}\left(\frac{\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci}}{1-\theta_c} + \Psi_{ik} + \sum^{F}\Psi_{if}\,\mathrm{d}\log\Lambda_f\right) + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f. \tag{79}$$

Since, by assumption i is entirely domestically produced $\Psi_{ik} = 0$, this simplifies to

$$\mathrm{d}\log Y_c = -\sum_{j=1}^{N} b_{cj}\left(\frac{\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci}}{1-\theta_c}\right) + \sum_{j=1}^{N} b_{cj}\sum^{F}\Psi_{if}\,\mathrm{d}\log\Lambda_f + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f. \tag{80}$$

The representativeness of the supply chain of i then implies

$$\mathrm{d}\log Y_c = -\sum_{j=1}^{N} b_{cj}\left(\frac{\mathrm{d}\log b_{cj} - \mathrm{d}\log b_{ci}}{1-\theta_c}\right). \tag{81}$$

Since $\sum b_{cj}\,\mathrm{d}\log b_{cj} = 0$, this leaves us with

$$\mathrm{d}\log Y_c = \sum_{j=1}^{N} b_{cj}\left(\frac{\mathrm{d}\log b_{ci}}{1-\theta_c}\right). \tag{82}$$

Integrate this expression to get the desired result.

Part III:

As above, we have

$$\begin{aligned}
\mathrm{d}\log Y_c &= -\lambda_k^c + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f - \sum_{f=1}^{F}\Lambda_f^c\,\mathrm{d}\log\Lambda_f,\\
&= -\sum_i b_{ci}\left(\sum_j \omega_{ij}\Psi_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f,\\
&= -\sum_i b_{ci}\left(\sum_j \omega_{ij}\left(\frac{\mathrm{d}\log\omega_{ij} - \mathrm{d}\log\omega_{ii_d}}{1-\theta_i} + \Psi_{i_dk} + \sum_f \Psi_{i_df}\,\mathrm{d}\log\Lambda_f\right)\right) + \sum_{f\in c}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f,\\
&= -\sum_i b_{ci}\left(\sum_j \omega_{ij}\left(\frac{\mathrm{d}\log\omega_{ij} - \mathrm{d}\log\omega_{ii_d}}{1-\theta_i}\right)\right),\\
&= \sum_i b_{ci}\left(\sum_j \omega_{ij}\left(\frac{\mathrm{d}\log\omega_{ii_d}}{1-\theta_i}\right)\right).
\end{aligned}$$

Integrate this, noting that b_{ci} is constant, to get the desired result. ■

Proof of Proposition 16. This follows from Proposition 3. Specifically, let θ be the elasticity of substitution for all $i \in \mathcal{R}$. Then, if $L \in \mathcal{F}$, then

$$\begin{aligned}
\lim_{\theta\to\infty}\frac{1}{1-\theta}\Lambda_L\,\mathrm{d}\log\Lambda_L = \lim_{\theta\to\infty}\Bigg(&\sum_{i\in R}\lambda_j Cov(\Psi_{(k)} + \sum_f \Psi_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(L)})\\
&+ \sum_{i\notin R}\left(\frac{1-\theta_j}{1-\theta}\right)\lambda_j Cov(\Psi_{(k)} + \sum_f \Psi_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(L)})\\
&+ \frac{1}{1-\theta}\sum_c \lambda_l^c\left(\sum_{f\in c}\Lambda_f\,\mathrm{d}\log\Lambda_f\right)\Bigg).
\end{aligned}$$

This gives the result.

■

Proof of Proposition 19. For each good,

$$\lambda_i = \sum_c b_{ci}\chi_c + \sum_i \Omega_{ji}\lambda_j. \tag{83}$$

This means

$$\lambda_i \,\mathrm{d}\log\lambda_i = \sum_c \chi_c b_{ci}\,\mathrm{d}\log b_{ci} + \sum_j \Omega_{ji}\lambda_j\,\mathrm{d}\log\Omega_{ji} + \sum_j \Omega_{ji}\,\mathrm{d}\,\lambda_j + \sum_c b_{ci}\chi_c\,\mathrm{d}\log\chi_c. \quad (84)$$

Now, note that

$$\mathrm{d}\log b_{ci} = (1-\theta_c)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_{y_c}\right) \quad (85)$$

$$\mathrm{d}\log\Omega_{ji} = (1-\theta_j)\left(\mathrm{d}\log p_i - \mathrm{d}\log P_j + \mathrm{d}\log\mu_j\right) - \mathrm{d}\log\mu_j \quad (86)$$

$$\mathrm{d}\log\chi_c = \sum_{f\in F_c^*}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f + \sum_{i\in c}\frac{\lambda_i}{\mu_i}\,\mathrm{d}\log\mu_i. \quad (87)$$

$$\mathrm{d}\log p_i = \tilde{\Psi}\left(\mathrm{d}\log\mu - \mathrm{d}\log A\right) + \tilde{\Psi}\tilde{\alpha}\,\mathrm{d}\log\Lambda. \quad (88)$$

$$\mathrm{d}\log P_{y_c} = b'\tilde{\Psi}\left(\mathrm{d}\log\mu - \mathrm{d}\log A\right) + b'\tilde{\Psi}\tilde{\alpha}\,\mathrm{d}\log\Lambda. \quad (89)$$

For shock $\mathrm{d}\log\mu_k$, we have

$$\mathrm{d}\log b_{ci} = (1-\theta_c)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \sum_j b_{cj}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right). \quad (90)$$

$$\mathrm{d}\log\Omega_{ji} = (1-\theta_j)\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right) - \theta_j\,\mathrm{d}\log\mu_j. \quad (91)$$

Putting this altogether gives

$$\begin{aligned}\mathrm{d}\,\lambda_l = &\sum_i\sum_c (1-\theta_c)\chi_c b_{ci}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \sum_j b_{cj}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\Psi_{il}\\ &+ \sum_i\sum_j (1-\theta_j)\lambda_j\mu_j^{-1}\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f - \tilde{\Psi}_{jk} - \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il}\\ &- \theta_k\lambda_k\sum_i \Omega_{ki}\Psi_{il} + \sum_c \chi_c\sum_i b_{ci}\Psi_{il}\,\mathrm{d}\log\chi_c.\end{aligned}$$

Simplify this to

$$\mathrm{d}\,\lambda_l = \sum_c (1-\theta_c)\chi_c\sum_i b_{ci}\left(\tilde{\Psi}_{ik} + \sum_f \tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il} - \left(\sum_i b_{ci}\left(\tilde{\Psi}_{jk} + \sum_f \Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)\right)\left(\sum_i b_{ci}\Psi_{il}\right)$$

$$+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\sum_i\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik}+\sum_f\tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il}-\left(\sum_i\tilde{\Omega}_{ji}\Psi_{il}\right)\left(\tilde{\Psi}_{jk}+\sum_f\Psi_{jf}\,\mathrm{d}\log\Lambda_f\right)$$
$$-\theta_k\lambda_k\left(\Psi_{kl}-\mathbf{1}(l=k)\right)+\sum_c\chi_c\sum_i b_{ci}\Psi_{il}\,\mathrm{d}\log\chi_c.$$

Simplify this further to get

$$\begin{aligned}\mathrm{d}\,\lambda_l=&\sum_c(1-\theta_c)\chi_c Cov_{b^{(c)}}\left(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f,\Psi_{(l)}\right)\\&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}\sum_i\tilde{\Omega}_{ji}\left(\tilde{\Psi}_{ik}+\sum_f\tilde{\Psi}_{if}\,\mathrm{d}\log\Lambda_f\right)\Psi_{il}\\&-\left(\sum_i\tilde{\Omega}_{ji}\Psi_{il}\right)\left(\sum_i\tilde{\Omega}_{ji}\tilde{\Psi}_{ik}+\sum_i\tilde{\Omega}_{ji}\sum_f\Psi_{if}\,\mathrm{d}\log\Lambda_f\right)\\&-\theta_k\lambda_k\left(\Psi_{kl}-\mathbf{1}(l=k)\right)+\sum_c\chi_c\sum_i b_{ci}\Psi_{il}\,\mathrm{d}\log\chi_c,\end{aligned}$$

Using the input-output covariance notation, write

$$\begin{aligned}\mathrm{d}\,\lambda_l=&\sum_c(1-\theta_c)\chi_c Cov_{b^{(c)}}\left(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f,\Psi_{(l)}\right)\\&+\sum_j(1-\theta_j)\lambda_j\mu_j^{-1}Cov_{\tilde{\Omega}^{(j)}}\left(\tilde{\Psi}_{(k)}+\sum_f\tilde{\Psi}_{(f)}\,\mathrm{d}\log\Lambda_f,\Psi_{(l)}\right)\\&-(1-\theta_k)\lambda_k(\Psi_{kl}-\mathbf{1}(l=k))-\theta_k\lambda_k\left(\Psi_{kl}-\mathbf{1}(l=k)\right)+\sum_c\chi_c\sum_i b_{ci}\Psi_{il}\,\mathrm{d}\log\chi_c,\end{aligned}$$

This then simplifies to give from the fact that $\sum_i b_{ci}\Psi_{il}=\lambda_l^c$:

$$\begin{aligned}\lambda_l\,\mathrm{d}\log\lambda_l=&\sum_{j\in N,C}(1-\theta_j)\lambda_j\mu_j^{-1}Cov(\tilde{\Psi}_{(k)}+\sum_f^F\mathrm{d}\log\Lambda_f,\Psi_{(l)})\\&-\lambda_k\left(\Psi_{kl}-\mathbf{1}(k=l)\right)+\sum_c\chi_c\lambda_l^c\,\mathrm{d}\log\chi_c.\end{aligned}$$

To complete the proof, note that

$$P_{y_c} Y_c = \sum_f w_f L_f + \sum_{i \in c} \left(1 - \frac{1}{\mu_i}\right) p_i y_i. \tag{92}$$

Hence,

$$\mathrm{d}(P_{y_c} Y_c) = \sum_{f \in c} w_f L_f \,\mathrm{d}\log w_f + \sum_{i \in c} \left(1 - \frac{1}{\mu_i}\right) p_i y_i \,\mathrm{d}\log(p_i y_i) + \sum_{i \in c} \frac{\mathrm{d}\left(1 - \frac{1}{\mu_i}\right)}{\mathrm{d}\log \mu_i} p_i y_i \,\mathrm{d}\log \mu_i. \tag{93}$$

In other words, since $P_y Y = 1$, we have

$$\mathrm{d}\,\chi_c = \sum_{f \in c} \Lambda_f \,\mathrm{d}\log w_f + \sum_{i \in c} \left(1 - \frac{1}{\mu_i}\right) \lambda_i \,\mathrm{d}\log \lambda_i + \sum_{i \in c} \frac{\mathrm{d}\left(1 - \frac{1}{\mu_i}\right)}{\mathrm{d}\log \mu_i} \lambda_i \,\mathrm{d}\log \mu_i. \tag{94}$$

Hence,

$$\mathrm{d}\log \chi_c = \sum_{f \in F_c^*} \frac{\Lambda_f}{\chi_c} \,\mathrm{d}\log \Lambda_f + \sum_{i \in c} \frac{\lambda_i}{\chi_c} \,\mathrm{d}\log \mu_i. \tag{95}$$

■

Proof of Proposition 6. Here we assume that there is only one factor in the domestic economy and normalize its price to one.

Define the "fictitious domestic" IO matrix

$$\widehat{\Omega}_{ij} \equiv \frac{\Omega_{ij}}{\sum_{k \in N_c} \Omega_{ik}}, \tag{96}$$

with associated Leontieff-inverse matrix

$$\widehat{\Psi} \equiv (1 - \widehat{\Omega})^{-1}. \tag{97}$$

Then for each producer $i \in C$, we have

$$d\log p_i = \sum_{j \in N_c} \widehat{\Omega}_{ij} d\log p_j + \frac{d\log \lambda_{ic}}{\theta_i - 1}, \tag{98}$$

where λ_{ic} is the domestic cost share of producer i. The solution of this system of equations

is

$$d\log p_i = \sum_{j\in N_c} \widehat{\Psi}_{ij} \frac{d\log \lambda_{jc}}{\theta_j - 1}. \tag{99}$$

From this we can get welfare gains

$$\mathrm{d}\log Y = -\sum_{i\in C} \widehat{b}_i \,\mathrm{d}\log p_i = -\sum_{i\in C}\sum_{j\in C} \widehat{b}_i \widehat{\Psi}_{ij} \frac{d\log \lambda_{jc}}{\theta_j - 1} = -\sum_{j\in C} \widehat{\lambda}_j \frac{d\log \lambda_{jc}}{\theta_j - 1}, \tag{100}$$

where

$$\widehat{\lambda}_i \equiv \sum_{j\in C} \widehat{b}_j \widehat{\Psi}_{ji}. \tag{101}$$

This can be thought of as hitting the fictitious domestic economy with productivity shocks $-d\log\lambda_{jc}/(\theta_j - 1)$. We should therefore be able to leverage our results from the Hulten paper to derive a second-order approximation as a function of these shocks. ■

Proof of Proposition 18. Assume that we observe changes in factor shares $d\log\Lambda_f$ in C. Then the changes in *nominal* wages are given by

$$d\log w_f - d\log w_g = d\log\Lambda_f - d\log\Lambda_g. \tag{102}$$

We get that

$$d\log p_i = \sum_{j\in C} \Psi^c_{ij} \frac{d\log\lambda_{jc}}{\theta_j - 1} + \sum_{f\in F_c} \Psi^c_{if} d\log w_f, \tag{103}$$

which implies that

$$\begin{aligned}
d\log Y^c &= \sum_{f\in F_c} \Lambda_f d\log w_f - \sum_{i\in C} b^c_i d\log p_i, \\
&= \sum_{f\in F_c} \Lambda_f d\log w_f - \sum_{i\in C}\sum_{f\in F_c} b^c_i \Psi^c_{if} d\log w_f - \sum_{i\in C}\sum_{j\in C} b^c_i \Psi^c_{ij} \frac{d\log\lambda_{jc}}{\theta_j - 1}, \\
&= \sum_{f\in F_c} (\Lambda_f - \Lambda^c_f) d\log w_f - \sum_{j\in C} \lambda^c_j \frac{d\log\lambda_{jc}}{\theta_j - 1}, \\
&= \sum_{f\in F_c} (\Lambda_f - \Lambda^c_f) d\log \Lambda_f - \sum_{j\in C} \lambda^c_j \frac{d\log\lambda_{jc}}{\theta_j - 1}, \\
&= -\sum_{f\in F_c} \Lambda^c_f d\log\Lambda_f - \sum_{j\in C} \lambda^c_j \frac{d\log\lambda_{jc}}{\theta_j - 1},
\end{aligned}$$

where

$$\Lambda_f^c \equiv \sum_{i \in C} b_i^c \Psi_{if}^c, \tag{104}$$

$$\lambda_i^c \equiv \sum_{j \in C} b_j^c \Psi_{ji}^c, \tag{105}$$

and where we have used the fact that $\sum_{f \in F_c} \Lambda_f = \sum_{f \in F_c} \Lambda_f^c = 1$ and $\sum_{f \in F_c} \Lambda_f d \log \Lambda_f = 0$. Note that I may have introduced some conflict with the notation in the paper...

