Efficient and Incentive Compatible Mediation: An Ordinal Market Design Approach*

Onur Kesten† Selçuk Özyurt‡

1 August 2018

Abstract

Mediation is an alternative dispute resolution method, which has gained increasing popularity over the last few decades and become a multi-million dollar industry. When two or more parties are in a disagreement, they can take the case to a court and let the judge make a binding final decision. Alternatively, the disputing parties can get assistance from an experienced, neutral third party, i.e., a mediator, who facilitates the negotiation and help them voluntarily reach an agreement short of litigation. The emphasis in mediation is not upon who is right or wrong, but rather on exploring mutually satisfactory solutions. Employment disputes, patent/copyright violations, construction disputes, and family disputes are some of the most common mediated disputes. The rising popularity of mediation is often attributed to the increasing workload of courts, its cost effectiveness and speed relative to litigation, and disputants' desire to have control over the final decision. Many traditional "cardinal" settings of bargaining and mechanism design, starting with the seminal work of Myerson and Satterhwaite (1983), have shown the incompatibility between efficiency and incentives, even in Bayesian sense. This paper uses an "ordinal" market/mechanism design approach, where the mediator seeks a resolution over (at least) two issues in which negotiators have diametrically opposed rankings over the alternatives. Each negotiator has private information about her own ranking of the outside option, e.g., the point beyond which the negotiator would rather take the case to the court. We construct a simple theoretical framework that is rich and practical enough allowing for optimal mechanisms that the mediators can use for efficient resolution of disputes. We propose and characterize the class of strategy-proof, efficient, and individually rational mediation mechanisms. A central member of this class, the "constrain-and-choose" mechanism stands out as the unique strategy-proof, efficient, and individually rational mechanism that minimizes rank variance. We also provide analogous mechanisms when the issues consist of a continuum of alternatives.

*We thank Johannes Hörner, George Mailath, Vijay Krishna, Herve Moulin, Tayfun Sönmez, Utku Ünver, Larry Ausubel, Bumin Yenmez, Ed Green, Ron Siegel, Luca Rigotti, Sevgi Yüksel, Alexey Kushnir, Alex Teytelboym, William Thomson, Ruben Juarez, Francis Bloch, and seminar participants at Rice, Maryland, Boston College, University of Pittsburgh, Carnegie Mellon, Penn State (PETCO), and Durham for many useful discussions and suggestions.

†Carnegie Mellon University (okesten@andrew.cmu.edu)

‡Sabanci University (ozyurt@sabanciuniv.edu)

"*Mediation has rapidly become, with precious little fanfare, the ocean we swim in and the air we breathe. It would now be hard to imagine a world where it wasn't.*" Jim Melamed (founder and CEO of Mediate.com, recipient of American Bar Association Institutional Problem Solver Award.)

Introduction

The best-seller book, "Getting to Yes," by Roger Fisher and William Ury is arguably one of, if not the most famous, works on the topic of negotiation. They identify conflict as a growth industry, and the last few decades proved them right. Courts in all US states offer some form of ADR (alternative dispute resolution) for the cases filed in state courts. 17 states require mandatory mediation: 11% of civil cases in Northern California courts in 2011, 35.6% of civil and 21.6% of divorce cases in New York state courts in 2016 have been mediated.[1] Total value of mediated cases in UK is estimated to be £10.5bn. in 2011, excluding mega-cases, family and community disputes.[2] In addition to face-to-face mediation practices, online dispute resolution, aiming to resolve disputes that arise online, has also gained increasing popularity over the last decade. These are small disputes in size but large in number. Dispute resolution centers of E-bay, PayPal, Uber and Amazon tackles more than a billion disputes a year. Empirical studies and mediation program evaluations suggest 60-90% success rate, 90-95% satisfaction by the disputants and higher rate of compliance relative to court-imposed orders.

Market design has been fruitful in many applications, most notably in auctions and matching theory. The goal of this paper is to offer a first market design setting to analyze dispute resolution via mediation, which is simple enough to be practically relevant while maintaining the informational richness and complexities faced in practical disputes. To this end, our modeling significantly departs from the traditional mechanism design approach to bargaining that builds on the seminal work of Myerson and Satterthwaite (1983) in the context of bilateral trade over a good for which traders have private valuations each drawn from pre-specified distributions and commonly known utility functions. This type of "cardinal approach" has however been the subject of the famous Wilson critique for it lacks "detail-freeness" and does not provide robust incentives to participants. In a similar vein, Ausubel, Crampton, and Denecker voice a similar concern:

"*... Despite these virtues, mechanism design has two weaknesses. First, the mechanisms depend in complex ways on the traders' beliefs and utility*

[1]Sources: dispute resolution centers of State of New York and California.
[2]The Seventh Mediation Audit, Centre for effective dispute resolution.

functions, which are assumed to be common knowledge. Second, it allows too much commitment. In practice, bargainers use simple trading rules—such as a sequence of offers and counteroffers—that do not depend on beliefs or utility functions." Handbook of Game Theory

The ordinal approach, whereby the designer elicits only ordinal preference information, has already lead to quite notable success in applications of matching and assignment such as medical residency, school choice, kidney exchange, and course assignment, where a plethora of strategy-proof and efficient mechanisms have been obtained, extensively studied, and even adopted in practice.

Our model assumes that two negotiators are in a dispute and aim to reach a resolution through a mediator. There is a main issue, issue X, consisting of a finite number of alternatives, which is relevant for both parties' welfare.[3] The negotiators have diametrically opposed preferences over alternatives in the sense that if one negotiator prefers one alternative over another, then the other negotiator has exactly opposite ranking of the two alternatives. However, not all alternatives are acceptable for any given negotiator. When offered one such alternative for her, a negotiator rejects the mediator's proposal and pursues alternative ways of resolution, e.g., litigation. We capture such circumstances by assuming an outside option whose ranking is each negotiator's private information. The mediator's objective is to truthfully elicit negotiators' private information about the position of their outside options and propose an efficient and mutually acceptable, i.e., individually rational, outcome.

We first show that if there is a single issue, i.e., no other issues than issue X, then there is no strategy-proof, efficient, and strategy-proof mechanism. Furthermore, we show that this impossibility extends to multiple issues if each issue has an outside option similarly defined, i.e., in each issue each negotiator has an outside option whose ranking is her private information. This motivates the need for a setting that asymmetrically treats different issues: Consider a second issue, issue Y, where the outside option is the least preferred alternative for both negotiators. In other words, litigation for the second issue is always inefficient. This asymmetric treatment of the outside options can be motivated by various employment, family, construction or patent/copyright infringement disputes. Litigation is naturally the default option if the issue is compensation or division of property and it is a very long and costly process, and so, inefficient relative to other potential divisions (alternatives). In this two-issue mediation problem, the mediator recommends a bundle (x, y) of outcomes from $X \times Y$. A mediation rule/mechanism is a systematic way of choosing an outcome for any given preferences of the negotiators.

[3] In the paper we later relax the finiteness and discreteness assumptions on X.

Since the mediator asks negotiators to report their outside options over alternatives in issue X (recall that there is no uncertainty regarding negotiators' preferences over alternatives in issue Y), one needs to invoke extension mappings to obtain the possible set of negotiators' underlying preferences over bundles. Alternatively, it is conceivable that the mediator elicits preferences over bundles of alternatives. This approach, which we do not pursue, however, has two drawbacks: First, the number of bundles to rank increases quadratically with the number of alternatives in each issue, which in turn makes asking for full-fledged rankings over bundles highly impractical. Second, a similar impossibility to the single-issue mediation would arise in this case.

In this paper we ask if there is an impartial and dominant strategy incentive compatible, i.e., strategy-proof, way of soliciting true preferences so that mediation outcomes are efficient and individually rational. A sufficient and almost necessary condition for obtaining a positive answer to this question is the so-called "logrolling (quid pro qu)" condition on negotiators' preferences which implies a form of substitution between issues X and Y. More specifically, logrolling requires preferences to be rich enough such that a negotiator is able to compromise issue X for a more preferred alternative in issue Y, e.g., for a given (x, y) bundle, there exists a (weakly) more preferred bundle which involves getting a worse alternative in X combined with a better alternative in Y. In the continuous version of our model, we show that many commonly used utility functions satisfy this assumption.

Our main result is a complete characterization of the class of strategy-proof, efficient, and individually rational mediation rules. These rules operate through an exogenously specified precedence order over a set of special bundles, which we call as the logrolling bundles and always make selections among these bundles. The logrolling bundles form a simple lattice structure with respect to the negotiators' preferences: given any set of mutually acceptable alternatives, for each negotiator there is always an optimal-logrolling bundle that she prefers over all other acceptable bundles; this bundle is the pessimal-logrolling bundle for the opposite negotiator. The characterized class of rules nest interesting extremal members. When the precedence order over coincides with the preference ranking of a given negotiator over the logrolling bundles, we obtain the corresponding negotiator-optimal rule.

In keeping with our main objective of finding impartial mediation rules, we search for members of this class of rules that satisfy sensible fairness criteria. To this end, we define the "rank variance" of an outcome as the sum of the square of each negotiator's ranking of each alternative in each issue. It turns out there is a unique member of the family of strategy-proof, efficient, and individually rational mediation rules that minimizes rank variance. This is the so-called "constrain-and-choose" rule, which recommends the median logrolling bundle when it is mutually acceptable, or the closest mutually acceptable

logrolling bundle to it when it is not mutually acceptable. This rule is intuitive and simple enough to be used as a standardized protocol for finding the middle ground between disputing parties in practice.

Related Literature

Our paper and modeling approach connects and spans four different types of literature:

1) Bargaining and Mechanism Design: Mediation is a part of the bargaining literature, which is primarily based on the cardinal approach discussed above. The more broadly-defined mechanism design approach to bargaining in the presence of private outside options, started with the classic paper by Myerson and Satterthwaite (1983) [MS henceforth], has generally emphasized the difficulty/impossibility of reaching efficient outcomes even in Bayesian settings let alone dominant strategies. Specifically, for the mediation context, there are very few papers: Bester and Warneryd (2006) show that the news are even worse than MS in this setting. The MS result depends on there being a positive probability of trade being inefficient ex post. Bester and Warneryd (2006), in a model featuring continuum of types, show that asymmetric information about relative strengths as an outside option in a conflict may cause agreement to be impossible even if the agreement is always efficient. In their model, conflict shrinks the pie and thus agreement on a peaceful settlement is always ex post efficient. Following Bester and Warneryd (2006), Horner et al (2015) compare the optimal mechanisms, with two types of negotiators, under arbitration, mediation and unmediated communication. For both models, there is no ex post efficient and Bayesian incentive compatible mechanism. Namely, optimal mechanism is necessarily inefficient.

In our model, we adopt an ordinal mechanism design approach in the sense that negotiators rank finitely many available options in opposite ways, which is common knowledge, but the outside option of each negotiator is her private information as in the Compte and Jehiel (2007) model, which adopts a cardinal utility approach much like the rest of this literature. In our benchmark model of single-issue mediation, a conflict situation which is defined as the mediators recommending negotiators to exercise their outside options, is also ex post inefficient so long as the mediators have a mutually acceptable outcome (clearly, when there is no mutually acceptable outcome, mediation is hopeless). Also, for the second issue Y, the conflict situation (outside option) is always inefficient in our model.

By contrast to this literature, our ordinal approach together with our modeling specifications enables us to obtain positive results: Indeed, we are able to achieve ex post

efficiency in dominant strategies and argue that the proposed rules can potentially be convenient and simple enough to use in practice.

2) Political Economy: Our benchmark model (but not the main, two-issue model where preferences over bundles are not necessarily single-peaked) resembles a voting model with single-peaked preferences where a number of voters have single-peaked preferences over the single-dimensional political spectrum and a voting rule aggregates individual preferences (Black 1948, Moulin 1980, Barbera, Gul and Stachetti 1991, and Ching 1997). In this type of models, the famous median voter theorem states that majority-rule voting system will select the outcome most preferred by the median voter.

