FAIR MATCHING UNDER CONSTRAINTS

YUICHIRO KAMADA AND FUHITO KOJIMA

Abstract. This paper studies a general model of matching with constraints. Observing that a stable matching typically does not exist, we focus on feasible, individually rational, and fair matchings. We characterize such matchings by fixed points of a certain function. Building on this result, we characterize the class of constraints on individual schools under which there exists a student-optimal fair matching (SOFM), the matching that is the most preferred by every student among those satisfying the three desirable properties. We study the numerical relevance of our theory using data on government-organized daycare allocation.

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Contents

1. Introduction 3
Related Literature 6
2. Model 8
3. Fixed-Point Characterization of Fair Matchings 10
4. General Upper-Bound and SOFM 13
4.1. General Upper-Bound 13
4.2. Existence of SOFM 17
5. Allocation of daycare center slots 19
5.1. Background and Institutional Detail 20
5.2. Data and Simulations 22
6. Discussion 29
6.1. Weak Priority Orders 29
6.2. Nonexistence of a Stable Matching 32
6.3. Strategic Issue 32
6.4. SOSM and SOFM 35
6.5. General Upper Bounds and Multidimensional Constraints 36
6.6. A Weaker Notion of Fairness 38
7. Conclusion 39
References 40
Appendix 46
Appendix A. Proofs 46
A.1. Proof of Theorem 3 46
A.2. Proof of Proposition 2 46
A.3. Proof of Theorem 4 47
A.4. Proof of Theorems 5 and 6 47
A.5. Proof of Proposition 3 48
A.6. Proof of Proposition 4 49
A.7. Proof of Proposition 5 49
A.8. Proof of Proposition 6 50
Appendix B. Additional Discussions 51
B.1. An alternative algorithm 51
B.2. Weak fairness and non-existence: An example 53

"Didn't Get a Slot in Day Care. Drop Dead, Japan!!!"
—An anonymous mother

1. Introduction

Daycare services for young children in Japan, as in many countries, are rationed by the government using matching mechanisms, and a large excess demand for daycare seats is a highly contentious political issue. A blog post by an anonymous mother who could not secure a slot in a daycare for her child became viral in 2016, leading to a large protest movement, a debate in Diet, and Prime Minister Abe's promise to improve the situation (Osaki, 2016).[1] In the midst of such a heated political environment, increasing allocation of slots at daycares in a fair manner is now among the very top priorities for many politicians.[2]

One notable complication of the daycare allocation problem is that the necessary teacher-child ratio varies across different ages. In the Japanese case, for instance, the national regulation requires at least one teacher for every three children of age 0 while the ratios are one teacher for every 6 children for ages 1 and 2, 20 children for age 3 and 30 children for ages 4 and 5 (Cabinet Office of Japan, 2017).[3] This means that each daycare center's feasibility constraint cannot be expressed by a mere capacity constraint. In fact, many matching markets are subject to constraints that are beyond the scope of the standard theory. In the medical match in the U.K. in the last century, for example, some hospitals required a certain composition of men and women in their workforce (Roth, 1991). In school choice, a school may be subject to diversity requirements (Abdulkadiroğlu and Sönmez, 2003). As discussed in more detail later, many more matching markets with constraints have recently become the subject of research, including problems of refugee resettlement, college admissions with budget constraint, and school choice under bullying concerns.

The aim of this paper is to study a general model of matching with constraints. Framing the problem as matching between students and schools, we seek to understand the theoretical properties of this problem (with an eye on market design—we apply our theory to the daycare allocation problem based on real data from Japan). Instead of considering various special cases one by one, we begin with a very general model. In that model, each school is subject to an arbitrary constraint, which is represented simply by an arbitrary

[1] A variety of policies for this goal are described by Prime Minister's Office (2017).

[2] For instance, mayors of all ten most populous municipalities list daycare policy as one of their major political agendas.

[3] Detailed institutional information is provided in Section 5.

family of subsets of students that are feasible at that school. The standard model with capacity constraints is a special case in which a set of students is feasible at a school if and only if its cardinality is no greater than the school's capacity, but many more cases fall into the scope of this model.

Given the generality of the constraints, a stable matching does not necessarily exist. In fact, we find that each school's constraint being a capacity constraint is a "necessary and sufficient" condition for guaranteeing existence (in a certain "maximal domain" sense). Faced with this impossibility result, we focus on weaker conditions than stability. More specifically, we consider a fairness notion which requires there be no justified envy, that is, there be no student who prefers a school to her outcome while another student with lower priority is matched to the school. We require this notion of fairness as well as feasibility and individual rationality as our basic requirements (by contrast, we do not require non-wastefulness, which together with the previous three conditions are equivalent to stability). Our first main result provides a characterization of matchings that satisfy these properties as fixed points of a certain mapping.

Equipped with the fixed-point characterization, we identify the class of constraints that admit a desirable matching. For this goal, we first define a student-optimal fair matching, SOFM. A matching is a SOFM if it satisfies fairness, feasibility, and individual rationality and, in addition, it is weakly preferred by every student to any matching that satisfies fairness, feasibility, and individual rationality. This is analogous to the well-known student-optimal stable matching but adapted to our more general environment where the latter may not exist. We identify a class of constraints on individual schools under which a SOFM exists. Specifically, if the constraint at each school is in a class that we call *general upper-bounds*, meaning that any subset of a feasible set of students is also feasible, then a SOFM exists. Moreover, if the constraint of even just one school is not a general upper-bound, then there exist student preferences and capacity constraints at other schools such that a SOFM fails to exist. In this sense, general upper-bound is the most permissive restriction on constraints that leads to the existence of a SOFM.

The family of general upper-bounds subsumes many constraints of interest. Standard cases such as capacities and type-specific quotas are examples of general upper-bounds, while floor (minimum) quotas and proportionality constraints are not. While excluding some cases, the family of general upper-bounds includes many less-known or more recent cases. Examples include college admissions with students with disabilities, refugee matching under multidimensional constraints, school choice under bullying concerns, and separation of conflicting groups in refugee matching.

The daycare allocation problem we discussed earlier provides an interesting example for which our theory can be applied. Daycare services are often highly subsidized and regulated in many countries such as most European and Asian countries. In Japan, the assignment of seats in daycare centers for children (of ages 0 to 5) is conducted by each municipality. It is a matching market: prices are set by the municipality, parents submit ordinal preferences over daycare centers, and the municipality assigns seats following an algorithm based on submitted preferences and pre-specified priorities over the applicants.[4]

As discussed earlier, heterogeneity of teacher-child ratio across age groups implies that the constraint at each daycare center cannot be described by a capacity constraint. Most municipalities, however, treat the number of seats for each age group at each daycare center as fixed and rigid. This leads to an artificial constraint that is more restrictive than the original daycare constraint since the former fails to take into account the inherent flexibility in the latter. We establish that the original daycare constraint and the rigid constraint are both general upper-bound, and that the SOFM under the former constraint is Pareto superior to the one under the latter.[5]

We supplement our theoretical investigation by analyzing the data we obtained from Yamagata City in Japan.[6] First, we compare SOFMs under the original daycare constraint and the rigid constraint. With 50 simulations for each mechanism, we find that the effect of allowing flexibility in constraints is substantial in our data.[7] For example, the average fraction of children who are matched with a strictly preferred daycare center is 60.3%, and the average number of unallocated children decrease by 43.5%. Second, we compare the SOFMs with the real allocations in Yamagata. The real allocation mechanism uses rigid constraints while eliminating envies only within the same age group and tolerating existence of envies across different ages. Although theoretically no clear general relation exists between the SOFMs and the real outcome, we find that the loss from the rigidity in the real allocation overwhelms the efficiency gain resulting from tolerating existence of certain envies. For example, the average fraction of children who are matched with a strictly preferred daycare center in the SOFM under flexible constraints is 16.6%, while the

[4]Priorities are based on characteristics of the children or their parents, such as whether parents have full-time jobs and whether the parent is a single parent. Most municipalities use either serial dictatorship or the "Boston" mechanism (also known as the immediate acceptance mechanism), with slight variations across municipalities. Almost all seats are allocated in the beginning of the academic year (April).

[5]In fact, some municipalities have been trying to make the allocation "more flexible" by relaxing the artificial capacity constraints, although in ad hoc manners. See Section 5.1 for more discussions.

[6]We obtained data from two other municipalities. We are currently conducting simulations using those data and plan to report the results in a future version of this paper.

[7] We are currently increasing the iterations of our simulations, and it seems the results are robust.

corresponding number for the real allocation is merely 5.0%. Moreover, the real allocation leaves many envies: the average proportion of applicants who have justified envy toward another applicant is 33.1%. These results suggest that, relative to the mechanism that is used in reality, our proposed mechanism may result in a mild improvement in efficiency while eliminating justified envy completely.

We study a variety of further issues, including the following topics. First, we show that our theory readily generalizes to the case in which school priorities are weak, which extends the scope of applications to real-life cases such as disaster relief and centralized college admissions. We also study the strategic issues. Although the SOFM mechanism is not strategy-proof unless all constraints are capacity constraints (in a maximal domain sense), we show that it is not the drawback of SOFM alone but is, in fact, shared by every fair mechanism which satisfies feasibility and a mild efficiency requirement. Moreover, we show that the SOFM mechanism is difficult to manipulate in large markets in a specific sense. Then we study the relationship between our general upper-bound and the "multidimensional constraints" studied by Delacrétaz, Kominers and Teytelboym (2016).

Related Literature. First and foremost, this paper contributes to the literature of matching with constraints. We discuss the most relevant related works in various parts of this paper. Other notable studies in this literature include Abdulkadiroğlu (2005), Ergin and Sönmez (2006), and Hafalir, Yenmez and Yildirim (2013) for school choice, Abraham, Irving and Manlove (2007) for a project allocation problem, and Westkamp (2013) and Aygun and Turhan (2016) for college admissions. Importantly, these papers study specific classes of constraints motivated by their intended applications. In a sharp contrast, the present paper starts with a fully general class of constraints and finds conditions on constraints under which desirable matchings and mechanisms exist. These approaches are complementary to each other.

There is another strand of literature on matching with certain constraints that shares a broad motivation with our work while differing in several crucial aspects. Inspired by medical residency matching in Japan, Kamada and Kojima (2015) study a stable matching problem in which the number of doctors who can be matched to hospitals in each region is subject to a constraint. Alternative solution concepts are studied by Kojima, Tamura and Yokoo (2015) and Goto et al. (2014), while more general constraints are studied by Biro et al. (2010*b*), Kamada and Kojima (2017, 2016) and Goto et al. (2016). In particular, general upper-bound may be reminiscent of a condition called *heredity* studied by Goto et al. (2016) and Kamada and Kojima (2017). While sharing broad motivation, there are at least two major differences between these studies and ours. First, they study

constraints imposed jointly on *subsets* of institutions (e.g., hospitals or schools) while our paper considers constraints imposed separately on individual institutions. Second, they restrict attention to constraints over the *numbers* of individuals in different institutions, while the present paper allows constraints to depend on the identity of the individuals. Due to these modeling differences, these two lines of works differ in many dimensions including scopes of applications, desirable properties considered, and obtained results.

Although not as closely related, there is also a recent literature on pure object allocation under constraints. Milgrom (2009) and Milgrom and Segal (2014) consider auction mechanisms under constraints, while Budish et al. (2013) analyze the problem of implementing lotteries for stochastic object allocation under constraints. The latter analysis has been extended in various directions by Che, Kim and Mierendorff (2013), Pycia and Ünver (2015), Akbarpour and Nikzad (2017), and Nguyen, Peivandi and Vohra (2016). These papers are different from ours in that they are primarily concerned with pure object allocation. Moreover, they do not consider elimination of justified envy, our central concern.

Elimination of justified envy is a standard fairness requirement in the literature of school choice (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Tolerating some waste is less standard but becoming more common, perhaps due to difficulty (or even impossibility) of eliminating both waste and justified envy. Feasible, individually rational, and fair (but possibly wasteful) matchings are studied by Sotomayor (1996) and Blum, Roth and Rothblum (1997) for one-to-one matching, Wu and Roth (2018) for many-to-one matching, Delacrétaz, Kominers and Teytelboym (2016) for matching with multidimensional constraints, Kesten and Yazici (2012) for a setting in which all students are in the same priority class with one another at all schools, and Biró (2008), Fleiner and Jankó (2014), and Biró and Kiselgof (2015) for many-to-one matching with weak priorities. Our paper subsumes the settings of all these papers, so our results directly apply to their environments.

A small but rapidly growing literature has recently analyzed allocation of daycare seats, one of the applications of the present paper. Motivated by the practice in Denmark, Kennes, Monte and Tumennasan (2014) study the issue of dynamic stability arising from the overlapping-generations structure of children's composition in daycare centers. Veski et al. (2017) study the effect of changes in priorities in the context of kindergarten allocation practices in Estonia. In a more descriptive work, Herzog and Klein (2018) discuss a variety of policy issues in childcare systems in several German municipalities. While sharing the interest in application of matching theory to childcare, the overlap with our

paper is rather tangential; none of these papers analyzes the problem of constraints, the primary focus of the present work. A recent work by Okumura (2016) is motivated by the daycare seat assignment in Japan and considers the issue of flexible allocation across ages, although his solution concept does not require our fairness concept and none of his results implies ours, and vice versa.

Lastly, this paper is part of the growing literature on matching theory and market design. Ever since the seminal contribution by Gale and Shapley (1962), matching theory proved to be a source of fruitful insights. What is especially remarkable is its use in applications to market design. Research in this field has been successfully applied to various problems such as medical match (Roth, 1984; Roth and Peranson, 1999), school choice (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003), organ donation (Roth, Sönmez and Ünver, 2004, 2005, 2007), and course allocation (Sönmez and Ünver, 2010; Budish and Cantillon, 2012), among others.

The remainder of this paper proceeds as follows. Section 2 introduces our model. In Section 3, we provide a fixed-point characterization of matchings that satisfy feasibility, individual rationality, and fairness. Based on that result, Section 4 identifies a necessary and sufficient condition for the existence of a SOFM. Turning to numerical analysis, in Section 5 we conduct simulations to quantify the welfare gain of relaxing constraints based on real data on the allocation of slots at daycare centers. Then, Section 6 provides a number of discussions. We conclude in Section 7. All proofs that are not in the main text are found in the Appendix.

2. Model

Let there be a non-empty finite set of students I and a non-empty finite set of schools S. Each student i has a strict preference relation $\succ_i$ over the set of schools and being unmatched (being unmatched is denoted by $\emptyset$). For any $s, s' \in S \cup \{\emptyset\}$, we write $s \succeq_i s'$ if and only if $s \succ_i s'$ or $s = s'$. Each school s has a strict priority order $\succ_s$ over the set of students. For any $i, i' \in I$, we write $i \succeq_s i'$ if and only if $i \succ_s i'$ or $i = i'$.[8] We denote by $\succ_I = (\succ_i)_{i \in I}$ the profile of all students' preferences, and by $\succ_S = (\succ_s)_{s \in S}$ the profile of all schools' priority orders. When there are three students i, i', and i'', for example, we write

$$\succ_s : i, i', i''$$

to mean that student i is of the highest priority, i' is of the second-highest priority, and i'' is of the lowest priority at s. School s is said to be **acceptable** to student i if $s \succ_i \emptyset$.

