Equilibrium Subprime Lending

Igor Makarov* and Guillaume Plantin†‡

October 29, 2009

Abstract

This paper develops an equilibrium model of a subprime mortgage market. The model is analytically tractable and delivers plausible orders of magnitude for borrowing capacities, loan-to-income ratios, home prices, and default and trading intensities. We offer simple explanations for several phenomena in the subprime market, such as the prevalence of "teaser rates" and the clustering of defaults. In our model, the degree of income co-movement among households plays an important role. We find that both systematic and idiosyncratic income risks reduce debt capacities, although through quite distinct channels, and that debt capacities and home prices need not be higher when a larger fraction of income risk is idiosyncratic.

*London Business School. E-mail: imakarov@london.edu.

†London Business School, Toulouse School of Economics, and CEPR. E-mail: guillaume.plantin@tse-fr.eu.

‡We thank Andrea Buffa, Joao Cocco, Bernard Dumas, Harold Cole, Stéphane Guibaud, Nagpurnanand Prabhala, Oleg Rytchkov, Andrei Shleifer, Harald Uhlig, Vikrant Vig, and participants in various seminars for very helpful comments.

Introduction

The U.S. subprime mortgage market has come under close scrutiny after a surge in subprime delinquencies sparked a severe financial crisis. An already sizeable empirical literature investigates the extent to which the evolution of subprime lending practices over the last decade explains this default crisis.[1] Many observers share the view that banks supplied more credit than individuals could afford to repay. This leads to important questions: what is the right amount of credit and what factors affect household borrowing capacities? To investigate these questions this paper develops an equilibrium model of a subprime mortgage market. The model is analytically tractable, and thus offers clear insights. It also delivers plausible orders of magnitude for many variables of interest such as loan-to-income ratios, home prices, and default and trading intensities.

We study an economy in which banks competitively grant mortgages to households so as to finance a fixed supply of homes. Several features of our model are worth noting. Households have two defining features of subprime borrowers. First, consistent with the findings of Mian and Sufi (2009), they face severe liquidity constraints. They always exhaust their mortgage capacities. Second, they have low financial planning skills, which we capture with a high discount rate. Banks try to accommodate their needs by offering them financial contracts that maximize their borrowing capacities. We incorporate two frictions that make the contracting problem between banks and households non-trivial. First, households privately observe and consume their income. The second friction is that households cannot commit to a contract and are free to terminate at any time. These two frictions imply that the optimal contract is non-recourse secured debt and force banks to commit to foreclosures in order to provide an incentive for households to repay their loans. This foreclosure process, however, is costly and takes a fraction of the house market value. In addition to foreclosures, home supply at a given date stems from sales motivated by the acquisition of a larger home, and sales that follow exogenous moving decisions. Households' aggregate debt capacity drives the aggregate demand for homes. At the same time, market-clearing home prices affect aggregate debt capacity. Thus, household borrowing capacities and home prices are jointly determined in the equilibrium.

Given the equilibrium nature of our model, the degree of income co-movement among households plays an important role. Systematic income risk has a negative impact on borrowing capacity because it implies that foreclosures are more likely to take place when home demand is low. This lowers the endogenous collateral value of homes, which in turn generates low current debt capacities and home prices. This

[1] Recent contributions include Foote et al. (2008, 2009), Gerardi et al. (2009), Gorton (2008), Keys et al. (2010), Mian and Sufi (2009, 2010), Piskorski et al. (2008), and Rajan et al. (2009).

mechanism is different from one of financial amplification in which a low collateral value of assets generates productive or allocative inefficiencies (see, e.g., Bernanke and Gertler (1989), or Kiyotaki and Moore (1997)). In our economy, agents do not create real externalities for each other through the balance-sheet channel and the competitive equilibrium is allocationally efficient.

Perhaps more surprisingly, all else being equal, when a larger fraction of individual income risk is idiosyncratic, then equilibrium debt capacities and home prices are not necessarily higher. If income risk is idiosyncratic, then the income of a particular household is less correlated with home prices. On one hand, this raises banks' proceeds from foreclosures, which mitigates the aforementioned problem of endogenous low liquidation values. On the other hand, diversification of income risk makes households' lack of commitment power more costly. Households with positive idiosyncratic income shocks have a low probability of default. But they eventually seek to move for exogenous reasons, or to climb up the property ladder since they moved up in the cross-section of incomes. As a result, they exit the contractual relationship when the net present value of continuation is the highest to the incumbent bank. Households' lack of commitment is less costly in the presence of more systematic income risk. In this case, a household which experiences positive income shocks competes in mortgage and housing markets with households with similarly high income realizations. Thus, termination is less valuable to the household, and less costly to its bank.

Another somewhat unexpected finding is that contracts with initial "teaser rates" that gradually increase over time are optimal for a wide range of parameters. These types of contracts were very common in the design of subprime mortgages and are often linked to the roots of the crisis. Indeed, banks have been portrayed as villains who misled naive households into taking too much debt using "teaser rates". Leaving aside any normative issues, we show that these contracts do maximize household borrowing capacities. The intuition is quite simple: even though the income process has no drift and there is no inflation risk in our model, then conditional on the household continuing to make payments, its expected income increases over time. This lowers the conditional probability of default, which in turn makes it optimal for banks to ask for higher payments over time.

We obtain a number of additional insights from the case in which household income risk is purely idiosyncratic. First, despite the fact that our model is a rather stylized description of the housing market we obtain plausible orders of magnitude of loan-to-income ratios, house prices, and default and trading intensities, which suggest it can be useful in empirical application as well. Second, we demonstrate that partial equilibrium arguments may often lead to incorrect conclusions. For example, we show that a

reduction in pre-payment penalties reduces both the size and the ex ante probability of delinquency of a given mortgage. It is tempting to conclude that it would lead to a smaller number defaults and smaller aggregate debt levels. The model, however, predicts the opposite effects. As it becomes easier to refinance, households optimize their leverage ratio more frequently. As a result, the economy becomes more leveraged with a higher number of defaults.

Third, we also solve for the dynamics of home prices and default intensities along the path to the steady-state after an initial exogenous loan supply shock, which can be, for example, an exogenous shift in securitization practice. We show that convergence may take quite a long time and is generally non-monotonic, creating "boom and bust" dynamics. This happens because of the "term structure of defaults": defaults on the initial loans cluster around the same dates, leading to depressed home prices.

Finally, we study equilibria in which income shocks are systematic and home prices do not satisfy the transversality condition. We show that such "bubbly" equilibrium price paths imply countercyclical equilibrium repayment-to-income ratio and lower equilibrium mortgage payments than those with non-bubbly prices. Being unable to invest directly in the housing market, banks try to ride the bubble and this incentive becomes stronger the more severe the bubble is. As a result, aggregate income shocks have much more persistent impact on the default rates.

This paper brings together two strands of literature - the literature on the microeconomics of mortgages and the literature on endogenous incomplete markets.

Mayer, Piskorski, and Tchistyi (2008) and Piskorski and Tchistyi (2008) also study optimal mortgage design in the presence of ex post informational asymmetry. Our papers are complementary. They solve for the optimal recursive contract between a lender and a long-sighted borrower in an exogenous environment. We focus on the aggregate implications of contracting frictions between many pairs of lenders and short-sighted borrowers.

Stein (1995) and Ortalo-Magné and Rady (2006) study the impact of credit constraints on prices and trading volume in housing markets. Both papers assume realistic but exogenous credit constraints, and write a detailed model of households' decision-making under such constraints. Our broad-brush approach is more stylized. We abstract from important features of housing markets such as interest rate risk and the role of downpayments. However, constraints and contracts are equilibrium consequences of primitive frictions in our environment.

A second body of work seeks to endogenize market incompleteness with commitment problems.[2] Closest to our approach, Krueger and Uhlig (2005) introduce the idea that

[2]See, e.g., Alvarez and Jermann (2000), Kehoe and Levine (2001), Krueger et al. (2008), and

households' outside options after terminating a contract are competitively supplied by a financial sector, which is an important ingredient of our model. We have a more applied focus than these papers, but share a broader goal of characterizing the equilibrium interaction between individual contracting problems and asset prices. Our main departure from these papers is that our primitive contracting friction is private information. This friction assumes that contractual repayments are noncontingent, and that borrowers with bad outside options unwittingly default in equilibrium. Thus, our setup offers a more realistic picture of actual debt markets than the ones in which borrowers default only voluntarily in order to exercise desirable outside options.

The paper is organized as follows. Section 1 studies a baseline model without aggregate uncertainty. Income risk is perfectly diversifiable across households. This enables us to identify the impact of purely idiosyncratic income shocks on steady-state debt capacities, default rates, and trading volume. Section 2 studies the effect of aggregate income shocks on this economy. We outline our conlusions in Section 3 concludes. The majority of our proofs are relegated to an appendix.

1 Baseline Model

Time is continuous and is indexed by $t \in [0, +\infty)$. There is a single perishable consumption good which serves as the numéraire. There is a unit mass of assets - housing units. There are two types of agents: a unit mass of households and several non-atomistic banks.

Households

Households derive utility from occupying homes and consuming. They are myopic, and their primary objective is to maximize the size of the homes that they currently occupy. More precisely, the preferences of household j at date t are defined over bundles $(q_{j,t}, n_{j,t}) \in [0, +\infty) \times [0, +\infty)$, where $q_{j,t}$ is the number of housing units that the household occupies, and $n_{j,t}dt$ its consumption for the period $[t, t + dt)$. Household j ranks such bundles as follows.

1. If $q > q'$, then household j prefers (q, n) to (q', n') for all $n, n' \geq 0$.

2. If $n > n'$, then household j prefers (q, n) to (q, n') for all $q \geq 0$.

That is, households prefer to occupy the largest possible home, and maximize consumption when indifferent.

Hellwig and Lorenzoni (2008) for recent contributions.

Two remarks on these non standard preferences are in order. First, similar to Becker and Mulligan (1997), we interpret a high discount rate as the inability to think through future implications of current actions. This way we hope to capture low financial-planning abilities and limited access to financial experts of households in subprime markets. The limiting case of an infinite rate considerably simplifies the analysis of optimal contracts. This in turn enables us to study the equilibrium impact of contracting frictions in a tractable framework. In any case, optimal recursive contracts obtained in the presence of dynamically rational borrowers would converge to the ones that we obtain here when the discount rate of the borrowers become large. Second, we are not wedded to these particular lexicographic preferences. Their sole role is to ensure that binding liquidity constraints drive mortgage and housing demand. We consider this to be a natural feature of subprime lending markets. Any set of myopic preferences for which mortgage constraints bind in equilibrium would yield equivalent results.

For all $j \in [0,1]$, there exists a Poisson process $(N_{j,t})_{t\geq 0}$ with intensity $\delta > 0$ such that at each jump date, household j vacates its current home, and re-enters the housing market for unmodelled reasons. These dates capture trades in the housing market that are not primarily driven by the evolution of the real estate market, but rather by occupational changes, changes in household size, etc. In the remainder of the paper, we refer to these dates as exogenous termination dates, or ET dates. We denote $\aleph_j$ the set of jump dates of $(N_{j,t})_{t\geq 0}$ for all $j \in [0,1]$. Each household $j \in [0,1]$ is endowed with an income stream $(I_{j,\,t})_{t\geq 0}$ such that

$$\begin{cases} \forall t \in \aleph_j, I_{j,t} = 1 \\ \frac{dI_{j,\,t}}{I_{j,\,t}} = \sigma dW_{j,\,t}, \text{ for } t \notin \aleph_j \end{cases}, \tag{1.1}$$

where $W_{j,t}$ is a standard Brownian motion, and $\sigma > 0$. This exponential re-setting of idiosyncratic income processes implies that the cross-section of incomes has a constant distribution. That exogenous termination decisions and re-setting of idiosyncratic incomes are contemporaneous simplifies the analysis but is not crucial to our results. All stochastic processes $\left((W_{j,t})_{t\geq 0}, (N_{j,t})_{t\geq 0}\right)$ are pairwise independent.[3] Thus there is no aggregate uncertainty in this baseline model.

[3]We will apply the exact law of large numbers to this continuum of independent processes. As is well known, doing so in a mathematically correct fashion requires the construction of an extension of the product of the Lebesgue unit interval and the state space such that the continuum of random variables be measurable with respect to this extension. Sun and Zhang (2008) show the existence of such extensions in which the exact law of large numbers applies for any variety of distributions. For expositional simplicity, we will informally invoke the exact law of large numbers throughout without explicitly constructing such complex mathematical objects.

Banks

Banks care only for consumption. They are infinitely lived, risk-neutral, and discount the future at the rate $r > 0$. Banks are not financially constrained: We assume that their aggregate endowment is always larger than households' aggregate debt capacity.