In our model, preferences can also be thought to be single-peaked with each negotiator preferring the opposite extremes of the spectrum. There are a number of differences in our model from a voting model. In a voting model, there are several voters whose bliss point (peak value) is their private information. In our model, peaks are publicly known. What is private information is the two negotiators' outside options, which do not have any analogues in a voting model. Because of this difference, the voting model admits a class of strategy-proof rules for which efficiency and individual rationality vacuously hold. In our setup, however, such rules are either inefficient or violate individual rationality. In our setup, even dictatorship rules, despite being efficient, violate individual rationality.

In our benchmark model with single issue, there is no strategy-proof, efficient and individually rational rule. This in turn motivates to consider multi-issue models where single-peakedness does not necessarily hold and there is asymmetry in terms of outside options are treated for each issue. In fairness, there are also multi-dimensional voting models where people vote on multiple issues but this literature also concludes that strategy-proofness effectively requires each dimension to be treated independently for other. In our setup, by contrast we exploit a kind of exchangeability between the two issues, together with an asymmetric treatment of outside options, to arrive at strategy-proof rules

A logically independent but similar result that we find to the median voter theorem is that although the family of strategy-proof, efficient and individually rational rules we characterize are much different than those strategy-proof rules characterized generalized Condorcet rules, our family also nests a central rule dubbed the "constrain-and-choose rule" that chooses the "median bundle" when preferences of the negotiators are symmetric and tries to choose outcomes as close to the median bundle as possible. In our setup, however, the class of rules need not even include the impartial median-type rule. Indeed, we also identify rules that may also be partial toward either negotiator.

3) Matching/Assignment: Matching models and applications have championed the ordinal mechanism design approach (see, for example, Gale and Shapley 1962, Shapley and Shubik 1971, Crés and Moulin 2001, and especially recent applications of ordinal

assignment mechanisms Balinski and Sönmez 1998, Abdulkadiroglu and Sönmez 2003, Roth, Sönmez and Ünver 2005.) Ordinal rankings over objects together with an outside option is a common feature of matching/assignment models. Given that both negotiators end up consuming the same bundle, our model with ordinal preferences can be though to be a "public good" assignment version of a matching problem. This connection to matching is important in two regards: 1. Ordinal mechanisms may be more practical and convenient than cardinal ones as supported by experimental work. In this regard ordinal mechanisms coupled with strategy-proofness can help avoid the Wilson's critique often imposed on the "cardinal/Bayesian" mechanism design approach.

A second connection that surfaces to matching type models as a result of our analysis is that we find that the class of strategy-proof, efficient and individually rational rules also contain the negotiator-optimal rules, much in the same spirit as the proposing-side optimal deferred acceptance mechanisms or the buyer/seller optimal core assignments in the Shapley-Shubik assignment game.

4) Non-dictatorial strategy-proof mechanisms escaping the Arrow - Gibbard - Satterthwaite impossibilities: With the hope of arriving at possibility results, there is a tradition of identifying strategy-proof rules in restricted economic environments: see, for example, Vickrey (1961), Groves (1973), Clarke (1971) [VCG] for public goods and private assignment with transfers, uniform rule (Benassy 1982, Sprumont 1991) for division of divisible private good under single-peaked preferences, generalized median-voters (Moulin 1980), proportional-budget exchange rules (Barbera and Jackson 1995) that allow for trading from a finite number of pre-specified proportions (budget sets), deferred acceptance (Gale and Shapley, 1962) and top trading cycles (David Gale, 1974 and Abdulkadiroglu and Sönmez, 2003); hierarchical exchange and brokerage (Papai 2001 and Pycia and Ünver 2015). We also add to this literature in the sense that one may draw a conceptual parallel with the VCG mechanisms, though our rules look nothing like the above rules. In the VCG model, preferences over objects are private info and the preferences over money is common knowledge. This is much like negotiator's preferences over issue X versus issue Y. This connection is only superficial since VCG mechanisms are cardinal, and assignments and transfers depend on reported utilities.

A simple example: single-issue mediation

Negotiators 1 and 2 are in a dispute over a single issue that is important for both. Let x_1 and x_2 denote the available alternatives (solutions) for the dispute. The negotiators are also entitled to the outside option, o, in case one or both of them reject to accept one of the alternatives. Therefore, the set $X = \{x_1, x_2, o\}$ denotes the set of all possible outcomes of the dispute.

It is common knowledge that negotiator 1 (strictly) prefers alternative x_1 to x_2 and negotiator 2 prefers x_2 to x_1. That is, the negotiators have diametrically opposed preferences over the alternatives x_1 and x_2. The ranking of the outside option, however, is the negotiators' private information. Therefore, each negotiator has two types[4]:

$\theta_1^{x_1}$	$\theta_1^{x_2}$
x_1	x_1
o	x_2
x_2	o

$\theta_2^{x_2}$	$\theta_2^{x_1}$
x_2	x_2
o	x_1
x_1	o

Consider the mediation process, denoted by f, as a mechanism with veto rights that maps the negotiators' private information to an outcome in X. Then, it would be represented by the following matrix

	$\theta_2^{x_1}$	$\theta_2^{x_2}$
$\theta_1^{x_1}$	$f_{1,1}$	$f_{1,2}$
$\theta_1^{x_2}$	$f_{2,1}$	$f_{2,2}$

where $f_{i,j} \in X$ for all $i, j \in \{1, 2\}$.

We can assign $f_{1,2} = o$, without loss of generality, because there is no mutually acceptable alternative when the negotiators' types are $\theta_1^{x_1}$ and $\theta_2^{x_2}$, and thus, the outside option o is effectively the only result in all voluntary mediation processes. If the outcomes of the mediation process are (Pareto) efficient, then $f_{1,1}$ should be x_1 or x_2. Moreover, if the process produces individually rational outcomes, then we must have $f_{1,1} = x_1$. Likewise, an efficient and individually rational mediation process dictates $f_{2,2} = x_2$ and $f_{2,1} \in \{x_1, x_2\}$.

Therefore, we can construct several efficient and individually rational mechanisms for this simple example. However, none of these processes are immune to strategic manipulation (strategy-proofness). To prove this point, suppose that $f_{2,1} = x_1$. In this case, type $\theta_2^{x_1}$ of negotiator 2 would deviate and declare his type as $\theta_2^{x_2}$ to obtain x_2, contradicting with strategy-proofness. Alternatively, if $f_{2,1} \neq x_1$, then type $\theta_1^{x_2}$ of negotiator 1

[4]We suppose, without loss of generality, that there is at least one acceptable alternative for each negotiator.

would deviate and declare his type as $\theta_1^{x_1}$ to obtain x_1, again contradicting with strategy-proofness.

It is easy to extend this example to the case with more than two alternatives, and so extrapolate that there exists no efficient, individually rational and strategy-proof single-issue mediation process.

The Model: Multi-Issue Mediation

This section proves that we may escape from the impossibility result akin to Myerson and Satterthwaite (1983) if the negotiators are in dispute over multiple issues and two issues are enough to make our point. There are two agents, $N = \{1, 2\}$, in a dispute who aim to reach a resolution through mediation. There are two **issues**, for the sake of expositional simplicity, that are important for the negotiators. Let the sets $X = \{x_1, ..., x_m, o_X\}$ and $Y = \{y_1, ..., y_m, o_Y\}$ denote the finite sets of potential **outcomes** for each issue. The sets $X \setminus \{o_X\}$ and $Y \setminus \{o_Y\}$ are the available **alternatives**. The negotiators are entitled to an **outside option** (disagreement point) for each issue, o_X and o_Y, in case one or both of them reject to accept an alternative that is available for that issue. Negotiators have at least two available alternatives for each issue, and so $m \geq 2$.

Preferences over Outcomes: The negotiators' preferences over outcomes for each individual issue satisfy the following three condition:

1. The negotiators' preferences over alternatives (not including the outside option) for each individual issue are conflicting and public information.

2. Both negotiators' rankings of the outside option (relative to other alternatives) are private information in one of the issues.

3. It is public information that both negotiators rank the outside option as their worst outcome in one of the issues,.

More formally, for any issue $Z \in \{X, Y\}$, where $Z = \{z_1, ..., z_m, o_Z\}$, it is public information that $z_k \ \theta_1^Z \ z_{k+1}$ and $z_{k+1} \ \theta_2^Z \ z_k$ for all $k = 1, ..., m-1$. That is, the negotiators' preferences over the alternatives for each issue are diametrically opposed (the first condition). Let Θ_i^Z denote the set of all complete, transitive and antisymmetric preference relations of negotiator $i \in N$ over issue Z. We also suppose that the ranking of the outside option in issue X, o_X, is the negotiators' private information (the second condition). Furthermore, it is common knowledge that $y \ \theta_i^Y \ o_Y$ for all i and $y \in Y \setminus \{o_Y\}$ (the third condition). Therefore, the set of acceptable alternatives for issue X is privately known

by the negotiators, and it is unknown to them whether there is a mutually acceptable alternative for that issue. However, all alternatives in issue Y are acceptable and efficient. Therefore, let $\Theta_i = \Theta_i^X$ denote the set of all **types** for negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ be the set of all type profiles.

This asymmetric treatment of the outside options can be motivated by various employment, family, construction or patent/copyright infringement disputes. Litigation would naturally be the default option if the issue is compensation or division of property. It usually is the case in such disputes that litigation is a very long and costly process, and so, inefficient relative to other potential divisions (alternatives). Such issues would be mapped into the issue Y in our framework. Although money is an important component in disputes, it is not the only issue: In employment disputes, for example, the quality of the reference letter that the former employer would be willing to write could be another issue for the disputants, or child custody or visitation would be the alternative issues in family disputes. Such issues, where the disputants' ranking of the outside option is not clear to all the parties, would be represented by the issue X in our setup. Nonetheless, one can provide various examples where, naturally, the ranking of the outside option in all issues are the disputants' private information. For that reason, the symmetric treatment of the outside option is formally investigated at the end of this section.

Preferences over Bundles: A **bundle** (x, y) is a vector of outcomes, one for each issue, and the set $X \times Y$ denotes the set of all bundles. Let $\Re$ denote the set of all complete and transitive binary relations over the bundles. R is a standard element of the set $\Re$ and for any two bundles $b, b' \in X \times Y$, $b \; R \; b'$ means "b is at least as good as b'." We denote P for the strict counterpart of R.[5] An extension map is a rule Λ which assigns to every negotiator i and type $\theta_i \in \Theta_i$ a non-empty set $\Lambda(\theta_i) \subseteq \Re$ of admissible orderings over bundles.

For any negotiator i and type $\theta_i \in \Theta_i$, let $A(\theta_i) = \{x \in X | \; x \; \theta_i^X \; o_X\}$ denote the set of **acceptable** alternatives in issue X. For any type profile $(\theta_1, \theta_2) \in \Theta$, the set $A(\theta_1, \theta_2) = \{x \in X | \; x \; \theta_i^X \; o_X \text{ for all } i \in N\}$ denote the set of **mutually acceptable** alternatives in issue X. Whenever we need to specify a type's acceptable alternatives, let $\theta_i^x \in \Theta_i^X$ denote the preference relation (type) of negotiator i in which alternative $x \in X$ is the worst acceptable alternative: that is, for any $x' \in X \setminus \{o_X\}$, $x \; \theta_i^x \; x' \Longrightarrow o_X \; \theta_i^x \; x'$.

[5] That is, $b \; P \; b'$ if and only if $b \; R \; b'$ holds but $b' \; R \; b$ does not.