[8] In what follows the interpretation of our analysis is that we assume that all schools regard all students as acceptable.

We write, for example,

$$\succ_i\colon s, s'$$

to mean that school s is the most preferred, s' is the second most preferred, and s and s' are the only acceptable schools under preferences $\succ_i$ of student i.

Each school s is subject to a *constraint*. A **constraint** at school s is a nonempty collection $\mathcal{F}_s \subseteq 2^I$ of sets of students. Denote $\mathcal{F}_S = (\mathcal{F}_s)_{s\in S}$. We say that a subset $I' \subseteq I$ is **feasible** at s if $I' \in \mathcal{F}_s$ and it is **infeasible** otherwise.

We refer to a tuple $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$ as a **problem**.

A **matching** μ is a mapping that satisfies (i) $\mu_i \in S \cup \{\emptyset\}$ for all $i \in I$, (ii) $\mu_s \subseteq I$ for all $s \in S$, and (iii) for any $i \in I$ and $s \in S$, $\mu_i = s$ if and only if $i \in \mu_s$. That is, a matching simply specifies which student is assigned to which school (if any).

Let us define a few basic terms. First, a matching μ is **feasible** if $\mu_s \in \mathcal{F}_s$ for each $s \in S$. Second, a matching μ is **individually rational** if $\mu_i \succeq_i \emptyset$ for each $i \in S$. That is, no student is matched with an unacceptable school. Third, we say that i **has a justified envy toward** i' if there exists $s \in S$ such that $s \succ_i \mu_i$, $i' \in \mu_s$ and $i \succ_s i'$. We say that a matching μ is **fair** if there exist no students i and i' such that i has a justified envy toward i'.[9] Fourth, a matching μ is **non-wasteful** if there is no pair $(i, s) \in I \times S$ such that $s \succ_i \mu_i$ and $\mu_s \cup \{i\}$ is feasible at s.[10] Finally, a matching μ is said to be **stable** if it is feasible, individual rational, fair, and non-wasteful.

As we will see in Example 1, there may not exist a stable matching. For this reason, we will weaken our desiderata by dropping non-wastefulness, while still requiring fairness. The following concept will be of special interest.

Definition 1. A matching μ is the **student-optimal fair matching (SOFM)** if (i) μ is feasible, individually rational, and fair, and (ii) $\mu_i \succeq_i \mu'_i$ for each $i \in I$ and every μ' that is feasible, individually rational, and fair.

Given $(I, S, \mathcal{F}_S)$, a **mechanism** φ is a function that maps preference profiles to matchings for each profile of priority orders. The matching under φ at students' preference profile $\succ_I$ and priority profile $\succ_S$ is denoted $\varphi^{\succ_S}(\succ_I)$, and student i's match is denoted by $\varphi_i^{\succ_S}(\succ_I)$ for each $i \in I$.

[9] For detailed discussions on this concept, see Remark 1 and Section 6.6.

[10] An alternative definition of non-wastefulness would require there be no pair $(i, s) \in I \times S$ such that $s \succ_i \mu_i$, $\mu_s \cup \{i\}$ is feasible at s, *and* $\mu_{\mu_i} \setminus \{i\}$ *is feasible at* μ_i *if* $\mu_i \neq \emptyset$. None of the results of this paper changes if we adopt this definition, so we do not consider it. Similarly, an alternative definition of fairness may require justified envy to satisfy feasibility of $\mu_{\mu_i} \setminus \{i\}$ at μ_i if $\mu_i \neq \emptyset$, but we do not consider such a definition.

In $(I, S, \mathcal{F}_S)$, a mechanism φ is said to be **strategy-proof** if there do not exist a profile of priority orders $\succ_S$, a profile of students' preferences $\succ_I$, a student $i \in I$, and preferences $\succ_i'$ of student i such that

$$\varphi_i^{\succ_S}(\succ_i', \succ_{I\setminus\{i\}}) \succ_i \varphi_i^{\succ_S}(\succ_I).$$

That is, no student has an incentive to misreport her preferences under the mechanism.[11]

3. Fixed-Point Characterization of Fair Matchings

This section studies matchings that satisfy the desirable properties introduced in the last section. More specifically, we characterize all the matchings that satisfy fairness, individual rationality, and feasibility by fixed points of a certain function on a finite (and hence complete) lattice. We later use this result to study the existence and structure of matchings that satisfy these properties.

Our first observation is that there does not necessarily exist a stable matching.

Example 1 (Non-existence of a stable matching)**.** Suppose that there are one school s and ten students $i_1, i_2, \ldots, i_{10}$. Every student prefers to be matched to s rather than being unmatched. The school's priority is:

$$\succ_s\colon i_1, i_2, \ldots, i_9, i_{10}.$$

Each student with an odd index costs the school 3 units of money while each student with an even index costs 4 units of money. School s is subject to a budget constraint of 20 in the sense that a set of students is feasible at s if and only if the sum of the costs associated with them does not exceed 20 units of money.[12] Then, for example, the matching μ such that $\mu_s = \{i_1, i_2, i_3, i_4, i_5\}$ is fair but wasteful because $\mu_s \cup \{i_7\}$ is feasible and i_7 prefers to be matched to s rather than being unmatched. Meanwhile, the matching μ' such that $\mu_s' = \mu_s \cup \{i_7\}$ satisfies non-wastefulness but violates fairness because $s \succ_{i_6} \emptyset$, $i_7 \in \mu_s'$, and $i_6 \succ_s i_7$. In fact, there exists no stable matching in this example. To see this, first note fairness requires that the set of students who are matched to s should be of the form $I^l := \bigcup_{k=1}^{l} \{i_k\}$ for some $l \in \{1, \ldots, 10\}$, or $I^0 := \emptyset$. For any $l \leq 5$, the set I^l leads to

[11] In this paper we do not consider incentive problems of the school side. This is because in the applications we have in mind, the priorities are exogenous to schools; for example, priorities are usually given by the local government in school choice and daycare allocation. Note also that Roth (1982) shows that there is no mechanism that produces a stable matching for all possible preference profiles and is strategy-proof for both students and schools even in a market in which all constraints are capacity constraints.

[12] This constraint is not the usual capacity constraint. It satisfies a restriction of "general upper-bound" that we introduce in Section 4.

wastefulness because $I^l \cup \{i_7\} \in \mathcal{F}_s$ and $s \succ_{i_7} \emptyset$. Meanwhile, for any $l \geq 6$, the set I^l is infeasible. □

Remark 1. This non-existence example depends on our definition of fairness. Although our definition coincides with the standard one, in our general environment a matching can be deemed unfair even if there is no feasible way to satisfy students who have justified envy. In the above example, μ' is not fair even though i_6 cannot replace i_7 in a feasible manner ($(\mu' \cup \{i_6\}) \setminus \{i_7\}$ is infeasible at s). An alternative weaker definition would call μ' fair (a formal definition is provided in Section 6.6). However, there are at least two justifications for our choice of fairness notion.

First, which definition is more reasonable depends on applications. While there are cases in which the weaker notion may make more sense, we think there are a variety of markets in which our notion is more suitable. For example, universities around the world declare that they will not discriminate against students with disabilities in admissions. For instance, University of Oxford's admissions website (University of Oxford, 2018) claims to "view applications from students with disabilities on the same grounds as those from other candidates, which are assessed purely on academic merit and potential." At the same time, it is widely recognized that students with disabilities incur higher cost, which may result in a budget constraint similar to the one in Example 1. In this case, fairness requires that there be no situation in which a student is denied admission while a student with lower "academic merit and potential" is accepted, even if the former has disability and thus replacing the latter student with the former violates the budget constraint. This appears to be the stated policy of universities such as Oxford. Similarly, in allocation of relief material in the wake of major earthquakes in Japan, organizers of some disaster shelters regarded fairness as so important that they refrained from assigning the resource to anyone when it is not feasible to satisfy everyone. We come back to these examples in Sections 4.1 and 6.1, respectively.

Second, even if we weaken the fairness notion, the non-existence problem persists. Specifically, Section 6.6 provides an example in which there exists no matching satisfying feasibility, individual rationality, non-wastefulness and the weaker version of fairness. □

Given the non-existence of a stable matching, our approach is to refrain from insisting on non-wastefulness while maintaining fairness as well as feasibility and individual rationality. One could consider an alternative approach in which non-wastefulness is required while fairness is not. As we discussed in the Introduction, which approach is more reasonable depends on applications. In the wide range of examples discussed in the Introduction, some wastes are tolerated while fairness is considered primarily important.

Our approach is to characterize the matchings that satisfy fairness, individual rationality, and feasibility by fixed points of a function on a finite lattice. To establish this characterization, consider the space of cutoff profiles $P := \{1, \ldots, |I|, |I|+1\}^S$, endowed with a partial order $\leq$ such that $p \leq p'$ if and only if p_s is weakly smaller than p'_s for all $s \in S$. This space is a finite (hence complete) lattice. For each school s, let $i^{(s,l)}$ be the student whose rank is the l^{th} from the bottom according to the priority of s (for example, the best student for s is $i^{(s,|I|)}$, and the worst is $i^{(s,1)}$). Also, consider a hypothetical student $i^* \notin I$ such that $i^* = i^{(s,|I|+1)}$ for all $s \in S$, and expand the domain of $\succ_s$ for each $s \in S$ so that for any $i \in I$, $i^* \succ_s i$ holds. Given a cutoff profile $p \in P$, define the "demand" at each $s \in S$ as

$$D_s(p) := \{i \in I | i \succeq_s i^{(s,p_s)} \text{ and } s \succ_i \emptyset; i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow s \succeq_i s'\}.$$

In this definition, the first part "$i \succeq_s i^{(s,p_s)}$ and $s \succ_i \emptyset$" says that student i is as good as the cutoff student $i^{(s,p_s)}$ and finds s acceptable, while the second part "$i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow s \succeq_i s'$" says that s is the most preferred school among the ones at which i passes the cutoff. The demand for s is a collection of students who meet those two criteria.

We consider a mapping $T : P \to P$, called the **cutoff adjustment function**, defined as follows.[13]

$$T_s(p) = \begin{cases} p_s + 1 & \text{if } D_s(p) \notin \mathcal{F}_s \\ p_s & \text{if } D_s(p) \in \mathcal{F}_s, \end{cases} \tag{3.1}$$

where we set $(|I|+1)+1 = 1$.[14] That is, for each s and cutoff profile, this mapping raises the cutoff at s by one if the demand for s is infeasible, and leaves the cutoff unchanged if it is feasible. This is reminiscent of the standard price-adjustment process in the general equilibrium theory except that the cutoff does not decrease even when s is "under-demanded." For each $p \in P$, let μ^p be the matching such that

$$\mu^p_s = D_s(p) \text{ for each } s \in S. \tag{3.2}$$

That is, μ^p_s simply matches all students who demand s given cutoff profile p.

Theorem 1. *If a cutoff profile $p \in P$ is a fixed point of the cutoff adjustment function T, then μ^p is feasible, individually rational, and fair. Moreover, if μ is a feasible, individually rational, and fair matching, then there exists a cutoff profile $p \in P$ with $\mu = \mu^p$ that is a fixed point of T.*

[13] Another possible specification of a function is discussed in Appendix B.1.

[14] That is, we use the "modulo" operation. This definition ensures that the range of T is P.

Proof. The first part: If p is a fixed point of T, then $T_s(p) = p_s$ must hold for all $s \in S$. The definition of T then implies that $D_s(p) \in \mathcal{F}_s$ must hold for each $s \in S$. Hence, feasibility follows. Individual rationality follows from the definition of $D_s(\cdot)$ for each $s \in S$.

To show fairness, assume for contradiction that μ^p is not fair. Then there exists a triple $(i, i', s) \in I^2 \times S$ such that $s \succ_i \mu_i^p$, $i' \in \mu_s^p$ and $i \succ_s i'$. Because $i' \in \mu_s^p$, by definition of $D_s(\cdot)$ and μ_s^p, it follows that $i' \succeq_s i^{(s,p_s)}$. Since $i \succ_s i'$, we have that $i \succ_s i^{(s,p_s)}$. This and the definition of $D_{\mu_i^p}(\cdot)$ imply $\mu_i^p \succeq_i s$, which is a contradiction.

The second part: Take a feasible, individually rational and fair matching μ. For each s, let $p_s = \min\{l | i^{(s,l)} \in \mu_s\}$ if $\mu_s \neq \emptyset$ (where the minimum exists because P is finite), and $p_s = |I| + 1$ otherwise. Then, individual rationality and fairness of μ and the definition of $D_s(\cdot)$ imply $\mu = \mu^p$. Also, feasibility of μ and the definition of T imply that $p = (p_s)_{s \in S}$ is a fixed point of T. □

4. General Upper-Bound and SOFM

This section studies the existence and structure of matchings that satisfy fairness, individual rationality, and feasibility. More specifically, we establish a tight relationship between the existence problem and the nature of the constraints. We show that the class of general upper-bounds characterizes the situations in which a SOFM is guaranteed to exist.

4.1. **General Upper-Bound.**

Definition 2. A constraint $\mathcal{F}_s$ is a **general upper-bound** if $I' \in \mathcal{F}_s$ and $I'' \subseteq I'$ imply $I'' \in \mathcal{F}_s$.

That is, a constraint at school s is a general upper-bound if, for any set of students that is feasible at s, every subset of it is also feasible. Note that if a constraint $\mathcal{F}_s$ is a general upper-bound then $\emptyset \in \mathcal{F}_s$ must hold. This is because non-emptiness of $\mathcal{F}_s$ implies existence of some $I' \subseteq I$ such that $I' \in \mathcal{F}_s$, and we have $\emptyset \subseteq I'$.

A special case of a general upper-bound is a *capacity constraint*: a constraint $\mathcal{F}_s$ is a **capacity constraint** if there exists an integer $q \in \mathbb{N}$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $|I'| \leq q$. Let us provide real-life examples of constraints to discuss the applicability of general upper-bound as well as its limitation.

(1) **Diversity in school choice (type-specific quotas):** Many school districts require certain diversity of the student body at each school. A common way in the literature to formalize this requirement is to impose *type-specific quotas*

(Abdulkadiroğlu and Sönmez, 2003). Specifically, we require a constraint $\mathcal{F}_s$ to satisfy the following: There exist a partition of the students $I := \bigcup_{t \in T} I_t$ with an index set T (with $I_t \cap I_{t'} = \emptyset$ if $t \neq t'$) and integers $q \in \mathbb{N}$ and $q_t \in \mathbb{N}$ for every $t \in T$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $|I'| \leq q$ and $|I' \cap I_t| \leq q_t$ for every $t \in T$. Because $I'' \subseteq I'$ and $I' \in \mathcal{F}_s$ imply $|I''| \leq |I'| \leq q$ and $|I'' \cap I_t| \leq |I' \cap I_t| \leq q_t$ for every $t \in T$, this constraint is a general upper-bound.

(2) **College admissions with students with disabilities (budget constraints):** In college admissions, it is widely recognized that students with disabilities incur more cost to the university. For example, University of Oxford's admissions website mentioned in Remark 1 describes a variety of accommodation and financial assistance they offer to students with disabilities. One way to model this setup is through a budget constraint. Formally, assume that each student i is associated with cost $c_i \in \mathbb{R}_+$ and say that constraint $\mathcal{F}_s$ is a budget constraint if there exists $b \in \mathbb{R}_+$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $\sum_{i \in I'} c_i \leq b$. College admissions with students with disabilities could be modeled as a situation in which students with disabilities have higher cost than those without disabilities. Budget constraint is not necessarily a capacity constraint as seen in Example 1, but it is clearly a general upper-bound because $I'' \subseteq I'$ and $I' \in \mathcal{F}_s$ imply $\sum_{i \in I''} c_i \leq \sum_{i \in I'} c_i \leq b$.

