Commitment Problems, Welfare Costs and Sovereign Debt*

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July 8, 2018

Abstract

An impatient and risk-neutral borrower can sell bonds to a more patient group of competitive lenders. The key problem: the borrower cannot commit to either a particular financing strategy, or a default strategy. In equilibrium, lending occurs, but gains from trade end up entirely dissipated, as lenders compete with each other and the borrower competes with himself. We uncover this striking result by taking a standard sovereign default model and modifying it by (i) using a government with linear preferences, and (ii) shrinking to zero the time period during which such government can commit. We show that the financing policy of the government can be computed as the ratio of (i) the wedge between the government discount rate and the return required by investors, and (ii) the semi-elasticity of the bond price function w.r.t. the debt face value. We overturn an old result of Bulow and Rogoff (1989), which argues that a borrower should never buy back his own bonds. We analyze commitment devices that allow the borrower to recapture some of the gains from trade – sovereign debt ceilings and constant issuance policies.

*First draft September 2017. We would like to thank seminar participants at Northwestern University for helpful comments and suggestions.

1 Introduction

Since the seminal work of Eaton and Gersovitz (1981), a large number of articles have studied small open economies issuing defaultable sovereign debt. The theoretical building blocks of this literature include a government that makes financing and default decisions without being able to commit, and creditors that price the sovereign debt rationally. The underlying mathematical problem – viewed through the lens of game theory as a dynamic game with a continuous action space between one large player (the government) and a continuum of small players (the investors, acting competitively) – lacks sufficient monotonicity properties to be studied using standard tools. For this reason, this literature has struggled to address questions of existence and uniqueness of the Markov perfect equilibrium of interest, and has been unable to derive sharp theoretical characterizations of the equilibrium objects: the welfare of the optimizing government, debt prices, as well as the financing and default policies.

As a consequence, most recent articles analyzing defaultable sovereign debt have instead focused on their quantitative predictions for the average debt-to-income of small open economies, the level of sovereign credit spreads, and the behavior of the current account. These articles acknowledge that the government's inability to commit introduces welfare costs, but have not been able to derive any theoretical calculation of the magnitude of such losses. A separate but related debate has emerged over the optimality of short term vs. long term debt, with several articles suggesting that the use of short term debt provides welfare gains to an optimizing government over the use of long term debt.

In this paper, we make progress on those questions and debates by taking a standard model of sovereign default and modifying it along two dimensions. First, while most of the existing literature assumes a government with finite intertemporal elasticity of substitution, we will instead model a government that has a linear payoff over consumption streams. The motive for our government to take on debt purely stems from its impatience relative to its international creditors, whereas in most of the existing literature, defaultable debt is not only used for consumption "tilting" but also for consumption smoothing purposes. Second, we analyze an environment where the time-step between decisions – and effectively the length of time during which the government can commit – is equal to Δ, and analyze the continuous time limit of our model as $\Delta \to 0$.

In such an economic environment, we establish a striking result: when not indebted, a government that has the option to borrow from more patient lenders but cannot commit to a particular fiscal path or to a particular default policy does not achieve any welfare gain vs. the autarky benchmark. Moreover, the government, when indebted, has a welfare equal to the present value of future consumption flows computed *as if it was never issuing any*

more debt in international capital markets. This result echoes the conjecture made in Coase (1972) in the context of a durable goods' monopolist: under the assumption that (i) the time period during which it can commit to a particular path of sales is infinitesimally small, that (ii) marginal costs are constant, and that (iii) it faces a continuum of consumers with a downward sloping demand curve, the monopolist behaves competitively, selling its durable goods at marginal costs. In our model of defaultable sovereign debt, the government acts as a monopolist, unable to commit to a path of future bond sales, and unable to promise to always repay its debt. The flow payoff of issuing bonds in debt capital markets is linear in (a) the quantity of bonds issued, and (b) the price at which those bonds are issued, as in the context of a monopolist with constant marginal costs. The distribution of private valuations giving rise to a downward sloping demand curve in the context of the durable good monopoly problem is analogous to the downward sloping debt price (as a function of the face value of debt outstanding). Finally, the continuous time nature of our model is essential: we illustrate numerically that an optimizing government in the discrete time version of our model achieves positive welfare gains vs. autarky, since it has the power to commit not to issue any debt over time intervals with positive measure.

The result we establish is valid for a wide range of assumed income processes for the optimizing government, and holds whether international investors are risk-neutral, whether they are risk-averse with marginal utilities that co-move with the small open economy's income process, or whether they have different beliefs about the income process of such small open economy[1]. Since the welfare of the government can be computed as if such government was never issuing any more debt, we show that such equilibrium welfare does not depend on the state of international capital markets – more specifically, it does not depend either on international risk-free rates, nor on risk-prices, nor on differences in beliefs. In our equilibrium, since the marginal benefit of one more unit of debt issued is equal to the price of such bond, and since such benefit is equalized with its marginal cost – the decrease in the government continuation value due to an extra unit of debt outstanding – debt prices end up also being independent of the state of international capital markets.

While the government welfare and debt prices are independent of capital market conditions, we characterize the issuance policy of the government, and show that it is always equal to the ratio of (a) the wedge between the government rate of time preference and debt investors' required rate of return, over (b) the semi-elasticity of the bond price function w.r.t. the debt face value. This means that supply-side shocks in debt capital markets – whether they are interest rate shocks, risk-price shocks or beliefs shocks – lead to an adjustment of the government financing policy, with corresponding current account adjustments that are

[1]The result also holds when the default value is a random variable that is being hit by Brownian shocks.

qualitatively consistent with empirical studies: during periods of international capital market turbulences, small open economies tend to revert to running current account surpluses. In the particular case where capital markets' investors are risk-neutral, we recover an old result from the sovereign default literature, first established by Bulow and Rogoff (1989) in the context of a static model: it is never efficient for a government to buy back its own debt. This result is over-turned in the presence of risk-averse debt investors, or when there is sufficient disagreement between investors and the government over their beliefs about the small open economy's income growth rate: in such case, it is sometimes optimal for the government to buy back its own debt.

Our model also allows us to shed some light on two complex issues the sovereign debt literature had to deal with. First, the smooth Markov perfect equilibrium of our model, when it exists, is always unique[2]. While a similar result holds with discrete time models featuring one-period debt contracts, we are the first paper to our knowledge to establish such result in the context of long term defaultable debt. Second, in our economic environment, the duration of the debt contract that the government can issue when not indebted is irrelevant for welfare purposes: whether such contract structure has a short average life or a long average life, whether the contract is a bullet maturity bond or a sinking fund bond, and even if the contract has some state-contingency[3], the government does not realize any gains from trade. The key to our result is the inability for the government to commit, even over arbitrarily short time periods, to not issue any bonds. Giving the government the ability to issue shorter term bonds does not change this commitment problem.

While the government does not realize any welfare gains from being able to trade with more patient lenders, we also show that citizens of the small open economy, which might be more patient than their government who makes financing and default decisions, will be strictly worse off in this environment with open capital markets than in financial autarky. In other words, for the citizens of such small open economy, autarky is better than trade. The intuition behind this result is straighforward: since the government balances exactly the current benefits of high debt issuances and high consumption vs. the future default costs, citizens who discount consumption flows at a lower rate will weigh relatively more the default costs, making their welfare lower than the autarky benchark.

To provide a concrete illustration of the smooth Markov perfect equilibrium we focus on, we derive a complete analytical characterization of the government value function, debt prices, issuance policy and default policy in the particular case where the small open economy's income process follows geometric Brownian motion dynamics, where default entails output

[2]It is unique within the class of smooth Markov perfect equilibria defined in our paper.

[3]We consider for example GDP-linked bonds.

losses and where the small open economy emerges from default with a lower debt burden. This analytical characterization allows us to perform comparative statics that are typically unavailable in most articles of this literature, which have to rely on sometimes complex numerical procedures to compute the equilibrium of interest. We also provide analytical expressions for the bond credit spreads, for the consumption-to-income ratio, and show how to compute the average default rate while deriving the small open economy's ergodic debt-to-income distribution.

For this particular income process, we then study different possible commitment devices that could be used by the government to capture some of the welfare gains from trade. We show that a policy that would force the government to issue a constant fraction of the outstanding stock of debt achieves some welfare gains, but only if such fraction is below a certain threshold. We also analyze the extent to which a "debt-ceiling" policy, preventing the government from issuing any more debt once the debt-to-income limit is above a certain level, allows the country to recapture some of the welfare gains from trade. We show that it is the case if such debt-to-income limit is sufficiently low. However, issuance restrictions need to be structured carefully; if instead, the government is prevented from issuing bonds during time periods that have random lengths, but can otherwise issue bonds without restrictions, the country once again does not rip the benefits from being able to sell bonds to lenders that are relatively more patient.

Our paper is organized as follows. After reviewing the existing literature, we introduce a canonical model of sovereign default in discrete time, but shrink the time interval in order to give the reader an intuition for the results we obtain using continuous time. We then present our general "no-welfare" result, and show an application of such result for a particular income process for which we can obtain an analytical characterization of all equilibrium objects of interest. We then discuss alternative commitment devices that can allow the small open economy to recapture some of the gains from trade.

2 Related Literature

Our paper relates to the vast literature on sovereign credit risk, which includes the seminal papers of Eaton and Gersovitz (1981) and Cole and Kehoe (1996), and more recently Aguiar and Gopinath (2004) and Arellano (2008). All these articles focus on discrete-time economies, one-period debt contracts, income risk, and feature impatient governments with finite intertemporal elasticities of substitution. Those papers typically focus on the quantitative implications of this class of models for consumption, default probabilities, the behavior of the current account, but the authors rarely analyze the welfare costs incurred by a gov-

ernment that lacks a commitment technology.

A related literature has analyzed the properties of sovereign default models in the presence of long term debt. Chatterjee and Eyigungor (2010) establish rigorously the existence of a Markov perfect equilibrium of their model, and perform a numerical welfare comparison using different types of bond durations, concluding that short term debt leads to greater ergodic welfare than long term debt. Their existence proof restricts the government's action set to only a finite number of debt levels and relies on an application of Brouwer's fixed point theorem, while we establish existence and uniqueness by construction. Arellano and Ramanarayanan (2012) analyze a government that has the option to issue both short term and long term debt, and argue that the government policy balances the "hedging benefits" of long term debt (i.e. the fact that long term bond prices tend to be positively correlated with the small open economy's marginal utility, providing the government with an incentive to "short" them) vs. the "incentive benefits" of short term debt (i.e. the fact that bond prices tend to be less sensitive to short term debt than they are to long term debt, given the commitment problem that the government faces). Those hedging benefits are absent from our paper, given the linear preferences we assume. Finally, Aguiar et al. (2016a) study a model of sovereign default without income risk but with "outside option" risk. They use an arbitrary debt maturity structure and argue that the competitive equilibrium of the model leads to allocations that are efficient if one ignores existing lenders and only take into account the government and new lenders. This leads them to conclude that long term debt is never traded by the government in equilibrium, as issuing or buying back long term bonds reduces its budget set. In our paper, we argue instead that the lack of ability to commit over any possible time period renders the maturity stucture of debt irrelevant for welfare purposes.

Our result that a government facing risk-neutral lenders never buys back its own bonds echos a result obtained a long time ago by Bulow, Rogoff, and Dornbusch (1988) and Bulow and Rogoff (1989) in the context of a one period model. However, we show how to overturn this result in the presence of sufficiently risk-averse lenders (whose marginal utilities are positively correlated with the small open economy's income process), or when the government and its lenders have different beliefs about the growth rate of the small open ecomomy's income, and when lenders are sufficiently more pessimistic than the government.

A separate literature in corporate finance analyzes bond issuances and default in the presence of long term debt and a lack of commitment. Whereas the incentive to take on debt in the sovereign credit risk literature stems from impatience and consumption smoothing motives, firms' desire to issue debt in capital markets is typically linked to the tax benefits of debt. Dangl and Zechner (2016) study the dynamic capital structure decision of a firm that faces issuance costs and covenants that limit the issuance rate of new debt; they analyze

the complicated trade-off between (a) long term debt, which has low roll-over costs but poor incentive properties when a firm is close to defaulting, and (b) short term debt, which forces a firm to incur higher capital market issuances costs but that allows a faster deleveraging in bad times. He and Milbradt (2016) instead study a firm that has deterministic cashflows, that can commit to keeping a constant amount of debt outstanding, but has flexibility to issue short term or long term bonds. They show that "shorterning" equilibria – equilibria in which the firm, before defaulting, systematically chooses to issue short term as opposed to long term bonds – can be Pareto dominated by equilibria in which the firm can commit to a constant maturity mix, highlighting the potential costs of commitment problems. Finally, our paper is closest to Admati et al. (2013) and DeMarzo and He (2014), who show that a firm that can dynamically adjust its capital structure at no cost dissipates all the tax benefits from debt.

3 From Discrete to Continuous Time

We first study a standard discrete-time sovereign default model when the time period Δ becomes arbitrarily small. Time thus evolves on the grid $\{i\Delta\}_{i\in\mathbb{N}}$. The discussion that follows is intentionally heuristic, in order to provide the reader with the required intuition for the main results of our paper. In this section, we do not prove existence of the particular type of equilibrium we focus on, nor its uniqueness, but instead hope to shed some light on a particular aspect of sovereign default models that has been overlooked by the international macroeconomic literature.

3.1 The General Case

We focus our attention on a government that has preferences over consumption streams:

$$\mathbb{E}\left[\sum_{i=0}^{\infty} e^{-\delta i\Delta} C_{i\Delta}\Delta\right] \tag{1}$$

$C_{i\Delta}$ is the consumption *per unit of time* enjoyed at time $i\Delta$ by the small open economy of interest, while δ is the rate of time preference[4]. While most of the literature studying sovereign credit risk assumes a flow utility function that is concave in the consumption rate, we will study an environment with linear preferences. We make this modeling choice for the following reason: while it is complex to disentangle, in traditional sovereign default models,

[4]The attentive reader can already see that we are ruling out the ability for the government to consume in "lumpy" fashion. This is purely for pedagogical purposes. We will see in the next sections an example where such smooth consumption strategy is not optimal for the government.

whether debt is mostly useful (a) for consumption tilting purposes (i.e. the desire to front-load consumption due to the impatience of the government relative to its creditors) or (b) for consumption smoothing purposes, we argue that the former motive dominates the latter. Indeed, usual calibrations of these models feature an equilibrium consumption process that is more volatile than the assumed endowment process[5], suggesting that impatience is a force that dominates any consumption smoothing motive.

In our endowment economy, income *per unit of time* evolves according to:

$$Y_{(i+1)\Delta} = Y_{i\Delta} + \mu\left(Y_{i\Delta}\right)\Delta + \sigma\left(Y_{i\Delta}\right)\sqrt{\Delta}\tilde{\omega}_{(i+1)\Delta}$$

In the above, $\tilde{\omega}_{(i+1)\Delta}$ is a standard normal random variable measurable at time $(i+1)\Delta$. μ and σ are two smooth functions representing the drift rate and some measure of local income uncertainty, conditional on the level of income (per unit of time) being equal to $Y_{i\Delta}$. At the limit, when $\Delta \to 0$, the income process has continuous sample paths[6].

The government only has exponentially amortizing debt at its disposal: at each time period, if F represents the aggregate principal amount of debt outstanding, the government must pay $mF\Delta$ corresponding to principal amortizations to its creditors, in addition to a coupon payment $\kappa F\Delta$. The parameter m controls the weighted average life of the debt contract; if $m = \frac{1}{\Delta}$, the financial contract is equivalent to a one-period debt contract, while if $m = 0$, the financial contract is equivalent to a console bond. The incentive for the government to take on debt stems from its high impatience relative to its creditors, who are risk neutral but discount cash-flows at a rate $r < \delta$.

Let $F_{i\Delta}$ be the stock of government debt at the end of period $i\Delta$. Let $D\left(F_{(i+1)\Delta}, Y_{i\Delta}\right)$ be the price of one unit of debt if the government plans to have, at the end of date $i\Delta$, $F_{(i+1)\Delta}$ units of debt outstanding when entering period $(i+1)\Delta$. The government resource constraint at time $i\Delta$ is as follows:

$$C_{i\Delta}\Delta = \underbrace{Y_{i\Delta}\Delta}_{\text{income}} + \underbrace{D\left(F_{(i+1)\Delta}, Y_{i\Delta}\right)\left(F_{(i+1)\Delta} - (1-m\Delta)F_{i\Delta}\right)}_{\text{debt issuance proceeds}} - \underbrace{(\kappa+m)F_{i\Delta}\Delta}_{\text{interest and principal due}} \tag{2}$$

The government has two interrelated commitment problems. First, it cannot commit to always repaying its debt, forcing creditors to bear default risk and justifying a debt price $D\left(F_{(i+1)\Delta}, Y_{i\Delta}\right)$ that is lower than its credit-risk-free value. Second, it cannot commit to

[5] In the benchmark simulations of Aguiar and Gopinath (2006), the (targeted) ratio of income volatility to consumption volatility is 0.84; Arellano (2008) calibrates her model using Argentina, and obtains a ratio of income volatility to consumption volatility of 0.91; most other calibrated models of this literature obtain similar ratios.

[6] Subject to certain technical conditions, it will in fact be an Itô process.

a future financing policy – in other words, at time $i\Delta$, when choosing how much bonds to auction, the government cannot credibly promise to issue a particular amount of bonds in the future. If the government elects to default, it achieves a default value $V_d(Y)$, for some smooth function $V_d(\cdot)$. The sequence of events follows Eaton and Gersovitz (1981): at the beginning of period $i\Delta$, the government has a stock of debt $F_{i\Delta}$, flow income $Y_{i\Delta}$, and first decides whether to default. If the government elects not to default, it chooses to issue a net amount of bonds $F_{(i+1)\Delta} - (1 - m\Delta)F_{i\Delta}$, and receives a price (per unit of debt face value issued) $D(Y_{i\Delta}, F_{(i+1)\Delta})$. The government then consumes according to the resource constraint (2). Between the end of period $i\Delta$ and the beginning of period $(i+1)\Delta$, $\tilde{\omega}_{(i+1)\Delta}$ is realized. The Bellman equation for the government is as follows:

$$V(Y,F) = \max_{F'} \left[Y\Delta + D(Y,F')\left(F' - (1-m\Delta)F\right) - (\kappa+m)F\Delta \right.$$
$$\left. + e^{-\delta\Delta}\mathbb{E}\left[\max\left(V_d(Y'), V(Y',F')\right)|Y\right]\right] \quad (3)$$

V represents the continuation value of a government that has elected to repay its debt in the current period. This maximization problem results in a next-period debt balance policy $F^*(Y,F)$, as well as a repayment set $\mathcal{R} := \{(Y,F) : V(Y,F) \geq V_d(Y)\}$. Creditors are competitive, risk-neutral and they discount cash-flows at the constant interest rate r. If the government defaults on its debt, creditors do not recover anything from their defaulted debt claim. The bond price must thus satisfy:

$$D(Y,F) = e^{-r\Delta}\mathbb{E}\left[1_{\{(Y',F)\in\mathcal{R}\}}\left[(\kappa+m)\Delta + (1-m\Delta)D(Y',F^*(Y',F))\right]|Y\right] \quad (4)$$

Except for our linear preference specification, equations (3) and (4) are the two canonical equations of most sovereign default models. A Markov perfect equilibrium is typically defined as a pair of functions (V,D) that satisfies these equations. Assuming that an equilibrium exists, and assuming that the value function V and the debt price D are differentiable w.r.t. F, we can derive the first order condition that the policy function $F^*(Y,F)$ must satisfy:

$$\left[F^* - (1-m\Delta)F\right]D_F(Y,F^*) + D(Y,F^*) + e^{-\delta\Delta}\int_{(Y',F^*)\in\mathcal{R}} V_F(Y',F^*)\,dG_\Delta(Y'|Y) = 0 \quad (5)$$

In the above, we have noted G_Δ the cumulative distribution function for a normal random variable with mean $Y + \mu(Y)\Delta$ and variance $\sigma^2(Y)\Delta$. Now assume that we can express the next-period debt policy function as an issuance policy of the form:

$$I^*(Y,F)\Delta := F^*(Y,F) - (1-m\Delta)F \quad (6)$$

$I^*(Y,F)$ represents the face amount of bonds issued *per unit time* in state (Y,F). Loosely speaking, this means that the next period debt balance, as the time step becomes smaller and smaller, becomes arbitrarily closer to the current period debt balance. Equation (5) above has 3 terms. Heuristically, when $\Delta \to 0$, if we consider bounded issuance policies and if we assume that the partial derivative D_F is bounded, the first term of the equation above converges to zero. Since the conditional distribution $G_\Delta(\cdot|Y)$ has a second moment that vanishes as $\Delta \to 0$, we can also heuristically write that the third term in the equation above converges to $V_F(Y,F)$, whenever $Y > Y_d(F)$. This means that equation (5) admits a limit, as $\Delta \to 0$, that can be expressed as follows:

$$D(Y,F) + V_F(Y,F) \underset{\Delta \to 0}{=} o(1) \tag{7}$$

In other words, the risk-neutral government's financing decision is such that the marginal cost of an extra unit of debt, $-V_F(Y,F)$, is equated with its marginal benefit $D(Y,F)$. We then perform a first order Taylor expansion of the Bellman equation (3), evaluating such equation at its optimum $F' = F^*(Y,F)$, and assuming that V is smooth enough that it is twice differentiable in the direction Y and once differentiable in the direction F. To do this, we leverage equation (6), and we replace $D(Y,F')$ using equation (7). When $(Y,F) \in \mathcal{R}$, since the distribution function G_Δ becomes degenerate as $\Delta \to 0$, the measure of points $Y' : (Y', F^*) \in \mathcal{R}^c$ – i.e. the next period states where the government elects to default – converges to zero, and equation (3) becomes:

$$V(Y,F) = Y\Delta - V_F(Y,F)\, I^*(Y,F)\, \Delta - (\kappa+m)F\Delta + e^{-\delta\Delta} \int_{-\infty}^{\infty} \Big[V(Y,F) + (Y'-Y)V_Y(Y,F) + \frac{(Y'-Y)^2}{2} V_{YY}(Y,F) + V_F(Y,F)\,(I^*(Y,F) - mF)\,\Delta \Big] dG_\Delta(Y'|Y) + o(\Delta)$$

Collecting the zero-order terms of this Taylor expansion results in a trivial equation $V(Y,F) = V(Y,F)$. Remembering that $\mathbb{E}[Y'-Y] = \mu(Y)\Delta$ and that $\mathbb{E}[(Y'-Y)^2] = \sigma^2(Y)\Delta + o(\Delta)$, the first order terms yield:

$$\delta V(Y,F) = Y - (\kappa+m)F - mFV_F(Y,F) + \mu(Y)V_Y(Y,F) + \frac{\sigma^2(Y)}{2} V_{YY}(Y,F) \tag{8}$$

In equation (8), the strategic interactions between the government and its creditors has vanished: the debt price D no longer appears. We also note that this equation is the Hamilton-Jacobi-Bellman equation describing a government that is never issuing any debt, and that is allowing the existing stock of debt to amortize. Finally, the value function V evaluated at $F = 0$ must represent the autarky value. Taking stock, this heuristic analysis suggests the

following, when the time increment $\Delta \to 0$:

- If an equilibrium exists in which the issuance policy is "smooth", the government value function can be computed as if the government was never issuing any new debt;
- In such case, the government's value function, when not indebted, is the same as the autarky value;
- The debt price function in such case can be computed from the identity $D(Y, F) = -V_F(Y, F)$.

In section A.1, we illustrate the heuristic result above by focusing on an income process that follows a geometric random walk, and by plotting the value function v and the debt price d over debt-to-income levels[7] for different choices of time-step Δ. Figure 15a and Figure 15b make it clear that the welfare v of a non-indebted sovereign decreases as the length of the time period of commitment decreases.

4 The General Result

We now generalize the results presented in the previous section. To do this, we consider a broad class of income processes for our small open economy of interest, and also introduce creditors that are no longer risk-neutral, and whose marginal utility process might co-vary with the small open economy's income process.

4.1 Small Open Economy's Income Process

Our small open economy is now endowed with strictly positive real income Y_t per unit of time. International capital market conditions, which might affect the dynamics of the small open economy's income process, are described by the state variable $s_t \in \mathcal{E} \subset \mathbb{R}$. We model Y_t and s_t as Itô processes:

$$dY_t = \mu_Y(Y_t, s_t)dt + \boldsymbol{\sigma}_Y(Y_t, s_t) \cdot d\boldsymbol{B_t} \tag{9}$$

$$ds_t = \mu_s(s_t)dt + \boldsymbol{\sigma}_s(s_t) \cdot d\boldsymbol{B_t} \tag{10}$$

We assume standard conditions to guarantee the existence of a strong solution to the stochastic differential equations (9) and (10)[8]. Our notation will use bold letters for vectors. $\{\boldsymbol{B_t}\}_{t\geq 0}$

[7] In this example, given the homotheticity of preferences and linearity of the resource constraint, the state variables (Y, F) collapse into a unique state variable – the debt-to-income level x.

[8] Imposing for example that the drift rates and volatility vectors are uniformly Lipschitz suffices.

is a multi-dimensional Brownian motion on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$; the multi-dimensional nature of the Brownian shocks might for example allow us to distinguish between idiosyncratic country-specific shocks, and aggregate shocks – in other words shocks that will hit not only the country's income process but also the marginal utility of international financial market participants. We will refer to $\mathbb{P}$ as the physical probability measure, and note $\mathcal{F}_t$ the σ-algebra generated by the Brownian motion $\boldsymbol{B_t}$.

The state variable s_t will be the key variable describing the state of the creditors' stochastic discount factor, as will be discussed in Section 4.2. Given our small open economy assumption, the income level Y_t of our country of focus does not affect the dynamics of the international capital market conditions s_t. We keep however the flexibility to introduce a feedback loop between financial market conditions and the country's income process, and allow the growth rate and volatility of our country of focus to depend on s_t. Finally, we note that the results presented in this paper are robust to more general specifications of the process s_t; we could have assumed that s_t is a multi-dimensional Itô process, or that its dynamics include a jump component, with a jump measure only dependent on s_t, without changing any of the results of our paper[9].

Noting Γ_t the small open economy's cumulative consumption process[10], the government of the small open economy maximizes the life-time utility function:

$$J_t = \mathbb{E}\left[\int_t^{+\infty} e^{-\delta(s-t)} d\Gamma_s\right] \tag{11}$$

The notation $\mathbb{E}$ denotes expectations under the measure $\mathbb{P}$. We assume that the government autarky value is finite.

Assumption 1. The impatience parameter δ, income drift rate $\mu_Y(\cdot,\cdot)$, income volatility $\boldsymbol{\sigma}_Y(\cdot,\cdot)$, and the stochastic process $\{s_t\}_{t\geq 0}$ are such that for all values of $(Y, s) \in \mathbb{R}_+ \times \mathcal{E}$,

$$\mathbb{E}^{Y,s}\left[\int_0^{+\infty} e^{-\delta t} Y_t dt\right] < +\infty \tag{12}$$

In equation (12), we use the superscript notation to condition on the initial value of the relevant stochastic processes. As before, the government does not have a full set of Arrow-Debreu securities at its disposal, and instead can only use non-contingent debt contracts with amortization rate m and coupon rate κ. During each time period $(t, t+dt]$, the government

[9] In fact, section A.10 presents such an example, where s_t is a pure jump process.

[10] We do not presume, at this point, that cumulative consumption will be absolutely continuous.

decides to issue a dollar face amount dH_t of bonds, where the cumulative bond issuance process H_t will be constrained to be progressively measurable. The face value process thus satisfies:

$$dF_t = dH_t - mF_t dt \tag{13}$$

Between t and $t + dt$, the small open economy's cumulative consumption increases with (a) total per-period income, decreases with (b) debt interest and principal repayments due, and increases with (c) proceeds (in units of consumption goods) raised from capital markets:

$$d\Gamma_t = (Y_t - (\kappa + m)F_t)\, dt + D_t dH_t \tag{14}$$

In equation (14), D_t is the debt price per unit of face value, taken as given by the government and determined in equilibrium. Our government still faces a two-pronged commitment problem: neither can it commit to a particular path of future bond issuances, nor can it commit to always repaying its bonds, which are thus credit risky. In other words, the government will choose a sequence of default times[11]. Upon a default at time τ, the small open economy's income jumps down, from $Y_{\tau-}$ to $Y_\tau = \alpha\left(Y_{\tau-}, s_{\tau-}\right) Y_{\tau-}$, with α being measurable with image in the interval $(0, 1)$. One could think about such income drop as resulting from disruption of trade and financial flows occurring in connection with a sovereign default. The government renegotiates its debts immediately with creditors, such that the post-default debt-to-income of the small open economy is a fraction $\theta\left(Y_{\tau-}, s_{\tau-}\right) \in (0, 1)$ of its pre-default value:

$$\frac{F_\tau}{Y_\tau} = \frac{F_{\tau-}}{Y_{\tau-}} \theta\left(Y_{\tau-}, s_{\tau-}\right) \tag{15}$$

Since the income level of the small open economy jumps down by a factor α, this means that each dollar of face value of sovereign debt is haircut to $\alpha\theta$ dollars upon a sovereign default. One can think of the function θ as the outcome of a bargaining game between creditors and the sovereign government, once such government has elected to default. However, for simplicity and since the strategic interactions between the government in default and its creditors are not a focus of this paper, we elect to model the outcome of this renegotiation exogenously. Finally, we note that we could have instead assumed that upon a default at time

[11] The continuous time setting of this model allows us to abstract from the specific timing assumption of the government bond auction. In discrete time models, Cole and Kehoe (1996), Aguiar and Amador (2013) and Aguiar et al. (2016b) (for example) all assume that the bond auction happens before the default decision is made by the government, while Aguiar and Gopinath (2006), Arellano (2008) and many other quantitative models of sovereign debt assume that the government makes its default decision before the bond auction takes place. The former timing convention allows, in discrete time, for the existence of potentially multiple equilibria, induced by the creditor's self-fulfilling belief that the government will default immediately after debt has been issued, leading to a low auction debt price and a rational decision by the government to default. Those considerations are absent from the continuous time environment.

τ, the government receives a default payoff $V_d(Y_\tau, F_\tau, s_\tau)$[12], for example by assuming that the small open economy's income drops upon default and the country is stuck in financial autarky forever. This slightly different assumption would not change the spirit of any of the results to follow.

4.2 Creditors

International investors purchase the debt issued by the government. We model their marginal utility process M_t (which we will also refer to as the stochastic discount factor, or "SDF") as a random walk:

$$\frac{dM_t}{M_t} = -r(s_t)dt - \boldsymbol{\nu}(s_t) \cdot d\boldsymbol{B_t} \tag{16}$$

The international investors' risk free rate is $r(s)$, while $\boldsymbol{\nu}(s)$ is the international risk price vector in state s. The marginal utility specification (16) for creditors is a generalization of the risk-neutral creditors we studied in section 3. The j^{th} coordinate of $\boldsymbol{\nu}(s)$ represents the expected excess return compensation per unit of j^{th} Brownian shock earned by investors in state s. We note that we could have used a more general specification of the stochastic discount factor, by introducing for example jumps, with a jump measure purely dependent on the state variable s_t. Such addition would not change any of the results discussed in this paper.

