Monetary Policy in a World of Global Supply Chains*

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Abstract

We study the implications of global supply chains for the design of monetary policy. We build a small-open economy New Keynesian model with multiple stages of production. Within the family of simple monetary policy rules, a rule that allows for targeting production-stage-specific producer price inflation outperforms alternative policy rules. The optimal weights on different stages of production are in general not proportional to the sales volume of the stages. If the price is less sticky in upstream production, the relative weight on upstream would be smaller. If we have to choose among aggregate price indicators, targeting PPI inflation is significantly better than targeting CPI inflation alone. As the production chain becomes larger, the weight on PPI inflation should also rise since PPI index occupies a larger proportion in the welfare loss function in contrast to CPI index. Finally, trade frictions such as tariff war also affect the optimal monetary policy rule by altering the weights on the prices in different stages of production.

Keywords: Taylor rule; CPI; PPI; global supply chains; monetary policy; multi-stage production

JEL classification: E31; E52; E58; F11; F12; F41; F62

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1 Introduction

We live in a world of supply chains. From the data of World Input-output Tables, in 2000, the world gross output was 1.97 times that of the world value added, suggesting a large role of intermediate inputs in production and supply chains in the modern economy. The supply chains are global. ... The importance of supply chains has grown over time; the ratio of gross output to value added has increased to 2.18 by the end of 2014. In this paper, we study the implications of global supply chains for the design of optimal monetary policy.

There is an active research on outsourcing and offshoring in the field of international trade (e.g., Feenstra, 1999; Antràs, 2016), where firms purchasing intermediate inputs from other firms, sometimes foreign firms, for further processing. A rise in the importance of global supply chains means a greater fraction of output is produced as intermediate inputs rather than final consumption. As important, it is accompanied by an increase in the number of production stages in many sectors (e.g., Wang et al., 2017). While there is a large existing literature on monetary economics which studies the optimal price index to be targeted by the central bank in both closed-economy (e.g., Huang and Liu, 2005; Strum, 2009; Woodford, 2010) and open-economy environment (e.g., Benigno and Benigno, 2003, 2006; Corsetti and Pesenti, 2005; Gali and Monacelli, 2005; Faia and Monacelli, 2008; De Paoli, 2009; Corsetti, Dedola, and Leduc, 2011; Gong, Wang, and Zou, 2016),[1] the discussion for the role of global supply chains is still under explored in the sense that trade aspect like changes in trade costs is missing, and the design of the monetary policy might need to evolve with lengthening of the production process.[2]

If countries are more tightly connected through global value chains, their inflation dynamics may be altered (e.g., Auer, Levchenko, and Saurè, 2016; Auer, Borio, and Filardo, 2017; Forbes, 2018) and the trade-offs in inflation targeting policies could also be changed for central banks.

As documented by Wei and Xie (2019) and explained via the lens of a lengthening of production chains, the two most commonly discussed price indices in monetary literature, i.e., consumer price index (CPI) and producer price index (PPI), used to move in tandem before 2000, which suggests that a policy targeting CPI was almost harmless in practice. But the two price indices start to diverge in many countries since 2001. Therefore, multi-stage production and trade in intermediate inputs across countries, are essential in understanding desirable monetary policy and appropriate manner of inflation targeting.

[1] Among others, Woodford (2011) provides a good survey on the closed-economy discussion, while Corsetti, Dedola, and Leduc (2011) provide an exhaustive survey on the literature for the new open-economy macroeconomics ("NOEM").

[2] Huang and Liu (2005) compare alternative monetary rules through a two-stage production model in closed economy, while Strum (2008) adopts a similar two-stage production framework but studies optimal monetary policies under commitment and discretion. Shi and Xu (2007) build a two-country New Keynesian model with trade in vertical production and focus on the discussion on the optimal money supply policy. Gong, Wang, and Zou (2016) exploit a two-country New Keynesian model with two stages of production, and focus on how the degree of price stickiness in upstream and downstream stages affects the design of optimal Taylor-type policy rules.

In this paper, we study optimal monetary policy (within the family of simple rules) in a small-open economy with multi-stage of production.[3] Trade can happen at any stage of production or final consumption. In other words, there is not only trade in final consumption goods, but also trade in intermediate inputs along a sequential production chain. In the literature on "New Open Economy Macroeconomics" (NOEM), the small-open economy is either modeled by a two-country setup while taking a limit to zero on domestic part (e.g., Benigno and Benigno, 2003; De Paoli, 2009), or a continuum of small-open economies (e.g, Gali and Monacelli, 2005). We instead model the small-open economy by taking foreign inputs price and foreign demand for outputs as given, where the foreign demand is price elastic. This approach allows us to study the design of monetary policy without assuming trade balance to hold in steady-state, which is necessary for the analysis of implications of trade policies.[4]

Since the trade-offs between producer currency pricing (PCP) and local currency pricing (LCP) is not our focus of this paper, we assume producer currency pricing (PCP) in this paper.

We evaluate alternative simple monetary policy rules such as targeting CPI, PPI, exchange rate peg, or stage-specific producer price index. Furthermore, motivated by a rise in recent international trade tensions, and since we study internationally traded intermediate inputs, it is useful to incorporate trade barriers (which we call tariffs for short) in our framework. We study how a change in trade policies like import tariffs, which alters the cost of supply chain trade, may affect the design of a desirable monetary policy. In addition, since the duration of price contract in upstream production stages may be smaller than downstream stages (e.g., Cornille and Dossche, 2008; Nakamura and Steinsson, 2008), we also study the implications of different price stickiness along the production chain on the performance of alternative monetary policy rules in numerical analysis.[5]

With a lengthening of the production chain, intermediate goods production becomes more important, and the terms measuring intermediate input inflation and wedges between prices along a production chain can account for a larger proportion of the total welfare loss, and thus may need to receive higher weights in an optimal monetary policy. Meanwhile, due to supply chain trade across national borders, exchange rate fluctuation also plays a greater role in the welfare loss as compared to models featuring a single-stage production.

Two distortions are important for welfare in our framework. One is a distortion due to labor re-allocation caused by sticky-prices along the production chain, which is first discussed in a closed-economy New Keynesian model with two-stage production in Huang and Liu (2005). With price

[3] There is a debate about targeting rules versus instrument rules (Taylor-type) for monetary policy (e.g., Svensson, 2002; McCallum and Nelson, 2004). We focus on the Taylor-type instrument rules in this paper.

[4] For the two approaches modeling small-open economy in literature, there is always an assumption of symmetry across countries in steady state, which implies trade balance to be held in steady state. It basically assume away the effects of trade policies on steady state equilibrium.

[5] Gong, Wang, and Zou (2016) argue that the degree of price stickiness along the production chain is a key factor to determine which price index to be included in optimal simple monetary policy rules.

stickiness in each stage, output prices along the production chain would not be perfectly co-moving, and thus, if labor is an input in each stage of production, the price wedges across adjacent stages will lead to inefficient unemployment and output gap. A second distortion is caused by terms of trade externality, as emphasized by Benigno and Benigno (2003) and De Paoli (2009), etc. The terms of trade externality arises in an open economy when the elasticity of demand for export is finite, and the social planner wishes to exploit a domestic monopoly power in trade. Intuitively, an improvement in the terms of trade can increase households welfare by inducing them to consume more imported goods, and thus reducing labor efforts without a fall in consumption.[6] Due to the terms of trade externality, flexible price equilibrium is no longer Pareto optimal in contrast with the closed-economy scenario.

The two distortions have been discussed independently of each other. In our framework, these two distortions interact with each other through the intermediate goods along the production chain. In addition, motivated by the observed lengthening of the production process, we study the trade-offs in welfare loss function as the objective of central bank as the number of production stages changes. We show that, on the one hand, given the number of production stages and price stickiness being the same across stages, coefficients before inflation in downstream stages are larger compared with those in upstream stages in welfare loss function. It suggests that monetary policy should react more to inflation in downstream stages. On the other hand, as the number of stages increases, aggregate PPI occupies larger fraction in welfare loss compared with CPI.

Building on this discussion, we suggest an optimal price index, which includes a system of price indices based on production chains. In practice, the US Bureau of Labor Statistics (BLS) has considered a four-stage production process and accordingly constructed a stage-of-processing price indices in the PPI Final Demand-Intermediate Demand indice.[7] Our theory highlights the importance of such production-stage-specific price indices in guiding optimal monetary policy.

Quantitatively, our numerical analysis in a small-open economy with two stages of production shows a simple monetary policy rule that targets stage-specific producer price indices outperforms all other alternative policy rules, and the weight on the downstream producer price index is larger. If only aggregate price indices are available, targeting PPI instead of only targeting CPI is significantly welfare improving, and it even does not lose much in terms of welfare compared with the case of targeting stage-specific producer price indices (in a two-stage production exercise).[8] Furthermore, trade policies matter for the conduct of monetary policy in the sense that, given the central bank has already implemented an optimal policy rule, the welfare loss in the counter-factual situation with high import tariff is always larger compared with the low tariff situation. The loss induced by higher

[6] In addition, there exists an internal monopolistic distortion which is assumed to be corrected by a subsidy tax.

[7] Details about PPI Final Demand-Intermediate Demand indices can be found at https://www.bls.gov/ppi/fdidsummary.htm.

[8] The benefit of targeting stage-specific producer price indices compared with just targeting aggregate PPI could be larger if the production chain becomes longer.

import tariff cannot simply be corrected by an appreciation in domestic currency. Nevertheless, this loss in welfare is not an obligation of the central bank, and the discussion about alternative policy rules in the baseline situation still applies. In addition, we find that, as the price is less sticky in upstream production, the relative weight on upstream producer price index versus downstream producer price index, when an optimal simple policy rule targets both simultaneously, becomes smaller.

Our paper expands on the growing open-economy New-Keynesian literature by introducing global supply chains, i.e., multi-stage of production, with an emphasis on intermediate goods, and also introduces a discussion of the consequences of trade policy. We also pay special attention to the lengthening of the supply chains (and expansion of the set of intermediate goods), and its implications in the objective function of the central bank. Corsetti, Dedola and Leduc (2010) provide an exhaustive survey on early literature. More specifically, Gali and Monacelli (2005) build a small-open economy New Keynesian model, and compare three alternative simple policy rules: PPI-based Taylor rules, CPI-based Taylor rules, and an exchange rate peg. De Paoli (2009) demonstrates a model with more general parameterization and focuses on terms of trade externality on driving optimal monetary policy, and analyzes optimal stabilization policy through utility-based welfare loss function. In both papers, there is only one stage of production, and PPI is defined as domestic inflation (based on domestic produced final goods), while CPI is defined as inflation index on domestic consumption goods. In other words, their discussion about CPI and PPI captures the difference of these two indices through the perspective of imported final goods, but fails to capture the role of intermediate goods. In our paper, and also in practice, PPI index includes both domestic produced intermediate goods and final goods, while CPI includes final consumption goods. A recent paper by Matsumura (2018) studies optimal inflation targeting policy in small-open economy by introducing multi-sector production and imported commodity goods (as production inputs), but still only has one stage of production and misses the role of intermediate goods.

The most relevant paper is Gong, Wang, and Zou (2016) in which they study optimal simple monetary policy rules in a two-country New Keynesian model with two stages of production, and focus on the effects of price stickiness along the production chain in affecting optimal policy rules. In their model, labor is only used in the first stage of production, and thus the distortion of labor re-allocation is missed, while, in our paper, the terms of trade externality and labor re-allocation distortion are captured simultaneously. We also differ from their paper and extend the literature on two further aspects: how monetary policies change with the variation of production structure (i.e., change in the length of supply chain), and how monetary policy responds to trade policy changes by the analysis of comparative statics on import tariffs.

Our work also contributes to the discussion in closed-economy monetary literature by focusing on the distortion of labor-reallocation across stages caused by sticky price and terms of trade extenality rised in open-economy. More importantly, we study the implications of a varying production

structure. Woodford (2010) provides a good survey for existing work. Huang and Liu (2001) suggest that, going from one stage to multiple-stage production, the effect of monetary shock becomes amplified and the responses of output and inflation become more persistent than desired**.** Huang and Liu (2005) build a two-stage production model in closed-economy and discuss the distortion of labor-reallocation in determining optimal targeting inflation index between CPI and PPI, while Strum (2008) adopts a similar two-stage production model but focusing on optimal commitment policies and the design of discretionary monetary policies. In addition, Huang and Liu (2007) build a multi-stage production model with staggered price to fit with observed international correlations in aggregate quantities, and emphasize the importance of multi-stage processing production on explaining international business cycles.

Moreover, this paper facilitates an interaction between the trade literature and monetary literature, and provides a theoretical support for those studies exploring inflation dynamics motivated (at least partially) from monetary literature (e.g., Monacelli and Sala, 2009; Auer, Levchenko, and Saurè, 2016; Forbes, 2019; Wei and Xie, 2019).

The rest of the paper proceeds as follows: Section 2 introduces basic settings of the small-open economy New Keynesian model with multiple stages of production; Section 3 characterizes the steady-state, flexible-price, and sticky-price equilibrium with two stages of production in a small open economy, and derives the expression for welfare loss function, and further discusses the comparative statics for the effects of changes in import tariff;[9] Section 4 reports all the numerical results on optimal simple policy rules and its welfare performances; and Section 5 concludes the paper.

2 The model setting

Consider a small-open economy New Keynesian model with an infinitely lived representative household. The households maximizes its utility through consumption and leisure. The household owns all domestic firms and receive dividends from them.

The production of a final good requires N-stages of production, which constitutes a vertical production chain. In each stage of production, a large number of domestic firms produce a unit continuum of differentiated outputs, i.e., $u \in [0, 1]$. In the first stage of production, firms use domestic labor as the only input. In each of the subsequent stages of production, intermediary inputs purchased from the previous stage (from both domestic and foreign sources) together with labor are used together for production. All production features constant returns to scale.

The firms and households take international prices of inputs and foreign demand of outputs as given (the small open economy assumption). While the firms are price-takers in the factor markets,

[9] Appendix E shows the welfare loss function in a closed-economy case with multiple N-stage of production, and focuses on the implications of lengthening production chain on welfare loss function.

they are assumed to be monopolistic competitive in their outputs and set the output prices in their own currency (the producer currency pricing assumption, or PCP).

2.1 Household

The representative household has the following utility function and budget constraint:

$$E\Sigma_{t=0}^{\infty}\beta^t[U(C_t) - V(L_t)]$$

$$s.t.\, P_tC_t + E_t\{D_{t,t+1}B_{t+1}\} \leq W_tL_t + \Pi_t + B_t$$

where C_t is the final consumption good, L_t is the supply of labor, $D_{t,t+1}$ is the price of a one-period nominal bond paying off in domestic currency, W_t is the nominal wage, Π_t includes both firm profits and a lump-sum transfer of any government tax revenue, and B_t denotes the holding of a riskless one-period bond.

The consumption good is a composite of both domestically produced and imported final goods, i.e.,

$$C_t = \Theta\bar{Y}_{NH,t}^{\gamma}\bar{Y}_{NF,t}^{1-\gamma}$$

where $\bar{Y}_{NH,t} = [\int_0^1 Y_{NH,t}(u)^{\frac{\theta-1}{\theta}}du]^{\frac{\theta}{\theta-1}}$ is a bundle of domestically produced differentiated final goods, and $\bar{Y}_{NF,t}$ is a bundle of foreign produced differential final goods. γ is the share of the household expenditure on domestically produced final goods, $1-\gamma$ is the share of the expenditure on imports, and θ is the elasticity of substitution among the differentiated final goods. $\Theta = [\gamma^{\gamma}(1-\gamma)^{1-\gamma}]^{-1}$ is a constant for normalization.

By the household's expenditure minimization problem, the demand function for the final goods are:

$$Y_{NH,t}^d(u) = [\frac{P_{NH,t}(u)}{\bar{P}_{NH,t}}]^{-\theta}\frac{\gamma P_t}{\bar{P}_{NH,t}}C_t$$

$$Y_{NF,t}^d = \frac{(1-\gamma)P_t}{\bar{P}_{NF,t}}C_t$$

where the aggregate price index for the final consumption is $P_t = \bar{P}_{NH,t}^{\gamma}\bar{P}_{NF,t}^{1-\gamma}$, the aggregate price index for the domestic produced final goods is $\bar{P}_{NH,t} = (\int_0^1 P_{NH,t}(\mu)^{1-\theta}du)^{\frac{1}{1-\theta}}$, and the aggregate price index for the foreign produced final goods is $\bar{P}_{NF,t} = T_t\mathcal{E}_tP_{NF,t}^*$. The term $\mathcal{E}_t$ is the price of foreign currency in units of domestic currency, $P_{NF,t}^*$ is the exogenous foreign price in foreign currency, and T_t is a uniform tariff on imports. An upper star $*$ is used to denote variables in the foreign country (denominated in the foreign currency).