Bottom line, we get the formula

$$d \log Y^c = - \sum_{f \in F_c} \Lambda_f^c d \log \Lambda_f - \sum_{j \in C} \lambda_j^c \frac{d \log \lambda_{jc}}{\theta_j - 1}. \tag{106}$$

Now the question is whether this formula is satisfactory or whether we can/want to express the changes in factor shares as a function of changes in other shares.

Note that we can always normalize nominal GDP so that

$$\sum_{f \in F_c} \Lambda_f d \log w_f = 0. \tag{107}$$

With this normalization, we have

$$d \log p_i = \sum_{j \in C} \Psi_{ij}^c \frac{d \log \lambda_{jc}}{\theta_j - 1} + \sum_{f \in F_c} \Psi_{if}^c d \log \Lambda_f. \tag{108}$$

And the cost function satisfies

$$d \log C^c = \sum_{f \in F_c} \Lambda_f^c d \log \Lambda_f + \sum_{j \in C} \lambda_j^c \frac{d \log \lambda_{jc}}{\theta_j - 1}. \tag{109}$$

This can be thought of as hitting the fictitious domestic economy with shocks $-d \log \Lambda_f$ to factor prices and productivity shocks $-d \log \lambda_{jc} / (\theta_j - 1)$. We should therefore be able to leverage our results from the Hulten paper to derive a second-order approximation as a function of these shocks. I think that this should be done using the cost function rather than the production function. ■

Proof of Proposition 8. Proof of Part(1):

The expression for $\mathrm{d}^2 \log Y$ follows from applying part (2) to the whole world. The equality of real GDI and real GDP at the world level completes the proof.

Proof of Part (2):

Denote the set of imports into country c by M_c. Then, we can write:

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log\mu_i} = \sum_{f\in F_c}\Lambda_{cf}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i} + \sum_j \frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_i}\frac{\left(1-\frac{1}{\mu_j}\right)}{P_{Y_c}Y_c} + \frac{\lambda_{ci}}{\mu_i} - \frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log\mu_i},$$

where

$$\frac{\mathrm{d}\log P_{Y_c}}{\mathrm{d}\log\mu_i} = \sum_{f\in F_c}\tilde{\Lambda}_{cf}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i} + \sum_{m\in M_c}\tilde{\lambda}_{cm}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_i} - \tilde{\lambda}_{ci} - \sum_{m\in M_c}\Lambda_{cm}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_i},$$

and

$$\tilde{\lambda}_{ci} = \sum_j F_{cj}\tilde{\Psi}_{ji}.$$

Combining these expressions, we get

$$\begin{aligned}\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log\mu_i} = &\sum_{f\in F_c}\left(\Lambda_{cf}-\tilde{\Lambda}_{cf}\right)\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_i} + \sum_{m\in M_c}\left(\lambda_{cm}-\tilde{\lambda}_{cm}\right)\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_i}\\ &+\sum_{j\in N_c}\lambda_{cj}\frac{\mathrm{d}\log\lambda_j}{\mathrm{d}\log\mu_i}\left(1-\frac{1}{\mu_j}\right)+\frac{\lambda_{ci}}{\mu_i}-\tilde{\lambda}_{ci}.\end{aligned}$$

At the efficient point,

$$\begin{aligned}\frac{\mathrm{d}^2\log Y_c}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} = &\sum_{f\in F_c}\left(\frac{\mathrm{d}\,\Lambda_{cf}}{\mathrm{d}\log\mu_i}-\frac{\mathrm{d}\,\tilde{\Lambda}_{cf}}{\mathrm{d}\log\mu_i}\right)\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\\ &+\sum_{m\in M_c}\left(\frac{\mathrm{d}\,\lambda_{cm}}{\mathrm{d}\log\mu_i}-\frac{\mathrm{d}\,\tilde{\lambda}_{cm}}{\mathrm{d}\log\mu_i}\right)\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_{ck}}{\mathrm{d}\log\mu_i}\\ &+\lambda_{ck}\left(\frac{\mathrm{d}\log\lambda_{ck}}{\mathrm{d}\log\mu_i}-\delta_{ki}\right)+\frac{1}{P_{Y_c}Y_c}\frac{\mathrm{d}\,\lambda_{ci}}{\mathrm{d}\log\mu_k}\end{aligned}$$

Using Lemma 12,

$$\begin{aligned}\frac{\mathrm{d}^2\log Y_c}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} = &-\sum_{f\in F_c}\lambda_{ci}\Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}-\sum_{m\in M_c}\lambda_{ci}\Psi_{im}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k}-\lambda_{ci}\left(\Psi_{ik}-\delta_{ik}\right)\\ &-\lambda_{ck}\delta_{ik}+\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{1}{P_{Y_c}Y_c},\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{f\in F_c} \lambda_{ci}\Psi_{if}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \sum_{m\in M_c}\lambda_{ci}\Psi_{im}\frac{\mathrm{d}\log p_m}{\mathrm{d}\log\mu_k} - \lambda_{ci}\Psi_{ik}\\
&+ \lambda_{ci}\left(\frac{\mathrm{d}\log p_i}{\mathrm{d}\log\mu_k} + \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}\right),\\
&= \lambda_{ci}\frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k}.
\end{aligned}
$$

Proof of Part(3):

$$
\frac{\mathrm{d}\log W^{BS}}{\mathrm{d}\log\mu_k} = \sum_h \overline{\chi}_h \frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k} = \sum_h \overline{\chi}_h\left(\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k}\right).
$$

$$
\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_k} = \sum_{f\in F_c}\frac{\Lambda}{\chi_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \sum_{i\in N_h}\frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{(1-\frac{1}{\mu_i})}{\chi_h}.
$$

$$
\frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} = \sum_{f\in F}\tilde{\Lambda}^h_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \tilde{\lambda}^h_k.
$$

Hence, assuming the normalization $P_Y Y = 1$ gives

$$
\begin{aligned}
\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} = \sum_h \overline{\chi}_h \Bigg(&\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{1}{\chi_h} + \sum_f\frac{\Lambda_f}{\chi_h}\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\\
&-\sum_f\frac{\Lambda_f}{\chi_h}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} + \frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i}\frac{1}{\chi_h\mu_k} - \frac{\lambda_k}{\chi_h\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} - \frac{\lambda_k}{\chi_h\mu_k}\delta_{ki}\\
&\sum_i\frac{\mathrm{d}^2\,\lambda_j}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log\mu_k}\frac{1}{\mu_i\chi_h} + \sum_j\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_k}\frac{1-\frac{1}{\mu_j}}{\chi_h}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}\\
&-\sum_f\frac{\mathrm{d}\,\tilde{\Lambda}^h_f}{\mathrm{d}\log\mu_i}\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} - \sum_f\tilde{\Lambda}^h_f\frac{\mathrm{d}^2\log\Lambda_f}{\mathrm{d}\log\mu_i\,\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}^h_k}{\mathrm{d}\log\mu_i}\Bigg).
\end{aligned}
$$

At the efficient point, this simplifies to

$$
\begin{aligned}
\frac{\mathrm{d}^2\log W^{BS}}{\mathrm{d}\log\mu_k\,\mathrm{d}\log\mu_i} = &\sum_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\left(\frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_i} - \sum_h\overline{\chi_h}\frac{\mathrm{d}\,\tilde{\Lambda}^h_f}{\mathrm{d}\log\mu_i}\right)\\
&+\frac{\mathrm{d}\,\lambda_k}{\mathrm{d}\log\mu_i} - \sum_h\overline{\chi_h}\frac{\mathrm{d}\,\tilde{\lambda}^h_k}{\mathrm{d}\log\mu_i} - \sum_{f,h}\Lambda_f\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k}\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}
\end{aligned}
$$

$$- \lambda_k \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} - \lambda_k \delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k}.$$

By Lemma 11, at the efficient point,

$$\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log \mu_i} - \sum_h \overline{\chi}_h \frac{\mathrm{d}\,\tilde{\lambda}_j^h}{\mathrm{d}\log \mu_i} = \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \tilde{\lambda}_j^h - \lambda_i \left(\Psi_{ij} - \delta_{ij}\right).$$

Whence, we can further simplify the previous expression to

$$\begin{aligned}
\frac{\mathrm{d}^2 \log W^{BS}}{\mathrm{d}\log \mu_k \,\mathrm{d}\log \mu_i} &= \sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \tilde{\Lambda}_f^h - \lambda_i \Psi_{if} \right) \\
&\quad + \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \tilde{\lambda}_k^h - \lambda_i (\Psi_{ik} - \delta_{ik}) - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&\quad - \frac{\lambda_k}{\mathrm{d}\log \chi_h} \mathrm{d}\log \mu_i - \lambda_k \delta_{ki} + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k}, \\
&= \sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \tilde{\Lambda}_f^h - \lambda_i \Psi_{if} \right) \\
&\quad + \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \tilde{\lambda}_k^h - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&\quad - \frac{\lambda_k}{\mathrm{d}\log \chi_h} \mathrm{d}\log \mu_i + \frac{\mathrm{d}\,\lambda_i}{\mathrm{d}\log \mu_k},
\end{aligned}$$

and using $\mathrm{d}\log \lambda_i = \mathrm{d}\log p_i + \mathrm{d}\log y_i$,

$$\begin{aligned}
&= \sum_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \left(\sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \tilde{\Lambda}_f^h - \lambda_i \Psi_{if} \right) \\
&\quad + \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log \mu_i} \tilde{\lambda}_k^h - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \\
&\quad - \frac{\lambda_k}{\mathrm{d}\log \chi_h} \mathrm{d}\log \mu_i + \lambda_i \frac{\mathrm{d}\log p_i}{\mathrm{d}\log \mu_k} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log \mu_k}, \\
&= \sum_{f,h} \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \tilde{\Lambda}_f^h \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} - \lambda_i \sum_f \Psi_{if} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k} \\
&\quad + \sum_h \chi_h \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \tilde{\lambda}_k^h - \lambda_i \Psi_{ik} - \sum_{f,h} \Lambda_f \frac{\mathrm{d}\log \chi_h}{\mathrm{d}\log \mu_i} \frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}
\end{aligned}$$

$$
\begin{aligned}
&- \lambda_k \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} + \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} \\
&+ \lambda_i \left(\sum_f \Psi_{if} \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \Psi_{ik} \right), \\
&= \sum_{f,h} \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \left(\chi_h \tilde{\Lambda}_f^h - \Lambda_f \right) \\
&+ \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \tilde{\lambda}_k^h - \lambda_k \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \left(\tilde{\Lambda}_f^h \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} + \tilde{\lambda}_k^h \right) \\
&- \sum_{f,h} \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \Lambda_f - \lambda_k \sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + \sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} \\
&- \left(\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \Lambda_f \right) \left(\sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} \right) - \lambda_k \sum_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \\
&= \lambda_i \frac{\mathrm{d}\log y_i}{\mathrm{d}\log\mu_k} + Cov_\chi \left(\frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i}, \frac{\mathrm{d}\log P_{cpi,h}}{\mathrm{d}\log\mu_k} \right),
\end{aligned}
$$

since

$$
-\sum_f \frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \Lambda_f = -\sum_f \frac{\mathrm{d}\,\Lambda_f}{\mathrm{d}\log\mu_k} = \frac{\mathrm{d}\left(1 - \sum_j \lambda_j (1 - \frac{1}{\mu_j})\right)}{\mathrm{d}\log\mu_k} = -\lambda_k
$$

at the efficient point, and

$$
\sum_h \chi_h \frac{\mathrm{d}\log\chi_h}{\mathrm{d}\log\mu_i} = 0.
$$

■

Lemma 11.

$$
\frac{\mathrm{d}\,\lambda_j}{\mathrm{d}\log\mu_k} - \sum_h \overline{\chi}_h \frac{\mathrm{d}\log\tilde{\lambda}_j^h}{\mathrm{d}\log\mu_k} = \sum_h \frac{\mathrm{d}\,\chi_h}{\mathrm{d}\log\mu_i} \tilde{\lambda}_j^h - \lambda_i \left(\Psi_{ij} - \delta_{ij} \right).
$$

Proof. Let μ be the diagonal matrix of μ_i and I_{μ_k} be a matrix of all zeros except μ_k for its kth diagonal element. Then for each h,

$$
\tilde{\lambda}^h = b^{(h)} + \tilde{\lambda}^h \mu \Omega.
$$

Hence,

$$\frac{\mathrm{d}\,\tilde{\lambda}^h}{\mathrm{d}\log\mu_k} = \frac{b^{(h)}}{\mathrm{d}\log\mu_k} + \frac{\mathrm{d}\,\tilde{\lambda}^h}{\mathrm{d}\log\mu_k}\mu\Omega + \tilde{\lambda}^h I_{\mu_k}\Omega + \tilde{\lambda}^h\mu\frac{\mathrm{d}\,\Omega}{\mathrm{d}\log\mu_k}.$$

Hence,

$$\overline{\chi}'\frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log\mu_k} = \chi'\frac{b}{\mathrm{d}\log\mu_k} + \chi'\frac{\mathrm{d}\,\tilde{\lambda}}{\mathrm{d}\log\mu_k}\mu\Omega + \chi'\tilde{\lambda}I_{\mu_k}\Omega + \chi'\tilde{\lambda}\mu\frac{\mathrm{d}\,\Omega}{\mathrm{d}\log\mu_k}.$$