Given our assumed investor pricing kernel, any $\mathcal{F}_{t+s}$-measurable amount A_{t+s} received at time $t+s$ will be valued by investors by weighting such future cash-flow by the investors' future marginal utility, and taking expectations. One can also use a standard tool of the financial economics literature, and instead discount this future cashflow A_{t+s} at the risk-free rate, while distorting the probability distribution of such future cashflow via the following change in measure:

$$\text{Price}_t(A_{t+s}) = \mathbb{E}\left[\frac{M_{t+s}}{M_t}A_{t+s}|\mathcal{F}_t\right] := \hat{\mathbb{E}}\left[e^{-\int_0^s r(s_{t+u})du}A_{t+s}|\mathcal{F}_t\right]$$

$\hat{\mathbb{E}}$ is the risk-neutral expectation operator. It implicitly defines the risk-neutral measure $\mathbb{Q}$, under which $\hat{\boldsymbol{B}}_{\boldsymbol{t}} := \boldsymbol{B_t} + \int_0^t \boldsymbol{\nu}(s_u)du$ is a standard multi-dimensional Brownian motion.Using Girsanov's theorem, there is a separate interpretation for the behavior of our investors. In that interpretation, investors are risk-neutral, with time-varying rate of time preference $\{r(s_t)\}_{t\geq 0}$, and with beliefs about the income growth rate that are different from the beliefs of the government of the small open economy. When $\boldsymbol{\nu}(s_t) \cdot \boldsymbol{\sigma}_Y(Y_t, s_t) < 0$, investors are more pessimistic about the income growth prospects of the small open economy

[12]With V_d increasing in Y and decreasing in F.

than the government, whereas when $\boldsymbol{\nu}(s_t) \cdot \boldsymbol{\sigma}_Y(Y_t, s_t) > 0$, they are more optimistic. Both investors and the government are aware of each other's probability measure, and they simply agree to disagree. This second interpretation will be useful for some of the results to come. We end this section by introducing a restriction that will guarantee that the government will always have an incentive to borrow from international lenders.

ASSUMPTION 2. The international risk free rate $r(\cdot)$ satisfy:

$$r(s) < \delta \quad \forall s \in \mathcal{E} \tag{17}$$

4.3 Debt Valuation, Government Problem and Equilibrium

In this section, we focus on a Markovian setting. All technical details are relegated to the appendix, in section A.4; a reader less interested in the technical definition of admissible policies and equilibrium can skip this section. We restrict ourselves to "smooth" issuance policies (i.e. of order dt), and show that this restriction is without loss of generality for the class of equilibria we are considering. The payoff-relevant variables for the sovereign government and creditors are Y_t, F_t and s_t. The state space will be $\mathbb{R}^2 \times \mathcal{E}$, or a subset thereof. An admissible cumulative debt issuance policy H will be an absolutely continuous measurable function of the state variables, in other words it is uniquely defined by a measurable function $I : \mathbb{R}^2 \times \mathcal{E} \to \mathbb{R}$ such that:

$$dH_t = I\left(Y_t, F_t, s_t\right) dt$$

We will require I to satisfy a particular integrability condition, and will note $\mathcal{I}$ the set of admissible flow issuance policies. An admissible default policy $\boldsymbol{\tau}$ will be a sequence of increasing stopping times[13] $\boldsymbol{\tau} := \{\tau_k\}_{k \geq 1}$ that can be written as first hitting times of particular subsets of the state space:

$$\begin{aligned} \tau_0 &:= 0 \\ \tau_{k+1} &= \inf\{t \geq \tau_k : (Y_t, F_t) \in \mathcal{O}\left(s_t\right)\} \end{aligned}$$

$\{\mathcal{O}\left(s\right)\}_{s \in \mathcal{E}}$ is a family of open sets representing the default regions of the state space. We will also note $N_{d,t}^{(\boldsymbol{\tau})}$ the counting process for default events. We will note $\mathcal{T}$ the set of admis-

[13] With respect to the filtration $\mathcal{F}_t$.

sible default policies. Our formulation of admissible policies leads to a controlled face value process $F_t^{(I,\boldsymbol{\tau})}$ that is absolutely continuous whenever the government is performing under its contractual obligations:

$$F_t^{(I,\boldsymbol{\tau})} = F_0 + \int_0^t \left(I_u - mF_u^{(I,\boldsymbol{\tau})}\right) du + \int_0^t \left(F_u^{(I,\boldsymbol{\tau})} - F_{u-}^{(I,\boldsymbol{\tau})}\right) dN_{d,u}^{(\boldsymbol{\tau})} \tag{18}$$

The superscript notation $F_t^{(I,\boldsymbol{\tau})}$ emphasize the fact that the face value process F_t is altered by both the issuance policy I and the default policy $\boldsymbol{\tau}$. Similarly, the resulting cumulative controlled consumption process is absolutely continuous; the small open economy does not consume in "lumpy fashion", but rather always in "flow" fashion. If we note C_t the consumption *rate* of the small open economy, we have:

$$\begin{aligned}
\Gamma_t^{(I,\boldsymbol{\tau};D)} &= \int_0^t C_u^{(I;D)} du \\
C_t^{(I,\boldsymbol{\tau};D)} &:= Y_t^{(\boldsymbol{\tau})} + I\left(Y_t^{(\boldsymbol{\tau})}, F_t^{(I,\boldsymbol{\tau})}, s_t\right) D\left(Y_t^{(\boldsymbol{\tau})}, F_t^{(I,\boldsymbol{\tau})}, s_t\right) - (\kappa+m)F_t^{(I,\boldsymbol{\tau})} \\
Y_t^{(\boldsymbol{\tau})} &:= \left(\Pi_{k=1}^{N_{d,t}^{(\boldsymbol{\tau})}} \alpha_{\tau_k}\right) Y_t
\end{aligned} \tag{19}$$

$Y_t^{(\boldsymbol{\tau})}$ is the *controlled* income process (while Y_t is the uncontrolled income process) – this notation is meant to capture the fact that the small open economy's income drops by a proportional factor α_τ at each time τ the government elects to default. Creditors price the sovereign debt rationally. Upon a sovereign default at time τ, their principal balance suffer a haircut $\alpha_\tau\theta_\tau$. Thus, if they anticipate that the government will follow admissible policy $(I,\boldsymbol{\tau}) \in \mathcal{I}\times\mathcal{T}$, they will value one unit of sovereign debt as follows:

$$D\left(Y,F,s;(I,\boldsymbol{\tau})\right) := \hat{\mathbb{E}}^{Y,F,s}\left[\int_0^{+\infty} e^{-\int_0^t (r(s_u)+m)du}\left(\Pi_{k=1}^{N_{d,t}^{(\boldsymbol{\tau})}}\left(\alpha_{\tau_k}\theta_{\tau_k}\right)\right)(\kappa+m)dt\right] \tag{20}$$

We use a notation that makes the dependence of the debt price function on the anticipated issuance and default policies explicit[14]. Equation (20) can be interpreted as follows: creditors receive cash-flows $\kappa+m$ per unit of time on a debt balance that amortizes exponentially at rate m, and that suffers a haircut $\alpha_{\tau_k}\theta_{\tau_k}$ at each default time τ_k. The expectations are taken under the risk-neutral measure $\mathbb{Q}$.

We then focus on the government life-time utility. Given a debt price schedule $D(\cdot,\cdot,\cdot)$ that the government faces, and given admissible issuance and default policies $(I,\boldsymbol{\tau})$ used by the government (where $(I,\boldsymbol{\tau})$ might not necessarily be consistent with the debt price D),

[14] We have also used the short notation $\theta_{\tau_k} := \theta\left(Y_{\tau_k}, F_{\tau_k}, s_{\tau_k}\right)$, and a similar notation for α_{τ_k}.

there is a controlled flow consumption process $C_t^{(I,\boldsymbol{\tau};D)}$ which satisfies equation (19). This leads to the following government life-time utility:

$$J\left(Y,F,s;(I,\boldsymbol{\tau});D\right)=\mathbb{E}^{Y,F,s}\left[\int_0^{+\infty}e^{-\delta t}C_t^{(I,\boldsymbol{\tau};D)}dt\right] \tag{21}$$

The expectations are taken under the probability measure $\mathbb{P}$. The government takes as given the debt price function D and chooses its issuance and default policies in order to solve the following problem:

$$V(Y,F,s;D):=\sup_{(I,\boldsymbol{\tau})\in\mathcal{I}\times\mathcal{T}}J\left(Y,F,s;(I,\boldsymbol{\tau});D\right) \tag{22}$$

When choosing its issuance policy, the government takes into account the debt price schedule and the impact that such schedule has on flow consumption, via the resource constraint. Consistent with Maskin and Tirole (2001), we then define a "smooth Markov perfect equilibrium" as follows.

Definition 1. A smooth Markov perfect equilibrium is a set of Markovian issuance and default policies $(I^*,\boldsymbol{\tau}^*)\in\mathcal{I}\times\mathcal{T}$ such that for any initial state (Y,F,s),

$$(I^*,\boldsymbol{\tau}^*)=\arg\max_{(I,\boldsymbol{\tau})\in\mathcal{I}\times\mathcal{T}}J\left(Y,F,s;(I,\boldsymbol{\tau});D\left(\cdot,\cdot,\cdot;(I^*,\boldsymbol{\tau}^*)\right)\right)$$

For a given equilibrium $(I^*,\boldsymbol{\tau}^*)$, we will note $V(\cdot,\cdot,\cdot)$ the government's equilibrium value function, and $D(\cdot,\cdot,\cdot)$ the debt price across the state space.

4.4 Risk Premia

In an equilibrium of our economy, to the extent the country's income process exhibits non-zero local correlation with investors' pricing kernel, investors will earn a risk-premium. We show in section A.2 how to compute such risk-premium as a function of the debt price. This formula will turn out to be handy when we look at the optimal bond issuance policy for the class of equilibria of focus. For notational convenience, we note $X_t:=(Y_t,s_t)'$, $\boldsymbol{\mu}_X:=(\mu_Y,\mu_s)'$, and $\boldsymbol{\sigma}_X:=(\boldsymbol{\sigma}_Y',\boldsymbol{\sigma}_s')'$.

Lemma 1. Let $\{R_t\}_{t\geq 0}$ the cumulative return earned by investors when buying the bonds issued by the government of the small open economy. The instantaneous expected excess return $\mathbb{E}\left[dR_t-r(s_t)dt|\mathcal{F}_t\right]:=\pi\left(Y_t,F_t,s_t\right)dt$ earned by investors can be characterized as

follows:

$$\pi\left(Y,F,s\right)=\left(\boldsymbol{\sigma}_{X}\left(Y,s\right)\boldsymbol{\nu}(s)\right)\cdot\partial_{X}\ln D\left(Y,F,s\right) \tag{23}$$

Thus, sovereign bond investors are compensated for taking Brownian risk. The expected excess return can be read as (minus) the local covariance between (a) sovereign debt returns and (b) the creditors' pricing kernel. This risk compensation is similar to a standard multi-factor asset pricing compensation. Indeed, we can interpret $\partial_X \ln D$ as the market beta of sovereign debt w.r.t. the shock vector $\boldsymbol{B_t}$, while $\boldsymbol{\sigma}_X\boldsymbol{\nu}$ is the small open economy's income claim's risk premium earned in connection with such shock.

4.5 Optimality of Smooth Issuance Policies

We focus our attention on a smooth Markov perfect equilibrium of our game and derive necessary conditions for such an equilibrium to exist. In the continuation region (i.e. when the government is performing), the government value function satisfies the following HJB equation:

$$\delta V=\sup_{I}\left[Y+ID-\left(\kappa+m\right)F+\boldsymbol{\mu}_{X}\cdot\partial_{X}V+\frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_{X}'\partial_{XX'}V\boldsymbol{\sigma}_{X}\right)+\left(I-mF\right)\partial_{F}V\right] \tag{24}$$

In the default region $(Y,F)\in\mathcal{O}(s)$, the small open economy's income drops down by a factor α and the sovereign debt face value is haircut by a factor $\alpha\theta$:

$$V\left(Y,F,s\right)=V\left(\alpha(Y,s)Y,\alpha(Y,s)\theta(Y,s)F,s\right),\quad\left(Y,F\right)\in\mathcal{O}(s) \tag{25}$$

Default optimality gives a condition that is imposed on the boundaries of the default region:

$$\partial_{X}\left[V\left(Y,F,s\right)\right]=\partial_{X}\left[V\left(\alpha(Y,s)Y,\alpha(Y,s)\theta(Y,s)F,s\right)\right] \tag{26}$$

Loosely speaking, this condition imposes a minimum amount of "smoothness" of the value function at the boundaries of the default region, and is essential when using verification theorems that establish the optimality of the government decisions. For a solution to equation (24) with I finite to exist, we must have:

$$D(Y,F,s)+\partial_{F}V(Y,F,s)=0 \tag{27}$$

Equation (27) is a necessary condition that needs to hold in equilibrium. It turns out that an absolutely continuous face value policy (in other words, a "smooth" issuance policy) is

optimal in equilibrium if and only if the debt price function D is decreasing in the face value F and equation (27) holds. The proof is identical to the proof in DeMarzo and He (2014) and is thus omitted. Reinjecting the optimality condition (27) into equation (24) leads to:

$$\delta V = Y - (\kappa + m) F + \boldsymbol{\mu}_X \cdot \partial_X V + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}_X' \partial_{XX'} V \boldsymbol{\sigma}_X\right) - mF\partial_F V \tag{28}$$

Using Feynman-Kac, equation (28), in conjunction with equations (25-26), has the following integral representation:

LEMMA 2. Let I_0 be the issuance policy consisting in never issuing any new debt. In any smooth Markov perfect equilibrium, the government value function is identical to its value if it was allowing its debt to amortize, without ever re-issuing new debt or buying back existing debt:

$$V(Y, F, s) = \sup_{\tau \in \mathcal{T}} J(Y, F, s; (I_0; \tau); D)$$

In addition, the life-time utility function of a government without debt oustanding is equal to its autarky value:

$$V(Y, 0, s) = \mathbb{E}^{Y,s}\left[\int_0^\infty e^{-\delta t} Y_t dt\right]$$

The surprising result of lemma 2 is that the welfare value of a government without any debt outstanding is exactly equal to the autarky welfare[15]. While there should be gains from trade in this economic environment (since the government is more "impatient" than its creditors), those gains are entirely dissipated by default costs. When the government is indebted, the welfare of the government can be expressed as the sum of (a) the welfare of a debt-free government whose country suffers a downward income drop each time the government defaults, minus (b) the aggregate value of sovereign debt, computed *as if creditors were risk-neutral with a discount rate* δ:

$$\begin{aligned} V(Y, F, s) &= \mathbb{E}^{Y,F,s}\left[\int_0^{+\infty} e^{-\delta t}\left[Y_t^{(\boldsymbol{\tau}^*)} - (\kappa + m) F_t^{(I_0, \boldsymbol{\tau}^*)}\right] dt\right] \\ &= \underbrace{\mathbb{E}^{Y,F,s}\left[\int_0^{+\infty} e^{-\delta t}\left(\Pi_{j=1}^{N_{d,t}^{(\boldsymbol{\tau}^*)}} \alpha_{\tau_j}\right) Y_t dt\right]}_{\text{value of credit-risky endowment}} - F \underbrace{\mathbb{E}^{Y,F,s}\left[\int_0^{+\infty}\left(\Pi_{j=1}^{N_{d,t}^{(\boldsymbol{\tau}^*)}} \alpha_{\tau_j}\theta_{\tau_j}\right) e^{-(\delta+m)t}(\kappa + m) dt\right]}_{\text{credit-risky debt price, valued at discount rate } \delta} \end{aligned}$$

This result is related to the conjecture made in Coase (1972), and formally proven by Stokey

[15]Note that this indifference between (i) financial autarky and (ii) starting to take on debt leads to an equilibrium indeterminacy at the point $F = 0$. Indeed, the usual "trivial" equilibrium in which the government never borrows, and debt prices are equal to zero if $F > 0$, still exists.

(1981) and Gul, Sonnenschein, and Wilson (1986), who show that a monopolist with constant marginal costs selling a durable good to a continuum of consumers will actually behave competitively, in the continuous-time limit, and not extract any monopoly rent. In the context of our model, the government acts as a monopolist over a durable good – the exponentially amortizing sovereign debt. Default risk embedded in the sovereign debt creates a downward sloping bond price schedule, analogous to the downward sloping demand curve arising from the distribution of consumer's private valuations in Coase's model. Without commitment, no matter how many bonds the government sold in the past, the government will sell more bonds if there are marginal gains from doing so (in other words if $D(Y,F) > -\partial_F V(Y,F)$). In equilibrium, it must thus be the case that $D(Y,F) = -\partial_F V(Y,F)$, which makes the government indifferent between any amount of bond issuances (per unit of time), stripping away any potential welfare gain that the government may extract from facing financiers that discount cash flows at a rate strictly lower than the government discount rate. Investors in our model are competitive, and thus do not extract any welfare gains either, leading to our main result that trades occur in equilibrium, but for different reasons, none of our economic agents capture any of the potential gains from trade.

Our no-welfare result does not depend on the assumed maturity profile of the sovereign debt contract. In other words, irrespective of the parameter m governing the average life of long term bonds issued, a small open economy without any debt outstanding does not reap any welfare gains from selling bonds to more "patient" lenders. But our result goes even further. Indeed, it does not depend on the repayment profile of the bonds issued: those bonds could be "bullet" as opposed to exponentially amortizing, or they could have an arbitrary "sinking fund" schedule. They could even be state-contingent, with a face value indexed to the Brownian vector $\boldsymbol{B}_t$ (we investigate this set-up in section A.8), with an identical outcome for the small open economy.

The government has a two-pronged commitment problem: it cannot commit either to a particular financing policy, nor to a default policy. Our no-welfare result stems from both commitment problems taken together. In other words, imagine that the government could credibly commit to defaulting whenever the state (Y_t, F_t) was reaching some set $\tilde{\mathcal{O}}(s_t) \neq \mathcal{O}^*(s_t)$, but still was unable to commit to a future path of bond issuances – this case is analyzed in section 6, where we show for a particular income process that the government incurs a first-order welfare gain (resp. loss) if it can commit to defaulting at a higher debt level (resp. lower debt level) than in our smooth Markov perfect equilibrium. Similarly, imagine that the government can credibly commit to a particular financing policy, but cannot commit to always repaying its debt – a particular example of this situation is investigated in section 6.1, where we show that the government, in such case, does extract welfare gains

from being able to finance itself with more patient lenders.

Our "no-welfare gain" result is also due to the continuous-time nature of our model; as highlighted by Stokey (1981), and as illustrated in section 3 in the geometric Brownian motion case, the discrete time counterpart to our model yields strictly positive welfare gains for the risk-neutral government, since in such case, the government can commit not to issue bonds and not to default during a strictly positive measure of time.

Similarly, the linear preferences of the government are analogous to the constant marginal cost assumption in the durable goods' monopoly problem. The critical assumption is that the flow payoff of the government is an affine function of the product of (a) amount of bonds sold, times (b) the price of such bonds. In our smooth equilibrium, the government ends up perfectly indifferent as to the notional amount of bonds sold per unit of time; the costs and benefits of the marginal and infra-marginal units of debt issued are equal. If instead the government exhibits some degree of risk aversion or has a finite intertemporal rate of substitution, the result above no longer holds: the benefit of the marginal unit of debt sold is no longer equal to the benefit of inframarginal units, and the government will extract welfare gains from issuing bonds to investors whose implied interest rate is lower than the government's rate of time preference. This result is analogous to what is showed theoretically in Kahn (1986) in the context of the durable goods monopoly problem: rents can be extracted by the monopolist if its marginal production costs are increasing.

One additional result comes out of our analysis of equation (28). Note indeed that neither risk-free rates $\{r(s)\}_{s\in\mathcal{E}}$, nor risk-prices $\{\boldsymbol{\nu}(s)\}_{s\in\mathcal{E}}$ appear in equation (28). They appear neither in equation (25), nor in equation (26), the value-matching and smooth pasting conditions at default. In other words, the life-time utility function of the government is entirely independent of the characteristics of investors in international debt markets; whether risk-free rates are high or low, or whether risk-prices are high or low, the welfare of the government is identical. This also allows us to derive one more implication: if the income drift function $\mu_Y(\cdot,\cdot)$, the income volatility vector $\boldsymbol{\sigma}_Y(\cdot,\cdot)$, and the punishment functions α, θ are independent of the state s, then the value function of the government does not depend on the state s. We will study such a case in section 5.

Finally, we conclude this section by discussing the issue of equilibrium multiplicity. The sovereign default literature has struggled with the possibility that the Markov perfect equilibria studied are not unique. In discrete time, while equilibrium uniqueness obtains with one-period defaultable debt[16], we are not aware of any paper establishing uniqueness in the presence of long term debt. In fact, a simple reasoning suggests the possibility that we might

[16]This result was proven rigorously by Auclert and Rognlie (2016) when the default decision is taken before the bond auction.

find multiplicity in this class of models: if creditors price the sovereign bonds issued at a low level, it will be optimal for the government to default "early", i.e. at debt levels that are relatively low and/or income levels that are relatively high. Instead, if creditors price the sovereign bonds issued at a level close to "par", it will be optimal for the government to default at debt levels that are relatively high and/or income levels that are relatively low. Instead, in our particular modeling environment, we can use lemma 2 to conclude the following:

COROLLARY 1. In our class of equilibria of focus, if a smooth Markov perfect equilibrium exists, then it must be unique.

The proof is immediate, once we notice that in our equilibria of focus, the problem solved by the government is equivalent to a single-agent default problem that is independent of the pricing of debt – neither the HJB equation in the continuation region, nor the value matching condition at the default boundary or the smooth-pasting default optimality condition contain the debt price function.

4.6 The Optimal Financing Policy

We now characterize the financing policy of the government. We use $D(Y,F,s)+\partial_F V(Y,F,s)=0$ and differentiate equation (28) w.r.t. F to obtain:

$$(\delta+m)D=\kappa+m+\boldsymbol{\mu}_X\cdot\partial_X D+\frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}_X'\partial_{XX'}D\boldsymbol{\sigma}_X\right)-mF\partial_F D$$

Note that $\int_0^t e^{-\int_0^v(r(s_u)+m)du}(\kappa+m)\,dv+e^{-\int_0^t(r(s_u)+m)du}D_t$ represents the cumulative debt investors' gain rate. It must be a $\mathbb{Q}$-martingale, which allows us to derive the partial differential equation satisfied by D:

$$(r+m)\,D=\kappa+m+(\boldsymbol{\mu}_X-\boldsymbol{\sigma}_X\boldsymbol{\nu})\cdot\partial_X D+\frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}_X'\partial_{XX'}D\boldsymbol{\sigma}_X\right)+(I(Y,F,s)-mF)\,\partial_F D$$

Subtracting one equation from the other allows us to obtain a formula for the optimal Markov issuance policy, as a function of the debt price (which itself can be computed via equation (27)).

LEMMA 3. In our class of equilibria of focus, the optimal bond issuance policy of the government is:

$$I^*(Y,F,s)=\frac{\delta-(r(s)+\pi(Y,F,s))}{-\partial_F\ln D(Y,F,s)}\tag{29}$$

Upward shocks to risk-free rates $r(s_t)$ or risk-premia $\pi(Y_t, F_t, s_t)$ cause the government to adjust its bond issuance policy downwards, creating an upward current account adjustment[17].

In other words, the bond issuance rate is (a) proportional to the wedge between (i) the rate of impatience of the government and (ii) the required rate of return of international bond investors, and (b) inversely proportional to the semi-elasticity of the bond price function w.r.t. the aggregate face value F. The optimal Markov policy uncovered in equation (29) delivers additional insights. First, when the probability measure $\mathbb{Q}$ of investors and the probability measure $\mathbb{P}$ of the government correspond to each other (in other words, when $\boldsymbol{\nu}(s) = \mathbf{0}$ for all s), the issuance policy takes the simple form:

$$I^*(Y, F, s) = \frac{\delta - r(s)}{-\partial_F \ln D(Y, F, s)}$$

This means that bond issuances are always positive: in this particular case, it is never efficient for the government to buy back its own debt. This result echoes an insight from Bulow, Rogoff, and Dornbusch (1988) who show, in the context of a one-period model of sovereign default with a risk-neutral government and risk-neutral lenders, that it is never welfare-improving for a country to buy back its own debt.

This result breaks down in the presence of risk-averse lenders, whose price of risk has a positive correlation with the country's endowment process: in such case, equation (29) shows that when sovereign bond risk premia are sufficiently high, the country might find it optimal to buy back its own debt. This result stems from the fact that the probability measure under which investors discount cash-flows (the "risk-neutral" measure) is different from the probability measure (the "physical" measure) under which the government optimizes. A different interpretation of this result can be put as follows: persistent differences in beliefs about the growth rate of the country's income (where investors would be more "pessimistic" than the government) would also lead the government to buy back debt when investors are sufficiently pessimistic compared to the government.

4.7 When Autarky is better than Trade

Our no-welfare result is applicable when the preferences of the government correspond to those of the citizens of the small open economy. One could instead imagine, for political economy reasons, that the goverment has an effective discount rate greater than the discount rate of the citizens of the small open economy; this assumption could be rationalized for

[17] We define the current account balance as $Y_t - C_t$

example by observing that government officials are elected for short durations. It turns out that in such case, the commitment problem faced by the government has an even worse outcome for citizens' welfare, as we discuss in the following lemma.

LEMMA 4. Suppose that government officials have an effective discount rate δ, while the citizens of the small open economy are more "patient", with a discount rate $\hat{\delta} < \delta$. Consider a simplified economic environment in which, upon default, the entire small open economy's income stream is lost and creditors incur a full loss on their investment (i.e. $\alpha = \theta = 0$). Assume also that in our smooth Markov perfect equilibrium, the optimal issuance policy $I(Y, F, s)$ is always weakly positive[18]. Let $\hat{V}(Y, F, s)$ be the indirect utility function of the citizens of the small open economy when the government makes financing and default decisions, and $\hat{V}_a(Y, F, s)$ the indirect utility function of the citizens of a country in financial autarky, and whose default decisions follow an identical stopping rule. Then $\hat{V}(Y, F, s) < \hat{V}_a(Y, F, s)$: the citizens are strictly worse off when their country has access to international debt markets (and the government makes borrowing and default decisions) than when their country is in financial autarky. In addition, in all states, $\partial_F \hat{V} + D < 0$, meaning that if citizens were able to, they would want to buy back outstanding bonds and reduce the country's indebtedness.

The proof is straightforward and is discussed in section A.3. While the government is exactly indifferent between (a) financial autarky and (b) having access to international debt market, its citizens are strictly worse off, suggesting an economic environment where autarky is better than trade. This result, while surprising, stems from the following intuition: the government, using discount rate δ, balances the upside of taking on debt (frontloading consumption by selling a lot of bonds today) vs. its downside (suffering a permanent income loss upon default in the future). Since the benefits of debt issuances are incurred today, whereas the costs are incured in the future, citizens of the country, who discount cash flows at a lower rate, will weigh those benefits less than their related cost, and will thus incur a welfare loss compared to the autarky benchmark.

5 Geometric Brownian Motion Income Process

The results discussed until now are general to an entire class of income and stochastic discount factor processes, but they assume the existence of a smooth equilibrium. In this section, we focus on a particular income process, show the existence and uniqueness of a smooth equilibrium, characterize fully such equilibrium, and provide some comparative static results.

[18] This is the case for example when creditors and the government use identical probability measures to discount cashflows, or when the risk-price vector $\boldsymbol{\nu}(s)$ is sufficiently small to keep $I(Y, F, s) \geq 0$.

To achieve this, we assume an income process of the small open economy that is independent of the discrete state s and follows:

$$dY_t = Y_t \left(\mu dt + \boldsymbol{\sigma} \cdot d\boldsymbol{B_t}\right) \tag{30}$$

To insure that assumption 1 is satisfied, we need to impose that:

$$\delta > \mu \tag{31}$$

We also assume that the small open economy's income drop upon default α, and the debt-to-income reduction upon default θ, are constant across the state space. In section A.10, we relax these assumptions and study a more general income process – namely Markov-modulated geometric Brownian motion, for which the equilibrium can also be solved analytically. Given the notational complexity of this case, we restrict ourselves in the main body of the paper to the simple geometric Brownian motion case.

5.1 Equilibrium Characterization in GBM Case

Using our remark at the end of section 4.5, the government life time utility function V must be independent of the state s_t. Given the optimality condition (27), the bond prices must also be independent of the discrete state s_t. Since the flow payoff and the state dynamics are linear in the state variables (Y, F), we notice immediately that the value function V and the optimal issuance policy I must be homogeneous of degree 1 in (Y, F), and the default policy will be cutoff. Given equation (27), the debt price function must be homogeneous of degree 0 in (Y, F). Let $x := F/Y$ be the debt-to-income ratio of our small open economy. Since the optimal issuance policy must be homogeneous of degree 1 and the optimal default policy must be cutoff, we can write such policies as follows:

$$\begin{aligned} I(Y, F, s) &= \iota(x, s)Y \\ \tau_{k+1} &= \inf\{t \geq \tau_k : F_t/Y_t \geq \bar{x}\} \end{aligned}$$

In other words, the government will elect to default as soon as the small open economy's debt-to-income ratio is greater than or equal to a certain threshold $\bar{x}$. The function ι satisfies $\iota(x, s) := I(1, x, s)$. The life time utility function for the government can be expressed as $V(Y, F) := Yv(x)$, and the debt price can be written $D(Y, F) := d(x)$. The functions v and d satisfy $v(x) := V(1, x)$ and $d(x) = D(1, x)$. The debt-to-income ratio $x_t^{(\iota,\boldsymbol{\tau})}$ is a controlled

stochastic process that evolves as follows on $(0, \bar{x})$:

$$dx_t^{(\iota,\boldsymbol{\tau})} = \left(\iota(x_t^{(\iota,\boldsymbol{\tau})}, s_t) - \left(m + \mu - |\boldsymbol{\sigma}|^2\right) x_t^{(\iota,\boldsymbol{\tau})}\right) dt - x_t^{(\iota,\boldsymbol{\tau})} \boldsymbol{\sigma} \cdot d\boldsymbol{B_t} + (\theta - 1)\, x_t^{(\iota,\boldsymbol{\tau})} dN_{d,t}^{(\boldsymbol{\tau})}$$

Note that while the life time utility function, the default cutoff, and the debt prices must be independent of the state s_t, it is not the case for the issuance policy, which, according to equation (29), depends on the level of international interest rates and risk-prices. In this setup, we have a complete characterization of the government value function, debt prices, and issuance policy.

