Neoclassical Growth with Long-Term One-Sided Commitment Contracts*

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June 26, 2018

Abstract

This paper characterizes the stationary equilibrium of a continuous-time neoclassical production economy with capital accumulation in which households seek to insure against idiosyncratic income risk through long-term insurance contracts. Insurance companies operating in perfectly competitive markets can commit to future contractual obligations, whereas households cannot. For the case in which household income takes two values, one of which is zero, and where households have CRRA preferences we provide a complete analytical characterization of the optimal consumption insurance contract as well as the stationary consumption distribution. Under parameter restrictions, we show that there is a unique stationary equilibrium with partial consumption insurance if households have log-preferences, and that there are two equilibria with non-log CRRA preferences. We also demonstrate analytically that the stationary consumption distribution has a Pareto form, truncated by a lower and an upper mass point. For the logarithmic case the unique equilibrium interest rate (capital stock) is strictly decreasing (increasing) in income risk. Thus the paper provides an analytically tractable alternative to the standard incomplete markets general equilibrium model developed in Aiyagari (1994) by retaining its physical structure, but substituting the

*We thank Leo Kaas and Matt Rognlie, as well as seminar participants at Minnesota, Princeton, UCL and the Wharton Macro lunch for useful comments and suggestions, and the National Science Foundation as well as the the MFM group of the Becker-Friedman Institute for generous financial support.

assumed incomplete asset markets structure with one in which limits to consumption insurance emerge endogenously, as in Krueger and Uhlig (2006). Thus our model provides an alternative, analytically tractable stationary general equilibrium model with idiosyncratic income income risk

1 Introduction

This paper develops a new, analytically tractable general equilibrium macroeconomic model with idiosyncratic income risk and endogenous financial contracts, and therefore, inequality in household incomes, consumption and wealth. To do so, long-term contractual arrangements between risk-averse households and risk-neutral competitive financial intermediaries are embedded into a stationary version of the neoclassical growth model. We seek to integrate two foundational strands of the literature on macroeconomics with household heterogeneity. The first strand has developed and applied the standard incomplete markets model with uninsurable idiosyncratic income shocks and neoclassical production, as Bewley (1986), Imrohoroglu (1989), Huggett (1993) and Aiyagari (1994). In that model, households can trade assets to self-insure against income fluctuations, but these assets are not permitted to pay out contingent on a household's individual income realization, thereby ruling out explicit insurance against income risk.

The second branch is the broad literature on recursive contracts and endogenously incomplete markets which permits explicit insurance, but whose scope is limited by informational or contract enforcement frictions. More specifically, we incorporate dynamic insurance contracts offered by competitive financial intermediaries (as analyzed previously in Krueger and Uhlig, 2006) into a neoclassical production economy. Financial intermediaries can commit to long term financial contracts, whereas households can not. The project thereby seeks to provide the macroeconomics profession with a novel, fully micro founded, analytically tractable model of neoclassical investment, production and the cross-sectional consumption and wealth distribution, where the limits to cross-insurance are explicitly derived from first principles of contractual frictions.

We aim to make two contributions, one substantive and one methodological in nature. On the substantive side, we provide a model that links the accumulation of the aggregate capital stock in the economy to the extent to which financial intermediaries can provide insurance to households against their idiosyncratic income risk, and their resulting demand for assets fund their contractual obligations. On the methodological side we construct and

analytically (as well as numerically) characterize a dynamic optimal insurance model with one-sided limited commitment and production as well as capital accumulation.

In a seminal paper, Aiyagari (1994) analyzed an economy in which households self-insure against idiosyncratic income fluctuations by purchasing shares of the aggregate capital stock. His model has become the canonical macro model with household heterogeneity. Variants of the model differ in the precise set of assets that households can trade, but the common assumption is that agents do not have access to financial instruments that provide direct insurance against the idiosyncratic income risk, despite the fact that such insurance would be mutually beneficial, given the underlying physical environment. There is now a large literature building on that model to link microeconomic inequality to macroeconomic performance, including applied policy (reform) analysis.[1] Any analysis of welfare in such models then necessarily comes with the caveat, that households may already be able to do better for themselves, if only the model builder allowed them to do so. As parameters or policies change, one may be concerned that these missing gains from trade shift too. Alternative general equilibrium workhorse models are therefore needed, in which households are allowed to pursue all contractual possibilities, limited only by informational or commitment constraints. The purpose of this paper to provide such an alternative model.

The contractual friction in our model arises from the inability of households to commit to future obligations implied by full insurance, risk sharing contracts. More precisely, we postulate financial markets in which perfectly intermediaries offer long-term insurance contracts to households. These financial intermediaries receive all incomes from a customer that has signed a contract, and can commit perfectly to future state-contingent consumption payments. Competition among intermediaries implies that the present discounted value of profits from these contracts is zero at the time of contract signing. The crucial friction that prevents perfect consumption insurance in the model is that *households,* at any moment, can costlessly switch to another intermediary, signing a new contract there. That is, we model relationships between financial intermediaries and private households as long-term contracts with one-sided limited commitment (the intermediary is fully committed, the household is not). This structure of financial markets is identical to the one assumed in the discrete-time, partial equilibrium model of Krueger and Uhlig (2006), which in turn builds on the seminal work of Harris and Holmstrom (1986), Thomas and Worrall (1989), Kehoe and Levine (1993) and Kocherlakota (1996).

In that paper, we showed that the one-sided limited commitment induces contracts with

[1] See recent surveys by Heathcote, Storesletten and Violante (2011) and Krueger, Mitman and Perri (2016).

payments from the household to the intermediary that are front-loaded: when income is high, the household effectively builds up a stock of savings with the intermediary, which then finances the insurance offered by the intermediary against low income realizations down the road. In this paper we embed these contracts into a dynamic production economy, as in Aiyagari (1994). Now these contractual savings implied by back-loaded insurance contracts finance the aggregate capital stock of the economy. Effectively, financial intermediaries buy shares of the capital stock to fund their future liabilities from the insurance contracts they have signed with households. As in a standard neoclassical growth model, aggregate capital itself is accumulated linearly and used together with inelastically supplied in an aggregate Cobb-Douglas production function by a competitive sector of production firms.

Households supply their labor inelastically to these firms, but as in Bewley (1986), Imrohoroglu (1989), Huggett (1993) and Aiyagari (1994) their labor productivity and thus earnings are subject to idiosyncratic risk. This risk induces household insurance needs and thus generates a savings motive, which in turn finances the capital stock. Our model therefore provides a third (and intermediate) alternative neoclassical production economy with capital, relative to the self-insurance framework of Aiyagari (1994) and the full insurance framework (a.k.a. the standard neoclassical growth model with complete markets and implied full consumption insurance).

As a methodological innovation to the limited commitment general equilibrium literature we describe our model in continuous time. This is useful since, as we demonstrate, the optimal insurance contract is akin to an optimal stopping problem, and the use of continuous time avoids integer problems (the optimal stopping time falling in between two period) that arise in a discrete time setting. Households are potentially infinitely lived, but face a positive and constant probability of dying in every period. To keep the population constant, at every instant a new mass of households without assets is born.[2] Households are assumed to have CRRA utility, and, in order to obtain a sharp analytical characterization of the equilibrium we focus on the case where household income takes two values, one of which is zero. For this case, we provide a complete analytical characterization of the optimal consumption insurance contract as well as the stationary consumption distribution. Under mild restrictions on the parameters, we show that there is a unique equilibrium

[2]This model element is not needed for the production economy, but allows us to study the endowment economy version of the model as a limiting case, and to potentially obtain non-trivial equilibria even in this case. In Krueger and Uhlig (2006) we showed that absent this model feature the only stationary equilibrium is autarkic.

if households have log-preferences, and that there are two equilibria with non-log CRRA preferences. We provide explicit formulas for the calculation of these equilibria, including the steady state level and return on the capital stock. We can also analytically calculate the stationary consumption distribution, and show that this distribution has a Pareto in shape, truncated by a lower and an upper mass point. Comparative statics with respect to the deep parameters of the model (and specifically, the parameters determining income risk, preferences and production technologies) deliver unambiguous results for the unique equilibrium in the log-utility case. We submit that this full analytical characterization of a stationary equilibrium is an additional, attractive benefit of our model, and a welcome methodological advance, noting that Aiyagari-type models (as standard limited commitment economies with a continuum of households, as in Krueger and Perri, 2006) typically require numerical solutions. We therefore hope that our model structure can serve as an analytically tractable framework for applied work in macroeconomics that connects idiosyncratic risk to aggregate phenomena such as consumption distributions and the size of the economy.

1.1 Relation to the Literature

As discussed above, our broad aim in this paper is to connect the dynamic contracting literature with income risk and limited commitment to the quantitative, general equilibrium literature in macroeconomics with household heterogeneity discussed above. Our dynamic limited commitment risk sharing contract model builds on the theoretical work characterizing optimal contracts in such environments. Especially relevant is the subset of the literature that has done so in continuous time.

Specifically, Zhang (2013) studies a consumption insurance model with limited commitment similar to that in Krueger and Uhlig (2006), but permits the income process to be serially correlated finite state Markov chain, rather than a sequence of iid random variables. He also allows the outside option of the household to be a general function of the current income state, rather than simply autarky. The author derives the optimal consumption insurance contract. Grochulski and Zhang (2012) characterize the optimal contract in continuous time, under the assumption that the market return equals the discount rate, the outside option is autarky, and the income process follows a general geometric Brownian motion. The work by Miao and Zhang (2013) contains related results. Turning to general equilibrium treatments, Hellwig and Lorenzoni consider an endowment economy, in which two agents optimally share their risky income stream over time, subject to contrac-

tual constraints. The market return in their economy is shown to be zero under appropriate assumptions.

Returning to the substantive contribution of the paper, one interpretation of the contractual arrangements is that of firms who provide workers with long-term employment-wage contracts. A recent literature, building on the early work of Harris and Holmstrom (1986), emphasizes that firms provide insurance to its workers and their productivity fluctuations. Lamadon (2016) has calculated the optimal within-firm insurance mechanism, in the presence of a variety of sources of risk, including firm-specific risk, worker productivity risk and unobservable effort. Guiso, Pistaferri and Schivardi (2005) also argue, empirically, that the insurance of worker productivity by firms is an important mechanism to insulate workers from idiosyncratic shocks. Finally, Saporta-Eksten (2014) has shown that wages are lower after a spell of unemployment, which he interprets as a loss in productivity. In the context of our model this observation can alternatively be rationalized as part of the optimal consumption insurance contract, in the event productivity of the worker has dropped temporarily.

2 The Model

2.1 Preferences and Endowments

Time is continuous. There is a population of a continuum of agents of mass 1, who die at rate $\gamma > 0$, replaced by a generation of newborns of equal size. Thus at any point in time a unit mass of households are alive in this economy. Agents have the period utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

and discount the future at rate $\rho > 0$, so that the expected utility, including the probability of death, of a household born in period t is given by

$$E\left[\int_t^\infty e^{-(\rho+\gamma)(\tau-t)} \frac{c(\tau)^{1-\sigma}}{1-\sigma} d\tau\right].$$

Labor productivity y_{it} of an individual agent i at time t is assumed to follow two-state Markov process that is independent across agents. More precisely, it can either be high, $y_{it} = y_h$ or low, $y_{it} = y_l$, with $y_h > y_l = 0$. Let $Y = \{y_l, y_h\}$. The transition from

high to low productivity occurs at rate $\pi > 0$, whereas the transition from low to high productivity occurs at rate $\nu > 0$. Households with low labor productivity $y_l = 0$ also have some nontradable endowment $\chi > 0$ that they can consume if they do not sign up for a consumption risk sharing contract.[3] Denote the utility from consuming the nontradable endowment by $\underline{u} = u(\chi) > -\infty$.

Given the stochastic structure of the endowment process the share of households with low and high income is equal to

$$(\psi_l, \psi_h) = \left(\frac{\pi}{\pi + \nu}, \frac{\nu}{\pi + \nu} \right)$$

We assume that newborn households draw their productivity from the stationary income distribution and that the average labor productivity in the economy is equal to 1. Thus we assume that

$$\frac{\nu}{\pi + \nu} y_h = 1.$$

For future reference we note that this assumption implies

$$\nu(y_h - 1) = \pi \tag{1}$$

2.2 Technology

There is a competitive sector of production firms which uses labor and potentially capital to produce the final output good according to the production function

$$F(K, L) = AK^{\theta} L^{1-\theta}.$$

where $\theta \in (0, 1)$ denotes the capital share. The capital depreciates at a constant rate $\delta \geq 0$. In our companion paper, Krueger and Uhlig (2017), we also consider the case $A = 1$ and $\theta = 0$ and $\delta = 0$, in which case our economy boils down to an endowment economy. Production firms seek to maximize profits, taking as given the market spot wage w per efficiency unit of labor and the market rental rate per unit of capital. Capital accumulation is linear and depreciates at rate δ. There is a resulting equilibrium rate of return or interest rate r for investing in capital. We dropped the subscript t to economize on notation, since

[3] This assumption avoids the complication that individuals that have not yet received the high income realization at least once and thus won't be provided with consumption insurance (as we will so) are forced to consume 0. For $\sigma < 1$, this assumption would be unnecessary since $u(0) < \infty$, but it is required for $\sigma \geq 1$.

we shall concern ourselves only with stationary equilibria and aggregate variables will be constant.