By the household's utility maximization problem, we obtain the labor supply and Euler Equa-

tion, respectively, as

$$\frac{W_t}{P_t} = \frac{V'_{N,t}}{U'_{c,t}} \tag{1}$$

and

$$U'_{c,t} = \beta R_t E_t[U'_{c,t+1}\frac{P_t}{P_{t+1}}] \tag{2}$$

where $R_t = \frac{1}{E_t D_{t,t+1}}$ is the gross return on a one-period risk-free nominal bond in domestic currency.

Assuming a CRRA utility function, i.e., $U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma}$ and $V(N_t) = \frac{L_t^{1+\psi}}{1+\psi}$, the labor supply (1) and Euler equation (2) can be written in log-linearized form as

$$w_t - p_t = \sigma c_t + \psi n_t \tag{3}$$

$$c_t = E_t(c_{t+1}) - \frac{1}{\sigma}[i_t - E_t(\pi_{t+1}) - \rho] \tag{4}$$

where lower-case letters denote the logarithm of the respective variables, $i_t = R_t - 1$ is the nominal interest rate in domestic currency, $\pi_{t+1} = p_{t+1} - p_t$ is the CPI inflation, and $\rho = \beta^{-1} - 1$.[10]

The household is assumed to have access to a complete set of state-contingent securities (both domestic and international) (the same assumption is used in Gali and Monacelli, 2005; De Paoli, 2009). Then, the intertemporal marginal rates of substitution must be equalized across countries, i.e.,

$$(\frac{C^*_{t+1}}{C^*_t})^{-\sigma}\frac{\mathcal{E}_t P^*_t}{\mathcal{E}_{t+1}P^*_{t+1}} = (\frac{C_{t+1}}{C_t})^{-\sigma}\frac{P_t}{P_{t+1}}$$

which implies

$$C_t = \theta^* C^*_t Q_t^{1/\sigma} \tag{5}$$

i.e.,

$$c_t = c^*_t + \sigma^{-1} q_t \tag{6}$$

where $Q_t = \frac{\mathcal{E}_t P^*_t}{P_t}$ is the real exchange rate, θ^* is a constant whose value depends on initial asset position. Without loss of generality, we set $\theta^* = 1$. Furthermore, we assume that the foreign consumption follows an $AR(1)$ process, i.e., $c^*_t = \rho_{c^*} c^*_{t-1} + \epsilon_{n,t}$ with $\rho_{c^*} \in (0,1)$ and $\epsilon_{c^*} \sim N(0, \sigma^2_{c^*})$. Note that a shock to foreign consumption also captures fluctuations in the foreign demand for domestically produced goods.

From the risk sharing condition (6), given exogenous foreign consumption, there is an increase in consumption if and only if the real exchange rate depreciates. Under the assumption of complete international financial markets, it also implies uncovered interest parity, i.e., $i_t - i^*_t = E_t(\Delta e_{t+1})$, where $e_t = ln\mathcal{E}_t$.

[10] It's equivalent to write $i_t = -logE_t D_{t,t+1}$ and $\rho = -log\beta$.

2.2 The production of firms

Each final good requires N-stages of production, with a large number of domestic firms producing a unit continuum of differentiated outputs and constant returns to scale at each stage. In the first stage, the production function for good $u \in [0,1]$ is given by

$$Y_{1H}(u) + Y_{1H}^X(u) = A_1 L_1(u)$$

where A_1 is the productivity in stage 1 and $L_1(u)$ is the quantity of labor employed in the production of good u. The output is either sold at home $Y_{1H}(u)$ or exported $Y_{1H}^X(u)$.

The stage-1 output sold at home and the corresponding price are given, respectively, by

$$Y_{1H} = [\int_0^1 Y_{1H}(u)^{\frac{\theta-1}{\theta}} du]^{\frac{\theta}{1-\theta}}$$

$$P_{1H} = [\int_0^1 P_{1H}(u)^{1-\theta} du]^{\frac{1}{1-\theta}}$$

In each subsequent stage, the production needs to have intermediate inputs. The production in stage n (for $n = 2, \ldots, N$) can be viewed as a two-step process. In the first step, a firm purchases all differentiated outputs produced in the previous stage $n-1$ from all global sources and form a bundle of intermediate inputs. Specifically, the intermediate input bundle to be used in the production stage n, $\bar{Y}_n$, is a bundle of two composites of stage $n-1$ outputs:

$$\bar{Y}_n = \Theta \bar{Y}_{(n-1)H}^{\gamma} \bar{Y}_{(n-1)F}^{1-\gamma}$$

$$\bar{Y}_{(n-1)H} = [\int_0^1 Y_{(n-1)H}(j)^{\frac{\theta-1}{\theta}} dj]^{\frac{\theta}{\theta-1}}$$

where $Y_{n-1,H}(j)$ is the amount of good j that is domestically produced in stage $n-1$ and purchased by the firm in stage n, and $\bar{Y}_{(n-1)F}$ is the amounts of composite good that foreign firms produced in stage $n-1$. In factor market, domestic firms are price takers in purchasing foreign composite goods $\bar{Y}_{(n-1)F}$, and because of the small-open economy setup, the supply of foreign composite goods is perfectly elastic in price.

The aggregate price index for the inputs in stage n is then given by

$$\bar{P}_n = \bar{P}_{(n-1)H}^{\gamma} \bar{P}_{(n-1)F}^{1-\gamma}$$

$$\bar{P}_{(n-1)H} = [\int_0^1 P_{(n-1)H}(u)^{1-\theta} du]^{\frac{1}{1-\theta}}$$

$$\bar{P}_{(n-1)F} = T_t \mathcal{E}_t P^*_{(n-1)F}$$

where $P^*_{(n-1)F}$ is the price of composite goods in foreign currency produced in stage $n-1$ by foreign firms. Note that the output price in stage n satisfies $P_{nH} = \bar{P}_{nH}$ for $\forall n = 1, 2, \ldots, N$.

In the second step, the firm combines the composite intermediate good with labor input to produce an output. The production function for good u in stage n is given by

$$Y_{nH}(u) + Y^X_{nH}(u) = \Theta^* A_n \bar{Y}_n(u)^{\phi} L_n(u)^{1-\phi}$$

where $\Theta^* = [(1-\phi)^{1-\phi}\phi^{\phi}]^{-1}$ is a constant for normalization. We assume the technology in each stage following the $AR(1)$ process $a_{n,t} = \rho_n a_{n,t-1} + \epsilon_{n,t}$ with $a_{n,t} = lnA_{n,t}$ and $\rho_n \in (0,1)$ for $n = 1, 2, \ldots, N$. Note that $\{\epsilon_n\}_{n=1}^N$ are i.i.d. normal shocks with the same distribution, i.e., $\epsilon_n \sim N(0, \sigma_a^2)$.

Since the production of any good in stage n needs a bundle of output from the previous stage as inputs, it captures a feature of a typical input-output table in which the output from all sectors might be used as inputs into the production. In the language of Baldwin and Venables (2013), the entire manufacturing production process follows a combination of a snake and a spider patterns. At a given stage, outputs from the previous stage from all over the world are purchased to form a composite intermediate input, resembling a spider pattern. Going from one stage of production to the next, the process resembles a snake pattern.

By the small-open economy set-up, the foreign demand for domestic output in stage $n = 1, 2, \ldots, N$ is given by

$$Y^{Xd}_{nH}(u) = \left(\frac{P_{nH}(u)}{P_{nH}}\right)^{-\theta} \frac{Y^*_{nH} P^*_{nH} \mathcal{E}_t}{P_{nH}}$$

where Y^*_{nH} and P^*_{nH} are exogenous foreign demand and price for domestic produced composite goods in foreign currency. This foreign demand function can be derived from the cost minimization problem of a foreign buyer who aggregates the composite of domestic produced goods.

Similarly, the domestic demand function in stage $n = 1, 2, \ldots, N$ is given by

$$Y_{nH}(u) = \left(\frac{P_{nH}(u)}{P_{nH}}\right)^{-\theta} \frac{\bar{Y}_{nH} \bar{P}_{nH}}{P_{nH}}$$

It is worth noting that nominal exchange rate is not a sufficient statistics for import tariffs in a world of multi-stage production. An increase in the nominal exchange rate (i.e., a depreciation of the domestic currency) raises the input costs and foreign demand for domestic goods simultaneously. But an increase in import tariffs only affects production cost through higher costs of imported inputs.

2.3 The firms pricing problem

Firms in each stage of production are price-takers in factor markets, but are monopolistic competitors in their outputs. We assume firms following Calvo pricing, and the probability that firms in stage n can adjust prices freely is $1-\alpha_n$, $n=1,\ldots,N$. Then, by the law of large numbers, in each period, a fraction $1-\alpha_n$ of firms in stage n can adjust prices while the rest of firms have to stay unchanged. For a firm producing good u in stage n, which can set a new price in period t, it chooses price $P_{nH}(u)$ in domestic currency for its product sold both at home and in the foreign market. Its maximization problem becomes

$$max_{P_{nH,t}(u)} E_t \Sigma_{k=t}^{\infty} \alpha_n^{k-t} D_{t,k}[(1+\tau)P_{nH,t}(u) - \Psi_{n,k}(u)][Y_{nH,k}^d(u) + Y_{nH,k}^{Xd}(u)]$$

where τ is the subsidy to firms that corrects the distortion from monopolistic competition, $\Psi_{n,k}(u) = \bar{P}_{n,k}^{\phi} W_k^{1-\phi}/A_{n,k}$ is the nominal unit production cost for $n=2,\ldots,N$ and $\Psi_{1,k}(u) = W_k/A_{1,k}$, $\bar{P}_{n,k}$ is the price for the composite of intermediate input goods at stage n, and $Y_{nH,k}^d(u)$ and $Y_{nH,k}^{Xd}(u)$ denote the output demand from both domestic and foreign market respectively.

The optimal pricing decision is given by

$$P_{nH,t}^o(u) = \frac{\mu}{1+\tau} \frac{E_t \Sigma_{\tau=t}^{\infty} \alpha_n^{\tau-t} D_{t,\tau} \Psi_{n,\tau}(u)[Y_{nH,\tau}^d(u) + Y_{nH,k}^{Xd}(u)]}{E_t \Sigma_{\tau=t}^{\infty} \alpha_n^{\tau-t} D_{t,\tau}[Y_{nH,\tau}^d(u) + Y_{nH,k}^{Xd}(u)]}$$

where $\mu = \frac{\theta}{\theta-1}$ is the markup in the market for producing outputs in each stage.[11] To be abstract from the distortion generated by monopolistic competition, we set $1+\tau = \mu$.

Taking input prices as given, the cost minimization problem for the firms at stage n for $n = 2,\ldots N$ in period t yields a factor demand function as

$$\bar{Y}_{n,t}^d = \phi \frac{\Psi_{n,t}}{\bar{P}_{n,t}} \int_0^1 [Y_{nH,t}^d(u) + Y_{nH,t}^{Xd}(u)]du \tag{7}$$

$$L_{n,t}^d = (1-\phi) \frac{\Psi_{n,t}}{W_t} \int_0^1 [Y_{nH,t}^d(u) + Y_{nH,t}^{Xd}(u)]du \tag{8}$$

$$\bar{Y}_{(n-1)H,t}^d = \frac{\gamma \bar{P}_{n,t}}{\bar{P}_{(n-1)H,t}} \bar{Y}_{n,t}^d \tag{9}$$

$$\bar{Y}_{(n-1)F,t}^d = \frac{(1-\gamma) \bar{P}_{n,t}}{\bar{P}_{(n-1)F,t}} \bar{Y}_{n,t}^d \tag{10}$$

[11] Since we assume the elasticity of substitution among differentiated goods to be the same across stages, it implies the same markup across different stages.

and

$$Y^d_{(n-1)H,t}(u) = (\frac{P_{(n-1)H,t}(u)}{\bar{P}_{(n-1)H,t}})^{-\theta}\bar{Y}^d_{(n-1)H,t} \tag{11}$$

In the first stage of production, the firm problem is simpler since labor is the only input. Specifically, the optimal pricing decision for a firm in stage 1 is

$$P^o_{1H,t}(u) = \frac{E_t\Sigma^\infty_{\tau=t}\alpha_1^{\tau-t}D_{t,\tau}\Psi_{1,\tau}(u)[Y^d_{1,\tau}(u) + Y^{Xd}_{1H,k}(u)]}{E_t\Sigma^\infty_{\tau=t}\alpha_1^{\tau-t}D_{t,\tau}[Y^d_{1,\tau}(u) + Y^{Xd}_{1H,k}(u)]}$$

where $\Psi_{1,\tau}(u) = W_\tau/A_{1,\tau}$ is the unit production cost in stage 1.

Since labor is the only input in the first stage, the labor demand is

$$L^d_{1,t} = \frac{\Psi_{1,t}}{W_t}\int_0^1 [Y^d_{1H,t}(u) + Y^{Xd}_{1H,t}(u)]du$$

As the goods are symmetric, we drop good index u in the price variable. The aggregate price index for the outputs in stage n, $n = 1, 2, \ldots, N$, is thus given by

$$P_{nH,t} = [\alpha_n P^{1-\theta}_{nH,t-1} + (1-\alpha_n)(P^o_{nH,t})^{1-\theta}]^{\frac{1}{1-\theta}} \tag{12}$$

2.4 The market clearing conditions and equilibrium definition

Equilibrium definition: given exogenous monetary policy (the rule of nominal interest rate or nominal exchange rate $\{i_t, \mathcal{E}_t\}$) and tariffs $\{T_t\}$, as well as exogenous foreign demand and foreign prices $\{C^*_t, P^*_{nH,t}, P^*_{nF,t}, Y^*_{nH,t}\}^N_{n=1}$, the market equilibrium consists of a set of stochastic processes – $\{C_t, L_t\}$ for domestic households, $\{L^d_{n,t}(u), Y_{nH,t}(u), Y^X_{nH,t}(u), P_{nH,t}(u)\}^N_{n=1}$ for firms $u \in [0,1]$ and prices index $\{P_{nH,t}\}^N_{n=1}$, and wages and real exchange rate $\{W_t, Q_t\}$, satisfying the following conditions:

1. Taking prices and wages as given, the representative household maximizes their utility.
2. Taking intermediate input goods prices, wages, and all outputs prices except their own's as given, firms in each stage maximize their profits
3. The inter-temporal trade balance condition hold.
4. The labor market clears, and the goods market clears in all production stages, i.e.,

$$L_t = \sum_{n=1}^N L^d_t, \quad Y_{nH} = Y^d_{nH}, \quad Y^X_{nH} = Y^{Xd}_{nH}$$

It is worth noting that the inter-temporal trade balance condition is derived from no-Ponzi game in households debt, which does not necessarily require balanced trade in each period.

3 Two-stage production in a small-open economy

For a small-open economy with two stages of production, we can obtain more analytical expressions. We now characterize the equilibrium of this economy. (The characterization of the general case of N-stage of production is presented in Appendix A.) We first characterize the steady-state, flexible-price, and sticky-price equilibria, where the steady-state equilibrium is efficient, and derive the second-order approximation of welfare loss function. Because the terms of trade externality and the labor re-allocation distortions across production stages (due to sticky prices) interact with each other, the real exchange rate and the relative price gaps between the production stages enter the welfare loss function.

The model lends itself well to thought experiments on how a change in the import tariff affects the welfare loss function. In addition, while we only consider domestic productivity shocks in this section, the numerical analysis in Section 4 will allow for a broader set of stochastic shocks.