Definition 1. *The* ***extension map*** Λ *is* ***consistent*** *if the followings hold for all* i, $\theta_i \in \Theta_i$ *and all* $R_i \in \Lambda(\theta_i)$*:*

i. [Monotonicity] For any $x, x' \in X$ *and* $y, y' \in Y$ *with* $(x, y) \neq (x', y')$,

$$(x, y) \; P_i \; (x', y')$$

whenever $\left[x \; \theta_i^X \; x' \text{ or } x = x'\right]$ *and* $\left[y \; \theta_i^Y \; y' \text{ or } y = y'\right]$.

ii. [Deal Breakers] For any $y, y' \in Y \setminus \{o_Y\}$,

$$(x, y) \; R_i \; (x', y')$$

whenever $x \in A(\theta_i)$ *and* $x' \notin A(\theta_i)$.

iii. [Logrolling] For any i*, there exists a one-to-one mapping* $t_i : X \to Y$ *such that for all* $\theta_i \in \Theta_i$*,* $R_i \in \Lambda(\theta_i)$ *and all* $x, x' \in A(\theta_i)$ *with* $x \; \theta_i \; x'$,

$$\big(x', t_i(x')\big) \; R_i \; \big(x, t_i(x)\big).$$

Monotonicity is a standard assumption. The second condition suggests that unacceptable alternatives in issue X are "deal-breakers" for the negotiators: regardless of the alternative in the second issue, a bundle with an unacceptable alternative is never preferred to a bundle with an acceptable alternative. Put differently, alternatives in issue Y are not "too important" for the negotiators, and so, an unacceptable alternative can never become a part of an acceptable bundle.

Logrolling or quid pro quo allows trading of favors. It requires two things. First, for any two acceptable alternatives x, x' in X where x is ranked above x' for type θ_i, there must exist two alternatives y, y' in Y s.t (x', y') is ranked at least as high as (x, y) at all consistent orderings over bundles, $R_i \in \Lambda(\theta_i)$. Second, types must be "consistent." Namely, order reversing mapping, t_i, is independent of types. Logrolling implies that alternatives in Y are "important enough" to reverse the rankings of (acceptable) alternatives in X when they are bundled with alternatives in the second issue. Logrolling rules out lexicographic preferences and many standard utility functions satisfy it; we discuss this point later in detail. Furthermore, it is sufficient and "almost necessary" for the possibility result. We prove this point next.

Example (logrolling): Suppose that $X = \{x_1, x_2, x_3, o_X\}$ and $Y = \{y_1, y_2, y_3, o_Y\}$. Because the number of alternatives in issues X and Y are equal, there is a unique one-to-one mapping t (which is the same for both negotiators), where $t(x_k) = y_{4-k}$ for $k = 1, 2, 3$,

which satisfies the requirements of Definition 1.[6] Therefore, logrolling implies that the type $\theta_1^{x_3}$ of negotiator 1 who deems all three alternatives in issue X acceptable, i.e., $x_1\ \theta_1^{x_3}\ x_2\ \theta_1^{x_3}\ x_3\ \theta_1^{x_3}\ o_X$, will rank (x_3, y_1) at least as high as the bundle (x_2, y_2) and rank (x_2, y_2) at least as high as the bundle (x_1, y_3) for all consistent orderings $R \in \Lambda(\theta_1^{x_3})$. The consistency of the mapping t over the types implies, for example, that type $\theta_1^{x_2}$ of negotiator 1 who deems only x_1 and x_2 acceptable, i.e., $x_1\ \theta_1^{x_2}\ x_2\ \theta_1^{x_2}\ o_X\ \theta_1^{x_2}\ x_3$, will rank (x_2, y_2) at least as high as the bundle (x_1, y_3). Logrolling imposes no restriction on consistent orderings $R \in \Lambda(\theta_1^{x_2})$ regarding how they rank the bundle (x_3, y_1) relative to the bundles (x_2, y_2) and (x_1, y_3). Let Z^* denote the set of logrolling bundles. Namely, $Z^* = \{(x_k, y_{m+1-k})\}$.

Direct Mechanisms with Veto Rights: Mediation would be a very complicated, multi-stage game between the negotiators and the mediator. The mediation protocol, whatever the details are, produces proposals for agreement that are always subject to unanimous approval by the negotiators. That is, before finalizing the protocol, each negotiator has the right to veto the proposal and the option to receive the outside options.

A version of the revelation principle that we prove in the appendix guarantees that we can stipulate the following direct mechanism with veto rights without loss of generality, when representing mediation. The direct mediation mechanism consists of two stages: an *announcement* stage and a *ratification* stage; and it is characterized by the mediation rule $f : \Theta \to X \times Y$. After being informed of its type, each negotiator i privately reports his type, $\widehat{\theta}_i$, to the mediator, who then proposes $f(\widehat{\theta}_1, \widehat{\theta}_2) \in X \times Y$. In the ratification stage, each party simultaneously decides whether to accept or veto the proposed bundle. In case both negotiators accept the proposed bundle, then it becomes the final outcome. In case one or both negotiators veto the proposal, each party gets the outside option for both issues, i.e., (o_X, o_Y). Such direct mechanisms will be called direct truthful mechanisms with veto rights.

Definition 2. *The mediation rule f is **strategy-proof** if for all i and all $\theta_i \in \Theta_i$, $f(\theta_i, \theta_{-i})\ R_i\ f(\theta_i', \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$.*

Definition 3. *The mediation rule f is **individually rational** if for all i and all $(\theta_i, \theta_{-i}) \in \Theta$, $f(\theta_i, \theta_{-i})\ R_i\ (o_X, o_Y)$ for all $R_i \in \Lambda(\theta_i)$.*

Definition 4. *The mediation rule f is **efficient** if there exists no $(\theta_i, \theta_{-i}) \in \Theta$ and $(x', y') \in X \times Y$ such that $(x', y')\ R_i\ f(\theta_i, \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$ and all $i \in N$, and for at least one $i \in N$, $(x', y')\ P_i\ f(\theta_i, \theta_{-i})$ for all $R_i \in \Lambda(\theta_i)$.*

[6] Note that logrolling is a well-defined concept only if the number of alternatives in issue Y is greater than or equal to the number of alternatives in issue X. In case the number of alternatives in Y greater than that of X, one of many one-to-one mappings suffices.

We seek direct mechanisms with veto rights in which, it is a *dominant strategy equilibrium* to report the true private information at the announcement stage, and in which, in equilibrium, proposals are not vetoed. It immediately follows from the definitions that such an equilibrium exists if and only if the mediation rule f is strategy-proof and (ex-post) individually rational.[7] For convenience, we present a mediation rule f as an $(m)\times(m)$ matrix $f=[f_{i,j}]_{(i,j)\in M^2}$, where $M=\{1,...,m\}$. The rows indicate all the types of negotiator 1 and the columns are for all the types of negotiator 2. We ignore, without loss of generality, the types that deems no alternative acceptable.

$$f = \begin{array}{c|ccc} & \theta_2^{x_1} & \cdots & \theta_2^{x_m} \\ \theta_1^{x_1} & f_{1,1} & \cdots & f_{1,m} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_1^{x_m} & f_{m,1} & \cdots & f_{m,m} \end{array}$$

In this matrix, row (column) i represents the preference of negotiator 1 (2) that finds all alternatives $\{x_k|k\leq i\}$ $\left(\{x_k|k\geq i\}\right)$ acceptable. Therefore, there is a unique mutually acceptable alternative in the main (first) diagonal of the matrix. There is no mutually acceptable alternative in the upper half of the matrix.

Theorem 1. *The mediation rule f is efficient, individually rational and strategy-proof if and only if the following hold:*

(*i*) *If $i<j$, then $f_{i,j}=(o_X,y)$ for some $y\in Y$.*

(*ii*) *If $i=j$, then $f_{i,j}=(x_i,y_{m+1-i})$.*

(*iii*) *(Adjacency) If $i>j$, then $f_{i,j}\in\{f_{i-1,j},f_{i,j+1}\}\subset Z^*$ and there exists a complete, transitive and strict precedence order $\rhd$ on Z^* such that*

$$f_{i,j}=\begin{cases} f_{i-1,j}, & \text{if } f_{i-1,j}\rhd f_{i,j+1} \\ f_{i,j+1}, & \text{oth.} \end{cases}$$

Example (adjacent rules): To picture the standard members of adjacent rules, consider the following example with five alternatives. Suppose that $X=\{x_1,x_2,x_3,x_4,x_5,o_X\}$ and $Y=\{y_1,y_2,y_3,y_4,y_5,o_Y\}$. The set Z^* consists of all the bundles that satisfy logrolling, i.e., $Z^*=\{(x_1,y_5),(x_2,y_4),(x_3,y_3),(x_4,y_2),(x_4,y_1)\}$. A member of this family is constructed with a strict precedence order: for example, let $\rhd$: $(x_5,y_1)\rhd(x_1,y_5)\rhd(x_4,y_2)\rhd(x_2,y_4)\rhd(x_3,y_3)$, then the following mediation rule is efficient, individually rational and strategy-proof:

[7] The proof is omitted as it directly follows from similar arguments in our revelation principle result.

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)	(o_X, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(o_X, y_3)	(o_X, y_3)
$\theta_1^{x_4}$	(x_1, y_5)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(o_X, y_3)
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

Special members of the adjacent rules family

There are some special members of the adjacent rules. Negotiator 1 (2)-optimal adjacent rule, for example, is constructed by picking the precedence order that is the strict counterpart of the preference order of negotiator 1 (2) over the logrolling bundles Z^*. For the same example above, the negotiator 1 adjacent rule takes

$$\rhd^1\text{: } (x_5, y_1) \ \rhd^1 \ (x_4, y_2) \ \rhd^1 \ (x_3, y_3) \ \rhd^1 \ (x_2, y_4) \ \rhd^1 \ (x_1, y_5)$$

whereas the negotiator 2 adjacent rule takes

$$\rhd^2\text{: } (x_1, y_5) \ \rhd^2 \ (x_2, y_4) \ \rhd^2 \ (x_3, y_3) \ \rhd^2 \ (x_4, y_2) \ \rhd^2 \ (x_5, y_1)$$

and they look as follow:

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)				
$\theta_1^{x_2}$	(x_2, y_4)	(x_2, y_4)			
$\theta_1^{x_3}$	(x_3, y_3)	(x_3, y_3)	(x_3, y_3)		
$\theta_1^{x_4}$	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	(x_4, y_2)	
$\theta_1^{x_5}$	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)	(x_5, y_1)

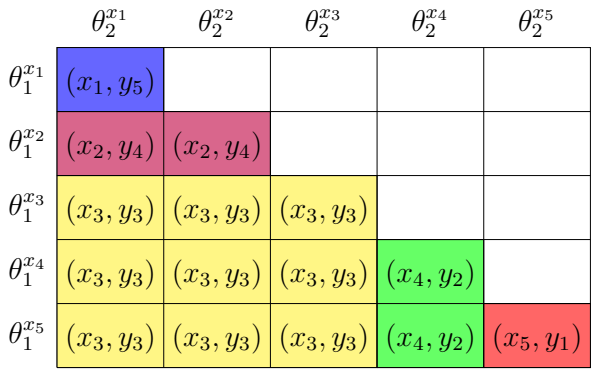
Negotiator 1-optimal rule

	$\theta_2^{x_1}$	$\theta_2^{x_2}$	$\theta_2^{x_3}$	$\theta_2^{x_4}$	$\theta_2^{x_5}$
$\theta_1^{x_1}$	(x_1, y_5)				
$\theta_1^{x_2}$	(x_1, y_5)	(x_2, y_4)			
$\theta_1^{x_3}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)		
$\theta_1^{x_4}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	
$\theta_1^{x_5}$	(x_1, y_5)	(x_2, y_4)	(x_3, y_3)	(x_4, y_2)	(x_5, y_1)

Negotiator 2-optimal rule

Negotiator 1-optimal rule always picks negotiator 1's most preferred bundle among the mutually acceptable logrolling bundles. Although these rules are efficient, individually rational and strategy-proof, they are not impartial (symmetric). There is another special member of this class that treat negotiators symmetrically if the mediation problem is symmetric.[8]

[8]The mediation problem is symmetric if the number of alternatives in issues is odd number. The



Constrain-and-choose rule

Constrain-and-choose rule and its characterization

In the **"constrain and choose" (CC)** rule the mediator picks one of the negotiators and asks his least acceptable alternative, say x_n. Then the mediator proposes three bundles to the other negotiator to pick. One of these bundles is the element of Z^* that corresponds to the bundle with the least preferred alternative x_n, i.e., (x_n, y_{m-n+1}). The other bundle is the one in Z^* where the alternatives in both issues are the median alternatives, i.e., (x_{k^*}, y_{k^*}) where k^* is the indices of the median alternative. Namely, $k^* \in \{\overline{k}^*, \underline{k}^*\}$, where $\overline{k}^* = \lceil \frac{m+1}{2} \rceil$ and $\underline{k}^* = \lfloor \frac{m+1}{2} \rfloor$. If m is odd, then $\overline{k}^* = \underline{k}^* = \frac{m+1}{2}$. If there are two median alternatives, then the mediator picks one of them at all times. Finally, the third bundle is the bundle with outside option in issue X, i.e., (o_X, y_{k^*}).