Cases of budget constraints in college admissions may also arise in different contexts (Biró et al., 2010*a*; Abizada, 2016). For example, a number of universities commit to so-called "need-blind admission," where admission decisions are made without regard to applicants' financial situations.[15] Yet, a college may be subject to a budget constraint. Because different students may need different amounts of financial aid, the constraint cannot be described as a capacity constraint in general, but it is a budget constraint and hence a general upper-bound.

(3) **Refugee match (multidimensional constraints):** Given the recent surge of refugees, authorities in many countries are faced with the task of matching refugees to different localities.[16] One of the requirements in this problem is to match all members of a family to the same place. In addition, different refugee families need different kinds of services such as job training, language class, and so on.

[15]Some forms of need-blind admission are adopted not only by elite schools but also by others. A list of institutions committed to such an admission policy is in `https://en.wikipedia.org/wiki/Need-blind_admission`.

[16]As of the end of 2016, there were 22.5 million refugees worldwide (United Nations High Commissioner for Refugees, 2017).

Given the available resources of a locality, variations in family size and needs for different types of resources imply that the constraint is not necessarily a capacity constraint. This type of constraint is called multidimensional constraints and studied by Delacrétaz, Kominers and Teytelboym (2016).[17] Section 6.5 formally analyzes this model and shows that, among other things, this constraint is a general upper-bound.

(4) **Anti-bullying school choice design:** Bullying has a negative effect on the victim academically as well as mentally. It is a concern in many countries, and governments take measures to address this issue. Motivated by such a concern, Kasuya (2016) considers anti-bullying policies in the context of school choice design. Specifically, he analyzes a requirement that, no pair of a bully and his or her victim be placed in the same school. To express this formally as a constraint, let $B \subseteq I \times I$ be the set of bullying incidents, meaning that $(i,j) \in B$ implies that "i has bullied j." The constraint at each school s (ignoring other types of constraints such as the school's capacity for simplicity) can be expressed as $\mathcal{F}_s = \{I' \subseteq I | (i,j) \in B \Rightarrow \{i,j\} \not\subseteq I'\}$. This is a general upper-bound because $I'' \subseteq I'$ and $\{i,j\} \not\subseteq I'$ imply $\{i,j\} \not\subseteq I''$, hence $I' \in \mathcal{F}_s$ implies $(i,j) \in B \Rightarrow \{i,j\} \not\subseteq I''$.

(5) **Separating conflicting groups in refugee match:** Separating different types of individuals may be important not only in the bullying context, but also in other applications. For example, in refugee match, authorities are concerned that refugees from conflicting religious or ethnic groups may fight with each other in refugee shelters if they live close to each other. The policy to separate them by placing them in different locations is used or being considered as a temporary, if not permanent, solution (Breitenbach, 2015). To model this policy formally as a constraint, assume that there exists a partition of the students (refugees in this context) $I := \bigcup_{t \in T} I_t$ with an index set T (with $I_t \cap I_{t'} = \emptyset$ if $t \neq t'$) and, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if there exists $t \in T$ such that $I' \subseteq I_t$. This is a general upper-bound because $I'' \subseteq I'$ and $I' \in \mathcal{F}_s$ imply there exists $t \in T$ such that $I'' \subseteq I' \subseteq I_t$.

(6) **Daycare allocation:** In many countries around the world, assignment of daycare seats for small children is organized by the (local) government through a matching mechanism. There are often legal restrictions on a teacher-child ratio at each

[17] Andersson and Ehlers (2016) also study the problem of refugee matching, but the model is different from the ones studied by Delacrétaz, Kominers and Teytelboym (2016) or us.

daycare center. Since such a ratio varies across different ages of children, the constraints implied by these regulations cannot be described by capacity constraints. In Section 5 we show they can be described as general upper-bounds and study the data of daycare allocation in Japan, where finding a desirable allocation in the face of this non-capacity general upper-bound has become a pressing practical issue (Okumura, 2016).

Of course, not all constraints of interest are general upper-bounds. The following are some of the examples.

(1) **Floor constraints:** Floor constraints may appear in a variety of environments. For example, in school choice, a school may need at least a certain number of students in order to organize group activities, have students interact with one another, or simply to operate efficiently enough given fixed cost. Formally, the constraint $\mathcal{F}_s$ at each school s (ignoring other types of constraints such as the capacity of the school) is that there exists an integer $q \geq 1$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $|I'| \geq q$ (Ehlers et al., 2011; Fragiadakis et al., 2012; Fragiadakis and Troyan, 2016). This is not a general upper-bound because any $I' \subseteq I$ with $|I'| \geq q$ is in $\mathcal{F}_s$ but, for instance, $\emptyset$ is a subset of I' but $|\emptyset| = 0 < q$. Similarly, a type-specific floor constraint, i.e., a constraint in which at least a certain number of students of specific types are needed for feasibility, is not a general upper-bound.

(2) **Proportionality constraints:** As mentioned earlier, a common requirement in matching markets is to achieve certain balance of workforce or student body. Such a requirement is sometimes expressed in terms of proportion. For instance, in the public school district of Cambridge, Massachusetts in 2003, the proportion of students from low socioeconomic status families is required to be within a range of 15 percent of the district-wide proportion (Nguyen and Vohra, 2017). A constraint $\mathcal{F}_s$ is a *proportionality constraint* (Nguyen and Vohra, 2017) if there exist a partition of the students $I := \bigcup_{t \in T} I_t$ with an index set T (with $I_t \cap I_{t'} = \emptyset$ if $t \neq t'$) and numbers $\alpha_t, \beta_t \in [0, 1]$ for every $t \in T$ with $\alpha_t \leq \beta_t$ and $\sum_{t \in T} \alpha_t \leq 1 \leq \sum_{t \in T} \beta_t$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $\alpha_t |I'| \leq |I' \cap I_t| \leq \beta_t |I'|$ for every $t \in T$. This is not necessarily a general upper-bound because any $I' \subseteq I$ with $\alpha_t |I'| \leq |I' \cap I_t| \leq \beta_t |I'|$ for every $t \in T$ is in $\mathcal{F}_s$ but, for instance, $I'' := I' \cap I_t$ for any $t \in T$ is a subset of I' while $|I'' \cap I_t| = |I''| > \beta_t |I''|$ if $\beta_t < 1$ and $I'' \neq \emptyset$, so $I'' \notin \mathcal{F}_s$.

We do not necessarily assert that real-market constraints are general upper-bounds in all or even most applications. In fact, the two examples of constraints shown just above are demonstrably *not* general upper-bounds. Instead, our study aims to characterize the situations, in terms of restrictions on constraints, under which a solution with desirable properties exists. As shown in Section 4.2, it turns out that general upper-bound is necessary and sufficient for guaranteeing the existence of such a solution.

4.2. **Existence of SOFM.** Equipped with Theorem 1, our approach is to study the desirable matchings by way of analyzing the fixed points of the cutoff adjustment function T. Our first observation is, though, that under general upper-bounds, the existence of a matching satisfying fairness as well as feasibility and individual rationality is trivial; an empty matching, i.e., a matching with $\mu_i = \emptyset$ for each $i \in I$, satisfies all the properties above.[18] Since an empty matching is typically highly inefficient, this example also suggests that only requiring the above three conditions may not lead to a desirable outcome for students. To the extent that we care about student outcomes, an interesting question is whether one can identify a fair matching that is desirable in terms of welfare. The next theorem answers this question in the affirmative.

Theorem 2. *If the constraint at each school is a general upper-bound, then there exists a SOFM.*

Proof. Fix a general upper-bound for each school s, $\mathcal{F}_s$.

Claim 1. *The mapping T has the smallest fixed point, i.e., there exists $p \in P$ such that $T(p) = p$ and $p \leq p'$ for all $p' \in P$ with $T(p') = p'$.*

Proof of Claim 1. Tarski's fixed point theorem implies that if a function from a finite lattice into itself, $f : X \to X$, is weakly increasing (i.e., for any $x, x' \in X$, $x \leq x'$ implies $f(x) \leq f(x')$), then the set of the fixed points of f is a finite lattice, and in particular it has the smallest fixed point. Letting $f = T$ and $X = P$, we only need to show that, for any $p, p' \in P$, $p \leq p'$ implies $T(p) \leq T(p')$.

To see that this holds, fix $s \in S$. We shall show $T_s(p) \leq T_s(p')$. Note first that the conclusion holds if $p'_s = |I| + 1$. This is because, as $\mathcal{F}_s$ is a general upper-bound, $D_s(p') = \emptyset \in \mathcal{F}_s$, so $T_s(p) \leq |I| + 1 = T_s(p')$. Thus, in the remainder, we suppose

[18] In addition, it is straightforward to show that $\emptyset \in \mathcal{F}_s$ for each $s \in S$ is necessary and sufficient to guarantee existence of a matching that is feasible, individually rational, and fair (in a maximal-domain sense).

$p'_s \neq |I|+1$. In this case, $T_s(p) \leq T_s(p')$ is immediate if $p_s < p'_s$ by the definition of $T(\cdot)$. Hence we consider the case with $p_s = p'_s$. Then $p_{-s} \leq p'_{-s}$ implies that

$$D_s(p) = D_s(p_s, p_{-s}) \subseteq D_s(p_s, p'_{-s}) = D_s(p'_s, p'_{-s}) = D_s(p')$$

by the definition of $D_s(\cdot)$.[19] Since $\mathcal{F}_s$ is a general upper-bound, if $D_s(p)$ is infeasible at s then $D_s(p')$ is infeasible at s, too. This implies that, whenever $T_s(p) = p_s + 1$, we have $T_s(p') = p'_s + 1 = p_s + 1 = T_s(p)$. Finally, if $T_s(p) = p_s$, it is immediate from the definition of $T(\cdot)$ and $p_s = p'_s$ that $T_s(p') \geq T_s(p)$.

Hence we have that $T_s(p) \leq T_s(p')$ for any $p, p' \in P$ with $p \leq p'$. Since this argument holds for every $s \in S$, we have $T(p) \leq T(p')$. □

To complete the proof of the theorem, let p^* be the smallest fixed point of the function T, whose existence is guaranteed by Claim 1. By the first part of Theorem 1, μ^{p^*} is a feasible, individually rational, and fair matching. Take an arbitrary matching μ that is feasible, individually rational, and fair. By the second part of Theorem 1, there exists a fixed point p of T such that $\mu = \mu^p$. By the relation $p^* \leq p$ and the definition of $D_s(\cdot)$ for each $s \in S$, the equality $\mu = \mu^p$ implies that $\mu_i^{p^*} \succeq_i \mu_i$ for each student $i \in I$. Hence we have shown that μ^{p^*} is a SOFM, completing the proof. □

The proof of Theorem 2 is based on Tarski's fixed point theorem. As such, a simple algorithm can be used to find the SOFM, building on the cutoff adjustment function T. Consider the following algorithm, called the **cutoff adjustment algorithm**:

- Step 0: Let $p^0 := (1, 1, \ldots, 1)$.
- Step $t \geq 1$: Let $p^t = T(p^{t-1})$. If $p^t = p^{t-1}$, terminate the algorithm and define the outcome as the matching μ^{p^t}.[20] Otherwise, go to step $t+1$.

The cutoff adjustment algorithm is well-defined (i.e., it terminates in a finite number of steps). To see this, first note that since $p \leq T(p)$ for any $p \in P$ by the definition of T and the fact that $\emptyset \in \mathcal{F}_s$ for each $s \in S$ under general upper-bound, we have $p^{t-1} \leq p^t$ for every positive integer t. Because P is a finite set, $p^{t^*-1} = p^{t^*}$ for some finite t^*.

To see that this algorithm produces the SOFM, note that $p^* := p^{t^*-1} = p^{t^*}$ satisfies $p^* = T(p^*)$, i.e., p^* is a fixed point of T. Moreover, for any fixed point p' of T, we have $p^0 \leq p'$ by definition of p^0, so $p^* = T^{t^*}(p^0) \leq T^{t^*}(p') = p'$, implying that p^* is the smallest fixed point of T. Since the matching corresponding to the smallest fixed point of T is the

[19] The set inclusion relation follows because the property $i \succeq_{s'} i^{(s', p_{s'})}$ that appears in the definition of $D_s(\cdot)$ is implied by $i \succeq_{s'} i^{(s', p'_{s'})}$. All the equalities are straightforward.

[20] For each $p \in P$, $T(p)$ and μ^p are as defined in Equations (3.1) and (3.2), respectively.

SOFM (see the proof of Theorem 2 for detail), the outcome produced by this algorithm is the SOFM. Thus, we have established the following result:

Proposition 1. *The outcome of the cutoff adjustment algorithm is the SOFM.*

Under capacity constraints, a standard way to find a stable matching is the deferred acceptance algorithm. In Appendix B.1, we present a generalization of the deferred acceptance algorithm which finds the SOFM under a general upper-bound and compare it with the cutoff adjustment algorithm.

As mentioned earlier, the class of general upper-bounds subsumes many practical cases, but there are some constraints that are not general upper-bounds, such as floor constraints. A question of interest, then, is whether the conclusion of Theorem 2 holds without the assumption of general upper-bound. The following result offers a sense in which the answer to this question is negative.

Theorem 3. *Fix a set of students I, a set of schools S with $|S| \geq 2$ and their priorities $\succ_S$, and a school $s \in S$ and its constraint $\mathcal{F}_s$. Suppose $\mathcal{F}_s$ is not a general upper-bound. Then there exist student preferences $\succ_I$ and a profile $\mathcal{F}_{-s}$ of capacity constraints such that a SOFM does not exist in the problem $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$.*

This result shows that the class of general upper-bounds is a "maximal domain," providing a sense in which Theorem 2 cannot be generalized further. In other words, general upper-bound is the most permissive restriction on constraints imposed on individual schools (as long as the capacity constraints are included) which guarantees the existence of a SOFM. Even if a SOFM is a desirable outcome, the policy maker cannot expect to always find a SOFM unless the constraints of each school is a general upper-bound. This finding may shed light on the non-existence and impossibility results in the literature. Settings with those negative results include floor constraints (Biro et al., 2010*b*; Ehlers et al., 2011; Fragiadakis et al., 2012; Fragiadakis and Troyan, 2016) and proportionality constraints (Nguyen and Vohra, 2017). These constraints are not general upper-bounds as pointed out in Section 4.1.

5. Allocation of daycare center slots

This section describes the problem of allocating slots at daycare centers, "daycare allocation," as an application of our theory. Then we conduct numerical analysis based on unique datasets on daycare allocation we obtained to study the numerical impact of allowing flexibility in constraints.

5.1. **Background and Institutional Detail.** Child care services are often highly subsidized and regulated in many countries. Some Scandinavian countries such as Sweden regard daycare service as parents' right and guarantee a seat in some, albeit not necessarily the parents' first choices, daycare center. At the other end of the spectrum are countries such as the United States, in which a typical daycare center is run by a private provider and financed by tuitions from parents, though with regulations. Some countries such as Japan lie between these extremes: daycare services are highly subsidized and regulated, but at the same time a slot at a daycare is not guaranteed for parents. In many of those countries, the supply of slots at daycares are very limited, leaving a large number of applicants unmatched and thus making allocation of slots at daycares an especially big concern for parents.