There is a competitive sector of intermediaries, who seek to maximize profits. Agents attempt to insure themselves against these income fluctuations with financial intermediaries. However, the commitment is one-sided only: while the intermediary can commit to the contract for all future, agents can leave the contract at any time they please and sign up with the next intermediary. Intermediaries compete for agents, and do not have resources on their own. Similar to Krueger-Uhlig (2006), newborn agents wait until their first time that they receive the high income. They then provide their chosen intermediary with a stream of "insurance premium payments", while in the high income state, to finance subsequent payments for a potential "dry spell" of low productivity, until they transit to high income again. We assume that the law of large numbers applies at each individual intermediary or, alternatively, that there is full mutual insurance among intermediaries, so that intermediaries are only exposed to aggregate risk. We only examine stationary equilibria in this paper, thus rendering these intermediaries risk neutral. The intermediaries invest the premium payments in capital and therefore discount future streams of payments and incomes at the rate of return r on capital.

2.3 Timing of Events

In each instant of time, birth and death occur first. A newborn household draws labor productivity y from the stationary income distribution and then signs a long-term consumption insurance contract with one of the many competing financial intermediaries, delivering lifetime utility $U^{out}(y)$. For surviving households, the current labor productivity y is realized from the household-level Markov process. The household has the option of sticking with the previous intermediary or signing up with another intermediary, in the latter case receiving a contract delivering lifetime utility $U^{out}(y)$. Consumption is then allocated to the household according to the consumption insurance contract this household has signed in the past.

2.4 Equilibrium

Intermediary contracts promise some lifetime utility U for the household per delivering a stochastic stream of future consumption. Given U and given the current labor productivity y of the household, the profit maximization objective of intermediaries is equivalent to

minimizing the net present value $V(y,U)$ of the contract costs, i.e. to minimize the net present value of the difference between the the stream of consumption of the household and its income. The income is given by the labor productivity $y(\tau)$ at future dates τ multiplied with the wage w. It will likewise be convenient to scale consumption by the wage level. In slight abuse of notation, let $c(\tau)w$ denote the consumption of the household at date τ. In designing the contract, the intermediary needs to take into account that the household will depart, should the residual lifetime utility drop below promises the outside option $U^{out}(y)$ of promises, available when signing a new contract with some other intermediary.

Definition 1 *For fixed outside options $U^{out}(y)$, with $y \in Y$, and a fixed wage w and rate of return on capital or interest rate r, an optimal consumption insurance contract $c(\tau; y, U)$, $V(y,U)$ solves*

$$V(y,U) = \min_{\langle c(\tau)\rangle \geq 0} \mathbf{E}\left[\int_t^\infty e^{-(r+\gamma)(\tau-t)}\left[wc(\tau) - wy(\tau)\right]d\tau \middle| y(t) = y\right]$$

subject to

$$\begin{aligned}
\mathbf{E}\left[\int_t^\infty e^{-(\rho+\gamma)(\tau-t)}u(c(\tau))d\tau \middle| y(t) = y\right] &\geq U \\
\mathbf{E}\left[\int_s^\infty e^{-(\rho+\gamma)(\tau-s)}u(c(\tau))d\tau \middle| y(s)\right] &\geq U^{out}(y(s)) \text{ for all } s > t
\end{aligned}$$

for all $t \geq 0$, for all $y \in Y$ and all $U \in \left[U^{out}(y), \frac{\bar{u}}{\rho+\gamma}\right)$.

Note that the stationary structure of the model insures that the optimal consumption insurance contract does not depend on calendar time, but rather only on the income y the household is born with.

Definition 2 *A Stationary Equilibrium consists of outside options $\{U^{out}(y)\}_{y\in Y}$, consumption insurance contracts $c(\tau, y, U) : \mathbf{R}_+ \times Y \times \left[U^{out}(y), \frac{\bar{u}}{\rho+\gamma}\right) \to \mathbf{R}_+$ and $V : Y \times \left[U^{out}(y), \frac{\bar{u}}{\rho+\gamma}\right) \to \mathbf{R}$, an equilibrium wage w and interest rate r and a stationary consumption probability density function $\phi(c)$ such that*

1. *Given $\{U^{out}(y)\}_{y\in Y}$ and r, the consumption insurance contract $c(\tau, y, U), V(y,U)$ is optimal in the sense of definition (1).*

2. *The outside options lead to zero profits of the financial intermediaries: for all* $y \in Y$

$$V(y, U^{out}(y)) = 0.$$

3. *The interest rate and wage* (r, w) *satisfy*

$$r = AF_K(K, 1) - \delta \tag{2}$$
$$w = AF_L(K, 1) \tag{3}$$

4. *The goods market clears*

$$\int wc\phi(c)dc + \delta K = AF(K, 1). \tag{4}$$

5. *The capital market clears*

$$\frac{w\left[\int c\phi(c)dc - 1\right]}{r} = K \tag{5}$$

6. *The stationary consumption probability density function is consistent with the dynamics of the optimal consumption contract as well as the stochastic structure of birth and death in the model.*

2.4.1 Discussion of Equilibrium Definition[4]

Several elements of this definition are noteworthy. The first two items are identical to Krueger and Uhlig (2006), accounting for the fact that the current model is cast in continuous time. Whereas item 3 contains the standard optimality conditions of the representative production firm, the statement of the capital market clearing condition in 5, as well as the inclusion of both the goods market clearing and the capital market clearing condition require further discussion.

In item 5, the right hand side $K = K^d$ is the demand for capital by the representative firm. The numerator on the left hand side is the excess consumption, relative to labor

[4]We thank Matt Rognlie for very helpful discussions leading to this section. Auclert and Rognlie (2016) argue that the same argument applies to the standard incomplete markets model (as originally described in Aiyagari, 1994). Our discussion here is simply an adaptation of their argument to our environment.

income, of all households, that is, the capital income required to finance the consumption that exceeds labor income. Dividing by the return to capital r gives the capital stock households, or financial intermediaries on behalf of households, need to own to deliver the required capital income. Thus we can think of

$$K^s = \frac{w\left[\int c\phi(c)dc - 1\right]}{r} \tag{6}$$

as the supply of capital by the household sector, intermediated by the financial intermediaries. By restating the capital market clearing condition as

$$K^s(r) = K^d(r)$$

where $K^s(r)$ is defined in (6) and $K^d(r)$ is defined through (2) we will be able to provide a graphical analysis of existence and uniqueness of stationary equilibrium in (K, r) space, analogously to the well-known figure contained in Aiyagari (1994) for the standard incomplete markets model.

Finally, we note that as long as $r \neq 0$, the usual logic of Walras law applies and one of the two market clearing conditions is redundant. To see this, note that the right hand side of equation (4) can be written as

$$AF(K,1) = AF_L(K,1) + AF_K(K,1)K$$

and from equations (2) and (3) it follows that

$$AF(K,1) = w + (r+\delta)K.$$

Using this in equation (4) and rearranging implies, for $r \neq 0$, the capital market clearing condition (5). Thus for all $r \neq 0$ we can use either of the market clearing conditions in our analysis. The case $r = 0$, however, will require special attention, and we will argue in section 5 that even though the goods market clears for $r = 0$ under fairly general conditions, the capital market generically does not, indicating that a) $r = 0$ is generically not a stationary equilibrium interest rate and b) at $r = 0$ we need to study both the goods and the capital market clearing condition when analyzing a stationary equilibrium.

In order to do so, in the next sections we now aim to characterize the entire steady state equilibrium, including the stationary consumption distribution whose cumulative dis-

tribution function we denote by Φ (with associated probability density function ϕ). First we characterize the optimal consumption contract under various assumptions on the relationship between the constant interest rate r and the constant time discount rate ρ of the household. Then we discuss aggregation and the equilibrium determination of interest rates.

3 The Optimal Risk-Sharing Contract

The nature of the optimal consumption insurance contract depends crucially on the relationship between the subjective time discount factor ρ and the endogenous stationary equilibrium interest rate r. We discuss the relevant cases in turn. First we discuss the case $r = \rho$ which will deliver a sharp and very simple characterization of the optimal consumption contract that features full consumption insurance of the household after the first instance of having received high income. We then analyze the case $r < \rho$ which will result in a partial consumption insurance, the relevant case for the general equilibrium in a wide range of model parameterizations. We conclude with a discussion why, in continuous time, the autarkic allocation can never be an equilibrium, in contrast to the situation in discrete time.

3.1 Full Insurance in the Long Run: $\rho = r$

We first characterize the optimal consumption insurance contract for $(y_l, U^{out}(y))$ and then discuss how it looks for other promised lifetime $U > U^{out}(y))$. We conjecture that the consumption contract has constant consumption $c_l = 0$ as long as $y(t) = y_l = 0$, and then consumption jumps up to c_h the instant income rises to y_h and remains there forever. Households born with income y_h instead consume c_h forever. As shorthand, denote as

$$\begin{aligned} V_l &= V(y_l, U^{out}(y_l)) \\ V_h &= V(y_h, U^{out}(y_h)) \end{aligned}$$

and let V_{hl} denote the cost of a contract for the financial intermediary in which the household had high income in some previous periods (and thus currently consumes c_h) but now has low income y_l. In what follows we characterize the net cost of this contract deflated by the wage level, and the let the wage-deflated cost be denoted by $v = V/w$.

These cost levels of the financial intermediary satisfy the Hamilton-Jacoby-Bellman

equations

$$\begin{aligned} rv_l &= c_l - y_l + \gamma(0 - v_l) + \nu(v_h - v_l) \\ rv_h &= c_h - y_h + \gamma(0 - v_h) + \pi(v_{hl} - v_h) \\ rv_{hl} &= c_h - y_l + \gamma(0 - v_{hl}) + \nu(v_h - v_{hl}) \end{aligned}$$

Due to perfect competition of financial intermediaries we have that $v_h = v_l = 0$. Using this in these equations yields

$$\begin{aligned} c_l &= y_l = 0 \\ \pi v_{hl} &= y_h - c_h \\ (r + \gamma + \nu)v_{hl} &= c_h \end{aligned} \tag{7}$$

and, solving the last two equations explicitly, and evaluating at $r = \rho$, delivers

$$\begin{aligned} c_h &= \frac{\rho + \gamma + \nu}{\rho + \gamma + \nu + \pi} y_h = c_h(\rho) \\ v_{hl} &= \frac{c_h}{r + \gamma + \nu} > 0 \end{aligned} \tag{8}$$

Thus the optimal risk sharing contract collects a net insurance premium

$$y_h - c_h = \frac{\pi y_h}{\rho + \gamma + \nu + \pi}$$

from households with high income realizations and uses it to pay consumption insurance

$$c_h = \frac{(\rho + \gamma + \nu)\, y_h}{\rho + \gamma + \nu + \pi}$$

to those households that have obtained insurance (those with previously high income realizations) and have currently low income.

3.2 Partial Insurance: $\rho > r$

We denote the expected discounted net cost to the financial intermediary from a consumption contract by v. With two income levels y_l and y_h and Poisson arrival probabilities of switching down (π) and up (ν) as well as constant death probability γ we can write the

consumption dynamics and profit value functions as follows. The consumption dynamics is such that whenever a household has the high income, she consumes c_h, and when income switches to y_l consumption drifts down according to the full insurance Euler equation

$$\frac{\dot{c}(t)}{c(t)} = -\frac{\rho - r}{\sigma} = -g < 0$$

where we have defined the growth rate of consumption as

$$g = \frac{\rho - r}{\sigma} > 0.$$

This consumption dynamics implies that

$$c(t) = c_h e^{-gt} \tag{9}$$

Furthermore, by perfect competition expected profits when entering the consumption contract with high income, $v_h = 0$, and similarly for entering the consumption contract with low income, $v_l = 0$.

Denote by t the time elapsed since having had the high income. Asymptotically, as $t \to \infty$, consumption converges to $c_l = 0$. The Hamilton-Jacobi-Bellman equations read as

$$\begin{aligned} rv_h &= c_h - y_h + \gamma(0 - v_h) + \pi(v(0) - v_h) && (10)\\ rv_l &= c_l + \gamma(0 - v_l) + \nu(v_h - v_l) && (11)\\ rv(t) &= c(t) - y_l + \gamma(0 - v(t)) + \nu(v_h - v(t)) + \dot{v}(t) && (12) \end{aligned}$$

with terminal condition

$$v(\infty) = v_l = 0.$$

Here $v(t)$ is the cost of the consumption contract of an agent that had high income last t periods ago, and had low income for time interval t since.