CC rule is a special member of the adjacent rule family. It acts as though it is a negotiator 1 or 2-optimal rule if the median alternative in issue X is not mutually acceptable and suggests the "median" bundle otherwise. In addition to being efficient, individually rational and strategy-proof, CC rule minimizes rank variance within the class of efficient, individually rational and strategy-proof rules. We will prove this last point next.

Given the negotiators' fixed preferences over alternatives (not including the outside option), let $r_i(z) \in M$ denote negotiator i's ranking of the alternative $z \in Z \in \{X, Y\}$.[9] Given a mediation rule $f = [f_{i,j}]_{(i,j)\in M^2}$, let $f_{i,j} = (f_{ij}^X, f_{ij}^Y)$ correspond to the bundle the mediation rule proposes when the negotiators types are $\theta_1^{x_i}$ and $\theta_2^{x_j}$. Therefore, the *rank*

problem is symmetric in this case because there is a unique median alternative in each issue, and thus, the number of alternatives (not including the outside option) better than the median alternative is the same for both negotiators. However, if there are two median alternatives, which is the case when the number of alternatives is even, then the problem is not symmetric.

[9]We ignore the outside option from the rank calculations without loss of generality because we will restrict our attention to individually rational and efficient rules.

variance of bundle $f_{i,j}$ is defined by[10]

$$var(f_{i,j}) \equiv \sum_{l=1,2} \left(r_l(f_{ij}^X)\right)^2 + \left(r_l(f_{ij}^Y)\right)^2.$$

Thus, the rank variance of the mediation rule f is

$$Var(f) = \sum_{i=1}^{m}\sum_{j=1}^{m} var(f_{i,j})$$

A bundle including a/the median alternative in both issues has the smallest rank variance and bundles (x_1, y_1) and (x_m, y_m) have the highest rank variance. Intuitively, rank variance of a bundle is a measure of the extent to which that bundle favors one negotiator over his counter. In this sense, the higher the rank variance of a bundle or a mediation rule, the more biased its treatment is. Normatively speaking, a rule that aims to minimize rank variance can be viewed as one choosing "the center of gravity" or the "middle ground" along the tradeoffs the negotiators are facing.

Definition 5. *For any $k \in M$, let $z_k = (x_k, y_{m-k+1}) \in Z^*$ denote the logrolling bundle in Z^*. A rule is a "Constrain and choose" rule, denoted f^{CC}, if it is a precedence rule (as described in Theorem 1) that is associated with a precedence order $\rhd$, where $z_{k^*} \rhd z_{k^*-1} \rhd \ldots \rhd z_1$ and $z_{k^*} \rhd z_{k^*+1} \rhd \ldots \rhd z_m$ with k^* being the indices of the median alternative, and $f_{i,j}^{CC} = (o_X, y_{k^*})$ whenever $i < j$.*

Note that if m is odd, There is a unique CC rule if m is odd. If m is even, however, a CC rule prescribes one of four types of outcomes depending on whether $z_{\overline{k}^*}$ or $z_{\underline{k}^*}$ has the highest precedence and whether $y_{\overline{k}^*}$ or $y_{\underline{k}^*}$ is chosen when no mutually acceptable outcome in issue X exists.

Theorem 2. *A mediation rule minimizes rank variance within the class of efficient, individually rational and strategy-proof rules if and only if it is a CC rule.*

Proof of Theorem 2: Clearly, a CC rule belongs to the precedence family. To see that the rank variance of a CC rule is lower than any other member of the precedence family, we simply consider two cases about the number of possible outcomes.

First, when m is odd, $var(z_{k^*}) = (m+1)^2$. For any $z_{k^*-t}, z_{k^*+t} \in Z^*$ with $t < k^*$, we have $var(z_{k^*-t}) = var(z_{k^*+t}) = 2(\frac{(m+1)}{2} - t)^2 + 2(\frac{(m+1)}{2} + t)^2 = (m+1)^2 + 4t^2$. Thus, $var(z_{k^*}) < var(z)$ for any $z \in Z^* \setminus \{z_{k^*}\}$.

Since any member of the precedence family must pick an element of Z^* whenever there is a mutually acceptable alternative in issue X (by cases (ii) and (iii) of Theorem

[10] One may assign different weights to the issues in the definition of rank variance. The results still go through without any loss.

1), minimization of rank variance requires that $z_{k^*} \rhd z$ for any $z \in Z^* \backslash \{z_{k^*}\}$. Also observe that $var(z_{k^*}) < var(z_{k^*-1}) < \ldots < var(z_1)$ and $var(z_{k^*}) < var(z_{k^*+1}) < \ldots < var(z_m)$. Thus, minimization of rank variance subsequently requires that $z_{k^*-1} \rhd \ldots \rhd z_1$ and $z_{k^*+1} \rhd \ldots \rhd z_m$. By case (i) of Theorem 1, the outcome for issue X is fixed to o_x whenever there is no mutually acceptable alternative in this issue. Therefore, (o_X, y_{k^*}) is the rank minimizing bundle. Note that when m is odd, rank variance of the unique CC rule is strictly less than any other member of the precedence family.

On the other hand, when m is even, $var(z_{\overline{k}^*}) = var(z_{\underline{k}^*}) = \frac{1}{2}(m^2 + (m+2)^2)$. For any $z_{\underline{k}^*-t}, z_{\overline{k}^*+t} \in Z^*$ with $t < k^*$, we have $var(z_{\underline{k}^*-t}) = var(z_{\overline{k}^*+t}) = 2(\frac{m}{2} - t)^2 + 2(\frac{(m+2)}{2} + t)^2 = \frac{1}{2}(m^2 + (m+2)^2) + 4t^2$. Hence, $var(z_{\overline{k}^*}) = var(z_{\underline{k}^*}) < var(z)$ for any $z \in Z^* \backslash \{z_{\overline{k}^*}, z_{\underline{k}^*}\}$. Note that we also have $var(z_{\underline{k}^*}) = var(z_{\overline{k}^*}) < var(z_{\underline{k}^*-1}) < \ldots < var(z_1)$ and $var(z_{\underline{k}^*}) = var(z_{\overline{k}^*}) < (z_{\overline{k}^*+1}) < \ldots < var(z_m)$. Then, minimization of rank variance subsequently requires that either $z_{\overline{k}^*} \rhd z_{\underline{k}^*}$ or $z_{\underline{k}^*} \rhd z_{\overline{k}^*}$ together with $z_{\underline{k}^*-1} \rhd \ldots \rhd z_1$ and $z_{\overline{k}^*+1} \rhd \ldots \rhd z_m$. By case (i) of Theorem 1, the outcome for issue X is o_X and both $(o_X, y_{\overline{k}^*})$ and $(o_X, y_{\underline{k}^*})$ are rank minimizing bundles. Consequently, any one of the four types of CC rules are rank minimizing. Note that when m is even, rank variance of a CC rule is weakly less than any other member of the precedence family.

Preferences that Satisfy Logrolling

The critical requirement for out results is logrolling. Simply put, it requires that alternatives in issue Y are "important enough" to reverse the ranking of the alternatives in issue X when they are bundled together. Absolute utility values of the alternatives are irrelevant for logrolling. To see this point, consider the following simple example:

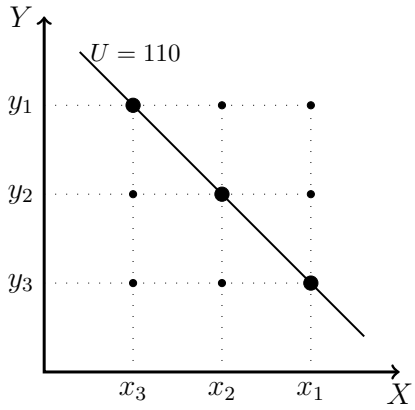
Example 1: Suppose that preferences over the bundles are additively separable and each issue has three alternatives. Let $u(.)$ and $v(.)$ represent preferences over issues X and Y, respectively. Therefore, $U(x, y) = u(x) + v(y)$ is the utility function over the bundles.[11]

X	$u(.)$	Y	$v(.)$	$X \times Y$	$U(.)$
x_1	100	y_1	20	(x_3, y_1)	110
x_2	98	y_2	12	(x_2, y_2)	110
x_3	90	y_3	10	(x_1, y_3)	110

The utility functions (preferences) in this simple example satisfy logrolling although the worst alternative in issue X is 4.5 times more valuable, in absolute terms, than the most valuable alternative in issue Y.

[11] For completeness, one may assume that all types get very large disutility from unacceptable alternatives, including the outside option.

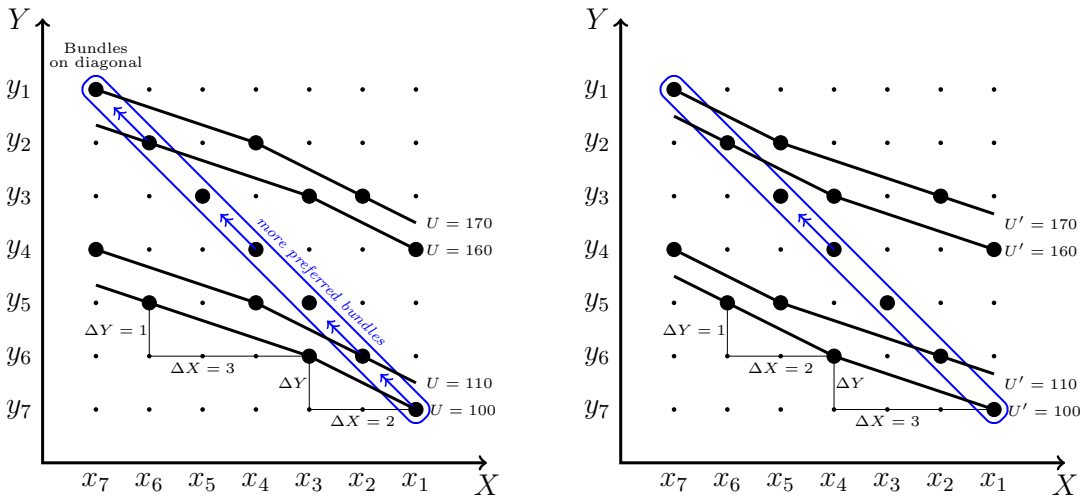
In standard consumer theory, we represent preferences over bundles by drawing corresponding indifference curves on commodity space, where each axis corresponds the quantity of a particular commodity. In the current model, issues serve the same role with commodities. However, distance between two alternatives is irrelevant in our setup as we abstract away from quantities. In our discrete setup, marginal rate of substitution is the rate at which a negotiator can give up some *number of alternatives* in one issue in exchange for the other issue while maintaining the same level of utility. Therefore, without loss of generality, we can place all alternatives equidistantly. Also, we place less preferred alternatives closer to the origin, implying (together with monotonicity) higher indifferent curves as we move northeastern direction. For the previous numerical example, therefore, preferences of the type that deems all alternatives acceptable can be pictured as follows:



