The Endowment Model and Modern Portfolio Theory*

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Abstract

We analyze a long-term investor's dynamic consumption and asset allocation decisions by incorporating an illiquid investment opportunity (alternative asset) into an otherwise standard modern portfolio theory framework. The alternative asset is subject to a lock-up period, but can be voluntarily liquidated by incurring a proportional liquidation cost prior to the expiration of the lock-up. The alternative asset's risk-return profile is not fully spanned by public equity, which provides diversification benefits. We compare the quantitative results of the model with data on university endowment fund spending and asset allocation choices. The results closely match aggregate endowment fund choices. Additionally, we can match the cross-sectional variation in endowment fund choices with reasonable variation in beliefs about alpha.

Our theory justifies the seemingly aggressive allocations to alternative assets, as advocated by Swensen's Endowment model, if the alternative asset can generate an expected risk-adjusted excess return of 2-3% (with public equity as the benchmark). However, investors with limited access to managers with sufficiently high alphas should hold conventional portfolios that follow the guidelines of MPT.

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1. Introduction

Modern portfolio theory (MPT) developed by Markowitz (1952) and Merton (1969, 1971) and the widely-used asset allocation rule by institutional investors, such as the 60-40 rule, frame the asset allocation problem within the set of perfectly liquid securities.[1] There are, however, many investable financial assets that are not liquid. The last several decades have seen a large increase in the popularity of illiquid "alternative assets," such as hedge funds, private equity, venture capital, and illiquid natural resources. For example, Figure 1 plots the aggregate portfolio allocations of university endowment funds over time, and shows how popular alternative assets have become, with alternative assets rising from only 7% of holdings in 1990 to 54.5% in 2015.

The widespread use of alternative assets was popularized, at least in part, by the success of David Swensen at the Yale University endowment fund, and is often referred to as the endowment model (see Swensen (2000)). The endowment model is an investment philosophy that advocates, among other things,[2] allocating a significant fraction of the portfolio to illiquid alternative assets. Swensen (2000) argues this approach can produce abnormal returns because there are greater opportunities in illiquid markets and because long-term investors are best able to earn the illiquidity premium.

We develop a model to analyze a long-term investor's dynamic consumption and asset allocation decisions by incorporating an illiquid investment opportunity into an otherwise

[1] MPT recommends that a mean-variance investor (or an investor with constant-relative-risk-averse preferences over consumption and wealth) should allocate a constant fraction given by $x = \eta/(\gamma\sigma)$ in public equity, where γ is the investor's coefficient of relative risk aversion, and η and σ are the Sharpe ratio and volatility of public equity, respectively. This implies that with commonly used parameter values for the U.S. stock market, i.e., 30% Sharpe ratio and 20% volatility, investors with standard risk aversion $\gamma = 2$ should invest 60% in public equity and the remaining 40% in bonds. This is how MPT provides a justification for the 60-40 rule.

[2] The endowment models also advocates careful selection of active portfolio managers, with a focus on aligning incentives and finding entrepreneurial managers. We do not consider this aspect of the endowment model in this paper.

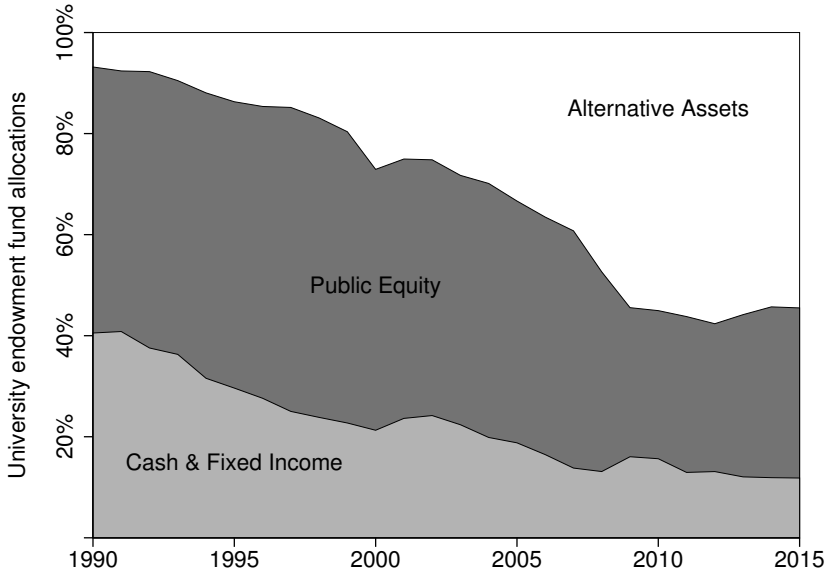


Figure 1: This figure shows the portfolio allocations for the "aggregate fund" by summing up the holdings of the endowments in the sample from 1990 to 2015. The light grey area at the bottom of the figure shows the allocations to cash and fixed income securities, the dark grey area shows the allocations to public equity, and the white area shows the allocations to alternative assets (hedge funds, private equity, venture capital, private real estate, and illiquid natural resources).

standard MPT framework. We capture the illiquidity of the alternative asset as follows. First, the alternative asset has a target finite duration based on the lock-up and holding period, meaning that only a fraction of the alternative asset can be liquidated at its full fundamental value at periodical maturity dates (e.g., every six years or so in our calibration). Second, we assume that the alternative asset's risk/return profile cannot be fully spanned by existing public equity. That is, we assume the alternative asset has unspanned risk that cannot be diversified away by public equity. Third, the investor can choose to voluntarily liquidate a part of the alternative asset holding on dates other than maturity dates, but doing so incurs a proportional liquidation cost. This cost captures the empirically observed discounts for alternative assets in the secondary markets, for

example, due to search and adverse selection costs. Fourth, for any voluntary acquisition of the alternative asset the investor pays an acquisition cost, which allows us to capture the costs of search, due diligence, legal, audit, and etc. Fifth, we show how the model can incorporate *liquidity diversification* by allowing the investor to stagger the maturities of portions of the alternative asset holding over time.

Our qualitative and quantitative results significantly differ from the standard predictions implied by MPT. In addition to deciding how much risk a long-term investor should take (i.e., the allocation between risky and risk-free assets), we show that liquidity considerations are also very important for the welfare of a long-term investor. That is, the liquidity ratio $w = W/A$ between the total market value of liquid wealth W and the fundamental value of the alternative asset A plays a critical role for the investor's asset allocation and welfare, or equivalently her certainty equivalent wealth, which is the natural welfare measure in dollars.

We show that the optimal portfolio holding involves a range of w depending on the history of realized shocks to public equity and the alternative asset, due to the transaction costs and the alternative asset's illiquidity. This is in sharp contrast with the single level for the optimal portfolio holding w^* when the alternative asset is fully liquid (no transaction costs and full spanning). Although the optimal w involves a range of values, we can still calculate the *target* portfolio holding that maximizes the investor's long-term welfare, which we use to compare with the frictionless case. Because the alternative asset is illiquid and its risk is incompletely spanned by publicly traded equity, the investor values liquid wealth more than its face value as it can be efficiently deployed to acquire alternative assets at any moment. However, to increase her liquid wealth holdings, the investor has to sell her alternative asset via costly liquidation.

Quantitatively, we show that the marginal (welfare) value of liquid wealth can be

significantly larger than one, especially when the investor's liquidity ratio w is low. This is because a long-term investor with low liquid wealth as a fraction of her net worth values liquid wealth more due to portfolio balancing considerations.