Japan provides a good example of the large excess demand for daycare seats. Recently, a blog post titled "Didn't Get a Slot in Day Care. Drop Dead, Japan!!!" by an anonymous mother became viral, leading to a large protest movement. The blog post made its way into debates in the National Diet, forcing prime minister Abe to respond to the complaint (Osaki, 2016), with him later promoting a policy to increase child care capacity by more than 300,000 in 5 years.[21] The above blog title was chosen as one of the top ten "New Words and Buzzwords" of 2016, although the organizers admitted to uneasiness about selecting what is essentially a swear word (Kikuchi, 2016). In the midst of such a heated political environment, increasing the number of allocated slots at daycares in a fair manner is now among the very top agendas for many politicians.[22] As we illustrate shortly, our theory may prove useful for daycare allocation problems with excess demand. For this purpose, we describe daycare allocation in Japan in some detail below.

In Japan, daycare centers serve children aged between 0 and 5 as of the beginning of a new academic year, and the assignment of seats in daycare centers is under the authority of each municipality.[23] A number of features of this market make it an unusually good subject of application of matching theory. First, daycares are heavily subsidized, with the

[21]A variety of policies for this goal are described by Prime Minister's Office (2017).

[22]For instance, mayors of all ten most populous municipalities list daycare policy as one of their major political agendas.

[23]In this paper, we focus on *Ninka Hoiku En*, which could be translated into accredited daycare centers. There are also daycare centers that are not accredited. Since non-accredited centers are more expensive (they are not subsidized by the central government) and considered to be of lower quality on average (Asai, Kambayashi and Yamaguchi, 2015), they are uncommon; of the children who are in daycare centers, as much as 93.5% go to accredited ones as of March 2016 (Ministry of Health, Labour and Welfare, 2017). Therefore, we focus on accredited daycare centers.

tuitions and fees set by each municipal government at a low level, leading to excess demand in many municipalities.[24] Given the excess demand, municipalities ration slots in daycare centers based on an algorithm. In fact, parents literally submit ordinal preferences over daycare centers.[25] Moreover, while in principle a child could start attending a daycare at any time of the year, almost all seats are allocated for the beginning of the academic year, which is April in Japan. Thus, unlike in some other matching problems such as organ allocation, it appears to be a good approximation to treat each year's allocation as a single static problem, just as is assumed in the present paper (as well as in most matching theory research in school choice and labor markets). Another aspect of daycare allocation in Japan is that, faced with high demand for daycare seats and resulting rationing, many municipalities follow formal assignment rules (the most popular are versions of serial dictatorship and the "Boston" mechanism).

For our purpose, three aspects of this market are especially important. First, the assignment rules are based on submitted preferences as well as priorities, where the latter are determined by applicant characteristics such as whether parents have full-time jobs and whether the parent is a single parent, among others. Second, the national regulation requires the teacher-child ratio be at least one teacher for every three children of age 0 while the ratios are one teacher for every 6 children for ages 1 and 2, 20 children for age 3 and 30 children for ages 4 and 5 (Cabinet Office of Japan, 2017).[26] While such a constraint is clearly not the traditional capacity constraint as per-capita resources needed to care for a child varies across the child's age, it is a general upper-bound and hence our theory implies a SOFM exists in this problem. Third, even though the true constraint itself is

[24]The average monthly fee per child is 2,0491 Japanese yen (about 200 U.S. dollars) as of 2012 (Ministry of Health, Labour and Welfare, 2014). In 2017, prime minister Abe and the Cabinet decided to make daycares free of charge beginning in 2020.

[25]The application form for Yamagata City (City of Yamagata, 2018*a*), for instance, offers space for listing a ranking of at most 5 desired daycare centers as a default, but allows for listing more centers if the applicant wants. All the municipalities that we are aware of provide application forms that are similar in that they ask a ranking over the applicants' desired daycare centers, although the number of centers that can be listed vary across municipalities.

[26]The national regulation also imposes certain age-dependent space-chid ratio requirements, and there are other exceptions and variations in rules. However, we ignore these in our analysis. While municipalities are allowed to place more stringent regulations than those imposed by the national government, many municipalities including Yamagata, on which our simulations in Section 5.2 are based, follow the national regulation on teacher-child ratios. We conducted simulations under specifications including both teacher-student ratio and space-child ratio and verified that the main conclusions we report in this draft are robust.

not a capacity constraint, most municipalities treat the number of seats at each daycare center for each age as fixed and non-transferable across different ages, thus effectively setting an artificial capacity constraint.

Faced with great pressure to improve daycare allocation, some municipalities have been experimenting with making the allocation "more flexible" by relaxing the artificial capacity constraint in one way or another. Specifically, they have introduced policies to have daycare centers with excess supply of seats for certain ages admit more children of ages with excess demand, although in ad hoc manners.[27] In fact, due to a variety of institutional features, it turns out that even highly demanded daycare centers often have some vacancies for children aged 3 to 5 while having excess demand for children aged 0 to 2.[28]

To proceed formally, let there be a set of ages T and for each $t \in T$, let there be a teacher-children ratio r_t. The interpretation is that if one teacher can watch up to n children of age t, then $r_t = 1/n$. A constraint $\mathcal{F}_s$ is a **daycare constraint** if there are a number m_s (representing the number of teachers) and a partition of the students (children in this context) $I := \bigcup_{t \in T} I_t$ (with $I_t \cap I_{t'} = \emptyset$ if $t \neq t'$) such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if

$$\sum_{t \in T} r_t \cdot |I' \cap I_t| \leq m_s. \tag{5.1}$$

Because $I'' \subseteq I'$ and $I' \in \mathcal{F}_s$ imply $|I'' \cap I_t| \leq |I' \cap I_t|$ for every $t \in T$, this constraint is a general upper-bound. We say that a constraint $\mathcal{F}'_s$ is a **rigid constraint** associated with $\mathcal{F}_s$ if there exists a number q^t for each age $t \in T$ such that $\sum_{t \in T} r_t q^t \leq m_s$, where for any $I' \subseteq I$, $I' \in \mathcal{F}'_s$ if and only if $|I' \cap I_t| \leq q^t$ for every $t \in T$. Note that the latter is a special case of the constraint of type-specific quotas discussed in Section 4.1, with an additional property that the capacity for the number of students in the entire school s is not binding.[29]

5.2. **Data and Simulations.** We use real data in Japan to test the effect of introducing the SOFM mechanism.[30] In particular, we are interested in the effect of allowing flexibility

[27]Municipalities with such policies include populous cities such as Yokohama, Kawasaki, Saitama, and Sendai. See Okumura (2016) for detail.

[28]Suzuki (2018) points out that about 190,000 seats (across all ages and including both accredited and non-accredited daycares) were vacant in Japan as of 2017, and attributes a significant part of this figure to the excess supply of seats for older children.

[29]Under the notations in Section 4.1, this corresponds to setting $q \geq \sum_{t \in T} q_t$.

[30]The daycare seat allocation problem in Japan has many aspects, but we focus on analyzing matching mechanisms under the given constraint. For example, some argue that price mechanisms may be effective in allocating daycare slots (Suzuki, 2018), but we do not consider them.

in seat allocation across different ages. Theoretically, the following observation about SOFM is of interest even though it is straightforward to show:

Proposition 2. *Fix a set of students I, a set of schools S, student preferences $\succ_I$, and school priorities $\succ_S$. Let $\mathcal{F}_S := (\mathcal{F}_s)_{s\in S}$ and $\mathcal{F}'_S := (\mathcal{F}'_s)_{s\in S}$ be profiles of general upper-bounds such that $\mathcal{F}_s \subseteq \mathcal{F}'_s$ for every $s \in S$, and μ and μ' be the SOFMs in the problems $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$ and $(I, S, \succ_I, \succ_S, \mathcal{F}'_S)$, respectively. Then, $\mu'_i \succeq_i \mu_i$ for every $i \in I$, and $\mu_i \in S$ implies $\mu'_i \in S$.*

In words, the first part of this proposition shows that relaxing constraints leads to a Pareto improvement of student outcomes. The second part shows that, in particular, the set of unmatched students becomes weakly smaller in the set inclusion sense as the constraints are relaxed. This proposition, while being a very simple (perhaps trivial) observation, may be of interest in practice. This shows that a policy maker may be able to improve the welfare of the market participants if she relaxes the constraints, say by devoting more resources or simply by lifting some regulations. While this seems intuitive and hardly surprising, it is worth noting that it is *Pareto improvement* for students. In other words, relaxing constraints do not create any distributional tradeoffs and makes *every* student weakly better off.

To put this proposition into perspective, let us mention a comparative statics result due to Konishi and Ünver (2006).[31] The latter result states that, under capacity constraints, when the capacity of each school increases, every student becomes weakly better off at the student-optimal stable matching. Because the SOFM reduces to the student-optimal stable matching under capacity constraints (see Section 6.4 for a formal definition and analysis), Proposition 2 is a generalization of the result of Konishi and Ünver (2006) to the cases in which constraints are not necessarily capacity constraints.

To apply Proposition 2 to our daycare allocation problem, we compare the SOFM under the daycare constraint (henceforth "flexible SOFM") and the SOFM under an arbitrary rigid constraint associated with that daycare constraint (henceforth "rigid SOFM"). Since the relaxation from "rigid" constraints to "flexible" ones satisfies the hypothesis of Proposition 2 in the context of daycare allocation, we obtain the following corollary.

Corollary 1. *In the daycare allocation problem, fix a set of children I, a set of daycares S, child preferences $\succ_I$, daycare priorities $\succ_S$, and a rigid constraint. Then, every child is*

[31] See also an earlier result by Crawford (1991) who considers comparative statics with respect to adding an agent.

weakly better off under the flexible SOFM than under the rigid SOFM. The set of children who are unmatched at the flexible SOFM is a subset of those at the rigid SOFM.

While this corollary is unambiguous about the direction of welfare change for students, it is silent about the magnitude of the welfare improvement. To quantify the magnitude of the change, we use data on daycare seat allocation we received from Yamagata City, Japan. Yamagata City is the prefectural capital of Yamagata prefecture in the northeastern part of Japan, with just over 250,000 residents as of 2015.[32]

Our data involve the applicants (who are anonymized), usually parents, representing children who would begin attending the daycare in April of 2018. There were 1437 applicants aged between 0 to 5 as of April 1, 2018 on which they would begin attending the daycare. For each applicant, the data show her submitted preferences over the daycare centers and priority ranking (the priorities are common across daycare centers). Regarding submitted preferences, we note that the mechanism in Yamagata is based on serial dictatorship (with no restriction on the number of daycares that can be listed).[33] Strategy-proofness of serial dictatorship appears to be fairly well understood, which gives some justification for treating submitted preferences as true preferences.[34] The priority order is based on the applicant characteristics such as parents' job status and the number of adults available for care at home (City of Yamagata, 2018*b*). There are 93 daycare centers in our dataset. For each daycare center, the data show how many seats are supplied for each age.

In our simulation, we made several modeling choices given data limitation. First, the data we have involve ties although the actual priority order is strict. This is because our data lack information on some characteristics used by Yamagata to determine the strict order, such as whether the child is currently in an alternative form of childcare and whether the family has a member with disability. Given the absence of further

[32]As mentioned in footnote 6, we are currently conducting simulations using data from two other municipalities and plan to report the results in a future version of this paper.

[33]The mechanism is slightly different from pure serial dictatorship, i.e., there are a few special rules, mainly regarding children with siblings. In our numerical analysis, however, this difference causes only a minor difference between the assignments from pure serial dictatorship and the actual one.

[34]A popular how-to book for parents (Habu, 2016), for example, compares strategic properties of serial dictatorship with those of Boston mechanism as follows (these two mechanisms are the most popular mechanisms for daycare allocation in Japan). "In the first mechanism [serial dictatorship], ... there is no advantage or disadvantage associated with your stated first choice, while in the second mechanism [Boston mechanism], ... if you list a competitive daycare center as your first choice, the probability that you will be admitted by no daycare center can increase."

information, we randomly break ties using a single tie-breaking (that is, the tie-broken priorities are common across daycare centers) according to the uniform distribution. For each mechanism that we consider, we conducted 50 runs of simulations using such a tie-breaking rule.[35]

The second limitation involves constraints. For daycare centers, our dataset does not tell the entire family of feasible sets of children or the number of teachers corresponding to the flexible constraints. Instead, it only shows the number of advertised seats at each daycare center for each age, which is exactly enough to specify the rigid constraints. To overcome this limitation, we define m_s for each s in the daycare constraints (Equation (5.1)) by $m_s := \sum_{t \in T} r_t \cdot q^t$, where r_t and q^t are those in the data (recall that r_t is the teacher-child ratio under the national regulation, and q^t is the number of advertised seats for age t at daycare center s). That is, m_s is the minimum of the number of teachers such that the constraint implied by the number of advertised seats in data is a rigid constraint associated with our daycare constraint.[36]

We find that the effect of allowing flexibility in constraints is substantial in our data: the average number of children who are matched with a strictly preferred daycare center in the flexible SOFM compared to the rigid SOFM is 866.26, which amounts to 60.3 percent of all applicants (Table 1).[37] By contrast, no applicant is made worse off, as implied by Proposition 2. The average number of children who are unallocated changes from 713 to 87.6, a 43.5 percent decrease (Figure 1). The average numbers of children who are matched to their first choice, first two choices, and the first three choices increase by 586.7, 540.2, and 496.2, respectively (Figure 2). Our analysis suggests that utilizing the flexible nature of the constraints can be substantial in some environments.

[35]As mentioned in footnote 7, we are currently increasing the iterations of our simulations, and it seems the results are robust.

[36]We set m_s as the bare minimum that is consistent with the data on advertised seats so that we do not overstate our estimate of the gain from removing the rigid constraint. In a similar spirit, we allow for non-integral values of m_s although the number of teachers is an integer in practice. With an alternative specification setting m_s to be the integer rounded up from our present definition, for instance, our estimate of the gain from removing rigid constraints would be larger.

[37]This table as well as others also report simulations of other mechanisms we discuss below.

From/To	rigid SOFM	flexible SOFM	actual allocation	flexible ETSD
rigid SOFM	0	866.26 (60.28%)	657.34 (45.74%)	880.76 (61.29%)
flexible SOFM	0	0	71.7 (4.99%)	49.24 (3.43 %)
actual allocation	13.24 (0.92%)	237.86 (16.55%)	0	248.56 (17.30%)
flexible ETSD	0	0	62.4 (4.34%)	0

TABLE 1. The number of applicants who are made strictly better off by a change of a mechanism.

