Strategic Sample Selection*

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Abstract

What is the impact of sample selection on the inference payoff of an evaluator facing a monotone decision problem? We show that anticipated selection increases or decreases the accuracy of a statistical experiment according to whether the reverse hazard rate of the data distribution is log-supermodular—as in location experiments with normal noise—or log-submodular. The results are applied to the analysis of strategic sample selection by a biased researcher and extended to the case of uncertain and unanticipated selection. Our theoretical analysis offers applied research a new angle on the problem of selection in empirical and experimental studies, by characterizing when sample selectivity, selective assignment to treatment, and strategic omission of variables benefit or hurt the evaluator.

Keywords: Strategic selection; Comparison of experiments; Accuracy; Persuasion; Welfare

JEL codes: D82, D83, C72, C90

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1 Introduction

Empirical data are often nonrandomly selected, due to choices made by the subjects under investigation or sample inclusion decisions by data analysts.[1] Suppose a new treatment is given to the healthiest patients rather than to random patients in a group. Because of selection, favorable outcomes are weaker evidence that the treatment is effective. But is the evaluator's assessment of the treatment effect more accurate with selective or with random assignment? When estimating a regression coefficient of interest, is a selected sample more or less informative than a random sample with the same sample size? In a different, yet similar vein, is a regression with a random missing covariate more or less informative than a regression with a strategically omitted variable?[2]

These comparisons are all instances of one and the same issue: assessing the impact of selection in a monotone decision problem. There is an unknown real-valued state θ, e.g. the effect of a new treatment, or the true value of a regression coefficient. An evaluator must choose an action, e.g. approve the new treatment, or estimate the coefficient, knowing that marginally increasing the action increases the payoff in low states and decreases it in high states. The evaluator decides after observing the realization of a statistical experiment, a random vector $X = (X_1, \ldots, X_n)$. For instance, in a location problem, observations have the form $x_i = \theta + \varepsilon_i$, where ε_i represents the baseline health of an individual, or the noise term in a regression.[3] Our main question is, in which of the following scenarios does the evaluator make better decisions:

- **Random Experiment**. The n observations are i.i.d. draws from a state-dependent cumulative distribution $F(\cdot|\theta)$. In a location problem, this means that the noise terms $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. draws from a known cumulative distribution $F(\cdot)$, and $F(x|\theta) = F(x-\theta)$.
- **Selected Experiment**. The n observations are selected—possibly strategically, by another party—as the n highest out of $k > n$ presampled i.i.d. draws from $F(\cdot|\theta)$. In a location problem, equivalently, $\varepsilon_1, \ldots, \varepsilon_n$ are the n highest of $k \geqslant n$ i.i.d. draws from F.

Following a standard approach pioneered by Blackwell (1951, 1953), we address this question

[1] For instance, Heckman (1979) refers from the outset to these two sources of selection.

[2] Similar questions arise in many other contexts. For instance, peremptory challenge gives a defendant the right to strike down a number of jurors. Given that the defendant selects the most favorable jurors, how is the quality of final judgement affected? When feeding consumer reviews to potential buyers with limited attention, should an e-commerce platform post random reviews or allow the merchant to cherry-pick them? When testing a student in an exam, should the teacher ask questions at random or allow the student to select the most preferred questions out of a batch?

[3] In other applications, θ may represent a defendant's level of guilt, the quality of a merchant's good, or a student's ability, and ε_i a juror's bias, a reviewer's leniency, or a student's specific knowledge of a certain topic.

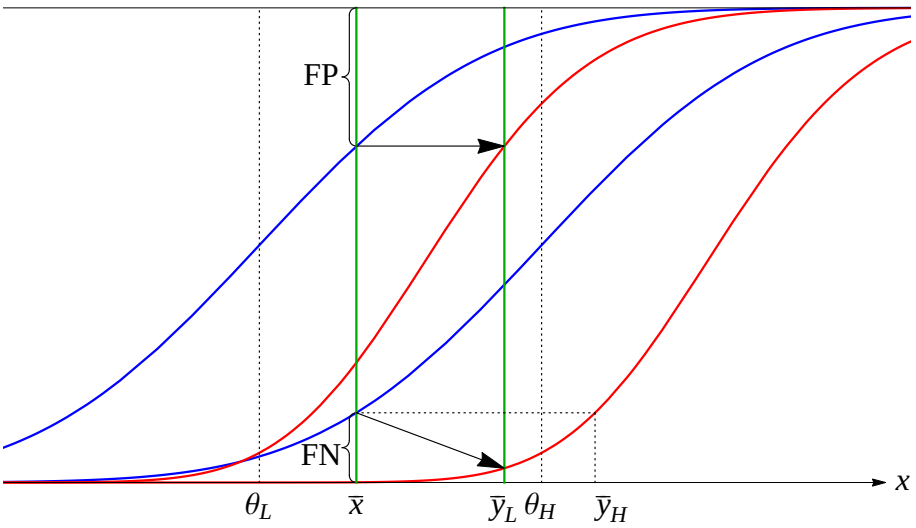


Figure 1: Simple hypothesis testing with a normal experiment: selection provides more accurate information, decreasing the false negative rate for any fixed false positive rate.

by comparing the probability distributions over actions that the evaluator can induce, in each state, in the two experiments. The approach is easy to illustrate in the special case of a simple hypothesis testing problem: two states, a low state θ_L and a high state θ_H, and two actions, rejection—the correct choice in the low state—and acceptance—the correct choice in the high state. In this case the evaluator's decision is the familiar trade-off between the probability of a false positive—accepting in the low state—and the probability of a false negative—rejecting in the high state.

Consider a one-dimensional (sample size $n = 1$) location problem with a normal noise distribution F. In the random experiment, the evaluator observes the realization of a random variable that is normally distributed with mean θ_L in the low state, and mean θ_H in the high state, as illustrated by the blue curves in Figure 1. The evaluator resolves the trade-off between false positives and false negatives by accepting if and only if the observed realization x exceeds some cutoff point $\overline{x}$.[4] The optimally chosen probability of a false positive is then $1 - F(\overline{x} - \theta_L)$, denoted by FP in the figure, while the probability of a false negative is $F(\overline{x} - \theta_H)$, denoted by FN.

How does a selected experiment with presample size $k > 1$ compare? In the selected experiment, the noise distribution becomes F^k, that is, the evaluator observes a random variable distributed according to $F^k(x - \theta_L)$ in the low state, and $F^k(x - \theta_H)$ in the high state—these distributions

[4] A location problem with normal noise, like every other experiment considered in this paper, satisfies the monotone likelihood ratio property: given any two states, the higher the realization x, the higher the relative odds of the higher state. This property implies that the evaluator's optimal decision is increasing in x. With two actions and sample size $n = 1$, this simply means choosing the higher action (acceptance) if and only if x is at least as large as some cutoff $\overline{x}$.

are represented by the red curves in Figure 1. The selected experiment turns out to be better: by adopting the (possibly suboptimal) cutoff point $\bar{y}_L$ in this experiment, the evaluator induces as many false positives, because $1-F^k(\bar{y}_L-\theta_L)=1-F(\bar{x}-\theta_L)$, but also induces fewer false negatives, because $F^k(\bar{y}_L-\theta_H)<F(\bar{x}-\theta_H)$.

What explains the beneficial impact of selection just described? To answer this question we start from an observation made by Lehmann (1988), who pointed out that an equivalent way to formulate the property that $\bar{y}_L$ induces as many false positives and fewer false negatives is to say that the selected experiment is *more accurate*.[5] This means that the cutoff point $\bar{y}_H$ that induces as many false *negatives*, defined by the equation $F^k(\bar{y}_H-\theta_H)=F(\bar{x}-\theta_H)$, is larger than $\bar{y}_L$.[6] Indeed, by adopting the smaller cutoff $\bar{y}_L$ the evaluator necessarily induces more acceptance—and in particular more acceptance in the high state—than by adopting the larger cutoff $\bar{y}_H$. Note that this is exactly what happens in the normal case depicted in Figure 1.

Our first main result identifies a necessary and sufficient condition for a one-dimensional location experiment with a larger presample size k to be more accurate than one with a smaller presample size. Theorem 1 shows that accuracy monotonically increases in presample size k if and only if the reverse hazard function of the basic noise distribution, $-\log F$, is logconcave. Likewise, accuracy monotonically decreases in k if and only if $-\log F$ is logconvex. Intuitively, our logconcavity criterion says that neither the top tail of the distribution should be thicker than in the Gumbel distribution, nor the bottom tail should be thinner—for otherwise the thickening of the upper tail or the thinning of the bottom tail created by selection would decrease accuracy. For example, the condition implies that the evaluator always gains from sample selection when F is normal or logistic, but always loses when F is exponential.

To assess the impact of selection in the general case, with possibly non-additive noise and any sample size $n\geqslant 1$, we employ a natural generalization of Lehmann's (1988) concept of accuracy, which we call *conditional accuracy*. This notion is new to our paper, and allows comparisons between any pair of n-dimensional experiments, with arbitrary patterns of correlation among observations. Conditional accuracy shares the basic intuition with (and, with $n=1$, it reduces to) Lehmann's (1988) original notion. To get this intuition in the most transparent way, consider the simple hypothesis testing setup. In an n-dimensional experiment X the evaluator again adopts a cutoff strategy, but now the cutoff is a more complicated object, an $(n-1)$-dimensional curve in

[5] The latter nomenclature is due to Persico (2000).

[6] Going beyond binary state, and comparing two arbitrary experiments X and Y with respective distributions $F(\cdot|\theta)$ and $G(\cdot|\theta)$, the property that Y is more accurate than X means that the function $(G(\cdot|\theta))^{-1}(F(x|\theta))$ is increasing in θ for every x. To keep our discussion here as simple as possible, we only discuss accuracy for the case of binary state.

$\mathbb{R}^n$.[7] For example, with a vector of i.i.d. observations $x = (x_1, \ldots, x_n)$ from a random location experiment with normal noise, the evaluator accepts if and only if the average observed value exceeds a certain threshold.[8] We say that another experiment Y is conditionally more accurate than X if the (appropriately defined) cutoff curve that induces as many false positives in Y as in X, lies entirely below the (analogously defined) curve that induces as many false negatives. By Lehmann's (1988) argument we can then show that in experiment Y the evaluator can achieve as many false positives as in X, but fewer false negatives.

The notion of conditional accuracy is the key technical tool needed to tackle the new issues arising in the multidimensional case. Indeed, the main difficulty in comparing multidimensional selected experiments is precisely due to the selected observations $x_1, \ldots, x_n$ being correlated with each other, even conditionally on the state θ. Our tool allows us to disentangle the net value of information added by each observation, and hence understand when selection adds or subtracts value to the evaluator's problem as the presample size k increases. Our second and more important main result, Theorem 2, shows that conditional accuracy monotonically increases or decreases in presample size k, according to whether the reverse hazard rate $f(x|\theta)/F(x|\theta)$ is log-supermodular or log-submodular. In a location experiment, log-supermodularity reduces to logconcavity of the noise distribution's reverse hazard rate f/F. This condition strengthens the logconcavity criterion in Theorem 1 by adding a regularity condition. Intuitively, the noise distribution must be *incresingly* thinner at the top and thicker at the bottom, compared to the Gumbel distribution.

Our results have important implications for applied research. While typically thought of exclusively as a threat to internal validity, selective assignment based on untreated outcomes—or, more generally, some unobservable characteristics correlated with untreated outcomes—can actually benefit an evaluator who properly anticipates selection. Actually, as we discuss at the end of the paper, selection may benefit even an unwary evaluator who does not anticipate it. Similar considerations apply in a regression context. Whereas other forms of selection—as occurring, for instance, in truncated regression—decrease accuracy, selection based on order statistics is beneficial (for a fixed sample size) under the conditions identified in our theorems. By the same token, a regression with a random missing covariate can be less informative than a regression with, say, a variable omitted in order to maximize selection bias.

Selection naturally arises when the evidence is provided by a strategic researcher who observes a presample $x_1, \ldots, x_k$ of size k and then reports the n most favorable realizations to the evaluator. Given the presample size k and the fact that the evaluator uses an acceptance threshold policy,

[7] With $n = 1$, this curve is a 0-dimensional object, namely a point $\bar{x}$ in $\mathbb{R}$.

[8] As is well known, in the normal case the average observation is a sufficient statistic for the whole vector of observations. In this case, the cutoff curve has the form $\sum_i x_i/n = \tilde{x}$ for some $\tilde{x}$.

it is indeed optimal for the researcher to report the highest realizations. We provide a strategic foundation for sample selection by introducing a researcher who aims at convincing the evaluator that the true state is high—e.g. that a new treatment is effective, or that a regression coefficient is large. The researcher's incentives to bias upward the evaluator's inference through sample selection are anticipated in equilibrium by the evaluator. Under the conditions in our theorems, equilibrium selective sampling benefits also the researcher in the empirically relevant case where the prior strongly favors rejection. In this case, equilibrium selection harms the researcher when the prior strongly favors acceptance.[9]

We also endogenize the amount of selection in terms of costly investment by the researcher in obtaining the k presample realizations. The evaluator's anticipation and resulting adjustment for selection partly frustrates the researcher's attempt to manipulate. If selection is fully anticipated in equilibrium—for example because the researcher's cost of presample collection is known—then presample collection and selection are a pure rat race when noise follows a Gumbel distribution. In that case, selection has no impact on the payoffs of the two parties. The result is a loss by the researcher exactly equal to the cost of presample collection. Thus, in Gumbel's pure rat race the researcher unambiguously benefits from commitment not to allow (or, equivalently, to disclose) presample collection.[10]

We also illustrate how an evaluator who can ex ante forbid sample selection may prefer not doing so, in order to incentivize the researcher to provide more evidence. Intuitively, consider a situation in which the researcher benefits from selection because of the increased acceptance rate that results from selective reporting. Given that the researcher's individual rationality constraint is relaxed, the researcher's incentives to collect a presample larger than the required sample ends benefitting the evaluator under our logconcavity condition on the reverse hazard rate. Through this mechanism, the evaluator can benefit from limiting the sample size, tolerating sample selection from a presample larger than the sample, and committing not to look at more data.

The final part of the paper discusses a number of extensions of our model. First, drawing on extreme value theory, we analyze the case of extreme selection, with k tending to infinity. We show that for a large class of noise distributions the evaluator achieves full information—in simple hypothesis testing, zero false positives and false negatives—in the limit. Second, we sketch how

[9] Instead, when the noise distribution has a thick right tail and selection is mild (small k), equilibrium selective sampling harms not only the evaluator but also the researcher when the prior strongly favors rejection—generating a credibility crisis. In this case, equilibrium selection benefits the researcher when the prior strongly favors acceptance.

[10] More generally, selection is not completely self defeating, even when the evaluator correctly anticipates the extent of selection k. Our results characterize the net impact of properly anticipated selection on acceptance probability (the researcher's decision payoff) and the evaluator's decision payoff.

to generalize our analysis to encompass comparisons of experiments that are not ranked by accuracy or conditional accuracy. Finally, we discuss the welfare impact of unanticipated and uncertain selection. Surprisingly, there are natural situations in which unanticipated selection benefits: intuitively, if the evaluator does not adjust for selection, acceptance occurs more often in every state, and the increase in false positives can be smaller than the reduction in false negatives. In addition we show that, while uncertainty about k tends to damage the evaluator,[11] a selected experiment with an uncertain k can be still better than a random experiment.

Related Literature. Concerns about data selection and manipulation have long been voiced by the science and medicine literature and have led to important policy responses.[12] However, there is a dearth of modeling in the area.[13] An early exception is Blackwell and Hodges (1957), who analyze how an evaluator should optimally design a sequential experiment to minimize *selection bias*, a term they coined to represent the fraction of times a strategic researcher is able to correctly forecast the treatment assignment.[14] However, they did not model the information available to the researcher at the assignment stage. Moreover, the ensuing literature focused on exogenous selection bias and on how to adjust for it, rather than on its strategic origin and its impact on the quality of inference. Once we explicitly model information, we characterize situations in which selection actually benefits the evaluator, contrary to what Blackwell and Hodges (1957) stipulate.