On the other hand,

$$\lambda = \chi' b + \lambda\Omega.$$

Form this, we have

$$\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} = \frac{\mathrm{d}\,\chi'}{\mathrm{d}\log\mu_k}b + \chi'\frac{b}{\mathrm{d}\log\mu_k} + \lambda\frac{\mathrm{d}\,\Omega}{\mathrm{d}\log\mu_k} + \frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k}\Omega.$$

Combining these two expressions

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} - \overline{\chi}'\frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right) = \left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} - \overline{\chi}'\frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right)\Omega + \frac{\mathrm{d}\,\chi}{\mathrm{d}\log\mu_k}b - \chi'\tilde{\lambda}^{(h)}I_{\mu_k}\Omega.$$

Rearrange this to get

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} - \overline{\chi}'\frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right) = \frac{\mathrm{d}\,\chi}{\mathrm{d}\log\mu_k}b\Psi - \chi'\tilde{\lambda}^{(h)}I_{\mu_k}(\Psi - I),$$

or

$$\left(\frac{\mathrm{d}\,\lambda}{\mathrm{d}\log\mu_k} - \overline{\chi}'\frac{\mathrm{d}\log\tilde{\lambda}}{\mathrm{d}\log\mu_k}\right) = \frac{\mathrm{d}\,\chi}{\mathrm{d}\log\mu_k}b\Psi - \lambda I_{\mu_k}(\Psi - I).$$

■

Lemma 12. *At the efficient steady-state*

$$\frac{\mathrm{d}\,\lambda_{cj}}{\mathrm{d}\log\mu_k} - \frac{\mathrm{d}\,\tilde{\lambda}_{cj}}{\mathrm{d}\log\mu_k} = -\lambda_{ck}\left(\Psi_{kj} - \delta_{kj}\right).$$

Proof. Start from the relations

$$\lambda_{cj} = F_j^c + \sum_i \lambda_{ci}\Omega_{ij},$$

and

$$\tilde{\lambda}_{cj} = F_j^c + \sum_i \tilde{\lambda}_{ci}\mu_i\Omega_{ij}.$$

Differentiate both to get

$$\frac{\mathrm{d}\,\lambda_{cj}}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_{cj}}{\mathrm{d}\log\mu_k}=\sum_i\left(\frac{\mathrm{d}\,\lambda_{cj}}{\mathrm{d}\log\mu_k}-\frac{\mathrm{d}\,\tilde{\lambda}_{cj}}{\mathrm{d}\log\mu_k}\right)\Omega_{ij}-\lambda_{ck}\Omega_{ki}.$$

Rearrange this to get the desired result. ■

Proof of Proposition 24. Loglinearizing this, we get

$$\Lambda_f\,\mathrm{d}\log\Lambda_f=\mathrm{d}\log w_f\sum_{h\in H}\delta_{hf}\chi_h\varepsilon_h-\sum_{h\in H}(\varepsilon_h-1)\chi_h\delta_{hf}\sum_{l\in F}\delta_{hl}\,\mathrm{d}\log w_l+\sum_{h\in H}\chi_h\delta_{hf}\,\mathrm{d}\log L^h.$$

We can put this back into familiar notation

$$\Lambda_f\,\mathrm{d}\log\Lambda_f=\mathrm{d}\log w_f\sum_{h\in H}\Phi_{hf}\Lambda_f\varepsilon_h-\sum_{h\in H}(\varepsilon_h-1)\Phi_{hf}\Lambda_f\sum_{l\in F}\frac{\Phi_{hl}\Lambda_l}{\chi_h}\,\mathrm{d}\log w_l+\sum_{h\in H}\Phi_{hf}\Lambda_f\,\mathrm{d}\log L^h.$$

Simplify this to get

$$\mathrm{d}\log\Lambda_f=\mathrm{d}\log w_f\sum_{h\in H}\Phi_{hf}\varepsilon_h-\sum_{h\in H}(\varepsilon_h-1)\Phi_{hf}\sum_{l\in F}\frac{\Phi_{hl}\Lambda_l}{\chi_h}\,\mathrm{d}\log w_l+\sum_{h\in H}\Phi_{hf}\,\mathrm{d}\log L^h.$$

We can beautify this a bit as

$$\mathrm{d}\log\Lambda_f=\sum_{h\in H}\varepsilon_h E_{\Phi^{(h)}}\left(E_{\delta^{(h)}}\left(\mathrm{d}\log w_f-\mathrm{d}\log w\right)\right)+\sum_{h\in H}E_{\Phi^{(h)}}\left(E_{\delta^{(h)}}(\mathrm{d}\log w)\right)+\sum_{h\in H}E_{\Phi^{(h)}}\left(\mathrm{d}\log L\right).$$

or

$$\mathrm{d}\log\Lambda_f=\sum_{h\in H}E_{\Phi^{(h)}}\left[\varepsilon_h\left(E_{\delta^{(h)}}\left(\mathrm{d}\log w_f-\mathrm{d}\log w\right)\right)+\left(E_{\delta^{(h)}}(\mathrm{d}\log w)\right)+(\mathrm{d}\log L)\right].$$

or

$$\mathrm{d}\log\Lambda_f=\sum_{h\in H}E_{\Phi^{(h)}}\left[E_{\delta^{(h)}}\left(\varepsilon_h\left(\mathrm{d}\log w_f-\mathrm{d}\log w\right)+(\mathrm{d}\log w)\right)+(\mathrm{d}\log L)\right].$$

The case with immobile labor is given by $\varepsilon_h=1$ for every $h\in H$, in which case $\mathrm{d}\log w_f=\mathrm{d}\log\Lambda_f$. Combine this with demand for the factors to finish the characteriza-

tion

$$\Lambda_l \frac{\mathrm{d}\log \Lambda_l}{\mathrm{d}\log A_k} = \sum_{i\in\{H,N\}} \lambda_j(1-\theta_j) Cov_{\Omega^{(j)}} \left(\Psi_{(k)} + \sum_f \Psi_{(f)} \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)} \right) + \sum_{h\in H} (\lambda_l^h - \lambda_l) \left(\sum_{f\in F_c} \Phi_{hf}\Lambda_f \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k} \right).$$

This means that we can also redo the welfare accounting and write

$$\mathrm{d}\log W_g = \mathrm{d}\log \chi_g - \mathrm{d}\log P_g^c,$$

where χ_g is the income of household g. This can be written as

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F} \left(\delta_{gs} - \Lambda_s^g\right) \mathrm{d}\log w_s + \lambda_k^g \,\mathrm{d}\log A_k + \mathrm{d}\log L^g,$$

or

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F} \left(\frac{\Phi_{gs}}{\chi_g}\Lambda_s - \Lambda_s^g \right) \mathrm{d}\log w_s + \lambda_k^g + \mathrm{d}\log L^g.$$

Using trick by Rodriguez-Clare and Feenstra, we note that

$$\mathrm{d}\log \chi_g = \mathrm{d}\log w_s + \frac{1}{\varepsilon_g}\,\mathrm{d}\log \delta_{gs}$$

for any s, and hence

$$\mathrm{d}\log \chi_g = \sum_f \overline{\Lambda}_f^g \left(\mathrm{d}\log w_f + \frac{1}{\varepsilon_g}\,\mathrm{d}\log \delta_{gf} \right),$$

where $\overline{\Lambda^g}$ and $\overline{\lambda}^g$ are the Domar weights under the closed-economy IO matrix. Then we can combine this with the fact that

$$\mathrm{d}\log P_g^c = \sum_f \overline{\Lambda}_f^g \,\mathrm{d}\log w_f + \sum_i \overline{\lambda}_i \frac{\mathrm{d}\log \lambda_{ii}}{\theta_i - 1}$$

to get

$$\mathrm{d}\log W_g = \sum_f \overline{\Lambda}_f^g \frac{\mathrm{d}\log \delta_{gs}}{\varepsilon_g} + \sum_i \overline{\lambda}_i^g \frac{\mathrm{d}\log \lambda_{ii}}{\theta_i - 1}.$$

■

B Appendix on Beyond CES

The quantitative and empirical trade literature has increasingly worked to weaken functional form assumptions. Thankfully, following Baqaee and Farhi (2018), all the results in this paper can be easily generalized to arbitrary neoclassical production functions simply by replacing the input-output covariance operator with the substitution operator instead. Define

$$\begin{aligned}\Phi_j(\Psi_{(k)},\Psi_{(l)}) &= \left(\sum_{\substack{x,y\\x\neq y}}\Omega_{jx}\Omega_{jy}(1-\sigma^j(x,y))\Psi_{xl}\Psi_{yk}\right),\\ &= \frac{1}{2}E_{\Omega^{(j)}}\left((1-\sigma^j(x,y))(\Psi_k(x)-\Psi_k(y))(\Psi_l(x)-\Psi_l(y))\right),\end{aligned}$$

where $\Psi_k(x) = \Psi_{xk}$. The substitution operator $\Phi_j(\Psi_{(k)},\Psi_{(l)})$ is a strict generalization of the input-output covariance operator, and reduces to the latter when the elasticities of substitution between any two input pairs x and y are constant (the CES special case). Although Φ_j is not strictly a covariance, it still retains many of its properties: namely, it is symmetric and bilinear in its arguments. Furthermore, Φ_j is zero whenever one of its arguments is a constant.

C Small Open Economy

Let Y denote GDP and P_y denote the ideal price index. Let

$$b_i = \frac{p_i c_i}{P_y Y},$$

where $p_i c_i$ is domestic final expenditures on good i. Let

$$\varepsilon_i = \frac{p_i E_i}{P_y Y},$$

where $p_i E_i$ is exports of good i.

Goods market clearing for each domestically produced good is

$$y_i^h = c_i + \sum_j x_{ji} + E_i,$$

and for each foreign produced good

$$m_i = c_i + \sum_j x_{ji},$$

where superscript h and f denote foreign and domestic goods.

Without loss of generality, model decreasing returns by using fixed factors. Then trade balance and household budget balance implies

$$P_y Y = \sum_f w_f L_f,$$

where w_f is the wage of each factor available to the household.

The price of each domestically produced good i is pinned down by the marginal cost function of that good

$$p_i = \frac{1}{A_i} C_i(\{p_j, w_f\}).$$

Note that since foreign prices are exogenous, that means prices are pinned down in levels (not just in relative terms). Everything is being expressed in units of the foreign prices.

Construct the $(N^h + N^f + F) \times (N^h + N^f + F)$ input-output matrix such that

$$\Omega_{ij} = \frac{p_j x_{ij}}{p_i y_i},$$

where the last $N^f + F$ rows and columns are zeros.

We define the domestic and foreign Domar weight of each good:

$$\lambda_k^h = b'(I - \Omega)^{-1} e_k, \tag{110}$$

and

$$\lambda_k^f = \varepsilon'(I - \Omega)^{-1} e_k. \tag{111}$$

By accounting, it has to be the case that

$$\lambda_k = \lambda_k^h + \lambda_k^f, \tag{112}$$

but the break-down between the two requires input-output data.

C.1 Foreign Tariff

Theorem 13. *Then, our generalization of Hulten is*

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\log p_k^f} = -\lambda_k^h + \sum_f (\Lambda_f - \Lambda_f^h)\frac{\mathrm{d}\log w_f}{\mathrm{d}\log p_k^f}.$$

Remark. Completely non-parametric. Doesn't require representative agent assumption. Can be used to do growth-accounting.

Remark. If we assume that final demand and export demand are homothetic, we can derive non-parametric counterfactual formulas. If we assume economy has nested CES form, then we can derive parametric counterfactuals.

Proposition 14. *If the economy belongs to the nested CES class. Then denote final demand by* 0 *and foreign demand by* $N+1$*, and we have*

$$\begin{aligned}\mathrm{d}\,\lambda_i = &\sum_{j=0}^{N+1} \lambda_j(1-\theta_j)Cov_{\Omega^{(j)}}(\Psi_{(k)} + \sum_f \Psi_{(f)}\,\mathrm{d}\log w_f, \Psi_{(i)}) \\ &+ \lambda_i^f \left(\mathrm{d}\log(P_E E) - \mathrm{d}\log(P_y Y)\right).\end{aligned}$$

Also,

$$\mathrm{d}\log P_y Y = \sum_f \Lambda_f\,\mathrm{d}\log w_f, \tag{113}$$

and

$$\mathrm{d}\log w_f = \mathrm{d}\log \Lambda_f + \sum_f \Lambda_f\,\mathrm{d}\log w_f. \tag{114}$$

We can rewrite this as

$$\Lambda_L\,\mathrm{d}\log w_L = \sum_{j=0}^{N+1} \lambda_j(1-\theta_j)Cov_{\Omega^{(j)}}(\Psi_{(k)} + \sum_f \Psi_{(f)}\,\mathrm{d}\log w_f, \Psi_{(L)})$$

$$+ \Lambda_L^f \, \mathrm{d}\log(P_E E) + (\Lambda_L - \Lambda_L^f) \, \mathrm{d}\log(P_y Y).$$

C.2 Domestic Tariff

Theorem 15. *Now suppose that home introduces a tariff on imports, labelled by τ_k^f. Then denote the revenues earned by tariffs on k as a share of GDP by R_k^f. Then we have*

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\,\tau_k^f} = -\lambda_k^h + \sum_f (\Lambda_f - \Lambda_f^h) \frac{\mathrm{d}\log w_f}{\mathrm{d}\log \tau_k^f} + \lambda_k (1 + \tau_k^f) + \sum_i \lambda_i^f \tau_i (\mathrm{d}\log \lambda_i + \mathrm{d}\log P_y Y). \tag{115}$$

At the zero-tariff steady state,

$$\frac{\mathrm{d}\log Y}{\mathrm{d}\,\tau_k^f} = -\lambda_k^h + \sum_f (\Lambda_f - \Lambda_f^h) \frac{\mathrm{d}\log w_f}{\mathrm{d}\log \tau_k^f} + \lambda_k. \tag{116}$$

The accounting convention here is to write the tax separately, so that total expenditures on imports inclusive of taxes is $p_j m_j (1 + \tau_j^f)$.