We calibrate our model using parameters drawn from the literature, and compare the calibration results with the portfolio allocations of U.S. university endowment funds. As Figure 1 shows, in 2015 the value weighted "aggregate endowment fund" allocated 54.7% of its holdings to alternative assets, 33.7% to public equity, and 11.8% to cash and fixed income. There is, however, substantial cross-sectional variation. For example, the Yale Endowment managed by Swensen for almost thirty years held 73.7% of its holdings in alternative assets (with 20.5% of net worth in hedge funds, 16.2% in private equity, 6.7% in natural resources, 14.0% in real estate, 16.3% in venture capital), 18.6% in public equity, and 7.7% in cash and fixed income. In contrast, 8.1% of endowments hold no alternative assets at all.

Our calibration results show that the seemingly aggressive allocations to alternative assets by Yale and other university endowment funds can be justified with reasonable levels of alpha[3] for the alternative asset. With an alpha of 3%, our model implies an alternative asset allocation of 60.2% of the portfolio's net worth, which is similar to the actual allocation found among the largest decile of endowment funds. The model matches the empirically observed variation in allocations across endowment funds with reasonable variation in beliefs about alphas. Expected alphas of 1% and 2%, imply alternative asset allocations of 12.7% and 34.5%, respectively, which approximately matches the interquartile range of actual endowment fund allocations. For small endowments, without access to good alternative investment opportunities, it is optimal to avoid alternative

[3]Note that we define alpha with public equity as the benchmark, and "alpha" may include illiquidity premiums as well as managerial skill.

assets. The model also implies a payout rate of 5.3%, which is close to the 5% payout rule used by many endowments (see Brown, Dimmock, Kang, and Weisbenner (2014)).

Related Literature. Our paper is closely related to Sorensen, Wang, and Yang (2014), henceforth SWY, which uses a dynamic portfolio choice model to value the cost of illiquidity and management compensation (including both management fees and carried interest) in private equity. There are several major differences between SWY and our paper. First, SWY uses constant-absolute-risk-averse (CARA) utility, which rules out wealth effects, while we use constant-relative-risk-averse (CRRA) utility. Our choice of CRRA allows us to focus on the quantitative implications for the portfolio rule as a time-varying percentage allocation of net worth, while CARA utility predicts a wealth-independent dollar allocation to the alternative asset. Second, the two papers model illiquidity very differently. SWY assumes that the investor must hold the alternative asset for a fixed T years without any option to exit. We allow for both costly liquidation and automatic free liquidity events at maturity for the alternative asset. Third, we allow for the investor to recurrently acquire and liquidate her alternative asset. Finally, our model quantitatively matches the Endowment Model for investors with access to sufficiently high alphas.

Our paper is also closely related to Ang, Papanikolaou, and Westerfield (2016), henceforth APW, which also models portfolio choice with an illiquid asset. APW assumes that the alternative asset cannot be traded for intervals of uncertain duration (modeled via a stochastic Poisson arrival process). Our approach is quite different. For example, in their model the asset can be fully illiquid for very long periods of time due to the long tails associated with exponential distributions. In contrast, our model has a secondary market that allows liquidation at all times by paying a proportional transaction cost.

Our paper is also related to Bollen and Sensoy (2016), which develops a method

for valuing the illiquidity of private equity when secondary markets exist. Bollen and Sensoy (2016) also solve for a limited partner's optimal portfolio allocations, however, our model is considerably more flexible than theirs. Bollen and Sensoy (2016) assumes that the investor must invest their entire private equity allocation in a single fund, and they exogenously fix the investor's allocation to the risk free asset and the spending rate. Further, in Bollen and Sensoy (2016) the investor follows exogenously pre-specified rebalancing rules rather than making rebalancing decisions based on utility maximization. By contrast, our model allows for liquidity diversification through staggering illiquid investments through time, and endogenously solves for allocations to the risk free asset, the spending rate, and rebalancing decisions.

Our model is related to the literature on portfolio choice with illiquid financial assets. Constantinides (1986), Davis and Norman (1990), Grossman and Laroque (1990), Vayanos (1998), Lo, Mamaysky, and Wang (2004), and Buss, Uppal, and Vilkov (2015) model illiquidity due to transaction costs. In these models, the illiquid asset is always tradable but at a cost. Kahl, Liu, and Longstaff (2003), Longstaff (2009), Dai, Li, Liu, and Wang (2010), De Roon, Guo, and ter Horst (2010), and Ang, Papanikoaou, and Westerfield (2016) model illiquidity from trading restrictions in which the asset can be freely traded at certain points in time but no trade is permitted at other times. Our model combines the features of both types of models; the alternative asset becomes fully liquid at maturity (e.g., when a private equity fund partnership is dissolved) but can be liquidated prior to maturity by paying a proportional cost (e.g., selling a private equity fund at a discount on the secondary market).

Our paper also contributes to the literature on university endowment funds. Merton (1992), Gilbert and Hrdlicka (2015), and Cejnek, Franz, and Stoughton (2017) model the spending and portfolio choice decisions of university endowment funds. These papers

consider only fully liquid securities, while our paper considers the addition of illiquid securities. Lerner, Schoar, and Wang (2008), Brown, Garlappi, and Tiu (2010), Barber and Wang (2013), and Ang, Ayala, and Goetzmann (2014) show there is a positive relation between allocations to alternative assets and return performance. Dimmock (2012) shows that non-tradable income risk affects university endowment fund portfolio choice. Brown, Dimmock, Kang, and Weisbenner (2014) study how endowment fund shocks affect payout rates and university resource allocations.

2. Model

We develop a model to analyze a long-term investor's dynamic consumption and asset allocation decisions by incorporating an illiquid investment opportunity into the otherwise standard modern portfolio choice framework developed by Merton (1969, 1971). We interpret the illiquid investment opportunity in our model as the representative portfolio of *alternative assets* including private equity, hedge funds, private real estate, infrastructure, and other illiquid assets. For ease of comparison with Merton's modern portfolio theory, we also develop our model in continuous time. Next, we summarize the standard investment opportunities in liquid assets and then introduce the alternative asset.

Liquid Investment Opportunities: Public Equity and Bonds. The risk-free bond pays interest at a constant (annualized) risk-free rate r. Public equity can be interpreted as the market portfolio for publicly traded securities, and its cum-dividend market value, S_t, follows a geometric Brownian motion (GBM):

$$\frac{dS_t}{S_t} = \mu_S dt + \sigma_S d\mathbb{B}_t^S \,, \tag{1}$$

where $\mathbb{B}_t^S$ is a standard Brownian motion, and μ_S and σ_S are the constant drift and volatility parameters. The Sharpe ratio for public equity is thus:

$$\eta_S = \frac{\mu_S - r}{\sigma_S}\,. \tag{2}$$

The liquid investment opportunity set in our model is, by construction, the same as in Merton (1971). Next, we introduce the alternative asset, which is the investor's third investment opportunity and the key building block in our model.

2.1. The Alternative Asset

Adding the alternative asset expands the investment opportunity set and thus makes the investor better off. Additionally, provided the alternative asset is not perfectly correlated with public equity, it provides diversification benefits to the investor. Unlike public equity, however, alternative assets are generally illiquid and involve some form of lock-up. For example, investments in private equity typically have a life span of 10 years with extension options, and hedge funds often have lock-up periods and gate provisions.