Banks own an eviction technology. This means that a bank can transform an occupied home that it has financed into a vacant home. Eviction comes at a cost equal to a fraction λ of the home market value, where $\lambda \in (0, 1]$. This captures the value loss implied by foreclosures.

Market for vacant homes

Vacant housing units are perfectly divisible. However, in order to modify its home size from q to $q' \neq q$, a household needs to move into q' new units. Households face relocation costs. If a household moves into q new units at date t, only a fraction χq of these units enters its date-t preferences. The relocation cost implied by $\chi \in [0, 1]$ captures all the costs associated with search in housing and mortgage markets, moving, etc. If household j does not move at date t, its entire real estate $q_{j,t}$ enters its preferences. Banks and households are home-price takers.

Contracts

Banks enter into individual financing contracts with households. There are two contracting frictions. First, households privately observe the realization of their income paths. They consume secretly, and can falsify public income reports at no cost.[4] Second, households cannot commit to a contract. Households are free to terminate a contractual relationship and re-trade in housing and mortgage markets as they see fit. How much borrowing can take place under such pure one-sided commitment is an interesting benchmark. Extensions could endow lenders with some recourse possibilities, and measure their impact on equilibrium debt capacities. Other than households' income realizations and consumption decisions, everything else in this economy is publicly observable.

Banks can commit to a contract with a household. If, however, a bank makes a strictly positive cash transfer to a household after the initial date of the contract, or trades its current eviction rights with other banks, then this qualifies as terminating the existing contract, and originating a new one. In this case, the borrower is entitled to auction off the eviction rights attached to his home and re-contract with the winning bank.[5]

[4]Lacker and Weinberg (1989) study optimal contracts when falsification comes at a cost.

[5]This ex post customer protection prevents banks from offering arbitrarily small additional loans to their captive (myopic) borrowers against the revelation of their entire current income.

Contracts map public history into paths of contractual payments, trades, and eviction decisions. We consider only eviction decisions that are a deterministic function of history. This restriction is commonly used in applications of costly state verification models to obtain realistic contracts. (see, e.g., Gale and Hellwig (1985)).[6]

Competition for contracts

At each date, banks simultaneously post contract menus. If several banks make competitive offers, they obtain equal market shares. We solve for steady-states in which there is no aggregate uncertainty. More precisely, a steady-state is characterized by a contract menu, a price per housing unit, and a distribution of housing units across households such that at each date:

1. A bank cannot make a strictly positive profit by offering a different menu of contracts.

2. The market for vacant units clears.

3. Each household has an optimal housing and consumption allocation given its contractual obligations, contract menus, and the housing price.

Solving for the Steady-States

Structure of optimal contracts. First, contracts are exclusive. A household uses its entire initial loan to acquire vacant units that it occupies until the contract is terminated by either party. The size of the loan is a strictly increasing function of the initial payment made by the household. This induces the household to pay its total income to the bank at this initial date. The household must then meet a deterministic repayment schedule. The bank commits to evict the household if it fails to meet the schedule. The bank retains the rights to sell the home once it is vacant, which is either due to an eviction, or to voluntary termination by the household.

To see why this contract is optimal, consider a deviation in which the bank writes a new and different covenant if the household reports an income strictly higher than the scheduled deterministic one at some future date. In order to be incentive-compatible, this new covenant must be such that the household consumes strictly more, and/or occupies a strictly larger place than when it reports the minimal repayment, or terminates the contract. This requires either a strictly lower repayment, and/or additional lending. A strictly lower repayment without additional lending violates incentive-compatibility:

[6] A key factor in the design of contracts that would allow for randomized eviction would be households' rates of substitution of consumption for housing services. Our model in which this rate is infinite is not appropriate for the analysis of such contracts.

The household would always issue a falsified large report. Additional lending from the incumbent bank or any other bank (possibly associated with a rescheduling of the future repayments) entitles the household to auction off the bank's eviction rights. Thus, it must have zero value to the incumbent bank and such a deviation can only strictly decrease the initial loan size. In equilibrium, households initially pay their whole income at the outset of a contract, and then meet the deterministic schedule until voluntary termination, an ET date, or eviction.

Second, loan size and contractual repayments are a linear function of the initial payment made by the household. This follows from the fact that income risk is multiplicative, households have linear preferences over housing, and over consumption given a housing allocation and pricing of housing units is linear (Walrasian housing market).

Thus contract menus in the steady-state are proportional to the loan and the deterministic repayment schedule offered to a household, which enters in the market with a unit income. Section 1.1 characterizes the steady-state under the additional restriction that banks quote constant repayment schedules. This particular case is simpler to analyze because two parameters - the loan-to-income ratio and the repayment ratio - are sufficient to characterize constrained-optimal contracts. It yields plausible orders of magnitude and interesting insights into the behavior of aggregate quantities that we discuss in Section 1.2. Subsection 1.3 studies price and default dynamics along paths to the steady-state. Subsection 1.4 then discusses steady-state optimal repayment schedules.

1.1 Steady-State With Fixed Repayment Contracts

Let $L \geq 0$ and $\kappa \in [0,1]$ denote the loan-to-income and the repayment ratios respectively. A household that makes an initial payment of i receives a loan equal to Li, and is asked to repay κi per unit of time from then on. Let $I > 0$ denote the average income of households in the market at a given date. Let

$$\alpha = -\frac{1}{2} + \sqrt{\frac{2(r+\delta)}{\sigma^2} + \frac{1}{4}},\ \rho = 1 - \lambda\left(1 + \frac{\delta}{r}\right),\ \omega = \sqrt{\frac{2\delta}{\sigma^2} + \frac{1}{4}}.$$

The following proposition shows that we can analytically solve for the steady-state up to the repayment ratio κ, which is the solution of an algebraic equation.

Proposition 1 *If banks quote a repayment ratio $\kappa \in [0,1]$ then*

$$I = \frac{1 - p_\kappa - p_\chi}{1 - \kappa p_\kappa - \frac{p_\chi}{\chi}},$$

where

$$p_\chi = \frac{\sqrt{\chi}\left(\kappa^\omega - \kappa^{-\omega}\right)}{\left((\kappa\chi)^\omega - (\kappa\chi)^{-\omega}\right)}, \tag{1.2}$$

$$p_\kappa = \frac{\left(\chi^\omega - \chi^{-\omega}\right)}{\sqrt{\kappa}\left((\kappa\chi)^\omega - (\kappa\chi)^{-\omega}\right)}. \tag{1.3}$$

The arrival intensity of households in the market is $\frac{\delta}{1-p_\kappa-p_\chi}$. *This arrival intensity is the sum of the arrival intensities of households i) who unwittingly default, ii) who voluntarily terminate their contract in order to relocate, and iii) who move for exogenous reasons at an ET date. These three intensities are respectively equal to* $\frac{\delta p_\kappa}{1-p_\kappa-p_\chi}, \frac{\delta p_\chi}{1-p_\kappa-p_\chi}$, *and* δ. *Further,*

$$L = l(\kappa) = \frac{\kappa}{r} \times \left[1 - \frac{1-\rho}{\frac{\kappa^{\alpha+1}(1-\chi^\alpha)+\kappa^{-\alpha}(\chi^{-\alpha-1}-1)}{(\chi^{-\alpha-1}-\chi^\alpha)} - \rho}\right]. \tag{1.4}$$

The equilibrium repayment ratio is $\arg\max l(\kappa)$ *and the price of a housing unit* $P = LI$.

Proof. A contract initiated by household j at date t is terminated at the date $t + T^{j,t}$, where

$$T^{j,t} = \min\left(T_\delta^{j,t}, T_\kappa^{j,t}, T_\chi^{j,t}\right). \tag{1.5}$$

The three stopping times $T_\delta^{j,t}, T_\kappa^{j,t}, T_\chi^{j,t}$ are defined as follows. First,

$$T_\delta^{j,t} = \min \aleph_j \cap (t, +\infty)$$

is the time elapsed until the first ET date after the contract is initiated. Second

$$\begin{aligned} T_\kappa^{j,t} &= \min\left\{\tau \geq 0 : I_{j,\,t+\tau} = \kappa I_{j,\,t}\right\} \\ &= \min\left\{\tau \geq 0 : W_{t+\tau}^j - W_t^j = \frac{\ln\kappa}{\sigma} + \frac{\sigma\tau}{2}\right\} \end{aligned}$$

is the time elapsed until the income of household j hits the value $\kappa I_{j,t}$ for the first time after t. Finally,

$$\begin{aligned} T_\chi^{j,t} &= \min\left\{\tau \geq 0 : \chi I_{j,\,t+\tau} = I_{j,\,t}\right\} \\ &= \min\left\{\tau \geq 0 : W_{t+\tau}^j - W_t^j = -\frac{\ln\chi}{\sigma} + \frac{\sigma\tau}{2}\right\} \end{aligned}$$

is the time elapsed until the income of household j hits the value $\frac{I_{j,t}}{\chi}$ for the first time after t. Equation (1.5) states that a contract is terminated for one of the following reasons: i) occurrence of an ET date, ii) the household cannot meet a repayment, iii)

the household voluntarily terminates the contract and vacates its place to acquire a larger one. Note that the distributions of $T_\delta^{j,t}, T_\kappa^{j,t}, T_\chi^{j,t}$ do not depend on j,t. Therefore, we will omit the superscripts j,t for notational simplicity. Now, let

$$\begin{aligned} p_\chi &= \text{Prob}\left(T_\chi < T_\kappa; T_\chi < T_\delta\right), \\ p_\kappa &= \text{Prob}\left(T_\kappa < T_\chi; T_\kappa < T_\delta\right). \end{aligned}$$

The proofs of (1.2) and (1.3) are in the Appendix. The income of a household which is in the market is reset to 1 if it is in the market because of an exogenous termination. If not, its income can be characterized with two integers m and n that count the respective numbers of times the household has defaulted and voluntarily relocated since the last time its income was reset to 1. Its income is then equal to $\kappa^m \chi^{-n}$. Thus,

$$I = (1-p_\kappa-p_\chi) + \sum_{\substack{m,n \\ m+n>0}} \left(\begin{array}{c} m \\ m+n \end{array} \right) (1-p_\kappa-p_\chi)\left(p_\kappa \kappa\right)^m \left(\frac{p_\chi}{\chi}\right)^n = \frac{1-p_\kappa-p_\chi}{1-\kappa p_\kappa - \frac{p_\chi}{\chi}}.$$

Furthermore, let Mdt denote the steady-state measure of households in the market between t and $t+dt$. A fraction $1-p_\kappa-p_\chi$ of these households trade because of the realization an ET date. Since such dates occur with intensity δ,

$$(1-p_\kappa-p_\chi)\, Mdt = \delta dt,$$

and the arrival intensity of households in the market M is $\frac{\delta}{1-p_\kappa-p_\chi}$. Thus the respective arrival intensities of households who default and voluntarily relocate are $\frac{\delta p_\kappa}{1-p_\kappa-p_\chi}$ and $\frac{\delta p_\chi}{1-p_\kappa-p_\chi}$ respectively.

Let us finally compute L and P. Let S denote the quantity of vacant units per household in the market. If household j purchases $q_{j,t}$ units at date t, then it must be that it receives a loan that satisfies

$$LI_{j,t} = E_t\left(\int_0^T e^{-rs}\kappa I_{j,t} ds + P \times q_{j,t} e^{-rT}(1-\lambda \mathbf{1}_{\{T_\kappa<T_c, T_\kappa<T_\delta\}})\right).$$

Integrating over the set of agents in the market at date t yields

$$LI = \frac{\kappa I}{r}\left(1-Ee^{-rT}\right) + PS \times E\left[e^{-rT}\left(1-\lambda \mathbf{1}_{\{T_\kappa<T_c, T_\kappa<T_\delta\}}\right)\right].$$

Market clearing implies $PS = LI$, so that

$$L = \frac{\kappa}{r} \times \frac{1-Ee^{-rT}}{1-Ee^{-rT}+\lambda E\left(e^{-rT}\mathbf{1}_{\{T_\kappa<T_\chi, T_\kappa<T_\delta\}}\right)}. \tag{1.6}$$

The computations of Ee^{-rT} and $E\left(e^{-rT_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi,T_\kappa<T_\delta\}}\right)$ that lead to expression (1.4) are relegated to the Appendix. The equilibrium repayment ratio must maximize the loan-to-income ratio given P and S. It is easy to see that it must also maximize the above expression for L. Finally, we show in subsection 1.3 that $S = 1$.■

Proposition 1 maps the five primitive parameters σ, δ, λ, r, and χ into a full characterization of the steady-state. Notice that as $\lambda \to 0$, the loan-to-income ratio L tends to the level $1/r$ that would prevail without contracting frictions. Absent eviction costs, committing to eviction upon default comes at no cost.