Relative to work on optimal persuasion following Rayo and Segal (2010) and Kamenica and Gentzkow (2011), in our setting information acquistion is costly and information manipulation is naturally constrained by the need of reporting a signal selected from the presample. With sample size $n = 1$, our researcher discloses a single observation, as in the limited-attention model first proposed by Fishman and Hagerty (1990).[15] Thus, we have a signal-jamming model of equilibrium

[11] This effect is most transparent in the Gumbel case, as indifference to selection hinges on the evaluator's exact knowledge of the extent of selection k.

[12] See, for example, the analysis of Schulz, Chalmers, Hayes, and Altman (1995) and the CONSORT statement, http://www.consort-statement.org.

[13] Glaeser (2008) discusses a number of issues in this regard. Di Tillio, Ottaviani, and Sørensen (2017) compare different types of selection in the context of an illustrative model with binary noise, which violates the logconcavity assumption maintained in this paper.

[14] Blackwell and Hodges (1957) argue that selection bias is minimized by a truncated binomial design, according to which the initial allocations to treatment and control are selected independently with a fair coin, until half of the subjects are allocated to either treatment or control; from that point on, allocation is deterministic. Efron (1971), instead, characterizes the selection bias resulting from a biased coin design, according to which the probability of current assignment to treatment is higher if previous randomizations resulted in excess balance of controls over treatments.

[15] See also Henry (2009), Dahm, Gonzàlez, and Porteiro (2009), Felgenhauer and Schulte (2014), Hoffmann, Inderst, and Ottaviani (2014) and Herresthal (2017) for persuasion models with endogenous information acquisition. Henry and Ottaviani (2015) analyze a dynamic model of persuasion with costly information acquisition à la Wald (1950),

persuasion through pre-sample collection and then sample selection. The researcher's choice of the size k of the presample is akin to the agent's effort choice in Holmström's (1999) classic career concern model. The twist here is that this effort results in private information, which the researcher then uses to select the reported information.

In a complementary approach to modeling conflicts of interest in statistical testing, Banerjee, Chassang, and Snowberg (2017) propose a theory of a researcher facing an adversarial evaluator who will challenge any prior information. In another complementary approach, Tetenov (2016) analyzes an evaluator's optimal commitment to a decision rule when privately informed researchers select into costly testing. Instead, we focus on the impact of a researcher's manipulation of data on the welfare of an uncommitted evaluator.

2 Statistical Setup

An evaluator chooses an action from a finite set $A = \{a_1, \dots, a_L\} \subseteq \mathbb{R}$ with $a_1 < \cdots < a_L$ under uncertainty about the true value of a state $\theta \in \Theta \subseteq \mathbb{R}$, where Θ is either a finite set or a (possibly unbounded) interval. The evaluator holds a prior belief on the state, represented by a density function λ, and a payoff function $u : \Theta \times A \to \mathbb{R}$ defining a *monotone decision problem*: there exist states $\theta_1 \leqslant \cdots \leqslant \theta_{L-1}$ such that, for every state θ and every $1 \leqslant \ell < L$, the difference $u(\theta, a_{\ell+1}) - u(\theta, a_\ell)$ is nonpositive if $\theta \leqslant \theta_\ell$ and nonnegative if $\theta > \theta_\ell$.

Information and Optimal Decision. Before deciding, the evaluator observes the realization of an *experiment*, an n-dimensional random vector $X = (X_1, \dots, X_n)$ taking values in some domain $D \subseteq \mathbb{R}^n$ and distributed according to a state-dependent density function $g(\cdot|\theta)$ satisfying the monotone likelihood ratio (MLR) property: for any two realizations $x, x' \in D$ with $x' \geqq x$, the ratio $g(x'|\theta)/g(x|\theta)$ is increasing in θ.[16] The MLR property has an important consequence in monotone problems: the evaluator can, without loss, limit attention to *monotone strategies*, where the chosen action increases with the observed realization of X.[17]

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where information is truthfully reported at the time of application.

[16] Here and in the remainder of the paper, given two vectors $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x'_n)$, we say that x' is larger than x, and write $x' \geqq x$, to indicate that $x'_i \geqslant x_i$ for every $i = 1, \dots, n$. Similarly, we say x' is smaller, and write $x' \leqq x$, to indicate that $x'_i \leqslant x_i$ for every $i = 1, \dots, n$.

[17] By Bayes' rule, the MLR property implies that the evaluator's posterior belief $\lambda(\cdot|x)$ increases with the observed realization x in the *likelihood ratio order*, that is, for every x and $x' \geqq x$ the ratio $\lambda(\theta|x')/\lambda(\theta|x)$ is increasing in θ. Thus, by our assumption on payoffs, the evaluator cannot lose by (weakly) increasing the chosen action in response to a higher observed realization of X. For a proof of this claim, see Quah and Strulovici (2009, Theorem 2).

Simple Hypothesis Testing. The most elementary instance of a monotone decision problem—our leading example in much of our discussion below—is a *simple hypothesis testing* problem: there are two states, a low state θ_L and a high state $\theta_H > \theta_L$, and two actions, rejection (a_1) and acceptance ($a_2 > a_1$). Payoffs are such that

$$u(\theta_L, a_2) \leqslant u(\theta_L, a_1) \quad \text{and} \quad u(\theta_H, a_1) \leqslant u(\theta_H, a_2), \tag{1}$$

so the evaluator would like to reject in the low state and accept in the high state. Note that, assuming that the evaluator observes the realization of a one-dimensional ($n = 1$) experiment, the optimal strategy takes the simplest and most familiar form: accept if and only if $x \geqslant \bar{x}$, where $\bar{x}$ is some cutoff point. More generally, with $n \geqslant 1$, monotonicity of the evaluator's decision means that the evaluator optimally accepts if and only if the signal realization lies in an *upper set* in D, i.e. a set $U \subseteq D$ containing every point of D that is larger than some point of U.

Random vs. Selected Experiments. Our main concern in this paper is the welfare comparison between experiments of a particular form: the evaluator observes the n highest of $k \geqslant n$ random variables that are conditionally i.i.d. given the state. Formally, given a family of distribution functions $(F(\cdot|\theta))_{\theta\in\Theta}$ with associated densities $(f(\cdot|\theta))_{\theta\in\Theta}$ satisfying the MLR property, a *selected experiment* is a random vector $X = (X_1, \ldots, X_n)$ where, for each state θ, the random variable X_1 is the highest of $k \geqslant n$ random draws from $F(\cdot|\theta)$, the random variable X_2 the second highest, and so on. Thus, the random vector X takes values in the domain

$$D = \mathbb{R}^n_> := \{x \in \mathbb{R}^n : x_1 \geqslant \cdots \geqslant x_n\},$$

and the density function of X in state θ is given by the following formula:

$$g(x|\theta) = \frac{k!}{(k-n)!} F^{k-n}(x_n|\theta) f(x_1|\theta) \cdots f(x_n|\theta).$$[18]

Note that the cumulative distribution function of X_1 is $F^k(\cdot|\theta)$ and, for every $i = 2, \ldots, n$, the conditional cumulative distribution function of X_i given that $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}$ depends only on the last of these conditions, and is given by

$$\frac{F^{k-i+1}(x_i|\theta)}{F^{k-i+1}(x_{i-1}|\theta)} \qquad (x_i \leqslant x_{i-1}). \tag{2}$$

We call n the *sample size* and k the *presample size* of the selected experiment. When $k = n$, we call the experiment *random*, because it is informationally equivalent to n i.i.d. draws from $F(\cdot|\theta)$.[19]

[18] The MLR property holds for this density because log-supermodularity is preserved by integration (Athey, 2002).

[19] Clearly, knowing in advance that in every state the random variables $X_1, \ldots, X_n$ are sorted so that X_1 is the highest, X_2 the second highest, etc. is of no value for the evaluator.

A particular case that often arises in applications is the case of a *location experiment*, where the evaluator knows the shape of the data distribution, but does not know where the distribution is located. Formally, the distributions in the family $(F(\cdot|\theta))_{\theta\in\Theta}$ are all shifted versions of one and the same cumulative distribution function F admitting a logconcave density function f, that is,

$$F(x|\theta) = F(x-\theta) \qquad \forall\theta\in\Theta,\ \ \forall x\in\mathbb{R}.$$

In this case we call F the *basic noise distribution*, because the random variables $X_1,\dots,X_n$ in a selected location experiment X with sample size n and presample size k have the following form: $X_1 = \theta+\varepsilon_{k:k},\dots,X_n = \theta+\varepsilon_{k-n+1:k}$, where the noise term $\varepsilon_{k:k}$ is the highest of $k\geqslant n$ noise terms $\varepsilon_1,\dots,\varepsilon_k$ randomly drawn from F, the noise term $\varepsilon_{k-1:k}$ the second highest, and so on.

Accuracy. Our comparison of random vs. selected experiments builds on the notion of accuracy, first investigated by Lehmann (1988) for the case of one-dimensional experiments.[20] Given two arbitrary one-dimensional experiments X and Y with respective distributions $F(\cdot|\theta)$ and $G(\cdot|\theta)$, say that Y as *more accurate* than X if the function $\zeta_\theta(x) = (G(\cdot|\theta))^{-1}(F(x|\theta))$ is increasing in θ for every x. This monotonicity criterion is a necessary and sufficient condition for Y to be preferred to X in every monotone decision problem: if Y is more accurate, then the evaluator can induce a state-by-state more favorable distribution over actions than with X. For example, in simple hypothesis testing, as discussed in the introduction and in our illustration of Theorem 1, the cutoff $\overline{y}_L = \zeta_L(\overline{x})$ inducing in Y the same false positives as in X, is smaller than the cutoff $\overline{y}_H = \zeta_H(\overline{x})$ matching the false negatives.[21] This implies that $\overline{y}_L$ induces fewer false negatives in Y than $\overline{x}$ in X, and a fortiori that the evaluator must prefer Y to X. Our analysis of multidimensional experiments in Section 3.2 builds on a natural generalization of accuracy that we will call conditional accuracy.

3 Welfare Impact of Selection

In this section we characterize the families of distributions $(F(\cdot|\theta))_{\theta\in\Theta}$ such that selection has a monotone welfare impact: the larger the presample size, the better (or the worse) the selected experiment. We begin our analysis with the simple case of one-dimensional location experiments.

[20] Accuracy can be defined as Blackwell's (1951,1953) sufficiency, restricted to monotone decision problems involving one-dimensional experiments satisfying the MLR property. This class of problems was first studied by Karlin and Rubin (1956). The term *accuracy* was introduced in Persico (2000) and later adopted by Quah and Strulovici (2009). For applications of the notion of accuracy to economic problems, see also Jewitt (2007).

[21] Here and in the sequel, to ease notation, we write ζ_L and ζ_H instead of the more cumbersome ζ_{θ_L} and ζ_{θ_H}. We adopt analogous notations for other objects as well.

Our characterization is tighter and simpler to illustrate in this case, and it will provide a useful starting point for introducing the characterization in the general case.

3.1 One-Dimensional Selected Location Experiments

A selected location experiment with sample size $n = 1$ and presample size $k \geqslant 1$ has the form $X = \theta + \varepsilon_{k:k}$ where $\varepsilon_{k:k} = \max\{\varepsilon_1, \ldots, \varepsilon_k\}$ and $\varepsilon_1, \ldots, \varepsilon_k$ are random draws from a basic noise distribution F with logconcave density f. Our first main result characterizes the basic noise distributions F for which the evaluator's optimal expected payoff is increasing, and those for which it is decreasing, in the presample size k.

Theorem 1. *Fixing the sample size to $n = 1$, an increase in the presample size makes the location experiment more (resp. less) accurate if and only if the reverse hazard function of the basic noise distribution, $-\log F(\varepsilon)$, is logconcave (resp. logconvex) in ε.*

To gain intuition for the role of logconcavity of the reverse hazard function in our characterization,[22] it is helpful to discuss the result in the context of simple hypothesis testing. Recall again that in this case there are two states, θ_L and $\theta_H > \theta_L$, two actions, rejection (a_1) and acceptance ($a_2 > a_1$), and payoffs satisfy (1). Let $X = \theta + \varepsilon_{k:k}$ and $Y = \theta + \varepsilon_{m:m}$ be selected experiments with sample size $n = 1$ and different presample sizes, k and $m \neq k$, respectively. Let $\bar{x}$ denote any cutoff point that the evaluator may set in experiment X, accepting if and only if $X \geqslant \bar{x}$. Then Y guarantees a larger payoff than X if, for any such cutoff point, there is a corresponding cutoff point $\bar{y}_L$ that, in experiment Y, leads to as many false positives and fewer false negatives, so that

$$F^m(\bar{y}_L - \theta_L) = F^k(\bar{x} - \theta_L) \quad \text{and} \quad F^m(\bar{y}_L - \theta_H) \leqslant F^k(\bar{x} - \theta_H). \tag{3}$$

Solving the equation in (3) for $\bar{y}_L$ and plugging the result into the inequality in (3), we conclude that Y guarantees a higher payoff in every simple hypothesis testing problem—that is, whatever values θ_L, θ_H and $\bar{x}$ may take—if and only if the function $\zeta_\theta(x) = (F^m)^{-1}\big(F^k(x-\theta)\big) + \theta$ is increasing in θ for every x. (We discuss this property below.) Taking the derivative of $\zeta_\theta(x)$ with respect to θ, for every x we must then have

$$mF^{m-1}(\zeta_\theta(x) - \theta) f(\zeta_\theta(x) - \theta) \geqslant kF^{k-1}(x-\theta) f(x-\theta). \tag{4}$$

Equivalently, changing variable from x to $u = F^k(x-\theta)$, it must be the case that for all $u \in [0,1]$ the slope of F^m computed at the quantile $\zeta_\theta(x) - \theta = (F^m)^{-1}(u)$ is greater than the slope of F^k computed at the corresponding quantile $x - \theta = (F^k)^{-1}(u)$.

[22] Marshall and Olkin (2007) define the hazard function as $\log F$. Since F ranges between zero and one, $\log F$ is necessarily negative. Our definition uses a minus sign, so that logconcavity of the function makes sense.

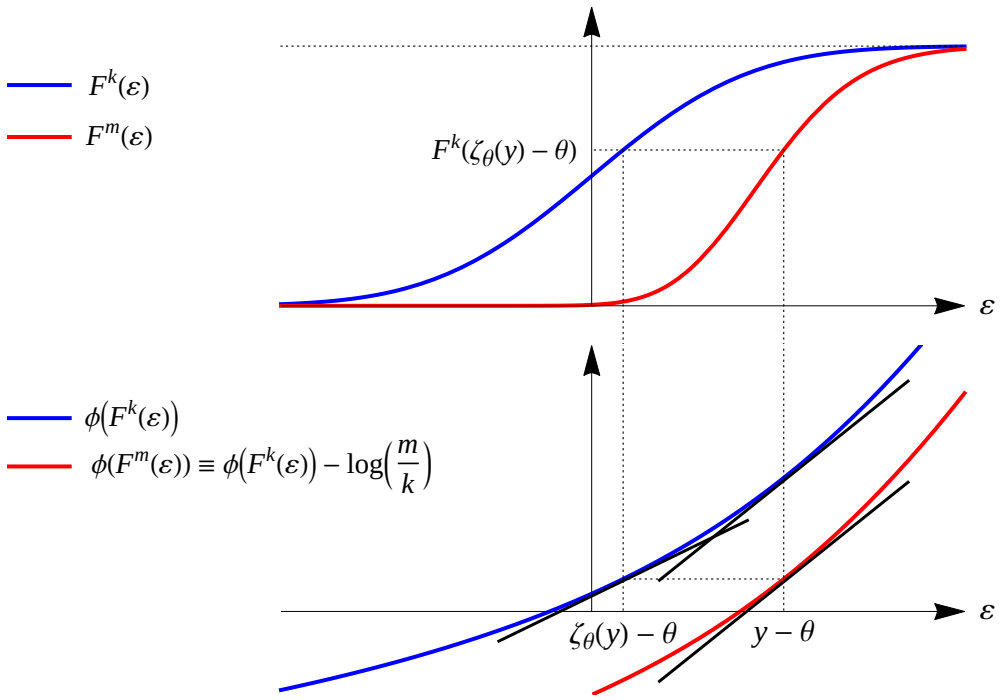


Figure 2: Normal one-dimensional location experiment: double-log transformation.