PROPOSITION 1. In the case where the small open economy's income process is a geometric Brownian motion, there exists a unique "smooth" equilibrium, where the life-time (normalized) government value function $v(x)$, the debt price $d(x)$ and the optimal default cutoff $\bar{x}$ have the following expressions, for $x \in (0, \bar{x}]$:

$$d(x) = \left(\frac{\kappa + m}{\delta + m}\right)\left[1 - \left(\frac{1 - \alpha\theta}{1 - \alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi - 1}\right] \tag{32}$$

$$v(x) = \frac{1}{\delta - \mu}\left(1 - \left(\frac{1 - \alpha}{1 - \alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi}\right) - x d(x) \tag{33}$$

$$\bar{x} = \frac{\xi}{\xi - 1}\left(\frac{\delta + m}{\kappa + m}\right)\left(\frac{1 - \alpha}{1 - \alpha\theta}\right)\frac{1}{\delta - \mu} \tag{34}$$

In the above, $\xi > 1$ is a constant that only depends on the model parameters $\delta, \mu, \boldsymbol{\sigma}, m$, and not on the level of interest rates or the prices of risk. For $x > \bar{x}$, $n(x) := 1 + \lfloor \frac{\ln x - \ln \bar{x}}{-\ln \theta} \rfloor$ represents the number of times the government will default consecutively in order to re-enter the continuation region. For $x > \bar{x}$, the life-time government value function $v(x)$ and the debt price $d(x)$ satisfy:

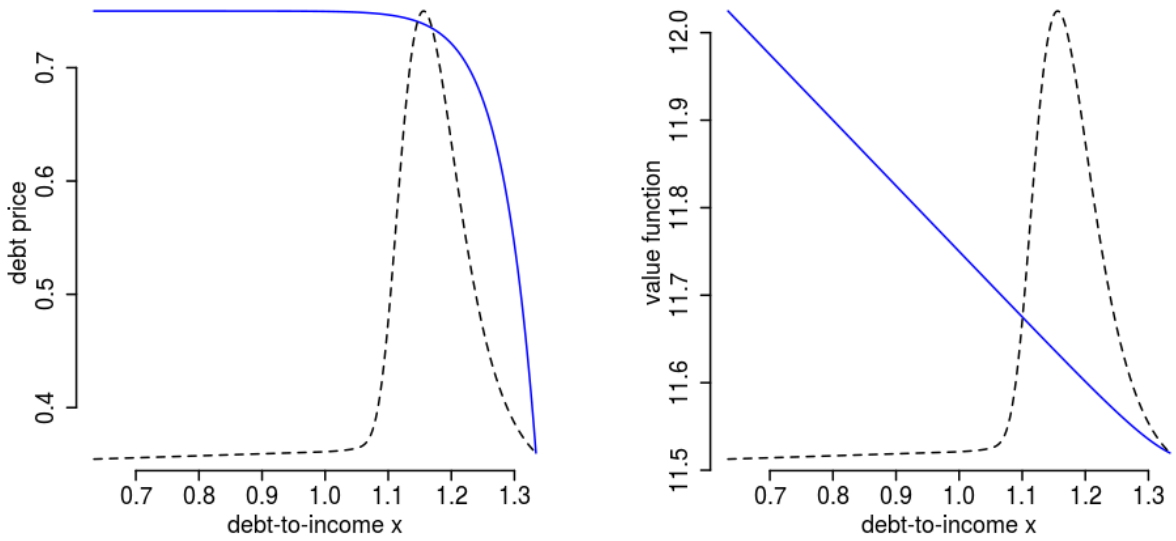
$$d(x) = (\alpha\theta)^{n(x)}\, d\left(\theta^{n(x)} x\right) \tag{35}$$

$$v(x) = \alpha^{n(x)} v\left(\theta^{n(x)} x\right) \tag{36}$$

Risk-premia required by investors are equal to:

$$\pi(x, s) := \frac{\xi - 1}{\left(\frac{1 - \alpha\theta^{\xi}}{1 - \alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi - 1} - 1} \boldsymbol{\sigma} \cdot \boldsymbol{\nu}(s) \tag{37}$$

The scaled optimal financing policy $\iota^*(x, s)$ depends on the discrete state s and has the



(a) Debt Price $d(x)$ (b) Value Function $v(x)$

Figure 1: Debt Price and Value Function

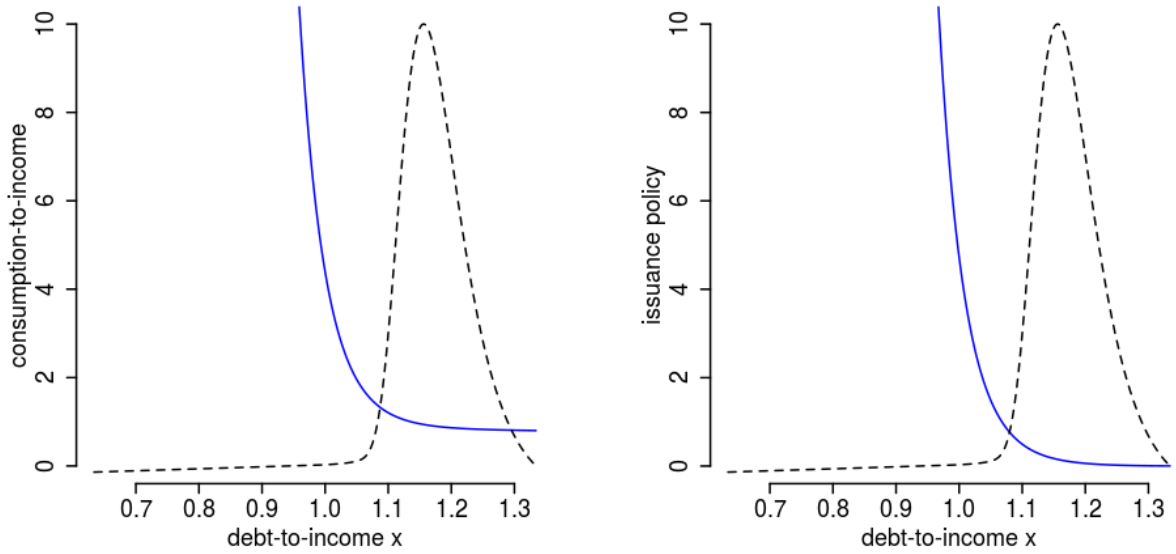
following expression:

$$\iota^*(x,s) = \frac{\delta - r(s)}{\xi - 1}\left[\left(\frac{1-\alpha\theta^\xi}{1-\alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi-1} - 1\right] x - \boldsymbol{\nu}(s)\cdot\boldsymbol{\sigma}x \tag{38}$$

The financing policy $\iota^*(x,s)$ is a strictly decreasing function of x if $\delta + m > |\boldsymbol{\sigma}|^2 - (m+\mu)$, and is otherwise hump-shaped.

Our proof is detailed in section A.4. In figure 1, we provide an illustrative example of a debt price function $d(x)$ and the life-time (income normalized) utility function $v(x)$ for specific model parameters[19]. The dotted black line shows the stationary distribution of the state variable x_t – such distribution is characterized in section 5.3.1. The debt price function is decreasing in the debt-to-income ratio, a necessary condition for the optimality of our smooth equilibrium. The value function v is $\mathcal{C}^2$ (and actually, $\mathcal{C}^\infty$) on $\mathbb{R}_+$ except at points of the type $x = \theta^k\bar{x}$, for $k \in \mathbb{N}$, where it is $\mathcal{C}^1$, guaranteeing the optimality of the default boundary (this is the so called smooth-pasting condition, discussed in our appendix). Figure 2 illustrates the resulting optimal consumption-to-income policy $c(x)$ and financing policy $\iota(x)$. Both are declining functions of the debt-to-income ratio x. The bond issuance policy stays positive (since we have assumed $\nu = 0$). It can be showed that the drift rate

[19] Our example assumes that $\mu = 2\%$ p.a., $\sigma = 10\%$ p.a., $\delta = 10\%$ p.a., $r = 5\%$ p.a., $\kappa = 5\%$ p.a., $1/m = 10$ years, $\theta = 50\%$, $\alpha = 96\%$ and $\nu = 0$.



(a) Consumption-to-income $c(x)$ (b) Issuance Rate $\iota(x)$

Figure 2: Consumption and Financing Policies

$\iota(x) - (m + \mu - \sigma^2)x$ of the state variable x_t is downward sloping and intersects zero, meaning that x_t is "mean-reverting".

The formulas above are a special case of the general results established previously: the life-time utility function of the government is independent of international financial markets, and converges to the autarky value as the debt-to-income ratio gets arbitrarily close to zero. The issuance policy reacts to interest rate shocks, as well as risk-price shocks. We notice immediately that in states $s \in \mathcal{E}$ where risk prices $\boldsymbol{\nu}(s)$ are sufficiently high and positively correlated with the income volatility vector $\boldsymbol{\sigma}$, it might become optimal for the government to buy back its own bonds.

Our proof establishes by construction the existence and uniqueness of the smooth Markov perfect equilibrium in the presence of long term debt. The closed-form expressions of Proposition 1 allow us to derive the following comparative static results for the optimal default boundary.

COROLLARY 2. The default boundary $\bar{x}$ and the life-time utility function $v(x)$ (keeping fixed the debt-to-income ratio x) admit comparative statics w.r.t. model parameters as disclosed in table 1; the (potentially infinite) constant $\bar{\kappa}$ depends on all other model parameters and is such that $\bar{\kappa} > \delta$.

Those results are established in section A.5 and section A.6. The comparative static result

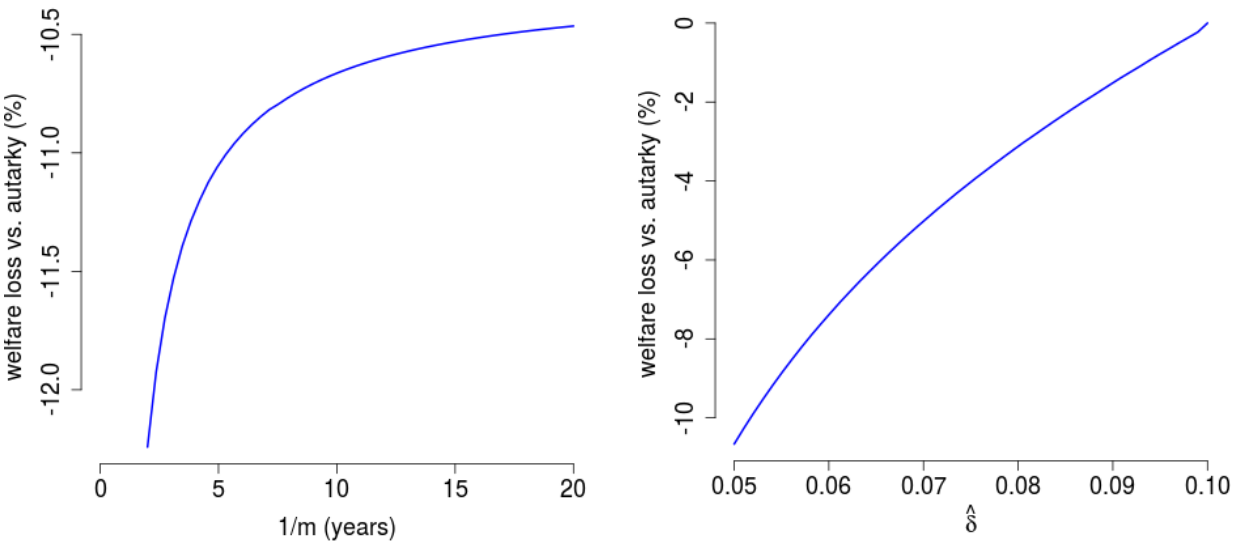
Parameter	$\bar{x}$	$v(x)$
$\lvert\boldsymbol{\sigma}\rvert$	$+$	$+$
μ	$+$	$+$
δ	$-$	$-$
κ	$-$	$-$
$1/m$	$+$ if $\kappa < \bar{\kappa}$	$+$ if $\kappa < \delta$
$1-\alpha$	$+$	$-$
θ	$+$	$-$

Table 1: Comparative Static Table

w.r.t. α and θ are easy to interpret. When one increases the punishment upon default (i.e. when $1-\alpha$ increases or θ increases), the incentive for the government to default decreases, making the optimal debt-to-income default boundary higher. This also makes the government unconditionally worse off. An increase in debt service payments (whether related to a higher coupon rate κ, or a higher rate of principal amortizations m) also leads to a lower debt-to-income default boundary. To see why, note that the flow utility depends negatively on the coupon rate κ, making the incentive to default unconditionally increasing in κ. Similarly, the flow utility depends negatively on the rate of principal amortizations m; however, there is an opposite effect, due to the fact that the drift of the debt-to-income ratio depends also negatively on m; however, it can be showed that for low values of the coupon rate κ, the former dominates the latter, and the debt-to-income optimal cutoff $\bar{x}$ is increasing in the average maturity of sovereign debt. Both an increase in coupon rate κ or the pace of principal amortizations m also lead to lower government welfare. More impatient governments will default at lower debt-to-income ratios; indeed, they will frontload consumption via large bond issuances. Since they discount utility flow at higher levels, this leads to lower welfare. Finally, the default boundary is increasing in the income volatility of the country; one can think about the indebted government as being long a put option, and using an analogy familiar to the option pricing literature, since the value of an option generally increases with volatility, the incentive for the government to default decreases with higher volatility, making the default boundary an increasing function of the small open economy's income volatility, and making the life-time utility of the government an increasing function of such parameter.

5.2 Citizens vs. Government

We now investigate the case discussed in section 4.7, in which citizens have a discount rate $\hat{\delta}$ that is lower than the government discount rate δ. As a reminder, we showed theoretically that the citizens' indirect utility function is strictly lower than the indirect utility function



(a) As a function of maturity $1/m$

(b) As a function of impatience $\hat{\delta}$

Figure 3: Welfare Losses $\hat{v}(0)/\hat{v}_a(0) - 1$

if the country was in financial autarky[20].

In figure 3a, we plot the percentage loss in welfare, when the goverment is not indebted, for the citizens of the country, as a function of the debt maturity choice, assuming that citizens discount consumption flows at the international risk free rate r. As the debt average maturity of the government bonds decreases, it turns out that the welfare loss incurred by citizens increases. The intuition behind this result is straightforward: with shorter term debt, the government defaults sooner (i.e. at a lower debt-to-income ratio) than with longer term debt, as was showed in corollary 2. For the government, the (future) costs of a default are perfectly equalized with the (current) benefits of greater debt issuances and greater consumption, but since citizens are more patient than the government, they weigh the future default costs more than the current consumption benefits. If those default costs increase (due to an earlier default linked to a shorter debt average life), they are worse off.

In figure 3b, we simply show citizens' welfare loss as a function of the discount rate $\hat{\delta}$. Not surprisingly, the closer citizens' discount rate is from the government discount rate, the lower the welfare loss vs. the autarky benchmark.

[20]Note that the proof we provide assumes a full loss of income at default, whereas our numerical calculations displayed show the case $\alpha = 96\%$ and $\theta = 50\%$.

5.3 Model-Implied Macroeconomic Predictions

The model presented above has a variety of economic implications that can be established rigorously. We present in this section a few of these implications when the state s_t is trivially equal to 1, and when $\boldsymbol{\nu} := \mathbf{0}$.

5.3.1 Ergodic Distribution and Average Default Rate

Our model admits a stationary distribution f that is relatively straightforward to characterize. Note $\mu_x(x)$ the drift rate of the state variable x, $\sigma_x(x)$ its volatility, and $J(x)$ the probability flux:

$$\mu_x(x) := \iota(x) - \left(m + \mu - \sigma^2\right) x \tag{39}$$

$$\sigma_x(x) := \sigma x \tag{40}$$

$$J(x) := \mu_x(x) f(x) - \frac{d}{dx}\left[\frac{\sigma_x^2(x)}{2} f(x)\right] \tag{41}$$

$J(x)$ can be interpreted as the probability current at x – i.e. the "mass of particles per unit of time" that crosses at x, if one were to interpret our stochastic differential equation for x_t as describing the movement of particles getting hit by idiosyncratic Brownian shocks. The debt-to-income ergodic distribution of our economy satisfies a standard Kolmogorov-Forward equation, valid for $x \in (0, \bar{x}), x \neq \theta\bar{x}$:

$$0 = \frac{dJ}{dx}$$

The above equation is a second order ordinary differential equation for the unknown density f, and two conditions are necessary to pin down its value across the state space:

$$f(\bar{x}) = 0 \qquad\qquad \int_0^{\bar{x}} f(x)dx = 1$$

The first condition is a standard condition for absorbing boundaries, while the second condition states that the measure f needs to be a density. In section A.7, we discuss a simple numerical method to integrate this differential equation, by approximating the stochastic process $\{x_t\}_{t\geq 0}$ by a discrete state discrete time Markov chain. Using the interpretation of J as a probability current, $J(\bar{x})$ directly gives the ergodic average default time for the sovereign government. Since $f(\bar{x}) = 0$, we obtain an ergodic default rate equal to:

$$-\frac{\sigma^2\bar{x}^2}{2} f'(\bar{x})$$

5.3.2 Credit Spreads

The sovereign bond spread $\varsigma(x)$ is the constant margin over the risk-free benchmark that is needed to discount the sovereign bond's cash flow stream assuming away any default risk. In other words, the credit spread must satisfy by definition:

$$d(x) := \hat{\mathbb{E}}\left[\int_0^\infty e^{-(r+m+\varsigma(x))t}\left(\kappa+m\right)dt\right]$$

In our smooth Markov perfect equilibrium, the credit spread $\varsigma(x)$ is equal to:

$$\varsigma(x) = \frac{m+\delta}{1-\left(\frac{1-\alpha\theta}{1-\alpha\theta^\xi}\right)\left(\frac{x}{\bar{x}}\right)^{\xi-1}} - (r+m)$$

Figure 4a provides an illustration of the credit spread as a function of the debt-to-income ratio. The credit spread is monotonically increasing with the debt-to-income ratio, and its smallest value is when the government has no debt outstanding: in such case, the credit spread is equal to the wedge $\delta - r$ between the government rate of time preference and the creditor risk-free rate.

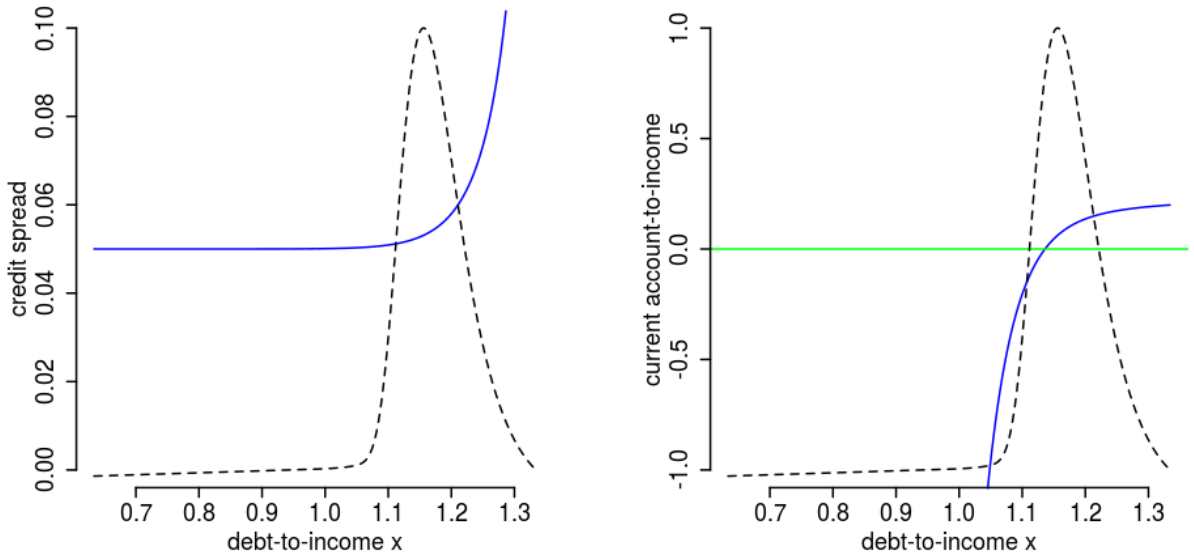
5.3.3 Current Account and Consumption Growth Volatility

Our analytical expressions for the debt price d and the issuance policy ι allow us to derive the consumption-to-output ratio $c(x) := 1 + \iota(x)d(x) - (\kappa+m)x$. The current account-to-income ratio is then simply $1 - c(x)$. Some algebra allows us to compute the derivative of the consumption-to-output ratio $c(x)$:

$$c'(x) = -\left(\frac{\delta-r}{\xi-1}\right)\left(\frac{\kappa+m}{\delta+m}\right)\left[\xi\left(\frac{1-\alpha\theta^\xi}{1-\alpha\theta}\left(\frac{\bar{x}}{x}\right)^{\xi-1} - \frac{1-\alpha\theta}{1-\alpha\theta^\xi}\left(\frac{x}{\bar{x}}\right)^{\xi-1}\right) - 2\left(\frac{1-\alpha\theta^\xi}{1-\alpha\theta}\left(\frac{\bar{x}}{x}\right)^{\xi-1} - 1\right)\right] - (\kappa+m)$$

This expression makes it clear that under the parameter restriction $\xi > 2$ (i.e. when $\delta + m > |\boldsymbol{\sigma}|^2 - (\mu+m)$), the consumption-to-output ratio is a strictly decreasing function of the debt-to-income ratio on $[0, \bar{x}]$. In other words, the current-account is a counter-cyclical variable in this model – a sequence of good income shocks will push the debt-to-income ratio downwards, causing the small open economy to run a current account deficit; instead, a sequence of bad income shocks will push the small open economy closer to its default boundary, forcing it to run a current account surplus, as illustrated in figure 4b.

One can also compute the ratio of consumption growth volatility divided by output growth



(a) Credit Spread $\varsigma(x)$ (b) Current Account $1 - c(x)$

Figure 4: Macroeconomic Implications

volatility as follows:

$$\frac{\text{stdev}_t\left(d\ln C_t\right)}{\text{stdev}_t\left(d\ln Y_t\right)} = 1 - \frac{x_t c'(x_t)}{c(x_t)}$$

Since $c'(x) < 0$, the consumption growth volatility is greater than the output growth volatility across debt-to-income ratios – the model is thus qualitatively consistent with the empirical evidence of Neumeyer and Perri (2005) or Aguiar and Gopinath (2004), who show that consumption growth volatility for emerging market economies is systematically greater than output growth volatility[21].

5.3.4 GDP-Linked Bonds

It is straightforward to extend our analysis and consider GDP-linked bonds. As discussed previously, our no-welfare result extends to any debt maturity structure, as well as environments where the debt contract has some state-contingency. Consider for example bonds whose principal balance goes up and down with income shocks – and note ς the sensitivity

[21] Neumeyer and Perri (2005) for example compute ratios of standard deviations of total consumption over GDP for Argentina (1.17), Brazil (1.24), Korea (2.05), Mexico (1.29) and the Philippines (2.78).

of the debt face value to such shocks. The debt face value in such case evolves as follows:

$$dF_t = (I(Y_t, F_t, s_t) - mF_t)\,dt + F_t \boldsymbol{\varsigma} \cdot d\boldsymbol{B}_t$$

We might for example consider a government issuing bonds that allow investors to "share" some of the small open economy's income risk with foreign investors – by for example setting $\boldsymbol{\varsigma} \propto \boldsymbol{\sigma}$. We treat GDP-linked bonds in details in section A.8, and summarize here the main result. In such a contractual environment, the value function of the government is identical to the value function of a government issuing non-state-contingent bonds but facing income growth risk with volatility vector $\boldsymbol{\sigma} - \boldsymbol{\varsigma}$ instead of $\boldsymbol{\sigma}$. A similar result holds for the default boundary, as well as debt prices. Therefore, as discussed in Corollary 2, the default boundary $\bar{x}$ *decreases* as the small open economy shares more income risk with foreign investors, and the welfare across the state space *decreases* – a result entirely attributable to the preference for this risk-neutral government for higher income volatility, given the option value of defaulting. Finally, the issuance policy of the government is identical to the issuance policy of a government issuing non-state-contingent bonds but facing a shadow interest rate $r(s) + \boldsymbol{\varsigma} \cdot \boldsymbol{\nu}(s)$, and income growth volatility vector $\boldsymbol{\sigma} - \boldsymbol{\varsigma}$.

6 The Welfare Benefit of a Commitment Technology

We now consider different commitment devices and study whether such devices enable the government to capture any of the welfare gains from being able to trade with more patient lenders. Our study will uncover that the specific commitment technology matters; certain of those technologies will not improve the sovereign's welfare, while others will. To simplify the exposition, we now assume that the discrete state variable s_t is trivially equal to 1, in other words that the international risk-free rate is constant equal to r, and that aggregate risk is not priced – i.e. $\boldsymbol{\nu} = \mathbf{0}$.

6.1 Constant Issuance Policy

In this section, we ask ourselves the following question: if a government without any debt outstanding was able to issue a lump amount of bonds in international capital markets, and then commit to a specific financing policy (without committing to a particular default policy), what would be its welfare gain vs. autarky? Following the initial bond issuance, we focus on simple financing policies – policies under which the government commits to issue,

per unit of time, a constant fraction γ of the stock of outstanding debt F_t:

$$I_t = \gamma F_t$$

The particular case $\gamma = 0$ for example would parametrize a government that commits never to issue any additional debt, past the initial lump debt issuance. The particular case $\gamma = m$ would parametrize a government that commits to always roll-over, at each time period, the maturing stock of debt; a similar case has been analyzed extensively in the corporate credit literature, for example in Leland (1998) and Goldstein, Ju, and Leland (2001)[22]. With this particular full-commitment issuance policy I_γ, the life-time utility function of an already-indebted government can be written:

$$V_\gamma(Y, F) = \sup_{\boldsymbol{\tau} \in \mathcal{T}} \mathbb{E}\left[\int_0^{+\infty} e^{-\delta t}\left[Y_t^{(\boldsymbol{\tau})} + \gamma F_t^{(\boldsymbol{\tau}, I_\gamma)} D_\gamma(Y_t^{(\boldsymbol{\tau})}, F_t^{(\boldsymbol{\tau}, I_\gamma)}) - (\kappa + m) F_t^{(\boldsymbol{\tau}, I_\gamma)}\right] dt\right]$$

Arguments similar to what was discussed previously allow us to reduce the dimensionality of the state space, and write the life-time utility function of the government as $V_\gamma(Y, F) := Y v_\gamma(x)$ and the debt price as $D_\gamma(Y, F) := d_\gamma(x)$. The optimal default decision is cutoff, with optimal debt-to-income default boundary $\bar{x}_\gamma$.

Proposition 2. Assume the small open economy's income process is a geometric Brownian motion, and assume that the government can commit to continuously issuing a constant fraction γ of the outstanding stock of debt, after having issued a lump sum amount when debt-free. Restrict $\gamma < m + \delta$. There exists a unique equilibrium of this economy, where the life-time (normalized) government value function $v_\gamma(x)$ is equal to, for $x \in [0, \bar{x}_\gamma]$:

$$v_\gamma(x) = \frac{1}{\delta - \mu}\left(1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi_v}}\right)\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_v}\right) - \left(\frac{\gamma}{r + \gamma - \delta}\right) x d_\gamma(x) \\ - \frac{(\kappa + m)(\delta - r)\, x}{(\delta + m - \gamma)(\delta - r - \gamma)}\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_v}}\right)\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_v - 1}\right] \quad (42)$$

In such equilibrium, the debt price $d_\gamma(x)$ and the optimal default cutoff $\bar{x}_\gamma$ have the following

[22] The debt dynamics in those papers is identical to the debt dynamics in our work when we consider $\gamma = m$. The motive to take on debt is however different: in the corporate credit literature, firms take on debt to take advantage of the tax deductibility of the interest payments, while in our paper, the government issues debt since it is relatively more impatient than the lenders it faces.

expressions, for $x \in [0, \bar{x}_\gamma]$:

$$d_\gamma(x) = \left(\frac{\kappa+m}{r+m}\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_d+1}}\right)\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_d}\right] \tag{43}$$

$$\bar{x}_\gamma = \frac{\xi_v}{\left(\frac{r+m-\gamma}{\delta+m-\gamma} - \frac{\gamma}{r+\gamma-\delta}\right)(\xi_v - 1) + \left(\frac{\gamma}{r+\gamma-\delta}\right)\xi_d}\left(\frac{r+m}{\kappa+m}\right)\left(\frac{1-\alpha}{1-\alpha\theta}\right)\frac{1}{\delta-\mu} \tag{44}$$

In the above, ξ_v, ξ_d are constants that depend on all the model parameters. The optimal initial debt-to-income ratio x^*_γ chosen by the government, when it is not indebted, is strictly positive, and strictly less than the default boundary $\bar{x}_\gamma$. When not indebted, a government who can commit to this particular financing policy achieves a welfare that is strictly greater than its autarky value:

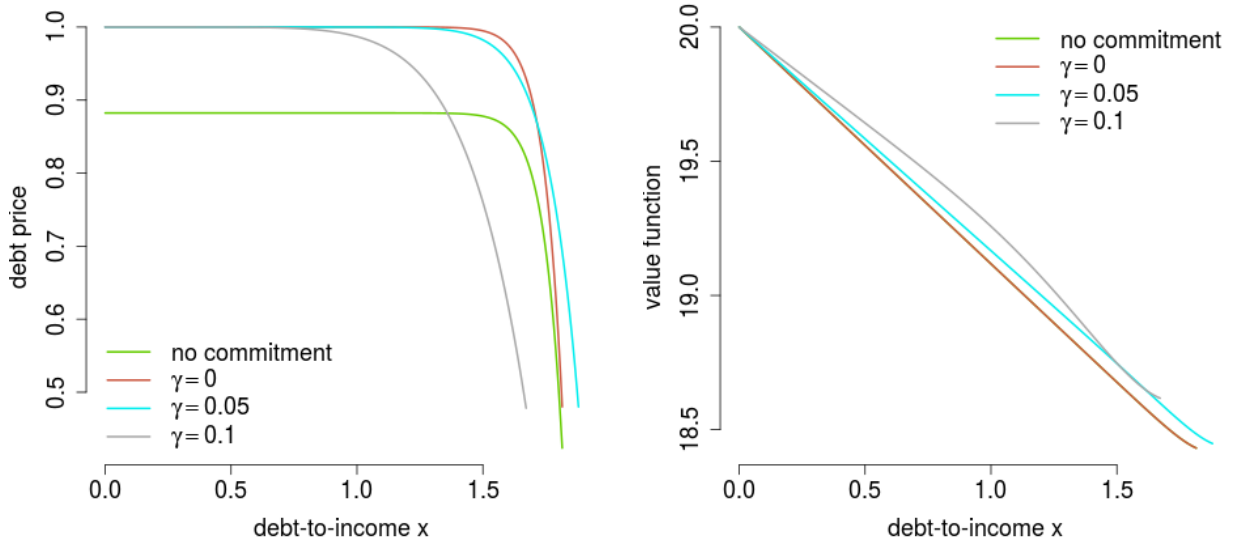
$$\max_F \left[FD_\gamma(Y,F) + V_\gamma(Y,F)\right] > \frac{Y}{\delta-\mu}$$

Our proof is exposed in section A.9. It is worth noting that the value function v_γ corresponds to the no-commitment value function v when $\gamma = 0$. Similarly, the optimal default cutoff $\bar{x}_0$ in that case is the same as the optimal default cutoff in the no-commitment equilibrium. However, with commitment never to issue any bonds again, the debt price function $d_0(x)$ is computed by discounting risky debt cash-flows at the interest rate $r < \delta$, whereas the no-commitment debt price is computed by discounting risky debt cash-flows at the rate δ. In other words, we have $d_0(x) > d(x)$[23].

In figure 5, we plotted the debt price d_γ and the income-normalized value function v_γ for a range of parameters γ, corresponding to the case where new debt issuances correspond to, respectively, 0%, ($\gamma = 0$), 50% ($\gamma = 0.05$) or 100% ($\gamma = 0.1$) of debt redemptions at each time. As discussed before, the value function v_0 is identical to the value function under which the government cannot commit.