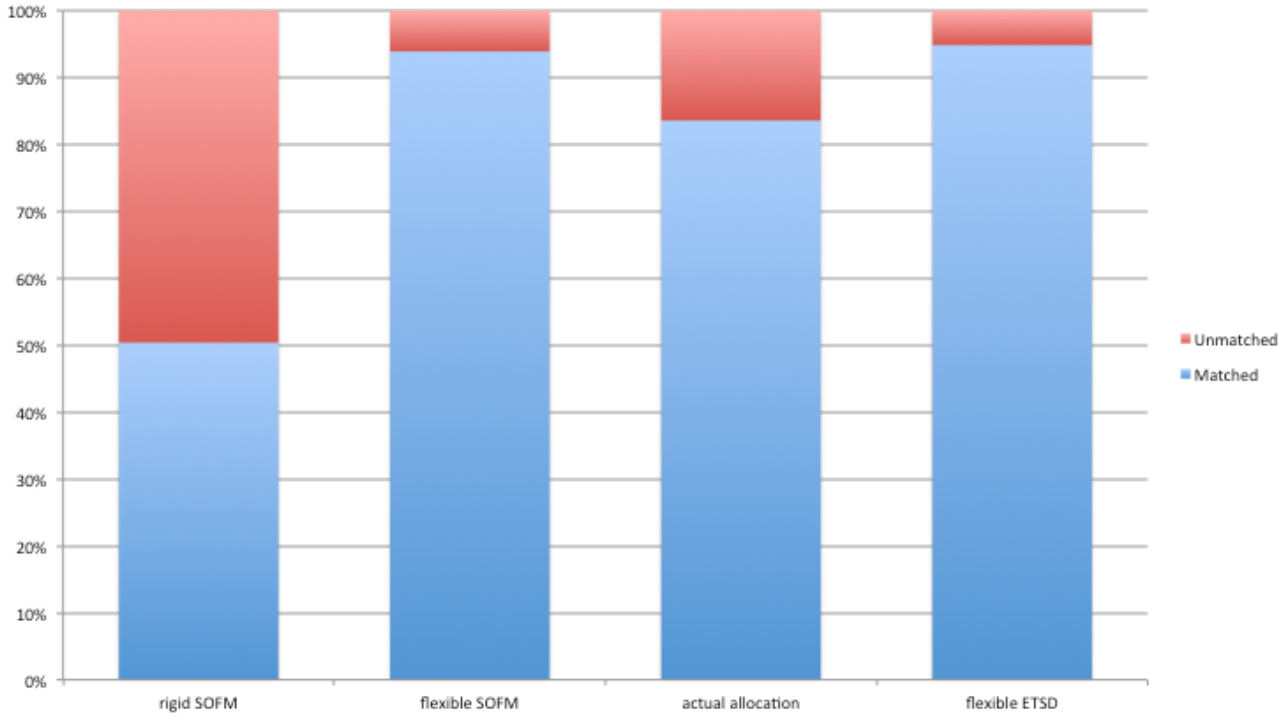


FIGURE 1. The fractions of matched applicants under different mechanisms.

Next, we compare the rigid and flexible SOFMs with Yamagata's actual assignment. Yamagata's mechanism is based on what we call **rigid envy-tolerating serial dictatorship (rigid ETSD)**. Rigid ETSD runs serial dictatorship, treating the problem for each age as separate from others. That is, for each age, it runs serial dictatorship for children of that age and the number of seats committed to that age in advance.[38] This means that,

[38]Yamagata's assignment mechanism has a few modifications to the serial dictatorship, most notably in the way it treats siblings. However, deviation of Yamagata's assignment from pure serial dictatorship seems very minor in magnitude.

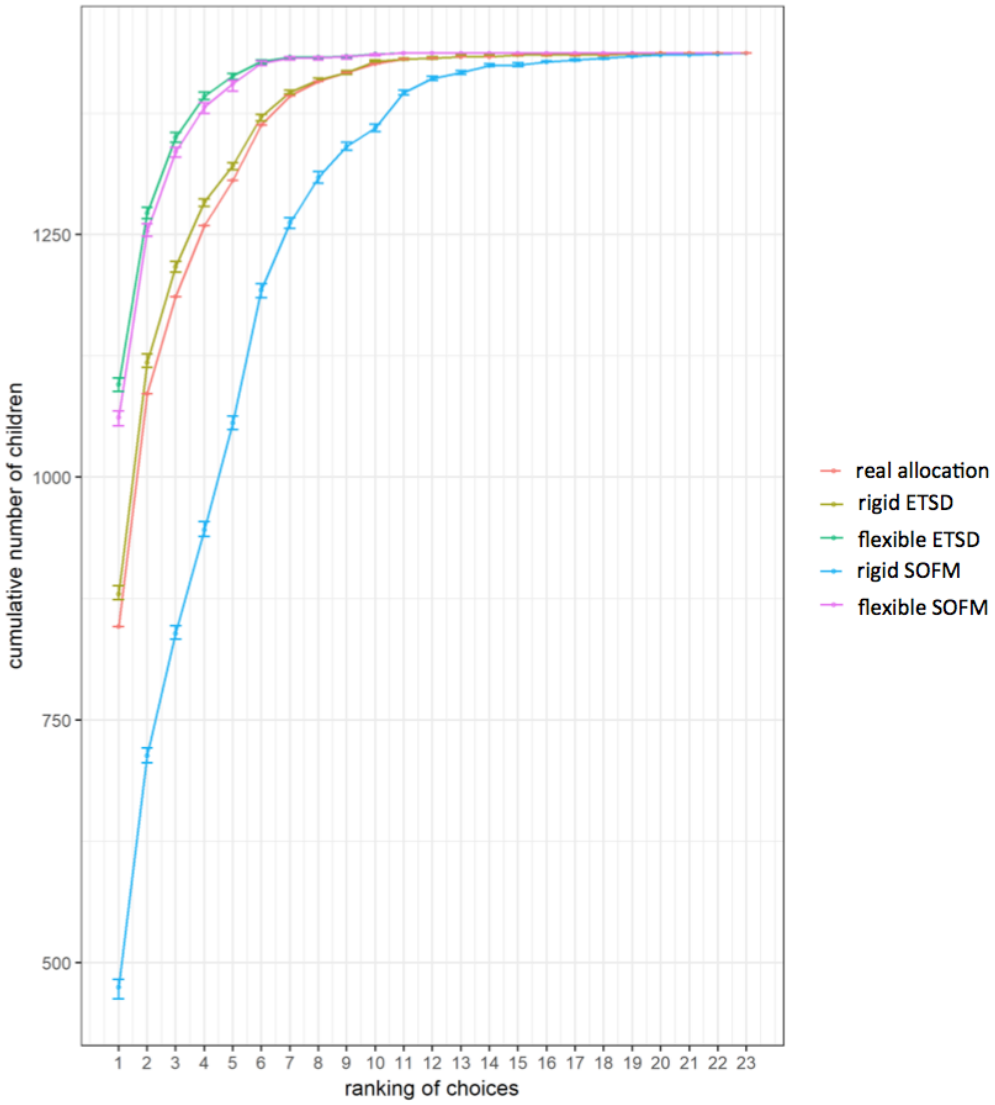


FIGURE 2. Rank distributions under different mechanisms: The graph reports the average cumulative number of children at each rank, as well as its range across all 50 realizations.

among other things, there may remain justified envy between two children i and i' if they are of different ages, while by construction there is no justified envy between children of the same age.[39] Yamagata's assignment is expected to have some efficiency advantage over the rigid SOFM since justified envy is tolerated across different ages, while the comparison with the flexible SOFM is theoretically indeterminate because Yamagata's assignment is

[39]This feature makes rigid ETSD similar in spirit to a mechanism proposed by Okumura (2016) in that both mechanisms require elimination of envy between children of the same age only. The same comment applies to serial dictatorship under daycare constraints discussed below.

based on the rigid constraint, 3which may or may not overwhelm the efficiency gain from tolerating justified envy across different ages.

We find that the flexible SOFM outperforms Yamagata's assignment not only in terms of fairness but also in terms of efficiency. Regarding efficiency, all our efficiency measures favor the flexible SOFM; the average fraction of unmatched children decreased by 10.3%, and 16.6% of children are matched with strictly preferred daycare under the flexible SOFM while only 5.0% are matched with strictly preferred daycare under the actual allocation. Turning our focus to fairness, Table 2 provides several measures of envy for Yamagata's assignment (note that all measures of envy are zero for the rigid and flexible SOFM). On average, there are about 1,000 pairs (i, s) such that i has a justifiable envy toward some i' matched to s under the actual allocations. Also, students involved in at least one of such pairs and daycares involved are 33.1% and 66.7% of the respective total numbers, on average.

	rigid SOFM	flexible SOFM	actual allocation	flexible ETSD
pairs with envy	0	0	989	157.64
students with envy	0	0	475 (33.05%)	130.46 (9.08%)
daycares with envy	0	0	62 (66.67%)	22.16 (23.83%)

TABLE 2. Numbers of justified envy under different mechanisms.

Another natural question is to see what happens in serial dictatorship if the rigid constraint is removed so that it is only subject to the daycare constraint. In the induced mechanism, called **flexible envy-tolerating serial dictatorship (flexible ETSD)**, the current dictator receives her most preferred daycare such that adding her to the current match does not lead to infeasibility. Since some envy is tolerated while the constraint is flexible in this mechanism, its efficiency is expected to be even higher than both Yamagata's actual assignment and SOFM under daycare constraint. Somewhat surprisingly, however, the magnitude of the improvement of this mechanism over the flexible SOFM seems rather small; The average fraction of unmatched children decreases only by 0.95%, and the average fraction of children who become strictly better off under the flexible ETSD is 3.4%. This difference is much smaller than improvement of the flexible SOFM over Yamagata's assignment, whose corresponding percentages are 10.3% and 16.6%, respectively. Meanwhile, the measures of envy show similar magnitudes to those for Yamagata's actual assignment. These numbers may suggest that the flexible SOFM may be a potentially useful mechanism in daycare allocation.

6. Discussion

6.1. **Weak Priority Orders.** As we have argued, there are practical problems in which fairness is so important that even some inefficiencies are tolerated. This is perhaps most vividly seen in the context of natural disasters, where the organizer of a disaster shelter needs to allocate the relief material such as food. For example, in the wake of the Kobe earthquake of 1995 which killed more than 6,400 people in Japan, the organizers of a disaster shelter who had 150 lunch boxes and two boxes of apples refrained from allocating them because they were not sufficient to allocate to everyone. In another shelter, the government instructed not to allocate relief supplies until there were enough to allocate to everyone in the shelter.[40] Even without such an instruction by the government, the organizers of yet another shelter made the same rule on their own (Hayashi, 2003).

Similar cases are repeatedly reported during natural disasters. For example, in the aftermath of Tohoku earthquake of 2011 which killed more than 15,800 people in Japan, the organizers of a disaster shelter in Fukushima refrained from allocating relief supplies, and they attributed their decision to fairness concerns, saying they "were worried about conflicts among disaster victims" (Town of Tomioka, 2015).

One of the key features of these problems is that priorities are weak. For instance, in the first of the aforementioned shelters in the Kobe earthquake, priority is given to the elderly and children before everyone else, but there were a large number of people in the same priority class (there were more than 1000 people in total). In the third shelter, there were only four priority classes (for instance, individuals whose houses were destroyed by the earthquake are given higher priority than those whose houses were not and who only need relief material).[41]

Motivated by these real cases, in this section we study fair matching under weak priorities. To do so, we generalize the model of Section 2 by assuming each school s has a weak priority order over the set of students. Specifically, the weak priority $\succeq_s$ is not required to have the property that $i \succeq_s i'$ and $i' \succeq_s i$ imply $i = i'$. In this context, we say that i has a justified envy toward i' if there exists $s \in S$ such that $s \succ_i \mu_i$, $i' \in \mu_s$ and $i \succeq_s i'$. Note that this condition reduces to the earlier definition when priorities are strict. Fairness and stability are analogously defined, while all other concepts are unchanged.

The following example shows a stable matching fails to exist even if the constraints are capacity constraints.

[40] Cases reported here are from Board of Education of Hyogo Prefecture (1996).

[41] The end of this section discusses other practical examples in which fairness issues are present and weak priority orders are involved.

Example 2 (Non-existence of a stable matching under weak priorities)**.** There is a disaster shelter which is endowed with 150 lunch boxes. In this shelter, there are three groups of individuals: 70 children, 70 elderly individuals, and 70 adults. Priorities are weak; children have the highest priority, elderly individuals have the next priority, and adults have the lowest priority, and individuals in the same group have the same priority as one another. Every individual can consume at most one lunch box and prefers to receive a lunch box than not.

In this problem, a stable matching does not exist. To see this, first note that non-wastefulness implies all lunch boxes are allocated. This fact and fairness imply that the only candidate for a stable matching assigns the lunch boxes to all children and elderly individuals while assigning remaining 10 to adults. This matching, however, is not fair because an adult without a lunch box has justified envy toward an adult with a lunch box. □

The above example shows that a stable matching based on fairness defined in this section does not necessarily exist, even if all schools have capacity constraints.[42] In fact, it is possible to show that strictness of the priorities is "necessary" in a maximal-domain sense. This suggests that requiring stability is too demanding.

The good news is that there exists a SOFM even under weak priorities if the constraint of each school is a general upper-bound. We can show this result by largely following the proof of Theorem 2, though with some care that we detail shortly: As in the proof of Theorem 2, consider the space of cutoff profiles $P = \{1, \dots, |I|, |I|+1\}^S$. For each $s \in S$, assign indices $1, 2, \dots, |I|$ to all the students, one index for each student, with the restrictions that (i) each index is assigned to exactly one student and (ii) if $i \succ_s i'$, then i has a higher index than i'. For each school s, let $i^{(s,l)}$ be the student whose index is l.[43] Intuitively, the indices represent a tie-breaking of the given weak priority, where 1 represents the lowest and $|I|$ the highest ranks among all the students in I breaking the ties.

With this modification, the rest of the proof for Theorem 2 needs little change. In particular, as in the original proof, we can show that if a cutoff profile p is a fixed point of T, then matching μ^p such that $\mu^p_s = D_s(p)$ for each $s \in S$ is fair. One might wonder why

[42]The existing literature often uses an alternative notion of fairness. Specifically, it requires there be no i, $i' \in I$ and $s \in S$ such that $s \succ_i \mu_i$, $i' \in \mu_s$ and $i \succ_s i'$ (instead of $i \succeq_s i'$). Under this definition of fairness, it is well-known that a stable matching exists. In applications we have in mind such as allocation of disaster relief material, however, our notion of fairness is more appropriate.

[43]As in the proof of Theorem 2, we also consider a hypothetical student $i^* \notin I$ such that $i^* = i^{(s,|I|+1)}$ for all $s \in S$.

such a proof works; we "break ties" when defining the indices of students while fairness requires there be no envy even between students with the same rank *according to the original weak priority.*[44]

To see the reason, recall the definition of the demand function $D_s(\cdot)$,

$$D_s(p) := \{i \in I | i \succeq_s i^{(s,p_s)} \text{ and } s \succ_i \emptyset; i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow s \succeq_i s'\}.$$

Here, the key is that all the relationships in $D_s(\cdot)$ involving school priorities are specified with respect to the original weak priority, not the one after tie-breaking. This allows that, for any given cutoff profile p, if multiple students have the same rank at a school, then either all of them can demand that school or none of them can, ensuring there be no envy among them.