3.1 The steady-state equilibrium

In the steady state, it satisfies $A_1 = A_2 = 1$ and foreign variables are kept constant. The price index satisfies $P_{1H} = \bar{P}_{1H} = P_{1H}(u)$ and $P_{2H} = \bar{P}_{2H} = P_{2H}(u)$. By $\bar{P}_2 = \bar{P}_{1H}^{\gamma}\bar{P}_{1F}^{1-\gamma}$, from Section 2.2, the price index of domestic produced goods across stages are given by

$$P_{1H} = W$$

and

$$P_{2H} = W^{1-\phi}[\bar{P}_{1H}^{\gamma}\bar{P}_{1F}^{1-\gamma}]^{\phi}$$

$$= W^{1-\phi+\gamma\phi}(T\mathcal{E})^{(1-\gamma)\phi}(P_{1F}^{*})^{(1-\gamma)\phi}$$

In the steady state, $P_{1H}(u) = P_{1H}$ and $P_{2H}(u) = P_{2H}$ imply $Y_{1H}^{d}(u) = \bar{Y}_{1H}^{d}$ and $Y_{2H}^{d}(u) = \bar{Y}_{2H}^{d}$. The goods market clearing condition requires $Y_{1H} = \bar{Y}_{1H}^{d}$ and $Y_{2H} = \bar{Y}_{2H}^{d}$. Therefore, the factor demand functions are given by

$$\bar{Y}_2^{d} = \phi W^{1-\phi}(\bar{P}_2)^{1-\phi}(Y_{2H}^{d} + Y_{2H}^{Xd})$$

$$L_2^{d} = (1-\phi)W^{-\phi}\bar{P}_2^{-\phi}(Y_{2H}^{d} + Y_{2H}^{Xd})$$

$$Y_{1H}^{d} = \frac{\gamma\bar{P}_2\bar{Y}_2^{d}}{\bar{P}_{1H}}$$

and

$$L_1^d = Y_{1H}^d + Y_{1H}^X$$

where $Y_{2H}^{Xd} = \frac{Y_{2H}^* P_{2H}^* \mathcal{E}}{P_{2H}}$, $Y_{1H}^{Xd} = \frac{Y_{1H}^* P_{1H}^* \mathcal{E}}{P_{1H}}$, $Y_{2H}^d = \gamma C \bar{P}_{2H}^{-(1-\gamma)} \bar{P}_{2F}^{1-\gamma}$, and $\bar{P}_{2F} = T P_{2F}^* \mathcal{E}$. By backward induction, we can get the labor demand in each stage of production.

The equilibrium in the steady state $\{C, L, W\}$ are then fully characterized by the labor supply equation (1), the risk sharing condition (5), and the labor demand function $L_1^d + L_2^d$ as derived above, where all price indices are a function of W. Following Huang and Liu (2005), we set $\psi = 0$ to simplify expressions, which can be justified by indivisible labor (e.g., Hansen, 1985).[12] Then, the equations (1) and (5) give

$$w = \sigma c^* + e + p^*$$

$$c = c^* + \frac{1}{\sigma}[e + p^* - p]$$

By substituting w into $p = \gamma \bar{p}_{2H} + (1-\gamma)\bar{p}_{2F}$, together with $\sigma c = \sigma c^* + e + p^* - p$, we get the expression of c, which includes neither the domestic price index nor nominal exchange rate. Similarly, by substituting w into price index, together with $L = L_1^d + L_2^d$, we obtain an expression of state-state labor l.

3.2 The flexible-price equilibrium

In the flexible price equilibrium, $\alpha_n = 0$ for $n = 1, 2$. The optimal pricing decision for firms at stage n becomes $P_{nH,t}^o = \Psi_{n,t}$ and thus $P_{nH,t} = \bar{P}_{nH,t} = P_{nH,t}^o$.

Note that $\bar{P}_{2,t} = \bar{P}_{1H,t}^{\gamma} \bar{P}_{1F,t}^{1-\gamma}$, we then obtain stage-specific price index as

$$P_{1H,t} = \frac{W_t}{A_{1,t}}$$

and

$$P_{2H,t} = W^{1-\phi+\gamma\phi} (T_t \mathcal{E}_t)^{(1-\gamma)\phi} (P_{1F}^*)^{(1-\gamma)\phi} A_{1,t}^{-\gamma\phi} A_{2,t}^{-1}$$

$$P_t = (P_{2H,t})^{\gamma} (T_t P_{2F,t}^* \mathcal{E}_t)^{1-\gamma}$$

where we have plugged in the expression of $\bar{P}_{2F,t}$ in the last equation.

Similar to the analysis in the steady-state equilibrium, we have $Y_{1H,t}^d(u) = \bar{Y}_{1H,t}^d$, $Y_{2H,t}^d(u) = \bar{Y}_{2H,t}^d$, $Y_{1H,t} = \bar{Y}_{1H,t}^d$ and $Y_{2H,t} = \bar{Y}_{2H,t}^d$. Then, the factor demand functions are given by

$$\bar{Y}_{2,t}^d = \phi \frac{W_t^{1-\phi} (\bar{P}_{2,t})^{1-\phi}}{A_{2,t}} (Y_{2H,t}^d + Y_{2H,t}^{Xd})$$

[12] In the numerical analysis in Section 4, we do not require $\psi = 0$ but take a more general calibrated value.

$$L^d_{2,t} = (1-\phi)\frac{W_t^{1-\phi}(\bar{P}_{2,t})^{1-\phi}}{A_{2,t}}(Y^d_{2H,t} + Y^{Xd}_{2H,t})$$

$$Y^d_{1H,t} = \frac{\gamma \bar{P}_{2,t}\bar{Y}^d_{2,t}}{\bar{P}_{1H,t}}$$

and

$$L^d_{1,t} = Y^d_{1H,t} + Y^X_{1H,t}$$

where $Y^X_{2H,t} = \frac{Y^*_{2H,t}P^*_{2H,t}\mathcal{E}_t}{P_{2H,t}}$, $Y^X_{1H,t} = \frac{Y^*_{1H,t}P^*_{1H,t}\mathcal{E}_t}{P_{1H,t}}$, $Y^d_{2H,t} = \gamma C \bar{P}^{-(1-\gamma)}_{2H,t}\bar{P}^{1-\gamma}_{2F,t}$, and $\bar{P}_{2F,t} = T_t P^*_{2F,t}\mathcal{E}_t$.

Similar to the steady-state equilibrium, the flexible-price equilibrium $\{C_t, L_t, W_t\}$ are then fully characterized by the labor supply equation (1), the risk sharing condition (5), and the labor demand function $L^d_1 + L^d_2$ as derived above. With the assumption of $\psi = 0$, the equations (1) and (5) gives

$$w_t = \sigma c^*_t + e_t + p^*_t$$

$$c_t = c^*_t + \frac{1}{\sigma}[e_t + p^*_t - p_t]$$

By substituting w_t into price index, we obtain the expressions of c_t and l_t, which does not include domestic price index or nominal exchange rate. Note that, by denoting $t_t = lnT_t$, we have the expression of CPI index p_t as

$$p_t = \gamma[(1-\phi+\gamma\phi)w_t + (1-\gamma)\phi(e_t + t_t) + (1-\gamma)\phi p^*_{1F,t} - \gamma\phi a_{1,t} - a_{2,t}]$$

$$+(1-\gamma)(e_t + t_t + p^*_{2F,t})$$

By substituting p_t into the risk sharing condition, we obtain the natural rate of interest rate as

$$r\bar{r} = \sigma E(c_{t+1} - c_t)$$

$$= \gamma[\gamma\phi\rho_1\Delta a_{1,t} + \rho_2\Delta a_{2,t}]$$

where we treat exogenous foreign variables and import tariff as constant.

3.3 The sticky-price equilibrium

We now derive the New Keynesian Phillips curves for each stage as a function of the relative price gap and output gap, and characterize the equilibrium with sticky prices. Similar to the derivation in Gali (2015), the Phillips curve for each stage is given by

$$\pi_{1H,t} = \beta E_t \pi_{1H,t+1} + \lambda_1 \hat{\gamma}_{1,t}$$

$$\pi_{2H,t} = \beta E_t \pi_{2H,t+1} + \lambda_2 \hat{\gamma}_{2,t}$$

where $\lambda_n = \frac{(1-\beta\alpha_n)(1-\alpha_n)}{\alpha_n}$ for $n = 1, 2$ and $\hat{\gamma}_n$ is the log-derivation of real marginal cost from steady-state equilibrium, i.e.,

$$\hat{\gamma}_{n,t} = ln(\Psi_{n,t}/P_{nH,t}) - ln(\Psi_n/P_{nH})$$

Since Ψ_n and P_{nH} are the marginal cost and aggregate price in stage n under steady state equilibrium, we have

$$\hat{\gamma}_{1,t} = \sigma\hat{c}_t + \frac{1-\gamma}{\gamma}\hat{q}_t - \hat{g}_{2H,t} - a_{1,t}$$

$$\hat{\gamma}_{2,t} = \gamma\phi\hat{g}_{2H,t} + \frac{1-\gamma}{\gamma}\hat{q}_t + (1-\phi)\sigma\hat{c}_t - a_{2,t}$$

where $\hat{g}_{2H,t}$ is the log-deviation of relative output price gap with respect to the input price in the steady-state equilibrium, i.e., $\hat{g}_{2H,t} = ln(\bar{P}_{1H,t}/P_{2H,t}) - ln(\bar{P}_{1H}/P_{2H})$.[13] In terms of notation, we use the variable with hat to denote deviations from the steady-state equilibrium, and use tilde to denote the deviation from the flexible price equilibrium.

After log-linearizing the Euler equation of the household around the steady state and subtracting the steady-state IS curve, we obtain the IS curve with sticky price as

$$\hat{c}_t = E_t\hat{c}_{t+1} - \frac{1}{\sigma}[\hat{\imath}_t - E_t(\pi_{t+1})]$$

The aggregate inflation π_t (CPI index) can be written as

$$\pi_t = \pi_{2H,t} + \frac{1-\gamma}{\gamma}\Delta\hat{q}_t$$

The derivation of the aggregate inflation can be found in Appendix B.

The law of motion for relative price gap between stage 1 and stage 2 is characterized by

$$\hat{g}_{2H,t} = \hat{g}_{2H,t-1} + \pi_{1H,t} - \pi_{2H,t}$$

The above equations together with the risk-sharing condition (6) fully characterize the sticky-price equilibrium.

3.4 A utility-based welfare loss function for optimal monetary policy

To evaluate monetary policy, we follow the standard approach in the New Keynesian literature (Rotemberg and Woodford, 1999; Benigno and Woodford, 2006; and Gali, 2015), and derive a

[13] The proofs for deriving the expression of $\hat{\gamma}_1$and $\hat{\gamma}_2$ are in Appendix A.3, where Appendix A.3 characterizes the sticky-price equilibrium with N stages of production.

utility-based objective function of the central bank to a second-order approximation. Note that, due to the simultaneous presence of multiple stages production and open economy, the welfare loss function of the household is different from the typical objective function in the literature. In particular, the objective function in our setting includes inflation for each stage of production, the relative price gap across adjacent stages, as well as the real exchange rate and output gap. In addition, the first order terms in the welfare loss function do not cancel out due to terms of trade externality. For the same reason, the flexible-price equilibrium is not Pareto-optimal. Only in the limit case of a closed economy would the first order terms cancel out and the welfare loss function is left only with the second order terms. In such a case, both the steady-state equilibrium and the flexible-price equilibrium are Pareto-efficient. Additional details about the closed-economy model with multi-stage of production are discussed in Appendix E.

Intuitively, since labor is present in all stages of production, the household welfare is affected by distortions in the labor allocation across production stages caused by sticky prices. The terms-of-trade externality emerges naturally in an open economy with the distortions of labor allocation since some of the domestically produced intermediate goods are exported and some imported intermediate inputs are used in domestic production.

The household's utility function is given by

$$E\Sigma_{t=0}^{\infty}\beta^t[U(C_t) - V(L_t)]$$

where $U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma}$ and $V(L_t) = \frac{L_t^{1+\psi}}{1+\psi}$.

A second-order Taylor expansion around steady state (C, L) for the period utility of consumption gives

$$U(C_t) - U = U_c C(\hat{c}_t + \frac{1-\sigma}{2}\hat{c}_t^2) + t.i.p.$$

where $\hat{c}_t$ denotes the log-deviation of consumption from steady state and $t.i.p.$ stands for "terms independent of policy" as in the literature (e.g., Woodford, 2003; Gali, 2015).

By the labor market clearing condition, we obtain the second order Taylor expansion around the steady state for the period utility of employment, i.e., $V(L_t)$, as

$$V(L_t) - V = V_L L\{\sum_{n=1}^{2}\frac{L_n}{L}[\hat{l}_{n,t} + \frac{1}{2}\sum_{n=1}^{N}\hat{l}_{n,t}^2]\} + t.i.p.$$

where L_n/L is the share of labor in stage n in total labor in the steady state, as described in Section 3.1, with the assumption of $\psi = 0$.[14]

[14] As specified for the steady-state variables in Section 3.1, we have $\frac{L_1}{L} = \frac{\gamma\phi}{\gamma\phi+(1-\phi)\bar{a}_1}$ and $\frac{L_2}{L} = 1 - \frac{L_1}{L}$, where $\bar{a}_1 = \frac{Y_{1H}^d}{Y_{1H}^d+Y_{1H}^X}$ and $\bar{a}_2 = \frac{Y_{2H}^d}{Y_{2H}^d+Y_{2H}^X}$ are the share of goods sold, respectively, to the domestic market in stages 1 and 2 in the steady state.

It is useful to rewrite the employment gap in the two production stages in terms of output gap and relative price gap:

$$\hat{l}_{1,t} = [-\bar{a}_1\sigma\phi + \bar{a}_1\bar{a}_2 + \bar{a}_1\sigma]\hat{c}_t + [\bar{a}_1\phi\gamma - 1]\hat{g}_{2H,t}$$

$$+\frac{1+\bar{a}_1-\bar{a}_1\gamma-\bar{a}_1\bar{a}_2\gamma}{\gamma}\hat{q}_t + \bar{a}_1 d_{2,t} + d_{1,t} - a_{1,t} - \bar{a}_1 a_{2,t}$$

$$\hat{l}_{2,t} = (\bar{a}_2 - \phi\sigma)\hat{c}_t + \phi\gamma\hat{g}_{2H,t} + \frac{1-\bar{a}_2\gamma}{\gamma}\hat{q}_t + d_{2,t} - a_{2,t}$$

where $d_{n,t} = ln(\int_0^1 (\frac{P_{nH,t}(u)}{P_{nH,t}})^{-\theta} du)$ for $n = 1, 2$ measures the price dispersion in stage n, $\bar{a}_1 = \frac{Y^d_{1H}}{Y^d_{1H}+Y^X_{1H}}$ and $\bar{a}_2 = \frac{Y^d_{2H}}{Y^d_{2H}+Y^X_{2H}}$ are the shares of goods sold, respectively, to the domestic market in stages 1 and 2 in the steady state. Details can be found in Appendix C.

Following Gali (2015), up to a second-order approximation around the steady state, the price dispersion term $d_{n,t}$ for $n = 1, 2$ can be written as

$$d_{n,t} = \frac{\theta}{2}\int_0^1 [p_{nH,t}(i) - p_{nH,t}]^2 di \equiv \frac{\theta}{2} var\{p_{nH,t}(i)\}$$

By Woodford (2003), the price dispersion can be re-written as a function of inflation in each stage of production, i.e.,

$$\sum_{t=0}^{\infty}\beta^t var\{p_{nH,t}(i)\} = \lambda_n^{-1}\sum_{t=0}^{\infty}\beta^t \pi^2_{nH,t} + t.i.p.$$

We then substitute $\hat{l}_{n,t}$ and $d_{n,t}$ into the period utility of employment. Since the total labor income of households are given by $WL = \frac{PC\gamma}{\bar{a}_2}(1-\phi) + \phi\frac{PC\gamma}{\bar{a}_2}\frac{\gamma}{\bar{a}_1}$, the steady state equilibrium implies $WL = PC(1-\Phi)$, where $1-\Phi = \frac{\gamma}{\bar{a}_2}(1-\phi) + \phi\frac{\gamma^2}{\bar{a}_2\bar{a}_1}$, and thus $U_c C = V_L L(1-\Phi)$.[15] In addition, in the steady state, the labor shares in the two stages are given, respectively, by $L_1/L = \frac{\gamma\phi}{\gamma\phi+(1-\phi)\bar{a}_1}$ and $L_2/L = \frac{(1-\phi)\bar{a}_1}{\gamma\phi+(1-\phi)\bar{a}_1}$.