D SOE as a limit

Divide the world into home, superscript h, and foreign, superscript f. Assume that $\Omega_{ji}^{fh} \to 0$. Then, we can write

$$\mathrm{d}\log p^h = (I - \Omega^{hh})^{-1} (\alpha^h \, \mathrm{d}\log w^h + \Omega^{hf} \, \mathrm{d}\log p^f - \mathrm{d}\log A^h) \tag{117}$$

$$\mathrm{d}\log p^h = (I - \Omega^{ff})^{-1} (\alpha^f \, \mathrm{d}\log w^f - \mathrm{d}\log A^f). \tag{118}$$

For domestic good $\lambda_i^h = \lambda_i / \chi_h$ and for a foreign good $\lambda_i^f = \lambda_i / \chi_h$. Then, market clearing condition for i produced in h is

$$\lambda_i^h = \sum_c \frac{\chi_c}{\chi_h} b_{ci} + \sum_i \Omega_{ji}^{hh} \lambda_j^h + \sum_i \Omega_{ji}^{fh} \lambda_j^f. \tag{119}$$

Differentiate this to get

$$\mathrm{d}\,\lambda_i^h = \sum_c b_{ci} \frac{\chi_c}{\chi_h} \mathrm{d}\log b_{ci} + \sum_c b_{ci} \frac{\chi_c}{\chi_h} \mathrm{d}\log(\chi_c - \chi_h)$$

$$
\begin{aligned}
&+\sum_j \Omega_{ji}^{hh}\,\mathrm{d}\,\lambda_j^h+\sum_j \Omega_{ji}^{hh}\lambda_j^h\,\mathrm{d}\log\Omega_{ji}^{hh}\\
&+\sum_j \Omega_{ji}^{fh}\,\mathrm{d}\,\lambda_j^f+\sum_j \Omega_{ji}^{fh}\lambda_j^f\,\mathrm{d}\log\Omega_{ji}^{fh}\\
&=\sum_l \Psi_{li}^{hh}\left[\frac{\chi_c}{\chi_h}\,\mathrm{d}\log b_{ci}+\sum_c b_{ci}\frac{\chi_c}{\chi_h}\,\mathrm{d}\log(\chi_c-\chi_h)\right.\\
&\left.\sum_j \Omega_{ji}^{hh}\lambda_j^h\,\mathrm{d}\log\Omega_{ji}^{hh}+\sum_j \Omega_{ji}^{fh}\,\mathrm{d}\,\lambda_j^f+\sum_j \Omega_{ji}^{fh}\lambda_j^f\,\mathrm{d}\log\Omega_{ji}^{fh}\right].
\end{aligned}
$$

Note that

$$
\mathrm{d}\log b_{ci}=(1-\theta_c)(\mathrm{d}\log p_i-\mathrm{d}\log P_{Y_c})=(1-\theta_c)(\mathrm{d}\log p_i-b'_{(c)}\,\mathrm{d}\log p) \tag{120}
$$

and

$$
\mathrm{d}\log\Omega_{ji}=(1-\theta_j)(\mathrm{d}\log p_i-\mathrm{d}\log p_j). \tag{121}
$$

Combining these equations, the ones determining prices, and the demand equations we get

$$
\begin{aligned}
\mathrm{d}\,\lambda_i^h&=\sum_l \Psi_{li}^{hh}\left[\frac{\chi_c}{\chi_h}(1-\theta_c)(\mathrm{d}\log p_i-b'_{(c)}\,\mathrm{d}\log p)+\sum_c b_{ci}\frac{\chi_c}{\chi_h}\,\mathrm{d}\log(\chi_c-\chi_h)\right.\\
&\left.\sum_j \Omega_{ji}^{hh}\lambda_j^h(1-\theta_j^h)(\mathrm{d}\log p_i^h-\mathrm{d}\log p_j^h)+\sum_j \Omega_{ji}^{fh}\,\mathrm{d}\,\lambda_j^f+\sum_j \Omega_{ji}^{fh}\lambda_j^f(1-\theta_j^f)(\mathrm{d}\log p_i^h-\mathrm{d}\log p_j^f)\right],\\
&=\sum_l \Psi_{li}^{hh}\left[\frac{\chi_c}{\chi_h}(1-\theta_c)(\Psi_{ik}^{hh}+\sum_f \Psi_{if}^{hh}\,\mathrm{d}\log w_f^h+\sum_m \Psi_{im}^{hf}\,\mathrm{d}\log p_m^f-b'_{(c)}\,\mathrm{d}\log p)\right.\\
&+\sum_c b_{ci}\frac{\chi_c}{\chi_h}\,\mathrm{d}\log(\chi_c-\chi_h)\\
&\sum_j \Omega_{ji}^{hh}\lambda_j^h(1-\theta_j^h)(\Psi_{ik}^{hh}-\Psi_{jk}^{hh}+\sum_f(\Psi_{if}^{hh}-\Psi_{jf}^{hh})\,\mathrm{d}\log w_f^h+\sum_m(\Psi_{im}^{hf}-\Psi_{ji}^{hf})\,\mathrm{d}\log p_m^f)\\
&\left.+\sum_j \Omega_{ji}^{fh}\,\mathrm{d}\,\lambda_j^f+\sum_j \Omega_{ji}^{fh}\lambda_j^f(1-\theta_j^f)(\Psi_{ik}^{hh}+\sum_f \Psi_{if}^{hh}\,\mathrm{d}\log w_f^h+\sum_m \Psi_{im}^{hf}\,\mathrm{d}\log p_m^f-\mathrm{d}\log p_j^f)\right].
\end{aligned}
$$

The expression above holds for every domestically produced good, including the factors, so letting i index the factors in the expressions above and using the fact that $\mathrm{d}\log\Lambda_f^h=$

$\mathrm{d}\log w_f + \mathrm{d}\log \chi_h$ allows us to write a system in $\mathrm{d}\,\Lambda^h$ that we can solve. This system can be simplified to be written in terms of the input-output covariance operator. We don't do that here, but we do note that this system will depend on $\mathrm{d}\log p^f$ *and* $\mathrm{d}\log \lambda_j^f$. In other words, knowing that foreign prices are exogenous is not enough, in general, to apply the usual SOE logic. Foreign price changes will change foreign demand for domestic goods, and these will have effects on output.

E Welfare Accounting

F Hecksher-Ohlin Results

Let R be the set of all nodes whose elasticity of substitution is infinity. Divide factors into groups two groups: $\mathcal{F}$ and its complement $\mathcal{F}^c$. The set of factors $\mathcal{F}$ such that for each $f \in \mathcal{F}$ there is some $j \in R$ and some factor g with

$$Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(f)}) \neq 0, \tag{122}$$

and, some $i \in R$

$$Cov_{\Omega^{(i)}}(\Psi_{(k)}, \Psi_{(f)}) \neq 0. \tag{123}$$

Then, we can take a limit of Proposition 3 to get

Proposition 16. *[Hecksher-Ohlin Models]*

$$\begin{aligned}\Lambda_L \,\mathrm{d}\log \Lambda_L = &\sum_{i \notin R} \lambda_j (1-\theta_j) Cov_{\Omega^{(j)}}(\Psi_{(k)} + \sum_f \Psi_{(f)} \,\mathrm{d}\log \Lambda_f, \Psi_{(L)}) \\ &+ \sum_c \Lambda_L^c \left(\sum_{f \in c} \Lambda_f \,\mathrm{d}\log \Lambda_f \right), \quad (L \notin \mathcal{F})\end{aligned}$$

and

$$0 = \sum_{i \in R} \lambda_j Cov_{\Omega^{(j)}}(\Psi_{(k)} + \sum_f \Psi_{(f)} \,\mathrm{d}\log \Lambda_f, \Psi_{(L)}), \quad (L \in \mathcal{F})$$

and

$$\Lambda' \,\mathrm{d}\log \Lambda = 0. \tag{124}$$

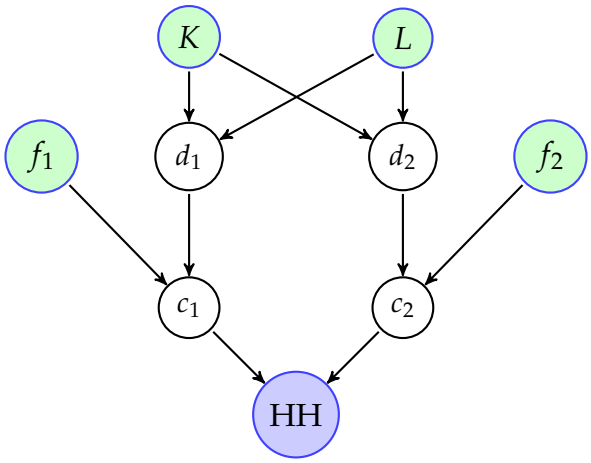


Figure 7: A Small Open Economy

If every factor belongs to $\mathcal{F}$, then we do not need to know the elasticities of substitution in order to predict the response of factor shares to shocks.

Generalized Rybczynski

Corollary 4. *Suppose that every factor $f \in \mathcal{F}$, then*

$$\begin{pmatrix} \sum_j \lambda_j Var_{\Omega^{(j)}}(\Psi_{(1)}) & \cdots & \sum_j \lambda_j Cov_{\Omega^{(j)}}(\Psi_{(1)}, \Psi_{(F)}) \\ & \ddots & \vdots \\ & & \sum_j \lambda_j Var_{\Omega^{(j)}}(\Psi_{(F)}) \end{pmatrix} \frac{\mathrm{d}\log\Lambda}{\mathrm{d}\log A_k} = - \begin{pmatrix} \sum_j \lambda_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(1)}) \\ \vdots \\ \sum_j \lambda_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(F)}) \end{pmatrix},$$

where $j \in R$. In addition, we have $\Lambda\,\mathrm{d}\log\Lambda = 0$. If the shock $\mathrm{d}\log A_k$ is to a primary factor, then this system pins down the change in relative factor prices in response to a change in factor quantity.

Intuitively, on impulse, the changes in factor demands are determined entirely by the demand of those nodes which are perfect substitutes, since they substitute much more aggressively to the impulse. Furthermore, the change in relative factor prices has to be such that it balances out the excess demand caused by the initial impulse since the quantity of factors is fixed. When the above system can be solved, there is a unique combination of factor price changes that would balance out the excess demand.

Stolper-Samuelson in 2×2 Model

The generalized Rybcynski result, Corollary 4, implies the Stolper and Samuelson (1941) result: in a two-good, two-factor small-open-economy, an increase in the price of good i will raise the returns to the factor that i uses relatively more intensively. To see this, consider the example depicted in Figure 7. There are two consumption goods c_1 and c_2. Each consumption good c_i could either be produced domestically d_i using capital K and labor L or purchased directly from foreign f_i. In other words, the elasticity of substitution between d_i and f_i is infinity. Now, consider an increase in the price of the foreign good i, then Corollary 4 implies

$$\frac{\mathrm{d}\log w}{\mathrm{d}\log p_{f_1}} = -\left(\sum_{j=c_1}^{c_2}\lambda_j\begin{bmatrix} Var_{\Omega^{(j)}}(\Psi_{(L)}) & Cov_{\Omega^{(j)}}(\Psi_{(L)},\Psi_{(K)}) \\ Cov_{\Omega^{(j)}}(\Psi_{(L)},\Psi_{(K)}) & Var_{\Omega^{(j)}}(\Psi_{(K)}) \end{bmatrix}\right)^{-1}\begin{bmatrix} \lambda_{c_1}Cov_{\Omega^{(c_1)}}(\Psi_{(i)},\Psi_{(L)}) \\ \lambda_{c_1}Cov_{\Omega^{(c_1)}}(\Psi_{(i)},\Psi_{(K)}) \end{bmatrix}, \tag{125}$$

where $\mathrm{d}\log w = (\mathrm{d}\log w_L, \mathrm{d}\log w_K) = \mathrm{d}\log\Lambda$, since world GDP is constant by the small open economy assumption. The above expression holds *regardless* of the elasticity of substitution between L and K or between c_1 and c_2. Specifically, the expression above simplifies to give the same answer as that given by the celebrated "hat" algebra of Jones (1965). Namely:

$$\frac{\mathrm{d}\log w}{\mathrm{d}\log p_{f_1}} = \begin{bmatrix} \Omega_{d_1L} & \Omega_{d_1K} \\ \Omega_{d_2L} & \Omega_{d_2K} \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{126}$$

The intuition for equation (125) is a little different to the one for Jones' algebra, even though the two equations give the same answer. In (125), the term

$$\begin{bmatrix} \lambda_{c_1}Cov_{\Omega^{(c_1)}}(\Psi_{(i)},\Psi_{(L)}) \\ \lambda_{c_1}Cov_{\Omega^{(c_1)}}(\Psi_{(i)},\Psi_{(K)}) \end{bmatrix}$$

measures how substitution by c_1 would redirect demand away from L and K, holding fixed their prices, whilst

$$\left(\sum_{j=c_1}^{c_2}\lambda_j\begin{bmatrix} Var_{\Omega^{(j)}}(\Psi_{(L)}) & Cov_{\Omega^{(j)}}(\Psi_{(L)},\Psi_{(K)}) \\ Cov_{\Omega^{(j)}}(\Psi_{(L)},\Psi_{(K)}) & Var_{\Omega^{(j)}}(\Psi_{(K)}) \end{bmatrix}\right)\mathrm{d}\log w$$

measures how a change in the price of the factors would redirect demand between L and K, due substitution by c_1 and c_2. In equilibrium, the sum of these two has to be zero, which gives rise to (125). On the other hand, Jones's algebra follows from the observation

that, in equilibrium, $\mathrm{d}\log p_{d_1} = \mathrm{d}\log p_{f_1} = 1$ and $\mathrm{d}\log p_{d_2} = \mathrm{d}\log p_{f_2} = 0$. The result then follows from combining these facts with $\mathrm{d}\log p_{d_i} = \Omega_{d_i L}\,\mathrm{d}\log w_L + \Omega_{d_i K}\,\mathrm{d}\log w_K$ for $i = 1, 2$. This means that individually, the components of (125) are not the same as the ones in (126), even though they ultimately imply the same thing.

Intuitively, Jones' algebra starts off by imposing that the relative price of domestic and foreign goods must not change, whilst Corollary 4 assumes that demand for factors is differentiable. Importantly, both approaches require *only* expenditure data.

G Welfare Accounting Per Capita

Our equations take the number of households as given. If one were to compute our welfare accounting decomposition using consumer data aggregated over a changing number of consumers, then one has to take into account the fact that the number of consumers is changing over time. To do this, suppose that a household type h actually consists of N_h identical agents indexed by i, consuming the same basket and in possession of the same factors of production. In that case, we can write

$$W_h(A, \mathcal{X}) = \sum_{i=1}^{N_h} W_{ih}(A, \mathcal{X}) = N_h W_{ih}(A, \mathcal{X}).$$

Therefore,

$$\frac{\partial \log W_h}{\partial \log A_k} = \frac{\partial \log W_{ih}}{\partial \log A_k} = \lambda_k^h,$$

whence

$$\frac{\partial \log W_{ih}}{\partial \mathcal{X}}\,\mathrm{d}\,\mathcal{X} = \frac{\partial \log W_h}{\partial \mathcal{X}}\,\mathrm{d}\,\mathcal{X} - \mathrm{d}\log N.$$

Using these formulas, we can transform our group-level welfare results into individual-level (or per capita) welfare results.