Simplifying equations (10) to (12) delivers

$$\begin{aligned} \pi v(0) &= y_h - c_h && (13) \\ c_l &= y_l = 0 && (14) \\ (r+\gamma+\nu)v(t) &= c(t) - y_l + \dot{v}(t) && (15) \end{aligned}$$

The first equation states that in the case of high income the household pays an insurance premium $y_h - c_h$ which has to compensate the financial intermediary for the cost incurred during the low spell in which the losses for the intermediary amount to $v(0)$. This equation relates the two endogenous variables c_h and $v(0)$ to each other.

The second equation directly yields

$$c_l = y_l = 0$$

and as before, individuals with initially low income do not obtain any consumption insurance in the risk sharing contract. Insurance would require prepayment by the insurance company, and perfect competition plus limited commitment on the household side imply that this prepayment cannot be recouped later.

Equation (15) can be integrated (for details see appendix A.1), using the fact that $c(t) = c_h e^{-gt}$ to obtain

$$\begin{aligned} v(t) &= \int_t^\infty e^{-(r+\gamma+\nu)(\tau-t)} c_h e^{-g\tau} d\tau \\ &= c_h e^{-gt} \int_t^\infty e^{-(r+\gamma+\nu+g)(\tau-t)} d\tau \\ &= \frac{c_h e^{-gt}}{r+\gamma+\nu+g} \end{aligned} \qquad (16)$$

We can evaluate (16) at $t = 0$ to obtain:

$$v(0) = \frac{c_h}{r+\gamma+\nu+g} \qquad (17)$$

The optimal consumption contract has consumption driving down at rate $-g = \frac{r-\rho}{\sigma}$ from c_h towards $c_l = y_l$, and asymptotically it reaches $y_l = c_l = 0$. Thus the consumption level c_h fully characterizes the consumption contract

Using equation (13) to substitute out $v(0)$ in equation (17) yields

$$\frac{c_h}{r+\gamma+\nu+g} = \frac{y_h - c_h}{\pi}$$

or

$$c_h(r) = \frac{r+\gamma+\nu+g}{r+\gamma+\nu+g+\pi} y_h = \frac{1}{1+\frac{\pi}{r\left(1-\frac{1}{\sigma}\right)+\gamma+\nu+\frac{\rho}{\sigma}}} y_h \tag{18}$$

We summarize the optimal consumption contract in the following

Proposition 1 *As long as $\rho > r$, there exists a unique consumption level $c_h(r)$, defined in (18), characterizing the optimal consumption risk sharing contract*

$$c(t) = c_h e^{-gt}$$

where $g = \frac{\rho - r}{\sigma}$. Households that never have had high income would consume

$$c_l = y_l = 0$$

until the first time they receive high income if they were to sign a consumption risk sharing contract. The upper bound is strictly increasing in y_h.

Corollary 1 *The upper support of the consumption distribution c_h increases in the interest rate for $\sigma > 1$, decreases in the interest rate if $\sigma < 1$ and is independent of the interest rate if $\sigma = 1$.*

It is noteworthy that the effect of an increase of the interest rate on the initial level of consumption in the contract depends on the $IES = \frac{1}{\sigma}$.

Corollary 2

$$\lim_{r \nearrow \rho} c_h(r) = \frac{\rho+\gamma+\nu}{\rho+\gamma+\nu+\pi} y_h = c_h(\rho)$$

where $c_h(\rho)$ is the full insurance consumption level.

3.3 Super-Insurance: $\rho < r$

In our model stationary equilibria with an interest rate exceeding the time preference rate are a possibility, in contrast to the standard incomplete markets model with infinitely lived

households.[5] Therefore we now characterize the optimal consumption insurance contract under the assumption that the interest rate exceeds the time discount factor.

Off constraints, as in the partial insurance case consumption grows at a constant rate,

$$c_h(t) = c_h(0)e^{-gt}$$

but now $g = \frac{1}{\sigma}(\rho - r) < 0$, that is, consumption grows at the positive rate $\frac{1}{\sigma}(r - \rho) > 0$. As in the full and partial insurance case, households born with the low income cannot obtain insurance until their income switches to y_h, at which it jumps to $c_h(0)$, as in the partial and full insurance case. From that point on the household obtains income insurance (as in the full insurance case), but now consumption grows at rate $-g$ (rather than remains constant), until the household dies. The level $c_h(0)$ is determined by the zero profit condition of the intermediary, equating the expected revenue from the household's income stream with the expected cost of the consumption contract. In Appendix A.4 we exploit the zero profit condition to determine the entry level of consumption $c_h(0)$ as

$$c_h(0) = \frac{(r + \gamma + g)}{(r + \gamma)} \cdot \frac{(r + \gamma + \nu)}{(r + \gamma + \nu + \pi)} y_h \tag{19}$$

As in the full insurance case, upon income increasing, so does consumption, but not as strongly as income. The household pays an insurance premium $y_h - c_h(0)$ in exchange for future consumption insurance and consumption growth. Note that since $g < 0$, the insurance premium is larger than in the full insurance case to finance future consumption growth, and as the interest rate r converges to the time discount rate ρ from above, the entry consumption level as well as the insurance premium converge to the full insurance consumption level from below.

4 The Invariant Consumption Distribution

In the previous section we have shown that the optimal consumption insurance contract depends on the relationship between the endogenous market interest rate r and the subjective discount factor ρ, which determines whether the contract is characterized by full or partial consumption insurance. The risk sharing contract in turn determines the stationary consumption distribution, which we now derive.

[5] Even with mortality risk, as long as there are perfect annuity markets, the stationary equilibrium interest rate has to be below the time discount rate.

4.1 Full Insurance in the Long Run: $\rho = r$

In this case the stationary consumption distribution places mass only on two points, $\{c_l, c_h\}$. We denote the probability masses as (ϕ_l, ϕ_h). We now derive the mass of households ϕ_l that have not received high income yet, and thus would consume $c_l = y_l = 0$. In a short time interval Δ a total mass $\Delta\gamma$ of households leave ϕ_l due to death. In addition, a flow Δv of households in ϕ_l transit to ϕ_h. Finally a flow $\Delta\gamma$ of new households is born, a share $\psi_l = \frac{\pi}{\pi+\nu}$ of which is born with low income and thus low consumption. Thus the stationary mass ϕ_l satisfies, for small Δ,

$$\phi_l = (1 - \Delta\gamma - \Delta\nu)\phi_l + \Delta\gamma\psi_l$$

and thus

$$(\gamma + \nu)\phi_l = \gamma\psi_l$$

and therefore the stationary consumption distribution is given by

$$\begin{aligned} \phi_h &= \frac{\gamma\nu + \nu(\pi + \nu)}{(\gamma + \nu)(\pi + \nu)} \in (0, 1) \\ \phi_l &= \frac{\gamma\pi}{(\gamma + \nu)(\pi + \nu)} \in (0, 1) \end{aligned} \tag{20}$$

4.2 Partial Insurance: $\rho > r$

In section 3.2 we characterized the optimal consumption contract under the parametric restriction that $r < \rho$. We showed that all households with high income consume $c_h(r)$, which is a function of the real interest rate, to be determined in equilibrium. Thus the stationary consumption distribution has a mass point at c_h with mass $\phi(c_h) = \frac{\nu}{\nu+\pi}$.

Households with currently low income have a consumption process that satisfies

$$\dot{c}(t) = -gc(t)$$

with

$$g = \frac{\rho - r}{\sigma} > 0$$

Finally, newborn households that have not yet received high income consume $c_l = y_l$, and the invariant consumption distribution has a second mass point at c_l equal to $\phi(c_l)$ whose size yet needs to be determined.

In $c \in (y_l, c_h)$ the consumption process follows a diffusion process with drift $-g$ (and no variance) and thus on $(0, c_h)$ the stationary consumption distribution satisfies the Kolmogorov forward equation (for the case of Poisson jump processes):

$$0 = -\frac{d\left[-gc\phi(c)\right]}{dc} - (\gamma + \nu)\phi(c)$$

where the second term comes from the fact that with Poisson intensities γ and ν the household dies and has a switch to high income, respectively. Since

$$-\frac{d\left[-gc\phi(c)\right]}{dc} = -\left[-g\phi(c) - gc\phi'(c)\right] = g\left[\phi(c) + c\phi'(c)\right]$$

we find that on $c \in (y_l, c_h)$ the stationary distribution satisfies

$$g\left[\phi(c) + c\phi'(c)\right] = (\gamma + \nu)\phi(c)$$

and thus

$$\frac{c\phi'(c)}{\phi(c)} = \frac{\gamma + \nu}{g} - 1$$

and thus on this interval the stationary consumption distribution is Pareto with tail parameter $\frac{\gamma+\nu}{g} - 1$, that is

$$\phi(c) = \phi_1 c^{\left(\frac{\gamma+\nu}{g} - 1\right)}$$

where ϕ_1 is a constant that needs to be determined. We immediately have the following

Proposition 2 *On the interval $(0, c_h)$ the consumption density has the form of a (truncated) Pareto distribution*

$$\phi(c) = \phi_1 (c)^{\kappa} \tag{21}$$

where the scale parameter $\phi_1 > 0$ has yet to be determined, and the Pareto parameter $\kappa = \frac{\gamma+\nu}{g} - 1 = \frac{\sigma(\gamma+\nu)-(\rho-r)}{\rho-r}$.

Remark 1 *Note that the sign of κ is indeterminate at this point, but since the support of the distribution is $(0, c_h)$ there is no issue that this distribution is not integrable.*

Now we need to determine the constant ϕ_1. Because of the mass point at c_h is is easier to think of the cdf for consumption on (y_l, c_h) given by $\Phi(c) = \frac{\phi_1 (c)^{\kappa+1}}{\kappa+1}$. The inflow mass into this range is given by the mass of individuals at c_h given by $\phi(c_h) = \frac{\nu}{\nu+\pi}$ times the

probability π of switching to the low income state, whereas the outflow is due to death and due to receiving the high income shock, and thus the stationary cdf has to satisfy

$$(\nu+\gamma)\Phi(c_h) = \frac{\pi\nu}{\nu+\pi}$$

and therefore

$$(\nu+\gamma)\frac{\phi_1 \left(c_h\right)^{\kappa+1}}{\kappa+1} = \frac{\pi\nu}{\nu+\pi}$$

Exploiting the fact that $\kappa+1 = \frac{\gamma+\nu}{g}$ we find

$$\phi_1 g \left(c_h\right)^{\frac{\gamma+\nu}{g}} = \frac{\pi\nu}{\nu+\pi}$$

and thus

$$\phi_1 = \frac{\pi\nu \left(c_h\right)^{-\frac{\gamma+\nu}{g}}}{g(\nu+\pi)}$$

and therefore the density on (y_l, c_h) is given by

$$\phi(c) = \frac{\pi\nu \left(c_h\right)^{-\frac{\gamma+\nu}{g}}}{g(\nu+\pi)} c^{\frac{\gamma+\nu}{g}-1}.$$

Finally we can determine the mass point ϕ_l at y_l from the requirement that the consumption distribution needs to integrate to 1. Thus we have

$$\phi_l + \phi_h + \int_{y_l}^{c_h} \phi(c)dc = 1$$

which determines the Dirac mass of people at y_l. Solving the integral yields

$$\int_0^{c_h} \phi(c)dc = \frac{1}{\frac{\gamma+\nu}{g}} \frac{\pi\nu \left(c_h\right)^{-\frac{\gamma+\nu}{g}}}{g(\nu+\pi)} c^{\frac{\gamma+\nu}{g}} \Bigg|_0^{c_h} = \frac{\pi\nu}{(\gamma+\nu)(\nu+\pi)} \left(1 - \left(\frac{0}{c_h}\right)^{\frac{\gamma+\nu}{g}}\right) = \frac{\pi\nu}{(\gamma+\nu)(\nu+\pi)}$$

and thus

$$\phi_l = \frac{\gamma\pi}{(\gamma+\nu)(\pi+\nu)}$$

Thus we have a complete analytical characterization of the stationary consumption distribution, summarized in the following

Proposition 3 *For any given $r \in (-(\gamma+\nu), \rho)$, the stationary consumption distribution is given by two mass points at $y_l = 0$ and $c_h(r)$ and a Pareto density in between:*

$$\phi_r(c) = \begin{cases} \frac{\gamma\pi}{(\gamma+\nu)(\pi+\nu)} & \textit{if} \quad c = 0 \\ \frac{\pi\nu(c_h(r))^{-\frac{\gamma+\nu}{g}}}{g(\nu+\pi)} c^{\frac{\gamma+\nu}{g}-1} & \textit{if} \quad c \in (0, c_h) \\ \frac{\nu}{(\nu+\pi)} & \textit{if} \quad c = c_h \end{cases}$$