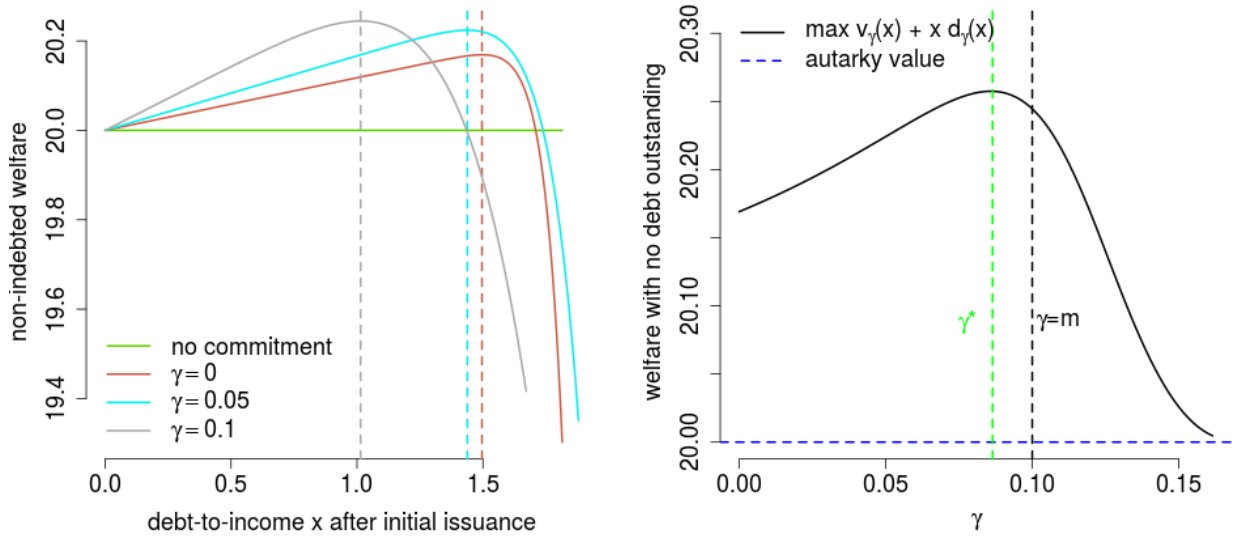
Finally, figure 6a analyzes the welfare of the government, when not indebted, for different choices of initial debt issuances. For each of the issuance rates γ under consideration, the initial welfare is single-peaked, and the maximal attainable welfare is plotted in figure 6b as a function of γ. We show in section A.9 that if the issuance rate is too high (i.e. if $\gamma > m + \delta$), no welfare gain can be achieved vs. autarky – this can be seen in figure 6b as the optimal welfare $\max_x v_\gamma(x) + xd_\gamma(x)$ converges to the autarky welfare $1/(\delta-\mu)$ as $\gamma \to m+\delta$.

[23]For other values of the parameter γ, there is no longer a ranking valid across the state space of $d_\gamma(x)$ vs. $d(x)$ – this is due to the fact that for low values of x, $\iota(x) > \iota_\gamma(x)$, while for high value of x, $\iota(x) < \iota_\gamma(x)$.



(a) Debt Price $d_\gamma(x)$ (b) Value Function $v_\gamma(x)$

Figure 5: Debt Price and Value Function



(a) $v_\gamma(x) + xd_\gamma(x)$ (b) $\max_{x \geq 0} [v_\gamma(x) + xd_\gamma(x)]$

Figure 6: Government Welfare

6.2 Restricted Issuance beyond Debt-to-GDP Limit

In this section, and in order to simplify the exposition, we assume that $\alpha = \theta = 0$: upon default, the small open economy's income falls to zero, and international investors lose their entire investment. Imagine for now that the government alters its constitution, such that it limits future governments from being able to issue additional bonds, if and when the debt-to-income ratio of the small open economy is above and beyond a certain limit x^*. When the debt-to-income ratio is below x^*, the government has full flexibility to issue additional bonds in international capital markets.

It turns out that this particular commitment technology does not always provide welfare gains for the government. When the issuance restriction is structured at a debt-to-income ratio x^* that is "too high", it turns out that the improvement in debt prices is not high enough to allow the government to benefit from gains from trade. In such case, a smooth equilibrium similar to what has been studied so far exists, and the value function v_c of the constrained government *across the entire state space* is identical to the value function v in an economy where the government faces no restriction.

Instead, if the debt-to-income ratio x^* at which the issuance restriction is introduced is low enough, the government is now able to monetize the welfare benefits from being able to borrow from patient lenders. In such case, a globally-smooth issuance strategy can no longer support a Markov perfect equilibrium of our economy. Instead, when the debt-to-income ratio is above the limit $x > x^*$, the government uses a singular control strategy, resulting in the debt-to-income ratio following a regulated diffusion: it goes up and down with income shocks, and once it reaches x^*, the government issues (via singular control) a positive measure of debt to insure that the debt-to-income ratio reflects at $x = x^*$.

PROPOSITION 3. Assume the small open economy's income process is a geometric Brownian motion, and assume that the government can commit not to issue any debt if its debt-to-income ratio x is above a cutoff x^*. Let $\bar{x}$ be the optimal default boundary when the government is unconstrained. There exists two limit cutoffs $\underline{x}^*, \bar{x}^*$, with $0 < \underline{x}^* < \bar{x}^* < \bar{x}$ such that:

1. if $x^* > \bar{x}^*$, there exists a unique smooth equilibrium of this economy, where the lifetime (income normalized) government value function $v_c(x)$ is equal to its unconstrained counterpart $v(x)$ and in which the debt price satisfies:

$$\begin{aligned} d_c(x) &= d(x) && x < x^* \\ d_c(x) &= \frac{\kappa + m}{r + m} + d_1 \left(\frac{x}{\bar{x}}\right)^{\eta_1} + d_2 \left(\frac{x}{\bar{x}}\right)^{\eta_2} > d(x) && x \geq x^* \end{aligned}$$

The constants η_1, η_2 are the roots of the characteristic polynomial related to the Feynman-Kac equation satisfied by d_c on $[x^*, \bar{x}]$, and d_1, d_2 are constants of integration derived in section A.11.1. In such equilibrium, the debt price $d_c(x)$ is continuously differentiable on $[0, \bar{x}]$ except at $x = x^*$, where it is continuous but exhibits a convex kink. When not indebted, the government does not achieve any welfare gain compared to its autarky value.

2. if $x^* < \bar{x}^*$, there exists a regulated equilibrium of this economy, where the life-time (income normalized) government value function $v_c(x)$ is greater than its unconstrained counterpart $v(x)$, and in which the default boundary is $\bar{x}_c > \bar{x}$. In such equilibrium, the government uses a singular issuance strategy to maintain its debt-to-income ratio above x^*. Two subcases arise:

   a. if $x^* > \underline{x}^*$, there exists a second endogenous cutoff $\hat{x} \in (0, x^*)$, such that if $x \in (0, \hat{x})$, the government follows a smooth debt issuance strategy, while if $x \in (\hat{x}, x^*)$, the government issues a lump amount of debt so as to reach the debt-to-income level x^*; the value function v_c is strictly greater than v, except at $x = 0$, where both values coincide;

   b. if $x^* < \underline{x}^*$, the government issues a lump amount of debt whenever the debt to income ratio is $x < x^*$, so as to reach the debt-to-income level x^*; the value function v_c is strictly greater than v everywhere; when not indebted, the government achieves a welfare gain compared to its autarky value.

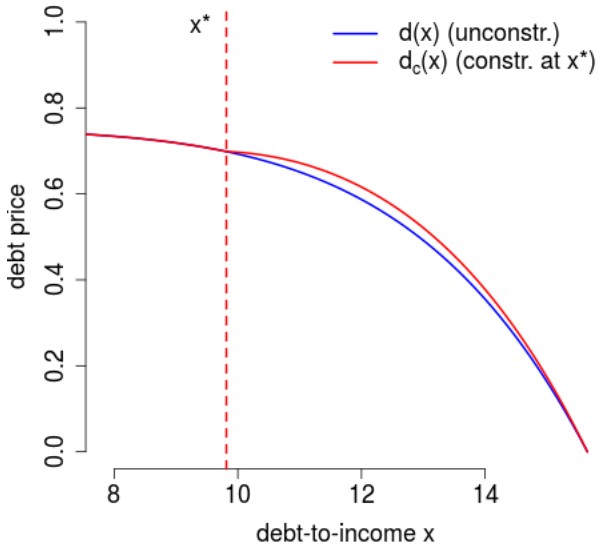
   Closed form expressions are provided in section A.11.2 for all cases.

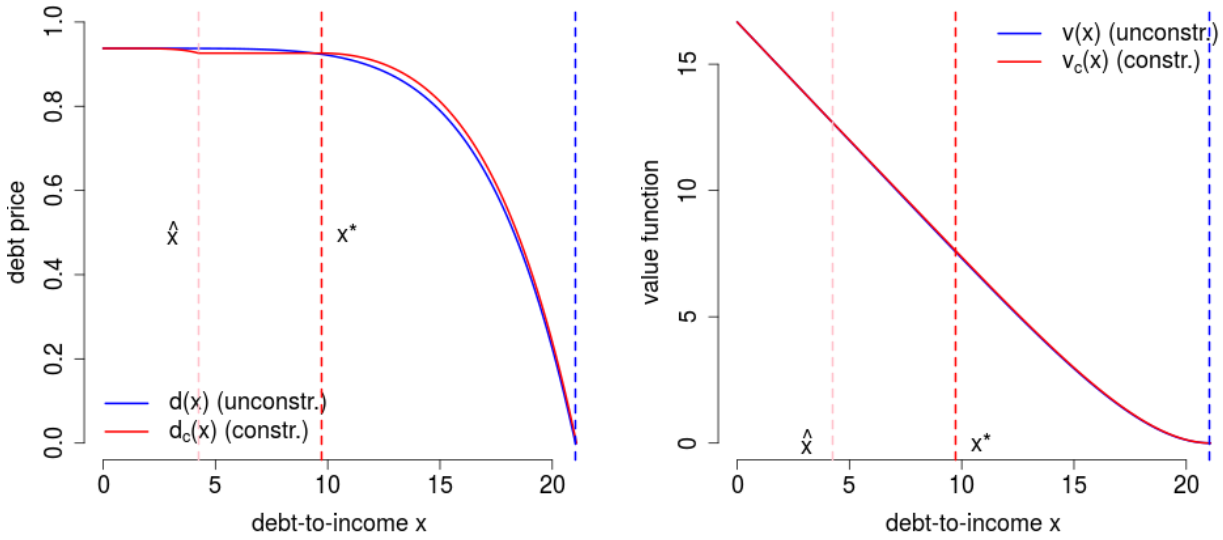
Figure 7 illustrates the shape of the debt price function $d_c(x)$ for a loose debt-ceiling policy (i.e. $x^* > \bar{x}^*$, the case where a smooth equilibrium still exists)[24]. The debt price $d_c(x)$ is identical to the unconstrained debt price $d(x)$ in the region $x < x^*$, while the constrained debt price $d_c(x)$ is strictly greater than the unconstrained debt price $d(x)$ in the region $x > x^*$. As discussed in proposition 3, the debt price exhibits a convex kink, and the right-derivative of d_c at $x = x^*$ is strictly greater than the left-derivative. The lack of arbitrage puts a restriction on the issuance policy at such boundary: as discussed in greater details in section A.11.1, the resulting debt-to-income process needs to be a skew-Brownian motion.

In figure 8, we give an illustration of the value function and debt prices in the regulated

[24] In this example, we use the following parameters: $\mu = 0\%$ p.a., $\sigma = 20\%$ p.a., $\delta = 10\%$ p.a., $r = 5\%$ p.a., $\kappa = 5\%$ p.a., $1/m = 10$ years, $\theta = 0\%$ and $\alpha = 0\%$.

Figure 7: Debt Price $d_c(x)$ in Smooth Constrained Equilibrium

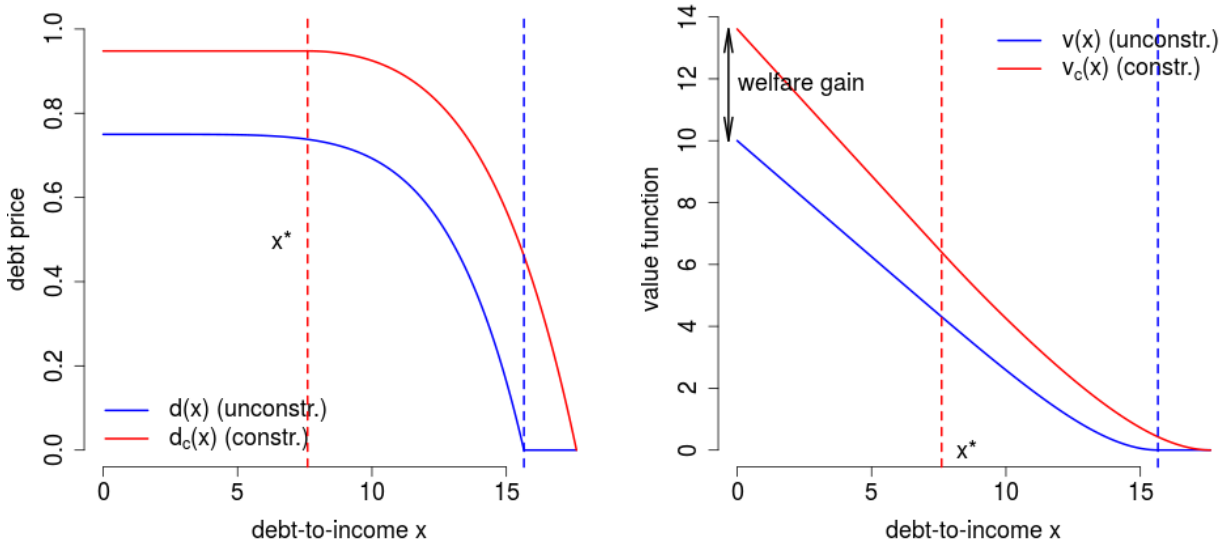




(a) Debt Price

(b) Value Function

Figure 8: Reflecting Equilibrium with "Moderate" Constraint x^*



(a) Debt Price

(b) Value Function

Figure 9: Reflecting Equilibrium with "Tight" Constraint x^*

equilibrium for an intermediate choice of constraint $x^* \in (\underline{x}^*, \bar{x}^*)$[25]. On the interval $x \in (\hat{x}, x^*)$, the value function is linear, the debt price function is flat, with a level $d(x^*)$ that is strictly below the risk-free debt price $\frac{\kappa+m}{\delta+m}$, as the government finds it optimal to jump immediately to the debt-to-income level x^*. On $x \in (0, \hat{x})$, the government follows a smooth debt financing policy, with a value function $v_c(x)$ that is weakly greater than $v(x)$ – both functions are equal when the government is not indebted (i.e. $v(0) = v_c(0)$).

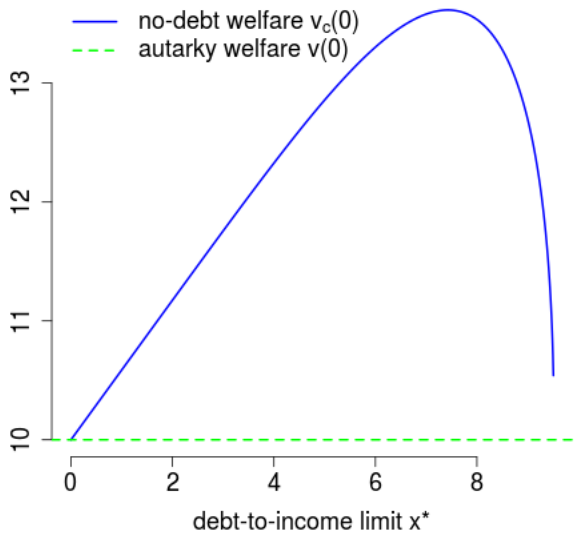
Finally, in figure 9, we give an illustration of the value function and debt prices in the regulated equilibrium, in the case of a tight debt ceiling policy $x^* < \underline{x}^*$[26]. The debt price d_c and the (income-normalized) government value function v_c are uniformly higher than in the unconstrained equilibrium, with a welfare gain without debt outstanding that can be seen graphically as the distance between $v_c(0)$ and $v(0)$. For $x < x^*$, the debt price satisfies $d_c(x) = d_c(x^*)$, since creditors anticipate that the government will issue a lump amount of debt in order achieve a debt-to-income ratio equal to x^*, while the income-normalized value function satisfies $v_c(x) = v_c(x^*) + (x^* - x)d_c(x^*)$ – in other words the value function is linear in the debt-to-income ratio when $x < x^*$.

Figure 10 shows the no-debt government welfare $v_c(0)$ for a range of choices of constraints x^*. The choice $x^* = 0$ corresponds to the case where the government commits to never issue

[25] This example uses the same parameters as in the example with loose debt-ceiling policy, except for $\delta = 6\%$.

[26] This example uses the same parameters as in the example with loose debt-ceiling policy.

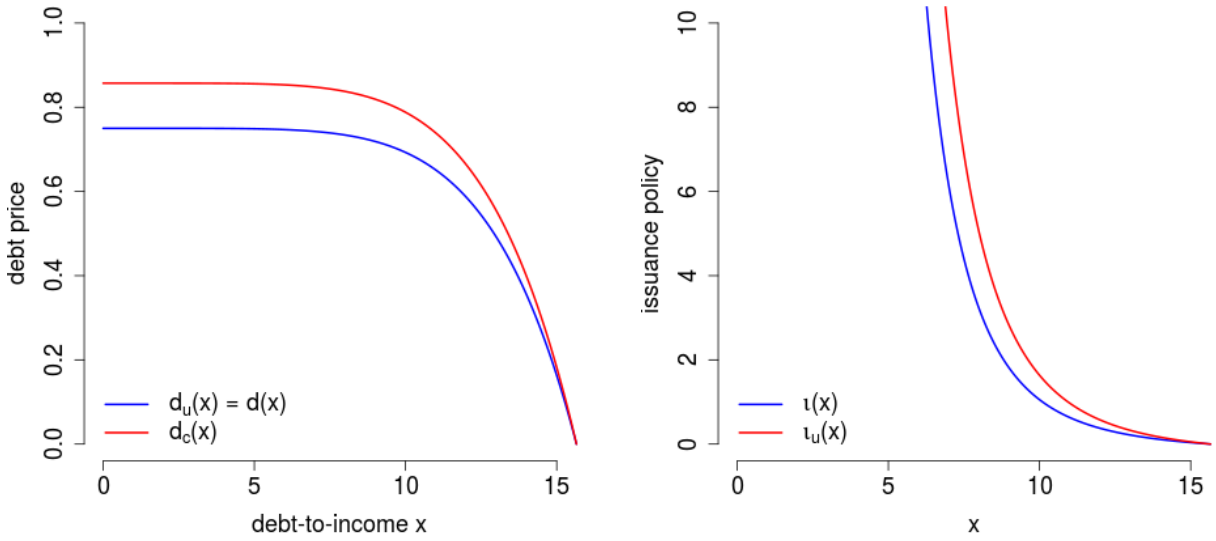
Figure 10: No-Debt Welfare $v_c(0)$ for Different Choices of x^*



any debt, and the welfare in that case corresponds to the autarky welfare. The choice $x^* = \underline{x}^*$ is the limit case above and beyond a smooth issuance policy region exists for $x \in (\underline{x}^*, x^*)$; at such limit, the no-debt welfare is also equal to the autarky welfare. This analysis suggests that there exists an optimal debt-to-income limit at which the initial government welfare is optimal, as illustrated in figure 10.

6.3 Markov-Switching Restricted Issuance

The previous analysis shows that a restriction on debt issuances if the debt-to-income ratio is above an exogenously-specified threshold can lead to welfare gains, but only if such threshold is sufficiently low. We now show that the institutional arrangement of such issuance restrictions are important in obtaining welfare gains. Specifically, for such potential gains from trade to be monetizable by the government, it is important that there is some "predictability" in the issuance restriction period. To illustrate our point, consider a 2-state Markov-switching economy in which, in the unrestricted state (state "u"), the government is entirely free to issue debt, while in the restricted state (state "c"), the government is prohibited from issuing any debt, and the Markov state switches back-and-forth at specific Poisson arrival times (with arrival intensities λ_u and λ_c respectively). One can also think about this economic environment from a credit supply standpoint: state "c" can be thought of as a sudden stop state, during which international capital markets are shut for the small open economy, while state "u" can be thought of as a normal state, during which international



(a) Debt Price

(b) Issuance Policy

Figure 11: Debt Price and Issuance Policy in Markov-Switching Economy

capital markets are functioning normally. In this particular economic environment, while the government is de-facto tying its hands in state "c", the length of the time period during which the government is restricted is uncertain. This turns out to be essential in explaining our next result.

PROPOSITION 4. Assume the small open economy's income process is a geometric Brownian motion; assume a 2-state Markov switching environment, with a constrained state "c" and an unconstrained state "u", with transition intensities out of such state equal to λ_c and λ_u, respectively. In state "c", the government cannot issue any debt, while in state "u", the government faces no restriction. There exists a smooth equilibrium of this economy, where the life-time (income normalized) government value function v_c in state "c" and v_u in state "u" are equal to the unconstrained value function v. In such economy, the debt price satisfies:

$$d_u(x) = d(x)$$
$$d_c(x) > d(x)$$

The exact expression for $d_c(x)$ is derived in section A.11.3. In such equilibrium, the government does not achieve any welfare gain compared to its autarky value.

Figure 11 illustrates what happens in our smooth Markov-switching equilibrium. Figure 11a highlights the fact that the debt price d_u in the unconstrained state "u" is strictly

lower than the debt price d_c in the restricted state – this is the case since creditors understand that in such state "c", no issuances occur, meaning that the drift rate of the debt-to-income ratio is strictly lower than in the unconstrained case. The government of the small open economy understands that it will be limited in its ability to issue bonds in state "c"; thus, it adjusts its issuance policy $\iota_u(x)$ *upward* in the unrestricted state, such that those issuances are strictly greater than in the unrestricted economy: $\iota_u(x) > \iota(x)$.

6.4 Cap on Bond Issuance Rate

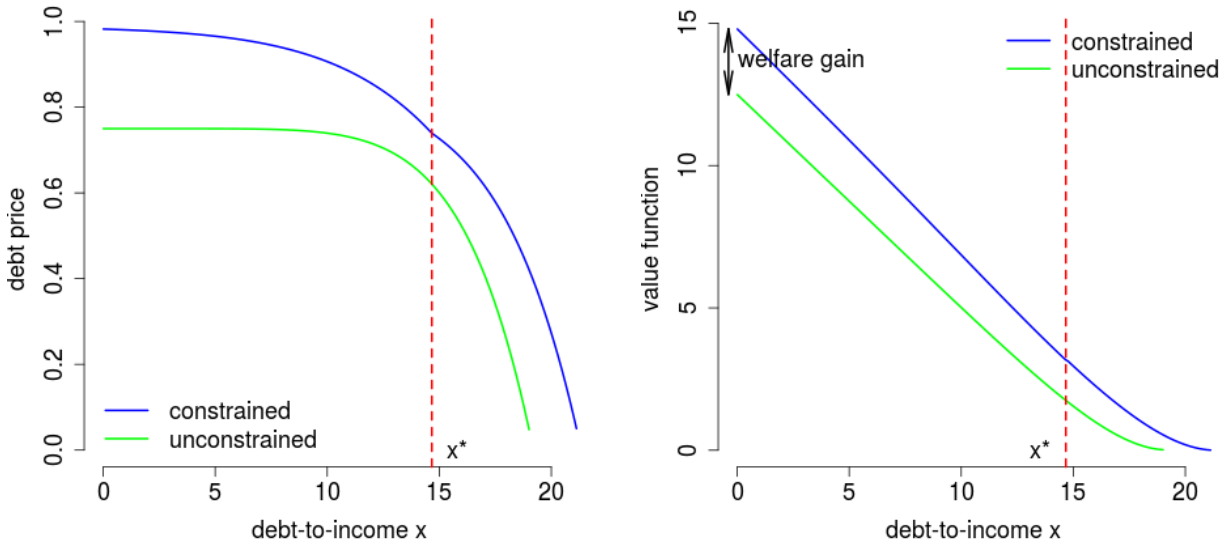
In this section, we analyze a commitment device that prevents the government from issuing bonds at an intensity greater than an exogenously specified cap. To accommodate the scale invariance of our model, we focus on caps of the form $I_t \leq \bar{\iota} Y_t$, in other words policies under which the bond issuance rate (per unit of income) $\iota(x)$ is capped below $\bar{\iota} > 0$. To simplify the exposition, we continue to assume that the discrete state variable s_t is trivially equal to 1, in other words that the international risk-free rate is constant equal to r, and that aggregate risk is not priced by international investors: $\boldsymbol{\nu}(s) = \mathbf{0}$. We also assume that the parameter $\alpha = 0$, in other words that upon a default, the small open economy's income is entirely lost, and investors lose their entire investment. In other words, the government solves the simplified normalized problem:

$$\begin{aligned} v(x) &= \sup_{(\iota,\tau)\in\mathcal{I}_{\bar{\iota}}\times\mathcal{T}} \tilde{\mathbb{E}}^x \left[\int_0^\tau e^{-(\delta-\mu)t} \left(1 + \iota_t d(x_t) - (\kappa+m)x_t\right) dt \right] \\ dx_t &= \left(\iota_t - (m+\mu)x_t\right) dt - \sigma x_t d\tilde{B}_t \end{aligned}$$

The set of admissible issuance policies $\mathcal{I}_{\bar{\iota}}$ is now the set of Markov controls such that $\iota(x) \leq \bar{\iota}$ for all debt-to-income ratios x.

Under the parameter restriction $\delta + m > \sigma^2 - (\mu + m)$, the issuance rate of our unconstrained economy is decreasing (as a function of x), and unbounded as $x \to 0$. This suggests that we focus on equilibria in which the issuance constraint is binding for low debt-to-income ratios (i.e. for $x \in [0, x^*]$, for some endogenously determined debt-to-income hurdle x^*), but slack for high debt-to-income ratios (i.e. for $x \in [x^*, \bar{x}_c]$, for some endogenously determined default boundary hurdle $\bar{x}_c$). We summarize below our key result.

PROPOSITION 5. Assume the small open economy's income process is a geometric Brownian motion, and assume that the government can commit to maintain its bond issuance rate (as a percentage of income) below $\bar{\iota}$. Assume that upon default, the small open economy's income stream is entirely lost, and creditors incur a full loss on their investment. Subject



(a) Debt Price $d_c(x)$

(b) Value Function $v_c(x)$

Figure 12: Constrained and Unconstrained Equilibrium

to the existence of a solution to a set of 2 algebraic equations in 2 unknown disclosed in section A.12, there exists two endogenous cutoffs $x^*, \bar{x}$, with $0 < x^* < \bar{x}$ such that:

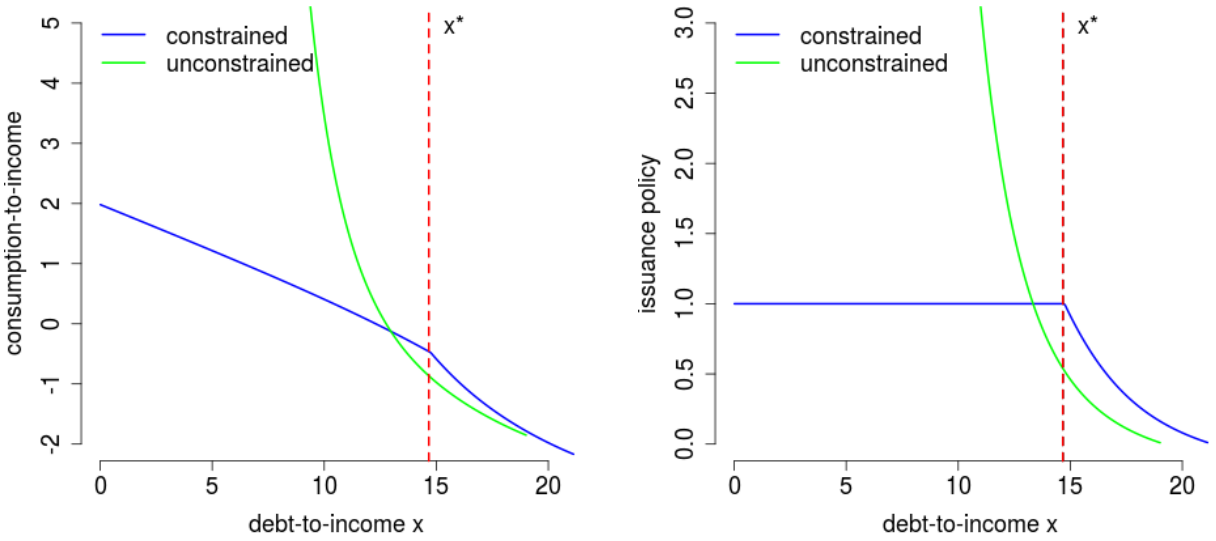
1. When $x \in (0, x^*)$, the government is constrained and uses an issuance rate (per unit of income) $\iota(x) = \bar{\iota}$. On this interval, debt prices and income-normalized value functions are analytic functions displayed in section A.12;

2. When $x \in (x^*, \bar{x})$, the government is unconstrained, uses a smooth issuance policy $\iota(x) < \bar{\iota}$, and defaults optimally when $x = \bar{x}$. On this interval, the debt price and income-normalized value function are analytic functions displayed in section A.12, and verifying $d_c(x) = -v_c'(x)$.

The no-debt government welfare is strictly greater than the autarky welfare for any choice of $\bar{\iota}$ that is finite. In the particular case where $\sigma = 0$ and $\mu + m < 0$, such equilibrium exists and is unique. The default cutoff in such case is $\bar{x} = 1/(\kappa + m)$. The issuance policy in such case follows:

$$\iota(x) = \begin{cases} \bar{\iota} & \text{if } x \leq x^* \\ \iota_u(x) := -\frac{(\delta-\mu)(\mu+m)}{\delta+m}\left[\left(\frac{\bar{x}}{x}\right)^{-\frac{\delta+m}{\mu+m}} - 1\right] x & \text{if } x > x^* \end{cases}$$

The cutoff x^* at which the regime switches from constrained to unconstrained is the unique solution to the equation $\iota_u(x) = \bar{\iota}$.

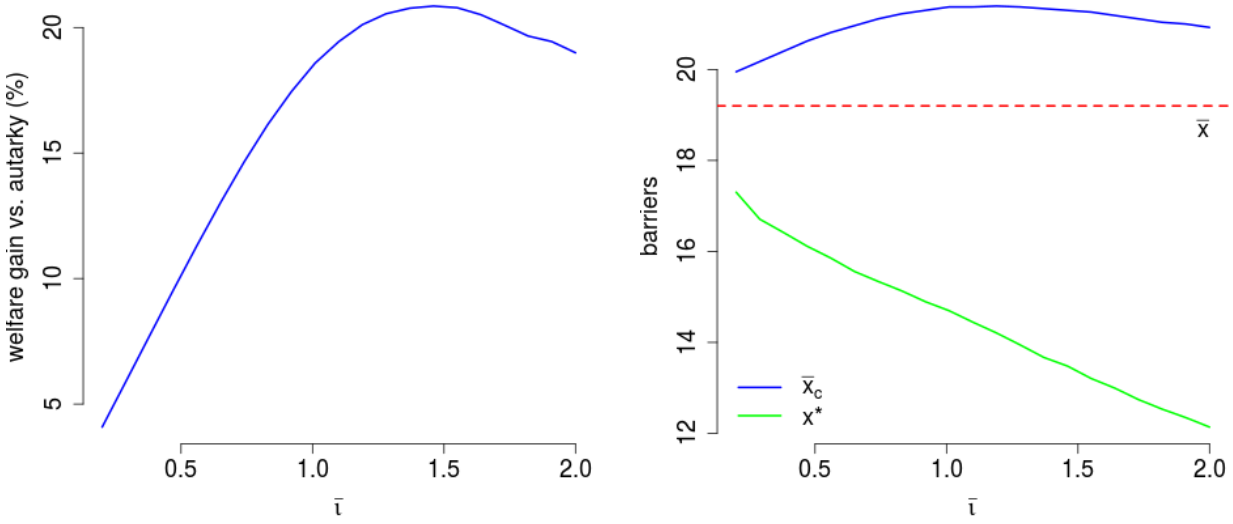


(a) Consumption-to-income $c(x)$ (b) Issuance Rate $\iota(x)$

Figure 13: Constrained and Unconstrained Equilibrium

In figure 12a and figure 12b, we plot an illustration of the debt price and value function in the constrained and unconstrained equilibrium. Both the debt price and value function in the constrained equilibrium are uniformly higher than in the unconstrained case, and default occurs at a higher debt-to-income ratio than in the in the unconstrained case. While the value function is $\mathcal{C}^1$ at $x = x^*$, the debt price feastures a kink. In figure 13a and figure 13b, we show the resulting consumption-to-income and issuance-to-income policies. The issuance rate is of course capped at $\bar{\iota}$ when $x < x^*$, and is unconstrained, decreasing (as a function of x) when $x > x^*$. At the default cutoff, the issuance rate is exactly zero, as the debt price at such point is also zero.