A key feature of alternative assets is that their illiquidity is not constant over time. For example, private equity funds are highly illiquid for much of their lives but eventually mature and return liquid capital to their investors. We model these liquidity events as follows. Let $\{A_t; t \geq 0\}$ denote the alternative asset's fundamental value process with a given initial stock A_0. The fundamental value refers to the fully realizable value of the asset if it is held to maturity. However, in general, with illiquidity or any other form of financial imperfections, the fundamental value at time t prior to its maturity is different from the asset's time-t market value. To capture the target finite duration of the lock-up and holding period, we assume every mT years, where m is an integer, a δ_T fraction of the stock of illiquid alternative asset A_{mT} automatically becomes liquid at no cost. That

is, in the absence of any active acquisition or disposition of the illiquid asset at mT, we have $A_{mT} = (1 - \delta_T)A_{mT-}$. Naturally, the investor's liquid asset value at time mT increases by $\delta_T A_{mT-}$.

The Fundamental Value Process A. We assume that the fundamental value A, in the absence of a scheduled automatic liquidity event (at time mT) or any interim acquisition or disposition, evolves via the following GBM:

$$\frac{dA_t}{A_{t-}} = \mu_A dt + \sigma_A d\mathbb{B}_t^A - \delta_A dt\,, \tag{3}$$

where $\mathbb{B}_t^A$ is a standard Brownian motion, μ_A is the cum-payout expected return (net of fees), σ_A is the constant volatility of returns, and δ_A is the alternative asset's payout rate. That is, the alternative asset pays dividends at the rate of $\delta_A A_t$ with an implied dividend yield of δ_A. Intuitively, δ_A can be interpreted as measuring regular cash disbursements or, alternatively, as a continuous time approximation of the annualized automatic liquidity as T becomes small. W use ρ to denote its correlation coefficient with $\mathbb{B}_t^S$, the shocks to public equity.

Note that we have purposefully written the drift of the value process A as $\mu_A - \delta_A$, because the payout δ_A will appear in the wealth accumulation process. In complete markets, the investor can frictionlessly and dynamically trade the alternative asset without restrictions or costs. Therefore, the alternative asset's market value is equal to its fundamental value and the Modigliani-Miller theorem holds, meaning that whether we explicitly write down the payout rate δ_A is irrelevant. In this ideal case, the alternative asset is conceptually no different than liquid public equity. In contrast, when the alternative asset is illiquid and not fully spanned by publicly traded equity, we must separately keep track of the dividend yield δ_A and expected capital gains $\mu_A - \delta_A$. That is, the

cum-dividend payout rate μ_A is no longer a sufficient measure of expected returns for the alternative asset as its dividend and expected capital gains influence the investor's portfolio optimization problem differently. Next, we introduce how the investor can actively change her illiquid alternative asset holdings.

Interim Acquisition and Liquidation of the Alternative Asset Holding. At any time, the investor can choose to change her alternative asset holdings through acquisitions or liquidations. Let dL_t denote the amount of the alternative asset that the investor liquidates at any time $t > 0$, and let dX_t denote the amount of the alternative asset that the investor purchases at time t. Then, we can incorporate the investor's liquidation and purchase options into the alternative asset's fundamental value process as follows:

$$dA_t = (\mu_A - \delta_A)A_{t-}dt + \sigma_A A_{t-}d\mathbb{B}_t^A - dL_t + dX_t - \delta_T A_{t-}\mathbb{I}_{\{t=mT-\}}\,, \tag{4}$$

where $\mathbb{I}_{\{.\}}$ is the indicator function. The first two terms correspond to the standard drift and volatility terms, the third and fourth terms give the acquisition and liquidation amounts, and the last term captures the lumpy payout to the investor at the scheduled liquidity event dates $t = mT$ where $m = 0, 1 \cdots$.

Although the acquisition and liquidation costs for the alternative asset do not appear in (4), they will appear in the liquid wealth accumulation process. We assume that the cost of voluntary liquidation is proportional. That is, by liquidating an amount of $dL_t > 0$, the investor realizes only the amount of $(1-\theta_L)dL_t$ in net, where the remaining amount $\theta_L dL_t$ accounts for the transaction cost paid to the third party and/or the market price discount/haircut on the asset's fundamental value. Similarly, if the investor acquires an amount of $dX_t > 0$, the transaction cost $\theta_X dX_t$ will be paid out of the liquid asset holding to the third party. Naturally, $0 \leq \theta_L \leq 1$ and $\theta_X \geq 0$. Higher values of θ_L or θ_X indicate the alternative asset is more illiquid.

Intuitively, θ_L can be interpreted as the illiquidity discount on secondary market sales of alternative assets (e.g., see Kleymenova, Tamor, and Vasvari (2012), Bollen and Sensoy (2016), Nadauld, Sensoy, Vorkink, and Weisbach (2017)). Such discounts can arise to compensate buyers for search costs, asymmetric information risks, or due to market power when there are few buyers. The parameter θ_X can be interpreted as the transaction costs of purchasing alternative assets, such as search costs, legal fees, placement agent fees, consultant fees, etc. The costs of interim liquidation (θ_L) and of purchases (θ_X) can be asymmetric (e.g., the cost of liquidation can be larger than the cost of purchases). In general, voluntary liquidation of the alternative asset can be costly, particularly when financing conditions are tough with few buyers and many sellers, as we saw during the recent financial crisis. Before we analyze the effect of illiquidity, we next construct measures to quantify the risk and return tradeoff for the alternative asset.

Alpha, Beta, and Epsilon (Unspanned Volatility). Suppose that the "instantaneous return" for the alternative asset, dA_t/A_{t-}, is perfectly measurable. We can then regress dA_t/A_{t-} on dS_t/S_t, and obtain the alternative asset's beta with respect to public equity, following the standard capital asset pricing model (CAPM) formula:

$$\beta = \frac{\rho\sigma_A}{\sigma_S}\,. \tag{5}$$

However, in reality, because investors cannot dynamically rebalance their holdings in the illiquid asset without incurring transaction costs, investors will demand an additional risk premium in addition to the standard risk premium implied by the CAPM.

We decompose the total volatility of the alternative asset, σ_A, into two orthogonal components: the part spanned by the public equity, $\rho\sigma_A$, and the remaining unspanned

volatility, ϵ, which is given by:

$$\epsilon = \sqrt{\sigma_A^2 - \rho^2 \sigma_A^2} = \sqrt{\sigma_A^2 - \beta^2 \sigma_S^2}\,. \tag{6}$$

This volatility, ϵ, which is unspanned by the public equity,[4] introduces an additional risk into the investor's overall portfolio, as markets are incomplete and adjusting the alternative asset holding is costly. We will show that the spanned and unspanned volatilities play distinct roles in the investors' dynamic asset allocation.

Anticipating our subsequent risk-return tradeoff analysis in the context of dynamic portfolio construction, we next introduce α implied by the CAPM where the public equity is used as the aggregate market portfolio. That is, α is defined as follows:

$$\alpha = \mu_A - (r + \beta(\mu_S - r))\,, \tag{7}$$

where β is the alternative asset's beta given by (5). In frictionless capital markets where investors can continuously rebalance their portfolio without incurring any illiquidity discounts or transaction costs, α measures the risk-adjusted excess return offered by this risky asset after benchmarking against the public equity index whose return is specified in (1). However, importantly, in our framework with illiquid assets, α also includes the part of the excess return that compensates investors for bearing an illiquidity premium (or for bearing any other systematic risk that is unspanned by public equity).