Under what conditions on F, m and k is condition (4) true for every y? To answer this question, it is convenient to first transform F^k and F^m in such a way that the transformed functions are parallel vertical shifts of each other. The suitable transformation is the strictly increasing function $u \mapsto \phi(u) = -\log(-\log u)$, because then $\phi(F^m(\cdot)) = \phi(F^k(\cdot)) - \log(m/k)$. The transformation is illustrated in Figure 2 for the case of a standard normal F and $m > k$.[23] Since our double-log transformation is strictly increasing, property (4) is equivalent to the slope of $\phi(F^k(\cdot))$ at $x-\theta$ being less than the slope of $\phi(F^m(\cdot))$ at $\zeta_\theta(x)-\theta$. Equivalently, the slope of $\phi(F^k(\cdot))$ itself is greater at $\zeta_\theta(x)-\theta$ than at $x-\theta$. Since $\phi(F^k(\cdot)) = \phi(F(\cdot)) - \log k$, this is satisfied when either $\phi(F(\cdot))$ is convex and $m > k$ (because $m > k$ implies $\zeta_\theta(x) \geqslant x$), or $\phi(F(\cdot))$ is concave and $m < k$ (because $m < k$ implies $\zeta_\theta(x) \leqslant x$). Thus, we conclude that a larger presample size benefits the evaluator when the reverse hazard function $-\log F$ is logconcave (as in the normal case illustrated in Figure 2) and hurts the evaluator when $-\log F$ is logconvex.

Dispersion and Accuracy. When X and Y are location experiments, so that $F(x|\theta) = F(x-\theta)$ and $G(y|\theta) = G(x-\theta)$ for some noise distributions F and G, we can also equivalently apply the notion of dispersion, instead of appealing to accuracy. Bickel and Lehmann (1979) define G as *less dispersed* than F if the quantile difference $G^{-1}(u) - F^{-1}(u)$ is decreasing in $u \in [0,1]$.

[23] The plots are drawn for $k = 1$ and $m = 8$, but any $k \geqslant 1$ and $m > k$ give the same qualitative result.

This notion also appeared in our discussion (and is, in fact, used in the proof) of Theorem 1, when we asked whether the slope of F^m computed at $\zeta_\theta(x) - \theta = (F^m)^{-1}(u)$ is greater than the slope of F^k computed at $x - \theta = (F^k)^{-1}(u)$. Accuracy and dispersion are equivalent for location experiments. A location experiment is more accurate if and only if the noise distribution is less dispersed (Lehmann, 1988, Theorem 5.2).

Gumbel Distribution. There is only one distribution F such that $-\log F$ is both logconcave and logconvex, i.e. loglinear, namely the Gumbel (maximum) extreme value distribution, $F(\varepsilon) = \exp(-\exp(-\varepsilon))$. This distribution, which plays a special role in the ensuing analysis, is such that for every k the selected experiment with presample size k is neither less nor more accurate than a random experiment—the evaluator is indifferent to selection. The following intuitive argument also leads to the same conclusion. With a presample size equal to k, the noise distribution is $F^k(\varepsilon) = \exp(-k\exp(-\varepsilon)) = F(\varepsilon - \log k)$. Thus, compared to a random sample, selection inflates noise by a constant, $\log k$. The evaluator adjusts for this inflation, and is back to square one.

Logistic and (Generalized) Exponential Distributions. Besides the normal case discussed earlier, another instance where more selection is better for the evaluator is the logistic case, $F(\varepsilon) = 1/(1+e^{-\varepsilon})$. (We prove this and the following claim below.) Our main example of the opposite case, where more selection hurts the evaluator, is the exponential distribution $F(\varepsilon) = 1 - e^{-\varepsilon}$ (for $\varepsilon \geqslant 0$). More generally, given any $a < -1$, the distribution F such that

$$F(\varepsilon) = \exp\left(\frac{1}{1+a}\left[(1-\exp(-\varepsilon))^{1+a} - 1\right]\right) \qquad (\varepsilon > 0) \tag{5}$$

is such that $-\log F$ is logconvex. (The exponential distribution is the special case $a \to -1$.)

Contribution to Stochastic Ordering of Order Statistics. Previous results in the literature on stochastic ordering of order statistics only covered basic noise distributions with decreasing hazard rate. Notably, Khaledi and Kochar (2000, Theorem 2.1) showed that for any distribution with decreasing hazard rate higher order statistics are less accurate.[24] Given that logconcavity implies increasing hazard rate by Prekopa's theorem, the only basic noise distribution with logconcave density for which Khaledi and Kochar's (2000) result applies is the exponential (loglinear) distribution, which has constant hazard rate.[25] The novel characterization in Theorem 1 applies more generally to the relevant case of basic noise distributions with logconcave densities.

[24] According to Khaledi and Kochar (2000, Theorem 2.1), if X_i's are i.i.d. with decreasing hazard rate, then $X_{i:n}$ is more accurate than $X_{j:m}$ whenever $i \leqslant j$ and $n - i \geqslant m - j$. Setting $i = n = 1$ and $j = m = k$, we have that the maximum of k i.i.d. variables with decreasing hazard rate is less accurate than the original variable.

[25] Theorem 1 also covers distributions with decreasing hazard rate, where $-\log(-\log F))$ is necessarily concave.

Real Presample Size. According to our definition, the presample size is a natural number k, but the interpretation—as well as the statement in the theorem—for real numbers $k > 1$ is equally valid. Increasing selection from k to $m > k$ changes the noise distribution from F^k to $F^m = (F^k)^{m/k}$. This is akin to having basic noise distribution F^k and a fractional presample size $m/k > 1$. Our comparative statics result in Theorem 1 characterizes when this increases accuracy. Note the implication that the basic noise distribution F is such that selection monotonically benefits (or hurts) the evaluator if and only if the basic noise distribution F^k has the same property for every real number $k > 1$. Indeed, both properties are equivalent to logconcavity of $-\log F$.

3.2 General Selected Experiments

We now turn to the comparison between selected experiments from a (not necessarily location type) family of distributions $(F(\cdot|\theta))_{\theta\in\Theta}$ with any sample size $n \geqslant 1$. As we shall see, the results obtained in the context of one-dimensional location experiments for the common distributions discussed earlier (normal, Gumbel, exponential, etc.) carry over to arbitrary sample sizes. However, the extension of the characterization in Theorem 1 to the multidimensional case is far from immediate, and poses some important challenges.

Conditional Accuracy. In order to deal with multidimensional experiments, we develop a generalization of Lehmann's (1988) notion of accuracy, which allows the comparison between any two experiments (not necessarily selected experiments). Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be experiments with respective domains D_X and D_Y. For every state θ, we define a function $\zeta_\theta : D_X \to D_Y$ such that in state θ the random vector $\zeta_\theta(X)$ has the same distribution as Y, as follows: $\zeta_\theta(x_1, \ldots, x_n) = (z_1, \ldots, z_n)$, where $z_1, \ldots, z_n$ are defined recursively by the equations below (for brevity, we write $< i$ for the indices $1, \ldots, i-1$):

$$\Pr_\theta(X_1 \leqslant x_1) = \Pr_\theta(Y_1 \leqslant z_1) \quad \text{and} \quad \Pr_\theta(X_i \leqslant x_i | X_{<i} = x_{<i}) = \Pr_\theta(Y_i \leqslant z_i | Y_{<i} = z_{<i}). \tag{6}$$

Then we say that Y is *conditionally more accurate* than X if $\zeta_\theta(x)$ is an increasing function of θ for every $x \in D_X$. Note that if $n = 1$, the function ζ_θ is defined by the first equation in (6). In this case, our definition reduces to Lehmann's (1988).

In what sense is a conditionally more accurate experiment better for the evaluator? To gain intuition, consider a simple hypothesis testing problem. In experiment X the evaluator optimally accepts if and only if $X \in U$, where U is some upper set. Now suppose that in Y the evaluator adopts the following, possibly suboptimal strategy: accept if and only if $Y \in \zeta_L(U)$. How does this strategy fare? In the low state θ_L it induces as many false positives, and hence the same payoff, for $\Pr_L(Y \in \zeta_L(U)) = \Pr_L(X \in U)$ by the very definition of ζ_L. In the high state θ_H it leads to

fewer false negatives, because $\Pr_H(Y \in \zeta_L(U)) \geqslant \Pr_H(Y \in \zeta_H(U)) = \Pr_H(X \in U)$, where the inequality follows Y being conditionally more accurate than X, given that U is an upper set, and the equality from the definition of ζ_H. But if a possibly suboptimal strategy for Y guarantees, in each state, at least as much payoff as the optimal strategy for X, a fortiori choosing optimally in Y must give at least as much expected payoff.

The analogous argument applies more generally to any monotone decision problem and any pair of n-dimensional experiments X and Y, as we show in the proof of the following:

Proposition 1. *Let X and Y be two n-dimensional experiments. If Y is conditionally more accurate than X, then for every prior λ the optimal expected payoff in experiment Y is greater than or equal to the optimal expected payoff in experiment X. If $n = 1$, then the converse also holds.*

We provide further intuition and discussion on our notion of conditional accuracy in the course of illustrating our second and more important main result, which we are now ready to state.

Comparing Multidimensional Selected Experiments. In a selected experiment with sample size n and presample size k, the evaluator observes the highest, second highest, ..., nth highest of k random draws from $F(\cdot|\theta)$. Toward a generalization of our analysis beyond one-dimensional experiments, one may therefore try to grasp intuition from the individual comparisons of each intermediate order statistic with a random draw. However, following this approach would be misleading, for two reasons. First, even in those cases where the n highest of $k > n$ realizations *are* conditionally more accurate than, say, n random draws, an intermediate order statistic (say, the second highest, or third highest, etc.) is, in isolation, generally *not* more accurate than a random draw.[26] Second, as is evident from (2), the n highest realizations of $k \geqslant n$ random draws are correlated among themselves, even conditionally on the state. Intuitively, correlation within the observed vector of observations tends to reduce information,[27] which creates further ambiguity about the sign of the net marginal value of information added by an intermediate order statistic.

[26] Consider, for example, a location experiment where the basic noise distribution is the *positive* exponential distribution: $F(x|\theta) = F(x-\theta) = \exp(x-\theta)$ for $x \leqslant \theta$. Let $X_1, \ldots, X_k$ be i.i.d. draws from $F(x|\theta)$, and let Y_1 and Y_2 be the highest and the second highest of these draws. The cumulative distribution functions of Y_1 and Y_2 are given by $\Pr_\theta(Y_1 \leqslant y) = \exp(n(y-\theta))$ and $\Pr_\theta(Y_2 \leqslant y) = n\exp((n-1)(y-\theta)) - (n-1)\exp(n(y-\theta))$, respectively. Thus, the function ζ_θ such that $\zeta_\theta(X_1)$ has the same distribution as Y_1 is $\zeta_\theta(x) = (x-\theta)/n + \theta$, which increases with θ. Indeed, by Theorem 1, experiment Y_1 is more accurate than X_1, because $\log(-\log F(\varepsilon)) = \log(-\varepsilon + \theta)$ is (strictly) concave in ε. Now consider the function ζ'_θ such that $\zeta'_\theta(X_2)$ has the same distribution as Y_2. This function is U-shaped in θ, because its reciprocal, $\log(n\exp((n-1)(y-\theta)) - (n-1)\exp(n(y-\theta))) + \theta$, is a bell-shaped function of θ. Thus, Y_2 is neither less nor more accurate than X_2. Yet, the basic noise distribution $F(\cdot) = \exp(\cdot)$ satisfies the condition of Corollary 1 below, which implies that the experiment (Y_1, Y_2) is conditionally more accurate than (X_1, X_2).

[27] The statistical literature on the comparison of multidimensional experiments with correlated observations is rather small. Shaked and Tong (1990, 1993) provide some results which, however, do not apply in our context.

Our notion of conditional accuracy provides a key tool to shed light on the problem, and unveil the crucial property of the family of distributions $(F(\cdot|\theta))_{\theta\in\Theta}$ which guarantees that the correlation structure of the entire vector of selected observations adds or subtracts value to the evaluator's problem as the presample size increases. This is the contribution of our second main result.

Theorem 2. *Between two selected experiments with the same sample size, the evaluator prefers the experiment with the larger (resp. smaller) presample size if the reverse hazard rate $f(\cdot|\theta)/F(\cdot|\theta)$ is log-supermodular (resp. log-submodular, with support of $f(\cdot|\theta)$ unbounded above for every θ), that is, if for all states θ and $\theta' > \theta$ the reverse hazard rate ratio*

$$\frac{f(\cdot|\theta')/F(\cdot|\theta')}{f(\cdot|\theta)/F(\cdot|\theta)}$$

is increasing (resp. decreasing).

Before illustrating the result, let us remark that, for location experiments, log-supermodularity of the reverse hazard rate is the same as logconcavity of the reverse hazard rate of the basic noise distribution. Thus, we obtain the following:

Corollary 1. *Between two selected location experiments with the same sample size, the evaluator prefers the experiment with the larger (resp. smaller) presample size if the basic noise distribution's reverse hazard rate, $f(\varepsilon)/F(\varepsilon)$, is logconcave (resp. logconvex, with support of f unbounded above) in ε.*

To reconcile this corollary with Theorem 1, observe that the reverse hazard function is the right-sided integral of the reverse hazard rate: $\int_\varepsilon^\infty (f(\varepsilon)/F(\varepsilon))d\varepsilon = -\log F(\varepsilon)$. Thus, the reverse hazard function inherits logconcavity (and logconvexity, if the support of f is unbounded above) of the reverse hazard rate (An, 1998, Lemma 3). Indeed, in all examples of location experiments listed in the previous section, Corollary 1 applies, with the Gumbel distribution again sitting at the boundary between the basic noise distributions for which more selection benefits, and those for which less selection does. In the normal case, the reciprocal of the reverse hazard rate, $F(\varepsilon)/f(\varepsilon) = \int_{-\infty}^{x} e^{\varepsilon^2/2}e^{-t^2/2}dt = \int_{-\infty}^{0} e^{-u^2/2}e^{-u\varepsilon}du$, is log-convex because $e^{-u\varepsilon}$ is log-convex (actually log-linear) and log-convexity is preserved under mixtures (An, 1998, Proposition 3).[28] Thus, the reverse hazard rate is logconcave. In the logistic case, the reverse hazard rate is $f(\varepsilon)/F(\varepsilon) = 1/(e^\varepsilon + 1)$, which is easily seen to be logconcave. In the Gumbel case, $f(\varepsilon)/F(\varepsilon) = \exp(-\varepsilon)$, a loglinear function. Finally, in the generalized exponential case, $f(\varepsilon)/F(\varepsilon) = (1-e^{-\varepsilon})^a e^{-\varepsilon}$, which is easily seen to be logconvex (as $a < -1$).

[28] Balakrishnan, Burkschat, Cramer, and Hofmann (2008) use the same argument to prove, in their Lemma A.2, that the hazard rate $f/(1-F)$ is logconcave in the normal case.