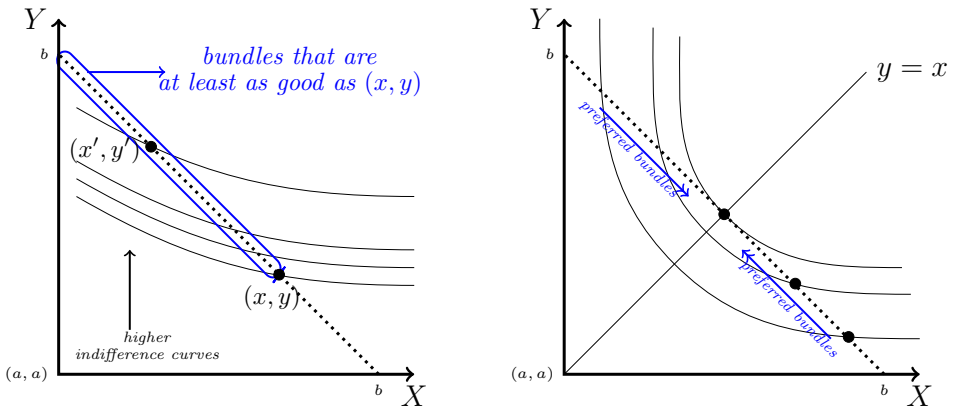
As it is evident from this graph, logrolling is a property of the bundles that are placed on the *diagonal*. Logrolling requires that these bundles are either lying on the same indifference curve or on higher indifference curves as we move along the diagonal in northwestern direction (and southeast direction for the other negotiator). The marginal rate of substitution at diagonal bundles is one for our example because it requires one alternative to trade between the issues to keep the negotiator's utility the same. The utility function we picked behaves *as if* issues are perfect substitutes. But it is hardly possible to make a concrete statement with only nine bundles. On the other hand, logrolling is consistent with all range of utility functions that have "convex" or "concave" indifference curves. Consider the following two utility functions, U and U':

X	$u(.)$	$u'(.)$	Y	$v(.)$	$X \times Y$	$U(.) = u(.) + v(.)$	$U'(.) = u'(.) + v(.)$
x_1	100	100	y_1	120	(x_7, y_1)	170	170
x_2	90	90	y_2	100	(x_6, y_2)	160	160
x_3	80	85	y_3	80	(x_5, y_3)	145	150
x_4	70	80	y_4	60	(x_4, y_4)	130	140
x_5	65	70	y_5	40	(x_3, y_5)	120	125
x_6	60	60	y_6	20	(x_2, y_6)	110	110
x_7	50	50	y_7	0	(x_1, y_7)	100	100

Both utility functions satisfy logrolling and their indifference curves are drawn in the following two graphs. As it is also clear from these graphs, the marginal rate of substitution of the utility function U (U') is increasing (decreasing) as we move to the right in the X axis. This implies that the indifference curves are concave for U and convex for U'.

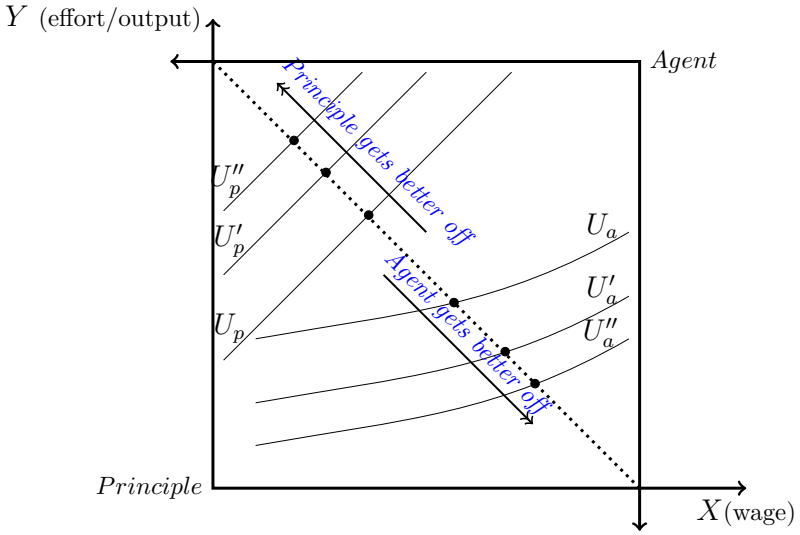


In a more standard setup where alternatives for issues X and Y represent quantities of two commodities, ranging over some interval $[a, b] \subseteq \mathbb{R}^2$, logrolling will be satisfied for all utility functions with marginal rate of substitution (MRS) that is less than or equal to one at all points of the diagonal. For example, a utility function $U(x, y) = \sqrt{x} + y$ does satisfy this condition whenever $1/4 \leq a$. This is true because upper counterset of a bundle (x, y) that is on diagonal includes all the other bundles (x', y') on the diagonal that are situated northwest of the original bundle (x, y).



In fact, all utility functions with convex indifference curves with $MRS_{x=y} \leq 1$ satisfy a weaker condition of logrolling, where logrolling holds only for the first half of the alternatives in issue X. This weaker condition is sufficient to guarantee strategy-proof rules. An example for these utility functions would be $U(x, y) = x^{\alpha} y^{\beta}$ where $\alpha/\beta \leq 1$. The second graph above demonstrates why such convex utility functions satisfy this weaker condition.

Alternatively, one may consider a moral hazard situation between a principle and an negotiator, where X denotes the domain for wage and Y denotes different levels of effort/output. Consistent to this framework, let the agent's and the principle's utility functions are $U_a(x, y) = u_a(x) - v_a(y)$ and $U_p(x, y) = v_p(y) - u_p(x)$, respectively, where all u's and v's are increasing functions. Along the diagonal, the principle's indifference curves increase as we move in northwestern direction (and the worker's indifference curves increase as we move in southeastern direction) as required by the logrolling condition. For a simple example, one may consider $U_w(x, y) = x - y^2$ and $U_p = y - x$, which we depict below.



Relating to the impossibility result of Myerson and Satterthwaite (83)

This section explores the underlying factors that are potentially absent in Myerson and Satterthwaite (83) model, which leads to the possibility result in our case. It is already well-known in the literature that denseness of the type space is one reason for the impossibility result in Myerson and Satterthwaite (83) (MS henceforth). However, this is not the driving force for our strategy-proof mediation rules in multi issue mediation case. The main factor seems to be that MS is effectively not a multi-issue negotiation problem.

MS considers a bilateral trade between a seller, who owns an indivisible good, and a buyer, who likes to buy this good, as a mechanism design problem. The mechanism (p, x) has two components; the probability of trade, p, and the transfer, x, both of which are functions of the players' reports. If no trade occurs, then $x = p = 0$ (the outside option), and so both players receive zero utility. The utility functions are $U_b = v_b p - x$ for the buyer and $U_s = x - v_s p$ where the valuations v_b, v_s are the players' private information. Consider for simplicity that both players' valuations are distributed over the unit interval $[0, 1]$ according to some probability distribution.

One may map this setup to our two-issue framework, with continuum of types, where the first issue is the probability of trade, i.e., p, and the second issue is the amount of transfer, i.e., x. It is clear from the utility functions that agents preferences over the individual issues are diametrically opposed. That is, for any fixed value of x, the buyer gets better off as p decreases from 1 to 0 and the seller gets worse off as p decreases from 1 to 0. Similarly, for any fixed value of p, the buyer gets better off as x decreases from 1 to 0 and the seller gets worse off as x decreases from 1 to 0. In addition to this, it is easy to verify that the preferences over the bundles satisfy logrolling.

In our setup, each issue has separate outside option whereas MS assumes joint outside option for the issues (no trade). However, this is not directly the main driving force for the difference between these two papers. Aside from this divergence of these two models, MS corresponds to our symmetric two issue case. Although the utility of the outside option in MS in each issue is 0, the ranking of the outside option is the negotiators' private information. Namely, the set of acceptable alternatives for the agents' is their private information. This is true because for the buyer, for example, the set of acceptable alternatives in issue p must satisfy $p \geq \frac{x}{v_b}$ for any fixed value of x, and this set is the buyer's private information as v_b is not common knowledge. Therefore, the set of acceptable outcomes (or the ranking of the outside option in each individual issue) are the players' private information, as it is the case in our symmetric case. We show, in the symmetric treatment of the outside option, that there is no individually rational and ex-post efficient strategy-proof mediation rules. We prove this point next.

Symmetric Treatment of the Outside Options

In this section, we relax the assumption that $y\ \theta_i^Y\ o_Y$ for all i and $y \in Y \setminus \{o_Y\}$. Instead, the negotiators' ranking of the outside option, o_Y, is their private information, as is the case for issue X. Thus, $\Theta_i = \Theta_i^X \times \Theta_i^Y$ denotes the set of all **types** of negotiator i, and $\Theta = \Theta_1 \times \Theta_2$ is the set of all type profiles. We also relax our assumption of consistency for the negotiators' ranking over the bundles, and suppose that they only satisfy conditions (i) and (ii) of Definition 1, i.e., monotonicity and deal breakers.

Proposition 1. *Under the symmetric treatment of the outside options, there is no mediation rule f that is strategy-proof, individually rational and efficient.*

Note that this impossibility result can easily be carried out to a single issue or more than two issue contexts. A rule that always picks the pair (o_X, o_Y) is strategy-proof but not efficient. A dictatorship is efficient and strategy-proof but not individually rational.

Proof of Proposition 1: Consider the (true) preference profile $(\theta_1, \theta_2) = (\theta_1^{x_m}, \theta_1^{y_m}, \theta_2^{x_1}, \theta_2^{y_1})$. That is, both negotiators find all alternatives acceptable. Let $(x, y) = f(\theta_1, \theta_2)$. Because negotiators preferences over alternatives are diametrically opposed for each single issue, there is at least one negotiator $i \in N$ and an issue for which negotiator i does not get his top alternative for that issue. Suppose, without loss of generality, that this negotiator is 1 and the issue is X: that is, $x \neq x_1$. Consider the new profile where only negotiator 1's preferences are different, $(\theta_1', \theta_2) = (\theta_1^{x_1}, \theta_1^{y_1}, \theta_2^{x_1}, \theta_2^{y_1})$.

We claim that $f(\theta_1', \theta_2) = (x_1, y_1)$. Suppose for a contradiction that $f(\theta_1', \theta_2) = (x', y') \neq (x_1, y_1)$. I will only show that $x' = x_1$ because similar arguments also prove $y' = y_1$, yielding the desired contradiction. To show $x' = x_1$, suppose for a contradiction that $o_X\ \theta_1^{x_1}\ x'$. Since Λ is consistent, $(o_X, o_Y)\ P_1\ (x', y')$ for some $R_1 \in \Lambda(\theta_1')$, and thus $f(\theta_1', \theta_2) = (x', y')$ contradicts with the individual rationality of f. Now suppose for a contradiction that $x' = o_X$. Then, since Λ is consistent, $(x_1, y')\ P_i\ (x', y')$ for $i = 1, 2$ and all $R_1 \in \Lambda(\theta_1^{x_1})$ and all $R_2 \in \Lambda(\theta_2^{x_1})$. Therefore, (x', y') is an inefficient bundle at (θ_1', θ_2), and thus $f(\theta_1', \theta_2) = (x', y')$ contradicts with the efficiency of f. Hence, we must have $x' = x_1$.

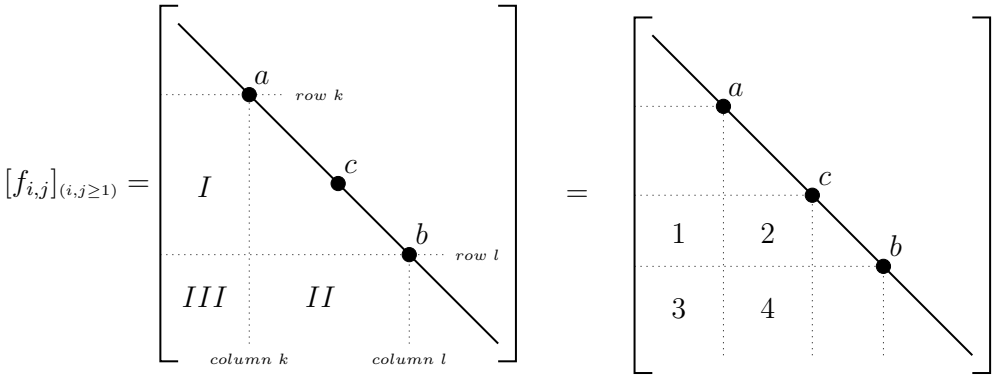
To conclude, we already know that $f(\theta_1, \theta_2) = (x, y)$ and $x \neq x_1$, which implies $x_1\ \theta_1^{x_m}\ x$. Because y_1 is negotiator 1's best alternative in issue Y, either $y = y_1$ or $y_1\ \theta_1^{y_1}\ y$ is true. In either case, consistency and transitivity implies $(x_1, y_1)\ P_1\ (x, y)$ for all $R_1 \in \Lambda(\theta_1)$. Finally, we showed in the previous paragraph that by misrepresenting his preferences at profile (θ_1, θ_2), negotiator 1 can achieve the bundle (x_1, y_1), which is strictly better than (x, y) for all $R_1 \in \Lambda(\theta_1)$, contradicting that f is strategy-proof.