Finally, figure 14a shows the no-debt welfare gain vs. the autarky benchmark as we vary the policy $\bar{\iota}$. For very small choices of $\bar{\iota}$, the government is basically not allowed to issue any debt, meaning that $v(0)$ in such case is close to the autarky benchark. Welfare gains increase as $\bar{\iota}$ increases, and those gain peak at around $\bar{\iota} = 1.5$. For higher $\bar{\iota}$, the constraint becomes more slack, and the welfare gains drop, to reach zero as $\bar{\iota} \to +\infty$, and as the equilibrium gets closer and closer to the unconstrained equilibrium originally studied. Finally, figure 14b shows how the default cutoff $\bar{x}$ and the regime-change cutoff x^* vary with the choice $\bar{\iota}$.



(a) Welfare Gain

(b) Barriers $x^*, \bar{x}_c$

Figure 14: Constrained and Unconstrained Equilibrium

6.5 First Best with Full Commitment

We end this section by considering the particular issuance policies that could in fact achieve the first best outcome for the government. Given our income process specification, the autarky value V_{aut} for the government and its first-best value V_{fb} are:

$$V_{aut}(Y) = \frac{Y}{\delta - \mu}$$
$$V_{fb}(Y) = \frac{Y}{r - \mu}$$

In order to avoid "Ponzi" schemes, in which the debt-to-income ratio of the government becomes unbounded, we consider only financing policies such that the debt-to-income ratio is a stationary state variable. In fact, we argue that a policy that guarantees that the debt-to-income ratio of the government stays constant at $x_t = \lambda$ (for a parameter λ carefully chosen) achieves the first-best outcome. To see this, note that such a policy leads to a debt face value process that satisfies:

$$dF_t = \mu F_t dt + \sigma F_t dZ_t$$

Since F_t satisfies equation (13), it means that the cumulative debt issuance policy H_t satisfies:

$$dH_t = (\mu + m)F_t dt + \sigma F_t dZ_t$$

In other words, we have moved away from dt-order financing policies, and instead are expanding on the set of admissible policies we focus on. The government value function satisfies equation (11), where Γ_t is the cumulative consumption process. Notice that the expected value of consumption flows received between t and $t + dt$ by the government are as follows:

$$\mathbb{E}_t \left[d\Gamma_t \right] = Y_t \left(1 - (\kappa + m)\lambda + D_t(\mu + m)\lambda \right] dt$$

If one picks λ such that $\mathbb{E}_t \left[d\Gamma_t \right] = 0$, the government never has any incentive to default, since its continuation value is always identical to zero (its "reservation" value). Thus, such choice would guarantee that the debt price D_t is equal to its risk-free value $D_{rf} := \frac{\kappa+m}{r+m}$. This is achieved by choosing $\lambda = \frac{r+m}{\kappa+m}\frac{1}{r-\mu}$. Notice finally that our financing policy needs to have an initial impulse, such that the debt-to-income ratio at $t = 0$ is equal to λ – in other words, $F_0 = \lambda Y_0$. This means that the value function for the borrower is equal to $F_0 D_{rf} = \frac{Y_0}{r-\mu}$.

7 Conclusion

Lack of commitment is a powerful force that can dissipate entirely gains from trade in particular economic environments. The original insight of Coase (1972) can be transported into the sovereign default framework, as we have showed throughout this paper. By viewing the government in this class of models as a monopolist selling a durable good (the long term bonds) without an ability to commit to a particular financing and default policies, by viewing debt investors as perfectly competitive consumers buying such durable good, and by assuming that the flow payoff function for the government is linear in the quantity and in the price of bonds issued, we recover a no-welfare result familiar to the durable goods' monopoly literature. Our result is general; for a very large class of income processes, so long as a smooth Markov perfect equilibrium exists, it must be unique, and in such equilibrium the small open economy does not rip the benefits of being able to finance itself with more patient lenders. We illustrate this result by using a standard model of income: the geometric brownian motion. We provide a complete analytical characterization of the equilibrium in this particular model, and demonstrate comparative statics that had historically only been obtained numerically by the sovereign default literature. To break our no-welfare result, one can (for example) consider different commitment devices, but the particular implementation matters; a debt-ceiling policy for example can enhance welfare, but only to the extent the

debt-to-income level at which such ceiling is introduced is low enough.

Our result raises new questions for the sovereign debt literature. It is no longer valid when preferences are not linear, but to what extent is it "approximately" valid when the intertemporal elasticity of substitution is low? In this class of models, if the small open economy's ability to commit to a particular financing policy destroys most of its potential gains from trade, to what extent should the literature spend anytime studying welfare-improving policies?

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A Appendix

A.1 An Example of a Discrete-Time Small Open Economy

In order to gain more intuition for the link between discrete time and continuous time models, and the economic phenomenon we want to uncover, we specialize the endowment process such that:

$$\frac{Y_{(i+1)\Delta}}{Y_{i\Delta}} = \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta + \sigma\sqrt{\Delta}\tilde{\omega}_{(i+1)\Delta}\right)$$

In other words, the endowment process follows the discrete time equivalent of a (μ, σ)-geometric Brownian motion. The default value for the government is simply set to $V_d(Y) = 0$. Given the linearity of the government preferences and of the dynamic resource constraint, if a Markov perfect equilibrium exists, it admits as state variable the debt-to-income ratio $x := Y/F$. In any such equilibrium, the value function of the government is homogeneous of degree 1 in (Y, F) (i.e. $V(Y, F) := Yv(x)$, where $v(x) := V(1, F)$), the price of one unit of debt is homogeneous of degree 0 in (Y, F) (i.e. $D(Y, F) := d(x)$, where $d(x) := D(1, F)$), the issuance policy is homogeneous of degree 1 in (Y, F) (i.e. $F^*(Y, F) := Yx^*(x)$, where $x^*(x) := F^*(1, F)$), and the optimal default policy is a barrier policy – in other words, $\{(Y, F) \in \mathbb{R}^2 : V(Y, F) \leq 0\} = \{(Y, F) \in \mathbb{R}^2 : F/Y \geq \bar{x}\}$, for some cutoff debt-to-income ratio $\bar{x}$. The Bellman equation for the government can be written[27]:

$$\begin{aligned} v_\Delta(x) = \max_{x'} \big[\Delta + d_\Delta(x')\left(x' - (1 - m\Delta)x\right) - (\kappa + m)x\Delta \\ + e^{-(\delta-\mu)\Delta}\mathbb{E}\left[e^{-\frac{\sigma^2}{2}\Delta + \sigma\sqrt{\Delta}\omega}\max\left(0, v_\Delta\left(x'e^{-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta - \sigma\sqrt{\Delta}\omega}\right)\right)\right]\big] \end{aligned}$$

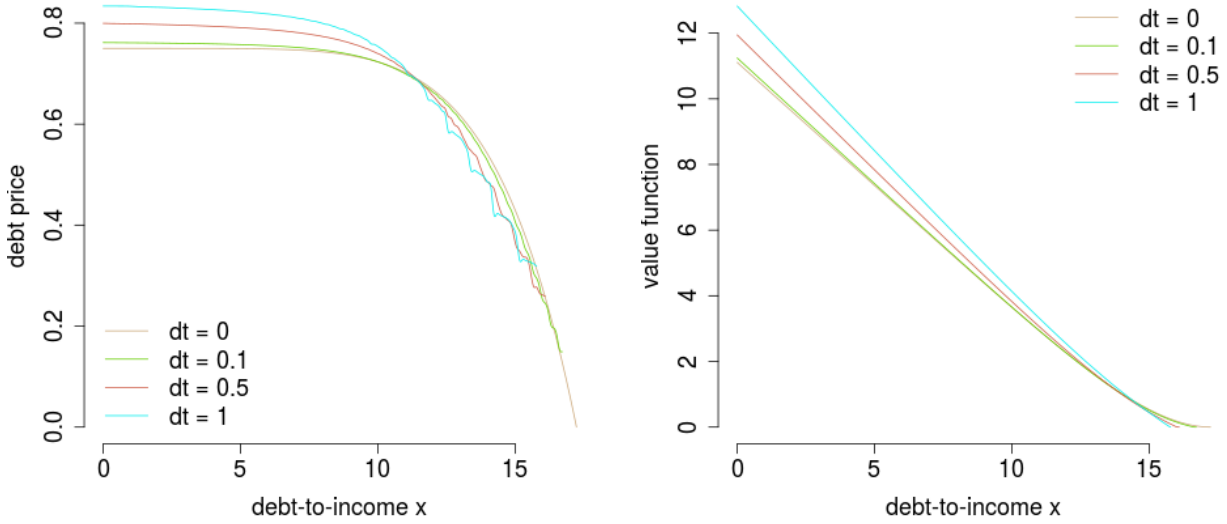
We note $x^*(x)$ the optimal next period debt-to-income choice given a current period debt-to-income ratio being equal to x. The debt-to-income ratio dynamics are such that:

$$x_{t+\Delta} = x^*\left(x_t\right)e^{-\left(\mu-\frac{\sigma^2}{2}\right)\Delta - \sigma\sqrt{\Delta}\omega}$$

The debt pricing equation can be written:

$$d_\Delta(x) = e^{-r\Delta}\mathbb{E}\left[1_{\left\{xe^{-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta-\sigma\sqrt{\Delta}\omega} \leq \bar{x}\right\}}\left[(\kappa + m)\Delta + (1 - m\Delta)\, d\left(x^*\left(xe^{-\left(\mu-\frac{1}{2}\sigma^2\right)\Delta-\sigma\sqrt{\Delta}\omega}\right)\right)\right]\right]$$

[27] The subscript Δ for the value function and the debt price are meant to emphasize that we consider a sequence of equilibria, indexed by the time-step Δ.



(a) Debt Price $d(x)$

(b) Value Function $v(x)$

Figure 15: Debt Price and Value Function

For time steps $\Delta = 1, 0.1, 0.01$, we solve for the equilibrium numerically and plot the resulting value function and debt prices in figure 15. In the same figure, we also plot the solution to the continuous time counterpart – the method to solve such model will be discussed in section 5. There are several take-aways from these plots. First, as the time step converges to zero, the equilibrium (income-normalized) value function and the equilibrium debt price converge to the continuous time counterpart. Second, when the government is not indebted, its (income-normalized) welfare $v_\Delta(0)$ declines monotonically with Δ, and reaches the continuous time autarky level $1/(\delta - \mu)$ when $\Delta = 0$.

A.2 Risk Premia

Remember that $D(Y_t, F_t, s_t)$ is the price per unit of face value at time t. Investors' cumulative gain rate can be expressed as follows:

$$\int_0^t e^{-\int_0^u (r(s_v)+m)dv} (\kappa + m)\, dt + e^{-\int_0^t (r(s_u)+m)du} D(Y_t, F_t, s_t)$$

Since this gain rate must be a $\mathbb{Q}$-martingale, we must have:

$$(r+m)\, D = \kappa + m + (\boldsymbol{\mu}_X - \boldsymbol{\sigma}_X \boldsymbol{\nu}) \cdot \partial_X D + \frac{1}{2} \mathrm{tr}\left(\boldsymbol{\sigma}_X' \partial_{XX'} D \boldsymbol{\sigma}_X\right) + (I(Y, F, s) - mF)\, \partial_F D$$

The excess return on holding sovereign bonds between t and $t+dt$ must reflect price changes dD_t, coupon and principal payments $(\kappa+m)dt$, as well as reinvestment costs $mD_t dt$. In other words, those excess returns can be computed as follows:

$$dR_t - r_t dt = \frac{dD_t + (\kappa+m)dt}{D_t} - (r_t + m)\, dt$$

We use Itô formula to compute dD_t as follows:

$$dD_t = \left[\boldsymbol{\mu}_{X,t} \cdot \partial_X D_t + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}'_{X,t}\partial_{XX'}D_t\boldsymbol{\sigma}_{X,t}\right) + (I_t - mF_t)\,\partial_F D_t\right] dt + \left(\boldsymbol{\sigma}'_{X,t}\partial_X D_t\right)\cdot d\boldsymbol{B}_t \quad (45)$$

Reinjecting equation (45) into our equation for excess returns, and using our martingale condition for D, we obtain the following formula for excess returns:

$$dR_t - r_t dt = \left(\boldsymbol{\sigma}'_{X,t}\partial_X \ln D_t\right)\cdot \boldsymbol{\nu}_t dt + \left(\boldsymbol{\sigma}'_{X,t}\partial_X \ln D_t\right)\cdot d\boldsymbol{B}_t$$

The second term has zero conditional expectations (under $\mathbb{P}$), which leads to the following formula for expected excess returns:

$$\begin{aligned}\mathbb{E}\left[dR_t - r_t dt|\mathcal{F}_t\right] &= \pi\left(Y_t, F_t, s_t\right) dt \\ \pi\left(Y, F, s\right) &:= \boldsymbol{\nu}'\boldsymbol{\sigma}'_X \partial_X \ln D\end{aligned}$$

□

A.3 Citizens vs. Government

Assume that the citizens of the small open economy have linear preferences with discount rate $\hat{\delta} < \delta$, where δ is the effective discount rate of the government. Assume for simplicity that upon default, the small open economy's income is zero forever after, and that creditors lose their entire investment – i.e. $\alpha = \theta = 0$. Let $C(Y, F, s)$ be the resulting country's consumption in state (Y, F, s) resulting from the government optimization outcome. Let V (resp. $\hat{V}$) be the indirect utility function of the government (resp. its citizens), and let $\hat{V}_a$ be the indirect utility function of citizens of a country in financial autarky, and which defaults

according to the government default stopping rule τ. We then have:

$$\begin{aligned}
V(Y,F,S) :&= \mathbb{E}^{Y,F,s}\left[\int_0^\tau e^{-\delta t}C(Y_t,F_t,s_t)dt\right]\\
\hat{V}(Y,F,S) :&= \mathbb{E}^{Y,F,s}\left[\int_0^\tau e^{-\hat{\delta} t}C(Y_t,F_t,s_t)dt\right]\\
\hat{V}_a(Y,F,S) :&= \mathbb{E}^{Y,F,s}\left[\int_0^\tau e^{-\hat{\delta} t}\left(Y_t-(\kappa+m)Fe^{-mt}\right)dt\right]
\end{aligned}$$

The equation defining $\hat{V}_a$ reflects the fact that in financial autarky, the existing stock of sovereign debt amortizes at rate m. In all these indirect utility functions, the stopping time τ is the same, and is pinned down by the government's optimal behavior. Remember that the equilibrium consumption is $C(Y,F,s) = Y + I(Y,F,s)D(Y,F,s) - (\kappa+m)F$. The value function for the government satisfies:

$$\begin{aligned}
\delta V &= C + \boldsymbol{\mu}_X \cdot \partial_X V + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X' \partial_{XX'} V \boldsymbol{\sigma}_X\right) + (I - mF)\,\partial_F V\\
&= Y - (\kappa+m)F + \boldsymbol{\mu}_X \cdot \partial_X V + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X' \partial_{XX'} V \boldsymbol{\sigma}_X\right) - mF\partial_F V
\end{aligned}$$

Since $\partial_F V + D = 0$, this also means that the debt price satisfies:

$$(\delta+m)\,D = (\kappa+m) + \boldsymbol{\mu}_X \cdot \partial_X D + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X' \partial_{XX'} D \boldsymbol{\sigma}_X\right) - mF\partial_F D \tag{46}$$

As a reminder, the (optimal) issuance policy is defined via:

$$I(Y,F,s) = \frac{\delta - \left(r(s) + \pi\left(Y,F,s\right)\right)}{\partial_F \ln D\left(Y,F,s\right)}$$

The indirect utility function $\hat{V}$ for citizens (who use a discount rate $\hat{\delta} < \delta$) satisfies:

$$\hat{\delta}\hat{V} = C + \boldsymbol{\mu}_X \cdot \partial_X \hat{V} + \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X' \partial_{XX'} \hat{V} \boldsymbol{\sigma}_X\right) + (I - mF)\,\partial_F \hat{V} \tag{47}$$

Differentiate w.r.t. F to yield:

$$\begin{aligned}
\left(\hat{\delta}+m\right)\partial_F \hat{V} = -(\kappa+m) + \left(D + \partial_F \hat{V}\right)\partial_F I + I\partial_F\left(D + \partial_F \hat{V}\right) + \boldsymbol{\mu}_X \cdot \partial_X\left(\partial_F \hat{V}\right)\\
+ \frac{1}{2}\mathrm{tr}\left(\boldsymbol{\sigma}_X' \partial_{XX'}\left(\partial_F \hat{V}\right)\boldsymbol{\sigma}_X\right) - mF\partial_F\left(\partial_F \hat{V}\right)
\end{aligned} \tag{48}$$

Add up equation (46) to equation (48), note $\hat{G} := D + \hat{V}$, and notice that:

$$\left(\hat{\delta} + m - \partial_F I\right)\hat{G} = (\hat{\delta} - \delta)D + \boldsymbol{\mu}_X \cdot \partial_X \hat{G} + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}'_X \partial_{XX'}\hat{G}\boldsymbol{\sigma}_X\right) + (I - mF)\,\partial_F \hat{G}$$

Note that at the default boundary $(Y, F) \in \mathcal{O}(s)$, we have:

$$V(Y, F, s) = \hat{V}(Y, F, s) = D(Y, F, s) = 0$$

It also must be the case that the function $\hat{V}$ is weakly decreasing in F, in other words it must be the case that at the default boundary $(Y, F) \in \mathcal{O}(s)$:

$$\hat{G}(Y, F, s) = \partial_F \hat{V}(Y, F, s) + D(Y, F, s) \leq 0$$

We have thus an integral representation of $\hat{G}(Y, F, s)$:

$$G(Y, F, s) = \mathbb{E}\left[\int_0^\tau e^{-\int_0^t \left(\hat{\delta} + m - \partial_F I_u\right)du}(\hat{\delta} - \delta)D_t dt + e^{-\int_0^\tau \left(\hat{\delta} + m - \partial_F I_u\right)du}\hat{G}_\tau\right]$$

In the above, we have used the "short" notation $D_t = D\left(Y_t, F_t, s_t\right)$ and $I_t = I\left(Y_t, F_t, s_t\right)$. Since $\hat{\delta} < \delta$, and since the debt price D is strictly positive on the interior of the continuation region, we must have $G(Y, F, s) < 0$ for all (Y, F, S) on the interior of the continuation region. Finally, we write down the PDEs statisfied by $\hat{V}$ and $\hat{V}_a$:

$$\hat{\delta}\hat{V} = Y - (\kappa + m)F + I\left(D + \partial_F \hat{V}\right) + \boldsymbol{\mu}_X \cdot \partial_X \hat{V} + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}'_X \partial_{XX'}\hat{V}\boldsymbol{\sigma}_X\right) - mF\partial_F \hat{V} \quad (49)$$

$$\hat{\delta}\hat{V}_a = Y - (\kappa + m)F + \boldsymbol{\mu}_X \cdot \partial_X \hat{V}_a + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}'_X \partial_{XX'}\hat{V}_a\boldsymbol{\sigma}_X\right) - mF\partial_F \hat{V}_a \quad (50)$$

Note $\Delta\hat{V} := \hat{V} - \hat{V}_a$, which then satisfies the following PDE:

$$\hat{\delta}\Delta\hat{V} = I\left(D + \partial_F \hat{V}\right) + \boldsymbol{\mu}_X \cdot \partial_X \Delta\hat{V} + \frac{1}{2}\text{tr}\left(\boldsymbol{\sigma}'_X \partial_{XX'}\Delta\hat{V}\boldsymbol{\sigma}_X\right) - mF\partial_F \Delta\hat{V} \quad (51)$$

At the default boundary $(Y, F) \in \mathcal{O}(s)$ (which, as you might recall, is optimal only from the point of view of the government, discounting cashflows at rate δ), we have $\hat{V}(Y, F, s) = \hat{V}_a(Y, F, s) = 0$. In other words, for $(Y, F) \in \mathcal{O}(s)$, $\hat{\Delta}V(Y, F, s) = 0$. This means that $\hat{\Delta}V(Y, F, s)$ admits the following integral representation:

$$\Delta\hat{V}(Y, F, s) = \mathbb{E}^{Y,F,S}\left[\int_0^\tau e^{-\hat{\delta}t} I_t\left(D_t + \partial_F \hat{V}_t\right)dt\right]$$

Since we have focused on an equilibrium where $I_t \geq 0$ almost surely, it must be the case that $\hat{V}(Y,F,s) < \hat{V}_a(Y,F,s)$ for all (Y,F,s) in the interior of the continuation region. $\square$

A.4 Geometric Brownian Motion Income Process

For a given admissible default policy $\boldsymbol{\tau} \in \mathcal{T}$, define $N_{d,t}^{(\boldsymbol{\tau})} := \max\{k \in \mathbb{N} : \tau_k \leq t\}$ to be the counting process for default events. Using this notation, the dynamic evolution of the controlled stochastic process $Y_t^{(\boldsymbol{\tau})}$ can be expressed as follows:

$$Y_t^{(\boldsymbol{\tau})} = \alpha^{N_{d,t}^{(\boldsymbol{\tau})}} Y_t \tag{52}$$

Similarly, the dynamic evolution of the controlled stochastic process $F^{(I,\boldsymbol{\tau})}$ can be expressed as follows:

$$F_t^{(I,\boldsymbol{\tau})} = \int_0^t \left(I\left(Y_u^{(\boldsymbol{\tau})}, F_u^{(I,\boldsymbol{\tau})}, s_u\right) - mF_u^{(I,\boldsymbol{\tau})}\right) du + \int_0^t \left(\theta\alpha - 1\right) F_{u-}^{(I,\boldsymbol{\tau})} dN_{d,u}^{(\boldsymbol{\tau})} \tag{53}$$

Armed with those notations, notice that V can be written as follows:

$$\begin{aligned} V(Y,F) &= \sup_{(I,\boldsymbol{\tau})\in\mathcal{I}\times\mathcal{T}} \mathbb{E}^{Y,F}\left[\int_0^{+\infty} e^{-\delta t}\left(Y_t^{(\boldsymbol{\tau})} + I\left(Y_t^{(\boldsymbol{\tau})}, F_t^{(I,\boldsymbol{\tau})}, s_t\right) D_t - (\kappa+m)F_t^{(I,\boldsymbol{\tau})}\right) dt\right] \\ &= Y \sup_{(\iota,\boldsymbol{\tau})\in\mathcal{I}\times\mathcal{T}} \mathbb{E}^{x}\left[\int_0^{+\infty} \alpha^{N_{d,t}^{\boldsymbol{\tau}}} e^{-\left(\delta-\mu+\frac{\sigma^2}{2}\right)t+\sigma B_t}\left(1 + \iota\left(x_t^{(\iota,\boldsymbol{\tau})}, s_t\right) D_t - (\kappa+m)x_t^{(\iota,\boldsymbol{\tau})}\right) dt\right] \\ &= Y v(x) \end{aligned}$$

In the continuation region $(0,\bar{x})$, the debt-to-income ratio $x_t^{(\iota,\boldsymbol{\tau})}$ is a controlled stochastic process that evolves as follows:

$$dx_t^{(\iota,\boldsymbol{\tau})} = \left(\iota(x_t^{(\iota,\boldsymbol{\tau})}, s_t) - \left(m + \mu - |\boldsymbol{\sigma}|^2\right) x_t^{(\iota,\boldsymbol{\tau})}\right) dt - x_t^{(\iota,\boldsymbol{\tau})} \boldsymbol{\sigma} \cdot d\boldsymbol{B_t} + (\theta - 1)\, dN_{d,t}^{(\boldsymbol{\tau})}$$

The normalized value function v is equal to:

$$\begin{aligned} v(x) &:= \sup_{(\iota,\boldsymbol{\tau})\in\mathcal{I}\times\mathcal{T}} \mathbb{E}^{x}\left[\int_0^{+\infty} \alpha^{N_{d,t}^{\boldsymbol{\tau}}} e^{-\left(\delta-\mu+\frac{|\boldsymbol{\sigma}|^2}{2}\right)t+\boldsymbol{\sigma}\cdot\boldsymbol{B}_t}\left(1 + \iota\left(x_t^{(\iota,\boldsymbol{\tau})}, s_t\right) D_t - (\kappa+m)x_t^{(\iota,\boldsymbol{\tau})}\right) dt\right] \\ &= \sup_{(\iota,\boldsymbol{\tau})\in\mathcal{I}\times\mathcal{T}} \tilde{\mathbb{E}}^{x}\left[\int_0^{+\infty} \alpha^{N_{d,t}^{\boldsymbol{\tau}}} e^{-(\delta-\mu)t}\left(1 + \iota\left(x_t^{(\iota,\boldsymbol{\tau})}, s_t\right) D_t - (\kappa+m)x_t^{(\iota,\boldsymbol{\tau})}\right) dt\right] \end{aligned} \tag{54}$$

In equation (54), we have introduced the measure $\tilde{\Pr}$, defined for any arbitrary Borel set $A \subseteq \mathcal{F}_t$ via $\tilde{\Pr}(A) = \mathbb{E}\left[\exp\left(-\frac{|\boldsymbol{\sigma}|^2}{2}t + \boldsymbol{\sigma}\cdot\boldsymbol{B}_t\right) A\right]$. Under such measure, in the continuation

region $(0, \bar{x})$, using Girsanov's theorem, the controlled debt-to-income ratio $x_t^{(\iota,\boldsymbol{\tau})}$ evolves as follows:

$$dx_t^{(\iota,\boldsymbol{\tau})} = \left(\iota(x_t^{(\iota,\boldsymbol{\tau})}, s_t) - (m+\mu)\, x_t^{(\iota,\boldsymbol{\tau})}\right) dt - x_t^{(\iota,\boldsymbol{\tau})} \boldsymbol{\sigma} \cdot d\tilde{\boldsymbol{B}}_{\boldsymbol{t}} + (\theta - 1)\, x_t^{(\iota,\boldsymbol{\tau})} dN_{d,t}^{(\boldsymbol{\tau})} \tag{55}$$

$\tilde{\boldsymbol{B}}_{\boldsymbol{t}} := \boldsymbol{B}_{\boldsymbol{t}} - \boldsymbol{\sigma} t$ is a standard Brownian motion under $\tilde{\Pr}$. As discussed in section 4.5, the government life-time value function can be computed as if the government was never issuing debt. Thus, for $x \in (0, \bar{x})$, v satisfies:

$$(\delta - \mu)\, v(x) = 1 - (\kappa + m)\, x - (\mu + m)\, x v'(x) + \frac{1}{2}|\boldsymbol{\sigma}|^2 x^2 v''(x) \tag{56}$$

This is a second order ordinary differential equation, whose general solutions are power functions of x. The exponent of the general solutions solves the quadratic equation:

$$\frac{1}{2}|\boldsymbol{\sigma}|^2 \xi^2 - \left(m + \mu + \frac{1}{2}|\boldsymbol{\sigma}|^2\right) \xi - (\delta - \mu) = 0 \tag{57}$$

Given the parameter restriction (31), this quadratic equation admits one positive, and one negative roots. Since v must be finite as $x \to 0$, we eliminate the negative root, and note ξ the positive one:

$$\xi := \frac{1}{2}\left(1 + \frac{2(m+\mu)}{|\boldsymbol{\sigma}|^2}\right)\left[1 + \left(1 + \frac{8(\delta - \mu)|\boldsymbol{\sigma}|^2}{(2(m+\mu) + |\boldsymbol{\sigma}|^2)^2}\right)^{1/2}\right]$$

We note that $\xi > 1$. We need one more boundary condition – we will use the fact that upon default, the small open economy suffers a discrete income drop by a factor α, and immediately emerges from autarky with a debt-to-income ratio that is a fraction θ of its pre-default debt-to-income ratio:

$$v(\bar{x}) = \alpha v(\theta \bar{x})$$

Using these, we can express v as follows on $[0, \bar{x}]$:

$$v(x) = \frac{1}{\delta - \mu}\left[1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi}\right] - \left(\frac{\kappa + m}{\delta + m} x\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi-1}\right]$$

For $x > \bar{x}$, let $n(x; \bar{x}) := 1 + \left\lfloor \frac{\ln x - \ln \bar{x}}{-\ln \theta} \right\rfloor$ be the number of times the government needs to default consecutively in order to re-enter the continuation region. For $x > \bar{x}$, the value

function satisfies:

$$v(x) = \alpha^{n(x;\bar{x})} v\left(\theta^{n(x;\bar{x})} x\right)$$

We now want to "paste" the solution for $x > \bar{x}$ with the solution for $x \leq \bar{x}$, in such a way that the function v is $\mathcal{C}^1$ on $\mathbb{R}^+$, so that we can use standard verification arguments. The determination of the optimal default boundary $\bar{x}$ relies on the observation that the government always has the option to default, in other words:

$$v(x_t) \geq \alpha v(\theta x_t) \quad \forall t$$

In particular, near the default boundary, for the inequality to be satisfied, we must have:

$$\lim_{t \nearrow \tau} \text{var}\left[d\left(v(x_t) - \alpha v(\theta x_t)\right)\right] = 0$$

This leads to the smooth pasting condition:

$$v'(\bar{x}) = \alpha\theta v'(\theta\bar{x})$$

Collecting these together, we compute the following default boundary $\bar{x}$:

$$\bar{x} = \frac{\xi}{\xi - 1}\left(\frac{\delta + m}{\kappa + m}\right)\left(\frac{1-\alpha}{1-\alpha\theta}\right)\frac{1}{\delta - \mu}$$

The debt price d per unit of debt outstanding can be computed by leveraging equation (27), which becomes in this particular case $d(x) = -v'(x)$. In other words, for $x \in [0, \bar{x}]$, we have:

$$d(x) = \left(\frac{\kappa + m}{\delta + m}\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi - 1}\right]$$

For $x > \bar{x}$, d is determined via the number of consecutive times the sovereign will default in order to reenter the continuation region:

$$d(x) = (\alpha\theta)^{n(x;\bar{x})}\, d\left(\theta^{n(x;\bar{x})} x\right)$$

Note that in the continuation region, the value function v takes the following form:

$$v(x) = \frac{1}{\delta - \mu}\left(1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi}}\right)\left(\frac{x}{\bar{x}}\right)^{\xi}\right) - xd(x)$$