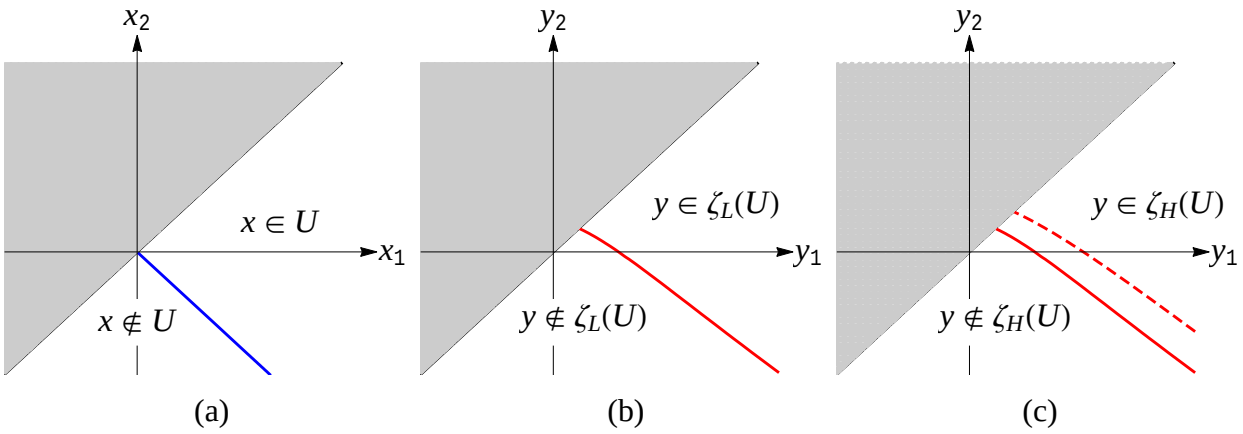


Figure 3: Normal location experiment: increasing presample size increases accuracy.

Let us now illustrate the idea behind Theorem 2, focusing again on simple hypothesis testing problems: two states, θ_L and $\theta_H > \theta_L$, two actions, rejection (a_1) and acceptance (a_2), and payoffs satisfying (1). For simplicity, assume a sample size $n = 2$. Let X be a selected experiment with presample size $k \geqslant 2$ from a distribution in some family $(F(\cdot|\theta))_{\theta \in \{\theta_L, \theta_H\}}$. In this experiment the evaluator optimally accepts after observing a realization $x \in U$, and rejects otherwise, where the acceptance region U is some upper set in $\mathbb{R}^2_>$. This is illustrated in Figure 3(a), where we assume that X is a random experiment ($k = 2$) from a normal location family—for every state θ, the distribution $F(\cdot|\theta)$ is normal with mean θ and variance one. The blue curve partitions the domain of X (the unshaded area in the diagram, $\mathbb{R}^2_>$) into the acceptance region U, the area above the curve, and the rejection region, the area below the curve.[29]

Consider now another selected experiment, Y, with presample size $m \neq k$. Following the same logic used for the one-dimensional case, in order to argue that Y gives a higher payoff than X we must define an acceptance region $V \subseteq \mathbb{R}^2_>$ for experiment Y, leading to as many false positives and fewer false negatives, that is,

$$\Pr_L(Y \in V) = \Pr_L(X \in U) \quad \text{and} \quad \Pr_H(Y \in V) \geqslant \Pr_H(X \in U). \tag{7}$$

But, unlike the one-dimensional case, where V is uniquely determined by U,[30] there are now many ways to define an upper set V satisfying the equality in (7). Moreover, the inequality in (7) may hold for some choices of V and not for others. How should we define the set V, then?

Our definition of accuracy proves crucial to answering this question. Recalling (2) and (6), for

[29] The blue curve is, in fact, a straight line. As is well known, the optimal strategy in a location experiment with i.i.d. normal observations (recall that we are assuming $k = n$ here) only depends on the average observed value, because this is a sufficient statistic for the whole vector of observations.

[30] Recall, going back to our illustration of Theorem 1, that equation (3) uniquely defines the cutoff point $\bar{y}$ from $\bar{x}$.

any state θ the function ζ_θ such that $\zeta_\theta(X)$ and Y have the same distribution (in state θ) is defined by $\zeta_\theta(x_1,x_2)=(z_1,z_2)$, where

$$F^k(x_1|\theta)=F^m(z_1|\theta) \quad \text{and} \quad \frac{F^{k-1}(x_2|\theta)}{F^{k-1}(x_1|\theta)}=\frac{F^{m-1}(z_2|\theta)}{F^{m-1}(z_1|\theta)}. \tag{8}$$

Letting $V=\zeta_L(U)$, as illustrated in Figure 3(b), the equality in (7) holds by construction. Moreover, since U is an upper set, checking the inequality in (7) simply requires checking that for every x we have $z:=\zeta_L(x)\leqslant\zeta_H(x)$ or, by (8),

$$\frac{F^m(z_1|\theta_H)}{F^k(x_1|\theta_H)}\leqslant 1 \qquad \text{and} \qquad \frac{F^{m-1}(z_2|\theta_H)/F^{m-1}(z_1|\theta_H)}{F^{k-1}(x_2|\theta_H)/F^{k-1}(x_1|\theta_H)}\leqslant 1.$$

At this point, log-supermodularity and log-submodularity enter the picture. Since both of the above inequalities hold in the limit as x_1 tends to the upper bound of the support of $F^k(\cdot|\theta_L)$ and x_2 tends to its largest possible value (namely x_1), they both hold if their left-hand sides are increasing functions of x_1 and x_2, respectively. As we show in the proof, taking derivatives and simplifying, this simply means that, for each $i=1,2$,

$$\frac{f(z_i|\theta_H)/F(z_i|\theta_H)}{f(z_i|\theta_L)/F(z_i|\theta_L)}\geqslant\frac{f(x_i|\theta_H)/F(x_i|\theta_H)}{f(x_i|\theta_L)/F(x_i|\theta_L)}, \tag{9}$$

revealing that Y is more accurate than X when either the reverse hazard rate is log-supermodular and $m\geqslant k$ (as then $z_i\geqslant x_i$), or the reverse hazard rate is log-submodular and $m\leqslant k$ (as then $z_i\leqslant x_i$).

As we argued while discussing Corollary 1, the inequality $\zeta_H(\cdot)\geqslant\zeta_L(\cdot)$ holds in the normal case depicted in Figure 3. In panel (c) of that figure, the dashed red curve defines the set $\zeta_H(U)$, the region above the curve. In state θ_H the probability of Y falling above the dashed curve equals the probability that X falls in U. The solid curve, which is the same as in panel (b), lies below the dashed curve, so the probability that Y falls above the solid curve is larger—in other words, the inequality in (7) is satisfied.

4 Applications

Selection bias is an important concern in observational studies, as well as in the practice of controlled experiments.[31] The received statistical and econometric literature on regression and treatment effects is typically concerned with identification issues—finding ways to avoid or at least

[31] See e.g. Schulz (1995) and Berger (2005) for extensive accounts and examples of subversion of randomization in clinical trials.

account for bias. Our analysis offers insights from a complementary angle: for a fixed sample size, does selection provide more or less precise information about the phenomenon of interest? In this section we discuss the implications of our results in three typical applied scenarios.

4.1 Subversion of Randomization in Randomized Controlled Trials

Following Neyman (1923) and Rubin (1974, 1978), consider a population of individuals and two alternative treatments—a default, known treatment 1 and a new treatment 2 whose benefit beyond the default is unknown. Let $X_{t,i}$ denote the potential outcome of individual i when receiving treatment $t \in \{1,2\}$. For simplicity, assume for now that the unknown treatment effect $X_{2,i} - X_{1,i}$ on individual i is the same for every individual i. Thus, the potential outcomes of individual i are

$$X_{1,i} = \varepsilon_i \quad \text{and} \quad X_{2,i} = \theta + \varepsilon_i$$

where the treatment effect θ is only known to belong to a subset $\Theta \subseteq \mathbb{R}$, and ε_i is drawn from a known distribution F with logconcave density f. The evaluator would like to approve the new treatment (action a_2) if the treatment effect exceeds some specific value, and stick with the traditional treatment (action a_1) otherwise.

Enter a researcher, who runs a randomized controlled trial with n treated individuals $i_1, \ldots, i_n$ and as many untreated individuals $i_{n+1}, \ldots, i_{2n}$. The evaluator then observes the following table of experimental results:

Treatment Group	Control Group
$X_{2,i_1} = \theta + \varepsilon_{i_1}$	$X_{1,i_{n+1}} = \varepsilon_{i_{n+1}}$
$\vdots$	$\vdots$
$X_{2,i_n} = \theta + \varepsilon_{i_n}$	$X_{1,i_{2n}} = \varepsilon_{i_{2n}}$

We are interested in comparing two scenarios. In the first scenario, the researcher picks $2n$ individuals at random, and randomly assigns n individuals to each treatment. In this case the control group adds no valuable information, because the distribution of outcomes under the first treatment, namely F, is known. Thus, the experiment boils down to the observation of the treatment group only—in the language used in this paper, the random experiment $X = (X_1, \ldots, X_n)$ where $X_1 = \theta + \varepsilon_{n:n}, \ldots, X_n = \theta + \varepsilon_{1:n}$.

In the second scenario, the researcher has information on the outcome of the first treatment (more generally, on characteristics correlated with this outcome) for $k > 2n$ individuals, and on this basis (i) selects $2n$ individuals for the experiment, and (ii) assigns n individuals to each treatment.

Out of the k presampled individuals, the researcher assigns the n individuals with the highest value of X_1 to the treatment group, and the n individuals with the lowest value of X_1 to the control group. Thus, the evaluator observes the following experiment:

$$\begin{array}{cc} \theta + \varepsilon_{k:k} & \varepsilon_{n:k} \\ \vdots & \vdots \\ \theta + \varepsilon_{k-n+1:k} & \varepsilon_{1:k} \end{array}$$

How do the two scenarios compare? Clearly, the control group can only add information—and in the second scenario it does, because the random vectors $\varepsilon_{1:k}, \ldots, \varepsilon_{n:k}$ and $\varepsilon_{k-n+1:k}, \ldots, \varepsilon_{k:k}$ are correlated. Thus, the experiment must be at least as accurate as the experiment consisting of the vector of observations in the treatment group—in our language, the random experiment $Y = (Y_1, \ldots, Y_n)$ where $Y_1 = \theta + \varepsilon_{k:k}$, …, $Y_n = \theta + \varepsilon_{k-n+1:k}$. By Corollary 1, we can therefore conclude that the evaluator is better off in the second scenario, provided that the reverse hazard rate f/F is logconcave.

4.2 Sample Selectivity in Regression

The estimation of a regression parameter is another prominent example of a monotone decision problem. Here, the set of actions A coincides with the state space Θ, and the evaluator would like to choose an action that is as close as possible to the true state.

Consider a linear regression setup $Y_i = \alpha + \theta X_i + \gamma Z_i + \varepsilon_i$ where X_i is a nonnegative treatment variable, Z_i a vector of covariates, and θ the unknown parameter of interest.[32] We assume that any sample of individuals must be representative of the population, in the sense that the distribution of X and Z in the sample is the same as in the population. More precisely, we assume that X and Z define S different strata—so that, across all individuals i in each stratum s, the values of X_i and Z_i are the same—and that any sample of n individuals must contain exactly n_s individuals from each stratum s, with $n_1 + \cdots + n_S = n$.

Our question here is whether a more accurate estimate of θ obtains with random sampling, where the noise terms $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. draws from some basic noise distribution F, or with the following form of selection: for each stratum s, the n_s individuals in the sample are selected as

[32] For simplicity, assume that the other two parameters, α and γ, are known. Our arguments can be easily extended to the case where α is unknown, using Proposition 2 below. The case where the parameter γ is also unknown is more complicated, but we conjecture that our arguments again apply.

those with the highest value of ε, among $k_s \geqslant n_s$ randomly drawn individuals from stratum s. Note that we allow stratum-specific presample sizes, reflecting the possibility that selection may be more or less severe depending on the values of X and Z.

Here, too, Corollary 1 provides the answer: the estimate of θ is more accurate in the selected scenario when f/F is logconcave, and less accurate when f/F is logconvex. To see this, consider first each stratum separately. For each individual i in the stratum, the evaluator observes

$$\tilde{Y}_i = \theta + \tilde{\varepsilon}_i,$$

where $\tilde{\varepsilon}_i = \alpha + \gamma Z_i/X_i + \varepsilon_i/X_i$. Since X_i and Z_i are constant within strata, in both the random and the selected case the noise terms $\tilde{\varepsilon}_i$ are drawn from the same basic noise distribution. Moreover, this distribution inherits logconcavity (or logconvexity) of the reverse hazard rate from the basic noise distribution of ε_i, because $\tilde{\varepsilon}_i$ is an affine transformation of ε_i (An, 1998, Corollary 2).

The above argument proves our claim for the case of a single stratum. The result for an arbitrary number S of strata obtains immediately from the fact that, conditional on θ, observations in different strata are independent.

Truncated Regression. It is instructive to contrast our findings with the common applied scenario in which observations on Y, X and Z are only available for individuals whose value of Y exceeds a certain (possibly stratum-specific) threshold—the case of *truncated regression*. In this scenario, for each stratum s the evaluator observes n_s i.i.d. draws from the truncated distribution $\tilde{Y}|\tilde{Y} \geqslant \bar{y}_s$, where $\bar{y}_s$ denotes the left-truncation point for individuals in stratum s. As we have emphasized earlier, in some important cases (e.g. with a normal basic noise distribution) this type of selection hurts the evaluator, even under the assumption that the truncation points $\bar{y}_s$ are known, while selection based on order statistics is beneficial. This striking contrast begs the question of what makes certain types of selection beneficial and others harmful. While providing a full answer to this question is beyond the scope of this paper, our results provide a contribution in that direction, and our notion of conditional accuracy can help in systematically addressing the question—both inside and outside regression contexts.

4.3 Selective Regression Specification

At least since Griliches (1957), economic researchers have sought to address a particular source of bias arising from missing variables. We illustrate here how our analysis can be applied to the situation where the omitted variable has been strategically selected to maximize the bias.[33] Before

[33] We provide an explicit game-theoretic foundation for this model in the next section.

doing so, we first need to develop a useful add-on to Theorem 1.

Correlated Draws. Our analysis of one-dimensional selected location experiments, which has so far focused on selection among conditionally independent draws, carries over to a relevant case of conditionally correlated draws. In this case, the k presample observations are subject to an identically distributed noise component. More precisely, the k presample observations have the form $\theta + \delta_i + \varepsilon_i$, where the vectors $(\delta_1, \ldots, \delta_k)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$ are mutually independent, and independent of θ. As before, $\varepsilon_1, \ldots, \varepsilon_k$ are i.i.d. draws from a basic noise distribution F with a logconcave density. The additional noise components $\delta_1, \ldots, \delta_k$ are identically (but not necessarily independently) distributed with a logconcave density. Note that $\delta_i + \varepsilon_i$ has a logconcave density, as recorded in An (1998, Corollary 1). Selection takes place on the basic noise terms $\varepsilon_1, \ldots, \varepsilon_k$. That is, we assume that the evaluator observes $\theta + \delta_{i^*} + \varepsilon_{i^*}$, where $i^* = \arg\max\{\varepsilon_1, \ldots, \varepsilon_k\}$, so that the correlation in the draws is introduced after selection.

The key result here is that the added noise components are without consequence for our main question. Whether the evaluator benefits from selection still turns on whether the basic noise distribution's reverse hazard function $-\log F$ is logconcave or logconvex.