For concreteness, consider an example with three students i_1, i_2, and i_3 and one school s, where i_3 has the highest priority while i_1 and i_2 are tied and below i_3. Assume that s is acceptable to all students. Then $D_s(p)$ reduces to $D_s(p) := \{i \in I | i \succeq_s i^{(s,p_s)}\}$. The indexing procedure would assign either 1 or 2 to i_1, the other index to i_2, and 3 to i_3. Suppose we assign 1, 2, and 3 to i_1, i_2, and i_3, respectively. In constructing $D_s(p)$, the only subtle case is when $p_s = 2$.[45] In that case, one might think that $D_s(p) = \{i_2, i_3\}$ holds because i_2 and i_3 have indices no smaller than 2 while i_1's index is 1, and it might interfere with fairness because i_1 would have a justified envy toward i_2. Fortunately, this concern is unfounded because $i^{(s,2)} = i_2$ and i_1 and i_2 have the same priority at s, so $i_1 \succeq_s i^{(s,2)}$ holds, and thus $D_s(p) = \{i_1, i_2, i_3\}$. The point is that even though the index of i_1 is strictly lower than that of i_2, that is only for convenience, and our construction of the demand $D_s(p)$ relies on the original weak priority, guaranteeing fairness of the induced matching.

Finally, let us mention that, although fairness concerns may be the most salient in allocation during disasters such as the major earthquakes discussed earlier, there are other examples in which fairness issues are present and weak priority orders are involved. For example, Hungarian college admissions are conducted by a central clearinghouse and a fair matching is produced. In their system, priorities at each college are based on test scores, and the applicants with the same score are ranked equally (Biró, 2008). Chilean college admissions share these features, although the mechanism in use differs from the

[44]Such a concern would not arise if fairness did not require nonexistence of an envy toward a student with the same rank, as defined in e.g., Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu, Pathak and Roth (2009).

[45]Just as in the case with strict priority, we have $i \succeq_s i^{(s,1)}$ for $i = i_1, i_2, i_3$ and $i \succeq_s i^{(s,3)}$ only for $i = i_3$, implying $D_s(p) = \{i_1, i_2, i_3\}$ if $p_s = 1$ and $D_s(p) = \{i_3\}$ if $p_s = 3$.

one in Hungary (Ríos et al., 2014). Similarly, fairness is explicitly taken into account in Turkish Navy, where officers in charge of logistical support are instructed to distribute resources homogeneously across units (Kesten and Yazici, 2012). Our analysis may be applied to each of these applications in practice.

6.2. **Nonexistence of a Stable Matching.** In Section 4 we showed that there is an environment in which no stable matching exists under a general upper-bound. In this section we provide a theorem clarifying the extent to which this impossibility result holds.

Theorem 4. *Fix a set of students I, a set of schools S, and a school $s \in S$ and its constraint $\mathcal{F}_s$. Suppose that $\mathcal{F}_s$ is not a capacity constraint while being a general upper-bound. Then there exist a school priority $\succ_s$ and student preferences $\succ_I$ such that, for any constraint profile $\mathcal{F}_{-s}$ and priority profile $\succ_{-s}$, there exists no stable matching in the problem $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$.*

Since a stable matching is guaranteed to exist if each school has a capacity constraint (Gale and Shapley, 1962; Roth and Sotomayor, 1990), Theorem 4 provides a tight characterization for the existence of a stable matching. Roughly, capacity constraints are the maximal domain for the existence of a stable matching, that is, a stable matching is guaranteed to exist if and only if each school has a capacity constraint. This result provides a justification for market designers faced with non-capacity constraints to seek a solution that may be unstable. The SOFM may be an appealing alternative because it always exists under general upper-bound and, while possibly unstable, it is most preferred by every student among all matchings that are fair, individually rational and feasible.

One might wonder why this impossibility result does not contradict the existence of a stable matching with substitutable preferences (e.g., responsive preferences with type-specific quota) which may not be associated with any capacity constraint. One major difference is that our definition of stability requires that there exist no envy even if satisfying such an envy is infeasible, which fits the applications we have in mind (see Remark 1 for a discussion of different fairness notions).

6.3. **Strategic Issue.** Assume that the constraint at each school is a general upper-bound. Consider a mechanism, called **the SOFM mechanism** which, for every input, produces the SOFM; of course this mechanism is well-defined by Theorem 2 because we assume each school's constraint is general upper-bound. One question of interest is whether the SOFM mechanism is strategy-proof. To answer this question, we begin by noting that the SOFM mechanism is strategy-proof if the constraint of each school is a capacity constraint. This is because the SOFM and the student-optimal stable matching

coincide under capacity constraints (Balinski and Sonmez 1999), and the mechanism that produces the student-optimal stable matching is strategy-proof in that setting (Roth 1982, Dubins and Freedman 1981).

Given this observation, the remaining question is whether strategy-proofness holds more generally. The following result offers a sense in which such a generalization is impossible.

Theorem 5. *Fix a set of students I, a set of schools S with $|S| \geq 2$, and a school $s \in S$ and its constraint $\mathcal{F}_s$. Suppose that $\mathcal{F}_s$ is not a capacity constraint while being a general upper-bound. Then there exists a profile $\mathcal{F}_{-s}$ of capacity constraints such that the SOFM mechanism is not strategy-proof in $(I, S, \mathcal{F}_S)$.*

Theorem 5 only relaxes $\mathcal{F}_s$ while keeping $\mathcal{F}_{-s}$ to be capacity constraints, and hence this is a maximal domain result. An implication is that the SOFM mechanism is no longer guaranteed to be strategy-proof once we go beyond the class of capacity constraints. Given this impossibility, one question of interest is whether any other reasonable mechanism satisfies strategy-proofness along with fairness, individual rationality, and feasibility. Of course, the mechanism that always produces an empty matching is strategy-proof and satisfies the other three properties, but that mechanism is hardly justifiable as it is extremely inefficient. Thus, the relevant question is whether there is a mechanism satisfying strategy-proofness and the other desiderata, while also satisfying at least some desirable efficiency property.

As it turns out, the lack of strategy-proofness is not the drawback of the SOFM mechanism alone, but is shared by a broad class of mechanisms that satisfy a very mild efficiency property. To state this finding formally, we say that mechanism φ satisfies **unanimity** if, for any $\succ_I$ and $\succ_S$, if a matching μ such that μ_i is the most preferred outcome for every $i \in I$ at $\succ_i$ is feasible, then $\varphi^{\succ_S}(\succ_I) = \mu$. In words, unanimity requires that if a matching in which every student is matched to her first choice is feasible, then the mechanism should produce that matching. This is arguably a very mild requirement and is satisfied by the SOFM mechanism and many other mechanisms. It turns out that this mild requirement is incompatible with strategy-proofness and other properties, as stated in the following generalization of Theorem 5.

Theorem 6. *Fix a set of students I, a set of schools S with $|S| \geq 2$, and a school $s \in S$ and its constraint $\mathcal{F}_s$. Suppose that $\mathcal{F}_s$ is not a capacity constraint while being a general upper-bound. Then there exists a profile $\mathcal{F}_{-s}$ of capacity constraints such that there*

exists no mechanism that satisfies feasibility, fairness, unanimity, and strategy-proofness in $(I, S, \mathcal{F}_S)$.[46]

This result shows that the lack of strategy-proofness is not a deficiency specific to the SOFM mechanism. Rather, this theorem establishes that there is a more fundamental incompatibility between strategy-proofness and other requirements once we go beyond the restrictive domain of capacity constraints.

Although the lack of strategy-proofness is inherent in the SOFM mechanism and even other mechanisms satisfying fairness, it does not necessarily imply that misreporting happens in practice. In fact, it may be difficult for students to precisely identify the case in which her misreporting is profitable, let alone what particular misreported preferences to submit. A simple but intuitive way to see this would be through the cutoff adjustment algorithm which by Proposition 1 implements the SOFM mechanism.

Proposition 3. *Fix a set of students* I*, a set of schools* S*, constraints* $(\mathcal{F}_s)_{s\in S}$*, student preferences* $\succ_I$*, and a misreported preference* $\succ_i'$ *of a student* $i \in I$*. Let* p *and* p' *be the cutoff profiles produced at the end of the cutoff adjustment algorithm under* $\succ_I$ *and* $(\succ_i', \succ_{I\setminus\{i\}})$*, respectively. If* i *is matched to* s *under* $(\succ_i', \succ_{I\setminus\{i\}})$ *and* i *prefers* s *to the outcome under* $\succ_I$*, then* $s \in S$ *and* $p_s > p_s'$*.*

That is, if a student's manipulation leads her to match with a more preferred school, then that school's cutoff has to be strictly lower under misreported preferences. Straightforward as this observation may be, it helps identify cases in which strategic manipulation is not profitable. For instance, in large markets, the cutoff of a school is determined by the *highest-priority applicant* who is rejected from it, and this depends on the entire distribution of student preferences as well as school priorities. Thus, it appears unlikely that any one particular student is in a position to influence the cutoff in any significant manner.

Based on the above discussion, we conjecture that in a wide class of large markets the SOFM mechanism has an approximate incentive compatibility property. We do not attempt to formally establish such a result because it will require us to add more structures and restrictions on the model, such as introducing cardinal utility to students and taking a particular stance on how the market with constraints "becomes large." Instead, here we consider a simple setup and show that in that environment, it is a low probability event that a student has any profitable misreport.

[46]This result is a generalization of Theorem 5 because the SOFM mechanism satisfies feasibility, fairness and unanimity. We do not require individual rationality in Theorem 6 because it turns out it is not necessary for establishing the result. Given that this is an impossibility result, not requiring a condition makes the claim stronger.

Specifically, suppose that there are two schools, s_1 and s_2. School s_1 is subject to a capacity constraint of $n > 0$. School s_2 has some total budget (which is arbitrary and could, but does not have to, depend on n), and a subset of students is feasible at s_2 if and only if the sum of the consumption of the budget by them is no greater than the total budget. There are a certain number of students (which again is arbitrary and could, but does not have to, depend on n). If a student is matched with s_2, she consumes a fixed positive amount of budget at s_2. Let the smallest unit of the budget consumed by a student be normalized at 1, and k be the maximum of a student's consumption of the budget. For any given n, the priority of s_1 is uniformly distributed over all possible strict orders over the students, while the priority of s_2 is arbitrarily fixed. All students prefer s_2 to s_1 to $\emptyset$.

For each student, we bound the probability that, under the realization of the priority, there exists a profitable misreporting for that student under the SOFM mechanism.

Proposition 4. *In the market described above with n, fix a student $i \in I$ arbitrarily. The probability of the realization of the priority such that there exists a profitable misreporting for i under the SOFM mechanism is less than $\frac{k}{n}$.*

Proposition 4 implies that, when k is bounded across all n, the probability that any given student has a profitable manipulation approaches zero as the market size n goes to infinity. The intuition for this is that, when the market is large, for any student whose priority at s_2 is close to the cutoff, it is unlikely for her priority at s_1 to be close to the cutoff as well. In Appendix A.6, we generalize this claim further by considering a more general class of constraints.

6.4. **SOSM and SOFM.** One may wonder whether the following claim might hold: If there exists a stable matching μ such that $\mu_i \succeq_i \mu'_i$ holds for each $i \in I$ and any stable matching μ' (the student-optimal stable matching, or the SOSM) in that market, it is the same as the SOFM. This claim turns out to be false. To see this, consider the following example.

Example 3. Suppose that there are four students i_1, i_2, i_3, and i_4, and two schools, s_1 and s_2. Let preferences and priorities be as follows:

$$\succ_{i_1}\colon s_2, s_1 \qquad\qquad \succ_{s_1}\colon i_1, i_2, i_3, i_4$$

$$\succ_{i_2}\colon s_1, s_2 \qquad\qquad \succ_{s_2}\colon i_2, i_1, i_3, i_4$$

$$\succ_{i_3}\colon s_1, s_2$$

$$\succ_{i_4}\colon s_1, s_2$$

The constraints are: $\mathcal{F}_{s_1} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_4\}, \{i_2, i_4\}\}$ and $\mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_4\}\}$. Note that the constraint of school s_2 is a capacity constraint while the constraint of school s_1 is not, and both are general upper-bounds.

Now, consider the following two matchings.

$$\mu = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_1 & i_2 & i_3, i_4 \end{pmatrix}, \qquad \mu' = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_2 & i_1 & i_3, i_4 \end{pmatrix}.$$

By inspection, one can verify that μ is the SOSM while μ' is the SOFM. Note that μ' is unstable because i_4 can be feasibly matched to s_1. Since $\mu' \neq \mu$, the SOFM and the SOSM are different.[47] □

Remark 2. (1) The difference between the SOSM and the SOFM is caused by the generality of general upper-bound. To see this, recall that, if each hospital's constraint is a capacity constraint, then the SOSM and the SOFM are identical to each other (Theorem 2 of Balinski and Sonmez (1999)). Combined with Theorem 3, this implies that, whenever we impose a condition on constraints over individual schools which guarantees the existence of a SOSM, it is identical to the SOFM.

(2) Since any stable matching is fair (by definition), for any problem with general upper-bounds, the SOFM is weakly preferred by every student to any stable matching (if the latter exists). Therefore, whenever there exists a SOSM and it is different from the SOFM, the SOFM is strictly preferred to the SOSM by some students while weakly preferred by every student. Example 3 shows that this can actually happen.

6.5. **General Upper Bounds and Multidimensional Constraints.** In a recent work, Delacrétaz, Kominers and Teytelboym (2016) study a model of matching with multidimensional constraints. This section investigates the relationship between our model of general upper-bound and their model.

In the model with multidimensional constraints, there is a finite set of **services**, Σ. Each student i is associated with **service needs** $\nu^i = (\nu^i_\sigma)_{\sigma\in\Sigma} \in \mathbb{R}^{|\Sigma|}_+$, and each school s is endowed with **service capacity profile** $\kappa^s = (\kappa^s_\sigma)_{\sigma\in\Sigma} \in \mathbb{R}^{|\Sigma|}_{++}$.[48] We say that a set

[47] We note that the existence of i_3 plays a major role in this example. For example, consider

$$\mu'' := \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_2, i_4 & i_1 & i_3 \end{pmatrix}.$$

We can see that μ'' is not fair (hence unstable), because i_3 likes s_1 better than $\emptyset$ and i_3 has higher priority than i_4 at s_1.

[48] Delacrétaz, Kominers and Teytelboym (2016) further assume that service needs and capacities are represented by integers. As they mention, none of the results in their paper or ours depends on this

of students I' is **DKT-feasible** at school s if $\sum_{i\in I'} \nu^i_\sigma \leq \kappa^s_\sigma$ for every $\sigma \in \Sigma$ and that matching μ is DKT-feasible if μ_s is DKT-feasible at every $s \in S$.

It is obvious that any constraint given as multidimensional constraints described above is a general upper-bound. The following proposition establishes that there is a specific sense in which these two classes of constraints are "equivalent" to each other, if one can specify any (possibly very large) set of services.

Proposition 5. *Fix I, s, and a constraint $\mathcal{F}_s$. The following two statements are equivalent.*

(1) *$\mathcal{F}_s$ is a general upper-bound.*

(2) *There exist a set of services Σ, a profile of students' service needs $(\nu^i)_{i\in I}$, and a service capacity profile κ^s such that a set of students I' is DKT-feasible at s if and only if $I' \in \mathcal{F}_s$.*

This proposition demonstrates that the class of constraints that can be described as general upper-bounds is the same as those that can be described by multidimensional constraints. This characterization exactly identifies what property is imposed on the types of constraints considered by Delacrétaz, Kominers and Teytelboym (2016) that use linear inequalities. Furthermore, this result is useful as it provides a potentially tractable "language" to code any general upper-bound using a number of linear inequalities. Related, the existence of a SOFM (our Theorem 2) can be obtained by exploiting the connection between these two models. More specifically, Proposition 6 of Delacrétaz, Kominers and Teytelboym (2016) shows the existence of a SOFM in the model with multidimensional constraints. This result and Proposition 5 provide an alternative proof of our existence result.