2.2. The Optimization Problem

Liquid Wealth and Net Worth. We use W to denote the investor's liquid wealth and Π to denote the amount allocated to public equity. The remaining liquid wealth,

[4] Although ϵ is unspanned by public equity, this does not necessarily imply that ϵ is purely idiosyncratic risk. We take no stand on whether public equity is the market portfolio in the sense of Roll (1977). Thus, ϵ may include systematic risks that are potentially compensated, but which are not correlated with the market portfolio.

$W - \Pi$, is allocated to the risk-free bond. Thus, liquid wealth evolves according to:

$$\begin{aligned} dW_t &= (rW_{t-} + \delta_A A_{t-} - C_{t-})\, dt + \Pi_{t-} \left((\mu_S - r)dt + \sigma_S d\mathbb{B}_t^S\right) \\ &\quad + (1-\theta_L)dL_t - (1+\theta_X)dX_t + \delta_T A_{t-}\mathbb{I}_{\{t=mT-\}}\,, \end{aligned} \tag{8}$$

where the first two terms in (8) are the standard ones in Merton's consumption/portfolio-choice problem. The third and fourth terms describe the effect on liquid wealth W due to the investor's interim liquidation and purchase of the alternative asset. Recall that θ_L and θ_X capture the proportional cost of interim liquidations and purchases of the alternative asset, respectively. Finally, the last term captures the lumpy payout to the investor at the automatic liquidity event dates $t = mT$.

In general, W and A are not perfect substitutes and the investor values them differently. Hence, simply adding them up generally does not make much economic sense. Having expressed our concerns, we still define the investor's net worth as the sum of the two sources of wealth, as conventionally done purely as an accounting measure:

$$N_t \equiv W_t + A_t\,. \tag{9}$$

It is convenient to use net worth N when demonstrating our model's solutions and linking to the empirical literature. We will show when net worth makes economic sense and when it does not. Finally, we close our model by introducing the investor's preferences and defining the value functions.

Preferences and Value Functions. The risk-averse investor is infinitely lived and has the following standard expected utility over a given consumption[5] process $\{C_t : t \geq 0\}$:

$$\mathbb{E}\left[\int_0^\infty e^{-\zeta t} U\left(C_t\right) dt\right], \tag{10}$$

[5]In the case of some investors, such as a university endowment, "consumption" would refer to the periodic payment from the investment fund to the university for expenditure.

where the discount rate $\zeta > 0$ is a constant and $U(C)$ is increasing and concave in C. We choose the widely-used constant-relative-risk-averse (CRRA) utility,

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma}\,, \tag{11}$$

where $\gamma > 0$ is the coefficient of relative risk aversion. Unlike Sorensen, Wang, and Yang (2014) who use constant-absolute-risk-averse (CARA) utility and hence ignore the wealth effect by assumption, our choice of CRRA utility allows us to incorporate the wealth effect on asset allocation, which is critical for asset allocation rules in practice and also ensures that consumption is positive. Another related desirable feature of CRRA utility is that it allows us to derive asset allocations and spending rates as percentages of net worth, which is the standard format practitioners use when setting policy rules.

The investor has three state variables: liquid wealth W_t, the alternative asset's value A_t, and calendar time t. Let $V(W_t, A_t, t)$ denote the corresponding value function. The investor chooses consumption C, public equity investment Π, and the alternative asset's cumulative liquidation L and cumulative acquisition X to maximize the value function $V(W_t, A_t, t)$ by solving:

$$V(W_t, A_t, t) = \max_{\{C,\,\Pi,\,L,\,X\}} \mathbb{E}\left[\int_t^\infty e^{-\zeta(s-t)} U\left(C_s\right) ds\right]. \tag{12}$$

Naturally, at each automatic liquidity event date mT, if $W_{mT} = W_{(m-1)T} = W$, and $A_{mT} = A_{(m-1)T} = A$, we must have

$$V(W, A, mT) = V(W, A, (m-1)T)\,. \tag{13}$$

Hence, it is sufficient for us to characterize our model over $(0, T]$, as the solution is stationary every T years.

3. Solution

We next provide solutions for the model. The state variables are the illiquid alternative asset holding A_t, calendar time t, as well as liquid wealth W_t.

Dynamic Programming and First-order Conditions (FOCs). Fix time t within the time interval $((m-1)T, mT)$, where m is a positive integer. Using the standard dynamic programming, we have the following standard HJB equation for the investor's value function $V(W, A, t)$ in this region:

$$\begin{aligned}\zeta V(W,A,t) &= \max_{C,\Pi} U(C) + (rW + \delta_A A + (\mu_S - r)\Pi - C)V_W + \frac{(\Pi\sigma_S)^2}{2}V_{WW} \\ &\quad + V_t + (\mu_A - \delta_A)AV_A + \frac{\sigma_A^2 A^2}{2}V_{AA} + \rho\Pi A\sigma_S\sigma_A V_{WA}\,. \end{aligned} \tag{14}$$

The left side of (14) is the flow value of the investor's value function. The first three terms on the right side of (14) capture the standard effects of consumption and asset allocation (both drift and volatility effects) on the investor's value function as in Merton (1971). The investor's opportunity to invest in the illiquid alternative asset generates three additional effects on asset allocation: 1) the effect of target holding horizon T captured by V_t; 2) the risk-return and diversification effects of changes in the value of the alternative asset A; and 3) the additional hedging benefits due to the correlation between public equity and the alternative asset. The investor optimally equates the two sides of (14) in the interior region where there is no interim liquidation and acquisition.

The optimal consumption C is characterized by the following standard FOC:

$$U'(C) = V_W(W, A, t), \tag{15}$$

which equates the marginal utility of consumption with the marginal value of wealth V_W.

The optimal investment in the public equity is given by:

$$\Pi = -\frac{\eta_S}{\sigma_S}\frac{V_W}{V_{WW}} - \frac{\rho\sigma_A}{\sigma_S}\frac{AV_{WA}}{V_{WW}}. \tag{16}$$

The first term gives the classical Merton's mean-variance demand and the second term captures the investor's hedging demand with respect to the illiquid alternative asset by using public equity.

Certainty equivalent wealth $P(W, A, t)$. We express the investor's value function $V(W, A, t)$ during the time period $t \in ((m-1)T, mT)$ as:

$$V(W, A, t) = \frac{(b_1 P(W, A, t))^{1-\gamma}}{1-\gamma}, \tag{17}$$

where b_1 is a constant given by:

$$b_1 = \phi_1^{\frac{1}{1-\gamma^{-1}}}, \tag{18}$$

and ϕ_1 is the constant marginal propensity to consume (MPC) given by

$$\phi_1 = \zeta + (1-\gamma^{-1})\left(r - \zeta + \frac{\eta_S^2}{2\gamma}\right). \tag{19}$$

We are then able to interpret $P(W, A, t)$ as the investors' certainty equivalent wealth, which is the minimal amount of total wealth required for the investor to permanently give up the opportunity to invest in the alternative asset. That is, in the interim period where $(m-1)T < t < mT$,

$$V(W, A, t) = J(P(W, A, t)), \tag{20}$$

where $J(\,\cdot\,)$, given by (A.2), is the value function with only liquid public equity and bonds as investable assets.