By summing up $U(C_t) - U$ and $V(L_t) - V$, the household's welfare loss as a fraction of the steady state consumption is given by

$$W = E_0\sum_{t=0}^{\infty}\beta^t \frac{U(C_t) - V(L_t) - (U-V)}{U_c C}$$

[15] In the small-open economy New Keynesian literature, Φ is normally assumed to be zero due to the symmetry assumption across countries in a two-country structure model, e.g., Faia and Monacelli (2008), De Paoli (2009), or in a model of continuum of small countries, e.g., Gali and Monacelli (2005).

$$= E_0 \sum_{t=0}^{\infty} \beta^t \{\hat{c}_t - \sum_{n=1}^{2}(1-\Phi)\frac{L_n}{L}\hat{h}_{n,t} - \frac{1}{2}\{-(1-\sigma)\hat{c}_t^2 + (1-\Phi)[\frac{L_1}{L}(\hat{h}_{1,t} - a_{1,t} - \bar{a}_1 a_{2,t})^2$$

$$+\frac{L_2}{L}(\hat{h}_{2,t} - a_{2,t})^2 + \frac{L_1}{L}\theta\lambda_1^{-1}\pi_{1H,t}^2 + (\frac{L_1}{L}\bar{a}_1 + \frac{L_2}{L})\theta\lambda_2^{-1}\pi_{2H,t}^2]\}\} + t.i.p. \quad (13)$$

where

$$h_{1,t} = (-\bar{a}_1\sigma\phi + \bar{a}_1\bar{a}_2 + \bar{a}_1\sigma)\hat{c}_t + (\bar{a}_1\phi\gamma - 1)\hat{g}_{2H,t} + \frac{1 + \bar{a}_1 - \bar{a}_1\gamma - \bar{a}_1\bar{a}_2\gamma}{\gamma}\hat{q}_t$$

$$h_{2,t} = (\bar{a}_2 - \phi\sigma)\hat{c}_t + \phi\gamma\hat{g}_{2H,t} + \frac{1 - \bar{a}_2\gamma}{\gamma}\hat{q}_t$$

The first-order terms can be eliminated by approximating the equilibrium conditions specified in Section 3.3 to a second-order expansion using the approach developed by Sutherland (2002) and Benigno and Woodford (2006). Though we do not present an explicit expression of the welfare loss function purely in second-order terms due to the complexity arising from the multi-stage production, the numerical analysis in Section 4 approximates the full equilibrium in second-order expansion.

It is worth noting that, in the special case of a closed economy with $N = 2$, $\gamma = 1$, and $\bar{a}_1 = \bar{a}_2 = 1$, the expression (13) reproduces the welfare loss function in Huang and Liu (2005). In a special case of a small-open economy with one stage of production ($N = 1$), $\bar{a}_1 = \gamma$, and the additional assumption of symmetry in the foreign country, the expression (13) reproduces the welfare loss function in De Paoli (2009).

The central bank is to choose a monetary policy to minimize welfare loss with endogenous variables $\{\hat{c}_t, \pi_{1H,t}, \pi_{2H,t}, \pi_t, \hat{q}_t, \hat{g}_{2H,t}, \hat{i}_t\}$ subject to

(1). IS curve: $\hat{c}_t = E_t\hat{c}_{t+1} - \frac{1}{\sigma}[\hat{i}_t - E_t(\pi_{t+1})]$

(2). Aggregate CPI inflation: $\pi_t = \pi_{2H,t} + \frac{1-\gamma}{\gamma}\Delta\hat{q}_t$

(3). Risk sharing: $\hat{c}_t = \frac{1}{\sigma}\hat{q}_t$

(4). The Phillips curve for production stage 1: $\pi_{1H,t} = \beta E_t\pi_{1H,t+1} + \lambda_1\hat{\gamma}_{1,t}$, where $\hat{\gamma}_{1,t} = \sigma\hat{c}_t + \frac{1-\gamma}{\gamma}\hat{q}_t - \hat{g}_{2H,t} - a_{1,t}$

(5). The Phillips curve for production stage 2: $\pi_{2H,t} = \beta E_t\pi_{2H,t+1} + \lambda_2\hat{\gamma}_{2,t}$, where $\hat{\gamma}_{2,t} = \gamma\phi\hat{g}_{2H,t} + \frac{1-\gamma}{\gamma}\hat{q}_t + (1-\phi)\sigma\hat{c}_t - a_{2,t}$

(6). Evolution of the relative price between the production stages: $\hat{g}_{2H,t} = \hat{g}_{2H,t-1} + \pi_{1H,t} - \pi_{2H,t}$

3.5 Discussion about the welfare loss function

There are two distortions in the model, i.e., the labor allocation distortion (caused by sticky prices along the production chain) and the terms of trade externality. Those terms measuring stage-specific unemployment gaps $\hat{h}_{1,t}$ and $\hat{h}_{2,t}$ show up in welfare loss function because of the distortion of labor allocation across different stages, while the stage-specific inflation shows up due to nominal rigidity

in each stage of production. Real exchange rate $\hat{q}_t$ appears due to a terms-of-trade externality.

In an open economy with a finite elasticity of foreign demand for export, the social planner wishes to exploit a domestic monopoly power in trade. This gives rise to a terms of trade effect. As the real exchange rate $\hat{q}_t$ and the relative price gap between production stage, $\hat{g}_{2H,t}$, jointly enter $\hat{h}_{1,t}$ and $\hat{h}_{2,t}$, we seen an interaction between the labor allocation distortion in a world of multiple stages of production and the terms of trade distortion in an open economy. This interaction suggests that the monetary policy discussion is not a simple sum of the results from an open-economy with one stage of production and a closed economy with two stages of production.

The second-order terms in the welfare loss function consist of three parts: (a) a consumption gap, and stage-specific unemployment gaps which can be written in terms of the consumption gap, (b) separate inflation terms for each production stage, and the relative price gap between production stages, and (c) the real exchange rate. The consumption gap is connected with the output gap and real exchange rate via $\hat{y} = \gamma\hat{c} + \frac{1-\gamma^2}{\gamma}\hat{q}$.

The welfare loss function indicates that targeting CPI and PPI are not satisfactory. Instead, the central bank needs to pay attention to stage-specific inflation terms along the production process as well as the price gap across the production stages. These terms will become more important as the number of production stages increase. This is elaborated in Appendix E.2 when we consider the case of N-stage production.

3.6 Effects of import tariff

Motivated by a rise in international trade tensions, we study how a change in trade policy, which alters the cost of supply chain trade, may affect the design of the monetary policy. We consider two counter-factual cases: a high tariff and a low tariff. In each case, the import tariff affects the welfare loss function through its impact on the steady-state shares of domestic demand in total demand for domestically produced goods in the two stages of production, i.e., $\bar{a}_1$ and $\bar{a}_2$.

It can be shown that

$$\frac{\bar{a}_2}{1-\bar{a}_2} = f_2(*) \cdot T^{(1-\gamma)(1+\phi\gamma)(1-\frac{1}{\sigma})}$$

and

$$\frac{\bar{a}_1}{1-\bar{a}_1} = f_2(*) \cdot \frac{1}{1-\bar{a}_2}$$

where $f_1(*)$ and $f_2(*)$ are functions of exogenous foreign variables. The explicit expression of $f_1(*)$ and $f_2(*)$ can be found in Appendix D. We then proceed with the following proposition.[16]

Proposition 1: If the relative risk aversion $\sigma = 1$, a higher import tariff does not affect the steady-state allocation, i.e., $\frac{\partial \bar{a}_1}{\partial T} = \frac{\partial \bar{a}_2}{\partial T} = 0$ and $\frac{\partial L_1/L}{\partial T} = \frac{\partial L_2/L}{\partial T} = 0$; if $\sigma > 1$, a higher import tariff

[16] We assume that $\gamma < 1$, i.e., the share of import is not zero.

will lead to a higher share of domestic demand for domestically produced goods, i.e., $\frac{\partial \bar{a}_1}{\partial T}, \frac{\partial \bar{a}_2}{\partial T} > 0$, and the labor share in the upstream production relative to the downstream decreases, i.e., $\frac{\partial L_1/L}{\partial T} < 0$ and $\frac{\partial L_2/L}{\partial T} > 0$.

4 Numerical analysis of monetary policy rules

We consider a family of simple monetary policy rules in the style of Taylor rules. As discussed in Section 3, the first order approximation for the equilibrium conditions is not enough for welfare analysis, and thus we estimate the general nonlinear model specified in Section 2 with $N = 2$ and approximate the equilibrium to the second order expansion. We will relax the assumption of $\psi = 0$ and include a broader set of stochastic shocks, i.e., stage-specific productivity shocks and shocks on foreign consumption (which are the two types of shocks most commonly studied in the literature).

We consider the following set of policy rules: (a) a classic Taylor rule that is based on CPI inflation (and output gap), (b) a Gali-Monacelli rule (2005) that replaces CPI inflation with PPI inflation, (c) a rule that targets separate inflation terms for each production stage (i.e., stage-specific producer price indices), (d) combinations of the above with the real exchange rate, and (e) an exchange rate peg. For each rule, we examine both the case with imposed coefficients as specified in the literature (such as 1.5 and 0.5 on CPI inflation and output gap in the classic Taylor rule) and optimally estimated coefficients.

With regard to the effects of trade frictions on monetary policy, we conduct two counter-factual analyses: (1) an increase in the import tariff (in a world with identical degrees of price stickiness in all production stages); (2) an increase in the import tariff (in a world with a higher degree of price stickiness in the downstream stage than the upstream stage). As discussed in Section 3, the import tariff affects the welfare loss function by altering steady-state allocation.

Asymmetric price stickiness along the production chain appears to be empirically relevant. Cornille and Dossche (2008) and Nakamura and Steinsson (2008) both suggest that the price contracts in more upstream production stages tend to have a shorter duration than those in the finished product sectors. Gong, Wang, and Zou (2016) argue that different degrees of price stickiness in different stages would affect which price index (i.e., CPI, final-goods-based PPI, or intermediate-goods-based PPI) should be included in a simple monetary policy rule.[17]

4.1 Model Parameters

We begin with the calibration of baseline model parameters. Each period in the model corresponds to a quarter. Following Gali and Monacelli (2005) and De Paoli (2009), the model economy is

[17] Instead of including all stage-specific price indices in a simple monetary rule, Gong, Wang, and Zou (2016) consider CPI-based Taylor rule, final-goods PPI-based Taylor rule, and intermediate-goods PPI-based Taylor rule, where the Taylor-type rule (only) includes one specific inflation index and output gap.

meant to resemble Canada in some key dimensions. The calibrated parameters are summarized in Table 1.

The subjective discount factor is set to be $\beta = 0.99$, which implies a 4% annual real interest rate in the steady state. Following Arellano (2008) and De Paoli (2009), the inverse of intertemporal elasticity of substitution is set to be $\sigma = 2$. The parameter in Calvo pricing in both production stages is set to be $\alpha_1 = \alpha_2 = 0.66$, implying an average contract duration of 3 quarters. Following Benigno and Woodford (2005), the elasticity of substitution in the consumption bundle is set to be $\theta = 10$. Consistent with Huang and Liu (2005), we set the share of intermediate goods in production to be $\phi = 0.6$.

We set the shares of goods sold to the domestic markets in both stages to be $\bar{a}_1 = \bar{a}_2 = 0.7$, implying a 30% export share of GDP (approximately the level observed for Canada). The parameters $\bar{a}_1, \bar{a}_2$ are the sufficient statistics for the (exogenous) foreign demand for output in the two production stages. Following Gali and Monacelli (2005), the process of productivity shock is set to follow an AR(1) process with persistence parameter $\rho_a = 0.66$ and standard deviation $\sigma_a = 0.0071$, which is calibrated from Canada data. Following De Paoli (2009), the foreign consumption shock is set to an AR(1) process with persistence $\rho_{C^*} = 0.66$ and standard deviation $\sigma_{C^*} = 0.0129$. We normalize the import tariff in the baseline numerical exercise to be $T = 1$ (implying a zero tariff).

Table 1: Parameter calibration

Parameter	Name	Value	Notes
β	Subjective discount factor	0.99	4% annual interest rate with a quarterly model
σ	Inverse of intertemporal elasticity of substitution	2	Standard value in literature, e.g., Arellano (2008)
α_1,α_2	Parameter in Calvo pricing	0.66	An average length of price contract of 3 quarters
γ	Share of goods purchased in domestic market	0.6	Implying 40% import share of GDP
θ	Elasticity of substitution in consumption bundle	10	Following Benigno and Woodford (2005)
ϕ	Share of intermediate goods in production	0.6	Following Huang and Liu (2005)
$\bar{a}_1, \bar{a}_2$	Share of goods selling to domestic market	0.7	Implying 30% export share of GDP
ρ_a	Persistency of productivity shock	0.66	Following Gali and Maonacelli (2005), De Paoli (2009)
σ_a	Standard deviation of productivity shock	0.0071	Following Gali and Monacelli (2005), De Paoli (2009)
ρ_{C^*}	Persistency of foreign consumption shock	0.66	Following De Paoli (2009)
σ_{C^*}	Standard deviation of foreign consumption shock	0.0129	Following De Paoli (2009)

4.2 Comparing policy rules and welfare losses

Since the numerical estimation is conducted based on the general nonlinear model specified in Section 2 with $N = 2$ and the equilibrium is estimated up to second order approximation, we need to define explicitly the welfare loss. We define the welfare loss χ in percentage term relative to the

steady-state consumption , i.e.,

$$E\Sigma_{t=0}^{\infty}\beta^t[\frac{[C(1-\chi)]^{1-\sigma}-1}{1-\sigma}-\frac{L^{1+\psi}}{1+\psi}]=V^a$$

where C and L are steady-state consumption and employment, and V^a is the welfare estimated from a given policy rule.

We define the aggregate PPI index as a sales-weighted average of producer prices index, [18]

$$\pi_{PPI}=(1-\omega)\pi_{1H}+\omega\pi_{2H}$$

where $\omega=\frac{P_{1h}(Y_{1h}+Y_{1h}^X)}{P_{1h}(Y_{1h}+Y_{1h}^X)+P_{2h}(Y_{2h}+Y_{2h}^X)}$ is the relative sale-weight in the upstream production stage.

The estimated optimal coefficients in simple rules are shown in Table 2. "Peg" in the table indicates a nominal exchange rate peg. By comparing the alternative policy rules P1-P9 and the policy of exchange rate peg, a simple interest rate rule that targets all stage-specific producer price indices outperforms any other policy rules. Adding output gap or real exchange rate does not visibly contribute to better performance (e.g., P1, P6, and P9).

When targeting stage-specific inflation indices, the estimated optimal coefficients in downstream producer price index are larger (e.g., P1, P5, and P9). Since targeting stage-specific price indices yields lower welfare loss than just targeting the aggregate PPI index (e.g., P3 and P6), in general, the optimal weights on different stages of production would not be proportional to the sales of production stages.[19] It is worth noting that, cost shocks (caused by productivity shocks in upstream stages) travel from upstream to downstream stages, potentially reversing the order of the optimal relative weights. In our calibration, since the labor weight is larger in downsteam stages (as a fraction of total employment), demand shocks (caused by final demand shocks or productivity shocks in downstream stages) dominate cost shocks.

If only aggregate price indices are available, then a simple interest rule that targets PPI (e.g., P3, P4, and P5) is significantly better than those rules that target just CPI. On the other hand, from P3 and P6, a simple interest rule that targets aggregate PPI captures much of the welfare gains from targeting all stage-specific inflation indices. A pure exchange rate peg performs the worst.

For each type of policy rule, besides optimally estimated coefficients, we also evaluate a version where the coefficients are imposed using the values suggested in the literature. Table 3 reports the welfare performance proposed by the original Taylor calibration (i.e., Taylor, 1993), alternative rules adopted in Gali and Monacelli (2005), and in Huang and Liu (2005). Evidently, simple monetary policy rules that target aggregate PPI, or stage-specific producer indices, outperform those targeting

[18] An alternative definition is a value-added-weighted average of producer prices. Most PPI measures in practice are sales weighted, e.g., US Bureau of Labor Statistics, or OECD.

[19] The same statement is true when we define the aggregate PPI index to be weighted by value-added.

Table 2: Optimal alternative simple rules of monetary policy

	π_{1H}	π_{2H}	π_{PPI}	π_{CPI}	$\tilde{c}$	q	i_{t-1}	Welfare loss (0.01%)
P1	6.3861	9.8675			0.7570	-1.5549	0.2048	0.319
P2				5.1882	0.0006		0.0215	0.577
P3			9.9999		0.1000		1.0441	0.329
P4			9.9888	0.0009	0.0004		0.8085	0.326
P5			9.1138	0.0747	0.1697	-0.6933	0.1432	0.320
P6	5.2870	9.9965			0.0001		0.6948	0.322
P7	4.2548				0.0000		0.8757	0.572
P8		2.6975			0.0003		0.8327	0.401
P9	5.2741	9.9825					0.7004	0.322
Peg								0.871

Notes: PPI index (sales-weighted): $\pi_{PPI} = (1-\omega)\pi_{1H} + \omega\pi_{2H}$ with $\omega = \frac{P_{1h}(Y_{1h}+Y_{1h}^X)}{P_{1h}(Y_{1h}+Y_{1h}^X)+P_{2h}(Y_{2h}+Y_{2h}^X)}$.
CPI index: $\pi_{CPI,t} = \pi_t$.

Table 3: Alternative simple rules of monetary policy in literature

	π_{1H}	π_{2H}	π_{CPI}	$\tilde{c}$	i_{t-1}	Welfare loss (0.01%)	Notes
P1			1.5	0.5		1.870	Taylor (1993)
P2	1.42	1.68		0.04	1.12	0.372	Huang and Liu (2005)
P3			1.5			0.849	Gali and Monacelli (2005) – CPI based
P4		1.5				1.226	Gali and Monacelli (2005) – PPI based

just the CPI index.