1.2 Orders of Magnitude

The orders of magnitude associated with the steady-state are quite plausible. Assume a (real) rate of $r = 2\%$, a volatility of households' income $\sigma = 20\%$ (see Dynan et al., 2008), and foreclosure costs $\lambda = 15\%$.[7] Consider a typical contractual mortgage duration of 20 years, which corresponds to $\delta = 5\%$. The only primitive parameter that is hard to pin down with a simple empirical counterpart is the relocation cost χ. Assume a relocation threshold $1/\chi = 1.5$. These parameter values yield a steady-state repayment ratio κ of 55%, an arrival intensity in the market $\frac{\delta}{1-p_\kappa-p_\chi}$ of 20%, and a default intensity of 7.7%. Fabozzi (2006) reports average front-end debt-to-income ratios of 40% for subprime borrowers, and back-end ratios averaging above 50%.[8] Considering that we do not explicitly model a fixed subsistence level of consumption, a 55% loan-to-income ratio is a reasonable first pass. The trading intensity of 20% corresponds to an effective (as opposed to contractual) mortgage lifespan of 5 years. This is below an average 7 year effective duration of U.S. mortgages but is plausible for subprime mortgages. Finally, the default intensity of 7.7% is in line with the default rates observed by Keys et al. (2006) for borrowers with low FICO scores over 2001-2006.

Comparative Statics. Figure 1 shows the evolution of the loan-to-income ratio as a function of δ and χ.

[Figure 1 About Here]

Debt capacity is not too sensitive to δ for low values of χ. Conversely, as the cost at which households climb the property ladder decreases (χ close to 1), the commitment problem becomes quickly very costly and hurts debt capacity. Notice that an increase in the ET intensity δ actually reduces debt capacity provided δ and χ are small. The broad intuition for this result is the following. An increase in δ makes both tails of

[7] Any λ in the 10% to 25% range yields very similar results.

[8] The front-end ratio divides mortgage payments, real estate taxes, and home insurance premia by gross income. The back-end ratio adds other obligations such as credit card and automobile debt to the numerator.

the cross-section of income thinner. A thinner left tail reduces expected eviction costs. On the other hand, a thinner right-tail implies that conditional on trading after an ET date, a household has a lower income. The positive effect overcomes this negative one only if δ is sufficiently large.

Figure 2 shows the equilibrium value of the steady-state default intensity $\frac{\delta p_\kappa}{1-p_\kappa-p_\chi}$.

[Figure 2 About Here]

Interestingly, equilibrium defaults increase w.r.t. χ and w.r.t. δ when δ is small. This is so even though one can check that the equilibrium repayment ratio κ decreases w.r.t. χ and w.r.t. δ for small values of δ. Thus an individual loan is ex ante less likely to default.[9] The intuition is the following. If δ is large, the dominant effect of an increase in δ is that it "completes contracts", or reduces the needs for enforcement through evictions. All else being equal, when δ is small, the dominant effect of an increase in χ or δ is a negative equilibrium effect on the distribution of households' solvency. As χ or δ increases, households enter new contracts and thus maximize their debt capacity more often. As a result, there is a higher fraction of households close to their default boundary in the steady-state. Otherwise stated, the dominant effect in this case is that an increase of δ or χ reduces the steady-state fraction of households that have become very safe borrowers over time. To see this, Figure 3 illustrates how the steady-state density of

$$\frac{I_{j,t+s} - \kappa I_{j,t}}{I_{j,t} - \kappa I_{j,t}}, \tag{1.7}$$

where t is the date at which the loan was originated, varies as a function of δ for $\chi = 0$.[10]

[Figure 3 About Here]

Figure 3 shows the density for values of (1.7) ranging from 0 (default) to 25%. This confirms that the dominant impact of an increase in δ for small δ is an increase in the number of risky borrowers.

Figure 4 shows comparative statics for the average income I of households in the market.

[Figure 4 About Here]

The average income of households in the market reflects the weights of each motive to trade. For small χ and δ, many trades follow from foreclosures, and thus there is a

[9] This contrasts with Mayer et al. (2008), where lower prepayment penalties lead to higher interest rates and defaults in partial equilibrium because the loan size is fixed.

[10] The derivation of a closed-form expression for this steady-state density is available upon request.

large majority of households whose income has done poorly in the market. Conversely, I becomes nearly 20% higher than unconditional average income 1 as the fraction of households which trade to climb up the property ladder rises. Finally, Figure 5 shows that the price increases with δ and χ. Here we have again the equilibrium effect: even though the individual debt capacity decreases, the aggregate income of households in the market increases, and the later effect is stronger.

[Figure 5 About Here]

1.3 Transition Dynamics

This subsection studies the dynamics of convergence to the steady state holding loan contracts constant. For simplicity, we set $\chi = 0$, so that loan termination occurs only at default and ET dates. We assume that banks start offering the steady-state contract (L, κ) defined in Proposition 1 at date 0. A possible interpretation is that a sudden regime shift in the securitization market boosts subprime loan supply at date 0. Before this date, all housing units are vacant and all households homeless. We solve for the dynamics of average income of households in the market, housing price, and default intensities along the path to the steady-state. Let I_t denote the average income of households which are in the market at date t, P_t the date-t price of one housing unit, and S_t the date-t vacant home supply per household. We have

Proposition 2 *For all $t > 0$, $S_t = S_0 = 1$ and $P_t = LI_t$.*

Proof. Market clearing implies that for all t

$$P_t S_t = LI_t.$$

Thus we only need to show that $S_t = 1$ for all $t > 0$. For $t > 0$, $\tau \in [0, t)$, let $\pi_{\tau,t}$ be the fraction of households in the market at date t whose previous market participation took place at date τ. Note that the probability to be in the market at date t, given that the previous participation occurred at date τ, does not depend on income at date τ. This is because both ET and default dates on a given loan are independent from income level at the loan origination. The average income of the above fraction at time τ therefore equals I_τ, and hence, the average supply of housing units of the above fraction is S_τ. Thus, S_t solves

$$S_t = \int_0^t \pi_{\tau,t} S_\tau d\tau, \quad S_0 = 1, \quad t \in (0, \infty). \tag{1.8}$$

We show in the Appendix that the only continuous solution to (1.8) is $S_t = 1$.■
We can now characterize the average income I_t, or equivalently the unit price $P_t = LI_t$.

Proposition 3 *Let $F(\cdot)$ and $F_\kappa(\cdot)$ denote the respective c.d.f. of T and T_κ. The average income of households in the market at date t, I_t, is the unique solution to the following Volterra integral equation of the second kind*

$$I_t = \kappa \int_0^t \frac{I_{t-\tau} e^{-\delta\tau}}{F(t)} dF_\kappa(\tau) + \delta \int_0^t \frac{(1 - F_\kappa(\tau))\, e^{-\delta\tau}}{F(t)} d\tau, \; I_0 = 1 \quad t \in (0, \infty), \tag{1.9}$$

where

$$F_\kappa(t) = \frac{1}{2}\left(\text{Erfc}\left(\frac{\ln\kappa}{\sigma} + \frac{\sigma t}{2}\right)\right) + \kappa^{-1}\text{Erfc}\left(-\frac{1}{\sqrt{2t}}\left(\frac{\ln\kappa}{\sigma} - \frac{\sigma t}{2}\right)\right), \tag{1.10}$$

$$F(t) = 1 - e^{-\delta t}(1 - F_\kappa(t)), \quad \text{Erfc}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dz. \tag{1.11}$$

Proof. See the Appendix.■

Figure 6 depicts the evolution over time of the price of a housing unit.

[Figure 6 About Here]

Recall that the price of a housing unit is proportional to the average income of market participants. Average income starts at a maximal level, when most trades are due to ET dates. Then an increasing fraction of households default and re-trade. This lowers average income in the market. Interestingly, there is overshooting. The average income reaches a minimum that is strictly below the steady-state value, and then increases slowly towards this long-term value. Thus, absent any aggregate shock on unconditional income, positive loan supply shocks lead to "boom and bust" dynamics. This happens because of the "term structure of defaults": defaults on the initial loans cluster around the same dates, leading to depressed home prices. This result suggests that even absent any housing "bubble" or inefficient lending, a sudden and efficient increase in the supply of well-priced subprime mortgages in the early 2000s would have mechanically created a decrease in home prices a few years later.

1.4 Towards Optimal Contracts

We restricted the analysis to constant repayment schedules because characterizing equilibrium contracts with two parameters simplifies the analysis. This section studies the impact of this restriction by allowing for repayment schedules that depend on time

elapsed since the origination of the loan. We focus on the case $\chi = 0$ for simplicity. The optimal repayment schedule is a continuous function of the time elapsed since origination $g(\cdot)$ such that if the contract has been originated at date t, then the $t+s$ contractual repayment is $e^{-g(s)}$. We have the following compact characterization of the optimal contract.

Proposition 4 *Let*

$$T_g = \min\left\{\tau \geq 0 \ : \ W_{j,\,t+\tau} - W_{j,\,t} = \frac{\sigma\tau}{2} - \frac{g(\tau)}{\sigma}\right\},$$

and

$$Q_g(\tau) = Prob(T_g \geq \tau),$$

the optimal contract $g(\cdot)$ *solves*

$$\sup_{g(\cdot)} \frac{\int_0^\infty Q_g(s) e^{-g(s)-(r+\delta)s} ds}{\lambda + \rho r \int_0^\infty Q_g(s) e^{-(r+\delta)s} ds}. \tag{1.12}$$

Proof. See Appendix.■

Problem (1.12) does not seem to be analytically solvable. Solving it numerically is also challenging.[11] We take the first steps in this direction by studying the equilibrium when the contract space is expanded to contracts with repayments that are log-linear, and log-linear with one kink. Namely we consider a space of contracts such that

$$g(s) = a + bs,$$

where constants a and b can take two different values on two intervals partitioning $[0, +\infty)$. The analytical expression of the loan-to-income ratio given such a contract is available upon request. Solving for the equilibrium contract is then a standard numerical search for extrema using this analytical expression. Figure 7 compares optimal repayment schedules for the cases in which repayments are constant, log-linear, and piecewise log-linear with one kink.[12]

[Figure 7 About Here]

[11] A numerical solution involves computing the distribution of the first hitting time $Q_g(\cdot)$ for a smooth boundary $g(\cdot)$, which is a notoriously difficult problem (see, e.g., Durbin and Williams (1992), and Wang and Pötzelberger (1997) for some advances).

[12] Closed-form solutions for the loan-to-income ratio with the piecewise linear contract are available upon request.

It is interesting to notice that for these parameter values, repayment ratios become optimally larger than 1 in the long run even though the income process has no drift and there is no inflation risk in our environment.[13] The log-linear contract features increasing repayments. So does the one with a kink after a short series of high and then decreasing initial repayments that can be interpreted as an initial downpayment. These repayment patterns are reminiscent of the "teaser rates" that often apply in practice. Such schedules are optimal because households' expected income at remote dates conditional on still honoring repayments increases over time.

Figure 8 illustrates equilibrium loan-to-income ratios with these three types of contracts as a function of δ.

[Figure 8 About Here]

As δ becomes small, the fixed-repayment contract does quite poorly. This is because suboptimal repayment schedules become more costly when contracting relationships become longer-lived. For δ around 5%, loan-to-income ratios increase by less than 5% when adding a drift, and less than 1% by allowing for a kink. This suggests that the restriction to fixed-repayment contracts is not a critical driver of our findings.

2 Systematic Income Risk

This section introduces systematic income shocks. The main goal is to show that idiosyncratic and systematic income risks dent equilibrium debt capacities through quite different channels in this economy. Consider a modification of the baseline model in which we replace the income process described in (1.1) with the following one. Each household $j \in [0,1]$ is now endowed with an income stream $(I_{j,t})_{t\geq 0}$ such that

$$I_{j,t} = I_t \times i_{j,t}.$$

The systematic component of income obeys

$$\frac{dI_t}{I_t} = \phi\sigma dW_t,$$

[13] Optimal repayment schedules are decreasing for larger values of δ.

where W_t is a standard Brownian motion[14], $\phi \in [0,1]$, and $\sigma > 0$. We set $I_0 = 1$. Idiosyncratic income follows:

$$\begin{cases} \forall t \in \aleph_j, i_{j,t} = 1 \\ \frac{di_{j,t}}{i_{j,t}} = \sqrt{1-\phi^2}\sigma dW_{j,t}, \text{ for } t \notin \aleph_j \end{cases},$$

where $W_{j,t}$ is a standard Brownian motion. All stochastic processes $(W_t)_{t\geq 0}$, $(W_{j,t})_{t\geq 0}$, and $(N_{j,t})_{t\geq 0}$ are pairwise independent. The baseline model is the particular case in which $\phi = 0$.