Thus, for a given interest rate r the invariant consumption distribution is completely characterized by the upper bound $c_h(r) = \frac{r+\gamma+\nu+g}{r+\gamma+\nu+g+\pi} y_h$. We also note that

$$\phi_{r\to\rho}(c) = \begin{cases} \frac{\gamma\pi}{(\gamma+\nu)(\pi+\nu)} & \text{if} \quad c = y_l \\ \lim_{g\to 0} \left(\frac{\pi\nu}{cg(\nu+\pi)} \left(c_h/c\right)^{-\frac{\gamma+\nu}{g}} \right) = 0 & \text{if} \quad c \in (y_l, c_h) \\ \frac{\gamma\nu+\nu(\pi+\nu)}{(\gamma+\nu)(\pi+\nu)} & \text{if} \quad c = c_h \end{cases}$$

and thus the invariant consumption distribution converges to full insurance distribution as r converges to ρ from below. Note that the shape of the consumption probability function in between the two mass points depends on the relative size of $(\gamma+\nu)$, which governs the hazard rate of moving out of this part of the distribution (either through death or a positive income shock), and g, the speed at which consumption drifts down. The growth rate of the pdf is given by

$$\frac{d\log\phi_r(c)}{d\log c} = \frac{\gamma+\nu}{g} - 1 = \frac{\sigma(\gamma+\nu)}{\rho-r} - 1$$

We therefore have the following

Corollary 3 *If $\gamma+\nu < g$, then the pdf is strictly decreasing in c. If $\gamma+\nu > g$ then the pdf is strictly increasing in c. If $\gamma+\nu \in (g, g+1)$, then the pdf is strictly increasing and strictly concave in c. Finally, if $\gamma+\nu > g+1$ then the pdf is strictly increasing and strictly convex in c.*

Figure 4 below shows consumption distributions for different interest rates r and thus different growth rates g. As expected, with a lower interest rate (a higher drift downwards in consumption), probability mass shifts towards lower consumption levels in the stationary distribution.

4.3 Superinsurance: $\rho < r$

The argument for deriving the invariant consumption distribution is conceptually very similar to the previous case, with mass points at $c_l = 0$ and $c_h(0)$ and a Pareto distribution driven by a diffusion process with drift $-g =$ above $c_h(0)$. In Appendix A.4 we show that for any $r > \rho$ and associated $g = \frac{1}{\sigma}(\rho - r) < 0$, the stationary distribution is given as

$$\phi_r(c) = \begin{cases} \frac{\gamma\pi}{(\pi+\nu)(\gamma+\nu)} & \text{if} \quad c = 0 \\ \frac{\nu}{(\nu+\pi)} \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)} & \text{if} \quad c = c_h(0) \\ \frac{\gamma\nu(1-\gamma)(\gamma+\nu+\pi)}{-g(\pi+\nu)(\gamma+\nu)c} \left(\frac{c_h(0)}{c}\right)^{\frac{\gamma}{-g}} & \text{if} \quad c \in (c_h(0), \infty) \end{cases}$$

5 General Equilibrium: The Market Clearing Interest Rate

In the previous sections we derived, as a function of the interest rate r, the optimal consumption risk sharing contract as well as the associated invariant consumption distribution. Denote by

$$C(r) = \int c\phi_r(c)dc$$

aggregate consumption (scaled down by the aggregate wage) implied by these two entities. Recall that the goods market clearing condition (4) reads as

$$w(r)C(r) + \delta K = AF(K, 1)$$

or

$$C(r) = \frac{AF(K,1) - \delta K}{w(r)} := G(r) \tag{22}$$

and the capital market clearing condition (5) can be written as

$$\frac{K^s(r)}{w(r)} := \frac{C(r) - 1}{r} = \frac{K^d(r)}{w(r)} \tag{23}$$

5.1 Supply of Consumption Goods and Demand for Capital

From (2) and (3), as in Aiyagari (1994) we can express the aggregate capital stock and wage as a function of the interest rate: $K = K(r), w = w(r)$. Thus the aggregate net

supply of goods is given by

$$G(r) = \frac{AF(K(r),1) - \delta K(r)}{AF_L(K(r),1)} = 1 + \frac{\left[AF_K(K(r),1) - \delta\right] K(r)}{AF_L(K(r),1)}$$

and the aggregate demand for capital $K^d(r)$ is implicitly defined by the marginal product of capital equation (2). If we assume a Cobb-Douglas production function, we can show

Proposition 4 *Let the production function be of the form*

$$Y = AK^{\theta}L^{1-\theta}.$$

Then

$$G(r) = 1 + \frac{\theta r}{(1-\theta)(r+\delta)} \tag{24}$$

$$\kappa^d(r) := \frac{K^d(r)}{w(r)} = \frac{\theta}{(1-\theta)(r+\delta)} \tag{25}$$

Corollary 4 *$G(r), \kappa^d(r)$ are continuously differentiable on $r \in (-\delta, \infty)$, and $G(r)$ is strictly increasing, with*

$$\begin{aligned} \lim_{r \searrow -\delta} G(r) &= -\infty \\ G(r &= 0) = 1 \\ \lim_{r \nearrow \infty} G(r) &= 1 + \frac{\theta}{1-\theta} > 1 \end{aligned}$$

and $\kappa^d(r)$ is strictly decreasing, with

$$\begin{aligned} \lim_{r \searrow -\delta} \kappa^d(r) &= \infty \\ \kappa^d(r &= 0) = \frac{\theta}{(1-\theta)\delta} \\ \lim_{r \nearrow \infty} \kappa^d(r) &= 0 \end{aligned}$$

Having very sharply characterized the goods supply side and capital demand side, the question of existence, uniqueness and characterization of a stationary equilibrium thus rests with the analysis of how aggregate consumption demand $C(r)$ and thus (normalized) aggregate capital supply $\frac{K^s(r)}{w(r)}$ depends upon the interest rate r.

5.2 Demand of Consumption Goods and Supply of Capital

In order to characterize stationary equilibria we now have to characterize the demand for consumption good in a stationary equilibrium, as given by the demand function $C(r)$

$$C(r) = \int c\phi_r(c)dc$$

and the associated supply of capital, which is, for $r \neq 0$,

$$\kappa^s := \frac{K^s(r)}{w(r)} = \frac{\int c\phi_r(c)dc - 1}{r} \tag{26}$$

For $r = 0$, we need to determine $\kappa^s(r = 0)$ through an application of L'Hopital's rule as we will show below that

$$\lim_{r\to 0} \int c\phi_r(c)dc - 1 = 0.$$

Depending on the relationship between the interest rate and the discount rate, the allocation and corresponding invariant consumption distribution $\phi_r(c)$ features full $(r = \rho)$ or partial $(r < \rho)$ insurance. Given that the consumption allocation and invariant distribution differs qualitatively in both cases, it is in general hard to establish general properties of $C(r)$ independent of the case being considered. Therefore we will directly move to the characterization of stationary equilibria, subsuming the discussion of consumption demand in the derivation of the determination of the equilibrium interest rate.

5.3 Equilibria with Full Insurance and $r = \rho$

In this section we will provide conditions under which a stationary general equilibrium with an interest rate $r = \rho > 0.$ exists, and thus provides full insurance. The invariant consumption distribution is given by (20) and the consumption levels by (7) and (8), so that, conditional on $r = \rho$ aggregate consumption demand is given by:

$$\begin{aligned} C(r &= \rho) = \phi_l y_l + \phi_h c_h(r = \rho) \\ &= \left(\frac{(\gamma+\nu)(\rho+\gamma+\nu+\pi) - \nu\rho}{(\gamma+\nu)(\rho+\gamma+\nu+\pi)}\right)\frac{\pi y_l}{(\pi+\nu)} + \left(\frac{\gamma+\pi+\nu}{\gamma+\nu}\right)\left(\frac{\rho+\gamma+\nu}{\rho+\gamma+\nu+\pi}\right)\frac{\nu y_h}{(\pi+\nu)} \\ &= 1 + \frac{\rho\pi}{(\gamma+\nu)(\rho+\gamma+\nu+\pi)} \end{aligned} \tag{27}$$

(see appendix A.5.2 for the detailed calculations showing the second equality). Thus conditional on full insurance and $\rho = r$, the demand and supply of goods read as

$$C(\rho) = 1 + \frac{\rho\pi}{(\gamma+\nu)(\rho+\gamma+\nu+\pi)} \tag{28}$$

$$\kappa^s(\rho) = \frac{\pi}{(\gamma+\nu)(\rho+\gamma+\nu+\pi)} \tag{29}$$

Recall that from equation (25)

$$\kappa^d(\rho) = \frac{\theta}{(1-\theta)(\rho+\delta)} \tag{30}$$

Thus there is a unique knife-edge time discount factor $\bar{\rho}$ such that $C(\bar{\rho}) = G(\bar{\rho})$ and $\kappa^s(\bar{\rho}) = \kappa^d(\bar{\rho})$, and it satisfies[6]

$$\bar{\rho} = -\frac{\pi(1-\theta)\delta - \theta(\gamma+\nu)(\gamma+\nu+\pi)}{\pi(1-\theta) - \theta(\gamma+\nu)} \tag{31}$$

In order to insure that full insurance (or superinsurance with $r > \rho$) is not a stationary equilibrium, Assumption 2 below will insure that full insurance results in too high a demand for consumption goods (equivalently, too high a supply of capital) to be a stationary equilibrium. Section A.3 in the Appendix shows that if $\delta < \gamma+\nu+\pi$ (Assumption 4), then for all $\rho > \bar{\rho}$, Assumption 2 below is also satisfied and thus every stationary equilibrium has to satisfy $r < \rho$, the case discussed in the next subsection.

5.4 Equilibria with Partial Insurance and $r < \rho$

Now consider an arbitrary $\rho > 0$. From the previous subsection we know that under assumption 4 any stationary equilibrium has to be a partial insurance equilibrium. We will first argue that if $\sigma = 1$ (log-utility) there exists a unique stationary equilibrium with partial insurance and $\rho > r > 0$. We then explore the question whether there are multiple stationary equilibria with partial insurance in case $\sigma > 1$. To do so we now derive aggregate consumption demand for the partial insurance case. Recall that

$$c_h(r) = \frac{r+\gamma+\nu+g(r)}{\pi+r+\gamma+\nu+g(r)} y_h$$

[6]For $\theta = 0$ we have $\kappa^d(\rho) = 0$ and thus $\bar{\rho} = \infty$.

and thus aggregate consumption demand is given by

$$\begin{aligned}
C(r) &= \frac{\nu}{\nu+\pi}c_h(r) + \int_0^{c_h(r)} c\frac{\pi\nu\left(c_h(r)\right)^{-\frac{\gamma+\nu}{g}}}{g(\nu+\pi)}c^{\frac{\gamma+\nu}{g}-1}dc \\
&= \frac{\nu}{\nu+\pi}\frac{\pi+\gamma+\nu+g(r)}{\gamma+\nu+g(r)}c_h(r) \\
&= \frac{\pi+\gamma+\nu+g(r)}{\gamma+\nu+g(r)}\frac{r+\gamma+\nu+g(r)}{\pi+r+\gamma+\nu+g(r)} \\
&= \left(1+\frac{\pi}{\gamma+\nu+g(r)}\right)\left(1-\frac{\pi}{\pi+\gamma+\nu+g(r)+r}\right) \\
&= 1+\frac{\pi}{\gamma+\nu+g(r)}-\frac{\pi}{\pi+\gamma+\nu+g(r)+r}-\frac{\pi^2}{(\pi+\gamma+\nu+g(r)+r)(\gamma+\nu+g(r))} \\
&= 1+\frac{r\pi}{(\pi+\gamma+\nu+g(r)+r)(\gamma+\nu+g(r))} \\
\kappa^s(r) &= \frac{\pi}{(\pi+\gamma+\nu+g(r)+r)(\gamma+\nu+g(r))}
\end{aligned}$$

where $g(r)=\frac{\rho-r}{\sigma}>0$.