4.3 Effects of higher import tariff

Now, we consider the case of a higher import tariff in the steady state. With all exogenous foreign variables unchanged as in the baseline case as in Section 4.2, the import tariff is doubled (to $T = 2$). In such a scenario, the shares of demand for domestic goods in the two production stages become $\bar{a}_1 = 0.73$ and $\bar{a}_2 = 0.74$, which is consistent with Proposition 1 and larger than the original calibration. While the central bank cannot undo directly the increase in tariff, it wishes to stabilize the economy given the (new) steady-state under a higher import tariff. We re-evaluate all alternative monetary policy rules in the new steady state.

The estimated results are showed in Table 4. Compared with the case of lower import tariff as reported in Table 2, in general, even though the central bank has already chosen their best response policy rule, a higher import tariff induces a greater welfare loss. Intuitively, a higher import tariff increases firm production cost and the prices of output. Given price-elastic foreign demand, a higher selling price will reduce the income of households. Not surprisingly, the simple monetary rule that targets stage-specific producer price index still outperforms other policy rules, and it is significantly

welfare-improving by not just targeting CPI index.

Table 4: Optimal alternative simple rules of monetary policy with higher imported tariff

	π_{1H}	π_{2H}	π_{PPI}	π_{CPI}	$\tilde{c}$	$\tilde{Q}$	i_{t-1}	Welfare loss (0.01%)
P1	5.0113	8.8786			0.2298	-0.7860	0.1629	0.377
P2				5.2378	0.0000		0.0007	0.684
P3			9.9988		0.1000		1.0509	0.390
P4			9.3569	0.0487	0.0766	-0.5579	0.1445	0.379
Peg								1.056

PPI index (sales-weighted): $\pi_{PPI} = (1-\omega)\pi_{1H} + \omega\pi_{2H}$ with $\omega = \frac{P_{1h}(Y_{1h}+Y_{1h}^X)}{P_{1h}(Y_{1h}+Y_{1h}^X)+P_{2h}(Y_{2h}+Y_{2h}^X)}$.

CPI index: $\pi_{CPI,t} = \pi_t$.

It is worth noting that the central bank cannot offset the effects of a higher import tariff by simply re-valuating the domestic currency. Appreciation in the domestic currency reduces the cost of imported intermediate inputs or imported final consumption goods, but also increases the price of domestically produced outputs of both intermediate goods and final goods. Since the foreign demand is price-elastic, firms obtain a smaller revenue from exporting.

It may be interesting to note that the relative importance of the downstream stage over the upstream stage has increased from the no-tariff baseline to the high-tariff scenario. The ratio of the weight on the downstream producer price inflation and that on the upstream inflation has changed from 1.55 (under Policy Rule P1 in Table 2) to 1.77 (under Policy Rule P1 in Table 4). As shown in Proposition 1, a higher import tariff reduces the relative labor weight in the upstream stage as a fraction of total employment, and thus reduces the relative importance of the upstream inflation index in the objective function of the central bank (i.e., welfare loss function).

4.4 Asymmetric price stickiness

We now consider the implications of uneven price stickiness in different stages of production. Cornille and Dossche (2008) and Nakamura and Steinsson (2008) argue that the duration of price contract in upstream production stages is shorter than downstream stages. For instance, Nakamura and Steinsson (2008) document that the median duration of price contract for finished producer goods in 1998-2005 is about 8.7 months, while the median duration of price contract for intermediate goods producer goods is about 7.0 months. We reduce the Calvo pricing parameter in the first stage of production to be $\alpha_1 = 0.5$, indicating an average of length of price contract of 2 quarters.

The estimated results are showed in Table 5. The welfare loss in general becomes smaller compared with the baseline case as in Table 2 since the price stickiness in aggregate becomes smaller. Furthermore, by comparing the coefficients of stage-specific producer price indices in Policy Rule P1 of both two tables, as the price is less sticky in upstream, the relative weight on

Table 5: Optimal alternative simple rules of monetary policy with lower price stickiness in upstream production

	π_{1H}	π_{2H}	π_{PPI}	π_{CPI}	$\tilde{c}$	$\tilde{Q}$	i_{t-1}	Welfare loss (0.01%)
P1	3.1846	9.8760			0.0100	-0.5776	0.0328	0.288
P2				5.0515	0.0001		0.0038	0.544
P3			9.9981		0.1001		1.0715	0.312
P4			9.8058	0.0240	0.0110	-0.6126	0.0174	0.299
Peg								0.893

PPI index (sales-weighted): $\pi_{PPI} = (1-\omega)\pi_{1H} + \omega\pi_{2H}$ with $\omega = \frac{P_{1h}(Y_{1h}+Y_{1h}^X)}{P_{1h}(Y_{1h}+Y_{1h}^X)+P_{2h}(Y_{2h}+Y_{2h}^X)}$.

CPI index: $\pi_{CPI,t} = \pi_t$.

upstream producer price index versus downstream producer price index becomes smaller.[20]

5 Concluding remarks

This paper studies the implications of global supply chains on the design of optimal monetary policy through a small-open economy New Keynesian model with multiple stages of production. We show that it is optimal for a simple monetary policy rule to target production-stage-specific producer price indices; if only aggregate price indices are available, targeting aggregate PPI price index compared with just targeting CPI price index is welfare improving. As the length of production chain increases, aggregate producer price index occupies a larger proportion in the objective function of central bank (i.e., welfare loss function) in contrast to consumer price index, which suggests that the central bank needs to pay more attention to the producer price indices.

Our framework allows us to study the implications of trade frictions such as an increase in the import tariffs on the design of monetary policy. Even in the case of central bank already implementing optimal monetary policy (in form of simple response rules), the welfare loss is higher compared with the situation of lower import tariff due to higher input prices along the production chain. In addition, we show that, as the price is less sticky in upstream production, the relative weight on upstream producer price index versus downstream producer price index in an optimal simple policy rule becomes smaller.

In practice, official statistics agencies in some countries such as the United States and Australia actually collect producer price information, and have a system of producer price indices in a manner of vertical production. For instance, US Bureau of Labor Statistics have a system of producer price indices featuring four-stage of vertical production chain, which is named under the PPI Final

[20] This is consistent with the findings in Gong, Wang, and Zou (2016). They argue that, when the degree of price stickiness for intermediate-goods production is high, the central bank should follow intermediate-goods PPI-based Taylor rule. To be noted, in their paper, there is no distortion from labor re-allocation along the production chain since labor is only used in the production of intermediate goods in their model.

Demand-Intermediate Demand indice (used to be called "stage-of-processing" system of producer price indices). It is basically a counterpart in reality for the stage-specific producer price indices as defined in our model. Nevertheless, to the best of our knowledge, the central banks have not really exploited these information in implementing monetary policies.

There are several other directions to further the discussion in this paper. The model adopted in our analysis assumes producer currency pricing, and as mentioned in the literature (e.g., Smets and Wouters, 2002; Corsetti, Dedola, and Leduc, 2011; Engel, 2011), local currency pricing may have a different implications on the trade-offs for monetary policy. Another direction could be to consider a more broad set of exogenous shocks, especially shocks on foreign input prices or foreign demand shocks along the production chain. One challenge in line with this extension would be how to appropriately calibrate the shocks to empirical data.

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A Equilibrium characterization with N-stage of production in a small-open economy

A.1 The steady state equilibrium

We first characterize the steady state with perfect foresight. The steady state is defined as the equilibrium under non-stochastic and constant exogenous variables. Since the whole economy does not change with timing, we can ignore the timing index t in all variables, and $A_n = 1$ for $n = 1, 2, \ldots, N$. The optimal pricing decision for firms at stage n, $n = 1, 2, \ldots, N$, becomes

$$P^o_{nH} = \Psi_n = P_{nH} = \bar{P}_{nH}$$

and for $n = 2, \ldots, N$, we have

$$\bar{P}_n = \bar{P}^{\gamma}_{(n-1)H} \bar{P}^{1-\gamma}_{(n-1)F}$$

where $\bar{P}_{(n-1)F} = T\mathcal{E}P^*_{(n-1)F}$.

Now, we solve for the price indices in terms of wages and derive the labor demand function. Note that $\Psi_n = \bar{P}^{\phi}_n W^{1-\phi}$ for $n = 2, 3, \ldots, N$ with $\Psi_1 = W$. By substituting $\bar{P}_n$, the relationship of output price index across adjacent stages is given by

$$P_{nH} = W^{1-\phi}(\bar{P}_n)^{\phi}$$

$$= W^{1-\phi} P^{\gamma\phi}_{(n-1)H} P^{(1-\gamma)\phi}_{(n-1)F}$$

for $n = 2, \ldots, N$ and $P_{1H} = W$.

By writing all price index in terms of wage and exogenous variables through forward induction, we get

$$P_{nH} = W^{(1-\phi)\frac{1-(\gamma\phi)^{n-1}}{1-\gamma\phi}+(\gamma\phi)^{n-1}} (T\mathcal{E})^{\phi(1-\gamma)\frac{1-[\phi(1-\gamma)]^{n-1}}{1-\phi(1-\gamma)}} \Pi^{n-1}_{i=1} (P^*_{nF})^{[\phi(1-\gamma)]^{n-i}} \tag{14}$$

for $n = 2, \ldots, N$ with $P_{1H} = W$.

Since $P_{nH}(u) = P_{nH}$ for $u \in [0, 1]$ in steady state, by Equation (11), for $n = 1, \ldots, N$, we have

$$Y^d_{nH,t}(u) = \bar{Y}^d_{nH,t}$$

Together with goods market clear condition $Y_{nH} = \bar{Y}^d_{nH}$, and factor market demand function (8) and (7), for $n = 2, \ldots, N$, we get

$$\bar{Y}^d_n = \phi \frac{\Psi_n}{\bar{P}_n} [\bar{Y}^d_{nH} + Y^X_{nH}]$$

$$L_n^d = (1-\phi)\frac{\Psi_n}{W}[\bar{Y}_{nH}^d + Y_{nH}^X]$$

$$\bar{Y}_{(n-1)H}^d = \frac{\gamma \bar{P}_n \bar{Y}_n^d}{\bar{P}_{(n-1)H}}$$

where $\bar{Y}_{NH}^d = \gamma C P_{nH}^{-(1-\gamma)} \bar{P}_{NF}^{1-\gamma}$, and $L_1^d = \frac{\Psi_1}{W}\bar{Y}_{1H}^d$. By substituting the price index and unit cost function in each stage, for $n = 1, \ldots, N$, and through backward induction, the factor demand functions for labor can be written in an implicit form as

$$L_n^d = f(W, C, \mathcal{E}, T, P_{1F}^*, \cdots, P_{NF}^*, P_{nH}^*, \cdots, P_{NH}^*, Y_{nH}^*, \cdots, Y_{NH}^*) \tag{15}$$

Therefore, by summing up labor demand across all stages, the total labor demand function becomes

$$L^d = \sum_{n=1}^{N} L_n^d \tag{16}$$

Therefore, the three equations (1), (5), and (16) fully characterize the steady-state wage, consumption and employment.

A.2 The flexible price equilibrium

In this subsection, we solve the flexible price equilibrium similarly as in the steady state equilibrium. In flexible price equilibrium, $\alpha_n = 0$ for $\forall n$, and then the optimal pricing decision for firms at stage n, $n = 1, 2, \ldots, N$, becomes

$$P_{nH,t}^o = \Psi_{n,t} = P_{nH,t} = \bar{P}_{nH,t}$$

and for $n = 2, \ldots, N$, we have

$$\bar{P}_{n,t} = \bar{P}_{(n-1)H,t}^{\gamma} \bar{P}_{(n-1)F,t}^{1-\gamma}$$

Similar to the steady state case, we solve for the price indices in terms of wages and productivity. Note that $\Psi_{n,t} = \bar{P}_{n,t}^{\phi} W_t^{1-\phi}/A_{n,t}$ for $n = 2, 3, \ldots, N$ with $\Psi_1 = W_t/A_{1,t}$. By substituting $\bar{P}_{n,t}$, the relationship of price index across adjacent stages is given by

$$P_{nH,t} = W^{1-\phi} P_{(n-1)H,t}^{\gamma\phi} P_{(n-1)F,t}^{(1-\gamma)\phi}$$

for $n = 2, \ldots, N$ and $P_{1H,t} = W_t/A_{1,t}$.

By writing all price index in terms of wage through forward induction, we similarly get

$$P_{nH,t} = W^{(1-\phi)\frac{1-(\gamma\phi)^{n-1}}{1-\gamma\phi} + (\gamma\phi)^{n-1}} (\mathcal{E}_t T_t)^{\phi(1-\gamma)\frac{1-[\phi(1-\gamma)]^{n-1}}{1-\phi(1-\gamma)}}$$

$$\cdot \Pi_{i=1}^{n-1}(P_{nF}^*)^{[\phi(1-\gamma)]^{n-i}} \cdot \Pi_{i=1}^{n}(A_{i,t})^{-(\gamma\phi)^{n-i}}$$

for $n = 2, \ldots, N$ with $P_{1H,t} = W_t/A_{1,t}$.

Due to flexible price, the expressions for factor market in each stage of production are exactly the same as in the steady state case. Therefore, we can derive labor demand function in each stage, i.e., $n = 1, \ldots, N$, as

$$L_{n,t}^{fd} = f(W_t, C_t, \mathcal{E}_t, T_t, P_{1F,t}^*, \cdots, P_{NF,t}^*, P_{nH,t}^*, \cdots, P_{NH,t}^*, Y_{nH,t}^*, \cdots, Y_{NH,t}^*, A_{1,t}, \cdots, A_{N,t})$$

where we denote the labor demand under flexible price with an upper symbol f.

The total labor demand function becomes

$$L_t^{fd} = \sum_{n=1}^{N} L_{n,t}^{fd} \tag{17}$$

The three equations (1), (5), and (17) fully characterize the wage, consumption and employment in flexible price equilibrium, where the consumption can be written in

$$C_t^f = f(T_t, P_{1F,t}^*, \cdots, P_{NF,t}^*, P_{nH,t}^*, \cdots, P_{NH,t}^*, Y_{nH,t}^*, \cdots, Y_{NH,t}^*, A_{1,t}, \cdots, A_{N,t})$$

which can be re-written in log-linearized form, i.e.,

$$c_t^f = f(t_t, p_{1F,t}^*, \cdots, p_{NF,t}^*, p_{nH,t}^*, \cdots, p_{NH,t}^*, y_{nH,t}^*, \cdots, y_{NH,t}^*, a_{1,t}, \cdots, a_{N,t})$$

By the Euler equation (4), the IS curve is characterized by

$$c_t^f = E_t(c_{t+1}^f) - \frac{1}{\sigma}[i_t - E_t(\pi_{t+1}) - \rho]$$

which implies the natural rate of interest as

$$r\bar{r}_t = \sigma E_t\{c_{t+1}^f - c_t^f\} \tag{18}$$

A.3 The sticky-price equilibrium

We now derive New Keynesian Phillips curves for each stage as a function of relative price gap and output gap, and characterize the equilibrium with sticky prices. Similar to the derivation in Gali (2015), in each stage of production $n = 1, 2, \ldots, N$, firms' optimal pricing decision gives

$$\pi_{n,t} = \beta E_t \pi_{n,t+1} + \lambda_n \hat{\gamma}_{n,t}$$

where $\lambda_n = \frac{(1-\beta\alpha_n)(1-\alpha_n)}{\alpha_n}$ and $\hat{\gamma}_n$ is the log-derivation of real marginal cost from steady-state equilibrium, i.e.,

$$\hat{\gamma}_{n,t} = ln(\Psi_{n,t}/P_{nH,t}) - ln(\Psi_n/P_{nH})$$

where Ψ_n and P_{nH} is the marginal cost and aggregate price in stage n under steady state equilibrium.