H Transfer Shocks

Next, we turn our attention to shocks to net transfers between countries.

Proposition 17. *[Transfer Shocks] For some feasible perturbation of transfers* $\sum_{i \in H} \mathrm{d}\,\tau_h = 0$,

$$\lambda_l \,\mathrm{d}\log\lambda_l = \sum_{i\in\{H,N\}} \lambda_j(1-\theta_j) Cov_{\Omega^{(j)}}\left(\sum_f \Psi_{(f)}\,\mathrm{d}\log\Lambda_f, \Psi_{(l)}\right) + \sum_{c\in H}(\lambda_l^c - \lambda_l)\left(\sum_{f\in F}\Phi_{cf}\Lambda_f\,\mathrm{d}\log\Lambda_f\right) + \sum_{h\in H}\lambda_l^h\,\mathrm{d}\,\tau_h.$$

Once again, when l is a factor, Proposition 17 gives us a linear system in $\mathrm{d}\log\Lambda$ whose fixed point pins down the equilibrium factor share movements:

$$\mathrm{d}\log\Lambda = \Gamma\,\mathrm{d}\log\Lambda + \Theta\,\mathrm{d}\log\Lambda + \lambda^h\,\mathrm{d}\,\tau, \tag{127}$$

where Γ and Θ are exactly the same as before, but now the impulse is simply the change in factor demands in partial equilibrium (holding fixed the price of factors). The model's propagation properties depend on Γ and Θ, and conditional on the initial impulse, are the same for transfers and productivity shocks. Once again, given $\mathrm{d}\log\Lambda$, and $\mathrm{d}\,\lambda$, we can combine these with Theorem 2

I Example: The International Quesnaysian Cross

Consider the effects of an improvement in productivity in one country on the distribution of income across both countries. Consider the economy depicted in Figure 8. Suppose that C_1 is China and C_2 is the USA. There are three goods in the world: d_1 and d_2 are "domestic" goods: produced from domestic labor and sold only locally. The good m is a multi-national good, produced from both types of labor and sold to both consumers. Finally, we assume that the elasticity of substitution between Chinese and American labor is σ for multinationals, while the elasticity of substitution between traded and non-traded goods is 1.[25]

Now, we consider an improvement in the productivity of Chinese labor L_1. Then, from Proposition 2, we know that Chinese labor's share of income has to satisfy the fixed

[25]The assumption that the consumption of households is Cobb-Douglas simplifies the algebra slightly, but it would be trivial to relax.

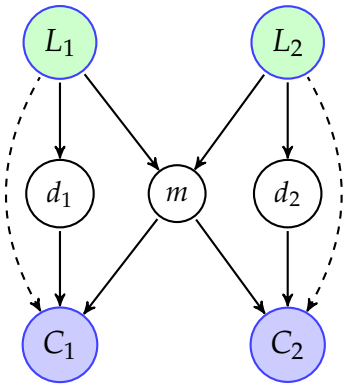


Figure 8: The solid arrows represent the flow of goods. The dashed arrows represent the flow of wages to households.

point

$$\frac{\mathrm{d}\,\Lambda_{L_1}}{\mathrm{d}\log L_1} = (\sigma-1)\lambda_m\omega_{mL_1}\omega_{mL_2} - \frac{(\sigma-1)}{\Lambda_{L_1}\Lambda_{L_2}}\lambda_m\omega_{mL_1}\omega_{mL_2}\frac{\mathrm{d}\,\Lambda_{L_1}}{\mathrm{d}\log L_1} + \sum_i \omega_{iL_1}(b_{c_1 i} - b_{c_2 i})\frac{\mathrm{d}\,\Lambda_{L_1}}{\mathrm{d}\log L_1},$$

This equation is plotted in Figure 9, which Baqaee and Farhi (2018) call a Quesnaysian cross. The first term corresponds to the substitution by m in response to the change in the productivity of Chinese labor, holding the relative wage constant. The second term corresponds to the fact that the relative wages will adjust in response to substitution by m, and this change in relative wages will cause m to substitute further. The final term corresponds to the fact that Chinese and American consumers have different preferences, due to home-bias, and hence, as Chinese consumers become richer, demand for Chinese labor increases, which bids up the wages of Chinese workers. The first term is the partial equilibrium effect, while the last two terms are general equilibrium terms. The general equilibrium response may amplify or mitigate the partial equilibrium effect depending on the sum of the substitution and distribution effect.

J Duality with Multiple Factors

The isomorphism between trade shocks in an open economy and productivity shocks in a closed economy extends beyond the one factor economy. In the multi-factor case, a trade shock is isomorphic to a series of productivity and factor endowment shocks in the closed economy. In this section, we establish this isomorphism, and as an example, we show how the model in Galle et al. (2017), which studies the distributional consequences

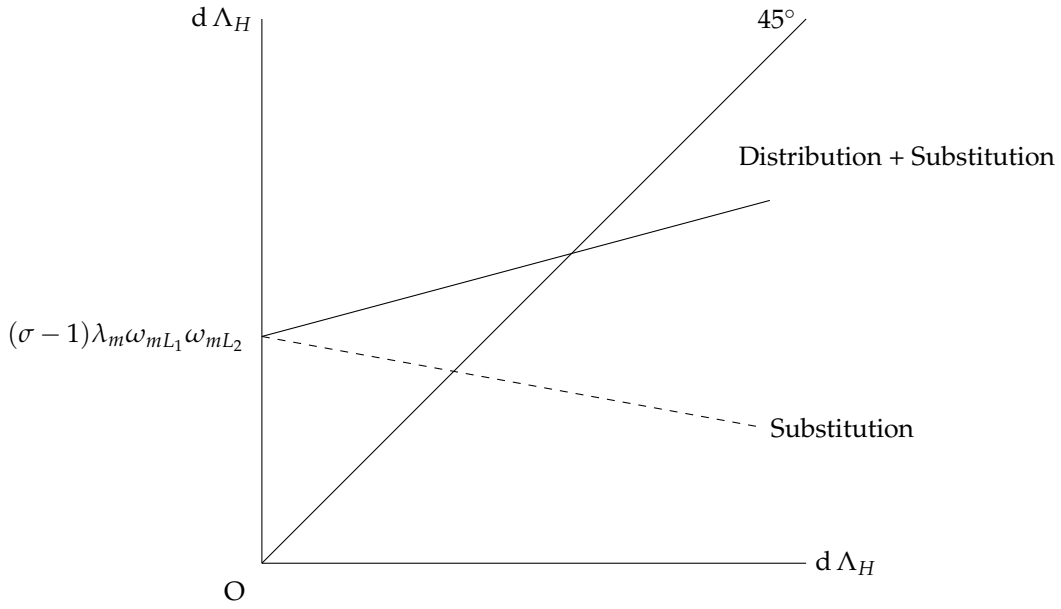


Figure 9: The "Quesnaysian Cross": the substitution line has slope $\frac{(\sigma-1)}{\Lambda_{L_1}\Lambda_{L_2}}\lambda_m\omega_{mL_1}\omega_{mL_2}$; and the distribution + substitution line has slope $\frac{(\sigma-1)}{\Lambda_{L_1}\Lambda_{L_2}}\lambda_m\omega_{mL_1}\omega_{mL_2} + \sum_i \omega_{iL_1}(b_{1i} - b_{2i})$.

of trade with a Roy model, can be generalized to economies with production networks.

Assume that the domestic economy has multiple factors of production. Let $W_c(z)$ denote real income of a country c as a function of iceberg trade costs z. Let $\check{P}_c(\check{A}, \check{L})$ denote the consumption price index of the closed-economy counterpart of c as a function of fictitious productivity parameters $\check{A}$ and factor prices $\check{w}$. Since the "caron" economy is closed, real income is equal to real output $\check{W}_c(\check{A}, \check{L}) = \check{Y}_c(\check{A}, \check{L})$.

Proposition 18 (Isomorphism)**.** *Consider a change in iceberg trade costs from $\overline{z}$ to z, then*

$$\Delta \log W_c = -\log\left(\frac{\check{P}_c(\check{A}, \check{w})}{\check{P}_c(\overline{\check{A}}, \check{w})}\right).$$

where

$$\Delta \log \check{A}_i = \log\left(\frac{\check{A}_i}{\overline{\check{A}}_i}\right) = \frac{1}{\zeta_i - 1}\log\left(\frac{\sum_{j\in N_c}\Omega_{ij}}{\sum_{j\in N_c}\overline{\Omega}_{ij}}\right),$$

and

$$\Delta \log \check{w}_f = \log\left(\frac{\check{w}_f}{\overline{\check{w}}_f}\right) = \log\left(\frac{\Lambda_{cf}}{\overline{\Lambda}_{cf}}\right).$$

In words, the effect of iceberg trade shocks on *income* in an *open economy* are isomor-

phic to the effects of productivity and factor price shocks on the *price level* in a fictitious *closed economy*. Given this isomorphism to a closed-economy, we can use results from the literature on closed-economy models to characterize the effects of trade shocks.

As before, we begin by looking at the first-order effects:

Corollary 5 (First-Order)**.** *When Proposition 18 holds, a first-order approximation to the change in welfare from iceberg shocks* $\Delta \log z$ *is*

$$\Delta \log W \approx \sum_{i \in N_c} \check{\lambda}_i \Delta \log \check{A}_i + \sum_{i \in F_c} \check{\Lambda}_f \Delta \log \check{w}_f, \tag{128}$$

where $\mathrm{d} \log \check{P}_c / \mathrm{d} \log \check{A}_i = \check{\lambda}_i$ *and* $\mathrm{d} \log \check{P}_c / \mathrm{d} \log \check{w}_f = \check{\Lambda}_f$ *by Hulten (1978).*

Hence, conditional on the size of the implied productivity shocks, the presence of intermediate inputs amplifies the effects of trade shocks much in the same way that the effect of intermediate inputs amplifies the effects of productivity shocks in closed economies. This is due to the intermediate input multiplier effect discussed by, among others, Jones (2011).

Next, we consider the second-order effect of these shocks.

Corollary 6 (Second-Order)**.** *When Proposition 18 holds, a second-order approximation to the change in welfare from iceberg shocks* $\Delta \log z$ *is*

$$\Delta \log W \approx \sum_{i \in N_c, F_c} \check{\lambda}_i \Delta \log \check{A}_i + \frac{1}{2} \sum_{i \in \{F_c, N_c\}} \sum_{j \in \{F_c, N_c\}} \frac{\partial \log \check{Y}_c}{\partial \log A_i \partial \log A_j} \Delta \log \check{A}_i \Delta \log \check{A}_j. \tag{129}$$

From Baqaee and Farhi (2017b),

$$\frac{d^2 \log \check{W}_c}{\mathrm{d} \log \check{A}_j \, \mathrm{d} \log \check{A}_i} = \frac{\mathrm{d} \check{\lambda}_i}{\mathrm{d} \log \check{A}_j} = \sum_{k \in N_c} (1 - \theta_k) \check{\lambda}_k Cov_{\check{\Omega}^{(k)}} \left(\check{\Psi}_{(i)}, \check{\Psi}_{(j)} \right),$$

whenever the model has a nested CES form. Here, we abuse notation, so that whenever $i \in F_c$*, then* $\lambda_i = \Lambda_i$ *and* $\Delta \log \check{A}_i = \Delta \log \check{w}_i$*.*

Hence, nonlinearities in consumption and production can act to amplify or mitigate the effects of the shock relative to the first-order approximation. Cobb-Douglas ($\theta_j = 1$ for all j) is the case that keeps the isomorphic closed economy log-linear.

K Network Irrelevance

A noteworthy special case is when i and j are made directly from factors, without any intermediate inputs. Then, we have the following

Corollary 7. *(Network Irrelevance) If λ_{ij} and λ_{ii} are only made from domestic factors, then*

$$\sum_{m\in C,N} \lambda_m Cov_{\Omega^{(m)}}(\Psi_{(ij)}, \Psi_{(ij)}/\lambda_{ij} - \Psi_{(ii)}/\lambda_{ii}) = 1. \tag{130}$$

Hence, if all microeconomic elasticities of substitution θ_m are equal to the same value $\theta_m = \theta$ then $\varepsilon_{ij,ij} = \theta$.

In the special case where ij and ii are both made only from domestic factors (no intermediate inputs are permitted), then the trade elasticity of imports from j to i with respect to trade costs between i and j become convex combinations of the underlying microelasticities. Of course, whenever all micro-elasticities of substitution are the same, the weights then become irrelevant, and this is the situation in most benchmark trade models with constant trade elasticities.

[XX: REMOVE THE FOLLOWING??] Baqaee and Farhi (2017b) show that in a closed-economy with a representative agent, the network-structure is irrelevant for studying aggregate outcomes as long as all micro elasticities of substitution are the same and shocks are factor-augmenting. For a closed-economy, this result fails due to heterogeneity of household preferences. Instead, we have the following:

Corollary 8 (Network Irrelevance)**.** *If $\theta_i = \theta$ for all i, and there is no home-bias, so that $\Lambda_l^h = \Lambda_l$ for every h, then for a factor augmenting shock* $\mathrm{d}\log L_k$

$$\Lambda_L \frac{\mathrm{d}\log \Lambda_L}{\mathrm{d}\log L_k} = (\theta - 1)\left(\Lambda_l(\mathbf{1}(l=k) - \Lambda_k)\right) + (1-\theta)\sum_{f\in F}\left(\Lambda_l(\mathbf{1}(l=f) - \Lambda_f)\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log L_k}\right).$$

Hence

L Nested-CES Models in Standard Form

Nested-CES models in written in what we call *standard form*, which is more convenient to study. For a formal description of this standard form see the discussion in Baqaee and Farhi (2017b) for more details. In short, a CES economy in standard form is one where we

treat each CES aggregator as a node in the production network, with an accompanying elasticity of substitution. Through a relabelling, this structure can represent any CES economy with an arbitrary pattern of nests and elasticities. Intuitively, by relabelling each CES aggregator to be a new producer, we can have as many nests as desired. For the rest of the paper, and without loss of generality, we restrict our attention to nested-CES models in standard-form.