The addition of systematic risk is clearly irrelevant absent any additional contracting restrictions. Since the mortgage market is active over all dates, banks observe the systematic component of income shocks. Equilibrium loans and repayment schedules are equal to those of the baseline model multiplied by the common component of income risk. Thus, equilibrium defaults and trading intensities are unchanged because idiosyncratic shocks remain their sole causes. Systematic income risk plays an important role only if it is not possible to index contracts on aggregate risk, as is the case in practice.[15] This is what we assume going forward.

Assumption The housing unit price and aggregate income are not contractible.

This assumption is strong given our environment, but is arguably realistic. Aspects of the housing market that are absent from our model limit mortgage indexation in practice. For instance, the cross-sectional variation in the structure of income risk is likely to be quite large, even in small areas. So is the cross-sectional variation in housing quality. Thus, defining and measuring the relevant index for subprime borrowers in a given local housing market seems practically difficult because model risk is quite large.

Under this assumption, the general case $\phi > 0$ is not analytically tractable because the average income in the market depends on history in an overly complex fashion in this case. A companion note (available upon request) tackles this general case in the limiting situation in which idiosyncratic incomes and housing allocations have an initially degenerate (diffuse) distribution. In what follows, we will focus instead on the case $\phi = 1$. This case is simple to analyze. It contains the same qualitative insights on the impact of systematic income shocks as our more general note does.

When $\phi = 1$, trades in the housing market happen because of only two motives: the occurrence of an ET date and default. Absent cross-sectional heterogeneity, there is no other reason for entering the market since there is no possibility to move along the property ladder. As in the idiosyncratic case, we will focus on contracts with

[14] It is easy to extend the whole analysis to the case of a Brownian motion with a constant drift.
[15] With the obvious exception of indexation to nominal short-term rates.

fixed repayment schedules. The next proposition characterizes the equilibrium with a linear housing unit price. Subsection 2.1 shows that this linear equilibrium is the only one in which the price satisfies the transversality condition. Recall the notations $\alpha = \sqrt{\frac{2(r+\delta)}{\sigma^2} + \frac{1}{4}} - \frac{1}{2}$, $\rho = 1 - \lambda\left(1 + \frac{\delta}{r}\right)$.

Proposition 5 *There is a unique equilibrium with fixed-repayment contracts and linear price paths. The repayment associated with a contract initiated at date t is $\kappa \times I_t$, where the constant κ is the unique solution within $[0,1]$ of:*

$$(\alpha + 1)\,\kappa^{\alpha} = 1 + \rho\alpha\kappa^{\alpha+1}. \tag{2.1}$$

The price of a home unit is:

$$P_t = \frac{1 - \kappa^{\alpha}}{1 - \rho\kappa^{\alpha+1}} \times \frac{\kappa I_t}{r} = \frac{\alpha}{\alpha + 1} \times \frac{\kappa I_t}{r}, \tag{2.2}$$

where $\frac{\kappa\alpha}{\alpha+1} \times \frac{1}{r}$ is also the loan-to-income ratio offered in the loan market.

Proof. See the Appendix.■

Equation (2.2) shows that when banks quote a repayment ratio κ, they discount promised repayments κI_t at a rate $r + \Delta r$, where the equilibrium spread Δr is equal to $\frac{r\kappa^{\alpha}(1-\rho\kappa)}{1-\kappa^{\alpha}}$. The spread Δr increases with respect to κ because default risk is increasing in κ. This increasing spread implies that the loan size as a function of κ has an inverted-U shape, and is maximal for the equilibrium value of κ defined by (2.1). The rising probability of default more than offsets the increase in promised repayments when κ increases beyond this equilibrium value. This model may be viewed as a dynamic equilibrium version of the model of credit rationing developed by Williamson (1987). The spread corresponding to the equilibrium value of κ is equal to r/α.

Assuming away eviction costs is instructive. From expressions (2.1) and (2.2), ρ tends to 1 when $\lambda \to 0$, so that

$$\lim_{\lambda \to 0} \kappa = 1, \quad \lim_{\lambda \to 0} P_t = \frac{\alpha}{\alpha + 1} \times \frac{I_t}{r}.$$

As in the baseline model, the repayment ratio tends to 1 when eviction costs vanish. Unlike in the baseline model, however, the equilibrium loan-to-income ratio is still strictly smaller than $1/r$ in the limit. This reflects the fact that negative realizations of systematic income risk imply a contemporaneous large supply of vacant units (due to foreclosures) and small demand (due to the low borrowing capacity of potential buyers).[16] Of course, this effect becomes marginal as $\delta \to +\infty$, in which case $\alpha \to +\infty$.

[16] Thus, trading volumes goes down when home prices go up in our model, which is counterfactual.

In this case most trades are for exogenous reasons, and supply and demand in the home market become therefore independent in the limit.

It is worthwhile noticing that the fact that banks do not internalize the impact of their choice of a repayment ratio κ on the value of collateral does not reduce their supply of funds. In other words, a social planner internalizing the impact of κ on home prices - namely, taking into account that P depends on κ - cannot impose a value of κ for which debt capacities or home prices are higher than in the competitive setting.[17] This is because our economy is always allocationally efficient: households collectively occupy the fixed supply of homes, and banks make no profits. This contrasts with models that mix financing constraints and investment decisions, such as Bernanke and Gertler (1989), or Kiyotaki and Moore (1997). In these models, systematic risk lowers debt capacities because lenders fail to internalize the negative balance-sheet externalities that they create for each other. This leads to suboptimal investment and possibly multiple equilibria. In our setup, systematic risk affects prices through contracting frictions without distorting allocations. Endogenizing the quantity of homes could lead to financial amplification in our setup as well.

2.1 Costs and Benefits of Diversification

It is interesting to compare loan-to-income ratios in the idiosyncratic and systematic cases when relocation costs become large.[18] In this case, contracts are terminated for two reasons only - ET dates or default - in both cases. As $\chi \to 0$, the limiting value of (1.4) is

$$\lim_{\chi \to 0} L = \frac{\kappa}{r} \times \frac{1 - \kappa^{\alpha}}{1 - \rho \kappa^{\alpha}}, \tag{2.3}$$

and the optimal repayment ratio - the one that maximizes L - is the root within $(0, 1)$ of:

$$\rho \alpha \kappa^{2\alpha} - (1 + \alpha) \kappa^{\alpha} + 1 = 0.$$

From (2.2), the loan-to-income ratio for a given repayment ratio κ if income risk is

This is because there is no rationale for downpayment constraints in our environment. If we introduced the exogenous constraint that households can voluntarily move at an ET date only if they have built-up a minimal amount of home equity at this date, then positive income realizations could entail a larger trading volume.

[17] To see this, note that solving for P as a function of κ in (A17), plugging the expression in (A16), and then maximizing over κ yields the same loan-to-income ratios and home prices as in the competitive case in which banks maximize over κ taking P constant.

[18] A higher debt capacity is not desirable *per se* in this simple environment. We find it relevant to compare debt capacities because a greater ability to borrow would be Pareto improving in simple extensions of the model. For instance it would be the case if banks made profits increasing in total lending, if housing supply was not inelastic, or if indivisibilities were restricting participation in the mortgage market.

fully systematic is $\frac{\kappa(1-\kappa^{\alpha})}{1-\rho\kappa^{\alpha+1}}$. Comparing with (2.3) yields the following result:

Proposition 6 *Assume $\chi = 0$. Equilibrium loan-to-income ratios are larger when $\phi = 0$ (diversifiable income risk) than when $\phi = 1$ (systematic income risk) if and only if*

$$\rho \geq 0 \Longleftrightarrow \lambda \leq \frac{r}{r+\delta}. \tag{2.4}$$

Proof. Let κ_S and κ_I denote the respective equilibrium repayment ratios when $\phi = 1$ and $\phi = 0$ respectively. If $\rho \geq 0$, then

$$\frac{\kappa_I\left(1-\kappa_I^{\alpha}\right)}{1-\rho\kappa_I^{\alpha}} \geq \frac{\kappa_S\left(1-\kappa_S^{\alpha}\right)}{1-\rho\kappa_S^{\alpha}} \geq \frac{\kappa_S\left(1-\kappa_S^{\alpha}\right)}{1-\rho\kappa_S^{\alpha+1}}.$$

If $\rho \leq 0$, then

$$\frac{\kappa_S\left(1-\kappa_S^{\alpha}\right)}{1-\rho\kappa_S^{\alpha+1}} \geq \frac{\kappa_I\left(1-\kappa_I^{\alpha}\right)}{1-\rho\kappa_I^{\alpha+1}} \geq \frac{\kappa_I\left(1-\kappa_I^{\alpha}\right)}{1-\rho\kappa_I^{\alpha}}. \blacksquare$$

The intuition for this result is as follows. The expected proceeds from a loan are the sum of three components: i) the promised repayments until default or an exogenous move, (ii) sale proceeds net of eviction costs in case of default, (iii) proceeds from selling the vacant home if the household moves for exogenous reasons. Whether income risk is diversifiable or not has no impact on the value of component (i). That income risk is idiosyncratic implies that the income of a given household is uncorrelated with home market values at default and ET dates. This has positive effect on component (ii), which represents situations in which individual income has done poorly, but negative one on component (iii) in which individual income has done rather well. As δ and λ increase, a larger fraction of total loan value stems from component (iii) than from component (ii). Thus, diversification has an adverse impact in this case. In sum, diversifiability of income risk implies that the home price that a bank receives upon termination of a given mortgage is not tied to the income of the borrower at the date of termination. This is beneficial to the bank when the borrower is unable to repay and is evicted because its income is low. This is detrimental to the bank when the borrower with high income voluntarily exits the contract.[19]

[19]Consistent with the one-sided commitment assumed here, households just vacate their current home when they seek to move. Thus, the bank's expected prepayment is the home market value, and this drives the negative impact of diversification shown in Proposition 6. In practice, upon moving, borrowers typically sell their house to prepay their mortgage according to pre-agreed terms. We believe that this departure from our model cannot fundamentally reduce the negative impact of diversification that we established. Banks can still not fully capture idiosyncratic income appreciation through high pre-agreed repayments. High repayments that exceed home values would create high incentives for strategic default, particularly so for non-recourse loans.

In the presence of a finite relocation cost, it is no longer clear that equilibrium loan-to-income ratios are larger with $\phi = 0$ than with $\phi = 1$, even for values of δ and λ that satisfy (2.4). In this case, the costs of diversification are more important because the cross-sectional mobility of incomes implies that good borrowers exercise their option to climb up the property ladder more often. Table 1 reports the values of χ above which equilibrium debt capacity with diversifiable income risk is lower than that of with systematic income risk for different values of δ.

δ	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
χ	0.97	0.88	0.83	0.76	0.71	0.68	0.65	0.62	0.59

Table 1: $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$.

The values of χ below which diversifiable income risk yields lower debt capacities become quickly very low as δ increases. Thus the costs of diversification become quickly as important as the benefits from higher collateral values.

In sum, this shows that in the presence of one-sided commitment, diversifiability of borrowers' income risk does not necessarily ease collateralized lending. The larger ex post heterogeneity of borrowers is a double-edged sword. On one hand, the assets seized from unlucky borrowers can always be sold to luckier borrowers standing ready to snap them up. But the flip side of these higher liquidation proceeds is that these same lucky borrowers receive more outside options in equilibrium. The exercise of these options also imposes more costs on lenders. The possible benefits from diversification of borrowers' risks on collateral liquidity have been well identified in the literature on endogenous debt capacities (see, e.g., Shleifer and Vishny (1992)). That the costs induced by ex post heterogeneity may more than offset these benefits when borrowers' outside options are an equilibrium outcome is a novel finding, to our knowledge.

2.2 Rational Bubbles and Contracting Terms

Proposition 7 *The linear equilibrium in Proposition 5 is the only symmetric equilibrium that satisfies the transversality condition:*

$$\lim_{s \to +\infty} E_t \left(e^{-rs} P_{t+s} \right) = 0. \tag{2.5}$$

Proof. See the Appendix.■

We now investigate whether rational housing bubbles can develop absent this transversality restriction. We assume that a large number of new banks enter the economy at each date, so that there are always sufficient resources in the economy to sustain a

bubbly equilibrium price path. We first look for simple deterministic bubbles of the form

$$P_t = PI_t + Be^{bt}, \tag{2.6}$$

where $(P, B, b) \in \mathbb{R}^3_+$. Absent any contracting friction, any price of the form $P_t = \frac{I_t}{r} + constant \times e^{rt}$ is an equilibrium. In the presence of noncontingent contracts, we have the following proposition.

Proposition 8 *Equilibria in which the price is of the form (2.6) are such that* $B = 0$.