Proposition 2. *The selected experiment $\theta + \delta_{i^*} + \varepsilon_{i^*}$ is more (resp. less) accurate than the random experiment $\theta + \delta_i + \varepsilon_i$ if the selected experiment $\theta + \varepsilon_i^*$ is more (resp. less) accurate than the random experiment $\theta + \varepsilon_i$.*

Strategic Variable Omission. Consider a researcher who privately collects a data set (x_i, y_i, z_i) for $i = 1, \ldots, n$. Here, x_i is a treatment variable, y_i an outcome variable, and z_i a vector of k covariates. The statistical relationship among these variable is captured by the linear model

$$y_i = \theta x_i + z_{1i} + \cdots + z_{ki} + v_i, \tag{10}$$

where v_i is a random noise term. We assume that the marginal effects of covariates are all known, say because the covariates are familiar. Covariates have been scaled so these effects are identical.

An evaluator is concerned that treatment need not be random. The evaluator correctly understands that a regression of z_j on x yields the population relationship

$$z_{ji} = \varepsilon_j x_i + u_{ji}, \tag{11}$$

where u_{ji} is noise. However, the evaluator does not know the realized regression coefficients $\varepsilon_1, \ldots, \varepsilon_k$.

Assume that $\theta, \varepsilon, u, v, x$ are independent stochastic variables, satisfying $E\left[u_{ji}\right] = E\left[v_i\right] = 0$. Moreover, $\varepsilon_1, \ldots, \varepsilon_k$ are i.i.d. draws from a distribution F admitting a logconcave density. Also,

the vectors $u_1, \ldots u_k$ are identically distributed. Furthermore, assume that one of the two following assumptions applies. (1) The treatment variable x is known and non-stochastic. Noise terms u_{ji} and v_i have logconcave densities. (2) The treatment variable x is stochastic, unknown to the evaluator. The term $\left(x^T x\right)^{-1} x^T \left(u_j + v\right)$ has a logconcave density.

One of the control variables is omitted when the researcher uses the realizations to compute an estimate of θ. We compare the case where this omission is of a random control variable to the case where the omitted variable is selected to maximize the estimate. Selection is performed when the researcher knows $\varepsilon_1, \ldots, \varepsilon_k$, but before obtaining the actual data set (x, y, z) for the estimation of θ. The bias thus results from the researcher's prior analysis of covariates, or from a non-random assignment procedure.

Proposition 3. *Omission of variable j results in estimate $\theta + \varepsilon_j + \delta_j$, where*

$$\delta_j = \left(x^T x\right)^{-1} x^T \left(u_j + v\right).$$

This proposition reduces the problem to our earlier analysis. Our technical assumptions mean that we can apply Proposition 2 in order to characterize whether selected omission benefits or harms the evaluator.[34] The answer turns on whether $-\log F$ is logconcave or logconvex.

5 Strategic Selection

Sample selection of the sort considered above naturally arises as an equilibrium phenomenon in a strategic setting where the experiment is carried out by a researcher who desires the evaluator to take higher actions. Taking the researcher's payoff into account, allows us to discuss the non-trivial impact on the researcher's own welfare when sample selection becomes easier. Initially, we exogenously fix both size n of the reported sample and size k of the researcher's presample. Subsequently, we endogenize these features, while also imposing more restrictive assumptions on signals and payoffs in order to facilitate the analysis.

5.1 Selective Sampling Game

In many settings, size n of the reported sample may be determined by forces outside our model. For instance, costly or limited attention from the evaluator may cap the number of items that the

[34] We have directly assumed that ε_j are i.i.d. with a logconcave density, independent of vector δ. We have also imposed two alternative assumptions to guarantee that the terms δ_j are identically distributed with logconcave density.

evaluator is willing to inspect. In some areas, a sample size may be guided by standards unrelated to the specific problem. Similarly, natural constraints may endow the researcher with a presample of fixed and known size $k \geqslant n$.

Consider the following timeline:

Stage 1. Researcher privately observes presample $(x_1, \ldots, x_k)$ and then chooses a subset $\mathscr{I} \subseteq \{1, \ldots, k\}$ of size n.

Stage 2. The evaluator observes the vector $(x_i)_{i \in \mathscr{I}}$ and then chooses an action in A.

Payoffs to the evaluator are as specified before. The researcher's payoff is $v(a)$ where v is a strictly increasing function from the ordered set A to the reals.

Proposition 4. *There exists a Bayes Nash equilibrium where the researcher always selects the n greatest realizations from presample* $(x_1, \ldots, x_k)$. *The evaluator's map from data to actions is as analyzed before.*

We use the Bayes Nash equilibrium concept since the researcher has private information. On the other hand, with the proposed strategy, no sample $(x_1, \ldots, x_n)$ is outside the support, so there is no reason to discuss any refinement of off-path beliefs.

Note that the evaluator's strategy is precisely the one we have analyzed until now. The researcher's strategy is a best response because it generates the highest possible realization of sample vector $(x_1, \ldots, x_n)$, and the evaluator then responds with no smaller action. It is technically possible to construct examples where other equilibria exist. These other equilibria are more intricate, and perhaps less natural in light of the researcher's goal to maximize the evaluator's response, so we choose to disregard them for now. When we later extend the model to take into account that the evaluator can be uncertain about k, we are forced to reconsider this in some cases where maximal selection is no longer an equilibrium.

If the researcher could somehow ex post choose to reveal the entire presample, there are cases when the researcher would wish to do so. Generally, these cases arise when the hidden part of the sample has a favorable realization, beating the posterior odds assumed by the evaluator. This unraveling effect is well known in the literature on strategic disclosure. Our assumption is that the evaluator can ex ante commit to study precisely n observations before making a decision, preventing such unraveling. We will endogenize n in the next subsection.

The result is robust to a natural modification of the game for the case of location problems, where $x_i = \theta + \varepsilon_i$. We can modify the first stage of the game by allowing the researcher to observe

the noise terms $(\varepsilon_1,\ldots,\varepsilon_k)$ rather than the outcomes. Of course, for any realization of the state, maximal selection in $(\varepsilon_1,\ldots,\varepsilon_k)$ is equivalent to maximal selection in $(x_1,\ldots,x_k)$. The result is therefore unchanged. The interpretation of this setup corresponds e.g. to subversion of randomization in randomized controlled trials, with known baseline.

Equilibrium Impact of Selection on Researcher's Welfare. Holding fixed the sample size n, does more selection improve the researcher's situation? If the evaluator were unaware of an increase in presample size k, a direct effect would benefit the researcher. Increasing k would directly raise the maximally selected sample in the sense of first-order stochastic dominance. Holding fixed the evaluator's response, higher actions would result, to the benefit of the researcher.

We are more interested in the equilibrium effect. When both the evaluator and the researcher know that selection is easier—that is, k is larger—is the researcher necessarily better off? Does this depend on whether the evaluator is better off?

We can immediately note that the researcher is entirely indifferent to k in the special case of a location experiment with Gumbel basic noise distribution. As we know, in this case all selected experiments (with the same sample size) are equally conditionally accurate. By definition of conditional accuracy, this means that any joint distribution on $\Theta \times A$ that the evaluator can induce with a strategy in a selected experiment, the evaluator can also induce in another selected experiment. Hence, in particular, the distribution over A induced by the evaluator's optimal strategy is independent of the presample size k.

For a less immediate example, let us consider a simple hypothesis testing problem and a one-dimensional location experiment with a normal basic noise distribution. Recall once again that in this type of problem we have two states θ_L and $\theta_H > \theta_L$, two actions a_1 and $a_2 > a_1$, and payoffs satisfying (1). For convenience, let us normalize the evaluator's payoffs, so that the payoff from acceptance in each state θ is simply equal to θ, and rejection leads to a safety payoff R. Thus,

$$\theta_L = u(\theta_L, a_2) \leqslant \underbrace{u(\theta_L, a_1) = u(\theta_H, a_1)}_{=R} \leqslant u(\theta_H, a_2) = \theta_H.$$

We investigate the comparative statics of the researcher's welfare gain from selection, as the evaluator's safety payoff R varies between θ_L and θ_H. Letting p denote the prior probability of state θ_H, the researcher's payoff is the ex ante probability of acceptance, that is,

$$p[1 - F(\bar{x} - \theta_H)] + (1-p)[1 - F(\bar{x} - \theta_L)],$$

where $\bar{x}$ is the cutoff point chosen by the evaluator.

Figure 4 illustrates the comparative statics result when $n = 1$ and $k = 2$, and the prior $p = 1/2$. The green curve represents the researcher's payoff difference between the selected and random

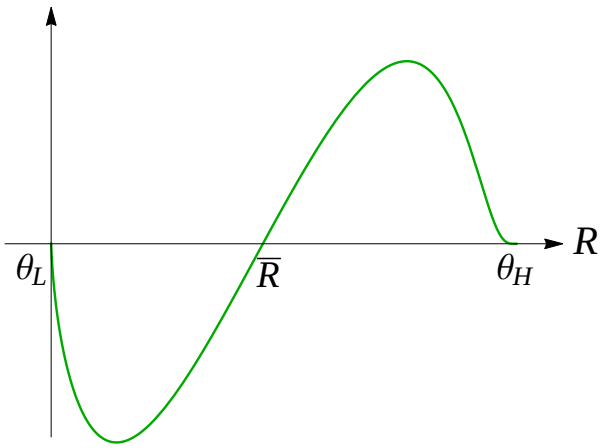


Figure 4: Researcher gain from selection, as evaluator's preference varies.

experiment. When R is low, the researcher suffers from selection. When R is high, the researcher benefits.[35] Intuitively, when R is low, the evaluator is more willing to accept. Providing more information (with higher k in the normal example), allows the evaluator to reject more often (in state θ_L). This is not in the researcher's interest. Conversely when R is high.[36]

5.2 Data Production Game

In the previous section we exogenously fixed sample size n and presample size $k \geqslant n$. Here, we endogenize the choice of n and k. Specifically, we consider two stages preceding the selection game studied before:

Stage -1. Evaluator chooses the sample size n and decides whether a presample is permitted.

Stage 0. Researcher privately chooses presample size $k \geqslant n$, or opts out of the game. If a presample is not permitted, the researcher only considers $k = n$ or opting out.

The researcher's gross payoff is still $v(a)$, but the researcher must bear cost $C(k)$ for obtaining the presample. We assume that this cost function is increasing and convex—if we restrict attention

[35] At $R = (\theta_L + \theta_H)/2$, the researcher benefits from selection. Without selection, in this symmetric case, the evaluator chooses the cutoff $\bar{x}$ inducing as many false positives as false negatives: $1 - F(\bar{x} - \theta_L) = F(\bar{x} - \theta_H)$. Maximal selection introduces an asymmetry in the distribution that leads the evaluator to optimally choose $\bar{x}$ such that $1 - F(\bar{x} - \theta_L) > F(\bar{x} - \theta_H)$, that is, more false positives are tolerated.

[36] This intuition is close to that in Johnson and Myatt (2006).

to natural numbers k, convexity means that $C(k+1)-C(k)$ is increasing in k. Opting out is free, so $C(0)=0$. The choice of k is private, and we assume that there is no credible way to directly signal any information about it. In equilibrium, the evaluator correctly anticipates k. We focus on pure-strategy equilibria, consistent with our statistical analysis so far where k was fixed.[37]

Payoffs to the evaluator are as specified before. We endow the evaluator with an option to completely forbid sample selection, assuming that such a rule can be perfectly enforced. This assumption allows the evaluator who stands to suffer from sample selection to completely rule it out. More interestingly, as we will show, the evaluator will in some circumstances prefer to tolerate sample selection, despite the availability of such a strong instrument. For simplicity, we ignore the evaluator's cost of studying larger samples. In case the researcher opts out, we assume that the evaluator must rely on an expensive alternative source of information—we will use this only to break ties in stages -1 and 0.

Following stages -1 and 0 above, we assume sequential rationality: the researcher selects the highest observations, and the evaluator best-responds to that selection, given the conjectured presample size, as in Proposition 4. Since the presample size k is privately chosen in stage 0, we need an expression for the researcher's expected payoff after deviating. With sample size n, actual presample size k, and $\hat{k}$ the presample size conjectured by the evaluator, we let $V\left(n,k,\hat{k}\right)$ denote this expected payoff. The distribution over observations here is governed by k and maximal selection by the researcher, but the evaluator best-responds to n out of $\hat{k}$.

Equilibrium Choice of Presample Size. Both parties observe the evaluator's choice of n in stage -1. We now consider stage 0 in the nontrivial case where greater presamples $k>n$ are permitted.

We restrict attention to simple hypothesis testing, and we also focus this first analysis on the case $n=1$. The evaluator accepts when the realization x exceeds threshold $\bar{x}$, an increasing function of the conjectured presample size $\hat{k}$. In stage 0, the researcher chooses k in order to maximize the payoff $V\left(n,k,\hat{k}\right)-C(k)$, that is,

$$p\left[1-F^k(\bar{x}-\theta_H)\right]+(1-p)\left[1-F^k(\bar{x}-\theta_L)\right]-C(k). \tag{12}$$

Considering a deviation from equilibrium, the researcher has a potential gain through the upward shift of the realized observation x. This is to be weighed against the cost of looking at more subjects, when already looking at k.

Proposition 5. *The researcher's objective function is concave in k. The first order condition (assuming $k\geqslant 1$ is a real number) is given by*

$$-p\log\left(F(\bar{x}-\theta_H)\right)F^k(\bar{x}-\theta_H)-(1-p)\log\left(F(\bar{x}-\theta_L)\right)F^k(\bar{x}-\theta_L)=C'(k). \tag{13}$$

[37] We later discuss the implications of uncertainty regarding k.

The researcher opts out if this k provides negative payoff in (12).

The equilibrium described in Proposition 5 exhibits a rat race effect: when the evaluator correctly anticipates a higher degree k of selection, the researcher's cost $C(k)$ to manipulate the experiment is partly wasted. To see the cleanest instance of this, consider the Gumbel example. The private choice of k tempts the researcher to choose $k > 1$ (we prevent opt-out by assuming that the evaluator is significantly less likely to accept without the researcher's sample). The evaluator's acceptance probability is independent of $k \geqslant 1$. The researcher's total payoff, having chosen $k > 1$, is then less than if the researcher could be tied to the mast, unable to augment k. Beyond the Gumbel example, costs of manipulation could even harm a researcher who already lost welfare due to sample selection.

The researcher's best response k may increase or decrease with the evaluator's cutoff $\bar{x}$, depending on parameters. The sign of this slope depends on the sign of the derivative of the left hand side in (13) with respect to $\bar{x}$,

$$-p\left(1+k\log\left(F\left(\bar{x}-\theta_H\right)\right)\right)F^{k-1}\left(\bar{x}-\theta_H\right)f\left(\bar{x}-\theta_H\right) \\ -\left(1-p\right)\left(1+k\log\left(F\left(\bar{x}-\theta_L\right)\right)\right)F^{k-1}\left(\bar{x}-\theta_L\right)f\left(\bar{x}-\theta_L\right),$$

which is positive when $F\left(\bar{x}-\theta_L\right)$ is sufficiently small, as happens when the safety payoff R is large. In that case, the best response k is an increasing function of $\bar{x}$. Conversely, when R is small, the best response k is a decreasing function of $\bar{x}$.[38]

This comparative statics result allows us to discuss the evaluator's optimal commitment to an ex-post suboptimal acceptance standard. If R is large, the researcher's best response is downward sloping. If the evaluator stands to gain welfare from selection (as depends on F), it is optimal for the evaluator to commit to reduce the acceptance standard below the Nash level in order to induce the researcher to increase k. Conversely, when the evaluator loses from selection.

Equilibrium Choice of Sample Size. We finally turn to stage -1. If the researcher ends up considering a presample of size k, why should the evaluator not request to see all the evidence? The evaluator could increase n until it hits k, or the evaluator could forbid presampling. The argument against this is that a freedom to presample can create value for the researcher, and hence make the researcher more willing to produce the sample. We will show by a simple example that this can be consistent with equilibrium. The example serves as proof of the more general concept.