Homogeneity Property. In our model, the certainty equivalent wealth $P(W, A, t)$ has the homogeneity property in W and A, and hence it is convenient to work with the liquidity ratio $w_t = W_t/A_t$ and the scaled certainty equivalent wealth function $p(w_t, t)$ defined as follows:

$$P(W_t, A_t, t) = p(w_t, t) \cdot A_t \,. \tag{21}$$

This homogeneity property is due to the CRRA utility and the value processes for public equity and the alternative asset. Importantly, this homogeneity property allows us to conveniently interpret the optimal portfolio rule and target asset allocation in the tradition of Merton (1971).

Endogenous Effective Risk Aversion γ_i. To better interpret our solution it is helpful to introduce the following measure of endogenous relative risk aversion for the investor, denoted by $\gamma_i(w, t)$ and defined as follows:

$$\gamma_i(w, t) \equiv -\frac{V_{WW}}{V_W} \times P(W, A, t) = \gamma p_w(w, t) - \frac{p(w, t) p_{ww}(w, t)}{p_w(w, t)} \,. \tag{22}$$

In (22) the first identity sign gives the definition of γ_i and the second equality follows from the homogeneity property.

What economic insights does $\gamma_i(w, t)$ capture and what is the motivation of introducing $\gamma_i(w, t)$? First, recall the standard definition of the investor's coefficient of absolute risk aversion is $-V_{WW}/V_W$. How do we convert this absolute risk aversion measure to a relative risk aversion measure? We need to multiply absolute risk aversion $-V_{WW}/V_W$ with an appropriate economic measure for the investor's total wealth. Under incomplete markets, although there is no market-based measure of the investor's economic well being, the investor's certainty equivalent wealth $P(W, A, t)$ is a natural measure of the

investor's welfare. This motivates our definition of γ_i in (22).[6] We will show that the illiquidity of alternative assets causes the investor to be effectively more risk averse, meaning $p_w(w,t) > 1$ and $p_{ww}(w,t) < 0$, so that $\gamma_i(w,t) > \gamma$. Intuitively, market incompleteness causes the investor to be effectively more risk averse. In contrast, if the alternative asset is publicly traded (and markets are complete), we have $\gamma_i(w,t) = \gamma$ as $p_w(w,t) = 1$ and $p_{ww}(w,t) = 0$.

Optimal Policy Rules. Again, by using the homogeneity property, we may express the scaled consumption rule $c(w_t,t) = C(W_t, A_t, t)/A_t$ as follows:

$$c(w,t) = \phi_1 \, p(w,t) \, p_w(w,t)^{-1/\gamma} \, . \tag{23}$$

Because illiquidity makes markets incomplete, the investor's optimal consumption policy is no longer linear and depends on both the certainty equivalent wealth $p(w,t)$ and also the marginal certainty equivalent value of liquid wealth $p_w(w,t)$.

The allocation to public equity is $\Pi_t = \pi(w_t,t)A_t$ where $\pi(w,t)$ is given by

$$\pi(w,t) \;\; = \;\; \frac{\eta_S}{\sigma_S} \frac{p(w,t)}{\gamma_i(w,t)} - \frac{\rho \sigma_A}{\sigma_S} \left(\frac{\gamma p(w,t)}{\gamma_i(w,t)} - w \right) , \tag{24}$$

where $\gamma_i(\,\cdot\,)$ is the investor's effective risk aversion given by (22). Intuitively, the first term in (24) reflects the mean-variance demand for the market portfolio, which differs from the standard Merton model in two ways: 1) risk aversion γ is replaced by the effective risk aversion $\gamma_i(w,t)$ and 2) net worth is replaced by certainty equivalent wealth $p(w,t)$. Because $\gamma_i(w,t) > \gamma$ and markets are incomplete, the investor is effectively more risk averse and also less wealthy, both of which cause the mean-variance demand to be lower than in the full-spanning case. The second term in (24) captures the dynamic hedging

[6] See Wang, Wang, and Yang (2012) and Bolton, Wang and Yang (2017) for similar definitions involving endogenous risk aversion but for very different economic applications.

demand, which also depends on $\gamma_i(w,t)$ and $p(w,t)$. Finally, we turn to the PDE for $p(w,t)$.

PDE for $p(w,t)$. Substituting the policy rules for c, and π and the value function (17) into the HJB equation (14) and using the homogeneity property and the definition of the investor's effective risk aversion, γ_i, given by (22), we obtain the following PDE for $p(w,t)$ at time t, for the liquidity ratio w_t in the interior region, and when $(m-1)T < t < mT$:

$$0 = \left(\frac{\gamma\phi_1 \left(p_w(w,t)\right)^{1-\gamma^{-1}} - \zeta}{1-\gamma} + \mu_A - \delta_A - \frac{\gamma\sigma_A^2}{2} \right) p(w,t) + p_t(w,t) + \frac{\epsilon^2 w^2}{2} p_{ww}(w,t) \qquad (25)$$
$$+ \left(\delta_A - \alpha + \gamma\epsilon^2\right) w p_w(w,t) - \frac{\gamma\epsilon^2 w^2}{2} \frac{\left(p_w(w,t)\right)^2}{p(w,t)} + \frac{(\eta_S - \gamma\rho\sigma_A)^2 p_w(w,t) p(w,t)}{2\gamma_i}.$$

Because of incomplete spanning (e.g., $\epsilon \neq 0$), unlike Black-Scholes, (25) is a nonlinear PDE, and moreover, $p_w(w,t) > 1$, as we will show. The numerical solution for $p(w,t)$ involves the standard procedure. Next, we analyze how the investor actively rebalances the allocation to the illiquid alternative asset.

Rebalancing the Illiquid Alternative Asset during the Interim Period. Although under normal circumstances the investor plans to hold the alternative asset until the target date mT when an automatic liquidity event occurs, under certain circumstances the investor may find it optimal to actively rebalance when $t \neq mT$.

Consider an investor whose pre-liquidation holdings in public equity and the alternative asset at time t are (W_t, A_t). Let $\Delta > 0$ denote the unit of the alternative asset that the investor is considering to voluntarily liquidate. As the cost of liquidating the alternative asset prematurely is θ_L for each unit of the asset, the investor's post-liquidation position, is then (W_t', A_t'), where $W_t' = W_t + (1-\theta_L)\Delta$ and $A_t' = A_t - \Delta$. We use the

prime superscript to denote the post-liquidation position.

One necessary condition when the investor liquidates a portion of the alternative asset is that the value function is continuous, which implies $P(W_t', A_t', t) = P(W_t, A_t, t)$, implicitly defining the lower liquidation boundary $\underline{W_t}$ as a function of A_t at time t. Intuitively, when the illiquid alternative asset position is too high compared with the liquid wealth W_t at time t, the investor may choose to voluntarily reduce the illiquid asset holding. In the differential form and after using the homogeneity property to simplify, we show in Appendix A that the following holds:

$$p(\underline{w_t}, t) = \left(1 - \theta_L + \underline{w_t}\right) p_w(\underline{w_t}, t)\,, \tag{26}$$

which defines $\underline{w_t}$, the lower liquidation boundary for the liquidity ratio w_t at time t. As $p \geq 0$ and $p_w \geq 0$, it is immediate to see that $\underline{w_t} \geq -(1-\theta_L)$. That is, the investor cannot borrow more than $(1 - \theta_L)$ fraction of the alternative asset's fundamental value. This ensures that the investor's liability can be fully repaid with probability one by liquidating the alternative asset. Note that the investor's debt capacity is endogenously determined by the liquidation value of the alternative asset holdings. Although the investor can borrow, in our numerical exercise, as in reality, the investor rarely does borrow.