Proof. See the Appendix.■

It is possible to show similarly that bubbles that burst with a fixed intensity as in Blanchard and Watson (1982) cannot be sustained in this environment. Following Froot and Obstfeld (1991), we now look at "intrinsic" bubbles of the form

$$P_t = PI_t + F(I_t), \tag{2.7}$$

where $P = \frac{1-\kappa^\alpha}{1-\rho\kappa^{\alpha+1}} \times \frac{\kappa}{r}$, and $F : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable. The following proposition characterizes equilibria with a price of the form (2.7).

Proposition 9 *If an equilibrium features a repayment ratio* κ *(depending on* I_t*) and a price of the form (2.7), then for all* I_t

$$F(I_t) = (1-\lambda)\,\kappa^\alpha F(\kappa I_t) + \frac{2\delta}{(2\alpha+1)\,\sigma^2}\begin{pmatrix} (1-\kappa^{2\alpha+1})\int_1^\infty F(I_t y) y^{-(2+\alpha)} dy \\ +\int_\kappa^1 F(I_t y)\left(y^{\alpha-1} - y^{-(2+\alpha)}\kappa^{2\alpha+1}\right) dy \end{pmatrix}, \tag{2.8}$$

and

$$\frac{1-(1+\alpha)\kappa^\alpha + \rho\alpha\kappa^{1+\alpha}}{(1-\rho\kappa^{1+\alpha})^2} \times \frac{I_t}{r} + (1-\lambda)\,\frac{\partial}{\partial\kappa}\left(\kappa^\alpha F(\kappa I_t)\right) - \frac{2\delta}{\sigma^2}\kappa^{2\alpha}\int_\kappa^\infty F(I_t y) y^{-(2+\alpha)} dy = 0. \tag{2.9}$$

Proof. See the Appendix.■

Equation (2.8) is the market-clearing condition. Equation (2.9) is the first-order condition associated with the banks' choice of an optimal κ. In an unconstrained economy with Brownian dividends similar to our income process, Froot and Obstfeld (1991) exhibit such "intrinsic" bubbles of the form $F(I_t) = CI_t^\gamma$ with $\gamma > 1$. Again, such functional forms for "intrinsic" bubbles do not correspond to equilibrium price paths in our model with contracting frictions.[20]

[20] To see this, note that with $F(I_t) = CI_t^\gamma$, equation (2.8) implies that an equilibrium κ should be a constant, while it is easy to see from (2.9) that κ must depend nontrivially on $I_t^{\gamma-1}$.

It is not clear whether or not there is a function $F(I)$ different from zero and a repayment ratio $\kappa(I)$ which solve the system of functional equations (2.8) and (2.9). Notwithstanding this open question, it is interesting to study the characteristics of such hypothetical equilibria. These equilibra display relationships between home price paths and contracting terms that are reminiscent of what has been recently observed in subprime markets. Consider the simple case in which the total value of a home is lost in the foreclosure process ($\lambda = 1$). We have the following result:

Proposition 10 *Consider a positive function $F(\cdot)$ and an associated repayment ratio $\kappa_F(.)$ that satisfy (2.8) and (2.9) for $\lambda = 1$. Then*

1. *If $F \neq 0$, then $\kappa_F(I) < \kappa_0$ for all $I > 0$, where κ_0 is the (constant) equilibrium repayment ratio in the linear equilibrium in which $F = 0$. Furthermore, 0 is an accumulation point of $\kappa_F(I)$ as $I \to +\infty$, and $+\infty$ is an accumulation point of the loan-to-income ratio as $I \to +\infty$.*

2. *If $F(I)$ is increasing and $\alpha \geq 1$, or if $\frac{F(I)}{I}$ is increasing, then $\kappa_F(.)$ decreases from κ_0 to 0 as I_t increases from 0 to $+\infty$.*

Proof. See the Appendix.■

Thus, the self-justified belief that the price path is explosive implies that banks set lower repayment ratios despite quoting larger loan-to-income ratios. The intuition is that defaults become quite costly in the presence of a bubble because they limit banks' ability to ride it. Future home value is a more important component of the loan than the current borrower's ability to repay. Interestingly, time variations in repayment ratios amplify the sensitivity of default intensities to income paths in the bubbly case (if such a case exists) compared with the linear equilibrium. After a negative shock to I_t, not only do some borrowers default on their current loans, but newly issued loans also feature higher repayment ratios and thus lower distances to default. This will generate more future defaults after a lag. The impact of income shocks on default intensities is therefore more persistent with bubbly equilibria than with the linear one.

Nonlinear dynamics are generated by the procyclicality of loan-to-income ratios that become very large when income is high. Almeida et al. (2006) and Lamont and Stein (1999) find evidence consistent with such an amplification through procyclical leverage in housing markets.

3 Concluding Remarks

We have developed an analytically tractable model of secured lending to tightly constrained borrowers. Primitive assumptions about contracting frictions drive debt ca-

pacities, asset prices, and equilibrium default and refinancing intensities. Our findings help us analyze the recent U.S. subprime crisis. First, trading volume, default rates, and repayment ratios in the steady-state give some sense of what should prevail in a stable subprime market. It is interesting to relate this stable benchmark to the behavior of these variables in the recent "boom and bust" episode. Second, comparative statics show that structural changes have interesting equilibrium effects. For instance, lower transaction costs imply a lower default risk per mortgage. Yet the higher equilibrium turnover may possibly lead to a higher equilibrium default intensity. Third, we have offered two ways to model and analyze unstable subprime markets. First, the transition dynamics in the baseline model illustrate how the propagation of a loan supply shock can significantly and durably affect price and then defaults. Second, nonlinear price paths in the presence of systematic risk offer novel insights into the relationship between bubbles and default cycles.

Our model can be estimated in principle, and its predictions can be tested using household-level data. A number of our simplifying assumptions were made for analytical tractability. They can be relaxed for the purpose of estimation. For example, we have assumed that households always exhaust their debt capacities, which results in a uniform repayment ratio for all households. We could instead indirectly specify the cross-section of household preferences by using a distribution of repayment ratios observed in the data, or any other distribution. Similarly, we could allow for heterogeneity in refinancing costs and in idiosyncratic income volatility.

Finally, our approach to equilibrium secured lending to difficult borrowers could be applied to situations other than mortgage markets, for example, to small businesses and entrepreneurs.

Appendix

We first state a number of auxiliary results that we use repeatedly throughout the proofs and that can be skipped at first reading.

Auxiliary results

Auxiliary result 1 *The density of* T_κ *is*

$$\varphi_\kappa(t) = -\frac{\ln \kappa}{\sigma\sqrt{2\pi t^3}} e^{\frac{-\left(-\frac{\sigma}{2}t - \frac{\ln \kappa}{\sigma}\right)^2}{2t}}. \tag{A1}$$

Its Laplace transform is

$$\mathcal{L}_\kappa(s) = e^{\frac{\ln\kappa}{\sigma}\left(-\frac{\sigma}{2}+\sqrt{2s+\frac{\sigma^2}{4}}\right)} = \kappa^{-\frac{1}{2}+\sqrt{\frac{2s}{\sigma^2}+\frac{1}{4}}}. \tag{A2}$$

Proof. See, e.g., Borodin and Salminen (2002).■

Auxiliary result 2 *Let X be a random variable independent from T_δ taking values in $[0,+\infty)\cup\{+\infty\}$ whose density has Laplace transform $\mathcal{L}(\cdot)$. Define $T_{min} = \min(X, T_\delta)$. Then the Laplace transform of the density of T_{min} is given by*

$$\frac{s\mathcal{L}(s+\delta)+\delta}{s+\delta}.$$

Proof. Straightforward computations.■

Auxiliary result 3

$$\mathcal{L}_d(s) = \int_0^{+\infty} e^{-st}\varphi(t)\,dt = \frac{s\kappa^{-\frac{1}{2}+\sqrt{\frac{2(s+\delta)}{\sigma^2}+\frac{1}{4}}}+\delta}{s+\delta}\ . \tag{A3}$$

Proof. From 1. and 2.■

Auxiliary result 4 *Let B_t be a standard Wiener process. Define the running minimum as*

$$M_t = \min_{0\le s\le t} B_{s.}$$

For $t>0$ and $x\ge y$, $y\le 0$,

$$Prob[B_t\in dx, M_t\in dy] = \frac{2(x-2y)}{\sqrt{2\pi t^3}}e^{-\frac{(x-2y)^2}{2t}}. \tag{A4}$$

Proof. See Borodin and Salminen (2002).■

3.1 Proof of Formulae in Proposition 1

We use the following mathematical results, which can be in Borodin and Salminen (2002) p. 627. Let $\omega(s) = \sqrt{\frac{2s}{\sigma^2}+\frac{1}{4}}$. For all $s>0$,

$$E\left(e^{-sT_\chi}\mathbf{1}_{\{T_\chi<T_\kappa\}}\right) = \frac{\sqrt{\chi}\left(\kappa^{\omega(s)}-\kappa^{-\omega(s)}\right)}{(\chi\kappa)^{\omega(s)}-(\chi\kappa)^{-\omega(s)}}, \tag{A5}$$

$$E\left(e^{-sT_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi\}}\right) = \frac{\chi^{\omega(s)}-\chi^{-\omega(s)}}{\sqrt{\kappa}\left((\chi\kappa)^{\omega(s)}-(\chi\kappa)^{-\omega(s)}\right)}. \tag{A6}$$

Formulae (1.2) and (1.3) follow from (A5) and (A6) by putting $s = \delta$.

Computation of $E\left(e^{-rT_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi,T_\kappa<T_\delta\}}\right)$ and $1-Ee^{-rT}$ in (1.6):

Let $\varphi_\kappa(\cdot)$ denote the density of T_κ. Note that

$$E\left(e^{-rT_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi,T_\kappa<T_\delta\}}\right) = \int_0^\infty e^{-(\delta+r)t}E\left(\mathbf{1}_{\{T_\chi>t\}}\right)\varphi_\kappa(t)dt = E\left(e^{-(r+\delta)T_\kappa}\mathbf{1}_{\{T_\kappa<T_\chi\}}\right), \tag{A7}$$

which can be computed using (A6) and putting $s = r+\delta$. Next, using Auxiliary result 2 we have that

$$1-Ee^{-rT} = \frac{r}{r+\delta}\left(1 - \frac{\sqrt{\chi}\left(\kappa^{\omega(r+\delta)} - \kappa^{-\omega(r+\delta)}\right)}{(\chi\kappa)^{\omega(r+\delta)} - (\chi\kappa)^{-\omega(r+\delta)}} - \frac{\chi^{\omega(r+\delta)} - \chi^{-\omega(r+\delta)}}{\sqrt{\kappa}\left((\chi\kappa)^{\omega(r+\delta)} - (\chi\kappa)^{-\omega(r+\delta)}\right)}\right). \tag{A8}$$

3.2 Proof of Proposition 2

Equation (1.8) is a Volterra integral equation of the second kind. Notice that,

$$\pi_{\tau,t}d\tau = dF(t-\tau)/F(t), \tag{A9}$$

where $F(\cdot)$ is the c.d.f. of the stopping time T that describes the realized duration of a loan. Define an operator $A^t : C[0,t] \to C[0,t]$ as

$$A^t f(s) = \int_0^s \pi_{\tau,s} f(\tau)d\tau \tag{A10}$$

Since

$$\forall\, t \quad \int_0^t \pi_{\tau,t}d\tau = 1$$

the operator A^t has a unit norm. To complete the proof, it suffices to show that there exists no eigenfunction which takes zero value at zero and has eigenvalue 1. Suppose conversely that there exists such a function $f(s)$. Let $t^{max} = arg\max_{s\in[0,t]}|f(s)|$. Since $f(0) = 0$ and $f \neq 0$, it must be that $t^{max} > 0$. However,

$$Af(t^{max}) = \int_0^{t^{max}} \pi_{\tau,t^{max}} f(\tau)d\tau < f(t^{max}),$$

a contradiction.■

3.3 Proof of Proposition 3

Let $\varphi_\kappa(.)$ and $F_\kappa(.)$ denote the p.d.f. and c.d.f. of T_κ. Let $\varphi(.)$ denote the p.d.f. of T. Consider again the set of households which had two consecutive market participations at dates τ and t, where $0 \leq \tau < t$. Among them those who participated at time τ because of default and because of an ET date constitute the respective fractions $\varphi_\kappa(t-\tau)e^{-\delta(t-\tau)}/\varphi(t-\tau)$ and $\delta e^{-\delta(t-\tau)}\left(1-F_\kappa(t-\tau)\right)/\varphi(t-\tau)$. Therefore, the average income of the market participants at time t whose previous trade took place at τ is

$$\frac{\kappa\varphi_\kappa(t-\tau)e^{-\delta(t-\tau)}I_\tau + \delta e^{-\delta(t-\tau)}\left(1-F_\kappa(t-\tau)\right)}{\varphi(t-\tau)}.$$