[38] It can be easily verified that the best reply of the researcher is increasing for $\bar{x} < \theta_L + F^{-1}(e^{-1/k})$ and decreasing for $\bar{x} > \theta_H + F^{-1}(e^{-1/k})$.

For this example, we consider again simple hypothesis testing with normal noise. To simplify the analysis, we consider parameters that satisfy *equipoise*: ex ante, the evaluator is indifferent among acceptance and rejection. Slightly stronger, we assume $p = 1/2$ and $R = (\theta_L + \theta_H)/2$.

Suppose that $\hat{k} = n$, that is, the evaluator anticipates no selection. Then the evaluator accepts if and only if $(1/n)\sum_{i=1}^{n} x_i \geqslant (\theta_L + \theta_H)/2$. By symmetry of this setting, the equilibrium probability of acceptance is $1/2$, independent of n. The researcher's equilibrium payoff (if $k = n$) from the evaluator's action is then invariant to k, that is, $V(n,n,n) = 1/2$. However, participation is costlier as n rises. In this example, we make the natural assumption that the evaluator accepts with probability $1/2$ in each state if the researcher does not produce evidence. The evaluator is of course better off, the greater is $k = n$, as more evidence allows acceptance to grow more likely in state H and less likely in state L (at equal rates).

Consider now the case $n = 1$ and $k = 2$. Here, the evaluator accepts with a chance strictly greater than $1/2$, to the interest of the researcher, so $V(1,2,2) > 1/2$. Selection determines a first-order stochastic dominance increase in the realization of the experiment, but the evaluator's cutoff point also shifts up. In net, the evaluator accepts more often in state θ_H and rejects more often in state θ_L, but the acceptance rate optimally grows by more than the rejection rate.

The researcher is willing to participate when $n = 1$ and presampling is allowed, provided that $C(2) < V(1,2,2) - 1/2$. Recall from Proposition 5 that the researcher's objective function is concave in k, so for $C(3)$ sufficiently large, the researcher will not deviate to choose $k = 3$ when $\hat{k} = 2$ is conjectured. Likewise, for $C(3)$ sufficiently large, the researcher will opt out if $n = 2$ or larger. Finally, since alternative information is very expensive, the evaluator will prefer the case $n = 1, k = 2$ over a researcher who opts out. We collect these facts in the following:

Proposition 6. *Consider a simple hypothesis testing problem with normal basic noise distribution. Assume equipoise, that $C(2)$ is not too large, but that $C(3)$ is large. In stage -1, the evaluator chooses to tolerate sample selection.*

The concept is of course more general, that allowing sample selection can incentivize the researcher to provide more evidence.

6 Extensions

In this section we sketch an extension of the analysis to situations where the evaluator may be uncertain about the correct extent of selection k, or even fail altogether to anticipate any selection. The supplementary appendix also analyzes the case of extreme selection, where the presample size

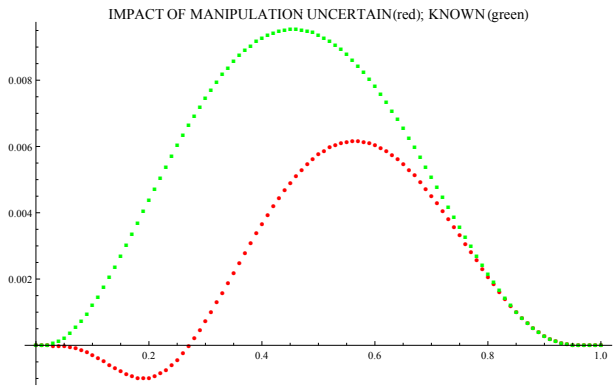


Figure 5: Evaluator's payoff gain over certain $k = 1$ in Normal hypothesis testing, as function of safety payoff R. Green curve: certain selection $k = 2$. Red curve: equal chance of $k = 1$ and $k = 2$.

grows unboundedly large and sketches the comparison between experiments that are not ranked by conditional accuracy.

6.1 Uncertain Selection

Sometimes, the extent of selection is known, for instance because a study has been conducted in n of k possible groups, or because a student has solved n of k given problems. Nevertheless, in other settings it seems natural that the evaluator is uncertain of the extent of section. Seeing a sample of size n, the presample size $k \geqslant n$ appears random. Uncertainty arises with endogenous sample selection when the evaluator doesn't know precisely the costs and preferences of the researcher.

Until now, we have identified conditions where the evaluator prefers grater selection to a known extent. It might then be natural to conjecture that the evaluator should prefer uncertain selection that attaches greater weight to greater selection. However, this idea is generally false. To illustrate this failure, consider hypothesis testing with sample size one in the Gumbel case. When F is Gumbel, F^k is also Gumbel, simply at a higher location. But the mixture of Gumbel distributions at different locations is not Gumbel. Intuitively, the evaluator would not know how to correct the given observation for the location change induced by selection. In fact, then the mixture must be worse in the strong sense of Blackwell (1951, 1953), for the evaluator has lost the information about the true extent of selection. The evaluator was indifferent to sample selection to any known extent k, but the uncertainty leads to a lower payoff.

More generally, we would expect the evaluator to be worse off with uncertain selection than certain selection. This is an important point when interpreting our earlier findings where the evaluator sometimes benefits from selection. Explicitly, Figure 5 shows that the evaluator in the Normal

example suffers from uncertainty. The gain from selection is wiped out when the evaluator is ex ante favorable to approval.

Our characterizations remain robust to the introduction of a small amount of of uncertainty, however. The evaluator can behave as if k is known. If the distribution over random k converges to this degenerate assumption, the evaluator's payoff converges to the payoff where it really is known, by continuity of expected payoffs. Responding optimally to uncertainty is even better. By implication, where we have derived a welfare gain from selection, it remains robust to a little bit of uncertainty.

Let us turn to an applied implication of uncertainty. Throughout, we have considered the case where the random experiment of size n satisfies the MLR property. We derived that this property extends to maximally selected samples of size n from a known presample size k. However, once k is uncertain, the selected data need not inherit the MLR property. Just consider the observation of a single data point n from a location experiment where the noise has a logconcave distribution. The mixture of logconcave distributions need not be logconcave, and then the mixture violates the MLR property.

For example, suppose that there is probability h that $k = n = 1$ and remaining probability $1-h$ that $k = 2$. At observation x, the evaluator's posterior likelihood ratio satisfies

$$\frac{\Pr(\theta_H|x)}{\Pr(\theta_L|x)} = \frac{q}{1-q}\frac{f(x-\theta_H)}{f(x-\theta_L)}\left(\hat{h}(x) + \left[1-\hat{h}(x)\right]\frac{F(x-\theta_H)}{F(x-\theta_L)}\right)$$

where $\hat{h}(x) = h/\left[h+(1-h)\,2F(x-\theta_L)\right]$ may be interpreted as a posterior weight on honesty. This weight is decreasing in x, so weight is gradually shifted towards selection as x grows. When x is very low, there is little correction in the likelihood ratio, as selection is unlikely. When x is very high, there is again little correction because F^2 is very close to F. In the middle, however, the change in weight towards selection can be so powerful that $\Pr(\theta_H|x)/\Pr(\theta_L|x)$ falls in x. In this region, better results indicate greater selection — essentially, the results are "too good to be true." Figure 6 illustrates this effect.

Once we enter this realm of non-monotone inference from the signal, also our monotone equilibrium in Proposition 4 is in jeopardy. The evaluator's action is increasing in belief $\Pr(\theta_H|x)$. If the researcher really has $k = 2$, and if this researcher observes two outcomes in this region, it would be a best response to report the lower of the two data points.

In turn, once the researcher's strategy is no longer monotone, the evaluator would react even more negatively to certain results, if they imply that the researcher was unable to report something a little lower. This can add a new dip to the evaluator's posterior curve. Once the researcher best responds, and the evaluator takes this into account, yet another dip can appear, and so on.

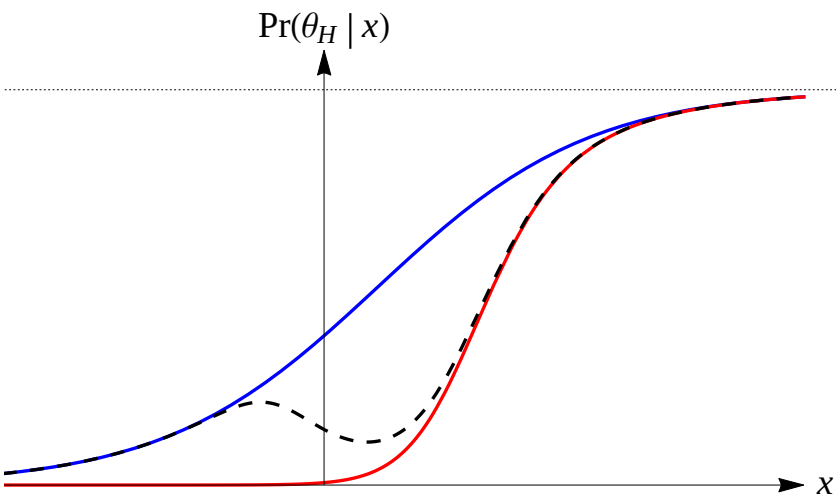


Figure 6: Evaluator's posterior expectation of the state as a function of the experimental observation (dashed curve). With known presample size $k = n = 1$ the expectation would be the blue curve, with known $k = 5$ the red curve.

It may be objected that the evaluator should be constrained to a monotone strategy, for normative or positive reasons. Interestingly, we have found that the evaluator sometimes prefers such a commitment over flexibility, since it induces a more informative response from the researcher.

6.2 Unanticipated Selection

So far, we considered situations in which the evaluator rationally predicts the correct extent of selection k and there is no uncertainty in k, for example because the parameters of the model (such as researcher's bias and cost of presampling) are known. This is the most optimistic scenario when evaluating the impact of selection. Here we relax these assumptions to consider more realistic scenarios.

Consider an unwary evaluator who wrongly anticipates a lower k than true. Holding fixed the true k, clearly the evaluator is generally worse off by being unwary than being rational. More interestingly, it is ambiguous whether an unwary evaluator who expects k gains or loses when the true signal has $k' > k$. If the rational evaluator would prefer k' to k, this gain might be greater than the cost of irrationality.

In an important benchmark case, we find that the unwary evaluator is exactly indifferent to an increase of selection from $k = 1$ to $k = 2$. Consider again a situation of equipoise, whereby

at the prior the evaluator is indifferent between accepting and rejecting.[39] Suppose that the basic noise distribution F is symmetric, so that for some ε_0 we have $F(\varepsilon_0+\varepsilon) = 1-F(\varepsilon_0-\varepsilon)$ for all ε. Start from the acceptance threshold that is optimal in the random experiment, namely $\bar{x} = \varepsilon_0 + (\theta_L+\theta_H)/2$, and consider how selection with $k=2$ affects an unwary evaluator who maintains the acceptance standard unchanged at $\bar{x}$. The probability of acceptance clearly increases. The resulting change in the evaluator's payoff equals

$$-(1-p)\underbrace{\left[F(\bar{x}-\theta_L)-F^2(\bar{x}-\theta_L)\right]}_{\text{increase in false positives}}(R-\theta_L)+p\underbrace{\left[F(\bar{x}-\theta_H)-F^2(\bar{x}-\theta_H)\right]}_{\text{reduction in false negatives}}(\theta_H-R).$$

By equipoise, $(1-p)(R-\theta_L) = p(\theta_H-R)$. Thus, false positives and false negatives are equally costly for the evaluator. By symmetry, $F(\bar{x}-\theta_L)+F(\bar{x}-\theta_H) = 1$. Thus, the increase in false positives exactly equals the reduction in false negatives. We conclude that the unwary evaluator, who anticipates no selection ($k=1$), is indifferent between no selection and selection with $k=2$.[40]

A fortiori, under symmetry and equipoise an increase in selection from $k=1$ to $k=2$ must necessarily benefit a rational evaluator.

7 Conclusion

Contrary to naive intuition, sample selection does not necessarily damage the evaluator. Increased selection benefits when the data distribution features a log-supermodular reverse hazard rate. For location experiments, this property corresponds to logconcavity of the noise distribution's reverse hazard rate—with a top tail thinner than the extreme value Gumbel distribution and the bottom tail thicker than Gumbel—a condition satisfied e.g. by normal noise. Sample selection is detrimental in the log-submodular case—for location experiments, a logconvex reverse hazard rate of the noise distribution. Adding uncertainty or unawareness of selection adds further damage. In addition, we characterize situations in which the researcher ends up suffering from information manipulation like in a rat race, even if we abstract away from the cost of acquiring information. At the same time, we exhibit cases in which the evaluator willingly chooses to allow sample selection to incentivize the researcher to provide more evidence. At a methodological level, we also develop a generally applicable method for comparing the value of information in multidimensional experiments with correlated observations.

[39] The condition of equipoise, requiring experimental subjects to be indifferent between treatment and control, is an ethical prerequisite that is often required for carrying out a randomized experiment.

[40] It is easy to verify that with a normal basic noise distribution an unwary evaluator who would reject at the prior (and who wrongly anticipates random data, $k=1$) benefits from observing selected data with $k=2$.

We leave to future work the design of experiments and policy responses in the presence of strategic selection. A natural starting point in this direction is Chassang, Padró i Miquel, and Snowberg's (2012) characterization of experimental design when outcomes are affected by experimental subjects' unobserved actions. Also, given the work by Allcott (2015) on site selection bias, another open question is a general characterization of the impact of selection challenging external validity in the presence of heterogeneous treatment effects.

A Proofs

Proof of Proposition 1. The second statement in the proposition was proved by Lehmann (1988, Theorem 5.1). We now prove the first statement. Since the evaluator's optimal strategy is monotone, the evaluator partitions D into a sequence of sets $(D_1,\dots,D_L)$ such that, for every ℓ, the set $U_\ell = D_\ell \cup \cdots D_L$ is an upper set in D, and chooses action a_ℓ if and only if the observed realization of X belongs to D_ℓ. For every $\ell = 1,\dots,L$ define $U_\ell = D_\ell \cup \cdots \cup D_L$. Then the optimal expected payoff, $\int_\Theta \sum_\ell \Pr_\theta(X \in D_\ell) u(a_\ell,\theta)\lambda(\theta)d\theta$, can be rewritten, summing by parts and disregarding constants, as

$$\int_\Theta \sum_{\ell<L} \Pr_\theta(X \in U_{\ell+1})\big[u(\theta,a_{\ell+1}) - u(\theta,a_\ell)\big]\lambda(\theta)d\theta.$$

In order to prove the proposition it therefore suffices to show that if Y is conditionally more accurate than X then we can exhibit nested upper sets $V_2 \supseteq \cdots \supseteq V_L$ such that, for every $\ell = 1,\dots,L-1$ and every state θ, the difference

$$\Pr_\theta(Y \in V_{\ell+1}) - \Pr_\theta(X \in U_{\ell+1}) \tag{14}$$

is nonpositive if $\theta \leqslant \theta_\ell$ and nonnegative if $\theta > \theta_\ell$. Define $V_{\ell+1} = \zeta_{\theta_\ell}(U_{\ell+1})$ for every $\ell = 1,\dots,L-1$. Then we can rewrite the difference in (14) as

$$\Pr_\theta(Y \in \zeta_{\theta_\ell}(U_{\ell+1})) - \Pr_\theta(Y \in \zeta_\theta(U_{\ell+1})).$$

For $\theta \leqslant \theta_\ell$ the difference is nonpositive, because $\zeta_\theta(\cdot) \leqslant \zeta_{\theta_\ell}(\cdot)$ in this case. For $\theta > \theta_\ell$ it is nonnegative, because then $\zeta_\theta(\cdot) \geqslant \zeta_{\theta_\ell}(\cdot)$. □

Proof of Theorem 1. Let $\phi(\varepsilon) = -\log(-\log(F(\varepsilon)))$ and observe that for all ε and $k \geqslant 1$ the horizontal distance between F^k and F, namely $(F^k)^{-1}(F(\varepsilon)) - \varepsilon$, is the same as the horizontal distance between the double-log transformations of F^k and F, namely $\varphi^{-1}(\varphi(\varepsilon) + \log(k)) - \varepsilon$. The derivative of the latter distance is

$$\frac{\varphi'(\varepsilon)}{\varphi'(\varphi^{-1}(\varphi(\varepsilon)+\log(k)))} - 1, \tag{15}$$

which is negative (resp. positive) for all ε and $k \geqslant 1$ if and only if φ is convex (resp. concave). By Theorem 5.2 in Lehmann (1988), F^k is more accurate than F if and only if φ is convex. □

Proof of Theorem 2. For each state θ, let ζ_θ be the function such that $\zeta_\theta(X)$ has the same distribution as Y. Thus, $\zeta_\theta(x_1,\dots,x_n) = (z_1,\dots,z_n)$, where $z_1,\dots,z_n$ are defined by

$$F^k(x_1|\theta) = F^m(z_1|\theta) \quad \text{and} \quad \frac{F^{k-i+1}(x_i|\theta)}{F^{k-i+1}(x_{i-1}|\theta)} = \frac{F^{m-i+1}(z_i|\theta)}{F^{m-i+1}(z_{i-1}|\theta)} \quad \text{for } i = 2,\dots,n.$$

Before proceeding with the proof, we make the following preliminary observation: since the densities of the distributions $F^k(\cdot|\theta)$ and $F^m(\cdot|\theta)$ have the same support, namely the support of $F(\cdot|\theta)$, as x_1 converges to the upper bound of this support, so does z_1. Similarly, for every $i = 2,\dots,n$ and every x_{i-1}, as x_i converges to its largest possible value, namely x_{i-1}, z_i converges to z_{i-1}.