M Propagation Equations with Distortions

So far, we have maintained the assumption of perfect competition. In this section, we extend our results to allow for tariffs, or more generally distortions (modelled via implicit or explicit taxes). We generalize our growth-accounting and propagation equations to distorted economies. Using these results, we briefly outline how disaggregation modifies the traditional "Harbgerger triangle" intuitions about the effects of tariffs,[26] and go through some examples motivated by recent events.

Since any wedge can be represented as a markup, without loss of generality, we assume that all wedges in the economy (including tariffs) have been represented as markups.[27] Denote the diagonal matrix of markups as μ and, following Baqaee and Farhi (2017c), define the *cost-based* HAIO matrix

$$\tilde{\Omega} = \mu\Omega. \tag{131}$$

If we use $\tilde{\Omega}$ in place of Ω, for each good k and country c and household h, we can define cost-based versions of the Domar weight $\tilde{\lambda}_k$, local Domar weights $\tilde{\lambda}_{ck}$, and household exposures $\tilde{\lambda}_k^h$. Simiarly, we can define these objects for the factors: $\tilde{\Lambda}$, $\tilde{\Lambda}_{ck}$ and $\tilde{\Lambda}_k^h$. Finally, it's convenient to introduce "fictitious" factors corresponding to the revenues earned by country c on the sale of each good as the result of the markup wedge. Denote the set of all factors (real and fictitious) by F^*. The fictitious factors have a cost-based Domar weight of zero, but a positive revenue-based Domar weight.

Now, we can state our comparative statics using ex-post sufficient statistics.

[26] See, for example, *Does Trade in Intermediate Goods Alter the Logic of Costs From Protectionism?* by Paul Krugman (2018).

[27] To represent a wedge on i's ability purchase inputs from k, we can introduce a new producer which buys from k and sells to i at a markup.

Proposition 19. *In response to shocks,*

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log \mu_k} = -\tilde{\lambda}_k^h - \sum_{f=1}^{F} \tilde{\Lambda}_f^h \,\mathrm{d}\log \Lambda_f + \sum_{f\in F_c^*} \Phi_{hf}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f. \tag{132}$$

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_k} = \tilde{\lambda}_k^h - \sum_{f=1}^{F} \tilde{\Lambda}_f^h \,\mathrm{d}\log \Lambda_f + \sum_{f\in F_c^*} \Phi_{hf}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f. \tag{133}$$

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log \mu_k} = -\tilde{\lambda}_{ck} - \sum_{f\in F_c} \tilde{\Lambda}_{cf}\,\mathrm{d}\log\Lambda_{cf} + \sum_{i\notin N_c}\left(\lambda_{ci}-\tilde{\lambda}_{ci}\right)\left(\mathrm{d}\log\lambda_{ci} - \mathrm{d}\log m_i\right), \tag{134}$$

$$\frac{\mathrm{d}\log Y_c}{\mathrm{d}\log A_k} = \tilde{\lambda}_{ck} - \sum_{f\in F_c}^{F} \tilde{\Lambda}_{cf}\,\mathrm{d}\log\Lambda_{cf} + \sum_{i\notin N_c}\left(\lambda_{ci}-\tilde{\lambda}_{ci}\right)\left(\mathrm{d}\log\lambda_{ci} - \mathrm{d}\log m_i\right), \tag{135}$$

where λ_{ci} and m_i for $i \notin N_c$ is the expenditure share and quantity of foreign imports i. Proposition 19 generalizes productivity and welfare accounting to open economies with distortions. These expressions maintain their interpretation as decompositions of welfare and output into changes due to "pure" technology and changes due to reallocation of resources. We discuss these equations in turn, starting with the welfare accounting equations (132) and (133).

The welfare equations are remarkably similar to the efficient economy versions in Theorem 19, the key difference is they use cost-based exposures rather than revenue-based exposures, and they account for the contributions of income from rents. As before, the first summand of (132) is the "pure" productivity effect (the size of the pie), while the rest of the terms account for changes in the shares of the pie (i.e. reallocations):

$$\mathrm{d}\log W_h = \underbrace{\tilde{\lambda}^h\,\mathrm{d}\log A}_{\frac{\partial\log W_h}{\partial\log A}\,\mathrm{d}\log A} + \underbrace{\left(\sum_{f\in F_c^*}\Phi_{hf}\frac{\Lambda_f}{\chi_c}\,\mathrm{d}\log\Lambda_f - \sum_{f=1}^{F}\tilde{\Lambda}_f^h\,\mathrm{d}\log\Lambda_f - \tilde{\lambda}^h\,\mathrm{d}\log\mu\right)}_{\frac{\partial\log W_h}{\partial\mathcal{X}}\,\mathrm{d}\,\mathcal{X}}.$$

Interestingly, if we aggregate these equations across all households, weighted by their income share, we recover precisely the Baqaee and Farhi (2017c) in the closed economy – that is a decomposition of the aggregate real income:

$$\sum_h \mathrm{d}\log W = \sum_h \chi_h\,\mathrm{d}\log W_h = \tilde{\lambda}\,\mathrm{d}\log A - \left(\tilde{\Lambda}\,\mathrm{d}\log\Lambda + \tilde{\lambda}\,\mathrm{d}\log\mu\right).$$

Now, let's our attention to the productivity equations (134) and (135). The productivity

equations are much more complicated than their efficient-economy counterpart (namely Hulten's theorem). This is because, even in a closed-economy, Hulten's theorem no longer holds in the presence of distortions. Changes in allocative efficiency (caused by $\mathrm{d}\,\mathcal{X}$) affects the evolution of aggregate productivity directly.[28] In this case, changes in real GDP are no longer measured by integrating a path integral in the vector field of the Domar weights. This is because (i) revenue-based local Domar weights λ_{ck} no longer capture the partial derivative of GDP with respect to productivity shocks (which are instead given by $\tilde{\lambda}_{ck}$, (ii) reallocations can have first-order effects on output since the first welfare theorem no longer holds.

If we aggregate real GDP across all countries, weighted by their share of output, we get

$$\mathrm{d}\log Y = \sum_c \frac{GDP_c}{GDP}\log Y_c = \tilde{\lambda}\,\mathrm{d}\log A - \left(\tilde{\Lambda}\,\mathrm{d}\log\Lambda + \tilde{\lambda}\,\mathrm{d}\log\mu\right) = \mathrm{d}\log W.$$

Hence, as we might expect, once we aggregate across everybody, real income and real output are, once again, the same — even in distorted economies. Interestingly, these formulas show how the reallocation of resources amongst consumers by tariffs come at the cost of productive efficiency, meaning that typically, the winners in a trade war gain less than the losers lose. There is one interesting, and perhaps obvious, exception to this dictum: at the efficient steady-state, tariffs are purely redistributive, so that the winners gain exactly the same amount that the losers lose.

Proposition 19 gives a unified framework for growth and productivity accounting in open, closed, distorted, and undistorted economies. In addition, it shows how we can construct counterfactuals conditional from information about changes in expenditure shares (namely factor shares). Now, we supplement Proposition 19 with propagation equations for the distorted economy.

Using results from Baqaee and Farhi (2018), we can determine how factor shares will respond to shocks as a function of the micro-economic elasticities of substitution and the cost-based and revenue-based input-output matrices $\tilde{\Omega}$ and Ω.

Proposition 20 (Baqaee and Farhi, 2018)**.** *In response to some increase in a markup/wedge, we*

[28]The growth-accounting formula in Baqaee and Farhi (2017c) is a trivial corollary of Theorem 2 when applied to the world economy.

have

$$\begin{aligned}\frac{\mathrm{d}\log\lambda_i}{\mathrm{d}\log\mu_k} &= \sum_j (1-\theta_j)\frac{\lambda_j}{\lambda_i}\mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(i)}) \\ &+ \sum_{g\in F}\sum_j (1-\theta_j)\frac{\lambda_j}{\lambda_i}\mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_g, \Psi_{(i)})\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log\mu_k} \\ &- \frac{\lambda_k}{\lambda_i}\left(\Psi_{ki} - \mathbf{1}(k=i)\right) + \frac{1}{\lambda_i}\sum_c (\lambda_i^c - \lambda_i)\chi_c \frac{\mathrm{d}\log\chi_c}{\mathrm{d}\log\mu_k}.\end{aligned}$$

Income changes are given by

$$\chi_c \frac{\mathrm{d}\log\chi_c}{\mathrm{d}\log\mu_k} = \left(\sum_{g\in F^*}\Phi_{cg}\Lambda_g \frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log\mu_k}\right), \tag{136}$$

where g sums over real and fictitious factors.[29] *The changes in Domar weight of real factors is given by*

$$\begin{aligned}\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log\mu_k} &= \sum_j (1-\theta_j)\frac{\lambda_j}{\Lambda_g}\mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(k)}, \Psi_{(g)}) \\ &+ \sum_{f\in F}\sum_j (1-\theta_j)\frac{\lambda_j}{\Lambda_g}\mu_j^{-1} Cov_{\tilde{\Omega}^{(j)}}(\tilde{\Psi}_{(f)}, \Psi_{(g)})\frac{\mathrm{d}\log\Lambda_f}{\mathrm{d}\log\mu_k} \\ &- \frac{\lambda_k}{\Lambda_g}\left(\Psi_{kg} - \mathbf{1}(k=g)\right) + \frac{1}{\Lambda_g}\sum_c \chi_c(\Lambda_g^c - \Lambda_g)\frac{\mathrm{d}\log\chi_c}{\mathrm{d}\log\mu_k}.\end{aligned}$$

The changes in the Domar weight of fictitious factors is given by

$$\frac{\mathrm{d}\log\Lambda_i^*}{\mathrm{d}\log\mu_k} = \mathrm{d}\log\lambda_i + \frac{1}{\mu_i - 1}\mathbf{1}(i=k), \tag{137}$$

as long as where Λ_{i^*} is the income earned by the markup placed on the sales of the ith good.

[29] This formula assumes that at steady-state, any markup that is perturbed is generating positive revenue otherwise there is a singularity. In that case we need to use $\chi_c \frac{\mathrm{d}\log\chi_c}{\mathrm{d}\log\mu_k} = \sum_{g\in F_c}\Phi_{cg}\Lambda_g\,\mathrm{d}\log\Lambda_g + \sum_{i\in F_c^*}\mathrm{d}(\lambda_i(1-1/\mu_i)) = \left(\sum_{g\in F}\Phi_{cg}\Lambda_g\frac{\mathrm{d}\log\Lambda_g}{\mathrm{d}\log\mu_k}\right) + \sum_{g^*\in F_c^*}\left(\Lambda_{g^*}\,\mathrm{d}\log\lambda_g + \lambda_{g^*}/\mu_{g^*}\right)$, where λ_{g^*} is the Domar weight of the set of goods which are taxed.

N Costs of Tariffs Continued

Proposition 21. *At the efficient (no wedge) equilibrium, up to a second-order approximation,*

1. *Country c real GDP:*

$$-\Delta \log Y_c \approx \mathcal{L}_I^c + \mathcal{L}_X^c + \mathcal{L}_H^c, \tag{138}$$

where

$$\mathcal{L}_{\mathcal{I}} = \frac{1}{2} \sum_{l \in N_c} \sum_{k \in N} (\mathrm{d} \log \mu_l)(\mathrm{d} \log \mu_k) \sum_j \lambda_{cj} \theta_j Cov_{\Omega^{(j)}}(\Psi_{(k)}, \Psi_{(l)}), \tag{139}$$

$$\mathcal{L}_X = -\frac{1}{2} \sum_{l \in N_c} \sum_g \mathrm{d} \log \mu_l \, \mathrm{d} \log \Lambda_g \sum_j \lambda_{cj} \theta_j Cov_{\Omega^{(j)}}(\Psi_{(g)}, \Psi_{(l)}), \tag{140}$$

$$\mathcal{L}_{\mathcal{H}} = -\frac{1}{2} \sum_{l \in N_c} \sum_{h \in H} (\mathrm{d} \log \mu_l)(d \log \chi_h) \chi_h (\lambda_{cl}^h - \lambda_{cl}). \tag{141}$$

2. *World welfare:*

$$-\Delta \log W^{BS} \approx \mathcal{L}_I + \mathcal{L}_X + \mathcal{L}_H + \mathcal{L}_W, \tag{142}$$

where

$$\mathcal{L}_{\mathcal{W}} = -\frac{1}{2} \sum_{h \in H} \chi_h \, \mathrm{d} \log \chi_h \, \mathrm{d} \log P_{cpi,h}. \tag{143}$$

O Case Study: Tariffs in Supply Chains

Recently, the Trump administration imposed tariffs on imports of steel and automobiles. Some commentators question the logic of imposing tariffs on imported steel. The concern is that since steel is an intermediate input, the tariff can hurt US manufacturers who use foreign steel as an input. Analyzing particular policy changes, like the one considered and enacted by the US, requires a detailed and calibrated model. However, to illustrate some of the forces that are going to matter for such a policy exercise, we study a simple example. In Section 7, we use a fully calibrated model to quantify these forces in a more realistic environment. The intuitions we glean from the simple example are helpful in understanding what drives the quantitative results to come.

Simple Example

Consider the economy depicted in Figure 11. For simplicity, we start at the efficient equilibrium with no tariffs.[30] Each country produces tradeable automobiles, which are consumption goods (A and A^*), and possesses an endowment of steel (S and S^*) which can be used to produce automobiles.

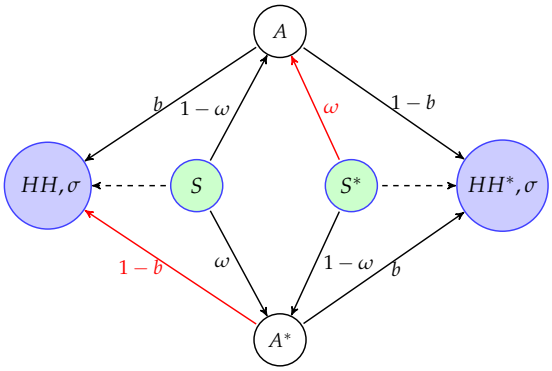


Figure 10: The solid lines show the flow of goods while the dashed lines show the flow of wage payments. Each node label gives the name of the node, as well as its elasticity of substitution. Potential tariffs are colored red.