Given $P_{nH,t} = \bar{P}_{nH,t}$ for $n = 1, 2, \ldots, N$ and the expression of production cost function, we have for stages $n = 2, \ldots, N$

$$\hat{\gamma}_{n,t} = \gamma\phi\hat{g}_{nH,t} + (1-\gamma)\phi\hat{g}_{nF,t} + (1-\phi)(\hat{w}_t - \hat{p}_{nH,t}) - a_{n,t}$$

where $\hat{g}_{nH,t}$ and $\hat{g}_{nF,t}$ are the log-deviation of relative output price gap with respect to input price from steady state equilibrium, i.e., $\hat{g}_{nH,t} = ln(\frac{\bar{P}_{(n-1)H,t}}{P_{nH,t}}) - ln(\frac{\bar{P}_{(n-1)H}}{P_{nH}})$, and $\hat{g}_{nF,t} = ln(\frac{\bar{P}_{(n-1)F,t}}{P_{nH,t}}) - ln(\frac{\bar{P}_{(n-1)F}}{P_{nH,t}})$. Since $\bar{P}_{(n-1)H,t} = P_{(n-1)H,t}$, $\hat{g}_{nH,t}$ also indicates the log-deviation of relative output price gap between adjacent stages n and $n-1$ from steady state equilibrium. By the definition of $\hat{g}_{nH,t}$ and $\hat{g}_{nF,t}$, we also have $\hat{p}_{nH,t} = \hat{p}_{NH,t} + \Sigma_{i=n+1}^{N}\hat{g}_{iH,t}$ for $n = 1, \ldots, N-1$.

Following Huang and Liu (2005), without losing generality, we assume $\psi = 0$. Then, from 3, we have $w_t - p_t = \sigma c_t$. By substituting $\hat{w}$ and $\hat{p}_{nH,t}$ into the real marginal cost function, for $n = 2, \ldots, N-1$, we get

$$\hat{\gamma}_{n,t} = \gamma\phi\hat{g}_{nH,t} + (1-\gamma)\phi\hat{g}_{nF,t}$$

$$+(1-\phi)[\sigma\check{c}_t - (1-\gamma)\hat{p}_{NH,t} + (1-\gamma)\hat{p}_{NF,t} - \Sigma_{i=n+1}^{N}\hat{g}_{iH,t}] - a_{n,t}$$

and

$$\hat{\gamma}_{N,t} = \gamma\phi\hat{q}_{NH,t} + (1-\gamma)\phi\hat{q}_{NF,t} + (1-\phi)[\sigma\hat{c}_t - (1-\gamma)\hat{p}_{NH,t} + (1-\gamma)\hat{p}_{NF,t}] - a_{N,t}$$

$$\hat{\gamma}_{1,t} = \sigma\hat{c}_t - (1-\gamma)\hat{p}_{NH,t} + (1-\gamma)\hat{p}_{NF,t} - \Sigma_{i=2}^{N}\hat{q}_{iH,t} - a_{1,t}$$

Note that $\hat{p}_{nF,t} = \hat{e}_t$ and $\hat{g}_{nF,t} = \hat{e}_t - \hat{p}_{nH,t}$ for $n = 1, \ldots, N$, and $\hat{q}_t = \gamma(\hat{e}_t - \hat{p}_{NH,t})$.

After log-linearizing the Euler equation of households around steady state and subtracting the natural rate IS curve, we get the IS curve with stick price as

$$\hat{c}_t = E_t\hat{c}_{t+1} - \frac{1}{\sigma}[i_t - E_t(\pi_{t+1}) - \bar{rr}_t]$$

where $\bar{rr}_t$ is the natural rate of interest.

The law of motion for relative price gap between stage n and stage $n-1$, for $n = 2, 3, \ldots, N$, is characterized by

$$\hat{q}_{nH,t} = \hat{q}_{nH,t-1} + \pi_{(n-1)H,t} - \pi_{nH,t}$$

Give the policy rule $\{i_t, \mathcal{E}_t\}$, risk-sharing condition 6, IS curve, the stage-specific Phillips curves, and the law of motion for relative price gap fully pin down the dynamic equilibrium under sticky prices.

A.4 Stage-specific employment in small open-economy with N-stage production

We derive the stage-specific employment gap in terms of output gap and relative price gap. By the factor demand function (7), (8), and (11) in each stage, and substituting with the unit cost, for $n = 2, 3, \ldots, N$, we have

$$lnL_{n,t} = ln(1-\phi) + \phi[ln\bar{P}_{n,t} - lnW_t] - lnA_{n,t} + ln[Y^d_{nH,t} + Y^X_{nH,t}] + d_{n,t} \tag{19}$$

where $d_{n,t} = ln(\int_0^1 (\frac{P_{n,t}(u)}{P_{n,t}})^{-\theta} du)$ and $lnL_{1,t} = -lnA_{1,t} + ln[Y^d_{1H,t} + Y^X_{1H,t}] + d_{1,t}$.

By the factor demand function for intermediate goods and labor in each stage, i.e., Expression (7) and (8), for $n = 2, 3, \ldots, N$, we get

$$lnL_{n,t} = ln(\frac{1-\phi}{\phi}) + ln\bar{P}_{n-1,t} - lnW_t + ln\bar{Y}^d_{n,t}$$

Also, not that

$$Y^d_{(n-1)H,t} = \frac{\bar{Y}^d_{n,t}\bar{P}_{n,t}\gamma}{\bar{P}_{(n-1)H,t}}$$

Then, by substituting $lnY^d_{nH,t}$ using $lnL_{n,t}$ and $ln\bar{Y}^d_{(n+1),t}$ into (19), we get backward induction relationship for the stage-specific employment, i.e., for $n = 2, 3, \ldots, N$,

$$l_{n,t} = l_{n+1,t} + d_{n,t} + g_n(\hat{c}_t, e_t, t_t, a_{1,t}, \ldots, a_{N,t}, p^*_{1F,t}, \ldots, p^*_{NF,t}, p^*_{nH,t}, \ldots, p^*_{NH,t}, y^*_{nH,t}, \ldots, y^*_{NH,t}, \hat{q}_{1,t}, \ldots, \hat{q}_{N,t})$$

where $l_{n,t} = lnL_{n,t}$.

B Aggregate price expression with two-stage production

Given exogenous foreign variables and import tariff constant, the aggregate inflation index (CPI index) can be written as

$$\pi_{t+1} = \gamma\pi_{2H,t+1} + (1-\gamma)\Delta e_{t+1}$$

Since $\hat{q}_t = \gamma(\hat{e}_t - \hat{p}_{2H,t})$, we have

$$\Delta e_{t+1} = \frac{\Delta\hat{q}}{\gamma} + \pi_{2H,t}$$

Then, the aggregate inflation index can be re-written as

$$\pi_{t+1} = \gamma\pi_{2H,t+1} + (1-\gamma)\Delta e_{t+1}$$

$$= \pi_{2H,t} + \frac{1-\gamma}{\gamma}\Delta\hat{q}_t$$

C Stage-specific employment with two-stage production

In this section, we derive explicit expression of the employment gap with two-stage of production, i.e, $N = 2$. As specified in Section A.4, and also note that $\hat{w}_t = \gamma\hat{p}_{2H,t} + (1-\gamma)\hat{p}_{2f,t} + \sigma\hat{c}_t$, $\hat{p}_{1H,t} = \hat{g}_{2,t} + \hat{p}_{2H,t}$, and $\hat{p}_{1F,t} = \hat{p}_{2F,t} = \hat{e}_t$,[21] we have for the second stage

$$\hat{l}_{2,t} = \phi[\gamma\hat{p}_{1H,t} + (1-\gamma)\hat{p}_{1F,t} - \hat{w}_t] + \bar{a}_2[\gamma\hat{p}_{2H,t} + (1-\gamma)\hat{p}_{2F,t} - \hat{p}_{2H,t} + \hat{c}_t]$$

$$+(1-\bar{a}_2)(\hat{e}_t - \hat{p}_{2H}) - a_{2,t} + d_{2,t}$$

$$= \phi[\gamma\hat{g}_{2,t} - \sigma\hat{c}] + \bar{a}_2[(1-\gamma)(\hat{e}_t - \hat{p}_{2H,t})] + (1-\bar{a}_2)(\hat{e}_t - \hat{p}_{2H,t}) - a_{2,t} + d_{2,t}$$

$$= (\bar{a}_2 - \phi\sigma)\hat{c}_t + \phi\gamma\hat{g}_{2,t} + \frac{1-\bar{a}_2\gamma}{\gamma}\hat{q}_t - a_{2,t} + d_{2,t}$$

where, in the first equality, we have used $Y^d_{1H,t} = \frac{\gamma\bar{P}_{2,t}\bar{Y}^d_{2,t}}{\bar{P}_{1H,t}}$, $Y^{Xd}_{1H,t} = \frac{Y^*_{1H}P^*_{1H}\mathcal{E}_t}{P_{1H,t}}$, $Y^d_{2H,t} = \frac{\gamma C_t P_t}{P_{2H,t}}$, and $Y^d_{2H,t} = \frac{Y^*_{2H}P^*_{2H}\mathcal{E}_t}{P_{2H,t}}$; in the last equality uses the condition that $\hat{q}_t = \gamma(\hat{e}_t - \hat{p}_{2H,t})$.

For the first stage, the employment is given by

$$\hat{l}_{1,t} = \bar{a}_1(\hat{\bar{p}}_{2,t} + \hat{\bar{y}}^d_{2,t} - \hat{p}_{1H,t}) + (1-\bar{a}_1)(\hat{e}_t - \hat{p}_{1H,t}) + d_{1,t} - a_{1,t}$$

$$= \bar{a}_1(\hat{\bar{p}}_{2,t} + \hat{w}_t + \hat{l}_{2,t} - \hat{\bar{p}}_{2,t} - \hat{p}_{1H,t}) + (1-\bar{a}_1)(\hat{e}_t - \hat{p}_{1H,t}) + d_{1,t} - a_{1,t}$$

$$= \bar{a}_1\hat{l}_{2,t} + \bar{a}_1[\gamma\hat{p}_{2H,t} + (1-\gamma)\hat{e}_t + \sigma\hat{c}_t - \hat{g}_{2,t} - \hat{p}_{2H,t}] + (1-\bar{a}_1)[\hat{e}_t - \hat{p}_{2H,t} - \hat{g}_{2,t}]$$

$$= [-\bar{a}_1\sigma\phi + \bar{a}_1\bar{a}_2 + \bar{a}_1\sigma]\hat{c}_t + [\bar{a}_1\phi\gamma - 1]\hat{g}_{2H,t}$$

$$+\frac{1+\bar{a}_1 - \bar{a}_1\gamma - \bar{a}_1\bar{a}_2\gamma}{\gamma}\hat{q}_t + \bar{a}_1 d_{2,t} + d_{1,t} - a_{1,t} - \bar{a}_1 a_{2,t}$$

where the second equality uses the condition that $\hat{l}_{2,t} = \hat{\bar{y}}^d_{2,t} - \hat{w}_t + \hat{\bar{p}}_{2,t}$.

[21] We have used the assumption of $\psi = 0$.

D Steady-state share of domestic demand in total demand with respect to import tariff

In this section, we characterize how the import tariff affects the steady-state share of domestic demand in total demand in both of two production stages, i.e., $\bar{a}_1$ and $\bar{a}_2$, as specified in Section 3.

By the definition of $\bar{a}_2$, we have

$$\frac{\bar{a}_2}{1-\bar{a}_2} = \frac{\gamma CP}{Y^*_{2H}P^*_{2H}\mathcal{E}}$$

$$= \frac{\gamma C^*(P^*_t)^{1/\sigma}}{Y^*_{2H}P^*_{2H}} P_t^{1-\frac{1}{\sigma}} \mathcal{E}_t^{1/\sigma}$$

$$= \frac{\gamma C^*(P^*_t)^{1/\sigma}}{Y^*_{2H}P^*_{2H}} \{[W^{1-\phi+\gamma\phi}(\mathcal{E}T)^{(1-\gamma)\phi}(P^*_{1F})^{(1-\gamma)\phi}]^{\gamma}[\mathcal{E}TP^*_{2F}]^{1-\gamma}\}^{1-1/\sigma}\mathcal{E}^{1/\sigma}$$

where the second equality uses $C = C^*(\frac{\mathcal{E}P^*_t}{P_t})^{1/\sigma}$ and the third equality uses the condition specified in Section 3.1. Since $W = \mathcal{E}P^*(C^*)^{\sigma}$ under the assumption of $\psi = 0$, by plugging W into the expression of $\bar{a}_2/(1-\bar{a}_2)$, domestic price index all cancel out (including nominal exchange rate) and thus we have

$$\frac{\bar{a}_2}{1-\bar{a}_2} = f_2(*) \cdot T^{(1-\gamma)(1+\phi\gamma)(1-\frac{1}{\sigma})}$$

where $f_2(*)$ are functions of exogenous foreign variables.

For the first stage, we have

$$\frac{\bar{a}_1}{1-\bar{a}_1} = \frac{\gamma \bar{P}_2 \bar{Y}^d_2}{\mathcal{E}P^*_{1H}Y^*_{1H}}$$

$$= \frac{\gamma\phi P_{2H}(Y^d_{2H} + Y^{Xd}_{2H})}{\mathcal{E}P^*_{1H}Y^*_{1H}}$$

$$= \frac{\gamma\phi}{P^*_{1H}Y^*_{1H}} \frac{P_{2H}Y^d_{2H}}{\mathcal{E}\bar{a}_2}$$

$$= \frac{\gamma\phi}{P^*_{1H}Y^*_{1H}} \frac{\gamma CP}{\mathcal{E}\bar{a}_2}$$

$$= \frac{\gamma\phi}{P^*_{1H}Y^*_{1H}} \frac{Y^*_{2H}P^*_{2H}}{1-\bar{a}_2}$$

where we have used $\frac{\bar{a}_2}{1-\bar{a}_2} = \frac{\gamma CP}{Y^*_{2H}P^*_{2H}\mathcal{E}}$ in the last equality. Therefore, for $\bar{a}_1$, we have

$$\frac{\bar{a}_1}{1-\bar{a}_1} = f_2(*) \cdot \frac{1}{1-\bar{a}_2}$$

where $f_1(*)$ are functions of exogenous foreign variables.

E N-stage production in a closed economy

In this section, we consider the case of N-stage production in a closed-economy, and focus on the effects of lengthening of production chain on welfare loss function. We can similarly characterize equilibrium as the case of the open-economy model with two stages of production, which is shown in Appendix F. In the closed-economy model, since the distortion from monopolistic competition is assumed to be corrected by a subsidy tax, the only distortion in the economy comes from sticky price. Thus, the flexible price equilibrium is Pareto optimal and we can write each variable in the deviation from flexible price equilibrium. The shocks considered in this section are stage-specific productivity shock.

E.1 A utility-based objective welfare loss function for optimal monetary policy

Similarly to the derivation in Section 3.4, the households utility function is given by

$$E\Sigma_{t=0}^{\infty}\beta^t[U(C_t) - V(L_t)]$$

where $U(C_t) = \frac{C_t^{1-\sigma}-1}{1-\sigma}$ and $V(L_t) = \frac{L_t^{1+\psi}}{1+\psi}$.

A second-order Taylor expansion around steady state (C, L) for the period utility of consumption gives

$$U(C_t) - U = U_c C(\hat{c}_t + \frac{1-\sigma}{2}\hat{c}_t^2) + t.i.p.$$

where $\hat{c}_t$ denotes the log-deviation of consumption from steady state. To write the output gap in terms of the gap between the output with sticky-price and natural output (flexible-price equilibrium), the period utility of consumption can be re-written as

$$U(C_t) - U = U_c C(\tilde{c}_t + \frac{1-\sigma}{2}\tilde{c}_t^2 + (1-\sigma)c_t^f\tilde{c}_t) + t.i.p.$$

where $\tilde{c}_t = c_t - c_t^f$ and c_t^f is the log-deviation of consumption in flexible-price equilibrium from steady state equilibrium.

By labor market clearing condition, we obtain the second order Taylor expansion around steady state for the period utility of employment, i.e., $V(L_t)$, as

$$V(L_t) - V = V_L L\{\sum_{n=1}^{N}\frac{L_n}{L}[\hat{l}_{n,t} + \frac{1}{2}\sum_{n=1}^{N}\hat{l}_{n,t}^2]\} + t.i.p.$$

where L_n/L is the share of labor demand in stage n in total labor demand under steady state, given by equations (17) and (23) in Appendix F, and we have used the assumption of $\psi = 0$. More

specifically, the stage-specific labor share under steady state is given by

$$\frac{L_n}{L} = (1-\phi)\phi^{N-n},\ n = 2, 3, \ldots, N$$

$$\frac{L_1}{L} = \phi^{N-1}$$

The period utility of employment can then be re-written as the gap between labor demand with sticky-price and natural rate of labor demand in each stage, i.e.,

$$V(L_t) - V = V_L L\{\sum_{n=1}^{N} \frac{L_n}{L}[\tilde{l}_{n,t} + \frac{1}{2}\tilde{l}_{n,t}^2 + l_{n,t}^f \tilde{l}_{n,t}]\} + t.i.p \tag{20}$$

where $\tilde{l}_{n,t} = l_{n,t} - l_{n,t}^f$.