Integrating over $[0,t]$ yields (1.9). That this equation has a unique continuous solution is a general result from the theory of Volterra integral equations with continuous kernels. ■

3.4 Proof of Proposition 4

Clearing the market for vacant homes in the presence of such contracts yields:

$$L = E\left(\int_0^{\min(T_\delta,T_g)} e^{-rs-g(s)}ds\right) + (1-\lambda)\,L\times E_t\left(e^{-rT_g}1_{\{T_\delta>T_g\}}\right) + L\times E_t\left(e^{-rT_\delta}1_{\{T_\delta\leq T_g\}}\right).$$

We have

$$\begin{aligned} E\left(\int_0^{\min(T_\delta,T_g)} e^{-rs-g(s)}ds\right) &= \int_0^\infty -\frac{d}{dt}\left(e^{-\delta t}Q_g(t)\right)\int_0^t e^{-rs-g(s)}ds \\ &= \int_0^\infty Q_g(s)\,e^{-(r+\delta)s-g(s)}ds, \\ E_t\left(e^{-rT_\delta}1_{\{T_\delta\leq T_g\}}\right) &= \int_0^\infty \delta e^{-(r+\delta)t}Q_g(t)\,dt, \end{aligned}$$

$$E\left(e^{-rT_g}1_{\{T_\delta>T_g\}}\right) = \int_0^\infty \frac{d}{dt}\left[1-Q_g(t)\right]e^{-(r+\delta)t}dt = (r+\delta)\int_0^\infty \left(1-Q_g(t)\right)e^{-(r+\delta)t}dt$$

Solving for L yields the proposition.■

3.5 Proof of Proposition 5

We need to solve for the loan-to-income ratio and for the repayment ratio κ that a bank chooses to offer at date t under the expectation that future home prices will satisfy

$$\forall u \geq 0, P_{t+u} = PI_{t+u} \tag{A11}$$

for some constant $P > 0$. With the notations introduced in the proof of Proposition 1, a loan taken at date t is terminated at the random date $t + T$, where

$$T = \min\left(T_\delta, T_\kappa\right),$$

If a household j accepts the offer from a bank that quotes a repayment ratio κ at date t, the bank expects the future flows from lending to household j to be equal to

$$L_{j,t} = E_t\left(\int_t^{t+T} e^{-rs}\kappa I_{j,t}ds + (1-\lambda)\,1_{\{T_\delta > T_\kappa\}}e^{-rT_\kappa}P_{t+T_\kappa} + 1_{\{T_\delta \leq T_\kappa\}}e^{-rT_\delta}P_{t+T_\delta}\right). \tag{A12}$$

Applying (A8) with $\chi = 0$ we have

$$E_t\left(\int_t^{t+T} e^{-rs}ds\right) = E_t\left(\frac{1-e^{-rT}}{r}\right) = \frac{1-\kappa^\alpha}{r+\delta}. \tag{A13}$$

Using Auxiliary result 1 we can see that

$$E_t\left(e^{-rT}1_{\{T_\delta > T_\kappa\}}I_{t+T}\right) = \kappa I_t\int_0^\infty e^{-(r+\delta)\tau}\varphi_\kappa(\tau)d\tau = \kappa^{\alpha+1}I_t. \tag{A14}$$

Finally, we use the following lemma, which we prove a bit later:

Lemma 1

$$E_t\left(e^{-rT}1_{\{T_\delta \leq T_\kappa\}}I_{t+T}\right) = \frac{\delta\left(1-\kappa^{\alpha+1}\right)}{r+\delta}I_t. \tag{A15}$$

Market clearing and (A13), (A14), and (A15) imply that

$$P = \left(\frac{\kappa\left(1-\kappa^\alpha\right)}{r+\delta} + \left(\left(1-\lambda\right)\kappa^{\alpha+1} + \frac{\delta\left(1-\kappa^{\alpha+1}\right)}{r+\delta}\right)P\right)I_t, \tag{A16}$$

or

$$PI_t = \frac{\kappa\left(1-\kappa^\alpha\right)}{r+\delta}I_t + PI_t \times \left(\frac{\delta}{r+\delta} + \frac{\rho r\kappa^{\alpha+1}}{r+\delta}\right). \tag{A17}$$

Solving for P we obtain (2.2). The equilibrium repayment ratio κ must maximize $L_{j,t}$ or in this case P. The equilibrium loan-to-income ratio yields zero profit to banks for

such a κ. From (A16) it is easy to see that the optimal κ is a constant such that

$$(1+\alpha)\left(1-r\rho P\right)\kappa^{\alpha}=1. \tag{A18}$$

Proposition 5 is then obtained by plugging (2.2) into (A18). That (2.1) has a unique solution over $[0,1]$ is easy to see with a monotonicity argument.

Proof of Lemma 1: Let us introduce the family of stopping times $\left(T_{\kappa,\tau}\right)_{\tau\geq 0}=\left(\min(T_{\kappa},\tau)\right)_{\tau\geq 0}$. We have

$$E_t\left(e^{-rT}\mathbf{1}_{\{T_{\kappa}\geq T_{\delta}\}}I_{t+T}\right)=\int_0^{\infty}\delta e^{-\delta\tau}e^{-r\tau}E_t[I_{t+\tau}\mathbf{1}_{\{T_{\kappa,\tau}=\tau\}}]d\tau.$$

Note that

$$\{T_{\kappa,\tau}=\tau\}=\{\min_{0\leq s\leq\tau}W_{t+s}-W_t-\frac{\sigma s}{2}>\frac{\ln(\kappa)}{\sigma}\}\quad a.s.$$

By Girsanov Theorem, $\left(W_s-\frac{\sigma s}{2}\right)_{0\leq s\leq\tau}$ is a standard Brownian motion under the measure Q_{τ} defined as

$$\forall A\in\mathcal{F}_{t+\tau},Q_{\tau}(A)=E_t\left(1_A\times e^{\frac{\sigma}{2}W_{\tau}-\frac{\sigma^2\tau}{8}}\right),$$

and

$$E_t^{Q_{\tau}}\left[I_{t+\tau}|W_{\tau}-\frac{\sigma\tau}{2}=y;\sigma(I_{t+s},0\leq s\leq\tau)\right]=e^{\sigma y}. \tag{A19}$$

By (A4) and (A19) we have

$$E_t\left(I_{t+\tau}\mathbf{1}_{\{T_{\kappa,\tau}=\tau\}}\right)=I_t\int_{\frac{\ln(\kappa)}{\sigma}}^{0}\int_y^{\infty}\frac{2(x-2y)}{\sqrt{2\pi\tau^3}}e^{-\frac{(x-2y)^2}{2\tau}}e^{\sigma x}e^{-\frac{\sigma^2 x}{2}-\frac{\sigma^6\tau}{8}}dxdy$$

$$=I_t\int_{\frac{\ln(\kappa)}{\sigma}}^{0}\int_0^{\infty}\frac{2(x-y)}{\sqrt{2\pi\tau^3}}e^{-\frac{(x-y)^2}{2\tau}}e^{\sigma(x+y)}e^{-\frac{\sigma^2(x+y)}{2}-\frac{\sigma^6\tau}{8}}dxdy.$$

Using (A2) we have

$$\int_0^{\infty}\delta e^{-(\delta+r+\frac{\sigma^6}{8})\tau}\frac{2(x-y)}{\sqrt{2\pi\tau^3}}e^{-\frac{(x-y)^2}{2\tau}}d\tau=2\delta e^{-\sqrt{2(r+\delta)+\frac{\sigma^2}{4}}(x-y)},\quad x\geq y.$$

By Fubini's theorem

$$E_t\left(e^{-rT}\mathbf{1}_{\{T_{\kappa}\geq T_{\delta}\}}I_{t+T}\right)=\delta I_t\int_{\frac{\ln(\kappa)}{\sigma}}^{0}\int_0^{\infty}e^{-\sqrt{2(r+\delta)+\frac{\sigma^2}{4}}(x-y)}e^{\frac{\sigma}{2}(y+x)}dxdy \tag{A20}$$

$$=\frac{\delta I_t}{r+\delta}\left(1-\kappa^{\frac{1}{2}+\sqrt{\frac{2(r+\delta)}{\sigma^2}+\frac{1}{4}}}\right).\blacksquare \tag{A21}$$

3.6 Proof of Proposition 7

Assume that the equilibrium price $(P_t)_{t\geq 0}$ is a positive process that satisfies the transversality condition (2.5). Let κ denote the equilibrium repayment ratio. Market clearing implies

$$P_t = \frac{\kappa\left(1-\kappa^{\alpha}\right)}{r+\delta} I_t + E_t\left[e^{-rT}\left(1-\lambda 1_{\{T_\delta > T_\kappa\}}\right) P_{t+T}\right],$$

where T, T_δ, T_κ are the stopping times introduced in the proof of Proposition 5. Further, we have seen in the proof of Proposition 5 that

$$E_t\left[e^{-rT}\left(1-\lambda 1_{\{T_\delta > T_\kappa\}}\right) I_{t+T}\right] = \frac{r\rho\kappa^{1+\alpha}+\delta}{r+\delta} I_t,$$

so that

$$\begin{aligned} P_t &= \frac{\kappa\left(1-\kappa^{\alpha}\right)}{r+\delta}\left(1+\frac{r\rho\kappa^{1+\alpha}+\delta}{r+\delta}\right) I_t + \\ &\quad E_t\left[e^{-r(T+T')}\left(1-\lambda 1_{\{T_\delta > T_\kappa\}}\right)\left(1-\lambda 1_{\left\{T'_\delta > T'_\kappa\right\}}\right) P_{t+T+T'}\right] \end{aligned}$$

where T', T'_δ, T'_κ are the stopping times associated with the contractual relationship after T, independent from (T, T_δ, T_κ). Clearly, iterating further and applying the transversality condition yields that the price must be linear with respect to I_t. Moreover, Proposition 5 establishes that there is a unique linear equilibrium.■

3.7 Proof of Proposition 8

We introduce the variable $s = r - b$ and stress the dependence of the coefficient α on r with the notation $\alpha(r) = \sqrt{\frac{2(r+\delta)}{\sigma^2} + \frac{1}{4}} - \frac{1}{2}$. Let κ denote the equilibrium repayment ratio. With a price of the form (2.6), the market-clearing condition in the linear case (A17) becomes

$$PI_t + B = \left[\frac{\kappa\left(1-\kappa^{\alpha(r)}\right)}{r+\delta} + P \times \frac{r\rho\kappa^{\alpha(r)+1}+\delta}{r+\delta}\right] I_t + B\left(\frac{s\kappa^{\alpha(s)}+\delta}{s+\delta} - \lambda\kappa^{\alpha(s)}\right). \quad \text{(A22)}$$

This is a straightforward consequence from Lemma A15 and equation (A3). Note that the right-hand side of (A22) is finite only for $b < r + \delta$. An equilibrium price with $B > 0$ requires that κ is such that

$$\frac{s\kappa^{\alpha(s)}+\delta}{s+\delta} = 1 + \lambda\kappa^{\alpha(s)}, \quad \text{(A23)}$$

and also that κ maximizes the right-hand side of (A22) taking all other parameters fixed. It is easy to see that such a κ cannot exist: solutions to equation (A23) do not depend on I_t while a κ that maximizes the right-hand side of (A22) does.■

3.8 Proof of Proposition 9

Proof of Equation (2.8). A price of the form (2.7) clears the market in the presence of such loan contracts if and only if:

$$F(I_t) = E_t\left[\mathbf{1}_{\{T_\kappa \geq T_\delta\}} e^{-rT_\delta} F\left(I_{t+T_\delta}\right) + \lambda \mathbf{1}_{\{T_\kappa < T_\delta\}} e^{-rT_\kappa} F\left(I_{t+T_\kappa}\right)\right].$$

A straightforward modification of (A14) yields:

$$E_t\left[\mathbf{1}_{\{T_\kappa < T_\delta\}} e^{-rT_\delta} F\left(I_{t+T_\delta}\right)\right] = F(\kappa I_t)\kappa^\alpha. \tag{A24}$$

Next, we compute $E_t\left[\mathbf{1}_{\{T_\kappa \geq T_\delta\}} e^{-rT_\delta} F\left(I_{t+T_\delta}\right)\right]$. A straightforward modification of (A20) yields:

$$E_t\left[\mathbf{1}_{\{T_\kappa \geq T_\delta\}} e^{-rT_\delta} F\left(I_{t+T_\delta}\right)\right] = 2\delta \int_{\frac{\ln(\kappa)}{\sigma}}^{0} \int_0^\infty e^{-\sqrt{2(r+\delta)+\frac{1}{4}\sigma^2}(x-y)} F(I_t e^{\sigma(x+y)}) e^{-\frac{1}{2}\sigma(x+y)} dxdy. \tag{A25}$$