The required expected excess return on the sovereign debt can be easily computed:

$$\begin{aligned}\pi(x,s) &= -\frac{xd'(x)}{d(x)}\boldsymbol{\sigma}\cdot\boldsymbol{\nu}(s)\\ &= \frac{\xi-1}{\left(\frac{1-\alpha\theta^{\xi}}{1-\alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi-1}-1}\boldsymbol{\sigma}\cdot\boldsymbol{\nu}(s)\end{aligned}$$

The issuance policy thus takes the following form:

$$\begin{aligned}\iota^*(x,s) &= \frac{d(x)}{-d'(x)}(\delta - r(s) - \pi(s))\\ &= \frac{\delta - r(s)}{\xi-1}\left[\left(\frac{1-\alpha\theta^{\xi}}{1-\alpha\theta}\right)\left(\frac{\bar{x}}{x}\right)^{\xi-1}-1\right]x - x\boldsymbol{\sigma}\cdot\boldsymbol{\nu}(s)\end{aligned}$$

We are not quite done with our proof. We still need to establish that no other admissible policy can achieve a higher welfare for the government, via a standard verification theorem. Let $(\iota,\boldsymbol{\tau})\in\mathcal{I}\times\mathcal{T}$ be an arbitrary issuance and default policy. We introduce the infinitessimal generator $\mathcal{L}^{(\iota)}$, defined for any function $f\in\mathcal{C}^2(\mathbb{R})$ as follows:

$$\mathcal{L}^{(\iota)}f(x) := (\iota(x,s)-(\mu+m)x)\,f'(x) + \frac{1}{2}x^2|\boldsymbol{\sigma}|^2 f''(x)$$

Note that the function v constructed above is defined on $\mathbb{R}_+$, and is $\mathcal{C}^2$ on $\mathbb{R}\setminus\{\theta^k\bar{x};k\in\mathbb{N}\}$. At $x=\theta^k\bar{x}$ $(k\in\mathbb{N})$, the function v is $\mathcal{C}^1$ by construction. The function v also satisfies the variational inequality:

$$0 = \max\left[\sup_{\iota}\left[-(\delta-\mu)v(x)+1+\iota d(x)-(\kappa+m)x+\mathcal{L}^{(\iota)}v(x)\right];\alpha v(\theta x)-v(x)\right] \tag{58}$$

Given the dynamic evolution of the controlled stochastic process $x_t^{(\iota,\boldsymbol{\tau})}$ (as described by equation (55)), we have the following Itô formula:

$$\begin{aligned}&\alpha^{N_{d,t}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)t}v(x_t^{(\iota,\boldsymbol{\tau})}) = v(x) - \int_0^t \alpha^{N_{d,z}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)z}x_z^{(\iota,\boldsymbol{\tau})}v'(x_z^{(\iota,\boldsymbol{\tau})})\boldsymbol{\sigma}\cdot d\tilde{\boldsymbol{B}}_{\boldsymbol{z}}\\ &+\int_0^t \alpha^{N_{d,z}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)z}\left[\mathcal{L}^{(\iota)}v(x_z^{(\iota,\boldsymbol{\tau})})-(\delta-\mu)v(x_z^{(\iota,\boldsymbol{\tau})})\right]dz + \int_0^t \alpha^{N_{d,z-}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)z}\left[\alpha v(\theta x_{z-}^{(\iota,\boldsymbol{\tau})})-v(x_{z-}^{(\iota,\boldsymbol{\tau})})\right]dN_{d,z}^{(\boldsymbol{\tau})}\end{aligned}$$

See for example Protter (2005). We then use our variational inequality (58) to obtain:

$$\alpha^{N_{d,t}^{(\boldsymbol{\tau})}} e^{-(\delta-\mu)t} v(x_t^{(\iota,\boldsymbol{\tau})}) \leq v(x) - \int_0^t \alpha^{N_{d,z}^{\boldsymbol{\tau}}} e^{-(\delta-\mu)z} \left[1 + \iota(x_z^{(\iota,\boldsymbol{\tau})}, s_z) - (\kappa+m) x_z^{(\iota,\boldsymbol{\tau})}\right] dz \\ - \int_0^t \alpha^{N_{d,z}^{(\boldsymbol{\tau})}} e^{-(\delta-\mu)z} x_z^{(\iota,\boldsymbol{\tau})} v'(x_z^{(\iota,\boldsymbol{\tau})}) \boldsymbol{\sigma} \cdot d\tilde{\boldsymbol{B}}_{\boldsymbol{z}}$$

The stochastic integral in the second line of the equation above is a martingale since $xv'(x)$ is bounded. Thus, taking expectations on both sides of this equality, we obtain:

$$\tilde{\mathbb{E}}^{x,s}\left[\int_0^t \alpha^{N_{d,z}^{(\boldsymbol{\tau})}} e^{-(\delta-\mu)z} \left[1 + \iota(x_z^{(\iota)}, s_z) - (\kappa+m) x_z^{(\iota)}\right] dz + \alpha^{N_{d,t}^{(\boldsymbol{\tau})}} e^{-(\delta-\mu)t} v(x_t)\right] \leq v(x)$$

When we take $t \to +\infty$, $\alpha^{N_{d,t}^{(\boldsymbol{\tau})}} e^{-(\delta-\mu)t} v(x_t) \to 0$. Using the dominated convergence theorem, we then obtain the desired result: $v(x) \geq J(1, x; (\iota, \tau))$ for any admissible control policy. The bound is achieved for our issuance policy ι^* and default policy $\boldsymbol{\tau}^*$, and the proof relies on steps identical to those described above, except that inequalities are now replaced by equalities. □

A.5 Comparative Statics – Default Boundary

For the comparative static with respect to σ, note that:

$$\frac{\partial \xi}{\partial \sigma^2} = -\left(1 + \frac{8(\delta-\mu)\sigma^2}{\left(2(m+\mu)+\sigma^2\right)^2}\right)^{-1/2} \left[\frac{m+\mu}{\sigma^2} + \frac{2(\delta-\mu)}{2(m+\mu)+\sigma^2}\right] - \frac{m+\mu}{\sigma^2} < 0$$

Since $\frac{\partial \bar{x}}{\partial \xi} < 0$, it means that the default boundary $\bar{x}$ is increasing as output volatility increases. For the comparative static w.r.t. μ, notice that:

$$\frac{\partial \xi}{\partial \mu} = \frac{\xi(\xi-1)}{\left(m+\mu+\frac{1}{2}|\boldsymbol{\sigma}|^2\right)\xi + 2(\delta-\mu)}$$

Thus, we can write:

$$\begin{aligned} \frac{d\bar{x}}{d\mu} &= \frac{\partial \bar{x}}{\partial \mu} + \frac{\partial \bar{x}}{\partial \xi}\frac{\partial \xi}{\partial \mu} \\ &= \bar{x}\left[\frac{1}{\delta-\mu} - \frac{1}{\left(m+\mu+\frac{1}{2}|\boldsymbol{\sigma}|^2\right)\xi + 2(\delta-\mu)}\right] > 0 \end{aligned}$$

In other words, $\bar{x}$ is increasing in μ. For the comparative static w.r.t. δ, notice that:

$$\frac{d\bar{x}}{d\delta} = \frac{\partial \bar{x}}{\partial \xi}\frac{\partial \xi}{\partial \delta} + \frac{\partial \bar{x}}{\partial \delta}$$

ξ is clearing increasing in δ, and $\bar{x}$ is clearing decreasing in ξ. Similarly, keeping ξ constant, $\bar{x}$ is decreasing in δ. This means that $\bar{x}$ is decreasing in δ. For the comparative static w.r.t. α and θ, notice that ξ does not depend on those parameters, while $\bar{x}$ is decreasing in α and increasing in θ, delivering the result stated. For the comparative static w.r.t. m, note that:

$$\frac{\partial \xi}{\partial m} = \frac{1}{\sigma^2}\left[1 + \left(1 + \frac{8(\delta-\mu)\sigma^2}{\left(2(m+\mu)+\sigma^2\right)^2}\right)^{-1/2}\right] > 0$$

This means that

$$\begin{aligned}\frac{d\bar{x}}{dm} &= \frac{\partial \bar{x}}{\partial \xi}\frac{\partial \xi}{\partial m} + \frac{\partial \bar{x}}{\partial m} \\ &= -\bar{x}\left[\frac{1}{\xi(\xi-1)}\frac{\partial \xi}{\partial m} + \frac{\delta-\kappa}{(\delta+m)(\kappa+m)}\right]\end{aligned}$$

Note that ξ does not depend on κ, meaning that the expression above is increasing in κ. In other words, there exists $\bar{\kappa} > \delta$ (with $\bar{\kappa}$ potentially infinite) such that $\frac{d\bar{x}}{dm} < 0$ if and only if $\kappa < \bar{\kappa}$. Finally, the threshold $\bar{x}$ is trivially decreasing in κ, since ξ is independent of κ. □

A.6 Comparative Statics – Value Function

To perform those comparative statics, we leverage extensively Feynman-Kac and the integral representation of second order differential equations. Let us look at the comparative static w.r.t. κ for example. Remember that the value function v satisfies the following:

$$\begin{aligned}(\delta-\mu)v(x;\kappa) &= 1 - (\kappa+m)x - (\mu+m)xv_x(x;\kappa) + \frac{1}{2}|\boldsymbol{\sigma}|^2x^2v_{xx}(x;\kappa) \\ v(\bar{x};\kappa) &= \alpha v(\theta\bar{x};\kappa) \\ v_x(\bar{x};\kappa) &= \alpha\theta v_x(\theta\bar{x};\kappa)\end{aligned}$$

In the above, we have used a notation that emphasizes that the value function depends on the parameter κ. Differentiate the first two equations above w.r.t κ to obtain:

$$(\delta-\mu)v_\kappa(x;\kappa) = -x-(\mu+m)xv_{\kappa x}(x;\kappa)+\frac{1}{2}|\boldsymbol{\sigma}|^2x^2v_{\kappa xx}(x;\kappa)$$
$$\frac{\partial\bar{x}}{\partial\kappa}v_x(\bar{x};\kappa)+v_\kappa(\bar{x};\kappa) = \alpha\theta\frac{\partial\bar{x}}{\partial\kappa}v_x(\theta\bar{x};\kappa)+\alpha v_\kappa(\theta\bar{x};\kappa)$$

Use the fact that $v_x(\bar{x};\kappa) = \alpha\theta v_x(\theta\bar{x};\kappa)$ to obtain the boundary condition $v_\kappa(\bar{x};\kappa) = \alpha v_\kappa(\theta\bar{x};\kappa)$. In other words, v_κ admits the following integral representation:

$$v_\kappa(x) = \tilde{\mathbb{E}}\left[\int_0^\infty \alpha^{N_{d,t}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)t}(-x_t)\,dt\right]$$

In other words, $v_\kappa(x) < 0$. A similar method leads to the other comparative statics:

$$v_{|\boldsymbol{\sigma}|^2}(x) = \frac{1}{2}\tilde{\mathbb{E}}^x\left[\int_0^\infty \alpha^{N_{d,t}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)t}x_t^2v''(x_t)dt\right] > 0$$
$$v_\delta(x) = -\tilde{\mathbb{E}}^x\left[\int_0^\infty \alpha^{N_{d,t}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)t}v(x_t)dt\right] < 0$$
$$v_\mu(x) = \tilde{\mathbb{E}}^x\left[\int_0^\infty \alpha^{N_{d,t}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)t}\left(v(x_t)-x_tv'(x_t)\right)dt\right] > 0$$
$$v_m(x) = -\tilde{\mathbb{E}}^x\left[\int_0^\infty \alpha^{N_{d,t}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)t}x_t\left(1+v'(x_t)\right)dt\right]$$

To sign v_m, it suffices to look at the behavior of the function $1+v'(x)$. Note that $v'(0) = -\frac{\kappa+m}{\delta+m}$, and since v is convex, we must have $v'(x) \geq -\frac{\kappa+m}{\delta+m}$ for all $x \in [0,\bar{x}]$. Thus, if $\kappa < \delta$, $v'(x)+1 > 0$ for all $x \in [0,\bar{x}]$, meaning that $v_m < 0$. Finally, a slight modification of our proof is needed for the comparative statics w.r.t. α and θ. For α for example, note that we have the following:

$$(\delta-\mu)v_\alpha(x;\alpha) = -(\mu+m)xv_{\alpha x}(x;\alpha)+\frac{1}{2}|\boldsymbol{\sigma}|^2x^2v_{\alpha xx}(x;\alpha)$$
$$\frac{\partial\bar{x}}{\partial\alpha}v_x(\bar{x};\alpha)+v_\alpha(\bar{x};\alpha) = \alpha\theta\frac{\partial\bar{x}}{\partial\alpha}v_x(\theta\bar{x};\alpha)+\alpha v_\alpha(\theta\bar{x};\alpha)+v(\theta\bar{x};\alpha)$$

Use the smooth-pasting condition $v_x(\bar{x};\alpha) = \alpha\theta v_x(\theta\bar{x};\alpha)$ to obtain the boundary condition $v_\alpha(\bar{x};\alpha) = \alpha v_\alpha(\theta\bar{x};\alpha)+v(\theta\bar{x};\alpha)$. In other words, v_α admits the following representation:

$$v_\alpha(x) = \tilde{\mathbb{E}}^x\left[\sum_{k=0}^\infty e^{-(\delta-\mu)\tau_k}\alpha^k v(\theta\bar{x})\right] > 0$$

Similarly, one can show that

$$v_\theta(x) = \tilde{\mathbb{E}}^x \left[\sum_{k=0}^{\infty} e^{-(\delta-\mu)\tau_k} \alpha^{k+1} \bar{x} v' \left(\theta \bar{x} \right) \right] < 0$$

A.7 Ergodic Distribution and Average Default Rate

The drift rate $\mu_x(x)$ of the state variable x and the volatility $\sigma_x(x)$ are equal to:

$$\begin{aligned}
\mu_x(x) :&= \iota(x) - \left(m + \mu - \sigma^2 \right) x \\
&= \left[\left(\frac{\delta - r}{\xi - 1} \right) \left(\frac{1 - \alpha\theta^\xi}{1 - \alpha\theta} \right) \left(\frac{\bar{x}}{x} \right)^{\xi - 1} - \left(m + \mu - \sigma^2 + \frac{\delta - r}{\xi - 1} \right) \right] x \\
\sigma_x(x) :&= \sigma x
\end{aligned}$$

We focus on the parameter restriction $\delta + m > \sigma^2 - (m + \mu)$, which implies that the issuance policy $\iota(x)$ is strictly decreasing on $(0, \bar{x})$, with $\iota(x) \to +\infty$ as $x \to 0$. This is also the parameter restriction that guarantees that $\xi > 2$. The stationary distribution f of the state variable x_t under the measure $\mathbb{P}$ solves the Kolmogorov-Forward equation:

$$0 = -\frac{d}{dx} \left[\mu_x(x) f(x) \right] + \frac{d^2}{dx^2} \left[\frac{\sigma_x^2(x)}{2} f(x) \right] \tag{59}$$

To compute numerically this distribution, we approximate the stochastic process for x_t as follows. Note $s_x(x) := \sigma_x^2(x)$, and $\bar{s} = \sigma_x^2(\bar{x}) = \max_{x \in [0, \bar{x}]} \sigma_x^2(x)$. Consider a discrete grid $G_h = \{h, 2h, ..., Nh\}$ where x_t^h will evolve, and construct the transition probabilities for the Markov chain $\{x_t^h\}_{t \geq 0}$:

$$\begin{aligned}
p_d^h(x) :&= \frac{s_x(x) - \mu_x(x) h}{\bar{s}} \\
p_d^h(x) :&= \frac{s_x(x) + \mu_x(x) h}{\bar{s}} \\
p_d^h(x) :&= 1 - \frac{s_x(x)}{\bar{s}}
\end{aligned}$$

Noting $\Delta_t^h := \frac{h^2}{\bar{s}}$, the usual consistency conditions are satisfied:

$$\begin{aligned}
\mathbb{E}_t \left[x_{t+1}^h - x_t^h \right] &= \mu_x(x_t) \Delta_t^h \\
\text{var}_t \left[x_{t+1}^h - x_t^h \right] &= s_x(x_t) \Delta_t^h + o\left(\Delta_t^h \right)
\end{aligned}$$

The Kolmogorov-Forward equation (59) can be obtained by using our Markov chain, and taking the limit, when $h \to 0$, of:

$$f(x) = p_d^h(x+h)f(x+h) + p_c^h(x)f(x) + p_u^h(x-h)f(x-h) \tag{60}$$

Once we have our Markov chain approximation $\{x_t^h\}_{\geq 0}$ of $\{x_t\}_{\geq 0}$, we construct its related Markov matrix and compute the positive eigen-vector of such matrix corresponding to the eigen-value 1, and normalize it so that the sum of its entries are equal to 1. □

A.8 GDP-Linked Bonds

Our proof follows closely section A.4. The bonds issued by the government are now GDP-linked, with weighting vector $\boldsymbol{\varsigma}$, such that the debt face value F_t follows:

$$F_t^{(I,\boldsymbol{\tau})} = \int_0^t \left(I\left(Y_u^{(\boldsymbol{\tau})}, F_u^{(I,\boldsymbol{\tau})}, s_u\right) - mF_u^{(I,\boldsymbol{\tau})}\right) du + \int_0^t F_u^{(I,\boldsymbol{\tau})} \boldsymbol{\varsigma} \cdot d\boldsymbol{B}_u + \int_0^t (\theta\alpha - 1) F_{u-}^{(I,\boldsymbol{\tau})} dN_{d,u}^{(\boldsymbol{\tau})} \tag{61}$$

The value function V can still be written $V(Y,F) = Yv(x)$. Under Pr and $\tilde{\Pr}$, the debt-to-income ratio follows:

$$dx_t^{(\iota,\boldsymbol{\tau})} = \left(\iota(x_t^{(\iota,\boldsymbol{\tau})}, s_t) - (m + \mu - \boldsymbol{\sigma}\cdot(\boldsymbol{\sigma}-\boldsymbol{\varsigma}))\, x_t^{(\iota,\boldsymbol{\tau})}\right) dt - x_t^{(\iota,\boldsymbol{\tau})} (\boldsymbol{\sigma}-\boldsymbol{\varsigma}) \cdot d\boldsymbol{B_t} + (\theta-1)\, x_t^{(\iota,\boldsymbol{\tau})} dN_{d,t}^{(\boldsymbol{\tau})}$$
$$dx_t^{(\iota,\boldsymbol{\tau})} = \left(\iota(x_t^{(\iota,\boldsymbol{\tau})}, s_t) - (m+\mu)\, x_t^{(\iota,\boldsymbol{\tau})}\right) dt - x_t^{(\iota,\boldsymbol{\tau})} (\boldsymbol{\sigma}-\boldsymbol{\varsigma}) \cdot d\tilde{\boldsymbol{B}}_{\boldsymbol{t}} + (\theta-1)\, x_t^{(\iota,\boldsymbol{\tau})} dN_{d,t}^{(\boldsymbol{\tau})}$$

In the continuation region $(0, \bar{x})$, v satisfies:

$$(\delta - \mu)\, v(x) = 1 - (\kappa + m)\, x - (\mu + m)\, xv'(x) + \frac{1}{2}|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2 x^2 v''(x)$$

This is a second order ordinary differential equation, whose general solutions are power functions of x. The exponent of the general solutions solves the quadratic equation:

$$\frac{1}{2}|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2 \xi_{\boldsymbol{\varsigma}}^2 - \left(m + \mu + \frac{1}{2}|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2\right)\xi_{\boldsymbol{\varsigma}} - (\delta - \mu) = 0$$

Given the parameter restriction (31), this quadratic equation admits one positive, and one negative roots. We also know that $\xi_{\boldsymbol{\varsigma}} > 1$. Since v must be finite as $x \to 0$, we eliminate the negative root, and note $\xi_{\boldsymbol{\varsigma}}$ the positive one. Our second boundary condition uses the

fact that upon default, the small open economy suffers a discrete income drop by a factor α, and immediately emerges from autarky with a debt-to-income ratio that is a fraction θ of its pre-default debt-to-income ratio:

$$v(\bar{x}_{\boldsymbol{\varsigma}}) = \alpha v(\theta \bar{x}_{\boldsymbol{\varsigma}})$$

Using these, we can express v as follows on $[0, \bar{x}]$:

$$v(x) = \frac{1}{\delta - \mu}\left[1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi_{\boldsymbol{\varsigma}}}}\right)\left(\frac{x}{\bar{x}_{\boldsymbol{\varsigma}}}\right)^{\xi_{\boldsymbol{\varsigma}}}\right] - \left(\frac{\kappa+m}{\delta+m}x\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_{\boldsymbol{\varsigma}}}}\right)\left(\frac{x}{\bar{x}_{\boldsymbol{\varsigma}}}\right)^{\xi_{\boldsymbol{\varsigma}}-1}\right]$$

The smooth-pasting condition takes the usual form:

$$v'(\bar{x}_{\boldsymbol{\varsigma}}) = \alpha\theta v'(\theta \bar{x}_{\boldsymbol{\varsigma}})$$

Collecting these together, we compute the following default boundary $\bar{x}_{\boldsymbol{\varsigma}}$:

$$\bar{x}_{\boldsymbol{\varsigma}} = \frac{\xi_{\boldsymbol{\varsigma}}}{\xi_{\boldsymbol{\varsigma}} - 1}\left(\frac{\delta+m}{\kappa+m}\right)\left(\frac{1-\alpha}{1-\alpha\theta}\right)\frac{1}{\delta-\mu}$$

The debt price d per unit of debt outstanding can be computed by leveraging equation (27), which becomes in this particular case $d(x) = -v'(x)$. In other words, for $x \in [0, \bar{x}_{\boldsymbol{\varsigma}}]$, we have:

$$d(x) = \left(\frac{\kappa+m}{\delta+m}\right)\left[1 - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_{\boldsymbol{\varsigma}}}}\right)\left(\frac{x}{\bar{x}_{\boldsymbol{\varsigma}}}\right)^{\xi_{\boldsymbol{\varsigma}}-1}\right]$$

Note that in the continuation region, the value function v takes the following form:

$$v(x) = \frac{1}{\delta-\mu}\left(1 - \left(\frac{1-\alpha}{1-\alpha\theta^{\xi_{\boldsymbol{\varsigma}}}}\right)\left(\frac{x}{\bar{x}_{\boldsymbol{\varsigma}}}\right)^{\xi_{\boldsymbol{\varsigma}}}\right) - xd(x)$$

The required expected excess return on the sovereign debt can be easily computed:

$$\begin{aligned}\pi(x,s) &= -\frac{xd'(x)}{d(x)}(\boldsymbol{\sigma} - \boldsymbol{\varsigma}) \cdot \boldsymbol{\nu}(s) \\ &= \frac{\xi_{\boldsymbol{\varsigma}} - 1}{\left(\frac{1-\alpha\theta^{\xi_{\boldsymbol{\varsigma}}}}{1-\alpha\theta}\right)\left(\frac{\bar{x}_{\boldsymbol{\varsigma}}}{x}\right)^{\xi_{\boldsymbol{\varsigma}}-1} - 1}(\boldsymbol{\sigma} - \boldsymbol{\varsigma}) \cdot \boldsymbol{\nu}(s)\end{aligned}$$

The issuance policy now takes a different form. Indeed, note that the debt price satisfies:

$$\begin{aligned} d(x) &= \hat{\mathbb{E}}^{x,s}\left[\int_0^\infty e^{-\int_0^t \left(r(s_u)+m+\frac{1}{2}|\boldsymbol{\varsigma}|^2\right)du+\int_0^t \boldsymbol{\varsigma}\cdot d\boldsymbol{B}_u}\left(\alpha\theta\right)^{N_{d,t}^{(\boldsymbol{\tau})}}(\kappa+m)dt\right] \\ &= \mathring{\mathbb{E}}^{x,s}\left[\int_0^\infty e^{-\int_0^t (r(s_u)+m+\boldsymbol{\varsigma}\cdot\boldsymbol{\nu}(s_u))du}\left(\alpha\theta\right)^{N_{d,t}^{(\boldsymbol{\tau})}}(\kappa+m)dt\right] \end{aligned}$$

In the above, we have introduced the measure $\mathring{\Pr}$, defined for any arbitrary Borel set $A \subseteq \mathcal{F}_t$ via $\mathring{\Pr}(A) = \mathbb{E}\left[\exp\left(-\frac{|\boldsymbol{\varsigma}|^2}{2}t + \boldsymbol{\varsigma}\cdot\tilde{\boldsymbol{B}}_t\right)A\right]$. $\mathring{\boldsymbol{B}}_{\boldsymbol{t}} := \hat{\boldsymbol{B}}_t - \boldsymbol{\varsigma}t$ is a standard Brownian motion under $\mathring{\Pr}$, and under such measure the debt-to-income ratio follows:

$$\begin{aligned} dx_t^{(\iota,\boldsymbol{\tau})} = &\left(\iota(x_t^{(\iota,\boldsymbol{\tau})},s_t) - \left(m+\mu-|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2-\boldsymbol{\nu}(s)\cdot(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\right)x_t^{(\iota,\boldsymbol{\tau})}\right)dt \\ &- x_t^{(\iota,\boldsymbol{\tau})}(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\cdot d\mathring{\boldsymbol{B}}_{\boldsymbol{t}} + (\theta-1)\,x_t^{(\iota,\boldsymbol{\tau})}dN_{d,t}^{(\boldsymbol{\tau})} \end{aligned}$$

The debt price thus satifies the following Feynman-Kac equation:

$$\begin{aligned} (r(s)+m+\boldsymbol{\varsigma}\cdot\boldsymbol{\nu}(s))\,d(x) = \kappa+m+&\left[\iota(x,s)-\left(m+\mu-|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2-\boldsymbol{\nu}(s)\cdot(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\right)x\right]d'(x) \\ &+\frac{|\boldsymbol{\sigma}-\boldsymbol{\varsigma}|^2}{2}x^2d''(x) \end{aligned}$$

The issuance policy thus takes the following form:

$$\begin{aligned} \iota^*(x,s) &= \frac{d(x)}{-d'(x)}(\delta-r(s)-\boldsymbol{\varsigma}\cdot\boldsymbol{\nu}(s)) - x\,(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\cdot\boldsymbol{\nu}(s) \\ &= \frac{\delta-r(s)-\boldsymbol{\varsigma}\cdot\boldsymbol{\nu}(s)}{\xi_{\boldsymbol{\varsigma}}-1}\left[\left(\frac{1-\alpha\theta^{\xi_{\boldsymbol{\varsigma}}}}{1-\alpha\theta}\right)\left(\frac{\bar{x}_{\boldsymbol{\varsigma}}}{x}\right)^{\xi_{\boldsymbol{\varsigma}}-1}-1\right]x - x\,(\boldsymbol{\sigma}-\boldsymbol{\varsigma})\cdot\boldsymbol{\nu}(s) \end{aligned}$$

□

A.9 Commitment Equilibrium

In this section, we use the notation introduced in section A.4. The normalized value function v_γ of the government and the debt price d_γ are equal to:

$$\begin{aligned} v_\gamma(x) :&= \sup_{\boldsymbol{\tau}\in\mathcal{T}}\tilde{\mathbb{E}}^x\left[\int_0^{+\infty}\alpha^{N_{d,t}^{(\boldsymbol{\tau})}}e^{-(\delta-\mu)t}\left(1+\gamma x_t d_\gamma(x_t)-(\kappa+m)x_t\right)dt\right] \\ d_\gamma(x) :&= \mathbb{E}^x\left[\int_0^{+\infty}\left(\alpha\theta\right)^{N_{d,t}^{(\boldsymbol{\tau})}}e^{-(r+m)t}\left(\kappa+m\right)dt\right] \end{aligned}$$

We assume that the parameter condition $\gamma < \delta + m$ holds. As a reminder, the dynamics of x_t under Pr and under $\tilde{\Pr}$ are as follows:

$$\begin{aligned} dx_t^{(\boldsymbol{\tau})} &= -\left(\mu - \gamma + m\right) x_t^{(\boldsymbol{\tau})} dt - x_t^{(\boldsymbol{\tau})} \boldsymbol{\sigma} \cdot d\tilde{\boldsymbol{B}}_t + (\theta - 1)\, x_t^{(\boldsymbol{\tau})} dN_{d,t}^{(\boldsymbol{\tau})} \\ dx_t^{(\boldsymbol{\tau})} &= -\left(\mu - \gamma + m - |\boldsymbol{\sigma}|^2\right) x_t^{(\boldsymbol{\tau})} dt - x_t^{(\boldsymbol{\tau})} \boldsymbol{\sigma} \cdot d\boldsymbol{B}_t + (\theta - 1)\, x_t^{(\boldsymbol{\tau})} dN_{d,t}^{(\boldsymbol{\tau})} \end{aligned}$$

In the continuation region $[0, \bar{x}_\gamma)$, the functions v_γ and d_γ are solutions to the following second order ordinary differential equations:

$$\begin{aligned} (\delta - \mu) v_\gamma(x) &= 1 + \gamma x d_\gamma(x) - (\kappa + m)x - \left(\mu - \gamma + m\right) x v_\gamma'(x) + \frac{1}{2}|\boldsymbol{\sigma}|^2 x^2 v_\gamma''(x) \\ (r + m) d_\gamma(x) &= (\kappa + m) - \left(\mu - \gamma + m - |\boldsymbol{\sigma}|^2\right) x d_\gamma'(x) + \frac{1}{2}|\boldsymbol{\sigma}|^2 x^2 d_\gamma''(x) \end{aligned}$$

Let us introduce ξ_v, ξ_d the positive roots of the quadratic equations:

$$\begin{aligned} \frac{1}{2}|\boldsymbol{\sigma}|^2 \xi_v^2 - \left(\mu - \gamma + m + \frac{1}{2}|\boldsymbol{\sigma}|^2\right) \xi_v - (\delta - \mu) = 0 \\ \frac{1}{2}|\boldsymbol{\sigma}|^2 \xi_d^2 - \left(\mu - \gamma + m - \frac{1}{2}|\boldsymbol{\sigma}|^2\right) \xi_d - (r + m) = 0 \end{aligned}$$

Note that $\xi_v, \xi_d > 0$. Since $\gamma < \delta + m$, notice that $\xi_v > 1$. Since $\frac{1}{2}\sigma^2(\xi_v - 1)^2 - \left(\mu - \gamma + m - \frac{1}{2}\sigma^2\right)(\xi_v - 1) - (r + m) = \delta - (r + \gamma)$, it is clear that $\xi_v - 1 > \xi_d$ if and only if $\delta > r + \gamma$. Using the boundary condition $d_\gamma(\bar{x}_\gamma) = \alpha\theta d_\gamma(\theta \bar{x}_\gamma)$, and the fact that $d_\gamma(x)$ must be finite as $x \to 0$, the function d_γ admits the following functional form:

$$d_\gamma(x) = \frac{\kappa + m}{r + m}\left(1 - \left(\frac{1 - \alpha\theta}{1 - \alpha\theta^{\xi_d + 1}}\right)\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_d}\right)$$

The function v_γ takes the following form:

$$v_\gamma(x) = k_\gamma x^{\xi_v} + a_0 + a_1 x + a_2 x^{\xi_d + 1}$$

The constant k_γ is a constant of integration that will be found using boundary conditions. The constants a_0, a_1, a_2 solve the following set of linear equations:

$$\begin{aligned} &(\delta - \mu) a_0 = 1 \\ &(\delta - \mu) a_1 = \gamma \frac{\kappa + m}{r + m} - (\kappa + m) - (m - \gamma + \mu)\, a_1 \\ &(\delta - \mu) a_2 = -\gamma \frac{\kappa + m}{r + m}\left(\frac{1 - \alpha\theta}{1 - \alpha\theta^{\xi_d + 1}}\right) \bar{x}^{-\xi_d} - (m - \gamma + \mu)(\xi_d + 1) a_2 + \frac{1}{2}\sigma^2 \xi_d (\xi_d + 1) a_2 \end{aligned}$$