However, we also note that our "equivalence" result is subtle, and we need caution when interpreting this result. To establish a given general upper-bound can be described by multidimensional constraints, the analyst needs to have the freedom to define the set of "services," as well as students' service needs and service capacities at each school. These services and related parameters defined in this attempt may not correspond to any physical services or other entities which one would regard as real services. In fact, in the proof of the direction "(1) $\Rightarrow$ (2)" of Proposition 5, we define a "service" corresponding to every single infeasible set of students.

A related problem is that the number of services needed to describe a given general upper-bound may be unreasonably large even if the underlying constraint is simple and

assumption. In a similar vein, they allow for zero service capacities but it does not affect any results in their paper or ours. Our assumptions are made only for convenience in proofs.

easily interpretable. To make this point in a simple setting, suppose that there is a school s, and the set of students I is partitioned into two groups, I_1 and I_2. Suppose $\mathcal{F}_s = \{I' | I' \subseteq I_1 \text{ or } I' \subseteq I_2\}$, that is, s can admit a set of students if and only if all of its members belong to a single group.[49] Now, suppose that each of I_1 and I_2 has n students. The following proposition demonstrates that even describing the above simple constraint requires an unboundedly large number of services as n grows.

Proposition 6. *Suppose that multidimensional constraints with the set of services Σ describe the above constraint for the problem with n students from each group. Then $|\Sigma| \geq n$.*[50]

This result calls for some caution when interpreting the "equivalence" result of Proposition 5. Although for any given general upper-bound one can find multidimensional constraints that describe it, the set of services–and hence the number of linear inequalities–needed to describe it may be large when there are many students. In such a case, the representation of a given general upper-bound by a system of linear inequalities may not be practical.

6.6. **A Weaker Notion of Fairness.** Our notion of fairness stipulates that a matching can be regarded as unfair even if there is no feasible way to satisfy a student who has justified envy. As illustrated in Remark 1, this definition seems to be appropriate in applications we have in mind. However, an alternative notion may be more appropriate in other contexts. This section formally defines an alternative notion and shows that the non-existence of a stable matching is robust to the change of the fairness notion.

We say that i **has a feasible justified envy toward** i' if there exists $s \in S$ such that (i) $s \succ_i \mu_i$, $i' \in \mu_s$ and $i \succ_s i'$ and (ii) $(\mu_s \cup \{i\}) \setminus \{i'\} \in \mathcal{F}_s$. We say that a matching μ is **weakly fair** if there exist no students i and i' such that i has a feasible justified envy toward i'. Note that the difference from the definition of fairness is the addition of condition (ii). Since we require a more stringent condition on the triples of the form (i, i', s) that cannot exist, the new notion is weaker than the original fairness concept.

We show by an example that the existence problem of a stable matching still persists under the weaker notion of fairness. More precisely, in the example in Appendix B.2, there exists no matching that is feasible, individually rational, non-wasteful, and weakly fair.

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[49] As detailed in Section 4.1, such a constraint is realistic in the context of refugee match.

[50] In Appendix A.8, we show that this bound is tight.

7. Conclusion

This paper studied a matching problem where institutions are subject to general constraints. Observing that a stable matching typically does not exist, our approach is to tolerate some waste while requiring fairness. Our first main result characterizes feasible, individually rational, and fair matchings by fixed points of a certain mapping on the space of cutoff profiles. Building upon this result, we find a necessary and sufficient condition for guaranteeing the existence of a student-optimal fair matching (SOFM). The condition is that the constraint of each school is a general upper-bound. Then we provide a constructive algorithm to find a SOFM based on our fixed-point mapping. Furthermore, we apply our findings to centralized allocation of daycare seats and find that the SOFM mechanism under flexible constraints performs substantially better than an alternative algorithm that treats age-specific capacities as rigid constraints.

This paper leaves various important questions for future research. For instance, our result shows the general upper-bound is exactly the condition that guarantees the existence of an unambiguously desirable fair matching, i.e., a SOFM. In particular, when the constraint of even one school fails to be a general upper-bound, there is no guarantee that a SOFM exists. In such a case, a natural question for researchers and policymakers alike is what kind of property to aim for. One possibility may be to find a feasible, individually rational, and fair matching that is *not dominated* by another according to student preferences. Although the existence of such a matching is rather straightforward given the finiteness of the environment, a constructive algorithm may be nontrivial and interesting.

Another direction for future research would involve data. For example, we conducted numerical analysis of a dataset on daycare allocation and found a large welfare gain. To what extent is such a finding generalizable to daycare allocation elsewhere? How about other applications such as school choice with diversity constraints, college admissions involving students with disabilities, and refugee matching? These questions are beyond the scope of our paper, but they seem to be important questions for future research.

Finally, it would be interesting to use our findings for design in practice. Such an experience may not only improve outcomes in real problems like daycare seat allocation, but it may also provide more insight about possible directions for future research. For instance, is the lack of exact strategy-proofness for students a major drawback in practice, or is an approximate incentive compatibility as shown in our analysis sufficient to eliminate strategic behavior? Is the SOFM mechanism transparent enough for applicants to understand or trust? Is there any unintended consequence of the change in the mechanism? These

are just a few of many possible questions that one may be able to investigate with the feedback from policy experience. We wait for future research to answer these questions.

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Appendix A. Proofs

A.1. **Proof of Theorem 3.**

Proof. Suppose the constraint $\mathcal{F}_s$ of s is not a general upper-bound. Consider the following two cases.

(1) Suppose that $\emptyset$ is infeasible at s. Then, assume all students find s to be unacceptable. It is clear that there is no feasible and individually rational matching in this case.

(2) Suppose that $\emptyset$ is feasible at s. Then there exist non-empty sets $I'' \subsetneq I' \subseteq I$ such that I' is feasible at s but I'' is infeasible at s. Let s' be a school different from s (note that such a school exists by the assumption that $|S| \geq 2$). Fix student preferences as follows:

$$\succ_i : s, s', \text{for every } i \in I'',$$
$$\succ_i : s', s, \text{for every } i \in I' \setminus I'',$$

and every other student finds all schools unacceptable. In addition, assume that each school other than s has a capacity constraint with a capacity of $|I|$.

In this problem, both of the following matchings are fair as well as feasible and individually rational:

(1) every student in I' is matched to s and every other student is unmatched.
(2) every student in I' is matched to s' and every other student is unmatched.

Therefore, if there is a SOFM, then it should match every student in I'' to s, every student in $I' \setminus I''$ to s', and leave every other student unmatched. But such a matching is infeasible because $I'' \notin \mathcal{F}_s$. □

A.2. **Proof of Proposition 2.**

Proof. Because μ is feasible in the problem $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$ by definition of SOFM and $\mathcal{F}_s \subseteq \mathcal{F}'_s$ for every $s \in S$ by assumption, μ is feasible in $(I, S, \succ_I, \succ_S, \mathcal{F}'_S)$ as well. Moreover, because μ is individually rational and fair in the problem $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$ by definition of SOFM, μ is individually rational and fair in $(I, S, \succ_I, \succ_S, \mathcal{F}'_S)$ as well. Therefore, since μ' is the SOFM in $(I, S, \succ_I, \succ_S, \mathcal{F}'_S)$, it follows by the definition of SOFM that $\mu'_i \succeq_i \mu_i$ for every $i \in I$, completing the first part of the proposition statement. This conclusion, together with individual rationality of μ and $\mu_i \in S$, imply $\mu'_i \succeq_i \mu_i \succ_i \emptyset$ and hence $\mu'_i \in S$, completing the second part of the proposition statement. □

A.3. **Proof of Theorem 4.**

Proof. Suppose the constraint $\mathcal{F}_s$ of s is not a capacity constraint while being a general upper-bound. Let k be the largest nonnegative integer such that all sets of students with cardinality k or smaller are feasible at s (note that k may be 0. Also note that k is well-defined, for $\emptyset$ is feasible at s because $\mathcal{F}_s$ is a general upper-bound, and I is finite).

Claim 2. *There exist $I_1 \in \mathcal{F}_s$ and $I_2 \notin \mathcal{F}_s$ such that $|I_1 \cap I_2| = k$ and $|I_1 \setminus I_2| = |I_2 \setminus I_1| = 1$.*

Proof of Claim 2. Let $\mathcal{I}_1 = \{I' \subseteq I || I'| = k+1, I' \in \mathcal{F}_s\}$ and $\mathcal{I}_2 = \{I' \subseteq I || I'| = k+1, I' \notin \mathcal{F}_s\}$. The former is nonempty because otherwise $\mathcal{F}_s$ would be a capacity constraint, and the latter is nonempty due to the definition of k. Let $l := \min\{|I'_1 \setminus I'_2| | I'_1 \in \mathcal{I}_1, I'_2 \in \mathcal{I}_2\}$ and assume for contradiction that $l > 1$; note that the minimum exists because $\mathcal{I}_1$ and $\mathcal{I}_2$ are nonempty finite sets. Fix arbitrarily $\bar{I}_1 \in \mathcal{I}_1$ and $\bar{I}_2 \in \mathcal{I}_2$ such that $|\bar{I}_1 \setminus \bar{I}_2| = l$. Then, fix $i_1 \in \bar{I}_1 \setminus \bar{I}_2$ and $i_2 \in \bar{I}_2 \setminus \bar{I}_1$ and define $\bar{I} := (\bar{I}_1 \setminus \{i_1\}) \cup \{i_2\}$. If $\bar{I} \in \mathcal{F}_s$, then $\bar{I} \in \mathcal{I}_1$ and $|\bar{I} \setminus \bar{I}_2| = l - 1 < l$, a contradiction to the minimality of l. If $\bar{I} \notin \mathcal{F}_s$, then $\bar{I} \in \mathcal{I}_2$ and $|\bar{I}_1 \setminus \bar{I}| = 1 < l$, again a contradiction to the minimality of l. □

In the remainder, we assume the condition in Claim 2 holds for I_1 and I_2. Denote by i_1 and i_2 the agents such that $\{i_1\} = I_1 \setminus I_2$ and $\{i_2\} = I_2 \setminus I_1$.

Now consider the following preference profile: every student in $I_1 \cup I_2$ finds only s acceptable; every other student finds all schools unacceptable. Also assume school s ranks all students in $I_1 \cap I_2$ first (in an arbitrary order), then the (unique) student $i_2 \in I_2 \setminus I_1$, then the (unique) student $i_1 \in I_1 \setminus I_2$, and then every other student (in an arbitrary order).

Suppose that μ is a stable matching. Now, note I_2 is infeasible at s by assumption, but because $k = |I_2| - 1$ is such that any set of doctors whose cardinality is at most k is feasible at s, non-wastefulness requires that at least k doctors are matched at s under μ. Because of the construction of $\succ_s$ and the requirement of fairness, all students in $I_1 \cap I_2$ should be matched at s. Because I_1 and I_2 satisfy the condition described by Claim 2, $(I_2 \cup \{i_1\}) \setminus \{i_2\} = I_1$ is feasible at s. By non-wastefulness, i_1 should be matched at s, and this implies i_2 is not matched to s and hence unmatched.[51] This is a contradiction to fairness because $i_2 \succ_s i_1$ and $s \succ_{i_2} \emptyset$. □

A.4. **Proof of Theorems 5 and 6.**

Proof. Because Theorem 5 is a special case of Theorem 6, we only provide a proof for the latter result. Suppose the constraint $\mathcal{F}_s$ of s is not a capacity constraint while being

[51] If i_1 is not matched to s but some student in $I \setminus (I_1 \cup I_2)$ is, then the matching violates fairness.

a general upper-bound. Let k be the largest nonnegative integer such that all sets of students with cardinality k or smaller are feasible at s (note that k may be 0. Also note that k is well-defined, for $\emptyset$ is feasible at s because $\mathcal{F}_s$ is a general upper-bound and I is finite). Then, Claim 2 implies that there exist subsets of students, I_1 and I_2, such that I_1 is feasible at s while I_2 is not, $|I_1 \cap I_2| = k$, and there exist $i_1, i_2 \in I$ such that $\{i_1\} = I_1 \setminus I_2$ and $\{i_2\} = I_2 \setminus I_1$. Now, fix a school $s' \neq s$ and consider the following preference and priority profiles as well as constraints: s ranks all doctors in $I_1 \cap I_2$ as the highest (in an arbitrary order), then i_2, then i_1, and then all other students (in an arbitrary order). School s' ranks i_1 first and i_2 second (while the ranking over all other students are arbitrary) and is subject to the capacity constraint with capacity of 1. Each student in $I_1 \cup I_2$ prefers s first and s' second (while preferences on all other schools are arbitrary), and all other students find all schools unacceptable.

Fix a mechanism φ that satisfies feasibility, fairness, unanimity, and strategy-proofness in $(I, S, \mathcal{F}_S)$. Under φ, i_2 is not matched to s. To see this, assume for contradiction that i_2 is matched to s. Then, since s is the most preferred by every student in $I_1 \cap I_2$, fairness implies that every student in $I_1 \cap I_2$ is matched to s, so every student in $(I_1 \cap I_2) \cup \{i_2\} = I_2$ is matched to s. But this is a contradiction to feasibility because $I_2 \notin \mathcal{F}_s$ by assumption and $\mathcal{F}_s$ is a general upper-bound.

Because i_2 is not matched to s and i_2 has higher priority than i_1 at s, it follows that i_1 is not matched to s. Given that, it also follows that i_2 is not matched to s' because i_1 has higher priority than i_2 at s' and s' has the capacity of one.

If i_2 misreports and declares that only s' is acceptable to her, then because φ satisfies unanimity, it matches all students in I_1 to s and i_2 to s', while leaving all other students unmatched. Thus, i_2 benefits from a misreport, and hence φ is not strategy-proof. □

A.5. **Proof of Proposition 3.**

Proof. Let $\tilde{s}$ be i's match under $\succ_I$. Since the cutoff adjustment algorithm implements the SOFM mechanism and the SOFM is individually rational by definition, $\tilde{s} \succeq_i \emptyset$ holds. Together with $s \succ_i \tilde{s}$, we have $s \in S$.

To show $p_s > p'_s$, recall the definition of demand:

$$D_{\bar{s}}(p) := \{i \in I | i \succeq_{\bar{s}} i^{(\bar{s}, p_s)} \text{ and } \bar{s} \succ_i \emptyset; i \succeq_{s'} i^{(s', p_{s'})} \Rightarrow \bar{s} \succeq_i s'\}.$$

First, suppose that $\tilde{s} = \emptyset$. Then, $i^{(s,p_s)} \succ_s i$ holds by the definition of $D_{\bar{s}}(\cdot)$ for all $\bar{s} \in S$, and $i \succeq_s i^{(s,p'_s)}$ holds by the definition of $D_s(\cdot)$. Thus, we have $i^{(s,p_s)} \succ_s i^{(s,p'_s)}$, and by the definition of $i^{(s,\cdot)}$, we obtain $p_s > p'_s$.