From the previous section recall that consumption demand and capital supply for the $r=\rho$ full insurance case were given by

$$C(r=\rho) = 1+\frac{\rho\pi}{(\gamma+\nu)(\rho+\gamma+\nu+\pi)} \tag{32}$$

$$\kappa^s(r=\rho) = \frac{\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu)} \tag{33}$$

and it follows that aggregate consumption demand and capital supply are continuous in the interest rate at $r=\rho$ since

$$\lim_{r\nearrow\rho} C(r) = 1+\frac{\rho\pi}{(\gamma+\nu)(\rho+\gamma+\nu+\pi)} = C(r=\rho) \tag{34}$$

$$\lim_{r\nearrow\rho} \kappa^s(r) = \frac{\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu)} = \kappa^s(r=\rho) \tag{35}$$

We now want to study the existence and uniqueness (or multiplicity) of a stationary equilibrium with partial consumption insurance. An interest rate $r<\rho$ is a stationary equilibrium interest rate with partial insurance if and only if equation (23) is satisfied, that

is, if and only if, exploiting equations (25) and (32)):

$$\frac{\theta}{(1-\theta)(r+\delta)} = \frac{\pi}{(\pi+\gamma+\nu+\frac{\rho-r}{\sigma}+r)(\gamma+\nu+\frac{\rho-r}{\sigma})}$$

This is a quadratic equation in the interest rate r which might have zero, one or two real roots $r^\star \in (-\min\{\delta,(\gamma+\nu+\pi)\},\rho)$.

5.4.1 Logarithmic Utility: Unique Stationary Equilibrium

In the case of logarithmic utility, $\sigma = 1$, we can demonstrate that there exists a unique stationary equilibrium with positive interest rate and partial insurance, under appropriate restrictions on parameters. We therefore make the following assumption for the rest of this section:

Assumption 1 *The utility function is logarithmic:* $\sigma = 1$

Under this assumption $g(r) = \rho - r$ and thus $g(r) + r = \rho$ and therefore

$$\begin{aligned} c_h(r) &= \frac{\rho+\gamma+\nu}{\pi+\rho+\gamma+\nu} y_h = c_h \\ C(r) &= 1 + \frac{r\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu+\rho-r)} \\ \kappa^s(r) &= \frac{\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu+\rho-r)} \end{aligned}$$

Thus the equilibrium interest rate satisfies the linear equation

$$\kappa^s(r) = \frac{\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu+\rho-r)} = \frac{\theta}{(1-\theta)(r+\delta)} = \kappa^d(r) \tag{36}$$

and since the supply of capital was derived under the assumption of partial insurance, it also has to satisfy $r \in (-\min\{\delta,(\gamma+\nu+\pi)\},\rho)$. We note that $\kappa^s(r)$ is continuous and strictly increasing on $r \in [-\delta,\rho)$ and $\kappa^d(r)$ is continuous and strictly decreasing on $r \in (-\delta,\rho]$, with

$$\infty = lim_{r \searrow -\delta} \kappa^d(r) > \kappa^s(-\delta)$$

Thus by the intermediate value theorem we obtain a unique equilibrium interest rate $r^\star \in (-\delta,\rho)$ if

$$\kappa^s(\rho) = \frac{\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu)} > \frac{\theta}{(1-\theta)(\rho+\delta)} = \kappa^d(\rho)$$

The following assumption insures that this is indeed the case.

Assumption 2 *Let the parameters be such that*

$$\frac{\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu)} > \frac{\theta}{(1-\theta)(\rho+\delta)} \tag{37}$$

We then have the following

Theorem 1 *Let assumptions 1 and 2 be satisfied. Then there exists a unique stationary equilibrium with partial consumption insurance, and with interest rate*

$$r^* = \frac{\theta(\pi+\gamma+\nu+\rho)(\gamma+\nu+\rho) - \pi\delta(1-\theta)}{\pi+\theta(\gamma+\nu+\rho)} \tag{38}$$

The equilibrium interest rate r^* is a strictly increasing function of $\rho+\gamma+\nu, \theta$ and a strictly decreasing function of π, δ. The equilibrium capital stock K^* is a strictly increasing function of π, θ and a strictly decreasing function of $\rho+\gamma+\nu, \delta$.

Thus far nothing guarantees that the unique partial insurance equilibrium interest rate r^* is positive. The following assumption guarantees that this is the case.

Assumption 3 *Let the parameters be such that*

$$\frac{\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu+\rho)} < \frac{\theta}{(1-\theta)\delta} \tag{39}$$

Corollary 5 *Let assumptions 1, 2 and 3 be satisfied. Then the unique stationary equilibrium interest rate satisfies $r^* \in (0, \rho)$ and induces a partial insurance risk consumption contract and associated distribution.*

Note that we can consolidate assumptions 2 and 3 into a single assumption on parameters only. We have

Corollary 6 *Assumptions 2 and 3 are jointly satisfied if and only if*

$$\frac{\delta}{\gamma+\nu+\rho} < \frac{\theta(\pi+\gamma+\nu+\rho)}{(1-\theta)\pi} < \frac{\delta+\rho}{\gamma+\nu} = \frac{\delta}{\gamma+\nu+\rho} \times \left(1+\frac{\rho}{\delta}\right) \times \left(1+\frac{\rho}{\gamma+\nu}\right) \tag{40}$$

which requires θ and ρ to be sufficiently large (relative to the other model parameters).

It should be noted that this last set of inequalities is stated purely in terms of parameters of the model, and it is straightforward to verify that the set of parameters satisfying these inequalities is not empty.

5.5 Superinsurance Equilibria

Similarly to the previous subsection, we can use the optimal consumption contract and the invariant consumption distribution to determine aggregate consumption demand $C(r)$ and capital supply $\kappa^s(r) = \frac{C(r)-1}{r}$. Aggregate consumption demand is given as (see again Appendix A.4) by

$$C(r) = \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)} \frac{(r+\gamma+g(r))}{(r+\gamma)} \cdot \frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)} \left[1 + \frac{(1-\gamma)}{g(r)+\gamma}\right] \tag{41}$$

with $g = \frac{1}{\sigma}(\rho - r)$, as long as

$$r < \rho + \gamma * \sigma. \tag{42}$$

In the appendix we also demonstrate that $C(r)$ is continuous from above at $r = \rho$. Characterizing superinsurance equilibria is in general hard, but under Assumption 1 (log-utility) we note that

$$C(r) = \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)} \frac{(\rho+\gamma)}{(r+\gamma)} \cdot \frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)} \left[1 + \frac{(1-\gamma)}{\rho-r+\gamma}\right] \tag{43}$$

and we have the following

Conjecture 1 *Let Assumption 1 be satisfied and Assumption 2 be strictly violated. There there exists a stationary equilibrium with $r > \rho > 0$ and superinsurance consumption contract, and associated invariant distribution.*

Conjecture 2 *Under the assumptions of the previous conjecture, the superinsurance equilibrium is unique.*

5.6 Why no Equilibrium with $r = 0$?

We should again emphasize that as long as $r \neq 0$, we can be sure that whenever the goods market clears, the capital market also clears. However, for $r = 0$, this is not necessarily the case. In this section we discuss why $r = 0$ is not a stationary equilibrium despite the fact

that

$$C(r=0) = 1 = G(r=0)$$

The problem is that at $r = 0$ goods market clearing does not necessarily implies that the capital market clears. To see this, consider the capital market clearing condition (23)

$$K^s(r) := \frac{w(r)\left[C(r)-1\right]}{r} = K^d(r)$$

Note that

$$\frac{w(r)}{r} = \frac{(1-\theta)AK(r)^\theta}{r} = \frac{(1-\theta)A}{r}\left(\frac{\theta A}{r+\delta}\right)^{\frac{\theta}{1-\theta}}$$

and thus we can state the capital market clearing condition (23) as

$$K^s(r) := (1-\theta)A^{\frac{1}{1-\theta}}\left(\frac{\theta}{r+\delta}\right)^{\frac{\theta}{1-\theta}}\frac{\left[C(r)-1\right]}{r} = \left(\frac{\theta A}{r+\delta}\right)^{\frac{1}{1-\theta}} := K^d(r) \tag{44}$$

Using L'Hopital's rule, that

$$\begin{aligned} K^s(r &= 0) = \lim_{r\to 0}\frac{w(r)\left[C(r)-1\right]}{r} = \lim_{r\to 0}\frac{\left[C(r)-1\right]}{r/w(r)} = \frac{\left[C'(r=0)\right]}{\lim_{r\to 0}\left.\frac{dr/w(r)}{dr}\right|_{r=0}} \\ &= \frac{(1-\theta)A^{\frac{1}{1-\theta}}\theta^{\frac{\theta}{1-\theta}}}{(\delta)^{\frac{\theta}{1-\theta}}}C'(r=0) = \frac{(1-\theta)A^{\frac{1}{1-\theta}}\left(\theta/\delta\right)^{\frac{\theta}{1-\theta}}\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu+\rho)} \end{aligned}$$

since

$$\begin{aligned} \lim_{r\to 0}\frac{dr/w(r)}{dr} &= \left[(r+\delta)^{\frac{\theta}{1-\theta}} + r\frac{\theta}{1-\theta}(r+\delta)^{\frac{\theta}{1-\theta}-1}\right] * \frac{1}{(1-\theta)A(\theta A)^{\frac{\theta}{1-\theta}}} \\ \lim_{r\to 0}\left.\frac{dr/w(r)}{dr}\right|_{r=0} &= \frac{(\delta)^{\frac{\theta}{1-\theta}}}{(1-\theta)A(\theta A)^{\frac{\theta}{1-\theta}}} \end{aligned}$$

Thus

$$K^s(r=0) = \frac{(1-\theta)A^{\frac{1}{1-\theta}}\theta^{\frac{\theta}{1-\theta}}}{(\delta)^{\frac{\theta}{1-\theta}}}C'(r=0)$$

and capital market clearing at $r = 0$ then requires

$$\frac{C'(r=0)}{\frac{(\delta)^{\frac{\theta}{1-\theta}}}{(1-\theta)A(\theta A)^{\frac{\theta}{1-\theta}}}} = \left(\frac{\theta A}{\delta}\right)^{\frac{1}{1-\theta}}$$

and thus

$$C'(r=0) = \frac{\theta}{(1-\theta)\delta} = G'(r=0),$$

in addition to

$$C(r=0) = G(r=0)$$

For $C'(r=0) = G'(r=0)$ the following knife edge condition needs to be satisfied (which is explicitly ruled out by assumption 3):

$$\frac{\pi}{(\pi+\gamma+\nu+\rho)(\gamma+\nu+\rho)} = \frac{\theta}{(1-\theta)\delta}$$

Furthermore, under the assumptions made we have

$$K^s(r=0) < K^d(r=0)$$

and thus at $r = 0$ the goods market clears, but there is insufficient capital demand. One way to implement $r = 0$ as an equilibrium is to have the government own just the right amount of capital $K^g > 0$ such that

$$K^s(r=0) + K^g = K^d(r=0)$$

Since $r = 0$, the government does not collect any revenue from this ownership that would need to be distributed, and thus a simple adjustment of the equilibrium definition that has the government own just the right amount of the capital stock would implement $r = 0$ as a second equilibrium, with associated partial insurance consumption allocation.

5.7 Graphical Depiction of Unique Stationary Equilibrium

For this case we can also easily generate the standard Aiyagari (1994) plot in (K, r) space. It is depicted in Figure 1 for the parameterization chosen in the welfare analysis conducted in the next section. As shown above, there is a unique equilibrium with a positive interest

rate that clears the capital market. Also note that, even though at $r = 0$ the goods market

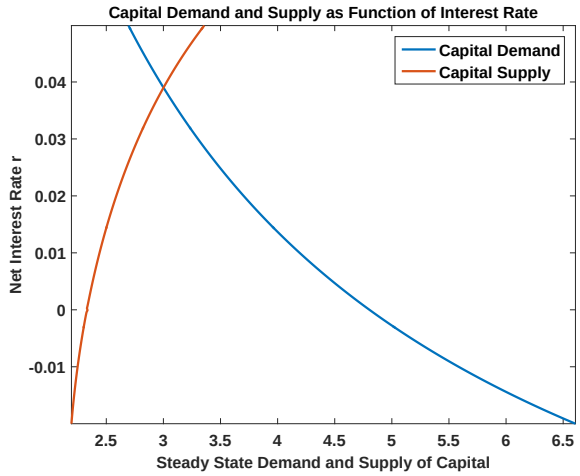


Figure 1: Capital Demand and Supply as a Function of the Interest Rate r

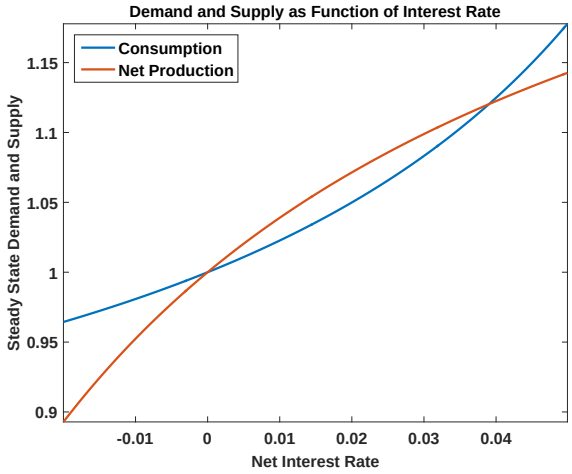


Figure 2: Goods Demand and Supply

clears (see Figure 2 which plots consumption demand and production net of depreciation), capital demand by firms exceeds capital supplied by households through the financial intermediaries, and thus $r = 0$ can only be implemented as a stationary equilibrium if the government (or some other outside entity) owns capital $K^d(r = 0) - K^s(r = 0) > 0$.