As showed in Appendix I, the stage-specific employment gap in terms of output gap and relative price gap for $n = 2, 3, \ldots, N-1$ are given by

$$\tilde{l}_{n,t} = \phi[\sum_{i=n}^{N} \tilde{g}_{i,t} - \sigma\tilde{c}_t] + \tilde{l}_{n+1,t} - [\sum_{i=n+1}^{N} \tilde{g}_{i,t} - \sigma\tilde{c}_t] + d_{n,t}$$

with

$$\tilde{l}_{N,t} = \phi[\tilde{g}_{N,t} - \sigma\tilde{c}_t] + \tilde{c}_t + d_{N,t}$$

$$\tilde{l}_{1,t} = \tilde{l}_{2,t} - [\sum_{i=2}^{N} \tilde{g}_{i,t} - \sigma\tilde{c}_t] + d_{1,t}$$

where $d_{n,t} = ln(\int_0^1 (\frac{P_{n,t}(u)}{P_{n,t}})^{-\theta} du)$ measures the price dispersion in stage n. Details can be found in Appendix I.

For simplicity, we denote

$$\tilde{l}_{n,t} = f_n(\tilde{g}_{n,t}, \ldots, \tilde{g}_{N,t}) + g(n)\tilde{c}_t + \sum_{g=n}^{N} d_{g,t}$$

where $g(n) = (N-n)(1-\phi)\sigma + 1 - \phi\sigma$ for $n = 2, 3, \ldots, N$, and $g(1) = (N-1)(1-\phi)\sigma + 1$.

Then, by summing up $U(C_t) - U$ and $V(L_t) - V$ and also noting that efficiency of steady state implies $U_c C = V_L L$ in closed-economy, the first order terms all cancel out, and only the second order terms are left. The welfare loss function as a fraction of steady state consumption is thus given by

$$W = E_0 \sum_{t=0}^{\infty} \beta^t \frac{U(C_t) - V(L_t) - (U - V)}{U_c C}$$

$$= -\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\{-(1-\sigma)\tilde{c}_t^2+\sum_{n=1}^{N}\frac{L_n}{L}[g(n)\tilde{c}_t+f_n(\tilde{g}_{n,t},\ldots,\tilde{g}_{N,t})]^2+\sum_{n=1}^{N}\theta\phi^{N-g}\lambda_n^{-1}\pi_{n,t}^2\} \quad (21)$$

where $\frac{L_n}{L}=(1-\phi)\phi^{N-n}$ for $n=2,3,\ldots,N$ and $\frac{L_1}{L}=\phi^{N-1}$.

Appendix (J) shows the welfare loss function for the case of $N=2$ and $N=3$ without abbreviation (i.e., expanding $\frac{L_n}{L}$, $g(n)$, and $f_n(\cdot)$ for n).

In general, monetary policy cannot attain Pareto optimal allocation except in special cases with restrictions on productivity shocks. We proceed with the following proposition.

Proposition 2: in the closed-economy model with N-stage of production and labor being used in each production stage (i.e., $0<\phi<1$), there is no monetary policy that can replicate flexible price equilibrium (Pareto-optimal allocation) unless the stage-specific productivity shocks satisfy $\sum_{i=1}^{n-1}\phi^{n-i-1}(\phi-1)\Delta a_{i,t}+\Delta a_{n,t}=0$ for $n=2,\ldots,N$ and for all t.

E.2 Discussion about the terms and coefficients in welfare loss function

There are three main parts in the welfare loss function: (a) output gap, and terms measuring stage-specific unemployment gap written in output gap, (b) the relative price gap, and (c) terms measuring stage-specific inflation. More specifically, as showed in the expression of welfare loss function (21), the coefficients before output gap $\tilde{c}_t^2$ and the stage-specific inflation, i.e., $\pi_{n,t}^2$ for $\forall n$, are all positive.[22] Therefore, similar to the standard welfare loss function (e.g., Rotemberg and Woodford, 1999; Woodford, 2003), the objective for a benevolent central bank still includes stabilizing output gap and inflation (i.e., the final stage inflation corresponding to typical "inflation" in literature).

Besides the output gap and final-stage inflation, there are many more terms included in the welfare loss function, classified by those measuring stage-specific unemployment gaps and stage-specific inflation. It suggests that the central bank should not only care about the output gap and CPI, but also need to pay attention to the variations in PPI inflation and the gaps of the real marginal cost in the production of intermediate goods.

Importantly, as showed in the expression of welfare loss function (21), by aggregating the terms of output gap $\tilde{c}_t^2$, the coefficient before output gap is $\sum_{n=1}^{N}\frac{L_n}{L}g(n)^2-(1-\sigma)$, which is a function of the production structure, and changes with the number of total production stage N. In contrast, the coefficient before CPI (i.e., the final stage inflation $\pi_{N,t}^2$) is a constant $\theta\lambda_N^{-1}$. That is to say, even the central bank follows the Taylor Rule, or the monetary rule suggested by Huang and Liu (2005), i.e., targeting both CPI and PPI, the optimal weights before output gap and CPI (or PPI) are changing with the production structure of the economy.

The welfare loss function with multi-stage production indicates that targeting both CPI and PPI are not satisfactory. Instead, the central bank needs to pay attention to all stage-specific

[22] Details about the proof can be found in Appendix K.

inflation during production process, especially in the case of lengthening production chain. Those terms measuring stage-specific inflation, i.e., $\sum_{n=1}^{N} \theta\phi^{N-n}\lambda_n^{-1}\pi_{n,t}^2$, in welfare loss function have two important implications. On the one hand, given the total number of production stages N and the price stickiness being the same across different stages, the coefficients before inflation in downstream stages are larger compared with those in upstream stages. On the other hand, as the number of total stages N increases, there are more terms of upstream inflation included in the welfare loss function, while the terms for downstream inflation do not change. In the latter case, the relative importance of final stage inflation (i.e., CPI) in welfare loss function becomes smaller; however, the inflation in upstream stages becomes more relatively important. That is to say, as the production length becomes longer, the central bank needs to care more about inflation in intermediate stages but less on final stage inflation (i.e, CPI).

From the perspective of practice, if the stage-specific inflation cannot be attained, PPI, as a sales-weighted price index for intermediate goods across all stages, can be a approximation. But, in general, if PPI index is available, the information used to construct PPI index is enough to construct the stage-specific inflation. For instance, in the PPI program of US Bureau of Labor Statistics, it not only constructs the aggregate PPI index, but also constructs the stage-specific inflation indices in a four-stage vertical production framework with the same original data.[23] Their idea in constructing this system of indices is to choose the total number of stages and assign industries to stages of production in such a manner that maximizes forward goods flow along vertical chain while minimizes backward flow and internal goods flow within the system.

F Equilibrium characterization with N-stage of production in closed economy

F.1 The steady state equilibrium

We first characterize the steady state with perfect foresight. Since the whole economy does not change with timing, we can ignore the timing index t in all variables, and $A_n = 1$ for $n = 1, 2, \ldots, N$. The optimal pricing decision for firms at stage n, $n = 1, 2, \ldots, N$, becomes

$$P_n^* = \mu\Psi_n$$

By the aggregate price expression (12), in steady state,

$$P_n = P_n^* = \mu\Psi_n$$

[23] Details for the stage-specific inflation indices constructed by US Bureau of Labor Statistics can be found at https://www.bls.gov/ppi/fdidsummary.htm, or Weinhagen (2011).

Now, we solve for the price indices in terms of wages and derive the labor demand function. Note that $P_n = \bar{P}_{n+1}$ for $n = 1, 2, \ldots, N-1$, and $\Gamma_n = \bar{P}_n^{\phi} W^{1-\phi}$ for $n = 2, 3, \ldots, N$ with $\Gamma_1 = W$. By substituting $\bar{P}_n$, the relationship of price index across adjacent stages is given by

$$P_n = \mu W^{1-\phi} (P_{n-1})^{\phi}$$

for $n = 2, \ldots, N$ and $P_1 = \mu W$.

By writing all price index in terms of wage, it comes

$$P_n = (\mu)^{1+\phi+\cdots+\phi^{n-2}} W^{1-\phi^{n-1}} (P_1)^{\phi^{n-1}}$$

$$= \mu^{\frac{1-\phi^n}{1-\phi}} W$$

for $n = 2, \ldots, N$ with $P_1 = \mu W$.

Since $P_n(u) = P_n$ for $u \in [0, 1]$ in steady state, we have

$$Y_{n-1,t}^d(u) = \bar{Y}_{n,t}^d$$

Together with goods market clear condition $Y_{n,t} = \bar{Y}_{n+1,t}^d$, and factor market demand function (8) and (7), for $n = 2, \ldots, N-1$, we get

$$\bar{Y}_n^d = \phi \frac{\Gamma_n}{\bar{P}_n} \bar{Y}_{n+1}^d$$

$$L_n^d = (1-\phi) \frac{\Gamma_n}{W} \bar{Y}_{n+1}^d$$

where $Y_N = C$, $\bar{Y}_N = \phi \frac{\Gamma_N}{\bar{P}_N} C$, $L_N^d = (1-\phi) \frac{\Gamma_N}{W} C$, and $L_1^d = \frac{\Gamma_1}{W} \bar{Y}_2^d$. By substituting the price index and unit cost function in each stage, for $n = 2, \ldots, N$, the factor demand functions for both labor and composite intermediate goods are given by

$$\bar{Y}_n^d = \phi^{N+1-n} (\mu)^{-(N+1-n) + \frac{\phi^{n-1} - \phi^N}{1-\phi}} C$$

$$L_n^d = (1-\phi) \phi^{N-n} (\mu)^{-(N+1-n) + \frac{1-\phi^N}{1-\phi}} C \tag{22}$$

with $L_1^d = \bar{Y}_2^d$.[24]

Therefore, by summing up labor demand across all stages, the total labor demand function

[24] A simple way of thinking labor demand in each stage can be viewed as backward deduction (which is helpful when taking log-linearization), i.e.,

$$L_n^d = L_{n+1}^d \cdot \phi \cdot \mu^{-1},\ n = 2, \ldots, N$$

$$L_1^d = (\frac{\phi}{1-\phi}) \mu^{-1} L_2^d$$

becomes

$$L^d = \sum_{n=2}^{N}[(1-\phi)\phi^{N-n}\mu^{-(N+1-n)+\frac{1-\phi^N}{1-\phi}}C] + \phi^{N-1}\mu^{-(N-1)+\frac{\phi-\phi^N}{1-\phi}}C \tag{23}$$

By the labor supply function (1) together with the price index, the labor supply in steady state becomes

$$L^{\psi}C^{\sigma} = (\mu)^{-\frac{1-\phi^N}{1-\phi}} \tag{24}$$

Given $L^d = L$, the two equations (23) and (24) fully characterize the steady state total consumption and total employment.

F.2 The flexible price equilibrium

In order to get the efficient allocation of the model, i.e., natural rate of the output, we solve the flexible price equilibrium similarly as in the steady state equilibrium. In flexible price equilibrium, $\alpha_n = 0$ for $\forall n$, and then the optimal pricing decision for firms at stage n, $n = 1, 2, \ldots, N$, becomes

$$P^*_{n,t} = \mu\Gamma_{n,t}$$

By the aggregate price expression (12), we have

$$P_{n,t} = P^*_{n,t} = \mu\Gamma_{n,t}$$

Similar to the steady state case, we solve for the price indices in terms of wages and productivity. Note that $P_{n,t} = \bar{P}_{n+1,t}$ for $n = 1, 2, \ldots, N-1$, and $\Gamma_{n,t} = \bar{P}^{\phi}_{n,t}W_t^{1-\phi}/A_{n,t}$ for $n = 2, 3, \ldots, N$ with $\Gamma_1 = W_t/A_{1,t}$. By substituting $\bar{P}_{n,t}$, the relationship of price index across adjacent stages is given by

$$P_{n,t} = \mu W_t^{1-\phi}(P_{n-1,t})^{\phi}/A_{n,t}$$

for $n = 2, \ldots, N$ and $P_1 = \mu W/A_{1,t}$.

By writing all price index in terms of wage, we get

$$\begin{aligned} P_{n,t} &= (\mu)^{1+\phi+\cdots+\phi^{n-2}}W^{1-\phi^{n-1}}(P_1)^{\phi^{n-1}}\Pi_{g=2}^{n}A_{g,t}^{-\phi^{n-g}} \\ &= \mu^{\frac{1-\phi^n}{1-\phi}}W \cdot \Pi_{g=1}^{n}A_{g,t}^{-\phi^{n-g}} \end{aligned} \tag{25}$$

for $n = 2, \ldots, N$.

Similar to the derivation for labor demand function in steady state, the labor demand function

in each stage in flexible price equilibrium is given by

$$L_{n,t}^d = (1-\phi)\phi^{N-n}\mu^{-(N+1-n)+\frac{1-\phi^N}{1-\phi}}\Pi_{g=1}^N A_{g,t}^{-\phi^{N-g}} C_t \tag{26}$$

for $n = 2, 3, \ldots, N$ with $L_{1,t}^d = (\frac{\phi}{1-\phi})\mu^{-1}L_{2,t}^d$. Details can be found in Appendix G.

Therefore, total labor demand is given by

$$L_t^d = \eta\Pi_{n=1}^N A_{n,t}^{-\phi^{N-n}} C_t \tag{27}$$

where η is a constant, and it is given by

$$\eta = \sum_{n=2}^{N}[(1-\phi)\phi^{N-n}\mu^{-(N+1-n)+\frac{1-\phi^N}{1-\phi}}C] + \phi^{N-1}\mu^{-(N-1)+\frac{\phi-\phi^N}{1-\phi}}$$

By the labor supply function (1) together with the price index, we know

$$L^\psi C^\sigma = \mu^{-\frac{1-\phi^N}{1-\phi}}\Pi_{n=1}^N A_{n,t}^{\phi^{N-n}} \tag{28}$$

After taking log-deviation from steady state for both the total labor demand (27) and labor supply (28), we get

$$l_t^f = c_t^f - [\sum_{n=1}^{N}\phi^{N-n}a_{n,t}] \tag{29}$$

and

$$\psi l_t^f + \sigma c_t^f = \sum_{n=1}^{N}\phi^{N-n}a_{n,t} \tag{30}$$

Therefore, the log-deviation of output from steady state under flexible price c_t^f is given by

$$c_t^f = \frac{1+\psi}{\psi+\sigma}[\sum_{n=1}^{N}\phi^{N-n}a_{n,t}]$$

By the Euler equation (4), the IS curve is characterized by

$$c_t^f = E_t(c_{t+1}^f) - \frac{1}{\sigma}[i_t - E_t(\pi_{N,t+1}) - \rho]$$

which implies the natural rate of interest as

$$\bar{rr}_t = i_t - E_t(\pi_{N,t+1})$$

$$= \rho + \sigma E_t\{c_{t+1}^f - c_t^f\}$$

Given the expression of output and the process of productivity shocks, we have

$$\bar{rr}_t = \rho + \frac{\sigma(1+\psi)}{\psi+\sigma} E_t[\sum_{n=1}^{N} \phi^{N-n} \Delta a_{n,t+1}] \tag{31}$$

where $\Delta a_{n,t} = a_{n,t} - a_{n,t-1}$ is the growth rate of productivity in stage n.

F.3 The sticky-price equilibrium

We now derive New Keynesian Phillips curves for each stage as a function of relative price gap and output gap, and characterize the equilibrium with sticky price. Similar to the derivation in Gali (2015), in each stage of production $n = 1, 2, \ldots, N$, firms' optimal pricing decision gives

$$\pi_{n,t} = \beta E_t \pi_{n,t+1} + \lambda_n \tilde{\gamma}_{n,t}$$

where $\lambda_n = \frac{(1-\beta\alpha_n)(1-\alpha_n)}{\alpha_n}$ and $\tilde{\gamma}_n$ is the log-derivation of real marginal cost from flexible price equilibrium, i.e.,

$$\tilde{\gamma}_{n,t} = ln(\Gamma_{n,t}/P_{n,t}) - ln(\Gamma^f_{n,t}/P^f_{n,t})$$

where $\Gamma^*_{n,t}$ and $P^*_{n,t}$ is the marginal cost and aggregate price in stage n under flexible price equilibrium.

Following Huang and Liu (2005), without losing generality, we assume $\psi = 0$. Then, together with labor supply function (28), for $n = 2, 3, \ldots, N-1$, the log-derivation of real marginal cost can be written as a function of relative price gap and output gap, i.e.,

$$\tilde{\gamma}_{n,t} = \phi \tilde{q}_{n,t} + (1-\phi)[\sigma \tilde{c}_t - \sum_{i=n+1}^{N} \tilde{g}_{i,t}] \tag{32}$$

$$\tilde{\gamma}_{1,t} = \sigma \tilde{c}_t - \sum_{i=2}^{N} \tilde{g}_{i,t}$$

$$\tilde{\gamma}_{N,t} = \phi \tilde{g}_{N,t} + (1-\phi)\sigma \tilde{c}_t$$

where $\tilde{g}_{n,t}$ is the relative price gap between stage n and stage $n-1$, i.e., $\tilde{g}_{n,t} = ln(\frac{P_{n-1,t}}{P_{n,t}}) - ln(\frac{P^f_{n-1,t}}{P^f_{n,t}})$. Details for Expression (32) can be found in Appendix H.