Appendix

Proof of Theorem 1: Consider two bundles $a = f_{i,j}$ and $b = f_{k,l}$. We say bundle a is above (below) bundle b whenever $i < k$ ($i > k$). Likewise, bundle a is on the right (left) of bundle b whenever $j > l$ ($j < l$).

Proof of 'if':

First note that the adjacency requirement, i.e., condition (iii), and condition (ii) of the theorem implies that for any bundle $a, b \in Z^*$ where $a = f_{k,k}$ and $b = f_{l,l}$ with $k < l$ the followings must hold: In the matrix $[f_{i,j}]$ where $i > j \geq 1$, bundle a can appear only at rows that are below row k, i.e., $i > k$, and only at columns that are left of column k, i.e., $j < k$. Similarly, bundle b can appear only at rows that are below row l, i.e., $i > l$, and only at columns that are left of column l, i.e., $j < l$. The next graph helps visualize these statements. Bundle a may appear only in regions I and III, whereas bundle b may appear only in regions II and III. Moreover, the existence of complete, transitive and strict order $\rhd$ on Z^* implies that we cannot observe both a and b in region III. The last claim does not immediately follow from the definitions, and so deserves a proof.



Suppose first that a and b appear on the same column in region III, say column r, and a is above bundle b. Starting from $f_{l,l} = b$, the adjacency and transitivity of $\rhd$ imply that all the bundles on row l (recall that we are considering only the lower half of the matrix) are either ranked higher than b with respect to $\rhd$ or equal to b. Once you hit the column r in Region III, now again by adjacency, all the bundles on column r and below row l are either ranked higher than b with respect to $\rhd$ or equal to b, including a. Thus, $a \rhd b$ must hold. Continue iteration on the same column, starting from bundle a, all bundles on column r and below a are either ranked higher than a with respect to $\rhd$ or equal to a, including b since b was below a on column r. The last argument implies we have $b \rhd a$, contradicting with the fact that $\rhd$ is strict. If bundle b is above bundle a on column r, then we start the iteration from $f_{k,k} = a$. Therefore, a and b cannot appear on the same column in region III. Symmetric arguments suffice to show that they cannot appear on the same row in region III either.

Therefore, suppose that a and b appear on different rows and columns. With similar arguments above, if we start iteration from $f_{l,l} = b$ and go left on the same row and then go down to bundle a in region III, we conclude that $a \rhd b$ by adjacency and transitivity of $\rhd$. However, when we start iteration from $f_{k,k} = a$ and go down on the same column and then go left to bundle b in region III, we conclude that $b \rhd a$, which yields the desired contradiction. Hence, either bundle a or b, whichever is ranked first with respect to $\rhd$, may appear in region III. Note that neither a nor b may appear in that region if both are ranked below another bundle, for example, c. We can use the same iteration reasoning backwardly and conclude that if bundles a and b ever appear on the same column (somewhere in regions $I - III$ combined) and if a is above b, then a must appear on the first diagonal above bundle b. We can make symmetric arguments for rows as well.

The mediation rule f never suggests an alternative for an issue that is worse than the outside option of that issue, and thus, it is individually rational by the consistency of preferences. Moreover, f always suggests one of the mutually acceptable alternatives in all issues, whenever one exists. Therefore, by the way preference rankings over alternatives are defined, consistency of preferences imply that any bundle other than $f_{i,i}$ makes at least one negotiator worse off. Thus, f is efficient.

We next show that f is strategy-proof. Without loss of generality, consider deviations of negotiator 1 only. If $i < j$ or $j = 0$, then $A(\theta_1^{x_i}) \cap A(\theta_2^{x_j}) = \emptyset$. Negotiator 1 may receive a different bundle by deviating to a type that is represented by a higher (numbered) row, say $\theta_1^{x_k}$ where $k > i$. Because negotiator 2's type is fixed, $A(\theta_2^{x_j})$ is fixed as well. Therefore, by individual rationality of f, the alternative in issue X at $(\theta_1^{x_k}, \theta_2^{x_j})$ will be unacceptable with respect to negotiator 1's true type, $\theta_1^{x_i}$. Thus, by deal-breakers property of preferences, the bundle at deviation will be worse than (o_X, y) for negotiator 1, implying that negotiator 1 has no profitable deviation. On the other hand, if $i = j \geq 1$, then negotiator 1 can deviate to (1) a lower row and receive (o_X, y), which is worse than $f_{i,i} = (x_i, \hat{y}_i)$ by deal-breakers, or (ii) a higher row and receive a bundle that suggests an unacceptable alternative in issue X. Thus, deal-breakers property and consistency of preferences imply that negotiator 1 has no profitable deviation in that case either.

Now suppose that $i > j \geq 1$. Let $c \in Z^*$ denote the bundle negotiator 1 gets if he truthfully reports his type. If negotiator 1 deviates to a row where f takes the value (o_X, y), then he clearly get worse off by deal-breakers property of preferences. If he deviates to a lower numbered row and receives bundle, say, a, then a appears on the first diagonal above bundle c. The last observation and logrolling property of the preferences imply that negotiator 1 prefers bundle c to a at all consistent preferences. Hence, there is no profitable deviation for negotiator 1 by declaring a lower numbered row. However, if he declares a higher numbered row and gets a different bundle, say, b, then c appears on

the first diagonal above bundle b. As it is clearly visible in the previous graph, negotiator 1's true preferences must give him the bundle c in region 1 or 2 and the deviation bundle b must be in region 3 or 4 since both bundles cannot appear on regions 3 and 4. However, the second part of the Theorem implies that bundle b includes alternative $x_l \in X$ which is an unacceptable alternative for all types that lie above row l, including negotiator 1's true type. Thus, by deal-breakers property of preferences negotiator 1 has no profitable deviation in that case either. Hence, f is strategy-proof.

Proof of 'only if':

Proof of Part i: By individual rationality and consistency of preferences, the alternative for issue X must be o_X whenever $i < j$ or $j = 0$. Then by efficiency, $f_{i,j} = (o_X, y)$ for some $y \in Y \setminus \{o_Y\}$. By strategy-proofness and monotonicity of preferences, $f_{i',j} = (o_X, y)$ for all $i' < j$ if $j \neq 0$ and for all i' if $j = 0$. Similarly, $f_{i,j'} = (o_X, y)$ for all $i < j'$ and if $j' = 0$. Fixing j (and i) and applying the same argument for all remaining rows and columns yield $f_{i,j} = (o_X, y)$ whenever $i < j$ or $j = 0$.

Proof of Part ii: Consider the main diagonal where $i = j = k \geq 1$. Row and column k correspond to preference profile $(\theta_1^{x_k}, \theta_2^{x_k})$ where the only mutually acceptable alternative in issue X is x_k. Therefore, by efficiency and individual rationality $f^X_{k,k} = x_k$. Now, we will show that $f_{k,k} = (x_k, \hat{y}_k)$ for every $k = 1, ..., m$ and $\hat{y}_k = y_{m+1-k}$. We start from $k = 1$. If $f^Y_{1,1} = y \neq \hat{y}_1$, then $f_{2,1}$ must also be (x_1, y) by strategy-proofness. This is true because (1) negotiator 1 is not able to unambiguously rank (x_1, y) against any other bundle (x', y') with $x' \neq x_1$ and $y' \ \theta_1^Y \ y$, and (2) negotiator 1 deviates from $\theta_1^{x_2}$ if $y \ \theta_1^Y \ y'$ by monotonicity. With exactly the symmetric arguments, we must have $f_{2,2} = (x_1, y)$. However, $f^X_{2,2} = x_1$ contradicts with individual rationality and efficiency of f. Hence, we must have $f_{1,1} = (x_1, \hat{y}_1)$.

By induction, suppose that the claim is true for all entries on the main diagonal up to $k - 1$. We now show that it must also hold for k. Suppose for a contradiction that $f^Y_{k,k} = y \neq \hat{y}_k$. Then $f_{k,k-1}$ must also be (x_k, y) by strategy-proofness. This is true because (1) by efficiency and individual rationality $f^X_{k,k-1} \notin \{x_{k+1}, ..., x_m, o_X\}$, (2) negotiator 2 is not able to unambiguously rank (x_k, y) against any other bundle (x', y') with $x_k \ \theta_2(x_k) \ x'$ and $y' \ \theta_2^Y \ y$ and (3) negotiator 2 deviates from $\theta_2^{x_k}$ if $y \ \theta_2^Y \ y'$ by monotonicity. With symmetric arguments, we must have $f_{k-1,k-1} = (x_k, y)$, contradicting with our induction hypothesis that $f_{k-1,k-1} = (x_{k-1}, \hat{y}_{k-1})$.

Proof of Part iii:

Lemma 2. *Suppose that adjacency holds for all bundles on all diagonals $2, ...k$ where $k \leq m$. That is, For all $m \geq i > j \geq 1$ where $i = j + k - 1$, $f_{i,j} \in \{f_{i-1,j}, f_{i,j+1}\} \subset Z^* = \{(x_i, \hat{y}_i) | i = 1, ..., m\}$. Consider two bundles $a, b \in Z^*$ that appear on some diagonal $t \in \{2, ..., k\}$. If bundle a lies on a higher row than b on the first diagonal, then a also*

lies on a higher row than b on all diagonals up to (and including) diagonal t.

Proof. Since both a and b appear on diagonal t, by adjacency, they both must also appear on every diagonal from the second through $(t-1)-$st diagonal. Suppose that a lies above b on the first diagonal. From the first diagonal to the second, adjacency implies that a bundle can either move by one cell horizontally to the left or drop by one cell down. If a moves horizontally, clearly it will remain above b on the second diagonal. If a drops by one cell, it remains above b or on the same row with b (which happens when a and b are diagonally adjacent on the first diagonal). In the former case, b is clearly below a on the second diagonal. In the latter case, for b to also appear on the second diagonal it must also have dropped one cell below, in which case it is again below a on the second diagonal. Iterating this argument for rows 3 through t, we have the desired result. □

STEP 1 (Adjacency): We first show the following: Take a bundle on some diagonal except the first one. This bundle coincides with the bundle either immediately above it or immediately to its right. Lemma 3 states this more formally.

Lemma 3. *For all $m \geq i > j \geq 1$, $f_{i,j} \in \{f_{i-1,j}, f_{i,j+1}\} \subset Z^* = \{(x_i, \hat{y}_i) | i = 1, ..., m\}$.*

Proof. Part *ii* proves that the bundles on the first diagonal constitute the set Z^*. We first claim our proof for the second diagonal. That is, take any $m \geq i > j \geq 1$ where $i = j + 1$ we have $f_{i,j} \in \{f_{i-1,j}, f_{i,j+1}\} \subset Z^*$. Suppose for a contradiction that this is not true for some such i and j. That is, $f_{i,j} \notin \{f_{i,j+1}, f_{i-1,j}\} = \{(x_i, \hat{y}_i), (x_{i-1}, \hat{y}_{i-1})\}$. Note that row i and column j correspond to the profile $(\theta_1^{x_i}, \theta_2^{x_j})$ and by efficiency and individual rationality $f_{i,j}^X \in \{x_i, x_{i-1}\}$.