5.8 Multiple Equilibria

The previous subsections argued that with log-utility th stationary equilibrium is unique, but might feature partial insurance, full insurance or superinsurance, depending on parameters. In this subsection we assess the possibility of multiple equilibria, which requires $\sigma \neq 1$. At this point we do not have a general result, but Figure 3 displays an example with two stationary equilibria, one with partial insurance and an interest rate $r < \rho$, and one with superinsurance and an interest rate $\rho < r < \rho + \gamma * \sigma$. The parameterization for this example is given in Table 1 below. The time discount rate for this example is 5%.

There is a nonempty region of the parameter space for which these two equilibria exist, but the region is not overly large, and requires risk aversion $\sigma > 2$ and thus a low intertemporal elasticity of substiution, so that the capital supply is a decreasing function of the interest rate.

Parameter	σ	θ	δ	A	γ	ν	π
Value	10	0.272	20%	1	2%	5%	1.37%

Table 1: Parameter Values for Example

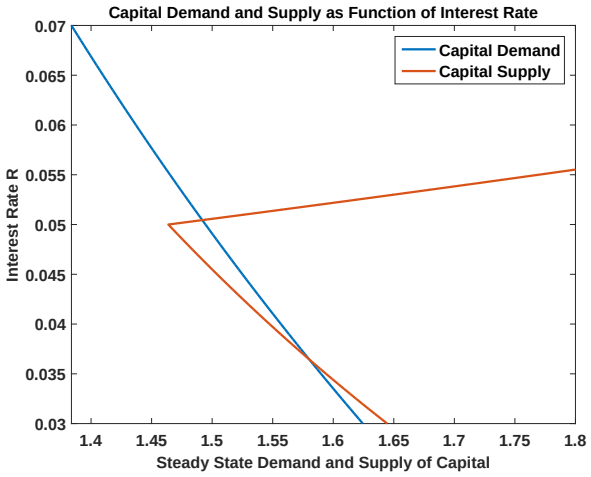


Figure 3: Capital Demand and Supply as a Function of the Interest Rate r

6 Welfare Properties of Stationary Equilibrium in the Log-Case

Suppose there is a government that owns part of the capital stock, and thus implements the stationary allocation associated with $r = 0$ as a stationary equilibrium. Can we rank the welfare properties of stationary allocations emerging under $r = 0$ and under $r = r^* > 0$. To do so, define social welfare as

$$\mathcal{U}_{\sigma=1}(r) = \frac{\pi}{\nu+\pi}U_l + \frac{\nu}{\nu+\pi}U_h$$

where $U_l := U^{out}(y_l)$ and $U_h := U^{out}(y_h)$ are the expected lifetimes under the equilibrium risk sharing contracts of individuals born into a steady state characterized by an equilibrium interest rate of r. We first note that for any household born with $y_l = 0$, the risk sharing contract would stipulate $c_l = 0$ and thus it is preferable to consume the nontradable endowment and obtain flow utility $\underline{u} = u(\chi)$ until income turns to $y = y_h$ at which an optimal consumption risk sharing contract is signed.

In the appendix we show that the difference in social welfare between any two stationary allocations associated with interest rates r_0 and r^* is given by

$$\mathcal{U}_{\sigma=1}(r_0) - \mathcal{U}_{\sigma=1}(r^*) = \frac{\nu(\pi+\rho+\gamma+\nu)}{(\nu+\pi)(\rho+\gamma+\nu)}\left(\frac{\log(w(r_0)/w(r^*))}{\rho+\gamma} + \frac{\frac{\pi(r_0-r^*)}{(\rho+\gamma+\nu)}}{(\rho+\gamma+\pi)(\rho+\gamma+\nu)-\nu\pi}\right)$$

with the sign determined by the sign of

$$\Delta = \frac{\log(w(r_0)/w(r^*))}{\rho+\gamma} + \frac{\pi(r_0-r^*)}{(\rho+\gamma+\pi)(\rho+\gamma+\nu)^2 - (\rho+\gamma+\nu)\nu\pi}.$$

The first term gives the effect of larger aggregate wages and thus larger aggregate consumption, whereas the second term gives the benefit of better consumption insurance. Now we note that

$$w(r) = (1-\theta)AK(r)^\theta = (1-\theta)A\left(\frac{\theta A}{r+\delta}\right)^{\frac{\theta}{1-\theta}}$$

and thus

$$\frac{\log(w(r_0)/w(r^*))}{\rho+\gamma} = \frac{\theta}{(1-\theta)(\rho+\gamma)}\log\left(\frac{r^*+\delta}{r_0+\delta}\right)$$

Parameter	σ	θ	δ	A	γ	ν	π
Value	1	0.3	10%	1	2%	5%	4%

Table 2: Parameter Values for Example

and therefore

$$\begin{aligned}\Delta &= \frac{\theta}{(1-\theta)(\rho+\gamma)}\log\left(\frac{r^*+\delta}{r_0+\delta}\right)+\frac{\pi(r_0-r^*)}{(\rho+\gamma+\pi)(\rho+\gamma+\nu)^2-(\rho+\gamma+\nu)\nu\pi}\\ &= \Delta_1+\Delta_2\end{aligned}$$

Finally we need to find conditions under which one term dominates the other. We are especially interested in conditions under which $\Delta < 0$. Consider the parameterization summarized in table 2 and consider two different time discount factors $\rho = 3\%$ and $\rho = 5\%$. In table 3 we collect the basic statistics for both configurations.

Thus for both time discount factors there exist exactly one stationary equilibrium with $r^* \in (0, \rho)$, and an allocation associated with $r_0 = 0$ that can be implemented as equilibrium with the government owning just the right amount of capital. Both allocations feature partial consumption insurance. The low interest rate $r_0 = 0$ is associated with higher wages (in fact at $r_0 = 0$ aggregate consumption is maximized). But the high interest rate allocation has better consumption insurance. As ρ increases, the high interest rate r^* also increases.

The aggregate welfare term Δ_1 depends on the distance between r^* and r_0 and thus is a bit larger under the high ρ. The insurance term Δ_2 depends on the difference between ρ and r and thus is roughly invariant to ρ. As a consequence as ρ gets larger the aggregate term dominates and welfare is higher under the low equilibrium interest rates. As ρ gets smaller, the aggregate welfare loss becomes smaller and is dominated by the risk sharing term. Crucially, table 2 demonstrates that depending on parameter either one of the effects can dominate, and thus it is possible that steady state welfare is higher in the low interest scenario or in the stationary equilibrium with $r^* > 0$. Figure 4 shows the resulting consumption distributions.

Variable	$\rho = 3\%$	$\rho = 5\%$
r_0	0%	0%
r^*	2.0%	3.9%
Δ_1	0.65	1.17
Δ_2	-1.14	-1.15
Δ	-0.49	0.02

Table 3: Welfare under two Interest Rates

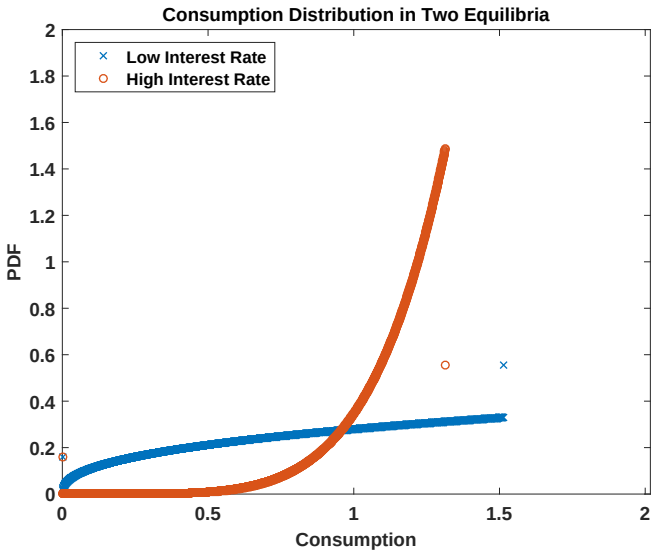


Figure 4: Consumption Distribution for Two Interest Rates

7 Conclusion

In this paper we have analytically characterized stationary equilibria in a neoclassical production economy with idiosyncratic income shocks and long-term one-sided limited commitment contracts. For an important special case (log-utility, two income state, zero income in the lower state) the equilibrium is unique and can be given in closed form, with complete comparative statics results. Given these findings, we would identify three immediately relevant next questions. First, on account of our use of continuous time, the endogenous optimal contract length is analytically tractable even outside the special case we have focused on thus far, and it will be important to generalize our findings about the stationary equilibria to the more general case.

Second, thus far we have focused on stationary equilibria, thereby sidestepping the question whether this stationary equilibrium is reached from a given initial aggregate stock, and what are the qualitative properties of the associated transition path. This also raises the conceptual question what is the appropriate initial condition for the distribution of outstanding insurance contracts.

Finally, thus far we have focused on an environment that has idiosyncratic, but no aggregate shocks, rendering the macro economy deterministic. Given our sharp analytical characterization of the equilibrium in the absence of aggregate shocks, we conjecture that the economy with aggregate shocks might be at least partially analytically tractable as well. We view these questions as important topics for future research.

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A Details of the Derivations

In this section we provide further details of the mathematical derivations in the paper. These are straightforward but tedious manipulations which were therefore excluded from the main text

A.1 Value Function for $r < \rho$ in Closed Form

The differential equation determining the cost function is given by

$$\begin{aligned}(r+\gamma+\nu)v(t) &= c(t) - y_l + \dot{v}(t) \\ \dot{v}(t) &= c(t) - y_l - (r+\gamma+\nu)v(t)\end{aligned}$$

Then, integrating the differential equation, we have

$$\begin{aligned}v(t) &= \int_t^\infty e^{-(r+\gamma+\nu)(\tau-t)}\left(c(t) - y_l\right)d\tau \\ &= \int_t^\infty e^{-(r+\gamma+\nu)(\tau-t)}\left(c_h e^{-g\tau} - y_l\right)d\tau\end{aligned}$$

One can of course check this by differentiating the solution to obtain back the differential equation. Solving the integral yields

$$v(t) = c_h e^{-gt}\int_t^\infty e^{-(r+\gamma+\nu+g)(\tau-t)}d\tau = \frac{c_h e^{-gt}}{r+\gamma+\nu+g}$$

which is the equation in the main text. Evaluating at $t = 0$ yields

$$v(0) = \frac{c_h}{r+\gamma+\nu+g}$$

which is equation (17) in the main text.

A.2 Invariant Consumption Distribution for $\rho = r$

The invariant consumption distribution satisfies

$$\phi_h = (1-\Delta\gamma)\phi_h + \Delta\nu(1-\Delta\gamma)(1-\phi_h) + \Delta\gamma\frac{\nu}{\pi+\nu}$$

Thus

$$\phi_h = \phi_h - \Delta\gamma\phi_h + \Delta\nu(1-\Delta\gamma) - \Delta\nu(1-\Delta\gamma)\phi_h + \Delta\gamma\frac{\nu}{\pi+\nu}$$

Simplifying and suppressing Δ^2 terms yields

$$\Delta\gamma\phi_h = \Delta\nu - \Delta\nu\phi_h + \Delta\gamma\frac{\nu}{\pi+\nu}$$

Dividing by Δ yields

$$\gamma\phi_h = \nu - \nu\phi_h + \frac{\gamma\nu}{\pi+\nu}$$

and solving for ϕ_h delivers

$$\begin{aligned}(\gamma+\nu)\phi_h &= \frac{\nu(\pi+\nu)+\gamma\nu}{\pi+\nu} \\ \phi_h &= \frac{\gamma\nu+\nu(\pi+\nu)}{(\gamma+\nu)(\pi+\nu)} \in (0,1)\end{aligned}$$

A.3 Existence of Full Insurance Equilibria

At this point it is not clear whether the threshold time discount factor $\bar{\rho}$ at which full insurance is exactly an equilibrium

$$\bar{\rho} = -\frac{\pi(1-\theta)\delta - \theta(\gamma+\nu)(\gamma+\nu+\pi)}{\pi(1-\theta)-\theta(\gamma+\nu)}$$

has $\bar{\rho} > 0$ or $\bar{\rho} < 0$, but note that for $\theta = 0$, we have $\bar{\rho} = -\delta < 0$. We now would like to establish conditions such that for all discount factors $\rho < \bar{\rho}$ and thus households are more patient than $\rho < \bar{\rho}$ we also have a full insurance, or superinsurance equilibrium, and if $\rho > \bar{\rho}$ (and thus households are relatively impatient) full insurance is too expensive a stationary equilibrium must feature partial insurance. These results would coincide with the general intuition from the limited commitment literature that if households are sufficiently patient, there is a full-insurance equilibrium.