Second, suppose that $\tilde{s} \neq \emptyset$. Letting $\bar{s} = \tilde{s}$, we have $i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow \tilde{s} \succeq_i s'$ for any s'. Since $s \succ_i \tilde{s}$, we have $i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow s \succeq_i s'$. Since we have already shown $s \succ_i \emptyset$, if $i \succeq_s i^{(s,p_s)}$ holds then we must have $i \in D_s(p)$ by letting $\bar{s} = s$. This contradicts $\tilde{s} \neq s$. Hence, $i^{(s,p_s)} \succ_s i$ must hold. Under $(\succ'_i, \succ_{I\setminus\{i\}})$, i is matched with s, hence $i \succeq_s i^{(s,p'_s)}$ must hold. Overall, we have $i^{(s,p_s)} \succ_s i^{(s,p'_s)}$, implying $p_s > p'_s$. □

A.6. **Proof of Proposition 4.** This section first states a generalization of Proposition 4 and then provides its proof. For this purpose, consider the same environment as the one considered in Proposition 4 except that we now suppose s_2 has a general upper-bound constraint that is not necessarily a budget constraint. For each n and $i \in I$, define

$$k_i = \max_{\substack{I', I'' \subseteq I \\ I' \cap I'' = \emptyset}} \{|I''| : I' \notin \mathcal{F}_{s_2},\ (I' \setminus \{i\}) \cup I'' \in \mathcal{F}_{s_2}\}.$$

Notice that, in the environment as the one considered in Proposition 4, $k_i \leq k - 1$ holds for all $i \in I$. The following result is a generalization of Proposition 4.

Proposition 7. *In the market described above with n, fix a student $i \in I$ arbitrarily. The probability of the realization of the priority such that there exists a profitable misreporting for i under the SOFM mechanism is no greater than $\frac{k_i}{n}$.*

Proof. It is clear that the only case in which a misreporting is beneficial is when i is unmatched under truthtelling while she is matched to s_1 under the misreporting, and the misreported preferences puts s_1 as her top choice. Let m and m' be the numbers of students matched at s_2 under true and misreported preferences, respectively. By the definition of k_i, we have $m' - m \leq k_i$. Because i is unmatched under truthtelling but matched to s_1 under the misreporting, the ranking of student i at s_1 among students who are not matched at s_2 under truthtelling should be at least $n + 1$ and at most $n + k_i$. The probability of such a realization of the priority order is at most $k_i/(n + 1)$. Since $k_i/(n + 1) \leq k_i/n$, the proof is complete. □

A.7. **Proof of Proposition 5.**

Proof. (2) ⇒ (1) is straightforward. To show (1) ⇒ (2), define the set of services Σ as $\Sigma := 2^I \setminus \mathcal{F}_s$. For each $\sigma \in \Sigma$, suppose

$$\nu^i_\sigma = \begin{cases} 1 \text{ if } i \in \sigma, \\ 0 \text{ otherwise,} \end{cases}$$

and let $\kappa^s_\sigma = |\sigma| - 1/2$.[52] We will show that the conclusion of the proposition holds with respect to these parameters for multidimensional constraints.

To show the "if" direction, suppose $I' \in \mathcal{F}_s$. Then, because $\mathcal{F}_s$ is a general upper-bound, it follows that, for any $\sigma \in \Sigma$, $I' \not\supseteq \sigma$ and hence $\sigma \cap I' \leq |\sigma| - 1/2$. So, $\sum_{i \in I'} \nu^i_\sigma \leq |\sigma| - 1/2 = \kappa^s_\sigma$. Therefore I' is DKT-feasible. To show the "only if" direction, suppose $I' \notin \mathcal{F}_s$. Then, the service $\sigma = I'$ is in Σ, and thus $\sum_{i \in I'} \nu^i_\sigma = |I'| > |I'| - 1/2 = \kappa^s_\sigma$. Thus, I' is not DKT-feasible. □

A.8. **Proof of Proposition 6.** As mentioned in footnote 50, we will show the following stronger result: *Given a problem with n students from each group as in Proposition 6, let $m \in \mathbb{N}$ be the minimum cardinality of the sets of services that describe the constraint.*[53] *Then $m = n$.*

Proof. We first show that in the problem of n students from each group, we need at least n services. To show this, suppose without loss of generality that each service capacity is normalized to 1.[54] Consider a partition of all students into pairs of students from different groups. More specifically, label the students from one group as $I_1 = \{i_1, i_2, \ldots, i_n\}$ and those from the other group as $I_2 = \{i'_1, i'_2, \ldots, i'_n\}$, and form n pairs by paring two students of the same index from the two groups, i.e., $I = \bigcup_{t=1}^n \{i_t, i'_t\}$. For each t, because the pair $\{i_t, i'_t\}$ is infeasible at s, there exists $\sigma \in \Sigma$ for which $\nu^{i_t}_\sigma + \nu^{i'_t}_\sigma > 1$. Choose such a service arbitrarily and denote it by σ_t. To prove our claim, it suffices to show $\sigma_t \neq \sigma_{t'}$ if $t \neq t'$. For this purpose, assume for contradiction that $\sigma_t = \sigma_{t'} =: \sigma$. Then $\nu^{i_t}_\sigma + \nu^{i'_t}_\sigma > 1$ and $\nu^{i_{t'}}_\sigma + \nu^{i'_{t'}}_\sigma > 1$, so $\nu^{i_t}_\sigma + \nu^{i'_t}_\sigma + \nu^{i_{t'}}_\sigma + \nu^{i'_{t'}}_\sigma > 2$. This implies $\nu^{i_t}_\sigma + \nu^{i_{t'}}_\sigma > 1$ or $\nu^{i'_t}_\sigma + \nu^{i'_{t'}}_\sigma > 1$. Hence $\{i_t, i_{t'}\} \notin \mathcal{F}_s$ or $\{i'_t, i'_{t'}\} \notin \mathcal{F}_s$ holds, a contradiction.

We next show that in the problem with n students from each group, there exist multidimensional constraints with n services that describe the given constraint. To do so, let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ and (as before) $I_1 = \{i_1, i_2, \ldots, i_n\}$. Set each service capacity as 1. For

[52]Note that the requirement $\kappa^s_\sigma \in \mathbb{R}_{++}$ is satisfied because $\emptyset \in \mathcal{F}_s$ by the assumption that $\mathcal{F}_s$ is a general upper-bound and hence $\emptyset \notin \Sigma$.

[53]Note that the minimum cardinality exists in $\mathbb{N}$ because of the construction of the set of services in the proof of Proposition 5.

[54]This is without loss of generality because, given any nonzero service capacity and service demands, one can normalize that service capacity to one while changing service needs of each student for that service in the same proportion.

each $t \in \{1, \ldots, n\}$, the service needs for σ_t are given by

$$\nu^i_{\sigma_t} = \begin{cases} 1 \text{ if } i = i_t, \\ 0 \text{ if } i \in I_1 \setminus \{i_t\}, \\ \frac{1}{n} \text{ otherwise.} \end{cases}$$

Suppose $I' \in \mathcal{F}_s$. Then either $I' \subseteq I_1$ or $I' \subseteq I_2$. In the former case, for any service σ_t, $\sum_{i \in I'} \nu^i_{\sigma_t} \leq \sum_{i \in I_1} \nu^i_{\sigma_t} = 1 + (n-1) \times 0 = 1 = \kappa^s_{\sigma_t}$, so I' is DKT-feasible. In the latter case, for any service σ_t, $\sum_{i \in I'} \nu^i_{\sigma_t} \leq \sum_{i \in I_2} \nu^i_{\sigma_t} = \frac{1}{n} \times n = 1 = \kappa^s_{\sigma_t}$, so I' is DKT-feasible.

Next suppose $I' \notin \mathcal{F}_s$. Then $I_1 \cap I' \neq \emptyset$ and $I_2 \cap I' \neq \emptyset$. Let $i_t \in I_1 \cap I'$ and $i' \in I_2 \cap I'$. Then, $\sum_{i \in I'} \nu^i_{\sigma_t} \geq \nu^{i_t}_{\sigma_t} + \nu^{i'}_{\sigma_t} = 1 + \frac{1}{n} > 1 = \kappa^s_{\sigma_t}$, so I' is not DKT-feasible. □

Appendix B. Additional Discussions

B.1. **An alternative algorithm.** Consider the following generalization of the deferred acceptance algorithm, called the **cumulative offer algorithm**:

- Step $t \geq 1$: Each student applies to her first choice school among those that have never rejected anyone whose priority is weakly higher than her if it is acceptable, while making no application otherwise. For each school s, let $\{i_1, i_2, \ldots, i_k\}$ be the set of students who have ever applied to it, with $i_1 \succ_s i_2 \succ_s \ldots \succ_s i_k$. If $\{i_1, i_2, \ldots, i_k\} \in \mathcal{F}_s$, then let s temporarily keep $\{i_1, i_2, \ldots, i_k\}$; otherwise, let s temporarily keep the set of students of the form $\{i_1, i_2, \ldots, i_{k'}\}$ such that $\{i_1, i_2, \ldots, i_{k'}\} \in \mathcal{F}_s$ and $\{i_1, i_2, \ldots, i_{k'+1}\} \notin \mathcal{F}_s$.[55] School s rejects all the remaining students who have ever applied to it, $\{i_{k'+1}, \ldots, i_k\}$. If no student is rejected by a new school, then terminate the algorithm and define the outcome as the matching in which each school is matched to the set of students who it currently keeps.[56] Otherwise, go to Step $t+1$.

The algorithm terminates in a finite number of steps because there is a new rejection in every step that is not terminal and there are only a finite number of student-school pairs. Therefore, the outcome of this algorithm is well-defined.

Note that this algorithm has a "cumulative" feature, that is, when a school temporarily keeps students, it considers all the students who have ever applied to it, even if they were rejected in an earlier step. Moreover, the school never rejects an applicant i while keeping another student i' with lower priority, even if keeping i is infeasible and keeping i' is feasible. These features are important for guaranteeing fairness of the resulting matching.

[55] Because $\mathcal{F}_s$ is a general upper-bound, k' is uniquely defined.

[56] As will be seen in Proposition 8, the outcome is indeed a matching.

Proposition 8. *The outcome of the cumulative offer algorithm is the SOFM.*

Proof. We use the same notations as in the proof of Theorem 2. Suppose that at steps $1, \ldots, t$ of the algorithm, if s has rejected at least one student, let the highest-priority student s who has been rejected be $i^{(s,l')}$ and $p_s^t = l' + 1$. Otherwise, let $p_s^t = 1$. Also, let $p_s^0 = 1$. By definition of p_s^t, the cumulative offer algorithm is equivalent to the following algorithm:

- Step $t \geq 1$: Each student i applies to her first choice school in $\{s \in S | i \succeq_s i^{(s,p_s^{t-1})}\}$, while making no application otherwise. Each school s keeps every student i such that $i \succeq_s i^{(s,p_s^t)}$ and rejects all the remaining students who have ever applied to it. If no student is rejected by a new school, then terminate the algorithm and define the outcome as the matching in which each school is matched to the set of students who it currently keeps. Otherwise, go to Step $t+1$.

For a profile $p = (p_s)_{s \in S}$, define $\tilde{T} : P \to P$ as follows:

$$\tilde{T}_s(p) = \begin{cases} \min\{p'_s | D_s(p'_s, p_{-s}) \in \mathcal{F}_s\} & \text{if } D_s(p) \notin \mathcal{F}_s \\ p_s & \text{if } D_s(p) \in \mathcal{F}_s \end{cases},$$

where the minimum exists because P is finite and $D_s(|I|+1, p_{-s}) = \emptyset \in \mathcal{F}_s$. By inspection, $p_s^t = \tilde{T}_s(p^{t-1})$ holds for each t. Hence, for each t, $p_s^t = \tilde{T}_s^t(p^0)$.

Observe that, by the definition of $D_s(\cdot)$ and the above algorithm, the set of students that each s keeps at the terminal step t is $D_s(p^t)$. Hence, it can be shown that the above algorithm produces the SOFM by a proof similar to those of Theorem 2 and Proposition 1. □

A corollary of Propositions 1 and 8 is that the outcomes of the cumulative offer algorithm and the cutoff adjustment algorithm coincide with each other. Given this fact, one might question the merit of our approach based on Tarski's fixed point theorem on the space of cutoffs. Our response is that there are at least two reasons to favor our approach. First, with our approach, it is easy to show that there exists a SOFM; this result is based on the well understood structure of fixed points of an increasing function on a complete lattice. By contrast, while it is possible to establish that the outcome of the cumulative offer algorithm is the SOFM without reference to this lattice structure, it would need a specific argument tailored to that particular algorithm.[57] Second, and related, the "right" way to generalize the deferred acceptance algorithm is not clear in

[57] Delacrétaz, Kominers and Teytelboym (2016) construct an equivalent algorithm to the cumulative offer algorithm using the language of multidimensional services. See Section 6.5 for detail.

the absence of the insight from our approach. For example, the "cumulative" nature of our cumulative offer algorithm, i.e., that each school considers the current *and previous* applicants, is not an ex ante obvious feature for the right generalization of the deferred acceptance algorithm.

B.2. **Weak fairness and non-existence: An example.** Delacrétaz, Kominers and Teytelboym (2016) consider a slightly different setting from ours and find an example to show that their concept of stability may lead to non-existence. The following example, which is a slight variation of theirs, shows that there does not necessarily exist a matching that is feasible, individually rational, non-wasteful, and weakly fair.[58]

Example 4. Suppose that there are three students i_1, i_2, and i_3, and two schools, s_1 and s_2. Their preferences and priorities are as follows:

$$\succ_{i_1}\colon s_2, s_1 \qquad\qquad \succ_{s_1}\colon i_1, i_2, i_3$$

$$\succ_{i_2}\colon s_1, s_2 \qquad\qquad \succ_{s_2}\colon i_3, i_1, i_2$$

$$\succ_{i_3}\colon s_1, s_2$$

The feasibility constraints are $\mathcal{F}_{s_1} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_1, i_3\}\}$ and $\mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}\}$. Note that the constraint of school s_2 is a capacity constraint while the constraint of school s_1 is not, and both are general upper-bounds.

In this market, there is no matching satisfying feasibility, individual rationality, non-wastefulness, and weak fairness. To see this, consider the following (exhaustive) cases:

(1) Suppose i_1 is matched with s_2. Then i_3 should be matched with s_1 because otherwise i_3 is unmatched and hence has a feasible justified envy toward i_1. Then i_2 is unmatched, but this means i_2 has feasible justified envy toward i_3.

(2) Suppose i_1 is matched with s_1. Then i_3 should be matched to s_1 because otherwise the allocation is wasteful (i_3 prefers s_1 most and $\{i_1, i_3\} \in \mathcal{F}_{s_1}$). This implies that i_2 is matched with s_2. But then i_1 has a feasible justified envy toward i_2.

(3) Suppose i_1 is unmatched. Then Neither i_2 nor i_3 can be matched to s_1 as otherwise i_1 has a feasible justified envy toward the student who matches with s_1. But this is wasteful because, by letting μ the resulting matching, we have $s_1 \succ_{i_1} \emptyset = \mu_{i_1}$ and $\mu_{s_1} \cup \{i_1\} \in \mathcal{F}_{s_1}$. □

[58] Adapted to our setting, their stability concept is slightly stronger than our requirements of feasibility, individual rationality, non-wastefulness, and weak fairness.