For simplicity, we consider the symmetric case. We consider two types of taxes, either a tariff on foreign automobiles or a tariff on foreign steel. In either case, since we start from an initially efficient equilibrium, the change in allocative efficiency is zero, and the policy is a *pure* distribution of resources (a zero sum game):

$$\mathrm{d}\log W + \mathrm{d}\log W^* = 0. \tag{144}$$

Ordinarily, tariff policy is a negative sum game due to the misallocation caused by the tariffs. To simplify the algebra, we assume that $\omega = 1/2$.

We start with the consumption tariff τ_{A^*}. In that case,

$$\mathrm{d}\log W = \mathrm{d}\log W^* = 0. \tag{145}$$

Intuitively, this follows from the fact that the tariff has no effect on the relative share of

[30] This simplifies the intuition since around the no-tariff equilibrium, distortions caused by tariffs (departures from the productive possibility frontier) are zero to a first-order. See Corollary 1 in Baqaee and Farhi (2017c).

income claimed by domestic factor versus the foreign factor:

$$\mathrm{d}\log\Lambda_S = \mathrm{d}\log\Lambda_{S^*} = -\frac{1}{2}(1-b) < 0. \tag{146}$$

Since both the foreign and domestic carmakers are symmetrically exposed to both factors, the effect on factor prices is symmetric. So, the foreign consumer is hurt by the reduction in the price of the factor they own and helped by the reduction in the price of the foreign factor (and these effects exactly cancel). The domestic consumer is symmetrical. On the other hand,

$$\mathrm{d}\log y_A = \frac{1}{2}(1-b)(2b(\sigma-1)+1) > 0, \tag{147}$$

as long as $\sigma > 1$.

Now consider a tax on intermediate imports of steel τ_{S^*}. In this case,

$$\mathrm{d}\log W = \frac{1}{2}(1-b) + \frac{1}{4} = -\,\mathrm{d}\log W^* > 0, \tag{148}$$

since

$$\mathrm{d}\log\Lambda_S = 0, \tag{149}$$

whereas

$$\mathrm{d}\log\Lambda_{S^*} = -\frac{1}{2}. \tag{150}$$

The tariff on steel shifts demand away from foreign steel, and so the price of foreign steel falls, because it is inelastically supplied. This helps the domestic consumer and hurts the foreign consumer. However,

$$\mathrm{d}\log y_A = -b(1-b)(\sigma-1) - \frac{1}{2}(1-b) - \frac{1}{4} < 0, \tag{151}$$

so that domestic automobile production always shrinks.

More Complex Example

To simplify the algebra, suppose now that production of cars is Cobb-Douglas between labor and steel, and that there is no home-bias in consumption of cars or steel, so that $\omega = b = 1/2$. Then, in response to a consumption tax τ_{A^*}

$$\mathrm{d}\log W = \frac{\alpha\sigma}{4(1+(\sigma-1)\alpha)} = -\,\mathrm{d}\log W^*. \tag{152}$$

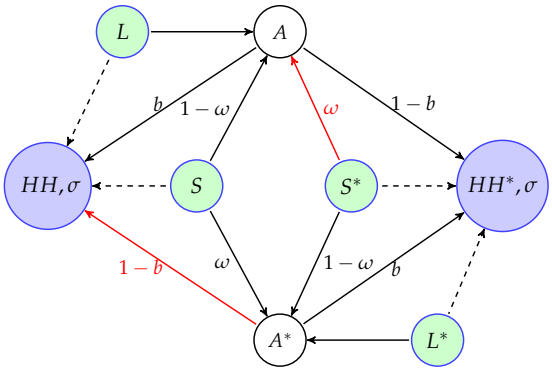


Figure 11: The solid lines show the flow of goods while the dashed lines show the flow of wage payments. Each node label gives the name of the node, as well as its elasticity of substitution. Potential tariffs are colored red.

Since the tax has no effect on consumer welfare holding fixed factor prices (the increase in tax revenue exactly offsets the increase in taxes on the margin), the only effect of the tax comes from the change in relative factor prices. For welfare, domestic workers gain

$$\mathrm{d}\log\Lambda_L = \frac{(\sigma-1)(1-\alpha)\alpha}{4+4(\sigma-1)\alpha} > 0 \tag{153}$$

and foreign labor loses

$$\mathrm{d}\log\Lambda_{L^*} = \frac{-(\sigma-1)\alpha-\sigma-1}{4+4(\sigma-1)\alpha} < 0. \tag{154}$$

On the other hand, both types of steel lose income

$$\mathrm{d}\log\Lambda_S = -\frac{1-b}{2} = \mathrm{d}\log\Lambda_{S^*}. \tag{155}$$

Since domestic workers are equally exposed to foreign and domestic steel, the two effects cancel. However, domestic consumers are long domestic labor and short foreign labor, so overall they benefit.

Now, we consider what happens if there is a tax on foreign steel instead

$$\mathrm{d}\log W = \frac{(2+\alpha(\sigma-1))(1-\alpha)}{4+4(\sigma-1)\alpha} > 0. \tag{156}$$

This time, the tax generates more revenue directly than it increases the price of consumption goods (holding factor prices constant). Specifically, the tax generates revenues equal

to $(1-\alpha)\omega$ but increases domestic prices by $b(1-\alpha)\omega$.

This is because the tax on foreign steel is re-exported. Next, we investigate the effect on the effect on the factor prices. As in the previous example, domestic steel is unaffected

$$\mathrm{d}\log\Lambda_S = 0, \tag{157}$$

while foreign steel prices fall

$$\mathrm{d}\log\Lambda_{S^*} = -\frac{1}{2}, \tag{158}$$

which is good for the domestic consumer and bad for foreign consumer. Finally, domestic labor loses whereas foreign labor gains

$$\mathrm{d}\log\Lambda_L = -\frac{(\sigma-1)(1-\alpha)}{4+4(\sigma-1)\alpha} = -\mathrm{d}\log\Lambda_{L^*} < 0. \tag{159}$$

O.1 Extra Ramblings on Tariffs

O.2 Harberger Triangles

- Some examples where Harberger triangle holds (so only the change in import share matters), in these cases only the change in net import share matters:
  - Case I: outside good pins down factor prices.
  - Case II: price of all imports and exports are exogenous (but domestic factor prices can change).
  - General details for I and II in the appendix.
- Cute examples where even though the only thing that matters is the net import share, the change in the net import share depends on the details (maybe a Kei Mu Yi type effect).
- Point out more generally, we don't have a triangle anyway — and details matter.
- Start putting in the quantitative stuff.

In this class of neoclassical economies, introducing tariffs can increase the welfare of consumers in a country by manipulating factor rewards. In general, Proposition 19 shows

that increasing a tariff has three effects (1) it increases the price of consumption goods, (2) it generates some revenue, and (3) it changes factor prices:

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log \mu_k} = \underbrace{\left(\frac{1}{\chi_h}\frac{\mathrm{d}\left(\sum_{i\in h}\left(1-\frac{1}{\mu_i}\right)\lambda_i\right)}{\mathrm{d}\log \mu_k} - \tilde{\lambda}_k^h\right)}_{\text{Goods Price Manipulation}} + \underbrace{\sum_{f=1}^{F}\left(\frac{\Lambda_f}{\chi_c}\Phi_{fc} - \tilde{\Lambda}_f^h\right)\frac{\mathrm{d}\log \Lambda_f}{\mathrm{d}\log \mu_k}}_{\text{Factor Price Manipulation}}.$$

$$-\tilde{\lambda}_k^h + \frac{1}{\chi_h}\frac{\mathrm{d}\left(\sum_{i\in h}\left(1-\frac{1}{\mu_i}\right)\lambda_i\right)}{\mathrm{d}\log \mu_k} = -\frac{\lambda_k}{\chi_h} + \frac{\lambda_k}{\chi_h} = 0.$$

Hence, around the no-tax equilibrium, holding fixed factor rewards $\mathrm{d}\log\Lambda$, tariffs have no effect on welfare whenever consumption exposure λ_k^h is the same as the import share in national income λ_k/χ_h. In models with an aggregate final good (consumed by domestic households and exported), this assumption always holds. This means that if a country is a price taker in factor markets, then the optimal tariff is zero.

To dig into whether or not this argument holds more generally, we abstract away from factor-price manipulation by assuming that (i) country h is infinitesimal so that foreign factor prices do not respond to trade policy by country h, and (ii) demand for the domestic factors owned by country h are infinitely elastic, so that their prices (in terms of foreign prices) are exogenous. This means that $\mathrm{d}\log\Lambda_f = 0$ and terms of trade are invariant to domestic trade policy. In this case, we call country h a price taker.

Corollary 9. *Whenever country h is a price-taker, and its consumption exposure λ_k^h to a foreign good is the same as that good's sales share in national income λ_k/χ_h,*

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log \mu_k} = 0,$$

where μ_k is a tariff place on imports.

This corollary can be beefed up considerably into the following familiar proposition.

Proposition 22. *Let country h be a price-taker, and suppose its consumption exposure λ_k^h to a foreign good is the same as that good's sales share in national income λ_k/χ_h. Up to a second-order*

approximation, around the no-tax equilibrium,

$$\Delta \log W_h = \int_1^{\overline{\mu}_k} \frac{\mathrm{d} \log W_h}{\mathrm{d} \log \mu_k} \mathrm{d} \log \mu_k \approx \frac{1}{2} \Delta\mu\Delta(\lambda_{hk}^{net}),$$

where μ_k is a tariff place on imports and $\lambda_{hk}^{net} = \lambda_k/(\mu_k\chi_h)$ is the expenditure on imports not counting the tariff.

Proposition 22 shows that we can measure the real income effects of a tariff, around the no-tax equilibrium, via a Harberger triangle. Hence, the change in welfare is approximated by the change in tariffs multiplied by the change in the net import share divided by two. Since the net import share always falls in response to the tariff, this means that tariffs reduce welfare for a price-taking economy.

Now, let's consider what happens if we allow the final good to be disaggregated, so that $\lambda_k^h \neq \lambda_k/\chi_h$. Around the no distortion equilibrium

$$\frac{\mathrm{d} \log W_h}{\mathrm{d} \log \mu_k} = -\lambda_k^h + \frac{\lambda_k}{\chi_h} + (\mu - 1)\frac{1}{\chi_h} \mathrm{d}(\lambda_k/\mu_k)\Big|_{\mu_k=1} = -\lambda_k^h + \lambda_k/\chi_h \neq 0.$$

Hence, if the domestic consumer is less exposed to good k than the world consumer is, then there is a reason to impose a tariff *even* if there is no way to manipulate factor rewards. This motive for imposing a tariff can be likened to introducing a toll for goods that pass through your borders.

More generally, the intuition for this type of tariff is not so different to any kind of optimal tariff: ceteris paribus, country h should tax those goods which contribute more to its income than its price index. For factors, this condition is

$$\frac{\Lambda_f}{\chi_h} > \tilde{\Lambda_f^h},$$

whereas for goods, this condition is

$$\frac{\lambda_f}{\chi_h} > \tilde{\lambda_f^h}.$$

For factors, this condition is obviously satisfied, since households earn all of their income through domestic factors, but are only partially exposed to the price of domestic factors (since they are also exposed to foreign factors). For goods, this condition is less obvious, but intuitively, it is the same basic idea.

Now, what role do the details of the production network play in all of this? Proposition 22 shows that, as long as country h is a factor-price taker, $\lambda_k^h = \lambda_k / \chi_h$, the welfare effects of the tariff are captured by the change in the import share. So, conditional on knowing the change in the import share, the network structure is irrelevant. Of course, as usual, different network structures will imply different changes in import shares, so without a priori knowledge of $\Delta\lambda_{hk}^{net}$, we still need a model of the production network. What's more, we also need to know the input-output matrix if we wish to check whether or not $\lambda_k^h = \lambda_k / \chi_h$.

Tariffs and Triangles in a Small Open Economy

In general, for a tariff, Proposition 19 implies that We abstract away from terms of trade manipulation by assuming that (i) country h is infinitesimal so that foreign factor prices do not respond to trade policy by country h, and (ii) demand for the domestic factors owned by country h are infinitely elastic, so that their prices (in terms of foreign prices) are exogenous. This means that $\mathrm{d}\log\Lambda_f = 0$ and terms of trade are invariant to domestic trade policy.

This means that we can write

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k} = -\tilde{\lambda}_k^h + \frac{1}{\chi_h}\frac{\mathrm{d}\left(\sum_{i\in h}\left(1-\frac{1}{\mu_i}\right)\lambda_i\right)}{\mathrm{d}\log\mu_k}.$$

Around the no distortion equilibrium, for an economy without intermediate inputs $\lambda_k^h = \lambda_k/\chi_h = \lambda_{hk}$, so we can write

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k} = -\tilde{\lambda}_k^h + \frac{1}{\chi_h}\frac{\mathrm{d}\left(\sum_{i\in h}\left(1-\frac{1}{\mu_i}\right)\lambda_i\right)}{\mathrm{d}\log\mu_k} = -\frac{\lambda_k}{\chi_h} + \frac{\lambda_k}{\chi_h} = 0.$$

Hence, locally, without networks or terms of trade effects, tariffs have no effect on welfare around the no distortion equilibrium.

Now consider the effects away from the no distortion equilibrium, when the gross tariff $\mu_k > 1$:

$$\begin{aligned}\frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k} &= -\left(1-\frac{1}{\mu_k}\right)\frac{\lambda_k}{\chi_h} + \left(1-\frac{1}{\mu_k}\right)\frac{\lambda_k}{\chi_h}\,\mathrm{d}\log\lambda,\\ &= \left(1-\frac{1}{\mu_k}\right)\frac{\lambda_k}{\chi_h}\left(\frac{\mathrm{d}\log\lambda}{\mathrm{d}\log\mu_k} - 1\right)\end{aligned}$$

$$= \left(1 - \frac{1}{\mu_k}\right) \frac{\lambda_k}{\chi_h} \left(\frac{\mathrm{d}\log(\lambda_k/\mu_k)}{\mathrm{d}\log\mu_k}\right).$$

Let $\lambda_{hk}^{net} = \lambda_{hk}/\mu_k$. Then, by the trapezoid rule, we can approximate the change in welfare to a second-order as

$$\Delta \log W_h = \int_1^{\overline{\mu}_k} \frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k} \mathrm{d}\log\mu_k \approx \frac{1}{2}(\overline{\mu_k} - 1)\Delta(\lambda_{hk}^{net}) = \frac{1}{2}\Delta\mu\Delta(\lambda_{hk}^{net}). \tag{160}$$

Hence, the change in welfare is approximated by the change in tariffs multiplied by the change in the net import share divided by two. Since the net import share always falls in response to the tariff, this means that absent input-output connections and terms of trade effects, tariffs reduce welfare.