Following essentially the same procedure, we obtain the following condition for the interim acquisition of alternative assets:

$$p(\overline{w_t}, t) = (1 + \theta_X + \overline{w_t})\, p_w(\overline{w_t}, t)\,, \tag{27}$$

which defines $\overline{w_t}$, the endogenous acquisition boundary for w_t at time t.

Next, we provide the conditions that describe the investor's *optimal* liquidation and acquisition decisions. By differentiating (26) with respect to $\underline{w_t}$ and (27) with respect to

$\overline{w_t}$, we obtain the following boundary conditions:

$$p_{ww}\left(\underline{w_t}, t\right) = 0\,, \tag{28}$$

$$p_{ww}\left(\overline{w_t}, t\right) = 0\,, \tag{29}$$

which are often referred to as the "super contact" conditions as in Dumas (1991). The intuition for the super-contact condition is as follows. For an optimally chosen liquidation boundary $\underline{w_t}$ or acquisition boundary $\overline{w_t}$, the investor's marginal (certainty equivalent) value of allocating a unit of wealth at the margin to either public equity or the alternative asset must be equal.

Value and Decisions at $t = mT$. The homogeneity property allows us to express the value-matching condition (13) in terms of $p(w,t)$ at $t = mT$:

$$p(w, mT) = p(w, (m-1)T)\,. \tag{30}$$

As acquisition and voluntary liquidation are costly, we have an inaction region at all time including $t = mT$.

Additionally, at $t = mT$, a fraction δ_T of the alternative asset automatically becomes fully liquid meaning that the alternative asset holding A changes discretely from A_t to $A_t - \delta_T A_t$ and the liquid wealth W correspondingly changes from W_t to $W_t + \delta_T A_t$ as $t \to mT$ in the absence of any acquisition or voluntary liquidation.

For expositional simplicity, we denote the corresponding levels of liquid wealth and the alternative asset: $\widehat{W}_{mT} = \lim_{t\to mT}(W_t + \delta_T A_t)$ and $\widehat{A}_{mT} = \lim_{t\to mT}(A_t - \delta_T A_t)$. Let $\widehat{w}_{mT}$ denote the corresponding liquidity ratio at mT:

$$\widehat{w}_{mT} \equiv \frac{\widehat{W}_{mT}}{\widehat{A}_{mT}} = \lim_{t\to mT} \frac{w_t + \delta_T}{1 - \delta_T}\,. \tag{31}$$

There are two cases to consider at $t = mT$: $(i.)$ $\widehat{w}_{mT} \leq \overline{w}_{mT}$ and $(ii.)$ $\widehat{w}_{mT} > \overline{w}_{mT}$. Case $(i.)$ means that, even after the automatic liquidity event, the liquid asset holding remains below the threshold $\overline{w}_{mT}$ at which the investor rebalances by voluntarily acquiring more of the illiquid asset. In contrast, Case $(ii.)$ means that the automatic liquidity event causes the investor's portfolio to be overly exposed to the liquid asset. In this latter case, the investor needs to acquire additional illiquid alternative asset to rebalance the overall portfolio. Finally, as the automatic liquidity event always increases the liquid asset holding relative to the illiquid asset holding, it thus will never cause $\widehat{w}_{mT}$ to fall below $\underline{w}_{mT}$ and hence we need only consider the two cases that may be triggered by the automatic liquidity event with a scheduled lump-sum payment $\delta_T A_{mT-}$.

- Case (i): $\widehat{w}_{mT} \leq \overline{w}_{mT}$. The continuity of the value function and the homogeneity property imply $\lim_{t\to mT} p(w,t)A_t = p(\widehat{w}_{mT}, mT)\widehat{A}_{mT}$. Simplifying this, we obtain:

$$\lim_{t\to mT} p(w,t) = p(\widehat{w}_{mT}, t)(1-\delta_T) = \lim_{t\to mT} p\left(\frac{w_t+\delta_T}{1-\delta_T}, mT\right)(1-\delta_T)\,. \tag{32}$$

- Case (ii): $\widehat{w}_{mT} > \overline{w}_{mT}$. The investor optimally purchases Δ units of the alternative asset such that $\widehat{W}_{mT} - (1+\theta_X)\Delta = \overline{W}_{mT}$. By purchasing Δ units, the investor pulls the liquidity ratio w back to the upper bound of the target asset allocation range $(\overline{w}_{mT}, \underline{w}_{mT})$, so that the following identity,

$$\overline{w}_{mT} = \frac{\widehat{W}_{mT} - (1+\theta_X)\Delta}{\widehat{A}_{mT} + \Delta} = \lim_{t\to mT} \frac{W_t + \delta_T A_t - (1+\theta_X)\Delta}{A_t - \delta_T A_t + \Delta}\,, \tag{33}$$

holds. Solving the above gives the following expression for the number of units for the alternative asset, Δ, to be purchased at time mT:

$$\Delta = \lim_{t\to mT} \frac{w_t + \delta_T - \overline{w}_{mT}(1-\delta_T)}{1+\theta_X+\overline{w}_{mT}}\, A_t \tag{34}$$

Again, the continuity of the value function and homogeneity property imply

$$\lim_{t\to mT} p(w,t) = p\left(\overline{w}_{mT}, mT\right)\left(1-\delta_T + \frac{w_{mT-} + \delta_T - \overline{w}_{mT}(1-\delta_T)}{1+\theta_X+\overline{w}_{mT}}\right). \tag{35}$$

Next, we summarize the main results of our model.

Proposition 1 *The scaled certainty equivalent wealth $p(w,t)$ in the interim period when $(m-1)T < t \leq mT$ solves the PDE (25) subject to the boundary conditions (26), (27), (28), (29), and (30). Additionally, $p(w_{mT-}, mT-)$ satisfies (32), if $\widehat{w}_{mT} < \overline{w}_{mT}$ where $\widehat{w}_{mT}$ is given by (31), and satisfies (35) if $\widehat{w}_{mT} \geq \overline{w}_{mT}$.*

4. Data and Calibration

4.1. Data and Summary Statistics

The data on university endowment fund sizes, asset allocations, and returns come from annual surveys conducted by the National Association of College and University Business Officers (NACUBO) and Commonfund. We focus on the cross-section of 774 university endowment funds[7] as of the end of the 2014-2015 academic year.

Asset allocation. The NCES provides annual snapshots of endowment funds' portfolio allocations. To link the NCES data to the model, we aggregate endowment allocations in the NCES data into the three asset classes featured in our model: (1) *the risk-free asset*, which aggregates cash and fixed income, (2) *public equity*, which aggregates public equity and REITs,[8] and (3) *the alternative asset*, which aggregates hedge funds (including managed futures), private equity, venture capital, private real estate, and illiquid natural resources. In the summary statistics, and for generating some of the calibration parameters, we use the disaggregated sub-asset classes (e.g., hedge funds, etc.).

[7] If there are multiple distinct endowment funds associated with a single university, we aggregate them to obtain a single fund per university.

[8] We group together public equity and REITs because they have similar levels of risk and of liquidity. Further, the NCES data set includes REITs in the public equity category.