To establish these results we first note that

$$\begin{aligned}
(\kappa^s)'(\rho) &= -\frac{\pi}{\gamma+\nu}\left[\rho+\gamma+\nu+\pi\right]^{-2} = -\frac{\kappa^s(\rho)}{\rho+\gamma+\nu+\pi} < 0 \\
(\kappa^s)''(\rho) &= \frac{2\pi}{\gamma+\nu}\left[\rho+\gamma+\nu+\pi\right]^{-3} > 0 \\
(\kappa^d)'(\rho) &= -\frac{\theta}{1-\theta}\left[(\rho+\delta)\right]^{-2} = -\frac{\kappa^d(\rho)}{\rho+\delta} < 0 \\
(\kappa^d)''(\rho) &= \frac{2\theta}{1-\theta}\left[(\rho+\delta)\right]^{-3} > 0
\end{aligned}$$

and thus both functions are strictly decreasing and strictly convex, with exactly one intersection at $\rho = \bar{\rho}$. To establish that $\kappa^s(\rho) > \kappa^d(\rho)$ for all $\rho > \bar{\rho}$ it therefore suffices to show that

$$0 > (\kappa^s)'(\bar{\rho}) > (\kappa^d)'(\bar{\rho}) \tag{45}$$

that is, we need to show that κ^s is flatter than κ^d at $\rho = \bar{\rho}$. But equation (45) holds if and only if

$$\begin{aligned}
-\frac{\kappa^s(\bar{\rho})}{\bar{\rho}+\gamma+\nu+\pi} &> -\frac{\kappa^d(\bar{\rho})}{\bar{\rho}+\delta} \\
\frac{\kappa^s(\bar{\rho})}{\kappa^d(\bar{\rho})} &< \frac{\bar{\rho}+\gamma+\nu+\pi}{\bar{\rho}+\delta} \\
\delta &< \gamma+\nu+\pi
\end{aligned}$$

We state this as

Assumption 4 *Assume parameters are such that*

$$\delta < \gamma+\nu+\pi \tag{46}$$

We now can state

Proposition 5 *Let assumption 4 be satisfied. Then $\kappa^s(\rho) > \kappa^d(\rho)$ for all $\rho > \bar{\rho}$ and $\kappa^s(\rho) < \kappa^d(\rho)$ for all $\rho < \bar{\rho}$*

This proposition also implies that under assumption 4, assumption 2 is satisfied for all $\rho > \bar{\rho}$, and thus for large enough time discount rates there cannot be a full insurance stationary equilibrium. We also conjecture that in that case there cannot be a super insurance stationary equilibrium either:

Corollary 7 *Let assumption 4 be satisfied. Then for all $\rho > \bar{\rho}$, full insurance cannot be a stationary equilibrium.*

Conjecture 3 *Let assumption 4 be satisfied. Then for all $\rho > \bar{\rho}$, super insurance cannot be a stationary equilibrium. Thus any stationary equilibrium has to feature partial insurance.*

A.4 Details of Superinsurance

First, characterize the cost for a household that starts with high income and consumes a profile

$$c_h(t) = c_h(0)e^{-gt}$$

where $g = \frac{1}{\sigma}(\rho - r) < 0$. For transparency, split the net cost into the gross cost $\kappa(t)$ and the revenue $a(t)$ from the contract. The gross cost satisfies

$$r\kappa(t) = c_h(t) + \gamma(0 - \kappa(t)) + \dot{\kappa}(t)$$

or

$$\kappa(t) = \int_t^\infty e^{-(r+\gamma)(\tau-t)} c_h(\tau) d\tau$$

Note that Leibniz rule implies that

$$\begin{aligned} \dot{\kappa}(t) &= -c_h(t) + (r+\gamma)\int_t^\infty e^{-(r+\gamma)(\tau-t)} c_h(\tau) d\tau \\ &= -c_h(t) + (r+\gamma)\kappa(t) \end{aligned}$$

Solving the integral yields

$$\begin{aligned} \kappa(t) &= c_h(0)e^{-gt}\int_t^\infty e^{-(r+\gamma-g)(\tau-t)} d\tau \\ &= c_h(0)e^{-gt}\int_0^\infty e^{-(r+\gamma-g)\tau} d\tau \\ &= \frac{c_h(0)e^{-gt}}{r+\gamma-g} \end{aligned}$$

The revenue satisfies

$$\begin{aligned} r a_h(t) &= y_h + \gamma(0 - a_h(t)) + \pi(a_l(t) - a_h(t)) + \dot{a}_h(t) \\ r a_l(t) &= \gamma(0 - a_l(t)) + \nu(a_h(t) - a_l(t)) + \dot{a}_l(t) \end{aligned}$$

Evidently these two functions do not depend on time and solve

$$\begin{aligned} ra_h &= y_h + \gamma(0 - a_h) + \pi(a_l - a_h) \\ ra_l &= \gamma(0 - a_l) + \nu(a_h - a_l) \end{aligned}$$

or

$$\begin{aligned} (r+\gamma)a_h &= y_h + \pi(a_l - a_h) \\ (r+\gamma)a_l &= y_l + \nu(a_h - a_l) \end{aligned}$$

and subtracting one from the other

$$\begin{aligned} (r+\gamma+\nu+\pi)(a_h - a_l) &= y_h \\ a_l &= a_h - \frac{y_h}{(r+\gamma+\nu+\pi)} \end{aligned}$$

and therefore

$$\begin{aligned} a_h &= \frac{r+\gamma+\nu}{(r+\gamma)(r+\gamma+\nu+\pi)} y_h \\ a_l &= \frac{\nu}{(r+\gamma)(r+\gamma+\nu+\pi)} y_h \end{aligned}$$

Therefore the net cost function satisfies

$$\begin{aligned} v_h(t) &= \kappa(t) - a_h \\ v_h(0) &= \kappa(0) - a_h = \frac{c_h(0)}{r+\gamma+g} - \frac{(r+\gamma+\nu)y_h}{(r+\gamma+\nu+\pi)(r+\gamma)} = 0 \end{aligned}$$

which determines the entry level of consumption as

$$\begin{aligned} c_h(0) &= \frac{(r+\gamma+g)}{(r+\gamma)} \cdot \frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)} y_h < y_h \\ &= \frac{(r+\gamma+g)}{(r+\gamma)} \cdot \frac{(r+\gamma+\nu)(\pi+\nu)}{(r+\gamma+\nu+\pi)\nu} \end{aligned}$$

since $g < 0$. Also note that since $\kappa(t)$ is strictly increasing in t, so is $v(t)$. Therefore the household eventually becomes a liability; initially the intermediary collects (in expectation) positive contributions from the household, and these pay for insurance and a rising

consumption profile later on. Finally, for $r \searrow \rho$ we obtain

$$\lim_{r \searrow \rho} c_h(0) = \frac{(\rho + \gamma + \nu)}{(\rho + \gamma + \nu + \pi)} y_h$$

which is identical to c_h from the full insurance case ($r = \rho$).

It remains to characterize the consumption dynamics and cost of the consumption contract for a household starting with income $y_l = 0$. Since consumption cannot fall below zero, prepayment for a rising consumption profile conditional on continuing with y_l cannot happen, and thus, as in the full insurance and partial insurance case households receive $c_l = 0$ until they switch to high income.

The invariant distribution thus has two mass points, one at $c_l = 0$ and one at $c_h(0)$, and a continuous distribution above $c_h(0)$. The lower mass point satisfies

$$\phi_l = (1 - \Delta\gamma - \Delta\nu)\phi_l + \Delta\gamma \frac{\pi}{\pi + \nu}$$

and thus

$$\phi_l = \frac{\gamma\pi}{(\pi + \nu)(\gamma + \nu)}$$

as in the full insurance case. We need to determine the mass point ϕ_h on $c_h(0)$ as well as the scale and power parameter for the pdf above $c_h(0)$, in the same way we did for the $r < \rho$ case.

In $c \in (c_h(0), \infty)$ the consumption process follows a diffusion process with drift $-g = \frac{r-\rho}{\sigma} > 0$ (and no variance) and thus on this interval the stationary consumption distribution satisfies the Kolmogorov forward equation (for the case of Poisson jump processes):

$$0 = -\frac{d\left[-gc\phi(c)\right]}{dc} - \gamma\phi(c)$$

where the second term comes from the fact that with Poisson intensity ν the household dies.

$$-\frac{d\left[-gc\phi(c)\right]}{dc} = -\left[-g\phi(c) - gc\phi'(c)\right] = g\left[\phi(c) + c\phi'(c)\right]$$

we find that on $c \in (c_h(0), \infty)$ the stationary distribution satisfies

$$0 = g\left[\phi(c) + c\phi'(c)\right] - \gamma\phi(c)$$

and thus the tail parameter of the truncated Pareto distribution is given by:

$$\kappa := \frac{c\phi'(c)}{\phi(c)} = \frac{\gamma}{g} - 1$$

Recall that $g = \frac{\rho - r}{\sigma} < 0$ and thus on $c \in (c_h(0), \infty)$ the stationary consumption distribution is Pareto with a tail parameter $\kappa < -1$, that is

$$\phi(c) = \phi_1 c^{\kappa}$$

where ϕ_1 is a constant that needs to be determined. The fact that $\kappa < -1$ will insure that expected and aggregate consumption are finite despite the fact that the consumption distribution has unbounded support. The two remaining unknown parameters characterizing the invariant consumption distribution are the scale parameter ϕ_1 and the mass at $c_h(0)$ denoted by ϕ_h.

In the case of superinsurance it is easier to determine ϕ_h first. The flow into $c_h(0)$ for a small time interval Δ is given by those individuals that experience an increase in their incomes, $\Delta\nu\phi_l$, plus the mass of newborns with high income, $\Delta\gamma\frac{\nu}{\nu+\pi}$. On the other hand, all individuals at $c_h(0)$ flow out, and thus the total outflow is $\Delta\phi_h$. Equating both flow gives

$$\phi_h = \nu\phi_l + \gamma\frac{\nu}{\nu + \pi}$$

and thus

$$\phi_h = \frac{\nu\gamma(\pi + \gamma + \nu)}{(\pi + \nu)(\gamma + \nu)}$$

The residual probability mass above $c_h(0)$ thus satisfies

$$\begin{aligned}
\phi_1 \int_{c_h(0)}^{\infty} c^{\kappa} &= 1 - \phi_l - \phi_h \\
\frac{\phi_1}{1+\kappa}\left((c = \infty)^{1+\kappa} - c_h(0)^{1+\kappa}\right) &= \frac{\nu(1-\gamma)(\gamma + \nu + \pi)}{(\pi + \nu)(\gamma + \nu)} \\
\frac{-\phi_1 g}{\gamma} c_h(0)^{\frac{\gamma}{g}} &= \frac{\nu}{(\nu + \pi)} \cdot \frac{(1-\gamma)(\gamma + \nu + \pi)}{(\gamma + \nu)} \\
\phi_1 &= \frac{\gamma\nu(1-\gamma)(\gamma + \nu + \pi)}{-g(\pi + \nu)(\gamma + \nu)} c_h(0)^{-\frac{\gamma}{g}}
\end{aligned}$$

and thus

$$\phi(c) = \frac{\gamma\nu(1-\gamma)(\gamma+\nu+\pi)}{-g(\pi+\nu)(\gamma+\nu)c}\left(\frac{c}{c_h(0)}\right)^{\frac{\gamma}{g}}$$

Therefore for any given $r > \rho$, the stationary consumption distribution is given by two mass points at $y_l = 0$ and $c_h(0)$ and a Pareto density above $c_h(0)$:

$$\phi_r(c) = \begin{cases} \frac{\gamma\pi}{(\pi+\nu)(\gamma+\nu)} & \text{if} \quad c = 0 \\ \frac{\nu}{(\nu+\pi)}\frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)} & \text{if} \quad c = c_h(0) \\ \frac{\gamma\nu(1-\gamma)(\gamma+\nu+\pi)}{-g(\pi+\nu)(\gamma+\nu)c}\left(\frac{c_h(0)}{c}\right)^{\frac{\gamma}{-g}} & \text{if} \quad c \in (c_h(0), \infty) \end{cases}$$