This yields:

$$
\begin{aligned}
a_0 &= \frac{1}{\delta - \mu} \\
a_1 &= -\left(\frac{r+m-\gamma}{\delta+m-\gamma}\right)\frac{\kappa+m}{r+m} \\
a_2 &= \frac{\gamma(\kappa+m)}{(r+m)(r+\gamma-\delta)}\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_d+1}}\right)\bar{x}_\gamma^{-\xi_d}
\end{aligned}
$$

The constant of integration k_γ is determined using the default boundary condition $v_\gamma(\bar{x}_\gamma) = \alpha v_\gamma(\theta\bar{x}_\gamma)$:

$$
\begin{aligned}
&k_\gamma \bar{x}_\gamma^{\xi_v} + \frac{1}{\delta-\mu} - \left(\frac{r+m-\gamma}{\delta+m-\gamma}\right)\left(\frac{\kappa+m}{r+m}\right)\bar{x}_\gamma + \frac{\gamma(\kappa+m)}{(r+m)(r+\gamma-\delta)}\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_d+1}}\right)\bar{x}_\gamma = \\
&\alpha\left[k_\gamma\theta^{\xi_v}\bar{x}_\gamma^{\xi_v} + \frac{1}{\delta-\mu} - \left(\frac{r+m-\gamma}{\delta+m-\gamma}\right)\left(\frac{\kappa+m}{r+m}\right)\theta\bar{x}_\gamma + \frac{\gamma(\kappa+m)}{(r+m)(r+\gamma-\delta)}\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_d+1}}\right)\theta^{\xi_d+1}\bar{x}_\gamma\right]
\end{aligned}
$$

This leads to a constant k_γ that is equal to:

$$
k_\gamma = \frac{-1}{1-\alpha\theta^{\xi_v}}\left[\frac{1-\alpha}{\delta-\mu} - \frac{(1-\alpha\theta)(r+m-\gamma)(\kappa+m)\bar{x}_\gamma}{(\delta+m-\gamma)(r+m)} + \frac{\gamma(\kappa+m)(1-\alpha\theta)\bar{x}_\gamma}{(r+m)(r+\gamma-\delta)}\right]\bar{x}_\gamma^{-\xi_v}
$$

This leads to the following value function:

$$
\begin{aligned}
v_\gamma(x) = \frac{1}{\delta-\mu}&\left[1-\left(\frac{1-\alpha}{1-\alpha\theta^{\xi_v}}\right)\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_v}\right] - \frac{(r+m-\gamma)(\kappa+m)x}{(\delta+m-\gamma)(r+m)}\left[1-\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_v}}\right)\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_v-1}\right] \\
&+ \frac{\gamma(\kappa+m)x}{(r+m)(r+\gamma-\delta)}\left[\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_d+1}}\right)\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_d} - \left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_v}}\right)\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_v-1}\right]
\end{aligned}
$$

The optimal default boundary satisfies the smooth pasting condition $v'_\gamma(\bar{x}_\gamma) = \alpha\theta v'_\gamma(\theta\bar{x}_\gamma)$, in other words:

$$
\bar{x}_\gamma = \frac{\xi_v}{\left(\frac{r+m-\gamma}{\delta+m-\gamma} - \frac{\gamma}{r+\gamma-\delta}\right)(\xi_v-1) + \left(\frac{\gamma}{r+\gamma-\delta}\right)\xi_d}\left(\frac{1}{\delta-\mu}\right)\left(\frac{r+m}{\kappa+m}\right)\left(\frac{1-\alpha}{1-\alpha\theta}\right)
$$

Finally, at time $t = 0-$, the government issues an impulse face amount of debt F, that solves the following maximization problem:

$$
\max_F \left[FD_\gamma(Y,F) + V_\gamma(Y,F)\right] = Y \max_x \left[xd_\gamma(x) + v_\gamma(x)\right] \tag{62}
$$

Note that the function $v_\gamma(x) + xd_\gamma(x)$ takes the value $\frac{1}{\delta-\mu}$ for $x = 0$, and a value strictly less than $\frac{1}{\delta-\mu}$ at $x = \bar{x}_\gamma$. Since $\xi_v > 1$, it also admits a derivative at the origin that is equal to:

$$\frac{d}{dx}\left[v_c(x) + xd_c(x)\right]_{x=0} = \frac{(\kappa+m)(\delta-r)}{(r+m)(\delta+m-\gamma)} > 0$$

In other words, since we have assumed that $\gamma < \delta + m$, the function $v_\gamma(x) + xd_\gamma(x)$ reaches a maximum at x^*_γ that is strictly inside the interval $(0, \bar{x}_\gamma)$, and such that $v_\gamma(x^*_\gamma) + x^*_\gamma d_\gamma(x^*_\gamma) > \frac{1}{\delta-\mu}$, where the right-handside of this inequality is the autarky value. We can also note that the first order condition for an interior solution to the maximization problem (62) can be written:

$$\left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_v - 1}\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_v}}\right)\left[\frac{r+m-\gamma}{\delta+m-\gamma} - \frac{\gamma(1+\xi_d)}{r+\gamma-\delta}\right] + \left(\frac{x}{\bar{x}_\gamma}\right)^{\xi_d}\left(\frac{1-\alpha\theta}{1-\alpha\theta^{\xi_d+1}}\right)(1+\xi_d)\frac{\delta-r}{r+\gamma-\delta} + \frac{\delta-r}{\delta+m-\gamma} = 0$$

It turns out that an interior solution exists only if $\gamma < m + \delta$. □

A.10 Income as Markov-Modulated GBM

In this section, we use the notation introduced in section A.4. The income process for the small open economy follows:

$$dY_t = Y_t\left(\mu(s_t)dt + \boldsymbol{\sigma}(s_t)\cdot d\boldsymbol{B}_t\right)$$

$\{s_t\}_{t\geq 0}$, taking values in $\{1, ..., n\}$, is a discrete state Markov process with a generator matrix $\Lambda = (\Lambda_{ij})_{1\leq i,j\leq n}$[28]. For simplicity, we assume that upon a sovereign default at time τ, the small open economy's income suffers a discrete downward jumps by a factor $\alpha(s_\tau) \in (0, 1)$, and the country is then stuck in financial autarky forever (as opposed to be allowed to re-enter capital markets with a smaller debt burden). We note W_t the country's autarky value:

$$W_t := \mathbb{E}\left[\int_t^\infty e^{-\delta(u-t)}Y_u du|\mathcal{F}_t\right]$$

Note that W_t satisfies $W_t = w(s_t)Y_t$, where the $n \times 1$ vector $\boldsymbol{w}$ is equal to:

$$\boldsymbol{w} = \left(\delta\boldsymbol{I}_n - \boldsymbol{\mu} - \boldsymbol{\Lambda}\right)^{-1}\mathbf{1}_n$$

[28] We normalized Λ_{ii} such that Λ is conservative – in other words $\sum_{j=1}^n \Lambda_{ij} = 0$ for all i.

In the above equation, $\boldsymbol{I}_n$ is the $n \times n$ identity matrix, and $\boldsymbol{\mu} = \text{diag}\,(\mu(i))$ is an $n \times n$ diagonal matrix, while $\mathbf{1}_n$ is an $n \times 1$ vector of 1. We will work with the log-debt-to-income ratio $z_t := \ln F_t/Y_t$. The life-time utility function V can be written as follows:

$$V(Y, F, i) = Y v(z, i)$$

We postulate a set of n default boundaries $\{\bar{z}_j\}_{1 \leq j \leq n}$ that are ordered as follows:

$$\bar{z}_n < ... < \bar{z}_1$$

By convention, we note $\bar{z}_{n+1} = -\infty$. In other words, we re-order the discrete states such that their corresponding default boundaries are ranked by decreasing order. Default regions are partitioned as follows, for $1 \leq k \leq n$:

$$\mathcal{D}_k := [\bar{z}_{k+1}, \bar{z}_k)$$

Thus, by convention, we note $\mathcal{D}_n = (-\infty; \bar{z}_n]$. We also define the index sets:

$$\begin{aligned} \mathcal{I}_k &:= \{1, ..., k\} \\ \mathcal{I}_k^c &:= \{k+1, ..., n\} \end{aligned}$$

Note that when $z \in \mathcal{D}_k$, the index set $\mathcal{I}_k$ is the set of "surviving" states, in other states in which the system can transition without triggering a default. Note that $|\mathcal{I}_k| = k$ and $|\mathcal{I}_k^c| = n - k$. The log-debt-to-income ratio $z_t^{(\iota,\tau)}$ is a controlled stochastic process that evolves as follows:

$$dz_t^{(\iota,\tau)} = \left(\iota(z_t^{(\iota,\tau)}, s_t) \exp\left(-z_t^{(\iota,\tau)}\right) - (m + \mu(s_t))\right) dt - \boldsymbol{\sigma}(s_t) \cdot d\boldsymbol{B_t}$$

The normalized value function v is equal to:

$$v(z, i) := \sup_{(\iota,\tau) \in \mathcal{I} \times \mathcal{T}} \tilde{\mathbb{E}}^{z,i} \left[\int_0^\tau e^{-\int_0^t (\delta - \mu(s_u))du} \left(1 + \iota\left(z_t^{(\iota,\tau)}, s_t\right) D_t - (\kappa + m) \exp\left(z_t^{(\iota,\tau)}\right)\right) dt \right.$$
$$\left. + e^{-\int_0^\tau (\delta - \mu(s_u))du} \alpha(s_\tau) w(s_\tau) \right]$$

We have introduced the measure $\tilde{\Pr}$, defined for any arbitrary Borel set $A \subseteq \mathcal{F}_t$ via $\tilde{\Pr}(A) = \mathbb{E}\left[\exp\left(-\int_0^t \frac{|\boldsymbol{\sigma}(s_u)|^2}{2} du + \int_0^t \boldsymbol{\sigma}(s_u) \cdot d\boldsymbol{B}_u\right) A\right]$. Under such measure, using Girsanov's theorem,

the controlled log-debt-to-income ratio $z_t^{(\iota,\tau)}$ evolves as follows:

$$dz_t^{(\iota,\tau)} = \left(\iota(z_t^{(\iota,\tau)}, s_t)\exp\left(-z_t^{(\iota,\tau)}\right) - \left(m + \mu(s_t) + |\boldsymbol{\sigma}(s_t)|^2\right)\right)dt - \boldsymbol{\sigma}(s_t)\cdot d\tilde{\boldsymbol{B}}_{\boldsymbol{t}} \tag{63}$$

$\tilde{\boldsymbol{B}}_{\boldsymbol{t}} := \boldsymbol{B}_{\boldsymbol{t}} - \int_0^t \boldsymbol{\sigma}(s_u)du$ is a standard Brownian motion under $\tilde{\Pr}$. For a log debt-to-income level $z \in \mathcal{D}_k$ and any state $i \in \mathcal{I}_k$, the value $v(z,i)$ satisfies the following:

$$\begin{aligned}(\delta - \mu(i))\, v\,(z,i) = 1 - (\kappa+m)e^z - \left(m + \mu(i) + |\boldsymbol{\sigma}(i)|^2\right) v_z\,(z,i) \\ + \frac{1}{2}|\boldsymbol{\sigma}(i)|^2 v_{zz}\,(z,i) + \sum_{j\in\mathcal{I}_k}\Lambda_{ij}v\,(z,j) + \sum_{j\in\mathcal{I}_k^c}\Lambda_{ij}\alpha(j)w(j)\end{aligned}$$

Note that this HJB equation takes into account the fact that for all i, $\sum_{j=1}^n \Lambda_{ij} = 0$. In matrix notation, for $k < n$, this is equivalent to:

$$\begin{pmatrix} v_z(z,1) \\ \vdots \\ v_z(z,k) \\ v_{zz}(z,1) \\ \vdots \\ v_{zz}(z,k)\end{pmatrix} = \boldsymbol{A}_k \begin{pmatrix} v(z,1) \\ \vdots \\ v(z,k) \\ v_z(z,1) \\ \vdots \\ v_z(z,k)\end{pmatrix} + \boldsymbol{b}_k(z)$$

The $2k \times 2k$ matrix $\boldsymbol{A}_k$ is the following:

$$\boldsymbol{A}_k := \begin{pmatrix} \boldsymbol{0}_{k,k} & \boldsymbol{I}_{k,k} \\ 2\boldsymbol{\Sigma}^{-1}_{\mathcal{I}_k,\mathcal{I}_k}\left(\delta\boldsymbol{I}_{k,k} - \boldsymbol{\mu}_{\mathcal{I}_k,\mathcal{I}_k} - \boldsymbol{\Lambda}_{\mathcal{I}_k,\mathcal{I}_k}\right) & 2\boldsymbol{\Sigma}^{-1}_{\mathcal{I}_k,\mathcal{I}_k}\left(\boldsymbol{\mu}_{\mathcal{I}_k,\mathcal{I}_k} + m\boldsymbol{I}_{k,k}\right) + 2\boldsymbol{I}_{k,k}\end{pmatrix}$$

The $2k \times 1$ vector $\boldsymbol{b}_k(z)$ is the following:

$$\boldsymbol{b}_k := -\begin{pmatrix} \vdots \\ \mathbf{0}_k \\ \vdots \\ \vdots \\ 2\boldsymbol{\Sigma}^{-1}_{\mathcal{I}_k,\mathcal{I}_k}\left[(1-(\kappa+m)e^z)\,\mathbf{1}_k + \boldsymbol{\Lambda}_{\mathcal{I}_k,\mathcal{I}_k^c}\left(\boldsymbol{\alpha}\odot\boldsymbol{w}\right)_{\mathcal{I}_k^c}\right] \\ \vdots \end{pmatrix}$$

In the above, $\boldsymbol{\Sigma} = \text{diag}(|\boldsymbol{\sigma}(i)|^2)$, and $\boldsymbol{\mu} = \text{diag}(\mu(i))$. $\odot$ denotes the element by element multiplication. We have also used $\boldsymbol{\Sigma}_{\mathcal{I}_k,\mathcal{I}_k}$ to represent a $k \times k$ matrix whose rows and columns are the $\mathcal{I}_k$ rows and columns of $\boldsymbol{\Sigma}$. As usual, the eigen-value problem that needs to be studied is:

$$\boldsymbol{A}_k \begin{pmatrix} \boldsymbol{g}_k \\ \boldsymbol{h}_k \end{pmatrix} = \beta_k \begin{pmatrix} \boldsymbol{g}_k \\ \boldsymbol{h}_k \end{pmatrix}$$

We will note $\{\beta_{k,j}\}_{j=1,\ldots,2k}$ the $2k$ eigen-values, and $\{\left(\boldsymbol{g}_{k,j}, \boldsymbol{h}_{k,j}\right)'\}_{j=1,\ldots,2k}$ the $2k$ eigen-vectors (with dimension $2k \times 1$) of the problem above. Thus, for a log debt-to-income level $z \in \mathcal{D}_k$, the solution to the homogeneous problem can be written:

$$\boldsymbol{v}(z)_{\mathcal{I}_k} = \sum_{j=1}^{2k} w_{k,j}\boldsymbol{g}_{k,j}e^{\beta_{k,j}z}$$

Similarly, for $z \in \mathcal{D}_n$, and for any state $i \leq n$, the value function satisfies:

$$(\delta - \mu(i))\,v\,(z,i) = 1 - (\kappa+m)e^z - \left(m + \mu(i) + |\boldsymbol{\sigma}(i)|^2\right)v_z\,(z,i) + \frac{1}{2}|\boldsymbol{\sigma}(i)|^2 v_{zz}\,(z,i) + \sum_{j=1}^{n}\Lambda_{ij}v\,(z,j)$$

The homogeneous solution to this equation takes the same form as the homogeneous solution of the previous HJB equation. For $z \in \mathcal{D}_k$ (and $k < n$) and $i \in \mathcal{I}_k$, we look for a particular solution of the form:

$$D(z,i) = \zeta^{(0)}_{k,i} + \zeta^{(1)}_{k,i}e^z$$

This guess is valid if the set of constants $\{\zeta_{k,i}^{(0)}, \zeta_{k,i}^{(1)}\}_{1\leq i\leq k}$ satisfy:

$$\begin{aligned}
(\delta - \mu(i))\,\zeta_{k,i}^{(0)} &= 1 + \sum_{j\in\mathcal{I}_k} \Lambda_{ij}\zeta_{k,j}^{(0)} + \sum_{j\in\mathcal{I}_k^c} \Lambda_{ij}\alpha(j)w(j) \\
(\delta - \mu(i))\,\zeta_{k,i}^{(1)} &= -(\kappa+m) - \left(\mu(i) + m + |\boldsymbol{\sigma}(i)|^2\right)\zeta_{k,i}^{(1)} + \frac{1}{2}|\boldsymbol{\sigma}(i)|^2\zeta_{k,i}^{(1)} + \sum_{j\in\mathcal{I}_k}\Lambda_{ij}\zeta_{k,j}^{(1)}
\end{aligned}$$

In matrix notation, this leads to $k\times 1$ vectors $\boldsymbol{\zeta_k^{(0)}}$ and $\boldsymbol{\zeta_k^{(1)}}$ that are equal to:

$$\begin{aligned}
\boldsymbol{\zeta_k^{(0)}} &= \left(\delta \boldsymbol{I}_k - \boldsymbol{\mu}_{\mathcal{I}_k,\mathcal{I}_k} - \boldsymbol{\Lambda}_{\mathcal{I}_k,\mathcal{I}_k}\right)^{-1}\left(\mathbf{1}_k + \boldsymbol{\Lambda}_{\mathcal{I}_k,\mathcal{I}_k^c}\left(\boldsymbol{\alpha}\odot\boldsymbol{w}\right)_{\mathcal{I}_k^c}\right) \\
\boldsymbol{\zeta_k^{(1)}} &= -(\kappa+m)\left((\delta+m)\boldsymbol{I}_k + \frac{1}{2}\boldsymbol{\Sigma}_{\mathcal{I}_k,\mathcal{I}_k} - \boldsymbol{\Lambda}_{\mathcal{I}_k,\mathcal{I}_k}\right)^{-1}\mathbf{1}_k
\end{aligned}$$

For $z\in\mathcal{D}_n$ and for all i, we look for a particular solution of the form:

$$D(z,i) = \zeta_{n,i}^{(0)} + \zeta_{n,i}^{(1)}e^z$$

This guess is valid if the set of constants $\{\zeta_{n,i}^{(0)}, \zeta_{n,i}^{(1)}\}_{1\leq i\leq n}$ satisfy:

$$\begin{aligned}
(\delta - \mu(i))\,\zeta_{n,i}^{(0)} &= 1 + \sum_{j\in\mathcal{I}_n} \Lambda_{ij}\zeta_{n,j}^{(0)} \\
(\delta - \mu(i))\,\zeta_{n,i}^{(1)} &= -(\kappa+m) - \left(\mu(i) + m + |\boldsymbol{\sigma}(i)|^2\right)\zeta_{n,i}^{(1)} + \frac{1}{2}|\boldsymbol{\sigma}(i)|^2\zeta_{n,i}^{(1)} + \sum_{j\in\mathcal{I}_n}\Lambda_{ij}\zeta_{n,j}^{(1)}
\end{aligned}$$

In matrix notation, this leads to $n\times 1$ vectors $\boldsymbol{\zeta_n^{(0)}}$ and $\boldsymbol{\zeta_n^{(1)}}$ that are equal to:

$$\begin{aligned}
\boldsymbol{\zeta_n^{(0)}} &= \left(\delta \boldsymbol{I}_n - \boldsymbol{\mu}_{\mathcal{I}_n,\mathcal{I}_n} - \boldsymbol{\Lambda}_{\mathcal{I}_n,\mathcal{I}_n}\right)^{-1}\mathbf{1}_n \\
\boldsymbol{\zeta_n^{(1)}} &= -(\kappa+m)\left((\delta+m)\boldsymbol{I}_n + \frac{1}{2}\boldsymbol{\Sigma}_{\mathcal{I}_n,\mathcal{I}_n} - \boldsymbol{\Lambda}_{\mathcal{I}_n,\mathcal{I}_n}\right)^{-1}\mathbf{1}_n
\end{aligned}$$

There are n^2 integration constants $\{w_{k,j}\}$ to be determined. Indeed, for each $k\leq n$, we need to find $\{w_{k,j}\}_{1\leq j\leq 2k}$. Those constants are determined via value matching and smooth-pasting conditions at the default boundary. At the default boundary, for $k\leq n-1$ and $i\in\mathcal{I}_k$, the life-time government value function and its first partial derivative w.r.t. z must be continuous, leading to:

$$\begin{aligned}
\lim_{z\searrow\bar{z}_{k+1}} v\left(z,i\right) &= \lim_{z\nearrow\bar{z}_{k+1}} v\left(z,i\right) \\
\lim_{z\searrow\bar{z}_{k+1}} v_z\left(z,i\right) &= \lim_{z\nearrow\bar{z}_{k+1}} v_z\left(z,i\right)
\end{aligned}$$

These represent $2\sum_{i=1}^{n-1} i = n(n-1)$ equations. Of course we must also have value matching at the default boundary $\bar{z}_k$ in state k, in other words:

$$v\left(\bar{z}_k, k\right) = \alpha(k)w(k)$$

These give us another n equations. This leads to a linear system of n^2 equations in n^2 to be solved. Finally, the n default boundaries $\{\bar{z}_i\}_{i\leq n}$ are determined via the smooth-pasting conditions:

$$\lim_{t\nearrow\tau} \text{var}_t\left[d\left(Y_t v\left(\ln\frac{F_t}{Y_t}\right) - \alpha_{s_t} w_{s_t} Y_t\right)\right] = 0$$

This gives us n smooth pasting conditions which, using the value matching condition at the default boundaries, can be expressed as follows, for $k \leq n$:

$$v_z\left(\bar{z}_k, k\right) = 0$$

□

A.11 Issuance Constraint

In this section, we assume that there is a debt-to-income limit $x^* < \bar{x}$ such that if the small open economy's debt-to-income is above such threshold, the government is prevented from issuing any debt.

A.11.1 Smooth Equilibrium

We derive a condition on x^* such that a smooth equilibrium still exists. When that is the case, the value function v_c is identical to the value function v in the unconstrained economy. The debt price d_c is such that when $x < x^*$, the debt price satisfies:

$$d_c(x) = \frac{\kappa + m}{\delta + m}\left[1 - \left(\frac{x}{\bar{x}}\right)^{\xi-1}\right]$$

When $x \in (x^*, \bar{x})$, the debt price satisfies the Feynman-Kac equation:

$$(r+m)d_c(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d_c'(x) + \frac{\sigma^2 x^2}{2} d_c''(x)$$

The boundary conditions are the following:

$$d_c(\bar{x}) = 0$$
$$d_c(x^*) = \frac{\kappa + m}{\delta + m}\left[1 - \left(\frac{x^*}{\bar{x}}\right)^{\xi - 1}\right]$$

Let $\eta_1 < 0 < \eta_2$ be the roots of the quadratic equation:

$$\frac{\sigma^2}{2}\eta^2 - \left(m + \mu - \frac{\sigma^2}{2}\right)\eta - (r + m) = 0$$

Since $\delta > r$, it is easy to verify that $\eta_2 < \xi - 1$. The debt price for $x \in (x^*, \bar{x})$ thus satisfies:

$$d_c(x) = \frac{\kappa + m}{r + m} + d_1\left(\frac{x}{\bar{x}}\right)^{\eta_1} + d_2\left(\frac{x}{\bar{x}}\right)^{\eta_2}$$

Since d is continuous at $x = x^*$ and at $x = \bar{x}$, the constants of integration d_1, d_2 must satisfy:

$$\frac{\kappa + m}{r + m} + d_1\left(\frac{x^*}{\bar{x}}\right)^{\eta_1} + d_2\left(\frac{x^*}{\bar{x}}\right)^{\eta_2} = \frac{\kappa + m}{\delta + m}\left(1 - \left(\frac{x^*}{\bar{x}}\right)^{\xi - 1}\right)$$
$$\frac{\kappa + m}{r + m} + d_1 + d_2 = 0$$

Note $\rho := x^*/\bar{x} \in (0, 1)$, we obtain the following expressions for d_1 and d_2:

$$d_1 = \frac{1}{\rho^{\eta_1} - \rho^{\eta_2}}\left[\frac{\kappa + m}{\delta + m}\left(1 - \rho^{\xi - 1}\right) - \frac{\kappa + m}{r + m}\left(1 - \rho^{\eta_2}\right)\right]$$
$$d_2 = \frac{1}{\rho^{\eta_1} - \rho^{\eta_2}}\left[-\frac{\kappa + m}{\delta + m}\left(1 - \rho^{\xi - 1}\right) + \frac{\kappa + m}{r + m}\left(1 - \rho^{\eta_1}\right)\right]$$

In order for this to be an equilibrium, a necessary and sufficient condition is that the debt price d_c is decreasing on $[x^*, \bar{x}]$. For this to be the case, a sufficient condition is that it is decreasing at $x = x^*$, in other words:

$$\eta_1 d_1 \rho^{\eta_1} + \eta_2 d_2 \rho^{\eta_2} < 0$$

We compute $\eta_1 d_1 \rho^{\eta_1} + \eta_2 d_2 \rho^{\eta_2}$ as follows:

$$\eta_1 d_1 \rho^{\eta_1} + \eta_2 d_2 \rho^{\eta_2} = \frac{\rho^{\eta_1 + \eta_2}}{\rho^{\eta_1} - \rho^{\eta_2}}\left[\left(\eta_2 \rho^{-\eta_1} - \eta_1 \rho^{-\eta_2}\right)\left(\frac{\kappa + m}{r + m} - \frac{\kappa + m}{\delta + m}(1 - \rho^{\xi - 1})\right) - \frac{\kappa + m}{r + m}(\eta_2 - \eta_1)\right]$$

Let $F(\rho)$ be the function in brackets above. It can be showed that F is convex, with limit $+\infty$ as $\rho \to 0$ and limit 0 when $\rho \to 1$. It can also be showed that $F'(\rho) \to -\infty$ as $\rho \to 0$ and that $F'(\rho) \to \frac{\kappa+m}{\delta+m}(\eta_2 - \eta_1)(\xi - 1) > 0$ as $\rho \to 1$. In other words, there is a unique ρ^* that satisfies $F(\rho^*) = 0$, with $F(\rho) > 0$ when $\rho < \rho^*$ and $F(\rho) < 0$ when $\rho > \rho^*$. In other words, our conjectured smooth equilibrium is an equilibrium of our economy if and only if $x^* > \rho^* \bar{x} := \bar{x}^*$. What remains to discuss is the fact that the debt price function d_c is continuous but not continuously differentiable at $x = x^*$. Note that usually, absent trading frictions and when x_t is an Itô process, the price function $d_c(x)$ must be differentiable everywhere – in particular, it must be differentiable at $x = x^*$. Indeed, imagine that it was not the case, and imagine instead that d_c exhibits a convex kink:

$$\lim_{x \nearrow x^*} d_c'(x) < \lim_{x \searrow x^*} d_c'(x)$$

Think about a discretization of the Itô process for x_t, such that x_t evolves on a discrete grid of size h, with probabilities of moving up and down equal to:

$$p_u^h(x) = \frac{1}{2}\left(1 + \frac{\mu(x)}{\sigma^2(x)}h\right) \tag{64}$$

$$p_d^h(x) = \frac{1}{2}\left(1 - \frac{\mu(x)}{\sigma^2(x)}h\right) \tag{65}$$

Note $dt^h(x) := \frac{h^2}{\sigma^2(x)}$, and note that our discrete state Markov chain approximation is "consistent" with the Itô process x_t – in other words, $\mathbb{E}_t\left[x_{t+1}^h - x_t^h\right] = \mu(x)dt^h$, and $\text{var}_t\left[x_{t+1}^h - x_t^h\right] = \sigma^2(x)dt^h + o(dt^h)$. Then at $x = x^*$, we have:

$$d_c(x^*) = \text{flow}(x)dt^h(x) + e^{-(r+m)dt^h(x)}\left[p_u^h d_c(x^* + h) + p_d^h d_c(x^* - h)\right]$$

A Taylor expansion of the expression above gives us, at the order h:

$$0 = \frac{1}{2}\lim_{x \searrow x^*} d_c'(x^*) - \frac{1}{2}\lim_{x \nearrow x^*} d_c'(x^*) + \frac{\mu(x^*)}{\sigma^2(x^*)}\left(\frac{1}{2}\lim_{x \searrow x^*} d_c(x^*) - \frac{1}{2}\lim_{x \nearrow x^*} d_c(x^*)\right)$$

Remember that d_c is continuous at $x = x^*$. Thus, with a convex kink, the right handside above is strictly positive, suggesting the exitence of arbitrage: one could purchase at time t an arbitrarily large amount of the bond, since the capital gains rate between t and $t + dt$ is larger than rdt. We rule out the existence of arbitrages by allowing the government to use an issuance policy that is such that the controlled debt-to-income ratio becomes a Skew

Brownian motion:

$$dx_t = \left[\iota(x_t) - \left(m + \mu - \sigma^2\right) x_t\right] dt - \sigma x_t dB_t + (2p-1) dL_t^{x^*}(x_t)$$

In the above, $L_t^{x^*}(x_t)$ is the local time at x^* of x_t:

$$L_t^{x^*}(x_t) := \lim_{\epsilon \searrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{x^* - \epsilon < x_s \leq x^* + \epsilon\}} ds$$

The probability $p \in (0,1)$ is equal to:

$$p = \frac{\lim_{x \nearrow x^*} d'(x)}{\lim_{x \nearrow x^*} d'(x) + \lim_{x \searrow x^*} d'(x)}$$

x_t is thus singular at x^* only, and one can think of the Skew Brownian motion as a way to distort probabilities of moving up or down at $x = x^*$, so that in expectations debt investors do not realize infinite (or minus infinite) capital gains' rates. □

A.11.2 Reflecting Equilibrium

Assume now that $x^* < \bar{x}^*$ such that a smooth equilibrium does not exist. We conjecture that there is an equilibrium in which the debt-to-income ratio is evolving "uncontrolled" on $[x^*, \bar{x}_c]$, and is controlled at $x = x^*$ via singular control. In other words, x_t is now a regulated Brownian motion, regulated at $x = x^*$. The default boundary $\bar{x}_c$ is now different from the smooth equilibrium default boundary $\bar{x}$. The government income-normalized value function and the debt price are then pinned down on $[x^*, \bar{x}_c]$, independently of what happens when $x < x^*$. For $x < x^*$, as we will see, two situations can arise. If $d_c(x^*) < \frac{\kappa+m}{\delta+m}$ – in other words if x^* is sufficiently close to $\bar{x}^*$ and if δ is not "too large" – there exists a jump region $[\hat{x}, x^*]$, in which the government finds it optimal to jump immediately to the debt-to-income level x^*, and a "smooth" region $[0, \hat{x}]$, where the government finds it optimal to follow a smooth debt issuance strategy. Instead, if $d_c(x^*) > \frac{\kappa+m}{\delta+m}$, only the jump region exists. In both cases, the value function v_c and the debt price d_c satisfy the following, for $x \in (x^*, \bar{x}_c)$:

$$(\delta - \mu) v_c(x) = 1 - (\kappa + m)x - (m + \mu) x v_c'(x) + \frac{\sigma^2 x^2}{2} v_c''(x)$$
$$(r + m) d_c(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d_c'(x) + \frac{\sigma^2 x^2}{2} d_c''(x)$$