If $f_{i,j} = (x_i, y)$ where $y \theta_2^Y \hat{y}_i$, then player 2 unambiguously prefers $f_{i,j}$ to $f_{i,j+1}$ by monotonicity, and so deviates from $\theta_2^{x_{j+1}}$, which contradicts with strategy-proofness. If $f_{i,j} = (x_i, y)$ where $\hat{y}_i \theta_2^Y y$, then player 2 cannot unambiguously rank $f_{i,j}$ against $f_{i,j+1}$, contradicting again with strategy-proofness. Similarly, if $f_{i,j} = (x_{i-1}, y)$ where $y \theta_1^Y \hat{y}_{i-1}$, then player 1 unambiguously prefers $f_{i,j}$ to $f_{i-1,j}$ by monotonicity, and so deviates from $\theta_1^{x_{i-1}}$, which contradicts with strategy-proofness. Finally, if $f_{i,j} = (x_{i-1}, y)$ where $\hat{y}_{i-1} \theta_1^Y y$, then player 1 cannot unambiguously rank $f_{i,j}$ against $f_{i-1,j}$, contradicting again with strategy-proofness. Hence, our claim holds for the second diagonal.

Now by induction, suppose that our claim holds for all diagonals up to k and show that the claim also holds for diagonal $k + 1$. That is, for any $m \geq i > j \geq 1$ where $i = j + k$ we have $f_{i,j} \in \{f_{i-1,j}, f_{i,j+1}\} \subset Z^*$. Once again, suppose for a contradiction that $f_{i,j} \notin \{f_{i-1,j}, f_{i,j+1}\}$. There are two exhaustive cases we need to think:

Case 1: Suppose that $f_{i,j} \in Z^*$. First note that by efficiency and individual rationality of f, all three bundles, $f_{i-1,j}$, $f_{i,j+1}$ and $f_{i,j}$, are unambiguously ranked over (o_X, o_Y) for

both negotiators at profile $(\theta_1^{x_i}, \theta_j^{x_j})$. Next we claim that $f_{i-1,j} \neq f_{i,j+1}$. Suppose not. Then by strategy-proofness $f_{i,j}\ R_1 f_{i-1,j}$ for all R_1 consistent with $\Lambda(\theta_1^{x_i})$. Because the negotiators' preferences over Z^* are just the opposite, then $f_{i-1,j} = f_{i,j+1}\ R_2\ f_{i,j}$ for all R_2 consistent with $\Lambda(\theta_2^{x_j})$. Therefore, negotiator 2 prefers to deviate from $\theta_2^{x_j}$, contradicting with strategy-proofness of f. Thus, we have $f_{i,j} \notin \{f_{i-1,j}, f_{i,j+1}\}$ and $f_{i-1,j} \neq f_{i,j+1}$. By Lemma 2, we have $f_{i,j+1}\ R_1 f_{i-1,j}$ for all R_1 consistent with $\Lambda(\theta_1^{x_i})$ because the former is below the latter in the $k-$th diagonal. There are three exhaustive subcases that we need to consider regarding how negotiator 1 ranks $f_{i,j}$ relative to $f_{i-1,j}$ and $f_{i,j+1}$:

Case 1A: Suppose that $f_{i,j}\ R_1\ f_{i,j+1}\ R_1\ f_{i-1,j}$ for all R_1 consistent with $\Lambda(\theta_1^{x_i})$. This implies that $f_{i,j+1}\ R_2\ f_{i,j}$. Therefore, negotiator 2 prefers to deviate from $\theta_2^{x_j}$, contradicting with strategy-proofness of f.

Case 1B: Suppose that $f_{i,j+1}\ R_1\ f_{i-1,j}\ R_1\ f_{i,j}$ for all R_1 consistent with $\Lambda(\theta_1^{x_i})$. Then, negotiator 1 prefers to deviate from $\theta_1^{x_i}$, contradicting with strategy-proofness of f.

Case 1C: Suppose that $f_{i,j+1}\ R_1\ f_{i,j}\ R_1\ f_{i-1,j}$ for all R_1 consistent with $\Lambda(\theta_1^{x_i})$. Therefore, by the deal-breakers property $\big(f_{i-1,j}^X = \big)x_l\ \theta_1^X\ x_r\big(\ = f_{i,j}^X\big)\ \theta_1^X\ x_h\big(\ = f_{i,j+1}^X\big)$, where $l < r < h$ as all these three bundles are different. Acceptable alternatives for negotiator 1 of type $\theta_1^{x_{i-1}}$ is $\{x_1, ..., x_{i-1}\}$ and by individual rationality x_l is in this set, i.e., $l \leq i-1$. Similarly, acceptable alternatives for negotiator 1 of type $\theta_1^{x_i}$ is $\{x_1, ..., x_i\}$ and by individual rationality both x_h and x_r are in this set, i.e., $r < h \leq i$. The last inequality implies that $r \leq i-1$, and thus x_r is an acceptable alternative for type $\theta_1^{x_{i-1}}$. Equivalently, $f_{i,j}$ is acceptable for type $\theta_1^{x_{i-1}}$. Therefore, type $\theta_1^{x_{i-1}}$ unambiguously prefers $f_{i,j}$ over $f_{i-1,j}$, the bundle that he receives when he truthfully reveals his type. Thus, negotiator 1 deviates from $\theta_1^{x_{i-1}}$, contradicting with strategy-proofness of f.

Case 2: Suppose now that $f_{i,j} \notin Z^*$. For any two alternatives $z, z' \in Z \in \{X, Y\}$ and negotiator i, we say $z\ \succeq_i^Z\ z'$ if and only if $z\ \theta_i^Z\ z'$ or $z = z'$. Because the bundles $f_{i-1,j}$ and $f_{i,j+1}$ are lying on diagonal k and $f_{i-1,j}$ is above $f_{i,j+1}$, we have $f_{i,j+1}\ R_1\ f_{i-1,j}$ for all R_1 consistent with $\Lambda(\theta_1^{x_i})$ by Lemma 2. Therefore, by deal-breakers $\big(f_{i-1,j}^X = \big)x_l\ \theta_1^X\ x_h\big(\ = f_{i,j+1}^X\big)$ where $l \leq h$. By strategy-proofness of f, $f_{i,j}\ R_1\ f_{i-1,j}$ for all R_1 consistent with $\Lambda(\theta_1^{x_i})$. By deal-breakers, $f_{i,j}$ is incomparable to $f_{i-1,j}$ because $f_{i,j} \notin Z^*$ unless $\big(f_{i,j}^X = \big)x_r\ \succeq_1^X\ x_l$ and $f_{i,j}^Y\ \succeq_1^Y\ f_{i-1,j}^Y$. Therefore, we have $r \leq l$. Similarly, strategy-proofness of f implies $f_{i,j}\ R_2\ f_{i,j+1}$ for all R_2 consistent with $\Lambda(\theta_2^{x_j})$, and thus $x_r\ \succeq_2^X\ x_h$ and $f_{i,j}^Y\ \succeq_2^Y\ f_{i,j+1}^Y$ by deal-breakers. That is, $r \geq h$. However, $r \leq l \leq h \leq r$ if and only if $f_{i,j}^X = f_{i-1,j}^X = f_{i,j+1}^X$. But then by deal-breakers we have $f_{i,j+1} = f_{i-1,j} = (x, y)$. Therefore, because the negotiators' preferences on issue Y are complete opposite of one another $f_{i,j}^Y\ \succeq_1^Y\ y$ and $f_{i,j}^Y\ \succeq_2^Y\ y$ hold simultaneously only if $f_{i,j}^Y = y$. Thus, we must have $f_{i,j} = (x, y)$, contradicting with the assumption that $f_{i,j} \notin Z^*$. □

STEP 2 (Construction of order $\rhd$): By step 1, we know that $f_{i,j} \in \{f_{i-1,j}, f_{i,j+1}\} \subset$

Z^* for all $i > j \geq 1$. For construct $\rhd$, perform a pairwise comparison for all the entries $f_{i,j}, f_{i-1,j}, f_{i,j+1}$. More formally, $f_{i-1,j} \rhd f_{i,j+1}$ whenever $f_{i,j} = f_{i-1,j}$ and $f_{i,j+1} \rhd f_{i-1,j}$ whenever $f_{i,j} = f_{i,j+1}$. We obtain a partial order $\rhd$ on Z^*, which may not be complete at this point. Next, we will show that $\rhd$ is asymmetric and transitive.

Lemma 4. *Order $\rhd$ is asymmetric. That is, if $a \rhd b$ for any pair $a, b \in Z^*$, then $\neg b \rhd a$.*

Proof. Suppose for a contradiction that there is $a, b \in Z^*$ such that both $a \rhd b$ and $b \rhd a$. Let $t \geq 1$ be the smallest diagonal on which a and b are diagonally adjacent and a is "chosen" according to $\rhd$. That is, let $f_{i-1,j} = a$, $f_{i,j+1} = b$, and so $f_{i,j} = a$. By lemma 2, bundle a must appear above bundle b at all diagonals, including the first one, which means $b \; R_1 \; a$ for all consistent $R_1 \in \Lambda(\theta_1^{x_i})$. Furthermore, because f is efficient, individually rational and taking values a and b when negotiator 1 announces his type as $\theta_1^{x_i}$, both bundles are acceptable for negotiator 1 of types $\theta_1^{x_k}$ where $k \geq i$. Because $b \rhd a$ by assumption, there must exist another diagonal $t' > t$ in which a and b are diagonally adjacent and b is chosen. By Lemma 2, bundles a and b cannot be adjacent to one another more than once on the same diagonal, and thus $t' > t$. Therefore, let $f_{k-1,l} = a$, $f_{k,l+1} = b = f_{k,l}$. By Lemma 2 and Lemma 3, we have $k > i$. Because negotiator 1 unambiguously ranks bundle b over a and both bundles are acceptable for type $\theta_1^{x_{k-1}}$, where negotiator 1 receives bundle a, he would profitably deviate to $\theta_1^{x_k}$ and get b. This contradicts the strategy-proofness of f. □

Let two bundles a and b be diagonally adjacent. If a lies on a higher row than b, the we say that a is *diagonally adjacent to b from below*. Equivalently, we say that b is *diagonally adjacent to a from above*.

Lemma 5. (*i*) *Let bundle $a = f_{i,j} \in Z^*$ be diagonally adjacent to some bundle $b \in Z^*$ from below and $a \rhd b$. Then, bundle b never appears on or below row i, i.e., there is no $k \geq i$ and l such that $f_{k,l} = b$. Additionally, bundle a never appears (strictly) above row i and (strictly) to the left of column j, i.e., there is no $i' < i$ and $j' < j$ such that $f_{i',j'} = a$.*

(*ii*) *Let bundle $c = f_{i,j} \in Z^*$ be diagonally adjacent to some bundle $d \in Z^*$ from above and $c \rhd d$. Then, bundle d never appears on column j or any lower column, i.e., there is no $k \leq j$ and l such that $f_{l,k} = d$. Additionally, bundle c never appears (strictly) below row i and (strictly) to the right of column j, i.e., there is no $i' > i$ and $j' > j$ such that $f_{i',j'} = c$.*

Proof. We prove the first part, i.e., (*i*), as symmetric arguments will suffice to prove part (*ii*). First part of (*i*): Because b is above a, then it must be on a on the main diagonal.

Hence, by deal-breakers $b\ R_2\ a$ for all types of negotiator 2. Moreover, negotiator 2 may receive bundles a and b when he declares his type $\theta_2^{x_{j-1}}$, and so by efficiency and individual rationality, both these bundles are acceptable for type $\theta_2^{x_{j-1}}$ of negotiator 2. Suppose for a contradiction that b occurs below row i. This implies by the adjacency property that this b should be coming all the way from the main diagonal, and hence b must also appear on row i. Let $f_{i,k} = b$ for some $k \neq j, j-1$. But then negotiator 2 would deviate from $\theta_2^{x_{j-1}}$ to $\theta_2^{x_k}$ to get the bundle b, contradicting with the strategy-proofness of f.