Table 1: Summary of Endowment Fund Asset Allocation

This table summarizes endowment fund portfolios as of the end of the 2014-2015 academic year for 774 endowments. The last column reports averages that are value weighted by endowment fund size. Panel A shows summary statistics for endowment fund size (reported in millions of dollars), asset class allocations and spending rates (reported in percentages), and the number of alternative asset management firms with which the endowment invests. *Cash & Fixed Income* includes cash, cash equivalents, and fixed income securities (except for distressed securities). *Public Equity* includes domestic and foreign equity as well as REITs. *Alternative Allocations* includes hedge funds, private equity, venture capital, private real estate, and illiquid natural resources. Panel B reports summary statistics for the more detailed sub- asset classes. *Hedge Funds* includes managed futures. *Real Estate* reports holdings of private real estate. *Natural Resources* includes illiquid natural resources, such as timberland and oil & gas partnerships.

	Mean	Std Dev	Min	5%	25%	Median	75%	95%	Max	VW Mean
Panel A										
Endowment Size ($M)	677	2,483	1	15	49	114	383	2,332	36,449	
Cash & Fixed Income	21.0%	11.3	0.0	6.6	12.9	19.0	27.4	40.7	91.2	12.7%
Public Equity	50.7%	14.6	0.0	25.4	42.0	51.2	59.9	73.5	100.0	35.6%
Alternative Allocations	28.3%	19.3	0.0	0.0	12.9	28.1	41.4	61.4	100.0	51.7%
Spending Rate	4.2%	2.2	0.1	2.0	3.5	4.1	4.7	6.0	51.0	4.4%
No. Alternative Funds	16.9	28.6	0.0	0.0	1	4	21	74	222	56.0
Panel B										
Cash & Equivalents	5.1%	8.3	-15.9	0.0	0.8	2.8	6.4	17.2	91.2	4.0%
Fixed Income	15.9%	9.3	0.0	2.2	9.3	14.5	21.9	32.7	54.0	8.7%
Public Equity	50.7%	14.6	0.0	25.4	42.0	51.2	59.9	73.5	100.0	35.6%
Hedge Funds	16.7%	12.2	0.0	0.0	6.9	17.2	24.5	36.9	83.0	23.4%
Private Equity	4.6%	6.6	0.0	0.0	0.0	3.0	6.9	14.6	95.6	10.9%
Venture Capital	1.7%	3.3	0.0	0.0	0.0	0.0	2.1	8.5	33.9	5.7%
Private Real Estate	2.7%	3.8	0.0	0.0	0.0	1.3	4.2	9.2	34.4	6.1%
Natural Resources	2.7%	3.5	0.0	0.0	0.0	1.4	4.5	9.2	29.7	5.9%

Table 1 shows summary statistics as of the end of the 2014-2015 academic year for the sample of endowment funds. The summary statistics are equal weighted, except for the last column which reports value weighted averages. Panel A shows the average *Endowment Size* is $677 million, but this is highly skewed and the median is $114 million. On an equal weighted basis, the largest asset class is *Public Equity*, with an average allocation of 50.7% and a value weighted average of 35.6%, implying that percentage allocations to public equity decrease with fund size. On a value weighted basis, *Alternative Allocations* is the largest asset class at 51.7%, compared with 35.6% for public equity and 12.7% for cash and fixed income. Comparing the equal and value weighted averages shows that allocations to alternative assets are considerably higher for larger endowments. Panel A also shows that the average endowment spends 4.2% of its assets in 2015. The final row shows that the average endowment invests with nearly 17 distinct alternative asset managers, but the median invests with only four. There is considerable dispersion, and larger endowments invest with significantly more distinct managers.

Panel B of Table 1 shows summary statistics for the more detailed sub-asset categories. Within *Alternative Allocations*, the largest sub-asset class is hedge funds with an average allocation of 16.7%. For all of the sub-asset classes, the value weighted averages are considerably larger than the equal weighted, particularly for the least liquid categories: private equity, venture capital, private real estate, and illiquid natural resources.

As seen in Figure 1, *Alternative Allocations* have increased over time.[9] The portfolio allocations for 2015 reported in Table 1 are substantially higher than the allocations reported in studies using earlier waves of the NCES data, such as Brown, Garlappi, and Tiu (2010) or Dimmock (2012).

[9]Note that to avoid conflating time-series changes and changes in the composition of endowments included in the NCES, Figure 1 uses only data for endowment funds that were in the sample in 1990.

Table 2: The Cross-Section of Endowment Fund Asset Allocation

This table summarizes the dispersion in asset allocation across the cross-section of endowment fund sizes as of the end of the 2014-2015 academic year. The columns show allocations for size-segmented groups of endowment funds. For example, the column "0-10%" shows the average portfolio allocation for the smallest decile of endowment funds. The column "45-55%" shows the average portfolio allocation for the decile of endowment funds centered around the median fund size. The final two columns show the average portfolio allocations for the largest 20 and 10 endowment funds, respectively. In all columns, the averages are value weighted by endowment fund size within the corresponding group. The target horizon variables summarized in Panel A are reported in years. *Alternative Target Horizon* reports the target horizon of the portion of the portfolio allocated to alternative assets. *Port. Target Horizon* reports the value weighted target horizon of the entire portfolio. *Average No. of Alt. Funds* (*Median No. of Alt. Funds*) is the average (median) number of different alternative asset management firms with which the endowment invests (we do not report these variable for the 10 largest endowment funds, as only three of the 10 largest funds report the number of managers).

	0-10%	20-30%	45-55%	70-80%	90-100%	Top 20	Top 10
Panel A							
Endowment Size ($M)	17	50.1	116	408	1,3409	18,585	22,506
Cash & Fixed Income	33.1%	21.7	22.4	15.2	10.8	9.7	9.4
Public Equity	60.1%	57.9	54.7	45.9	32.0	29.3	29.3
Alternative Allocations	6.3%	20.4	22.9	38.9	57.1	61.0	61.3
Alt. Target Horizon	3.6	4.1	4.0	4.6	5.9	6.3	6.6
Port. Target Horizon	0.19	0.80	0.89	1.8	3.4	3.8	4.1
Spending Rate	4.5%	3.7	3.9	3.9	4.5	4.5	4.6
Average No. of Alt. Funds	1.1	5.6	7.2	22.5	86.5	152.2	–
Median No. of Alt. Funds	1	3	5	23	83	166	–
Panel B							
Cash & Equivalents	7.2%	3.5	5.6	3.8	3.5	2.6	2.2
Fixed Income	25.8%	18.2	16.8	11.4	7.4	7.1	7.2
Public Equity	60.1%	57.9	54.7	45.9	32.0	29.3	29.3
Hedge Funds	4.6%	13.0	14.3	22.3	23.8	23.7	21.9
Private Equity	0.2%	3.1	3.1	7.1	12.3	14.0	15.0
Venture Capital	0.3 %	0.5	0.4	2.2	6.9	7.3	7.3
Private Real Estate	0.4%	1.8	2.7	3.1	7.0	8.4	9.3
Natural Resources	0.8%	1.9	2.0	4.2	6.7	7.5	7.9

Table 2 shows the cross-section of endowment fund asset allocation in 2015, with funds segmented by endowment size. For example, the first column shows the average allocations of endowment funds in the bottom size decile. Smaller endowments have higher allocations to cash, fixed income, and public equity; larger endowments have higher allocations to alternative assets. The differences in allocations are largest for private equity, venture capital, and natural resources, which are the least liquid categories.