Changing variables $w = x + y$ and $u = x - y$, we can rewrite (A25) as

$$E_t\left[\mathbf{1}_{\{T_\kappa \geq T_\delta\}} e^{-rT_\delta} F\left(I_{t+T_\delta}\right)\right] = \delta \left(\begin{array}{c} \int_{\frac{\ln(\kappa)}{\sigma}}^{0} \int_{-w}^{w-2\frac{\ln(\kappa)}{\sigma}} \\ + \int_0^\infty \int_w^{w-2\frac{\ln(\kappa)}{\sigma}} \end{array} \right) F(I_t e^{\sigma w}) e^{-\frac{1}{2}\sigma w} e^{-\sqrt{2(r+\delta)+\frac{1}{4}\sigma^2}u} dudw.$$

And

$$\delta \int_0^\infty \int_w^{w-2\frac{\ln(\kappa)}{\sigma}} F(I_t e^{\sigma w}) e^{-\frac{1}{2}\sigma w} e^{-\sqrt{2(r+\delta)+\frac{1}{4}\sigma^2}u} dudw$$

$$= \frac{2\delta\left(1-\kappa^{2\alpha+1}\right)}{\sigma\left(2\alpha+1\right)} \int_0^\infty F(I_t e^{\sigma w}) e^{-\sigma(1+\alpha)w} dw = \frac{2\delta}{\left(2\alpha+1\right)\sigma^2} \int_1^\infty F(I_t y) y^{-(2+\alpha)} dy. \tag{A26}$$

Finally,

$$\delta \int_{\frac{\ln(\kappa)}{\sigma}}^{0} \int_{-w}^{w-2\frac{\ln(\kappa)}{\sigma}} F(I_t e^{\sigma w}) e^{-\frac{1}{2}\sigma w} e^{-\sqrt{2(r+\delta)+\frac{1}{4}\sigma^2}u} dudw =$$

$$= \frac{2\delta}{\left(2\alpha+1\right)\sigma^2} \int_\kappa^1 F(I_t y)\left(y^{\alpha-1} - y^{-(2+\alpha)}\kappa^{2\alpha+1}\right) dy. \tag{A27}$$

Summing up (A24), (A26), and (A27) yields (2.8).■

Proof of Equation (2.9). A repayment ratio κ maximizes the right-hand side of

(2.8) and thus expected loan repayments κ if and only if

$$\kappa(x) = \operatorname{Arg}\max_{\kappa} \left\{ \begin{array}{c} \frac{1-\kappa^{\alpha}}{1-\rho\kappa^{\alpha+1}} \times \frac{\kappa x}{r} + (1-\lambda)\kappa^{\alpha} F(\kappa x) \\ + \frac{2\delta}{(2\alpha+1)\sigma^2} \left(\begin{array}{c} (1-\kappa^{2\alpha+1}) \int_1^{\infty} F(xy) y^{-(2+\alpha)} dy \\ + \int_{\kappa}^{1} F(xy) \left(y^{\alpha-1} - y^{-(2+\alpha)} \kappa^{2\alpha+1} \right) dy \end{array} \right) \end{array} \right\} \text{ for all } x > 0.$$

The first-order condition associated with this maximization problem is (2.9).■

3.9 Proof of Proposition 10

Assume $\lambda = 1$ and let $\Phi(I) = \frac{F(I)}{I}$ for a candidate solution F. Equation (2.9) is then

$$\frac{1 - (1+\alpha)\kappa_F^{\alpha}(I_t) - \frac{\delta\alpha\kappa_F^{1+\alpha}(I_t)}{r}}{\left(1 + \frac{\delta\kappa_F^{1+\alpha}(I_t)}{r}\right)^2} = \frac{2\delta r}{\sigma^2} \kappa_F^{2\alpha}(I_t) \int_{\kappa_F(I_t)}^{\infty} \Phi(I_t y) y^{-(1+\alpha)} dy. \tag{A28}$$

The left-hand side of (A28) is the derivative with respect to κ of the loan-to-income ratio $\frac{\kappa(1-\kappa^{\alpha})}{1-\rho\kappa^{\alpha+1}}$ that would prevail were the equilibrium linear. The numerator is a decreasing function of κ with one zero. Thus, in the presence of a bubble, it must be that the repayments are smaller than the ones that are optimal with a linear price in order to satisfy (A28). If $F \neq 0$, then $\Phi(I)$ cannot be bounded. Otherwise it would imply that the price satisfies the transversality condition. From (A28), this implies that 0 must be an accumulation point of $\kappa_F(\cdot)$ in $+\infty$. Also, the loan-to-income ratio $\frac{\kappa(I_t)(1-\kappa^{\alpha}(I_t))}{1+\frac{\delta\kappa^{\alpha+1}(I_t)}{r}} + r\Phi(I_t)$ is unbounded.

To establish that if $F(\cdot)$ is increasing and $\alpha \geq 1$, then the repayment ratio $\kappa_F(I_t)$ is decreasing from κ_0 to 0 when I_t goes from zero to infinity, it suffices to notice that the right-hand side of (A28) increases w.r.t. $\kappa_F(I_t)$ for $F(\cdot)$ increasing and $\alpha \geq 1$. To see this, note that

$$\begin{aligned} \frac{\partial}{\partial\kappa} \left(\kappa^{2\alpha} \int_{\kappa}^{\infty} \Phi(I_t y) y^{-(1+\alpha)} dy \right) &= \kappa^{\alpha-1} \left(2\alpha\kappa^{\alpha} \int_{\kappa}^{\infty} \frac{F(I_t y)}{I_t} y^{-(2+\alpha)} dy - \Phi(I_t\kappa) \right) \\ &> \kappa^{\alpha-1} \Phi(I_t\kappa) \left(\frac{2\alpha}{1+\alpha} \kappa^{-1} - 1 \right) \geq 0. \end{aligned}$$

That this result also holds when $\Phi(\cdot)$ is increasing is established similarly.■

References

[1] Heitor Almeida, Murillo Campello, Crocker Liu,"The Financial Accelerator: Evidence from International Housing Markets", The Review of Finance, Vol. 10, No.

3 (2006), pp. 321-352.

[2] Fernando Alvarez, Urban Jermann,"Efficiency, Equilibrium, and Asset Pricing with Risk of Default", Econometrica, Vol. 68, No. 4 (2000), pp. 775-797.

[3] Gary Becker, Casey Mulligan, "The Endogenous Determination of Time Preference", Quarterly Journal of Economics, Vol. 112, No. 3 (1997), pp. 729-758.

[4] Ben Bernanke, Mark Gertler,"Agency Costs, Net Worth, and Business Fluctuations", The American Economic Review, Vol. 79, No. 1 (1989), pp. 14-31.

[5] Olivier Blanchard, Mark Watson, "Bubbles, Rational Expectations and Financial Markets", NBER Working Paper No. W0945, (1982).

[6] Andrei Borodin, Paavo Salminen,"Handbook of Brownian Motion - Facts and Formulae", Second Edition, Birkhauser Verlag AG (2002).

[7] James Durbin, D. Williams,"The First-Passage Density of the Brownian Motion Process to a Curved Boundary", Journal of Applied Probability, Vol. 29, No. 2 (1992), pp. 291-304.

[8] Karen Dynan, Douglas Elmendorf, Daniel Sichel,"The Evolution of Household Income Volatility", Brooking Papers on Economic Activity, Spring (2008).

[9] Frank Fabozzi,"The Handbook of Mortgage-Backed Securities", McGraw-Hill Professional, 6^{th} edition (2006).

[10] Christopher Foote, Kristopher Gerardi, Lorenz Goette, Paul Willen, "Reducing Foreclosures: No Easy Answers", NBER Macroeconomics Annual (2009), forthcoming.

[11] Christopher Foote, Kristopher Gerardi, Lorenz Goette, Paul Willen, "Just the Facts: An Initial Analysis of the Subprime Crisis", Journal of Housing Economics, Vol. 17, No. 4 (2008), pp. 291-305.

[12] Kenneth Froot, Maurice Obstfeld,"Intrinsic Bubbles: The Case of Stock Prices", The American Economic Review, Vol. 81, No. 5 (1991), pp. 1189-1214.

[13] Douglas Gale, Martin Hellwig,"Incentive-Compatible Debt Contracts: The One-Period Problem", The Review of Economic Studies, Vol. 52, No. 4 (1985), pp. 647-663.

[14] Kristopher Gerardi, Harvey Rosen, Paul Willen, "The Impact of Deregulation and Financial Innovation on Consumers: The Case of the Mortgage Market", Journal of Finance, forthcoming.

[15] Gary Gorton,"The Subrpime Panic", NBER working paper #14398 (2008).

[16] Christian Hellwig, Guido Lorenzoni,"Bubbles and Self-Enforcing Debt", Econometrica, Vol. 44, No. 4(2009), pp. 1137-1164.

[17] Timothy J. Kehoe, David K. Levine,"Liquidity Constrained Markets versus Debt Constrained Markets", Econometrica, Vol. 69, No. 3 (2001), pp. 575-598.

[18] Ben Keys, Tanmoy Mukherjee, Amit Seru, Vikrant Vig,"Did Securitization Lead to Lax Screening? Evidence from Subprime Loans", Quarterly Journal of Economics, Vol. 125, No.1, (2010).

[19] Nobuhiro Kiyotaki, John Moore,"Credit Cycles", The Journal of Political Economy, Vol. 105, No. 2 (1997), pp. 211-248.

[20] Dirk Krueger, Harald Uhlig,"Competitive Risk Sharing Contracts with One-Sided Commitment", Journal of Monetary Economics, Vol. 53, No. 7 (2006), pp.1661-1691.

[21] Dirk Krueger, Hanno Lustig, Fabrizio Perri, "Evaluating Asset Pricing Models with Limited Commitment Using Household Consumption Data", Journal of the European Economic Association, Vol. 6, No. 2-3 (2008), pp. 715-726.

[22] Jeffrey Lacker, John Weinberg, "Optimal Contracts under Costly State Falsification", Journal of Political Economy, Vol. 97, No. 6 (1989),pp. 1345-1363.

[23] Owen Lamont, Jeremy Stein,"Leverage and House-Price Dynamics in U.S. Cities", The RAND Journal of Economics, Vol. 30, No. 3 (1999), pp. 498-514.

[24] Atif Mian, Amir Sufi, "The Consequences of Mortgage Credit Expansion: Evidence from the U.S. Mortgage Default Crisis", Quarterly Journal of Economics, Vol. 124, No. 4(2009).

[25] Atif Mian, Amir Sufi, "Securitization, House Prices and the Real Economy: Evidence From the U.S. Mortgage Default Crisis", American Economic Review, Papers and Proceedings (2010).

[26] Francois Ortalo-Magné, Sven Rady,"Housing Market Dynamics: On the Contribution of Income Shocks and Credit Constraints", The Review of Economic Studies, Vol. 73, No. 2 (2006), pp. 459-485.

bibitem Chris Mayer,Tomask Piskorski, Alexei Tchistyi, "The Inefficiency of Refinancing: Why Prepayment Penalties Are Good for Risky Borrowers?", working paper, Columbia Business School (2008).

[27] Tomask Piskorski, Alexei Tchistyi, "Stochastic House Appreciation and Optimal Mortgage Lending", working paper, Columbia Business School (2008).

[28] Tomask Piskorski, Amit Seru, Vikrant Vig, "Securitization and Distressed Loan Renegotiation: Evidence from the Subprime Mortgage Crisis", working paper, Columbia Business School (2008).

[29] Uday Rajan, Amit Seru, Vikrant Vig, "The Failure of Models that Predict Failure: Distance, Incentives and Defaults", working paper, University of Chicago Booth School of Business (2009).

[30] Andrei Shleifer, Robert Vishny,"Liquidation Values and Debt Capacity: A Market Equilibrium Approach", The Journal of Finance, Vol. 47, No. 4 (1992), pp. 1343-1366.

[31] Jeremy Stein,"Prices and Trading Volume in the Housing Market: A Model with Down-Payment Effects", The Quarterly Journal of Economics, Vol. 110, No. 2 (1995), pp. 379-406.

[32] Yeneng Sun, Yongchao Zhang,"Individual risk and Lebesgue Extension Without Aggregate Uncertainty", Journal of Economic Theory, forthcoming.

[33] Robert Townsend,"Optimal Contracts and Competitive Markets with Costly State Verification", Journal of Economic Theory, Vol. 21, No. 2 (1979), pp. 265-293.

[34] Liqun Wang, Klaus Pötzelberger, "Boundary Crossing Probability for Brownian Motion and General Boundaries", Journal of Applied Probability, Vol. 34, No. 1 (1997), pp. 54-65.

[35] Stephen Williamson,"Costly Monitoring, Loan Contracts, and Equilibrium Credit Rationing", The Quarterly Journal of Economics, Vol. 102, No. 1 (1987), pp. 135-145.