Second part of (i): Because b is above a, then by deal-breakers $a\ R_1\ b$ for all types of negotiator 1. Because $f_{i-1,j-1} = b$, the bundle b must appear on the first diagonal on column j or higher. Because a is below b on main diagonal as well, it also can appear on the main diagonal on column $j+1$ or higher. Therefore, if bundle a appears in the region, for contradiction, then by the adjacency property bundle a must appear on column $j+1$ as well. Let $f_{k,j+1} = a$ for some $k \leq i-1$. But if a is acceptable for type $\theta_1^{x_k}$ of negotiator 1, it must also be acceptable for type $\theta_1^{x_{i-1}}$ of negotiator 1, when he gets the bundle b. Because negotiator 1 prefers a to b, he would deviate from $\theta_1^{x_{i-1}}$ to $\theta_1^{x_k}$ to get the bundle a, contradicting with the strategy-proofness of f. □

Lemma 6. *Order $\rhd$ is transitive. That is, for any triple $a, b, c \in Z^*$ such that $a \rhd b$ and $b \rhd c$, we have $\neg c \rhd a$.*

Proof. Suppose, for a contradiction, that $a \rhd b$ and $b \rhd c$, but $c \rhd a$. Without loss of generality, suppose b is diagonally adjacent to a from above. Let $t \geq 1$ be the smallest diagonal on which a and b are adjacent where $f_{i,j} = a$, $f_{i-1,j-1} = b$ and $f_{i,j-1} = a$ because $a \rhd b$. By Lemma 5 part (i), b never appears on row i or below. Let t' be the smallest diagonal on which b and c are adjacent. We consider two cases:

Case 1 $t' \geq t$: This case has two subcases:

Case 1A. Suppose first that c is adjacent to b from below on diagonal t': Consider diagonal t. Clearly, c should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t' \geq t$. Then by Lemma 3, since c is adjacent to b from below on diagonal t', it must appear below b on row $i+1$ or below on diagonal t. Then by Lemma 2, c can appear only on $i+1$ or below on diagonal $t' \geq t$ as well. However, By Lemma 5 part (i), $f_{i,j} = a \rhd b$ implies that b can never appear on row i or below. This means b and c cannot be adjacent on diagonal t'. A contradiction.

Case 1B. Suppose now that c is adjacent to b from above on diagonal t'. Let $f_{p,q} = b$ and $f_{p-1,q-1} = c$. Because b never appears on row i or below, $p \leq i-1$. By Lemma 4, $b \rhd c$ implies $f_{p,q-1} = b$. By Lemma 5 part (i), $b \rhd c$ implies that c never appears on row p or below. Because b is diagonally adjacent to a from above and c is adjacent to b from above, by Lemma 3, $c \rhd a$ implies that c must be adjacent to a from above on some

diagonal t''. By Lemma 3, there is no b on diagonal t'' for otherwise it would be either below a or above c. Then $t'' > t'$. Thus, let $f_{r,s} = a$ and $f_{r-1,s-1} = c$ on diagonal t'', and so $f_{r,s-1} = c$ by $c \rhd a$. Because there is no c on or below row $p \leq i-1$, a and c must then be adjacent above row p on diagonal $t'' > t'$. That is, $r < p$. Then $t'' > t'$ implies that $s \leq q-2$. Because there is no c on row $p \leq i-1$ or below and $t' \geq t$, $f_{r,s} = a$ lies on row above row p, i.e., $r < p$ and on column $j-2$ or to the left, i.e., $s \leq j-2$. However, by Lemma 5 part (i), $f_{i,j} = a \rhd b$ implies that bundle a should never appear in the box (strictly) above row i and (strictly) to the left of column j. A contradiction.

Case 2 $t' < t$: This case also has two subcases.

Case 2A. Suppose c is adjacent to b from above on diagonal t'. Consider diagonal t'. Clearly, a should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t > t'$. Since a lies below b on diagonal t, it must again be below b on diagonal t'. Let k be the row on which b lies on diagonal t'. Clearly, a lies below row k on diagonal t' or any other diagonal $t'' > t$. Since c is adjacent to b from above on diagonal t' and $b \rhd c$, Lemma 5 part (i) implies that c never appears on row k or below. Thus, a and c cannot be diagonally adjacent on any diagonal $t'' > t'$. But they cannot be diagonally adjacent on any diagonal $t''' < t'$ either because that would mean that there is no b on diagonal t''' for otherwise b would be above c or below a, contradicting Lemma 3. A contradiction.

Case 2B. Suppose c is adjacent to b from below on diagonal t'. Consider diagonal t'. Clearly, a should also lie on this diagonal for otherwise, by Lemma 2 it cannot be on diagonal $t > t'$. Because a lies below b on diagonal t, it must lie below both b and c on diagonal t'. Suppose a and c are diagonally adjacent on some diagonal t''. Let $f_{p,q} = c$ on diagonal t''. Clearly, c must lie above a on diagonal t''. Because b is diagonally adjacent to a from above on diagonal t, there is no c on diagonal t (or on any higher numbered diagonal) for otherwise c would be above b or below a on diagonal t, contradicting Lemma 2. Thus, $t'' < t$. Since $a = f_{p+1,q+1}$ and $c = f_{p,q}$ are diagonally adjacent on t'' and $c \rhd a$, Lemma 5 part (ii) implies that a never appears on column q or any lower numbered column. Since $f_{i,j} = a$, we need $q < j-1$. Since $t'' < t$ and $q < j-1$, bundle $a = f_{p+1,q+1}$ must lie above row i. But, recall that Lemma 5 part (i) and $F_{i,j} = a \rhd b$ implies that a should never appear in the box (strictly) above row i and (strictly) to the left of column j. A contradiction. □

Finally, we stipulate that any incomplete portions of partial order $\rhd$ are chosen in any arbitrary manner without violating transitivity. This together with Lemmas 4-6 above give us a complete, asymmetric and transitive order $\rhd$ satisfying statement (*iii*) of Theorem 1.

The Revelation Principle

We prove the revelation principle for the symmetric treatment of the outside options. The same logic applies directly to the case with heterogeneous treatment of the outside options. A mediation mechanism $\Gamma = \big(S_1, S_2, g(.)\big)$ with veto rights is a collection of strategy sets (S_1, S_2) and an outcome function $g : S_1 \times S_2 \to X \times Y$. The mechanism Γ combined with possible types (Θ_1, Θ_2) and preferences over bundles (R_1, R_2) with $R_i \in \Lambda(\theta_i)$ for all i defines a game of incomplete information. A strategy for negotiator i in the game of incomplete information created by a mechanism Γ is a function $s_i : \Theta_i \to S_i$.

Lemma 1 (Revelation Principle in Dominant Strategies). *Suppose that there exists a mechanism $\Gamma = \big(S_1, S_2, g(.)\big)$ that implements the mediation rule f in dominant strategies. Then f is strategy-proof and individually rational.*

Proof. If Γ implements f in dominant strategies, then there exists a profile of strategies $s^*(.) = \big(s_1^*(.), s_2^*(.)\big)$ such that $g\big(s^*(\theta)\big) = f(\theta)$ for all $\theta \in \Theta$, and for all $i \in N$ and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}(\theta_{-i})\big) \; R_i \; g\big(s_i'(\theta_i'), s_{-i}(\theta_{-i})\big) \tag{1}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, $\theta_{-i} \in \Theta_{-i}$ and all $s_i'(.), s_{-i}(.)$. Condition 1 must also hold for s^*, meaning that for all i and all $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big) \; R_i \; g\big(s_i^*(\theta_i'), s_{-i}^*(\theta_{-i})\big) \tag{2}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last inequality implies that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i}) \; R_i \; f(\theta_i', \theta_{-i}) \tag{3}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$, and all $\theta_{-i} \in \Theta_{-i}$.

On the other hand, because the mechanism Γ always allows negotiators to veto proposed bundle before the mediation game ends, there exists a deviation strategy $\hat{s}_i(.)$ for any strategy $s_i(.)$ such that $g(\hat{s}_i(\theta_i), s_{-i}) = \big(o_X, o_Y\big)$ for all $\theta_i \in \Theta_i$ and all $s_{-i} \in S_{-i}$. The idea is that the negotiator i plays in $\hat{s}_i(.)$ exactly the same way in $s_i(.)$ (for all θ_i's) until the ratification stage and vetoes the proposed bundle.

Therefore, if $\hat{s}_i(.)$ is such a deviation strategy for $s_i^*(.)$, then condition 1 must also hold for $\hat{s}_i(.)$, implying that for all i and $\theta_i \in \Theta_i$,

$$g\big(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\big) \; R_i \; g\big(\hat{s}_i(\theta_i'), s_{-i}^*(\theta_{-i})\big) = \big(o_X, o_Y\big)$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Because $g(s^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, the last condition means that for all i and all $\theta_i \in \Theta_i$,

$$f(\theta_i, \theta_{-i}) \; R_i \; \left(o_X, o_Y\right) \tag{4}$$

for all $R_i \in \Lambda(\theta_i)$, $\theta_i' \in \Theta_i$ and all $\theta_{-i} \in \Theta_{-i}$. Hence, conditions 3 and 4 imply that f is strategy-proof and individually rational. □

References

[1] Abdülkadiroglu, A., and Sönmez, T. (2003). School choice: A mechanism design approach. American economic review, 93(3), 729-747.

[2] Balinski, M., and Sönmez, T. (1999). A tale of two mechanisms: student placement. Journal of Economic theory, 84(1), 73-94.

[3] Barbera, S., and Jackson, M. O. (1995). Strategy-proof exchange. Econometrica, 63(1): 51-87.

[4] Barbera, S., Gül, F., and Stacchetti, E. (1993). Generalized median voter schemes and committees. Journal of Economic Theory, 61(2), 262-289.

[5] Benassy, J. P. (1982). The economics of market disequilibrium. New York: Academic Press.

[6] Bester, H., and Warneryd, K. (2006). Conflict and the social contract. Scandinavian Journal of Economics, 108(2), 231-249.

[7] Black, D. (1948). On the rationale of group decision-making. Journal of political economy, 56(1), 23-34.

[8] Ching, S. (1997). Strategy-proofness and ?median voters?. International Journal of Game Theory, 26(4), 473-490.

[9] Clarke, E. H. (1971). Multipart pricing of public goods. Public choice, 11(1), 17-33.

[10] Compte, O., and Jehiel, P. (2009). Veto constraint in mechanism design: inefficiency with correlated types. American Economic Journal: Microeconomics, 1(1), 182-206.

[11] Crés, H., and Moulin, H. (2001). Scheduling with opting out: Improving upon random priority. Operations Research, 49(4), 565-577.

[12] Fisher, R. U., and Ury, W. (1981). Getting to Yes: Negotiating Agreement Without Giving In. A Perigee Book/Penguin Group

[13] Gale, D., and Shapley, L. S. (1962). College admissions and the stability of marriage. The American Mathematical Monthly, 69(1), 9-15.

[14] Groves, T. (1973). Incentives in teams. Econometrica, 617-631.

[15] Hörner, J., Morelli, M., and Squintani, F. (2015). Mediation and peace. The Review of Economic Studies, 82(4), 1483-1501.

[16] Moulin, H. (1980). On strategy-proofness and single peakedness. Public Choice, 35(4), 437-455.

[17] Myerson, R. B., and Satterthwaite, M. A. (1983). Efficient mechanisms for bilateral trading. Journal of economic theory, 29(2), 265-281.

[18] Pápai, S. (2001). Strategyproof and nonbossy multiple assignments. Journal of Public Economic Theory, 3(3), 257-271.

[19] Pycia, M., and Ünver, M. U. (2017). Incentive compatible allocation and exchange of discrete resources. Theoretical Economics, 12(1), 287-329.

[20] Roth, A. E., Sönmez, T., and Ünver, M. U. (2005). Pairwise kidney exchange. Journal of Economic theory, 125(2), 151-188.

[21] Shapley, L., and Scarf, H. (1974). On cores and indivisibility. Journal of mathematical economics, 1(1), 23-37.

[22] Shapley, L. S., and Shubik, M. (1971). The assignment game I: The core. International Journal of game theory, 1(1), 111-130.

[23] Sprumont, Y. (1991). The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica, 509-519.

[24] Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. The Journal of finance, 16(1), 8-37.