Now we determine aggregate consumption demand,

$$\begin{aligned}
C(r) &= \phi_h c_h(0) + \int_{c_h(0)}^{\infty} c\phi_r(c)dc \\
&= \frac{\nu y_h}{(\nu+\pi)}\frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)} \\
&\quad -\frac{\gamma\nu(1-\gamma)(\gamma+\nu+\pi)}{g(\pi+\nu)(\gamma+\nu)}\int_{c_h(0)}^{\infty}\left(\frac{c_h(0)}{c}\right)^{\frac{\gamma}{-g}}dc \\
&= \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)} - \frac{\gamma\nu(1-\gamma)(\gamma+\nu+\pi)}{g(\pi+\nu)(\gamma+\nu)}(c_h(0))^{\frac{\gamma}{-g}}\int_{c_h(0)}^{\infty} c^{\frac{\gamma}{g}}dc \\
&= \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)} - \frac{\gamma\nu(1-\gamma)(\gamma+\nu+\pi)}{g(\pi+\nu)(\gamma+\nu)}(c_h(0))^{\frac{\gamma}{-g}}\left[\frac{(c)^{1+\frac{\gamma}{g}}}{1+\frac{\gamma}{g}}\right]_{c_h(0)}^{\infty} \\
&= \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)} + \frac{\gamma\nu(1-\gamma)(\gamma+\nu+\pi)}{(\pi+\nu)(\gamma+\nu)}\frac{1}{(g+\gamma)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+} \\
&= \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)} + \frac{\gamma(1-\gamma)(\gamma+\nu+\pi)}{(\gamma+\nu)}\frac{1}{(g+\gamma)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+} \\
&= \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)}\left[1+\frac{(1-\gamma)}{g+\gamma}\right] \\
&= \frac{\gamma(\pi+\gamma+\nu)}{(\gamma+\nu)}\frac{(r+\gamma+g)}{(r+\gamma)}\cdot\frac{(r+\gamma+\nu)}{(r+\gamma+\nu+\pi)}\left[1+\frac{(1-\gamma)}{g\left(1+\frac{\gamma}{g}\right)}\right]
\end{aligned}$$

$$C(r \searrow \rho) = \frac{(\pi+\gamma+\nu)}{(\gamma+\nu)}\cdot\frac{(\rho+\gamma+\nu)}{(\rho+\gamma+\nu+\pi)} = 1 + \frac{\rho\pi}{(\gamma+\nu)(\rho+\gamma+\nu+\pi)} = C(r=\rho)$$

where we require that

$$\begin{aligned} 1+\frac{\gamma}{g} &< 0 \\ r-\rho &< \gamma\sigma \end{aligned}$$

That is, for aggregate consumption demand to be finite r cannot be too large and/or the probability of death and risk aversion have to be sufficiently large. Effectively, dying has to be more rapid than consumption growth. Check the algebra again, seems a bit messy.

A.5 Market Clearing Interest Rate

A.5.1 Supply of Consumption

The (normalized by wages) supply of consumption goods in the stationary equilibrium is given by

$$G(r) = \frac{\left[AF_K(K(r),1)-\delta\right]K(r)}{AF_L(K(r),1)}$$

Calculating the capital stock for Cobb-Douglas production yields

$$K(r) = \left(\frac{\theta A}{r+\delta}\right)^{\frac{1}{1-\theta}}$$

and thus

$$\frac{\left[AF_K(K(r),1)-\delta\right]K(r)}{AF_L(K(r),1)} = \frac{r}{(1-\theta)AK(r)^{\theta-1}} = \frac{r\theta}{(1-\theta)(r+\delta)} = \frac{\theta}{(1-\theta)(1+\delta/r)}.$$

A.5.2 Consumption Demand for $\rho = r$

Here we expand on the algebra in the main text for the calculation of aggregate consumption demand in the case of arbitrary $\rho = r$. The only requirement is that $\rho = r > -(\gamma+\nu+\pi)$.

$$
\begin{aligned}
C(r) &= \phi_l y_l + \phi_h c_h(r) \\
&= \left(1 - \frac{\nu r}{(\gamma+\nu)\,(r+\gamma+\nu+\pi)}\right)\left(1 - \frac{\nu y_h}{\pi+\nu}\right) + \left(\frac{\gamma+\pi+\nu}{\gamma+\nu}\right)\left(\frac{r+\gamma+\nu}{r+\gamma+\nu+\pi}\right)\frac{\nu y_h}{(\pi+\nu)} \\
&= 1 - \frac{\nu r}{(\gamma+\nu)\,(r+\gamma+\nu+\pi)} + \left(\frac{(\gamma+\nu+\pi)(r+\gamma+\nu) + \nu r - (\gamma+\nu)\,(r+\gamma+\nu+\pi)}{(\gamma+\nu)\,(r+\gamma+\nu+\pi)}\right)\frac{\nu}{(\pi+} \\
&= 1 - \frac{\nu r}{(\gamma+\nu)\,(r+\gamma+\nu+\pi)} + \frac{r\nu y_h}{(\gamma+\nu)\,(r+\gamma+\nu+\pi)} \\
&= 1 + \frac{r\nu(y_h-1)}{(\gamma+\nu)(r+\gamma+\nu+\pi)}
\end{aligned}
$$

and using equation (1) we have

$$C(r=\rho) = 1 + \frac{\rho\pi}{(\gamma+\nu)(\rho+\gamma+\nu+\pi)}$$

B Welfare

B.1 Full Insurance

In this case, in parallel to the cost calculation for the optimal contract the HJB equation for lifetime utility deflated by wages reads as

$$
\begin{aligned}
\rho u_h &= \log(c_h) + \gamma(0-u_h) + \pi(u_{hl}-u_h) \\
\rho u_{hl} &= \log(c_h) + \gamma(0-u_{hl}) + \nu(u_h-u_{hl})
\end{aligned}
$$

Simplifying this set of equations we obtain

$$
\begin{aligned}
(\rho+\gamma+\pi)u_h &= \log(c_h) + \pi u_{hl} \\
(\rho+\gamma+\nu)u_{hl} &= \log(c_h) + \nu u_h
\end{aligned}
$$

and thus $u_h = u_{hl}$ and

$$u_h = u_{hl} = \frac{\log(c_h)}{\rho+\gamma}$$

where

$$c_h = \frac{\rho+\gamma+\nu}{\rho+\gamma+\nu+\pi} y_h$$

Thus

$$U_h = \frac{w(r)}{\rho+\gamma} + \frac{\log(c_h)}{\rho+\gamma}$$

For lifetime utility conditional on being born with low income U_l we have

$$\rho U_l = \underline{u} + \gamma(0 - U_l) + \nu(U_h - U_l)$$

and thus

$$U_l = \frac{\underline{u} + \nu U_h}{\rho+\gamma+\nu}$$

Therefore lifetime utility is determined by

$$\begin{aligned}
\mathcal{U}_{\sigma=1}(r) &= \frac{\pi}{\nu+\pi}U_l + \frac{\nu}{\nu+\pi}U_h \\
&= \frac{\pi\underline{u}}{(\nu+\pi)(\rho+\gamma+\nu)} + \left(\frac{\nu(\rho+\gamma+\nu+\pi)}{(\nu+\pi)(\rho+\gamma+\nu)(\rho+\gamma)}\right)(\log w(r) + \log(c_h))
\end{aligned}$$

B.2 Partial Insurance

First we note that the calculation of U_l is exactly the same as before, and thus it remains to be true that

$$U_l = \frac{\underline{u} + \nu U_h}{\rho+\gamma+\nu}$$

Now of course the calculation for lifetime utility from the contract is more involved. As before we have

$$U_h = \frac{w(r)}{\rho+\gamma} + u_h$$

where u_h is lifetime utility from the deflated consumption contract.

Again in parallel to the cost calculation for the optimal contract we find as lifetime utility from the deflated consumption contract

$$\begin{aligned}
\rho u_h &= \log(c_h) + \gamma(0 - u_h) + \pi(u(0) - u_h) \\
\rho u(t) &= \log(c(t)) + \gamma(0 - u(t)) + \nu(u_h - u(t)) + \dot{u}(t)
\end{aligned}$$

and thus

$$\begin{aligned}
(\rho+\gamma+\pi)u_h &= \log(c_h) + \pi u(0) \\
(\rho+\gamma+\nu)u(t) &= \log(c(t)) + \nu u_h + \dot{u}(t)
\end{aligned}$$

where $c(t) = c_h e^{-(\rho - r)t}$ and thus

$$\log(c(t)) = \log(c_h) - (\rho - r)t$$

and

$$(\rho + \gamma + \nu)u(t) = \log(c_h) - (\rho - r)t + \nu u_h + \dot{u}(t)$$

Solving this ODE yields

$$u(t) = \int_t^\infty e^{-(\rho+\gamma+\nu)(s-t)} \left(-(\rho - r)s + \log(c_h) + \nu u_h\right) ds$$

and thus

$$\begin{aligned} u(0) &= \int_0^\infty e^{-(\rho+\gamma+\nu)(s-t)} \left(-(\rho - r)s + \log(c_h) + \nu u_h\right) ds \\ &= -\int_0^\infty e^{-(\rho+\gamma+\nu)(s-t)}(\rho - r)s ds + (\log(c_h) + \nu u_h) \int_0^\infty e^{-(\rho+\gamma+\nu)(s-t)} ds \\ &= \frac{r - \rho}{(\rho + \gamma + \nu)^2} + \frac{\log(c_h) + \nu u_h}{\rho + \gamma + \nu} \end{aligned}$$

where the last step uses integration by part (need to check algebra again!). Thus we have

$$\begin{aligned} (\rho + \gamma + \nu)u(0) &= \log(c_h) + \nu u_h + \frac{r - \rho}{(\rho + \gamma + \nu)} \\ (\rho + \gamma + \pi)u_h &= \log(c_h) + \pi u(0) \end{aligned}$$

Solving these two equations in two unknowns yields

$$\left[(\rho + \gamma + \pi)\right] u_h = \log(c_h) + \frac{\pi(r - \rho)}{(\rho + \gamma + \nu)^2} + \frac{\pi \log(c_h) + \nu\pi u_h}{\rho + \gamma + \nu}$$

and thus

$$\begin{aligned} \left[\frac{(\rho + \gamma + \pi)(\rho + \gamma + \nu) - \nu\pi}{\rho + \gamma + \nu}\right] u_h &= \left(\frac{\rho + \gamma + \nu + \pi}{\rho + \gamma + \nu}\right) \log(c_h) + \frac{\pi(r - \rho)}{(\rho + \gamma + \nu)^2} \\ u_h &= \frac{(\rho + \gamma + \nu + \pi) \log(c_h) + \frac{\pi(r-\rho)}{(\rho+\gamma+\nu)}}{(\rho + \gamma + \pi)(\rho + \gamma + \nu) - \nu\pi} \end{aligned}$$

Therefore social welfare is determined as

$$\begin{aligned}
\mathcal{U}_{\sigma=1}(r) &= \frac{\pi}{\nu+\pi}U_l + \frac{\nu}{\nu+\pi}U_h \\
&= \frac{\pi}{\nu+\pi}\frac{\underline{u}+\nu U_h}{\rho+\gamma+\nu} + \frac{\nu}{\nu+\pi}U_h \\
&= \frac{\pi\underline{u}}{(\nu+\pi)(\rho+\gamma+\nu)} + \frac{\nu(\pi+\rho+\gamma+\nu)}{(\nu+\pi)(\rho+\gamma+\nu)}U_h \\
&= \frac{\pi\underline{u}}{(\nu+\pi)(\rho+\gamma+\nu)} + \frac{\nu(\pi+\rho+\gamma+\nu)}{(\nu+\pi)(\rho+\gamma+\nu)}\left(\frac{\log(w(r)}{\rho+\gamma} + u_h\right) \\
&= \frac{\pi\underline{u}}{(\nu+\pi)(\rho+\gamma+\nu)} + \frac{\nu(\pi+\rho+\gamma+\nu)}{(\nu+\pi)(\rho+\gamma+\nu)}\left(\frac{\log(w(r)}{\rho+\gamma} + \frac{(\rho+\gamma+\nu+\pi)\log(c_h) + \frac{\pi(r-\rho)}{(\rho+\gamma+\nu)}}{(\rho+\gamma+\pi)(\rho+\gamma+\nu)-\nu\pi}\right)
\end{aligned}$$

The difference in welfare between two stationary equilibria is thus given by (noting that c_h is independent of r)

$$\mathcal{U}_{\sigma=1}(r_0) - \mathcal{U}_{\sigma=1}(r^*) = \frac{\nu(\pi+\rho+\gamma+\nu)}{(\nu+\pi)(\rho+\gamma+\nu)}\left(\frac{\log(w(r_0)/w(r^*))}{\rho+\gamma} + \frac{\frac{\pi(r_0-r^*)}{(\rho+\gamma+\nu)}}{(\rho+\gamma+\pi)(\rho+\gamma+\nu)-\nu\pi}\right)$$

with the sign determined by the sign of

$$\Delta = \frac{\log(w(r_0)/w(r^*))}{\rho+\gamma} + \frac{\pi(r_0-r^*)}{(\rho+\gamma+\pi)(\rho+\gamma+\nu)^2-(\rho+\gamma+\nu)\nu\pi}$$