The boundary conditions are the following:

$$v_c(\bar{x}_c) = 0 \qquad d_c(\bar{x}_c) = 0$$
$$v_c'(x^*) + d_c(x^*) = 0 \qquad d_c'(x^*) = 0$$

The first two boundary conditions are standard, as they correspond to value-matching conditions are $x = \bar{x}_c$. The last two boundary conditions are standard boundary conditions for regulated Brownian motions, and can also be derived using a discrete time approximation to the stochastic differential equation that governs x_t. Indeed, use for example the approximation introduced in the previous section, with probabilities $p_u^h(x)$ and $p_d^h(x)$ as defined in equations (64) and (65). Then at $x = x^*$, the debt price satisfies:

$$d_c(x^*) = (\kappa + m)dt^h(x^*) + e^{-(r+m)dt^h(x^*)}\left[p_u^h(x^*)d_c(x^* + h) + p_d^h(x^*)d_c(x^*)\right]$$

A Taylor expansion of the expression above yields, at the order h, the boundary condition $d_c'(x^*) = 0$. Similarly, at $x = x^*$, the value function satisfies:

$$v_c(x^*) = (1 - (\kappa + m)x^*)\, dt^h(x^*) + e^{-(\delta-\mu)dt^h(x^*)}\left[p_u^h(x^*)v_c(x^* + h) + p_d^h(x^*)\left(v_c(x^*) + hd_c(x^*)\right)\right]$$

A Taylor expansion of the expression above yields, at the order h, the boundary condition $v_c'(x^*) + d_c(x^*) = 0$. Let $\xi_1 < 0 < \xi_2$ be the roots of the quadratic equation:

$$\frac{\sigma^2}{2}\xi^2 - \left(m + \mu + \frac{\sigma^2}{2}\right)\xi - (\delta - \mu) = 0$$

Let $\eta_1 < 0 < \eta_2$ be the roots of the quadratic equation:

$$\frac{\sigma^2}{2}\eta^2 - \left(m + \mu - \frac{\sigma^2}{2}\right)\eta - (r + m) = 0$$

Since $\delta > r$, it is easy to verify that $\eta_2 < \xi_2 - 1$. Note $\rho := x^*/\bar{x}_c$. The debt price for $x \in (x^*, \bar{x}_c)$ thus satisfies:

$$d_c(x) = \frac{\kappa + m}{r + m} + d_1\left(\frac{x}{\bar{x}_c}\right)^{\eta_1} + d_2\left(\frac{x}{\bar{x}_c}\right)^{\eta_2}$$

The constants of integration d_1, d_2 satisfy:

$$d_1 = \frac{\kappa + m}{r + m}\left(\frac{\eta_2 \rho^{\eta_2}}{\eta_1 \rho^{\eta_1} - \eta_2 \rho^{\eta_2}}\right)$$
$$d_2 = \frac{\kappa + m}{r + m}\left(\frac{-\eta_1 \rho^{\eta_1}}{\eta_1 \rho^{\eta_1} - \eta_2 \rho^{\eta_2}}\right)$$

The value function v_c satisfies:

$$v_c(x) = \frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m}x + v_1\left(\frac{x}{\bar{x}_c}\right)^{\xi_1} + v_2\left(\frac{x}{\bar{x}_c}\right)^{\xi_2}$$

The constants of integration v_1, v_2 satisfy:

$$v_1 = \frac{\bar{x}_c}{\xi_1 \rho^{\xi_1 - 1} - \xi_2 \rho^{\xi_2 - 1}}\left[\frac{\kappa + m}{\delta + m}\left(1 - \xi_2 \rho^{\xi_2 - 1}\right) + \xi_2 \rho^{\xi_2 - 1}\frac{1}{\bar{x}_c(\delta - \mu)} - \frac{\kappa + m}{r + m}\left(1 + \frac{(\eta_2 - \eta_1)\rho^{\eta_1 + \eta_2}}{\eta_1 \rho^{\eta_1} - \eta_2 \rho^{\eta_2}}\right)\right]$$
$$v_2 = \frac{\bar{x}_c}{\xi_1 \rho^{\xi_1 - 1} - \xi_2 \rho^{\xi_2 - 1}}\left[\frac{\kappa + m}{\delta + m}\left(\xi_1 \rho^{\xi_1 - 1} - 1\right) - \xi_1 \rho^{\xi_1 - 1}\frac{1}{\bar{x}_c(\delta - \mu)} + \frac{\kappa + m}{r + m}\left(1 + \frac{(\eta_2 - \eta_1)\rho^{\eta_1 + \eta_2}}{\eta_1 \rho^{\eta_1} - \eta_2 \rho^{\eta_2}}\right)\right]$$

Finally, the default optimality condition $v_c'(\bar{x}_c)$ pins down $\bar{x}_c$:

$$-\frac{\kappa + m}{\delta + m}\bar{x}_c + v_1\xi_1 + v_2\xi_2 = 0$$

It is then clear that the value function v_c and the debt price d_c are uniquely pinned down on the interval $[x^*, \bar{x}_c]$, irrespective of what happens for $x < x^*$. Let us then focus on $x < x^*$.

Imagine first that $d_c(x^*) < \frac{\kappa + m}{\delta + m}$. In such case, one can construct an equilibrium in which the government follows a smooth issuance strategy for $x \in (0, \hat{x})$, and a jump strategy for $x \in (\hat{x}, x^*)$, for some cutoff $\hat{x}$ endogenously determined. In the jump region $[\hat{x}, x^*]$, the debt price must be constant and the value function must be linear in x:

$$v_c(x) = v_c(x^*) + (x^* - x)d_c(x^*)$$
$$d_c(x) = d_c(x^*)$$

On the interval $[0, \hat{x}]$, since we postulated that the government follows a smooth financing

policy, the value function must satisfy:

$$\begin{aligned}(\delta-\mu)v_c(x) &= 1-(\kappa+m)x-(m+\mu)\,xv_c'(x)+\frac{\sigma^2x^2}{2}v_c''(x)\\ v_c(0) &= \frac{1}{\delta-\mu}\\ \lim_{x\searrow\hat{x}} v_c(x) &= \lim_{x\nearrow\hat{x}} v_c(x)\end{aligned}$$

In other words, $v_c(x)$ is equal to:

$$v_c(x)=\frac{1}{\delta-\mu}\left[1-\left(\frac{x}{\hat{x}}\right)^{\xi_2}\right]-\frac{\kappa+m}{\delta+m}x\left[1-\left(\frac{x}{\hat{x}}\right)^{\xi_2-1}\right]+v_c(\hat{x})\left(\frac{x}{\hat{x}}\right)^{\xi_2}$$

Since $d_c(x)=-v_c'(x)$, we obtain the following expression for d_c:

$$d_c(x)=\frac{\kappa+m}{\delta+m}-\frac{\xi_2}{\hat{x}}\left[v_c(\hat{x})+\frac{\kappa+m}{\delta+m}\hat{x}-\frac{1}{\delta-\mu}\right]\left(\frac{x}{\hat{x}}\right)^{\xi_2-1}$$

$\hat{x}$ is then pinned down by the continuity of d_c at $\hat{x}$. Since $d_c(\hat{x}-)+v_c'(\hat{x}-)=0$ (due to the fact that the strategy is smooth on $[0,\hat{x}]$), and since $d_c(\hat{x}+)+v_c'(\hat{x}+)=0$ (due to the fact that the value function is linear, with slope $-d_c(\hat{x}+)$, on $[\hat{x},x^*]$), the requirement that d_c is continuous at $\hat{x}$ is identical to the requirement that v_c is $\mathcal{C}^1$ at such point. This condition can be showed to lead to:

$$\hat{x}=\frac{\xi_2}{1-\xi_2}\left(\frac{(1-\xi_1)\rho^{\xi_1}v_1+(1-\xi_2)\rho^{\xi_2}v_2}{\xi_1\rho^{\xi_1}v_1+\xi_2\rho^{\xi_2}v_2}\right)x^*$$

If instead $d_c(x^*)>\frac{\kappa+m}{\delta+m}$, the "smooth" region no longer exists, and one can construct an equilibrium in which the government follows a jump strategy for $x\in[0,x^*]$. On such interval, the debt price must be constant and the value function must be linear in x:

$$\begin{aligned}v_c(x) &= v_c(x^*)+(x^*-x)d_c(x^*)\\ d_c(x) &= d_c(x^*)\end{aligned}$$

□

A.11.3 Markov Switching Issuance Constraint

In this section, the government alternates between a constrained and unconstrained state at Poisson arrival times. When unconstrained, the regime transitions to the constrained regime

with intensity λ_u. When constrained, the regime transitions to the unconstrained regime with intensity λ_c. We note v_u (resp. d_u) the value function for the government (resp. the debt price) when unconstrained, and v_c (resp. d_c) the value function for the government (resp. the debt price) when constrained. We postulate that $v_c = v_u$, and in such case, we know the debt price when unconstrained:

$$d_u(x) = \frac{\kappa + m}{\delta + m}\left[1 - \left(\frac{x}{\bar{x}}\right)^{\xi - 1}\right]$$

In the constrained regime, the debt price d_c satisfies the following Feynman-Kac equation:

$$(r + m + \lambda_c)d_c(x) = \kappa + m - \left(m + \mu - \sigma^2\right) x d_c'(x) + \frac{\sigma^2 x^2}{2} d_c''(x) + \lambda_c d_u(x)$$

Note $\eta > 0$ the positive root of the quadratic equation:

$$\frac{\sigma^2}{2}\eta^2 - \left(m + \mu - \frac{\sigma^2}{2}\right)\eta - (r + m + \lambda_c) = 0$$

Note that $\eta < \xi - 1$ if and only if $\delta > r + \lambda_c$. Since d_c must be finite when $x = 0$ and since $d_c(\bar{x}) = 0$, we can compute the debt price d_c as follows:

$$\begin{aligned} d_c(x) = \frac{\kappa + m}{r + m + \lambda_c}\left(1 + \frac{\lambda_c}{\delta + m}\right) + \frac{\lambda_c}{\delta - (r + \lambda_c)}\frac{\kappa + m}{\delta + m}\left(\frac{x}{\bar{x}}\right)^{\xi - 1} \\ - \left[\frac{\kappa + m}{r + m + \lambda_c}\left(1 + \frac{\lambda_c}{\delta + m}\right) + \frac{\lambda_c}{\delta - (r + \lambda_c)}\frac{\kappa + m}{\delta + m}\right]\left(\frac{x}{\bar{x}}\right)^{\eta} \end{aligned}$$

A sufficient condition for d_c to be decreasing on $[0, \bar{x}]$ is for d_c to be decreasing at $x = 0$. One can show that irrespective of the parameter λ_c, $d_c'(x) < 0$ as $x \to 0$, guaranteeing the fact that d_c is downward sloping. Finally, the issuance policy, when unconstrained, can be written:

$$\iota_u(x) = (\delta - r)\frac{d_u(x)}{-d_u'(x)} + \lambda_u \frac{d_c(x) - d_u(x)}{-d_u'(x)}$$

This issuance policy is positive across the state space. Indeed, $\delta > r$, d_u is strictly in x and it is easy to verify that $\Delta d(x) := d_c(x) - d_u(x) > 0$, since $\Delta d(x)$ satisfies the ODE:

$$(r + m + \lambda_c)\,\Delta d(x) = (\delta - r)d_u(x) - \left(m + \mu - \sigma^2\right) x \Delta d'(x) + \frac{\sigma^2}{2} x^2 \Delta d''(x)$$

The boundary condition is $\Delta d(\bar{x}) = 0$, suggesting that $\Delta d(x)$ admits the Feynman-Kac integral representation:

$$\Delta d(x) = \mathbb{E}^x \left[\int_0^\tau e^{-(r+m+\lambda_c)t} (\delta - r) d_u(x_t) dt \right] \geq 0$$

□

A.12 Maximum Issuance Rate

In this section, the issuance rate (per unit of income) is capped at some arbitrary constant $\bar{\iota} > 0$. We look for an equilibrium where the constraint binds whenever the debt-to-income ratio is below an endogenously determined cutoff x^*. For $x \in (x^*, \bar{x}_c)$, the constraint is slack, where $\bar{x}_c$ is the optimal default boundary. We take $x^*, \bar{x}_c$ as given in the analysis below, and then discuss the two conditions that pin down both endogenous boundaries.

A.12.1 Constrained Region $[0, x^*]$

In the region $x \in [0, x^*)$, the issuance rate is bounded above by $\bar{\iota}$. On this interval, the (income-normalized) life-time utility function for the government and the debt price satisfy:

$$(\delta - \mu)\, v(x) = 1 + \bar{\iota} d(x) - (\kappa + m)x + \left[\bar{\iota} - (\mu + m)\, x\right] v'(x) + \frac{1}{2}\sigma^2 x^2 v''(x) \tag{66}$$

$$(r + m)\, d(x) = (\kappa + m) + \left[\bar{\iota} - \left(\mu + m - \sigma^2\right) x\right] d'(x) + \frac{1}{2}\sigma^2 x^2 d''(x) \tag{67}$$

Note that these ordinary differential equations are decoupled – we can solve for $d(\cdot)$ first, and reinject d into the ODE that v is solution of. The boundary conditions are as follows:

$$(\delta - \mu) v(0) = 1 + \bar{\iota}\,(d(0) + v'(0)) \qquad \lim_{x \nearrow x^*} v(x) = \lim_{x \searrow x^*} v(x)$$

$$(r + m) d(0) = \kappa + m + \bar{\iota} d'(0) \qquad \lim_{x \nearrow x^*} d(x) = \lim_{x \searrow x^*} d(x)$$

The boundary conditions at $x = 0$ are standard Robin boundary conditions, linking the value of the function to its derivative at that point. In what follow, we are going to treat $d(x^*)$ and $v(x^*)$ as parameters, and will eventually obtain equations that will tie $d(x^*)$ and $v(x^*)$ to the boundaries $x^*, \bar{x}_c$. We then have the following lemma.

LEMMA 5. Let $A, B, C \in \mathbb{R}^3_+$. Let $f\,(x; A, B, C)$ be a $\mathcal{C}^2$ function defined on $[0; x^*]$ (and thus

finite on that interval) that satisfies the second order ordinary differential equation:

$$x^2 f''(x) + (A - Bx) f'(x) - C f(x) = 0 \tag{68}$$

Then f takes the following form, for some coefficients $k_1, k_2 \in \mathbb{R}$:

$$f(x; A, B, C) = k_1 x^{-\eta_1} U\left(\eta_1; 2\eta_1 + B + 2; Ax^{-1}\right) + k_2 x^{-\eta_2} U\left(\eta_2; 2\eta_2 + B + 2; Ax^{-1}\right)$$

In the above, U is the Tricommi confluent hypergeometric function (see Abramowitz and Stegun (1964), chapter 13) and the constants $\eta_1 > 0 > \eta_2$ are the roots of the polynomial:

$$\eta^2 + (B+1)\eta - C = 0$$

The proof of the above lemma is straight-forward once we remember that Kummer's confluent hypergeometric function $M(a; b; z)$ and Tricommi's confluent hypergeometric function $U(a; b; z)$ are independent solutions to the Kummer differential equation:

$$zu''(z) + (b - z)\, u'(z) - au(z) = 0$$

It is then easy to check that $x^{-\eta} M\left(\eta; 2\eta + B + 2; Ax^{-1}\right)$ and $x^{-\eta} U\left(\eta; 2\eta + B + 2; Ax^{-1}\right)$ are solutions of equation (68). Note that M admits the asymptotic behavior $M(a; b; z) \sim e^z z^{a-b}/\Gamma(a)$ as $z \to +\infty$ and U admits the asymptotic behavior $U(a; b; z) \sim z^{-a}$ as $z \to +\infty$. In particular, f finite at $x = 0$ allows us to rule out the Kummer function and work with the Tricommi function only. □

Note $\eta_{d,1} < 0 < 1 < \eta_{d,2}$ the roots of:

$$\frac{1}{2}\sigma^2 \eta_d^2 + \left(m + \mu - \frac{1}{2}\sigma^2\right)\eta_d - (r + m) = 0$$

We can use the previous lemma to show that:

$$d(x) = \frac{\kappa + m}{r + m} + k_{d,1} x^{-\eta_{d,1}} U\left(\eta_{d,1}; 2\eta_{d,1} + \frac{2(m+\mu)}{\sigma^2}; \frac{2\bar{\iota}}{\sigma^2 x}\right) + k_{d,2} x^{-\eta_{d,2}} U\left(\eta_{d,2}; 2\eta_{d,2} + \frac{2(m+\mu)}{\sigma^2}; \frac{2\bar{\iota}}{\sigma^2 x}\right)$$

The boundary conditions at $x = 0$ and $x = x^*$ then allow us to pin down $k_{d,1}, k_{d,2}$ uniquely as a functions of the (yet unknown) value $d(x^*)$. Then, given the function d fully specified on $[0, x^*]$, equation (66) is a second order boundary value problem, and Baxley and Brown

(1981) provides for the existence and uniqueness of a solution to this ordinary differential equation. Note $\eta_{v,1} < 0 < 1 < \eta_{v,2}$ the roots of:

$$\frac{1}{2}\sigma^2\eta_v^2 + \left(m + \mu + \frac{1}{2}\sigma^2\right)\eta_v - (\delta - \mu) = 0$$

The function v takes the following form:

$$v(x) = \frac{1}{\delta - \mu}\left(1 - \frac{\kappa + m}{\delta + m}\bar{\iota}\right) - \frac{\kappa + m}{\delta + m}x + v_p(x) + k_{v,1}v_{g,1}(x) + k_{v,2}v_{g,2}(x)$$

In the above, the general solutions $v_{g,i}$ take the following form:

$$v_{g,i}(x) := v_i x^{-\eta_{v,i}} U\left(\eta_{v,i}; 2\eta_{v,i} + \frac{2(m+\mu)}{\sigma^2} + 2; \frac{2\bar{\iota}}{\sigma^2 x}\right)$$

v_p is a particular solution to the ordinary differential equation:

$$(\delta - \mu)\, v(x) = \bar{\iota}d(x) + [\bar{\iota} - (\mu + m)\, x]\, v'(x) + \frac{1}{2}\sigma^2 x^2 v''(x)$$

Some algebra can show that $v_p(x) = v_{g,1}(x)u(x)$, with the function $u(x)$ satisfying:

$$H(x) := \exp\left[\int_{x^*}^{x} \frac{(\bar{\iota} - (m+\mu)s)\, v_{g,1}(s) + \sigma^2 s^2 v'_{g,1}(s)}{\frac{\sigma^2 s^2}{2} v_{g,1}(s)} ds\right]$$
$$u(x) := \int_{x^*}^{x}\left(\int_{x^*}^{t} \frac{-2\bar{\iota}}{\sigma^2 s^2 v_{g,1}(s)} \frac{H(s)}{H(t)} ds\right) dt$$

It is also easy to prove that those solutions d and v are strictly decreasing on the interval $(0, x^*)$, under the assumption – to be verified numerically – that $d(x^*) < d(0)$ and $d'(0) < 0$ (for d) and under the assumption that $v(x^*) < v(0)$ and $v'(0) \leq 0$ for v. Indeed, assume for example by way of contradiction that d was not strictly decreasing on that interval. This means that there exists $0 < x_1 < x_2 < x^*$, such that $d(x_1) < d(x_2)$, $d'(x_1) = d'(x_2) = 0$, and $d''(x_1) > 0 > d'(x_2)$. But in that case, using equation (67), we have:

$$\frac{1}{2}\sigma^2 x_1^2 d''(x_1) = (r+m)d(x_1) - (\kappa + m) > 0$$
$$\frac{1}{2}\sigma^2 x_2^2 d''(x_2) = (r+m)d(x_2) - (\kappa + m) < 0$$

In other words, $d(x_1) > d(x_2)$, a contradiction. A similar proof holds for v. We thus have determined v and d on the interval $[0, x^*]$, subject to our knowledge of $x^*, d(x^*), v(x^*)$.

We can then verify that the issuance constraint is binding – in other words, that if the

government was allowed to issue a non-zero measure of debt, it would find it optimal to do so – this is identical to verifying that $d(x) + v'(x) \geq 0$. The unconstrained issuance policy $\iota_u(x)$ verifies $\iota_u(x) := \frac{d(x)}{-d'(x)}(\delta - r)$, and since in $(0, x^*)$ the government is constrained to issue an amount $\bar{\iota}$, we must have in this particular part of the state space $\bar{\iota} < \iota_u(x)$. Differentiate equation (66) to obtain:

$$(\delta + m)\, v'(x) = \bar{\iota} d'(x) - (\kappa + m) + \left[\bar{\iota} - \left(\mu + m - \sigma^2\right) x\right] v''(x) + \frac{1}{2}\sigma^2 x^2 v'''(x)$$
$$(r + m)\, d(x) = (\kappa + m) + \left[\bar{\iota} - \left(\mu + m - \sigma^2\right) x\right] d'(x) + \frac{1}{2}\sigma^2 x^2 d''(x)$$

Add those last two questions, introduce $g(x) := d(x) + v'(x)$, and note that g satisfies:

$$(\delta + m)\, g(x) = \left[\bar{\iota} - \iota_u(x)\right] d'(x) + \left[\bar{\iota} - \left(\mu + m - \sigma^2\right) x\right] g'(x) + \frac{1}{2}\sigma^2 x^2 g''(x)$$

Then use the boundary condition $(\delta - \mu)v(0) = 1 + \bar{\iota}\,(d(0) + v'(0))$, and remember that it must be the case that $v(0) \geq \frac{1}{\delta - \mu}$ (in other words, the welfare of a government that has no debt, but that has the option to borrow from more patient lenders must be at least as high as the autarky welfare) to conclude that $d(0) + v'(0) \geq 0$, in other words $g(0) \geq 0$. At $x = x^*$, v is $\mathcal{C}^1$ (to guarantee optimality of the endogenous threshold x^*) and d is continuous, meaning that we must have $d(x^*) + v'(x^*) = 0$, in other words $g(x^*) = 0$. Using Feynman-Kac, $g(x)$ admits the following integral representation:

$$g(x) = \mathbb{E}^x \left[\int_0^\tau e^{-(\delta+m)t} \left(\bar{\iota} - \iota_u(x_t)\right) d'(x_t) dt\right]$$

The stopping time τ is the first time the state variable x hits x^*. Since $\iota_u(x) \geq \bar{\iota}$ in that region of the state space, since d is a decreasing functions of x, it must be the case that $g(x) \geq 0$.

A.12.2 Unconstrained Region $[x^*, \bar{x}]$

Given our postulated behavior, in $x \in (x^*, \bar{x})$ the government financing policy is entirely unconstrained, meaning that the analysis we discussed in section 4.5 and section 4.6 is unchanged: the value function for the government behaves locally as if the government was commiting not to issue any debt. Thus, the life-time (income-normalized) utility function

for the government, the debt price and the issuance policy satisfy:

$$\begin{aligned}(\delta-\mu)\,v(x) &= 1-(\kappa+m)\,x-(\mu+m)\,xv'(x)+\frac{1}{2}\sigma^2x^2v''(x)\\ d(x) &= -v'(x)\\ \iota(x) &= \frac{d(x)}{-d'(x)}(\delta-r)\end{aligned}$$

These equations are derived using steps identical to those used in section 5.1. The debt price function is thus entirely pinned down by the equation $d(x) = -v'(x)$, and it can be showed that it satisfies the second order ordinary differential equation:

$$(\delta+m)\,d(x) = \kappa+m-\left(m+\mu-\sigma^2\right)xd'(x)+\frac{1}{2}\sigma^2x^2d''(x) \tag{69}$$

As discussed previously, equation (69) is the Feynman-Kac representation of the debt price computed using discount rate δ and under the assumption that the government never issues any additional bonds. Boundary conditions are as follows:

$$\begin{aligned}v(\bar{x}) &= 0 & \lim_{x\nearrow x^*} v(x) &= \lim_{x\searrow x^*} v(x)\\ d(\bar{x}) &= 0 & \lim_{x\nearrow x^*} d(x) &= \lim_{x\searrow x^*} d(x)\end{aligned}$$

We imposed continuity of the value function at $x = x^*$. The optimality of the endogenous boundary x^* will be determined by "smoothing" v at $x = x^*$ – i.e. by imposing that v is continuously differentiable at such point. The government value function, debt price and issuance policy take the following form on $x \in [x^*, \bar{x}]$:

$$\begin{aligned}v(x) &= \frac{1}{\delta-\mu}-\left(\frac{\kappa+m}{\delta+m}\right)x+v_1\left(\frac{x}{\bar{x}}\right)^{\xi_1}+v_2\left(\frac{x}{\bar{x}}\right)^{\xi_2}\\ d(x) &= \frac{\kappa+m}{\delta+m}+d_1\left(\frac{x}{\bar{x}}\right)^{\xi_1-1}+d_2\left(\frac{x}{\bar{x}}\right)^{\xi_2-1}\\ \iota(x) &= (r-\delta)x\frac{\frac{\kappa+m}{\delta+m}+d_1\left(\frac{x}{\bar{x}}\right)^{\xi_1-1}+d_2\left(\frac{x}{\bar{x}}\right)^{\xi_2-1}}{(\xi_1-1)d_1\left(\frac{x}{\bar{x}}\right)^{\xi_1-1}+(\xi_2-1)d_2\left(\frac{x}{\bar{x}}\right)^{\xi_2-1}}\end{aligned}$$

Since $-v'(x) = d(x)$, the constants v_1, v_2, d_1, d_2 are linked via $d_i = -\xi_i v_i/\bar{x}$. $\xi_1 < 0 < 1 < \xi_2$ are the roots of the polynomial:

$$\frac{1}{2}\sigma^2\xi^2-\left(\mu+m+\frac{1}{2}\sigma^2\right)\xi-(\delta-\mu) = 0$$

The boundary conditions for d and for v at $x = \bar{x}$ lead to:

$$d_1 + d_2 + \frac{\kappa + m}{\delta + m} = 0$$
$$v_1 + v_2 + \frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m}\bar{x} = 0$$

The boundary conditions for d and for v at $x = x^*$ lead to:

$$\frac{\kappa + m}{\delta + m} + d_1 \left(\frac{x^*}{\bar{x}}\right)^{\xi_1 - 1} + d_2 \left(\frac{x^*}{\bar{x}}\right)^{\xi_2 - 1} = d(x^*)$$
$$\frac{1}{\delta - \mu} - \left(\frac{\kappa + m}{\delta + m}\right) x^* + v_1 \left(\frac{x^*}{\bar{x}}\right)^{\xi_1} + v_2 \left(\frac{x^*}{\bar{x}}\right)^{\xi_2} = v(x^*)$$

Note that the boundary condition for d at $x = \bar{x}$ is identical to the smooth-pasting default optimality condition at such point (this latter condition is thus redundant). At the boundary $x = x^*$, the debt issuance rate of the small open economy is equal to $\bar{\iota}$. This gives us the following equation:

$$\bar{\iota} = (r - \delta)x \frac{\frac{\kappa+m}{\delta+m} + d_1 \left(\frac{x^*}{\bar{x}}\right)^{\xi_1 - 1} + d_2 \left(\frac{x^*}{\bar{x}}\right)^{\xi_2 - 1}}{(\xi_1 - 1)d_1 \left(\frac{x^*}{\bar{x}}\right)^{\xi_1 - 1} + (\xi_2 - 1)d_2 \left(\frac{x^*}{\bar{x}}\right)^{\xi_2 - 1}}$$

We need to make sure our initial choices $x^*, \bar{x}$ are such that $d(x^*) < d(0)$, which insures that the function d is monotone decreasing on $[0, x^*]$.

A.12.3 Determination of x^* and $\bar{x}$

It remains to discuss how the boundaries x^*, $\bar{x}$ are optimally set by the government. To be able to apply a standard verification theorem, we need to smooth the value function v, in other words, x^*, $\bar{x}$ are determined via the two smooth pasting conditions:

$$\lim_{x \nearrow x^*} v'(x; x^*, \bar{x}) = \lim_{x \searrow x^*} v'(x; x^*, \bar{x})$$
$$\lim_{x \searrow \bar{x}} v'(x; x^*, \bar{x}) = 0$$

Assuming that there exists a solution to this two-equation, two-unknown system, we then have our main result: when the government is constrained to use an issuance rate below a certain maximum level $\bar{\iota}$, an equilibrium exists, in which the issuance policy is unconstrained for $x > x^*$, and constrained at $\bar{\iota}$ when $x \in (0, x^*)$. It is optimal for the government to default as soon as x reaches $\bar{x}$. In that equilibrium the life-time utility function of a government that is not indebted is strictly greater than the autarky welfare.

A.12.4 A Simplification: the Case $\sigma = 0, \mu + m < 0$

In this particular case, we can solve for v and d in closed form. We have $\bar{x} = 1/(\kappa + m)$. When $x < x^*$, the debt price and government value function take the following expressions:

$$d(x) = \frac{\kappa + m}{r + m} - \left(\frac{\kappa + m}{r + m} - d(x^*) \right) \left(\frac{\bar{\iota} - (\mu + m)x}{\bar{\iota} - (\mu + m)x^*} \right)^{-\frac{r+m}{\mu+m}}$$

$$v(x) = a_0 + a_1 x + a_2 \left(\frac{\bar{\iota} - (\mu + m)x}{\bar{\iota} - (\mu + m)x^*} \right)^{-\frac{r+m}{\mu+m}} + (v(x^*) - a_0 - a_1 x^* - a_2) \left(\frac{\bar{\iota} - (\mu + m)x}{\bar{\iota} - (\mu + m)x^*} \right)^{-\frac{\delta-\mu}{\mu+m}}$$

In the above, the constants a_0, a_1, a_2 are equal to:

$$\begin{aligned} a_0 &:= \frac{1}{\delta - \mu} \left[1 + \frac{\bar{\iota}(\kappa + m)(\delta - r)}{(r + m)(\delta + m)} \right] \\ a_1 &:= -\frac{\kappa + m}{\delta + m} \\ a_2 &:= \frac{\bar{\iota}}{\delta - \mu - r - m} \left(d(x^*) - \frac{\kappa + m}{r + m} \right) \end{aligned}$$

When $x \in (x^*, \bar{x})$, the debt price, government value function and issuance policy take the following expressions:

$$\begin{aligned} d(x) &= \frac{\kappa + m}{r + m} \left[1 - \left(\frac{x}{\bar{x}} \right)^{-\frac{r+m}{\mu+m}} \right] \\ v(x) &= \frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m} x - \left[\frac{1}{\delta - \mu} - \frac{\kappa + m}{\delta + m} \bar{x} \right] \left(\frac{x}{\bar{x}} \right)^{-\frac{\delta-\mu}{\mu+m}} \\ \iota(x) &= -\frac{(\delta - \mu)(\mu + m)}{\delta + m} \left[\left(\frac{\bar{x}}{x} \right)^{-\frac{\delta+m}{\mu+m}} - 1 \right] x \end{aligned}$$

Since $\iota(x^*) = \bar{\iota}$, x^* is determined via:

$$\bar{\iota} = -\frac{(\delta - \mu)(\mu + m)}{\delta + m} \left[\left(\frac{\bar{x}}{x^*} \right)^{-\frac{\delta+m}{\mu+m}} - 1 \right] x^*$$