Figures

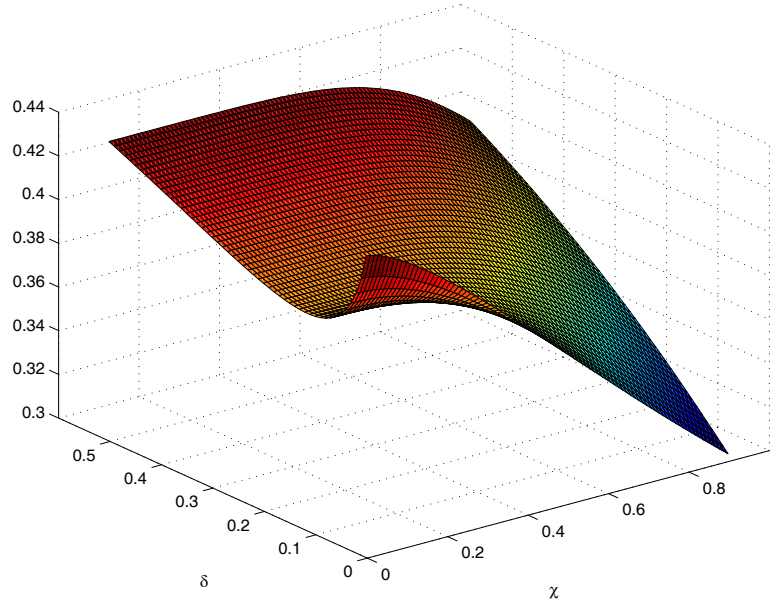


Figure 1: Equilibrium loan-to-income ratio for $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$.

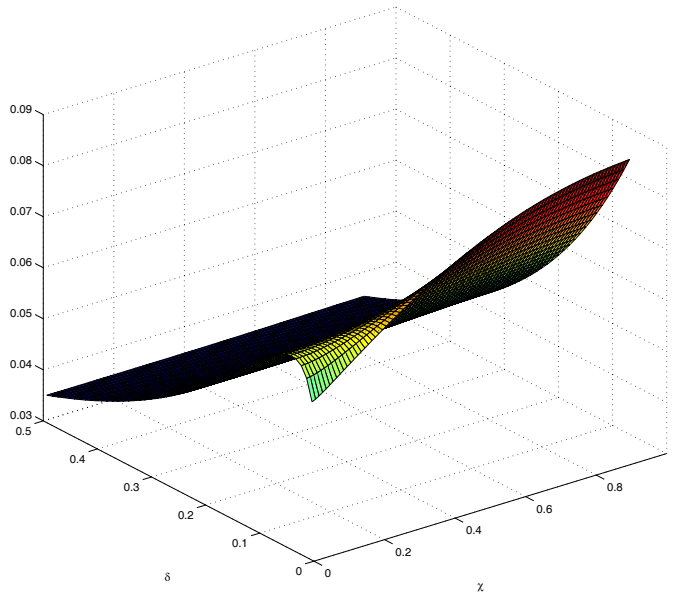


Figure 2: Equilibrium default intensity for $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$.

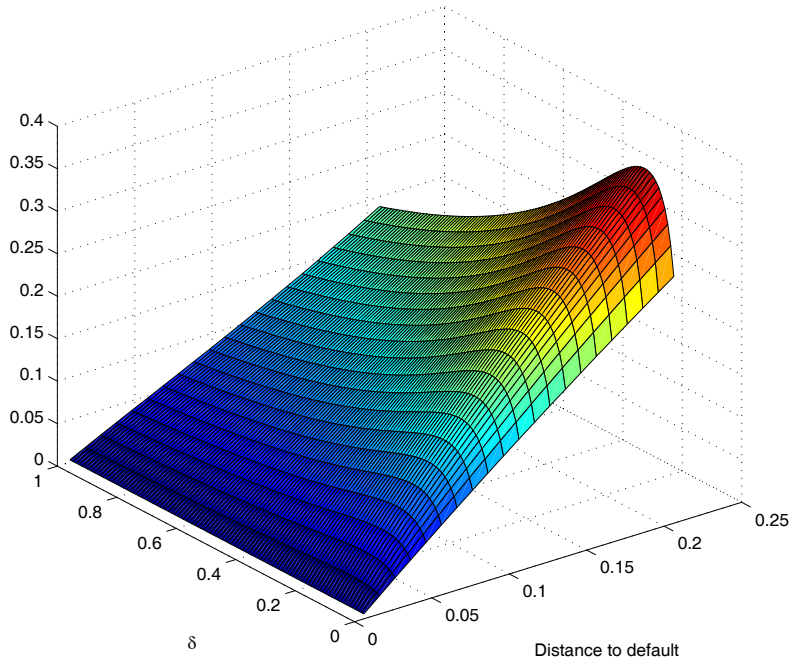


Figure 3: Density of the distance to default for $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$.

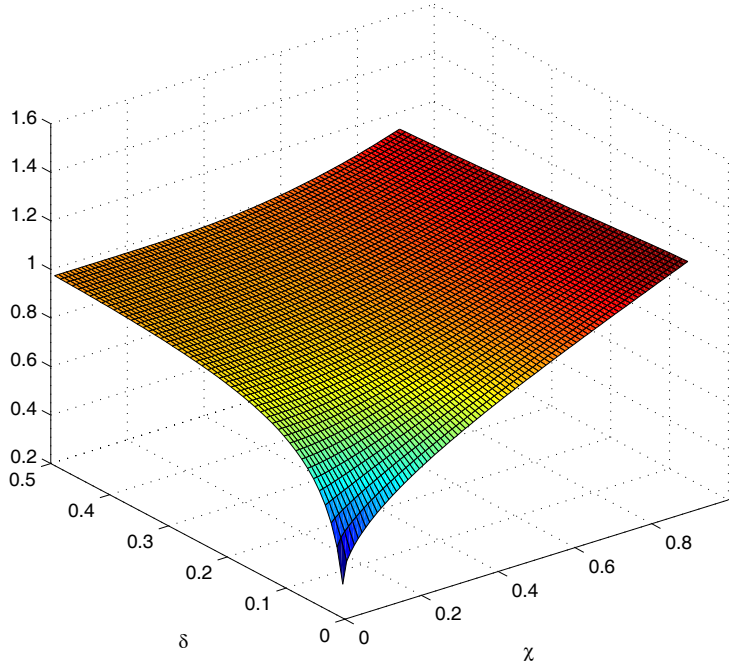


Figure 4: Average income of households in the market for $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$.

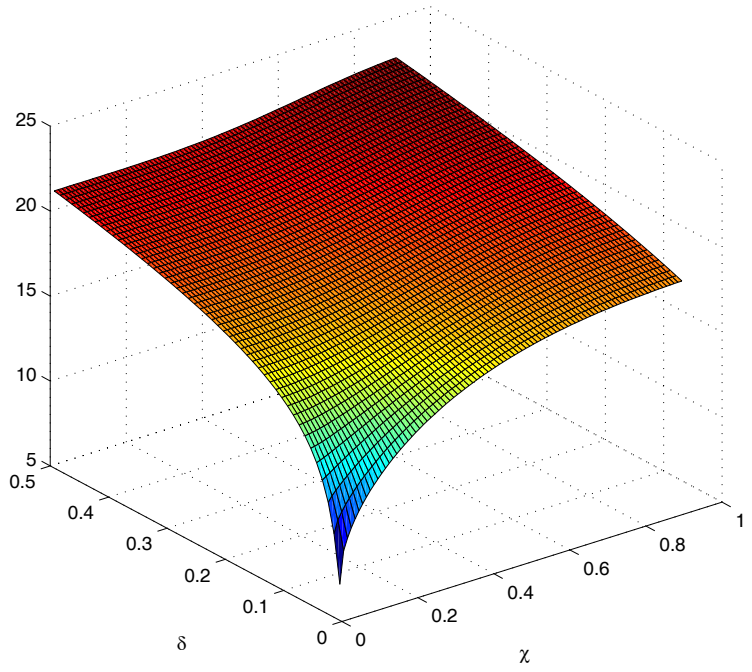


Figure 5: Equilibrium house price for $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$.

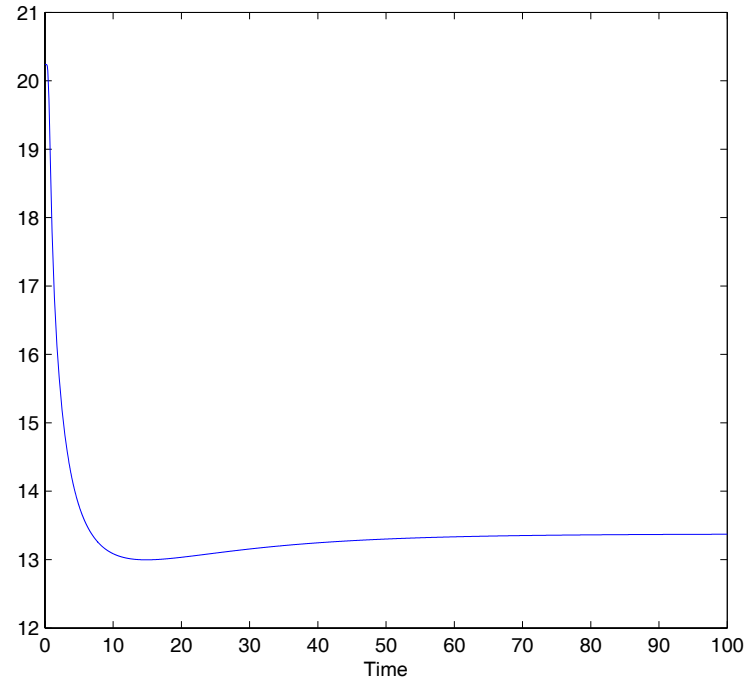


Figure 6: Price dynamics for $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$, $\chi = 0$.

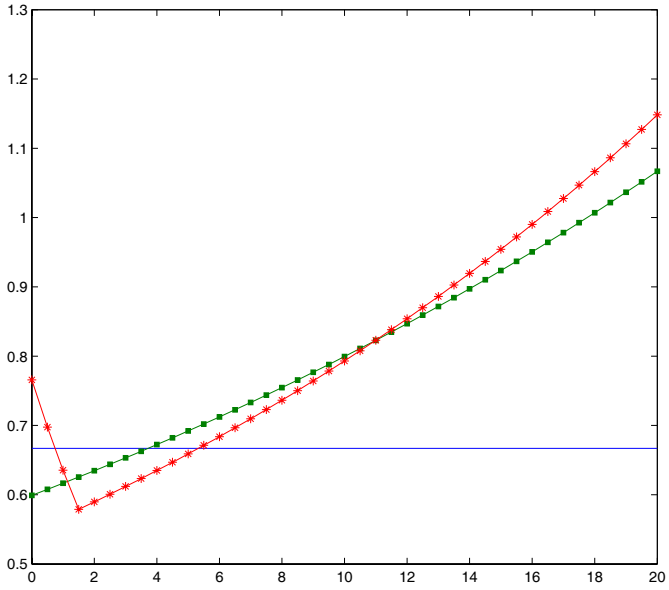


Figure 7: Optimal repayment schedules for $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$.

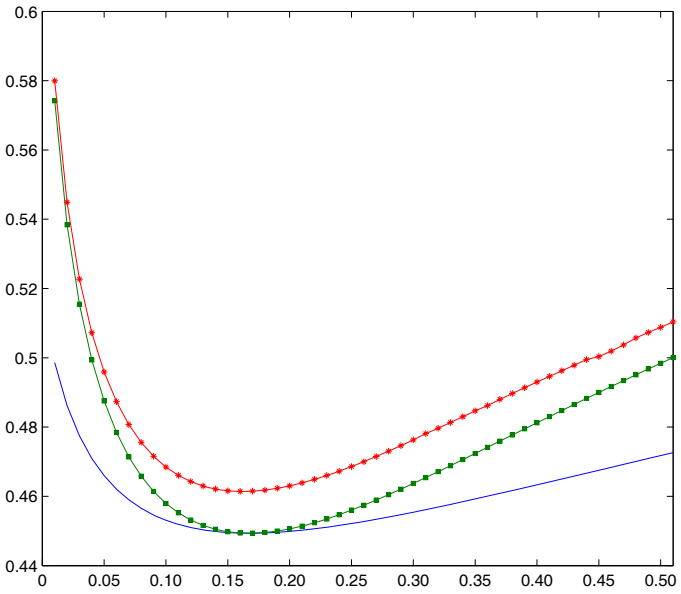


Figure 8: Loan-to-income ration for fixed, log-linear, and piecewise log-linear repayment schedules for $r = 2\%$, $\sigma = 20\%$, $\lambda = 15\%$.