Fix two states θ' and $\theta'' > \theta'$, and let $z' = \zeta_{\theta'}(x)$ and $z'' = \zeta_{\theta''}(x)$ for brevity. We must prove that under either condition in the theorem ($m \geqslant k$ and the reverse hazard rate is log-supermodular, or $m \leqslant k$ and the reverse hazard rate is log-submodular) for every x we have $z'' \geqq z'$, or equivalently

$$F^m(z_1'|\theta'') \leqslant F^m(z_1''|\theta'') \quad \text{and} \quad \frac{F^{m-i+1}(z_i'|\theta'')}{F^{m-i+1}(z_{i-1}''|\theta'')} \leqslant \frac{F^{m-i+1}(z_i''|\theta'')}{F^{m-i+1}(z_{i-1}''|\theta'')} \quad \text{for } i = 2,\dots,n.$$

Plugging the definition of z'', we can rewrite these inequalities as

$$F^m(z_1'|\theta'') \leqslant F^k(x_1|\theta'') \quad \text{and} \quad \frac{F^{m-i+1}(z_i'|\theta'')}{F^{m-i+1}(z_{i-1}''|\theta'')} \leqslant \frac{F^{k-i+1}(x_i|\theta'')}{F^{k-i+1}(x_{i-1}|\theta'')} \quad \text{for } i = 2,\dots,n.$$

But, for every $i = 2,\dots,n$, if $(z_1'',\dots,z_{i-1}'') \geqq (z_1',\dots,z_{i-1}')$ then the denominator of the left-hand side of the second inequality becomes smaller, and hence the left-hand side of the inequality larger, if we replace z_{i-1}'' with z_{i-1}'. Rearranging terms, we conclude that it suffices to prove that

$$\frac{F^m(z_1'|\theta'')}{F^k(x_1|\theta'')} \leqslant 1 \quad \text{and} \quad \frac{F^{m-i+1}(z_i'|\theta'')}{F^{k-i+1}(x_i|\theta'')} \leqslant \frac{F^{m-i+1}(z_{i-1}'|\theta'')}{F^{k-i+1}(x_{i-1}|\theta'')} \quad \text{for } i = 2,\dots,n. \tag{16}$$

By the preliminary observation, as x_1 tends to the upper bound of the support of the density associated to $F(\cdot|\theta')$, so does z_1. Thus, under either condition in the theorem ($m \geqslant k$, or $m \leqslant k$ and the support of $F(\cdot|\theta')$ is unbounded above), the left-hand side of the first inequality in (16) tends to a number no greater than one. This implies that the first inequality in (16) holds if the left-hand side of the inequality increases with x_1, that is, differentiating with respect to x_1 and dropping the positive denominator in the derivative,

$$mF^{m-1}(z_1'|\theta'')\,f(z_1'|\theta'')\frac{dz_1'}{dx_1}F^k(x_1|\theta'') \geqslant kF^{k-1}(x_1|\theta'')\,f(x_1|\theta'')F^m(z_1'|\theta''). \tag{17}$$

But, by definition of z',

$$\frac{dz_1'}{dx_1} = \frac{kF^{k-1}(x_1|\theta')f(x_1|\theta')}{mF^{m-1}(z_1'|\theta')f(z_1'|\theta')}.$$

Plugging the latter in (17) and simplifying, we conclude that the first inequality in (16) holds if

$$\frac{f(z_1'|\theta'')/F(z_1'|\theta'')}{f(z_1'|\theta')/F(z_1'|\theta')} \geqslant \frac{f(x_1|\theta'')/F(x_1|\theta'')}{f(x_1|\theta')/F(x_1|\theta')},$$

which in turn follows from log-supermodularity (resp. log-submodularity) of the reverse hazard rate when $m \geqslant k$ (resp. $m \leqslant k$), because $m \geqslant k$ implies $z_1' \geqslant x_1$ (resp. $m \leqslant k$ implies $z_1' \leqslant x_1$).

Again by the preliminary observation, for every $i = 2, \ldots, n$ and every x_{i-1}, as x_i converges to x_{i-1}, z_i' converges to z_{i-1}'. Thus, as before, under either condition in the theorem the left-hand side of the second inequality in (16) tends to a number no greater than the right-hand side. The second inequality in (16) then holds if its left-hand side increases with x_i. Differentiating with respect to x_i and simplifying, as before, we obtain

$$\frac{f(z_i'|\theta'')/F(z_i'|\theta'')}{f(z_i'|\theta')/F(z_i'|\theta')} \geqslant \frac{f(x_i|\theta'')/F(x_i|\theta'')}{f(x_i|\theta')/F(x_i|\theta')},$$

which again follows from log-supermodularity (resp. log-submodularity) of the reverse hazard rate when $m \geqslant k$ (resp. $m \leqslant k$), because $m \geqslant k$ implies $z_i' \geqslant x_i$ (resp. $m \leqslant k$ implies $z_i' \leqslant x_i$). □

Proof of Proposition 2. By independence of the vectors $(\delta_1, \ldots, \delta_k)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$, and by the identical distribution assumption on $\delta_1, \ldots, \delta_k$, it follows that $\delta_{i^*} + \varepsilon_{i^*}$ has the same distribution as $\delta_1 + \varepsilon_{i^*}$. Since δ_1 is independent of $(\varepsilon_1, \ldots, \varepsilon_k)$ and has a logconcave density, the convolution $\eta_{i^*} + \varepsilon_{i^*}$ is less (more) dispersed than $\eta_i + \varepsilon_i$ whenever ε_{i^*} is less (more) dispersed than ε_i, as shown in Theorem 7 of Lewis and Thompson (1981). See also Theorem 3.B.9 in Shaked and Shanthikumar (2007). The conclusion now follows from Theorem 5.2 in Lehmann (1988). □

Proof of Proposition 3. The researcher first corrects for control variables z_{-j} and then estimates θ with OLS. The first step uses (10) to adjust the outcome variable into

$$\hat{y} = y - \sum_{l \neq j} z_l = x\theta + z_j + v.$$

The second step provides the familiar OLS estimate

$$\left(x^T x\right)^{-1} x^T \hat{y} = \left(x^T x\right)^{-1} x^T \left(x\theta + z_j + v\right) = \left(x^T x\right)^{-1} x^T \left(x\theta + x\varepsilon_j + u_j + v\right),$$

where the algebraic manipulations used (10) and (11). This reduces to the claimed expression. □

Proof of Proposition 4. It follows from the analysis above that the evaluator's strategy satisfies the property: for any sample pair $x' \geqslant x$, if a is an optimal choice at x then any optimal choice a' at

a has $a' \geqslant a$. From presample $(x_1, \ldots, x_k)$, selection of the n greatest elements provides a sample vector that dominates any other sample after ranking their elements. The evaluator's strategy is independent of such re-ranking. The posited strategy is thus a best response for the researcher whose utility increases in action. □

Proof of Proposition 5. Since the cost function is convex, it suffices to check that the first two terms in (12) are concave in k. It suffices to take k to be a real number. The first derivative of a^k is $\log(a)\, a^k$ and the second derivative is $(\log(a))^2 a^k$ which is positive when $a \notin \{0, 1\}$. It is easy to see that the first terms are instead constant in k, if the base is zero or one. □

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B Additional Extensions

B.1 Other Forms of Selection: Truncated Data

The particular form of selection that is our focus in this paper is but one instance of lack of randomness in empirical or experimental data. Another kind of selection that is often relevant—for instance, in regression contexts, which we discuss in the Section —involves (independent) observations from a *truncated distribution*. Here we review this kind of selection and contrast it with the form of selection analyzed earlier.

Given a random variable with distribution $F(\cdot|\theta)$ and density $f(\cdot|\theta)$ satisfying the MLR property, and given two left-truncation points $-\infty \leqslant a < b < \infty$, define $Y_a := X|X \geqslant a$ and $Y_b := X|X \geqslant b$. Similarly, given two right-truncation points $-\infty < c < d \leqslant \infty$, define $W_c := X|X \leqslant c$ and $W_d := X|X \leqslant d$. Does the evaluator prefer the more left-truncated experiment Y_b or the less left-truncated experiment Y_a? And how do W_c and W_d compare?[41] Using the notion of accuracy, and variants of the arguments used to establish Theorem 2, we obtain a new proof of the following result, due to Goel and DeGroot (1992).

Proposition 7. *Experiment Y_a is more accurate than Y_b if the hazard rate $f(x|\theta)/[1-F(x|\theta)]$ is log-supermodular. Moreover, W_d is more accurate than W_c if the reverse hazard rate $f(x|\theta)/F(x|\theta)$ is log-supermodular.*

Proof of Proposition 7. Consider first the comparison between W_c and W_d. In any state θ, the cumulative distribution function of W_c is $F(w|\theta)/F(c|\theta)$, for $w \leqslant c$. Similarly, the cumulative distribution function of W_d is $F(w|\theta)/F(d|\theta)$, for $w \leqslant d$. Thus, the function ζ_θ such that $\zeta_\theta(W_c)$ has the same distribution as W_d is defined as follows: for every $w \leqslant c$,

$$F(w|\theta)/F(c|\theta) = F(\zeta_\theta(w)|\theta)/F(d|\theta).$$

Now, fixing two states θ' and $\theta'' > \theta'$, we must show that $\zeta_{\theta''}(w) \geqslant \zeta_{\theta'}(w)$ for all $w \leqslant c$. But, by definition of $\zeta_{\theta''}$, we have $F(\zeta_{\theta''}(w)|\theta'')/F(d|\theta'') = F(w|\theta'')/F(c|\theta'')$, so it suffices to show that

$$F(w|\theta'')/F(c|\theta'') \geqslant F(\zeta_{\theta'}(w)|\theta'')/F(d|\theta'').$$

The inequality holds (with equality) in the limit as w increases to the upper bound c, because both sides converge to one. Thus, all we need to prove is that the ratio between the right-hand side and the left-hand side of the inequality increases with w. Taking derivatives, this condition says that

$$\frac{f(w|\theta'')/F(w|\theta'')}{f(w|\theta')/F(w|\theta')} \leqslant \frac{f(\zeta_{\theta'}(w)|\theta'')/F(\zeta_{\theta'}(w)|\theta'')}{f(\zeta_{\theta'}(w)|\theta')/F(\zeta_{\theta'}(w)|\theta')},$$

[41] Here $a = -\infty$ means that $Y_a = X$, so that Y_a is a random draw from $F(\cdot|\theta)$. Similarly for W_d, when $d = \infty$.

which holds precisely when the reverse hazard rate is log-supermodular, given that $\zeta_{\theta'}(w) \geqslant w$ by the fact that W_d first-order stochastically dominates W_c.

Next, consider the comparison between Y_a and Y_b. In any state θ, the cumulative distribution function of Y_a is $[F(y|\theta) - F(a|\theta)]/[1-F(a|\theta)]$, for $y \geqslant a$. Similarly, the cumulative distribution function of Y_b is $[F(y|\theta) - F(b|\theta)]/[1-F(b|\theta)]$, for $y \geqslant b$. Thus, the function ζ_θ such that $\zeta_\theta(Y_b)$ has the same distribution as Y_a is defined as follows: for every $y \geqslant b$,

$$[F(y|\theta) - F(b|\theta)]/[1-F(b|\theta)] = [F(\zeta_\theta(y)|\theta) - F(a|\theta)]/[1-F(a|\theta)].$$

As before, fixing two states θ' and $\theta'' > \theta'$ we must show that $\zeta_{\theta''}(y) \geqslant \zeta_{\theta'}(y)$ for every $y \geqslant b$, and using the definition of $\zeta_{\theta''}$ it suffices to show that

$$[F(y|\theta'') - F(b|\theta'')]/[1-F(b|\theta'')] \geqslant [F(\zeta_{\theta'}(y)|\theta'') - F(a|\theta'')]/[1-F(a|\theta'')].$$

This inequality holds in the limit as y *decreases* to the lower bound b, as both sides converge to one, so we must prove that the ratio between right-hand and left-hand side *decreases* with y, or

$$\frac{f(y|\theta'')/[1-F(y|\theta'')]}{f(y|\theta')/[1-F(y|\theta')]} \geqslant \frac{f(\zeta_{\theta'}(y)|\theta'')/[1-F(\zeta_{\theta'}(y)|\theta'')]}{f(\zeta_{\theta'}(y)|\theta')/[1-F(\zeta_{\theta'}(y)|\theta')]}.$$

The latter inequality holds when the hazard rate is log-supermodular, given that $\zeta_{\theta'}(y) \leqslant y$ by the fact that Y_b first-order stochastically dominates Y_a. □

The proposition compares one-dimensional experiments. The extension of the result to an arbitrary number of independent observations is immediate. Indeed, it is easy to see that combining conditionally more accurate mutually independent experiments (in this case, more accurate one-dimensional experiments) results in a conditionally more accurate experiment.[42]

Left-truncation type selection bears a resemblance to the kind of selection considered earlier: with both forms of selection, probability mass is moved toward the upper tail of the distribution. In some important cases, however, the welfare consequences of the two types of selection are strikingly different. Consider, for instance, a normal location experiment. In this case the basic noise distribution has a both logconcave hazard rate (see Footnote 28) and a logconcave reverse hazard rate. Moreover, as we have already shown, more selection (of the form analyzed in this paper) benefits the evaluator. However, the more truncated experiment Y_b is worse than the less truncated experiment Y_a. We will return to this contrast in the next section, when we apply our results and discuss the implications of (different kinds of) data selection in a regression context.

[42] More precisely, take a pair of experiments X, X' with, say, sample sizes n and n', and suppose that X is independent of X' in each state. Let Y, Y' be another pair of experiments with sample sizes n and n', again with Y independent of Y' in each state. If Y is conditionally more accurate than X, and Y' conditionally more accurate than X', then the $(n+n')$-dimensional experiment (Y, Y') is conditionally more accurate than (X, X').