Now, let's consider what happens if we allow for input-output connections, but continue to abstract from terms of trade effects. In this case, $\lambda_k^h \neq \lambda_{hk}$, so the domestic household could be more or less exposed to the good than the world consumer. This means around the no distortion equilibrium

$$\frac{\mathrm{d}\log W_h}{\mathrm{d}\log\mu_k} = -\lambda_k^h + \frac{\lambda_k}{\chi_h} + (\mu - 1)\frac{1}{\chi_h}\mathrm{d}(\lambda_k/\mu_k) = -\lambda_k^h + \lambda_k/\chi_h \neq 0.$$

Hence, if the domestic consumer is less exposed to good k than the world consumer is, then there is a reason to impose a tariff *even* if there are no terms of trade effects.

Trade Barriers and Trapezoids in Small Open Economy

Now, let's consider the effects of an increase in trade barriers in the same type of environment. For an increase in trade barriers, assuming there are no wedges, we have

$$-\frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_k} = -\lambda_k^h. \tag{161}$$

Therefore, to a second-order, we can write

$$\Delta \log W_h = \int \frac{\mathrm{d}\log W_h}{\mathrm{d}\log A_k} \mathrm{d}\log A_k \approx \left(\lambda_k^h + \frac{1}{2}\left(\Delta\lambda_k^h\right)\right) \Delta\log A_k. \tag{162}$$

When there are no intermediate inputs $\lambda_k^h = \lambda_{hk}$, so

$$\Delta \log W_h \approx \left(\lambda_{hk} + \frac{1}{2}\Delta\lambda_{hk}\right) \Delta \log A_k. \tag{163}$$

Unlike the pervious example, where abstracting from input-output connections meant that losses were a triangle, now the losses are a trapezoid. Intuitively, the trade barrier destroys resources directly whereas the tariff also generates some revenues. Hence, where as input-output networks can imply that a tariff can locally improve the welfare of the domestic consumer, even abstracting from terms of trade effects, an increase in a trade barrier is generically bad for the domestic consumer (once we abstract from its terms of trade effects).

P Roy Models

Galle et al. (2017)(GRY) combine a Roy-model of labor supply with an Eaton-Kortum model of trade to study the effects of trade on different groups of workers in an economy. They prove an extension to the ACR result that accounts for the distributional consequences of trade shocks. In this section, we show how our framework can be adapted for analyzing models such models. We generalize our analysis to encompass Roy-models of the labor market, and show how duality with the closed economy can then be used to study the distributional consequences of trade.

Suppose that H_c denotes the set of households in country c. As in Galle et al. (2017), households consume the same basket of goods, but supply labor in different ways.[31] We assume that each household type has a fixed endowment of labor L_h, which are assigned to work in different industries according to the productivity of workers in that group and the relative wage differences offered in different industries.

Define δ_{hf} to be type h's share of income derived from earning wages f

$$\delta_{hf} = \frac{\Phi_{hf}\Lambda_f}{\chi_h}.$$

[31]Similar methods can also be used when households have different consumption baskets, although Borusyak and Jaravel (2018) suggest that at least in the US, households have similar imported consumption baskets.

Letting world GDP be the numeraire, then the Roy model implies that

$$\Lambda_f = \sum_{h \in H} \delta_{hf} \chi_h \left(\frac{w_f}{\chi_h / L^h} \right)^{\varepsilon_h} \chi_h,$$

where ε_h is the supply elasticity and L^h is a shock to the stock of labor h has been endowed with (since we analyze log changes, the value of the endowment is irrelevant). The Roy model implies that

$$\chi_h = \left(\sum_f \delta_{hf} w_f^{\varepsilon_h} \right)^{\frac{1}{\varepsilon_h}} L^h,$$

where $\varepsilon_h = 1$ represents the case where labor cannot be moved across markets by h. If $\varepsilon_h > 1$ then h can take advantage of wage differentials to redirect its labor supply and boost its income. When $\varepsilon \to \infty$, labor mobility implies that all wages in the economy are equalized (and the model collapses to a one-factor model). Galle et al. (2017) show that the above equations can be microfounded via a ROY model where homogenous workers in each group type draw their ability for each job from Frechet distributions, and choose to work in the job that offers them the highest return.

Proposition 23. *The change in the welfare of group $g \in H_c$, in response to iceberg trade shocks is given by*

$$\Delta \log W_g = -\log \left(\frac{\check{P}_g(\check{A}, \check{w})}{\check{P}_c(\overline{\check{A}}, \check{w})} \right).$$

where

$$\Delta \log \check{A}_i = \log \left(\frac{\check{A}_i}{\overline{\check{A}}_i} \right) = \frac{1}{\zeta_i - 1} \log \left(\frac{\sum_{j \in N_c} \Omega_{ij}}{\sum_{j \in N_c} \overline{\Omega}_{ij}} \right),$$

and

$$\Delta \log \check{w}_f = \log \left(\frac{\check{w}_f}{\overline{\check{w}}_f} \right) = \frac{1}{\varepsilon_g} \log \left(\frac{\delta_{gf}}{\overline{\delta}_{gf}} \right).$$

In the case where $\varepsilon \to \infty$, we recover the one-factor model.

Of course, due to the fact that factor shares δ_{gf} are now endogenously responding to factor prices, the results in Baqaee and Farhi (2017a) can no longer be used to determine how these shares will change in equilibrium. Therefore, we extend those propositions here.

Proposition 24. *The response of the factor prices to a shock* $\mathrm{d} \log A_k$ *is the solution to the following system:*

1. *Product Market Equilibrium:*

$$\Lambda_l \frac{\mathrm{d}\log\Lambda_l}{\mathrm{d}\log A_k} = \sum_{i\in\{H,N\}} \lambda_j(1-\theta_j) Cov_{\Omega^{(j)}}\left(\Psi_{(k)} + \sum_f \Psi_{(f)} \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}, \Psi_{(l)}\right) + \sum_{h\in H}(\lambda_l^h - \lambda_l)\left(\sum_{f\in F_c} \Phi_{hf}\Lambda_f \frac{\mathrm{d}\log w_f}{\mathrm{d}\log A_k}\right).$$

2. *Factor Market Equilibrium:*

$$\mathrm{d}\log\Lambda_f = \sum_{h\in H} E_{\Phi^{(h)}}\left[\varepsilon_h\left(E_{\delta^{(h)}}\left(\mathrm{d}\log w_f - \mathrm{d}\log w\right)\right) + \left(E_{\delta^{(h)}}(\mathrm{d}\log w)\right) + (\mathrm{d}\log L)\right].$$

Given this, the welfare of the gth group is

$$\frac{\mathrm{d}\log W_g}{\mathrm{d}\log A_k} = \sum_{s\in F}\left(\frac{\Phi_{gs}}{\chi_g}\Lambda_s - \Lambda_s^g\right)\mathrm{d}\log w_s + \lambda_k^g + \mathrm{d}\log L^g.$$

Note that Proposition 3 is a special case of this where $\varepsilon_h = 1$ for every group — which amounts to having fixed labor. In that case, the factor market equilibrium becomes trivial: $\mathrm{d}\log\Lambda_f = \mathrm{d}\log w_f$, and we recover Proposition 3. More generally, the method here can be extended to cover multi-product firms.

Q Additional Results

R Productivity Accounting Decomposition

This type of logic can also be applied to real GDP (i.e. output). Note that the quantity of each good i in GDP q_i^{GDP} is a function of both the vector of productivity levels A as well as the allocation matrix $\mathcal{X}$. Hence, discrete changes in output are given by integrating the definition in (12). Let t be a parameter that indexes changes in productivity and the allocation matrix, then

$$\log Y_c(t+\delta) - \log Y_c(t) = \int_t^{t+\delta} \sum_i F_{ci}(t)\left(\frac{\partial\log q_i^{GDP}}{\partial\log A}\frac{\mathrm{d}\log A(s)}{\mathrm{d}\,s} + \frac{\partial\log q_i^{GDP}}{\partial\mathcal{X}}\frac{\mathrm{d}\,\mathcal{X}(s)}{\mathrm{d}\,s}\right)\mathrm{d}\,s.$$

	AUS	AUT	BEL	BGR	BRA	CAN	CHN	CYP	CZE	DEU
Capital	-0.11%	0.01%	-0.01%	0.02%	-0.04%	-0.27%	-0.14%	0.02%	0.00%	0.00%
Low	0.07%	0.00%	-0.01%	0.00%	0.01%	0.30%	-0.17%	0.02%	0.01%	0.01%
Med	0.06%	0.01%	0.00%	0.02%	0.03%	0.20%	-0.08%	0.04%	0.02%	0.01%
High	0.05%	0.00%	0.03%	0.04%	0.02%	0.25%	-0.44%	0.05%	0.05%	0.02%
	DNK	ESP	EST	FIN	FRA	GBR	GRC	HUN	IDN	IND
Capital	-0.03%	0.00%	0.04%	0.02%	0.01%	-0.01%	-0.02%	-0.01%	-0.03%	-0.04%
Low	0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.02%	0.02%	0.05%	0.06%
Med	0.02%	0.01%	0.00%	0.00%	0.00%	0.01%	0.02%	0.02%	0.06%	0.07%
High	0.03%	0.03%	0.02%	0.01%	0.01%	0.03%	0.02%	0.04%	0.05%	0.08%
	IRL	ITA	JPN	KOR	LTU	LUX	LVA	MEX	NLD	NOR
Capital	-0.01%	0.00%	-0.01%	-0.11%	0.04%	0.02%	0.01%	-0.18%	0.04%	0.01%
Low	-0.05%	0.00%	0.00%	0.02%	-0.01%	-0.06%	0.00%	0.41%	-0.01%	0.01%
Med	-0.01%	0.01%	0.01%	0.01%	-0.02%	-0.02%	0.00%	0.63%	0.00%	0.01%
High	0.04%	0.02%	0.02%	0.03%	0.00%	0.02%	0.01%	0.10%	0.04%	0.02%
	POL	PRT	ROU	RUS	SVK	SVN	SWE	TUR	TWN	USA
Capital	0.01%	0.01%	0.01%	0.02%	0.01%	0.00%	0.00%	0.00%	-0.15%	-0.36%
Low	0.00%	-0.01%	-0.01%	-0.01%	0.00%	0.00%	0.00%	0.02%	-0.01%	0.08%
Med	0.00%	0.02%	0.00%	0.00%	0.01%	0.00%	0.01%	0.00%	0.02%	0.03%
High	0.02%	0.04%	0.02%	0.01%	0.04%	0.02%	0.01%	0.01%	0.07%	0.13%

Table 6: The change in factor rewards, in units of domestic consumption, for each primary factor in response to a 10% increase in USA taxes on Chinese imports.

	Overall	Factors	Transfers	Tariff Revenue
AUT	0.05%	-0.06%	0.01%	0.09%
BEL	0.07%	-0.19%	0.01%	0.25%
BGR	0.00%	-0.10%	-0.05%	0.14%
CYP	0.02%	-0.13%	-0.12%	0.27%
CZE	0.07%	-0.09%	0.02%	0.14%
DEU	0.04%	-0.13%	0.02%	0.15%
DNK	0.06%	-0.13%	0.02%	0.18%
ESP	0.00%	-0.04%	-0.02%	0.06%
EST	0.05%	-0.10%	-0.01%	0.16%
FIN	0.07%	-0.12%	0.02%	0.17%
FRA	0.01%	-0.06%	0.00%	0.08%
GBR	-0.58%	-0.59%	0.01%	0.00%
GRC	-0.02%	-0.03%	-0.03%	0.05%
HUN	0.05%	-0.09%	0.00%	0.14%
IRL	0.49%	-1.36%	0.13%	1.72%
ITA	0.02%	-0.04%	0.00%	0.06%
LTU	-0.01%	-0.04%	-0.02%	0.06%
LUX	3.12%	-0.68%	0.01%	3.80%
LVA	-0.01%	-0.05%	-0.03%	0.07%
NLD	0.15%	-0.55%	-0.06%	0.76%
NOR	0.05%	-0.23%	0.02%	0.26%
POL	0.02%	-0.04%	-0.04%	0.09%
SVK	0.07%	-0.04%	0.00%	0.11%
SVN	0.01%	-0.06%	-0.01%	0.08%
SWE	0.05%	-0.16%	0.02%	0.19%

Table 7: The change in real income for each EU country, as well as the contribution of each component of income (factor returns, transfers, and tariff revenue) to this overall change.

The power of Hulten's theorem, in its open-economy formulation of Theorem 1, is that the second summand is zero.[32] From this, it follows that

$$\log Y_c(t+\delta) - \log Y_c(t) = \int_t^{t+\delta} \frac{\mathrm{d}\log Y(A)}{\mathrm{d}\log A(s)}\,\mathrm{d}\log A(s) = \int_t^{t+\delta} \sum_{i\in N_c} \lambda_i(s)\,\mathrm{d}\log A_i(s). \tag{164}$$

Formally, Theorem 1 establishes that changes in real GDP are a path integral over the vector field given by the Domar weights.[33] Theorem 1 also implies that

$$\mathrm{d}^2 \log Y = (\mathrm{d}\,\lambda_1, \cdots, \mathrm{d}\,\lambda_N) = \begin{pmatrix} \frac{\partial \lambda_1}{\partial \log A_1} & \cdots & \frac{\partial \lambda_1}{\partial \log A_N} \\ & \ddots & \\ \frac{\partial \lambda_N}{\partial \log A_1} & \cdots & \frac{\partial \lambda_N}{\partial \log A_N} \end{pmatrix}. \tag{165}$$

In general $\mathrm{d}^2 \log Y$ will not be a symmetric matrix.[34]

[32] This follows from Pareto-efficiency of the equilibrium, which by the Envelope theorem, implies that reallocations have no effect on output.

[33] Note that Theorem 1 is false in the presence of distortions. See Baqaee and Farhi (2017c) and Section 6 for more information about how this result can be extended to inefficient economies.

[34] See Hulten (1973) and Baqaee and Farhi (2018) for more information on conditions under which symmetry holds.