After log-linearizing the Euler equation around steady state and subtracting the natural rate IS curve, we get the IS curve with stick price as

$$\tilde{c}_t = E_t \tilde{c}_{t+1} - \frac{1}{\sigma}[i_t - E_t(\pi_{N,t+1}) - \bar{rr}_t]$$

where $r\bar{r}_t$ is the natural rate of interest.

The law of motion for relative price gap between stage n and stage $n-1$, for $n = 2, 3, \ldots, N$, is characterized by

$$\tilde{g}_{n,t} = \tilde{g}_{n,t-1} + \pi_{n-1,t} - \pi_{n,t} - \Delta g^f_{n,t}$$

where $\Delta g^f_{n,t} = g^f_{n,t} - g^f_{n,t-1}$. By Equation (25), we have

$$\Delta g^f_{n,t} = \sum_{i=1}^{n-1} \phi^{n-i-1}(\phi - 1)\Delta a_{i,t} + \Delta a_{n,t}$$

Give the monetary policy rule, the Phillips curve, IS curve, and the law of motion for relative price gap fully pin down the dynamic equilibrium under sticky prices.

G Labor demand function in flexible price equilibrium in closed economy

Similar to the derivation in steady state equilibrium, we derive the labor demand function in flexible price equilibrium. Note that, in flexible price, $P_n(u) = P_n$ for $u \in [0, 1]$. We then get

$$Y^d_{n-1,t}(u) = \bar{Y}^d_{n,t}$$

Together with goods market clear condition $Y_{n,t} = \bar{Y}^d_{n+1,t}$, and factor market demand function (8) and (7), for $n = 2, \ldots, N-1$, we get

$$\bar{Y}^d_{n,t} = \phi \frac{\Gamma_{n,t}}{\bar{P}_{n,t}} \bar{Y}^d_{n+1,t}$$

$$L^d_{n,t} = (1-\phi)\frac{\Gamma_{n,t}}{W_t} \bar{Y}^d_{n+1,t}$$

where $Y_{N,t} = C_t$, $\bar{Y}_{N,t} = \phi \frac{\Gamma_{N,t}}{\bar{P}_N} C_t$, $L^d_{N,t} = (1-\phi)\frac{\Gamma_{N,t}}{W_t} C_t$, and $L^d_{1,t} = \frac{\Gamma_{1,t}}{W_t} \bar{Y}^d_{2,t}$.

Note that $P_{n,t} = \bar{P}_{n+1,t}$ for $n = 1, 2, \ldots, N-1$, and $\Gamma_{n,t} = \bar{P}^{\phi}_{n,t} W_t^{1-\phi} / A_{n,t}$ for $n = 2, 3, \ldots, N$ with $\Gamma_1 = W_t / A_{1,t}$. By substituting the unit cost function in each stage, for $n = 2, \ldots, N$, we get labor demand in each stage follows

$$\bar{Y}^d_{n,t} = \frac{\phi}{1-\phi} \frac{W_t}{P_{n-1,t}} L^d_{n,t}$$

and thus

$$L^d_{n,t} = \phi \frac{P_{n-1,t}}{P_{n,t}} \left(\frac{W_t}{P_{n-1,t}}\right)^{1-\phi} A^{-1}_{n,t} L^d_{n+1,t}$$

$$L_{1,t}^d = \frac{1}{A_{1,t}} \frac{\phi}{1-\phi} \frac{W_t}{P_{1,t}} L_{2,t}^d$$

A simple way of thinking labor demand in each stage can be viewed as backward deduction (which is helpful when taking log-linearization), i.e.,

$$L_{n,t}^d = L_{n+1,t}^d \cdot \phi \cdot \mu^{-1},\ n = 2, \ldots, N$$

$$L_{1,t}^d = (\frac{\phi}{1-\phi}) \mu^{-1} L_{2,t}^d$$

Note that $L_{N,t}^d = (1-\phi)\frac{\Gamma_{N,t}}{W_t} C_t$, which indicates

$$L_{N,t}^d = (1-\phi) \mu^{\frac{\phi - \phi^N}{1-\phi}} \Pi_{g=1}^N A_{g,t}^{-\phi^{N-g}} C_t$$

Therefore, for $n = 2, 3, \ldots, N$, we get labor demand function in each stage as

$$L_{n,t}^d = (1-\phi)\phi^{N-n} \mu^{-(N+1-n)+\frac{1-\phi^N}{1-\phi}} \Pi_{g=1}^N A_{g,t}^{-\phi^{N-g}} C_t$$

with $L_{1,t}^d = (\frac{\phi}{1-\phi}) \mu^{-1} L_{2,t}^d$.

H The log-derivation of real marginal cost from flexible price equilibrium in closed economy

Note that, for $n = 2, 3, \ldots, N$, $\Gamma_{n,t} = \bar{P}_{n,t}^{\phi} W_t^{1-\phi} / A_{n,t}$ and $\bar{P}_{n,t} = P_{n-1,t}$. The log-derivation of real marginal cost is then given by

$$\tilde{\gamma}_{n,t} = ln(\Gamma_{n,t}/P_{n,t}) - ln(\Gamma_{n,t}^*/P_{n,t}^*)$$

$$= \phi[ln(P_{n-1,t}/P_{n,t}) - ln(P_{n-1,t}^*/P_{n,t}^*)] + (1-\phi)[ln(W_t/P_{n,t}) - ln(W_t^f/P_{n,t}^f)]$$

Denote $g_{n,t} = ln(P_{n-1,t}/P_{n,t})$ and $\tilde{g}_{n,t} = ln(P_{n-1,t}/P_{n,t}) - g_{n,t}^f$. For $n = 1, 2, \ldots, N-1$, we have

$$lnP_{n,t} = \sum_{i=n+1}^{N} g_{i,t} + lnP_{N,t}$$

$$\Longleftrightarrow p_{n,t} = \sum_{i=n+1}^{N} g_{i,t} + p_{N,t}$$

Also, by the labor supply equation (28), by assuming $\psi = 0$, we have

$$w_t - p_{N,t} = \sigma c_t$$

Therefore, for $n = 2, 3, \ldots N-1$, the log-derivation of real marginal cost can be written as

$$\tilde{\gamma}_{n,t} = \phi \tilde{g}_{n,t} + (1-\phi)[\tilde{w}_t - \tilde{p}_{n,t}]$$

$$= \phi \tilde{g}_{n,t} + (1-\phi)[\sigma \tilde{c}_t + \tilde{p}_{N,t} - \tilde{p}_{n,t}]$$

$$= \phi \tilde{g}_{n,t} + (1-\phi)[\sigma \tilde{c}_t - \sum_{i=n+1}^{N} \tilde{g}_{i,t}]$$

with $\tilde{\gamma}_{N,t} = \phi \tilde{g}_{N,t} + (1-\phi)\sigma \tilde{c}_t$.

Similarly, for the first stage $n = 1$, since $\Gamma_1 = W_t / A_{1,t}$, we have

$$\tilde{\gamma}_{1,t} = \tilde{w}_t - \tilde{p}_{1,t}$$

$$= \sigma \tilde{c}_t - \sum_{i=2}^{N} \tilde{g}_{i,t}$$

I Stage-specific employment gap in closed-economy with N-stage production

We derive the stage-specific employment gap in terms of output gap and relative price gap. By the factor demand function (7), (8), and (11) in each stage, and substituting with the unit cost, for $n = 2, 3, \ldots, N$, we have

$$lnL_{n,t} = ln(1-\phi) + \phi[lnP_{n-1,t} - lnW_t] - lnA_{n,t} + ln\bar{Y}^d_{n+1,t} + d_{n,t}$$

$$\iff l_{n,t} = ln(1-\phi) + \phi[p_{n-1,t} - w_t] - a_{n,t} + ln\bar{Y}^d_{n+1,t} + d_{n,t}$$

where $d_{n,t} = ln(\int_0^1 (\frac{P_{n,t}(u)}{P_{n,t}})^{-\theta} du)$ and $l_{1,t} = -a_{1,t} + ln\bar{Y}^d_{2,t} + d_{1,t}$.

By the factor demand function for intermediate goods and labor in each stage, i.e., Expression (7) and (8), for $n = 2, 3, \ldots, N$, we get

$$l_{n,t} = ln(\frac{1-\phi}{\phi}) + p_{n-1,t} - w_t + ln\bar{Y}^d_{n,t}$$

Note that $\bar{Y}^d_{N,t} = C_t$. Then, by substituting $ln\bar{Y}^d_{n+1,t}$, we can get backward induction relationship

for the stage-specific employment, i.e., for the stage of $n = N$,

$$l_{N,t} = ln(1-\phi) + \phi[p_{N-1,t} - w_t] - a_{N,t} + c_t + d_{N,t}$$

for $n = 2, 3, \ldots, N-1$,

$$l_{n,t} = ln(\phi) + \phi[p_{n-1,t} - w_t] - a_{n,t} + l_{n+1,t} - [p_{n,t} - w_t] + d_{n,t}$$

for $n = 1$,

$$l_{1,t} = -a_{1,t} + l_{2,t} - [p_{1,t} - w_t] + d_{1,t}$$

As showed in Appendix H, for $n = 1, 2, \ldots, N$, we have $p_{n,t} = \sum_{i=n+1}^{N} g_{i,t} + p_{N,t}$, and, by assuming $\psi = 0$, $w_t - p_{N,t} = \sigma c_t$. The stage specific employment can be written in terms of relative price and output as, for $n = N$,

$$l_{N,t} = ln(1-\phi) + \phi[g_{N,t} - \sigma c_t] - a_{N,t} + c_t + d_{N,t}$$

for $n = 2, 3, \ldots, N-1$,

$$l_{n,t} = ln(\phi) + \phi[\sum_{i=n}^{N} g_{i,t} - \sigma c_t] - a_{n,t} + l_{n+1,t} - [\sum_{i=n+1}^{N} g_{i,t} - \sigma c_t] + d_{n,t}$$

for $n = 1$,

$$l_{1,t} = -a_{1,t} + l_{2,t} - [\sum_{i=2}^{N} g_{i,t} - \sigma c_t] + d_{1,t}$$

By subtracting the corresponding equations for flexible price equilibrium, the stage-specific employment gap in terms of output gap and relative price gap is given by, for $n = 2, 3, \ldots, N-1$,

$$\tilde{l}_{n,t} = \phi[\sum_{i=n}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t] + \tilde{l}_{n+1,t} - [\sum_{i=n+1}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t] + d_{n,t}$$

with

$$\tilde{l}_{N,t} = \phi[\tilde{g}_{N,t} - \sigma \tilde{c}_t] + \tilde{c}_t + d_{N,t}$$

$$\tilde{l}_{1,t} = \tilde{l}_{2,t} - [\sum_{i=2}^{N} \tilde{g}_{i,t} - \sigma \tilde{c}_t] + d_{1,t}$$

Therefore, by solving forward induction, the stage-specific employment gap is given by, for

$n = 2, 3, \ldots, N-1$,

$$\tilde{l}_{n,t} = \phi[\sum_{i=n}^{N} \tilde{g}_{i,t} - \sigma\tilde{c}_t] + \tilde{l}_{n+1,t} - [\sum_{i=n+1}^{N} \tilde{g}_{i,t} - \sigma\tilde{c}_t] + d_{n,t}$$

with

$$\tilde{l}_{N,t} = \phi[\tilde{g}_{N,t} - \sigma\tilde{c}_t] + \tilde{c}_t + d_{N,t}$$

$$\tilde{l}_{1,t} = \tilde{l}_{2,t} - [\sum_{i=2}^{N} \tilde{g}_{i,t} - \sigma\tilde{c}_t] + d_{1,t}$$

J The closed-form welfare loss function for the case of $N = 2$ and $N = 3$ in closed-economy

To illustrate the welfare loss function, we show the analytical welfare loss function for the case of $N = 2$ and $N = 3$ without abbreviation. For the case of $N = 2$, by Appendix I, the stage-specific employment gap in terms of output gap and relative price gap is given by

$$\tilde{l}_{1,t} = (1 + \sigma - \sigma\phi)\tilde{c}_t + (\phi - 1)\tilde{g}_{2,t} + d_{1,t} + d_{2,t}$$

$$\tilde{l}_{2,t} = (1 - \sigma\phi)\tilde{c}_t + \phi\tilde{g}_{2,t} + d_{2,t}$$

Since $\frac{L_1}{L} = \phi$ and $\frac{L_2}{L} = 1 - \phi$, by plugging into Equation (21), the welfare loss function with $N = 2$ is given by

$$W = -\frac{1}{2}E_0 \sum_{t=0}^{\infty} \beta^t \{\sigma\tilde{c}_t^2 + \phi(1-\phi)[\sigma\tilde{c}_t - \tilde{g}_{2,t}]^2 + \theta\lambda_2^{-1}\pi_{2,t}^2 + \theta\phi\lambda_1^{-1}\pi_{1,t}^2\}$$

which is exactly the same with the results in Huang and Liu (2005).

Similarly, for the case of $N = 3$, the stage-specific employment gap in terms of output gap and relative price gap is given by

$$\tilde{l}_{1,t} = (1 + 2\sigma - 2\sigma\phi)\tilde{c}_t + 2(\phi - 1)\tilde{g}_{3,t} + (\phi - 1)\tilde{g}_{2,t} + d_{1,t} + d_{2,t} + d_{3,t}$$

$$\tilde{l}_{2,t} = (1 + \sigma - 2\sigma\phi)\tilde{c}_t + (2\phi - 1)\tilde{g}_{3,t} + \phi\tilde{g}_{2,t} + d_{2,t} + d_{3,t}$$

$$\tilde{l}_{1,t} = (1 - \sigma\phi)\tilde{c}_t + \phi\tilde{g}_{3,t} + d_{3,t}$$

Since $\frac{L_1}{L} = \phi^2$, $\frac{L_2}{L} = \phi(1-\phi)$ and $\frac{L_3}{L} = 1 - \phi$, by plugging into Equation (21), the welfare loss

function with $N = 3$ is given by

$$W = -\frac{1}{2}E_0 \sum_{t=0}^{\infty} \beta^t \{-(1-\sigma)\tilde{c}_t^2 + \phi^2[(1+2\sigma-2\sigma\phi)\tilde{c}_t + 2(\phi-1)\tilde{g}_{3,t} + (\phi-1)\tilde{g}_{2,t}]^2$$

$$+(1-\phi)\phi[(1+\sigma-2\sigma\phi)\tilde{c}_t + (2\phi-1)\tilde{g}_{3,t} + \phi\tilde{g}_{2,t}]^2$$

$$+(1-\phi)[(1-\sigma\phi)\tilde{c}_t + \phi\tilde{g}_{3,t}]^2$$

$$+\theta\lambda_3^{-1}\pi_{3,t}^2 + \theta\phi\lambda_2^{-1}\pi_{2,t}^2 + \theta\phi^2\lambda_3^{-1}\pi_{3,t}^2\}$$

K The proof for positive coefficient of output gap in welfare loss function

The coefficient of output gap $\tilde{c}_t^2$ in the welfare loss function (21) is given by

$$-(1-\sigma) + \sum_{n=1}^{N} \frac{L_n}{L} g(n)^2 \equiv f$$

Note that the stage-specific labor share under efficient steady state yields

$$\frac{L_n}{L} = (1-\phi)\phi^{N-n},\ n = 2, 3, \ldots, N$$

$$\frac{L_1}{L} = \phi^{N-1}$$

and $g(n) = (N-n)(1-\phi)\sigma + 1 - \phi\sigma$ for $n = 2, 3, \ldots, N$, and $g(1) = (N-1)(1-\phi)\sigma + 1$. Also, since $\phi < 1$ and $\sigma \leq 1$, it is obvious that $g(n) \geq 1 - \sigma\phi > 0$ for $\forall n$.

Therefore,

$$f = -(1-\sigma) + \sum_{n=1}^{N} \frac{L_n}{L} g(n)^2$$

$$\geq -(1-\sigma) + \sum_{n=1}^{N} \frac{L_n}{L} g(n) \cdot (1-\sigma\phi)$$

$$= (1-\sigma\phi) - (1-\sigma)$$

$$= \sigma(1-\phi) > 0$$

In other words, the coefficient of output gap in the welfare loss function is positive.