B.2 Extreme Selection

Adding to our analysis of the impact of selection in one-dimensional location experiments, on the evaluator's welfare, we examine here the effect of extreme selection, with sample size $n = 1$ and presample size $k \to \infty$. We draw on the fundamental result in extreme value theory, which characterizes the limit distribution of the maximum of k i.i.d. random variables, properly normalized for location and scale inflation. Take a basic noise distribution F and suppose that, for some nondegenerate distribution $\bar{F}$ and some sequence of numbers $a_k > 0$ and b_k,

$$F^k(b_k + a_k\varepsilon) \to \bar{F}(\varepsilon)$$

for every continuity point ε of $\bar{F}$. The fundamental theorem of extreme value theory says that $\bar{F}$ must belong to one of the following three types: Gumbel, Extreme Weibull or Frechet.[43]

Since the distribution of the noise term is shifted upwards (in the sense of first-order stochastic dominance) as k increases, the location normalization sequence b_k is growing. However, the evaluator can adjust for any translation of the noise distribution without any impact on payoff. The limit impact of selection thus hinges on whether the scale normalization sequence a_k shrinks to zero or not. If $a_k \to 0$, then the noise distribution becomes more and more concentrated around b_k as k grows, providing the evaluator with arbitrarily precise information about the state—the value of the evaluator's problem converges to the full information payoff. If instead we can choose a constant sequence a_k, then we can also choose $a_k = 1$ for all k, and an extremely selected experiment is as accurate as a random experiment based on $\bar{F}$.

It is well known that many familiar distributions are in the domain of attraction of the Gumbel distribution. Specifically, when F is normal—or half-normal, which has the same right tails—then a_k must be decreasing to zero—the scale normalization sequence $a_k = (2\log k)^{-1/2}$ is appropriate in this case—and the limit distribution $\bar{F}$ is the Gumbel distribution. More generally, consider a distribution F in the *exponential power* family, with density

$$f(\varepsilon) = \frac{s}{\Gamma(1/s)} e^{-|\varepsilon|^s},$$

where s is a shape parameter and Γ is the Gamma function. This family includes the Laplace ($s = 1$), normal ($s = 2$), and uniform ($s = \infty$) distributions as special cases. Our next result shows that when the shape parameter s is strictly greater than 1, the scale normalization sequence a_k must be decreasing to zero, and the limiting distribution $\bar{F}$ is the Gumbel distribution.

[43] See e.g. Leadbetter, Lindgren, and Rootzén (1983) for a primer on extreme value theory. As shown in Müller and Rufibach (2008), every logconcave distribution F has a Gumbel or Extreme Weibull limiting distribution $\bar{F}$.

Proposition 8. *Let F be an exponential power distribution with shape parameter $s > 1$. Then $F^k(b_k + a_k\varepsilon) \to e^{-e^{-\varepsilon}}$ for some sequence of constants b_k and $a_k \to 0$. Thus, the value of the evaluator's problem converges to the full information payoff as $k \to \infty$.*

Proof of Proposition 8. We show that if $\varepsilon_1, \ldots, \varepsilon_k$ are i.i.d. exponential power with shape $s > 1$, location and scale parameters 0 and 1, then $M_k = \max\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ satisfies $\Pr(M_k \leqslant a_k\varepsilon + b_k) \to e^{-e^{-\varepsilon}}$ for all ε, where

$$a_k = (s\log k)^{-\frac{s-1}{s}} \qquad \text{and} \qquad b_k = (s\log k)^{1/s} - \frac{\frac{s-1}{s}\log\log k + \log(2\Gamma[1/s])}{(s\log k)^{\frac{s-1}{s}}}.$$

Start by noticing that $f(\varepsilon)/\left(\varepsilon^{b-1}[1-F(\varepsilon)]\right) \to 1$ as $\varepsilon \to \infty$. Fix ε and define y_k for each $k \geqslant 1$ by $1 - F(y_k) = e^{-\varepsilon}/k$, so that

$$\frac{e^{-\varepsilon}}{k}\frac{y_k^{s-1}}{f(y_k)} \to 1 \quad \text{as } k \to \infty. \tag{18}$$

We may assume $y_k > 0$ for all k. Then $f(y_k) = s^{\frac{s-1}{s}} e^{-y_k^s/s}/2\Gamma[1/s]$ and hence, by (18),

$$-\log k - \varepsilon + (s-1)\log y_k - \frac{s-1}{s}\log s + \log(2\Gamma[1/b]) + \frac{y_k^s}{s} \to 0. \tag{19}$$

From (19) we see that $-\log k + (s-1)\log y_k + y_k^s/s = -\log k + o\left(y_k^s/s\right) + y_k^s/s$ converges to a constant. Thus, $-s\log k/u_k^s + o\left(y_k^s/s\right)/(u_k^s/s) + 1$ converges to 0, i.e. $\log y_k = (1/s)(\log s + \log\log k) + o(1)$. Using this fact in (19), we obtain

$$\begin{aligned}\frac{y_k^s}{s} &= \log k + \varepsilon - \frac{s-1}{s}(\log s + \log\log k) + \frac{s-1}{s}\log s - \log(2\Gamma[1/s]) + o(1)\\ &= \log k + \varepsilon - \frac{s-1}{s}\log\log k - \log(2\Gamma[1/s]) + o(1).\end{aligned}$$

Equivalently,

$$\begin{aligned}y_k &= (s\log k)^{1/s}\left[1 + \frac{\varepsilon - \frac{s-1}{s}\log\log k - \log(2\Gamma[1/s])}{\log k} + o\left(\frac{1}{\log k}\right)\right]^{1/s}\\ &= (s\log k)^{1/s}\left[1 + \frac{\varepsilon - \frac{s-1}{s}\log\log k - \log(2\Gamma[1/s])}{s\log k} + o\left(\frac{1}{\log k}\right)\right]\\ &= (s\log k)^{1/s} + \frac{\varepsilon - \frac{s-1}{s}\log\log k - \log(2\Gamma[1/s])}{(s\log k)^{\frac{s-1}{b}}} + o\left(\frac{1}{(\log k)^{\frac{s-1}{s}}}\right) = a_k\varepsilon + b_k + o(a_k).\end{aligned}$$

Thus, $\Pr(M_k \leqslant a_k\varepsilon + b_k + o(a_k)) \to e^{-e^{-\varepsilon}}$, as was to be shown.

We find the conclusion in Proposition 8 striking, because it is known that when F is the exponential distribution—or the Laplace distribution, since the two distributions have the same right

tails—then F also converges to the Gumbel distribution, but we can take $a_k = 1$ for each k. (The same normalizing constants work for the generalized exponential distribution defined in (5).) Thus, while extreme selection leads to full information as $k \to \infty$ for any $b > 1$ in the exponential power family, the limit result is very different when $b = 1$. The negative impact of selection in the exponential case discussed earlier is, in this sense, non-generic, as any arbitrarily close distribution in the family reverses the conclusion.[44]

B.3 Local Accuracy

The notion of conditional accuracy provides a sufficient condition that is intuitive and easy to check, and requires no knowledge of the problem parameters: as long as the evaluator's payoff function defines a monotone decision problem, if an experiment Y is conditionally more accurate than experiment X, then the evaluator must prefer Y to X, as shown in Proposition 1. Moreover, as stated in that proposition, in the one-dimensional case considered by Lehmann (1988) (and the following papers based on Lehmann's article) the condition is sharp, for no weaker condition can guarantee that Y is always the preferred experiment—if Y and X are one-dimensional and Y is not more accurate than X, then there exists at least one monotone problem such that the evaluator strictly prefers X to Y.

Despite these convenient features, the accuracy (or conditional accuracy) criterion is often inapplicable—the ordering of experiments in terms of accuracy (or conditional accuracy) is only partial. If the function ζ_θ defined in (6) is not monotone, then the evaluator's preference over X and Y can depend on the specific problem at hand. But suppose now that we do know something about the problem, that is, we are interested in determining the evaluator's preference in some prespecified *subset* of monotone problems. Then, as the class of conceivable problems becomes smaller, more pairs of experiments become comparable.[45] In other words, global monotonicity of the function ζ_θ on the whole set Θ becomes unnecessarily strong a condition, and we may be able to compare experiments by looking at the *local* behavior of the function on a subset of Θ.

[44] Of course, as s approaches 1, the convergence to full information becomes slower.

[45] Lehmann's (1988) concept of accuracy, in turn, shares a similar motivation. By limiting attention to monotone decision problems (and one-dimensional experiments satisfying the MLR property), more pairs of experiments can be ranked than using Blackwell's (1951,1953) notion of sufficiency. Of course, in the extreme case where we focus attention on *one* decision problem, *any* two experiments become comparable, by simply computing which experiment gives the highest expected payoff in that problem. But this computation is, in general, not useful for comparing the same two experiments in a different decision problem. This is precisely why Blackwell's (or Lehmann's) ordering of experiments, despite being partial, is a valuable tool.

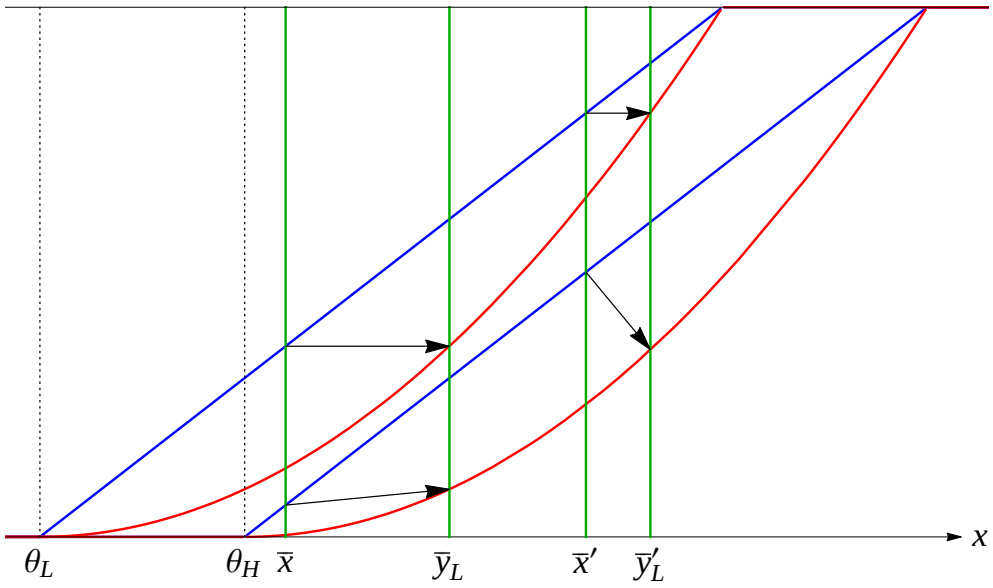


Figure 7: Simple hypothesis testing with uniform basic noise distribution: selection decreases (increases) accuracy for an evaluator strongly favoring acceptance (rejection) a priori.

To develop the idea a little further, take two experiments X and Y, and suppose that we are interested in checking whether Y is preferred to X in every simple hypothesis testing problem where the evaluator a priori strongly favors rejection, say, for all values of p, θ_L and θ_H but only for sufficiently large (not too smaller than θ_H) values of the safety payoff R. As illustrated in Figure 7, suppose X corresponds a single random observation drawn from a uniform F (the two blue curves with locations θ_L and θ_H), while Y corresponds to a single selected observation from a presample of size $k = 2$, with distribution F^2 (the two red curves). An evaluator choosing the high threshold $\bar{x}$ in experiment F is at least as concerned about false positives as an evaluator choosing $\bar{x}'$.

Now compare the two experiments. Suppose that R is sufficiently high, so that the evaluator optimally chooses threshold $\bar{x}$ in experiment X. The evaluator stands to gain from switching to Y: choosing threshold $\bar{y}$ in Y gives as many false positives, but fewer false negatives. This is because at $\bar{x}$ the horizontal difference between the selected and the random signal distribution is smaller in the low state than in the high state. Given that the signal distributions in the two states are just translated versions of the noise distributions F and G, the gain is due to the fact that G is *locally less dispersed* than F at the top.

For an evaluator more concerned about false negatives, who was choosing a lower threshold like $\bar{x}'$ in experiment X, the mechanics of the change to Y work just the opposite way, but the impact of the change is subtler. Getting as many false positive in Y as in X now requires increasing

the false negative rate, given that at the bottom, G is locally more dispersed than F. From this observation, however, we cannot conclude that changing experiment from X to Y damages the evaluator, because the change may lead to a new trade-off that might make the evalutor better off under Y. Had we started from the assumption that $\vec{x}$ were optimal for the selected experiment Y, we could have concluded that the random experiment X is better than Y.

To state our result, focus on the simple case with of the case with F such that, given any $k \geqslant 1$, F is first less and then more locally dispersed than F^k (or vice versa). Under simple hypothesis testing, recall that with a location experiment with distribution F the evaluator accepts when the likelihood ratio, $\ell(x) \equiv f(x-\theta_H)/f(x-\theta_L)$, is at least as large as the *acceptance hurdle* defined by

$$\bar{\ell} = \frac{1-p}{p}\frac{R-\theta_L}{\theta_H - R}. \tag{20}$$

Proposition 9. *Consider an experiment with hazard function $-\log F$ that is is first logconvex (resp. logconcave) and then logconcave (resp. logconvex). Then for every $k \geqslant 1$ there exists ℓ_k such that the evaluator prefers F to F^k (resp. F^k to F) for $\bar{\ell} \leqslant \ell_k$ and F^k to F (resp. F to F^k) for $\bar{\ell} \geqslant \ell_k$.*

When F is not everywhere double logconvex or everywhere double logconcave, random and selected experiment are incomparable in the sense of Lehmann (1988). The evaluator's preference for F or F^k then depends on the parameters of the specific problem at hand. However, we can qualify the impact of selection without actually solving the problem. Proposition 9 offers a particularly simple criterion for making this assessment: if F's reverse hazard function is first logconvex and then logconcave, then the quantile difference $(F^k)^{-1}(u) - F^{-1}(u)$ is first increasing and then decreasing in u. Thus, the evaluator suffers or benefits from selection according to whether the acceptance hurdle is, respectively, below or above a certain critical value. The opposite is true if F is instead first if F's reverse hazard function is first logconcave and then logconvex—in this case, the quantile difference is first decreasing and then increasing. Proposition 9 applies to the following two important special cases:

Uniform Example. Suppose we are interested in comparing F with F^k when F is uniform, $F(\varepsilon) = \varepsilon$ for $\varepsilon \in [0,1]$. Thus, the double-log transformation of F is $-\log(-\log\varepsilon)$, which is concave for ε smaller and convex for ε greater than the inverse of Euler's number, $1/e$. The evaluator is hurt by selection when more concerned about type II errors—the acceptance hurdle $\bar{\ell}$ is low—and benefits from selection when more concerned about type I errors—the acceptance hurdle $\bar{\ell}$ is high. Equivalently, since $F^k(\varepsilon) = \varepsilon^k$, the quantile difference $(F^k)^{-1}(u) - F^{-1}(u) = u^{1/k} - u$ is easily seen to be bell-shaped, so F is locally first less then more dispersed than F^k.

Laplace Example. Noise drawn from a Laplace distribution, where $F(\varepsilon) = (1/2)e^{\varepsilon}$ for $\varepsilon < 0$ and $F(\varepsilon) = 1 - (1/2)e^{-\varepsilon}$ for $\varepsilon \geqslant 0$, provides our second illustration of Proposition 9. In this case, the double-log transformation of F is convex for $\varepsilon < 0$ and concave for $\varepsilon > 0$. Thus, the evaluator's preference for selection is reversed compared to a uniform experiment. Here, the evaluator prefers F^k to F for low values of the acceptance hurdle, and F to F^k for large values. Plotting the distribution functions F and F^k reveals a U-shaped quantile difference.