Portfolio Illiquidity and Target Horizons. We estimate endowment fund target holding periods based on allocations to alternative assets combined with the horizons of the asset classes. For hedge funds, we assume a horizon of six months, which is approximately equal to the sum of the average redemption, advance notice, and lockup periods reported in Getmansky, Liang, Schwarz, and Wermers (2015). For private equity and venture capital, we assume a horizon of 10 years, based on the average commitment period reported in Metrick and Yasuda (2010). For private real estate and illiquid natural resources, we also assume a horizon of 10 years, based on the holding periods reported in Collet, Lizieri, and Ward (2003) and Newell and Eves (2009).

Panel B of Table 2 also summarizes the calculated target horizons for the cross-section of endowment funds. *Alternative Target Horizon* reports the value weighted average target horizon for the alternative assets held in the portfolio. *Portfolio Target Horizon* reports the target horizon of the entire portfolio. The portfolios of the larger endowment funds are substantially less liquid, both because they have higher allocations to alternative assets and because they allocate proportionally more of their alternative asset holdings to the least liquid categories.

Table 2 also reports the average number of alternative asset management firms with which the endowment invests, which is an important component of liquidity manage-

ment. Consider two examples. Suppose Endowment A invests its entire alternative asset allocation into a single private equity fund, with a 10 year partnership agreement. Suppose Endowment B invests its alternative asset allocation across 120 different private equity funds, each of which has a 10 year partnership agreement. Further suppose that Endowment B staggers its investments across time, so that one partnership agreement expires every month for the next 10 years. Although both endowments might have the same portfolio allocation to private equity, they have very different liquidity exposures. For the next 10 years, Endowment A can only reduce its private equity exposure through the secondary market. In contrast, Endowment B can costlessly reduce its exposure to private equity as partnership agreements expire each month. Thus, by holding multiple funds and staggering their maturity over time, the endowment can enhance the liquidity of the whole portfolio, which we refer to as *liquidity diversification*.

As Panel A of Table 2 shows, there is a strong positive relation between endowment size and the number of alternative asset funds. On average, endowments in the largest decile hold 86.5 alternative asset funds; endowments in the smallest decile own only a single fund. In the full sample, 8.1% of endowments do not own any alternative assets at all, and as expected, zero-holdings are far more common for smaller endowments.

4.2. Parameter Choices and Calibration

Calculating Unspanned Volatility. Calibrating the model requires the standard deviation, beta, and unspanned volatility of the representative alternative asset. To obtain these parameters, we build up from the standard deviations and correlations of the sub-asset classes comprising the representative alternative asset. For each sub-asset class a, we combine its β_a and R_a^2 with the standard deviation of the market $\sigma_S = 20\%$ to obtain the implied standard deviation for the asset class: $\sigma_a = \sqrt{\beta_a^2 \sigma_S^2 / R_a^2}$.

Panel A of Appendix Table A1 shows the β_a, R^2_a, and σ_a for each of the alternative sub-asset classes. For hedge funds, the β and R^2 are taken from Getmansky, Lo, and Makarov (2004) and account for return smoothing; other studies that account for return smoothing find similar values (e.g., Asness, Krail, and Liew (2004) and Jurek and Stafford (2015)). For private equity and venture capital, the βs and R^2s are taken from Ewens, Jones, and Rhodes-Kropf (2013).[10] For private real estate and illiquid natural resources, the variables are based on Pedersen, Page, and He (2014) and account for return smoothing.

Panel B of Appendix Table A1 shows the pairwise correlations between the asset classes. These are calculated using index returns over the period 1994-2015.[11] The correlations with private equity, venture capital, private real estate, and illiquid natural resources use quarterly returns; the other correlations use monthly returns.

We combine the asset allocations from Table 2 with the βs, σs, and correlations from Appendix Table A1 to impute portfolio βs, σs, and unspanned volatilities (ϵ). Table 3 shows the imputed variables for the cross-section of endowment funds. Although large endowments substitute alternative assets for public equity and fixed income, the implied portfolio β are remarkably similar across the size groups. The implied portfolio σs are higher for larger endowments, because they have greater exposure to the idiosyncratic risk of alternative assets, but the implied ϵ are similar across size groups.

Parameter choices. We base several of our parameters on the endowment fund data discussed above. Specifically, we set the alternative asset $\beta_A = 0.6$ and the unspanned volatility of the alternative asset to $\epsilon = 15\%$ with the objective of closely targeting the

[10] Empirical estimates for the β of venture capital vary widely, ranging from 1.2 (Kraussl, Jegadeesh, and Pollet (2015) to 2.73 (Driessen, Lin, and Phalippou (2012)).

[11] The index returns used to calculate the correlations in Panel B are: Bloomberg/Barclays US Aggregate Bond Index, CRSP value weighted index, Credit Suisse/Tremont Aggregate Hedge Fund Index, Cambridge Associates U.S. Private Equity Index, Cambridge Associates U.S. Venture Capital Index, NCREIF Property Index (unsmoothed), and the S&P Global Timber and Forestry Index.

Table 3: Implied Endowment Fund Risk and Unspanned Volatility

This table shows the implied βs, σs, and unspanned volatilities of the endowment fund portfolios whose allocations are shown in Table 2. Note that β_A is the beta of the alternative asset holdings, σ_A is the standard deviation of alternative assets, and ϵ is the volatility of the endowment fund's alternative asset holdings that is unspanned by public equities. *Portfolio* β and *Portfolio* σ are the overall endowment portfolio β and standard deviation, respectively. The columns show allocations for size-segmented groups of endowments. e.g., the column "0-10%" shows the value-weighted statistics for the smallest decile of endowment funds. The last column shows summary statistics for the value weighted endowment fund portfolio.

	0-10%	20-30%	45-55%	70-80%	90-100%	Top 20	Top 10	VW
β_A	0.53	0.55	0.54	0.57	0.62	0.62	0.62	0.61
σ_A	17.7%	17.7	17.3	18.2	18.9	18.8	18.8	18.7
ϵ	14.1%	13.9	13.5	14.2	14.3	14.2	14.1	14.2
Portfolio β	0.63	0.69	0.67	0.68	0.67	0.67	0.67	0.67
Portfolio σ	12.9%	14.5	14.1	15.3	16.1	16.4	16.4	15.8

value weighted endowment fund portfolio values reported earlier ($\beta_A = 0.61$ and $\epsilon = 14.2\%$). We set the horizon of the representative alternative asset $H = 6$ years based on the Alternative Target Horizons summarized earlier.

Following the literature, we choose the following standard parameter values: the annual risk-free rate $r = 4\%$ and the investor's coefficient of relative risk aversion $\gamma = 2$. We also set the investor's discount rate equal to the risk-free rate, $\zeta = r$. For public equity, we use an annual volatility of $\sigma_S = 20\%$, with the widely-used aggregate equity risk premium of $\mu_S - r = 6\%$, which implies a Sharpe ratio of $\eta_S = 0.3$.

In our model, the alpha of the illiquid alternative investment includes compensation for skill, liquidity risk, and other risks unspanned by public equities. We set $\alpha = 2\%$, which we view as conservative given the empirical findings in the literature. For example, Franzoni, Nowak, and Phalippou (2012) find that private equity earns a net-of-fees liquid-

ity risk premium of 3% annually. Aragon (2007) and Sadka (2010) find similar net-of-fees liquidity risk premiums for hedge funds. Empirical studies of endowment funds also find a positive relation between performance and allocations to illiquid assets (e.g., Lerner, Schoar, and Wang (2008), Brown, Garlappi, and Tiu (2010), Barber and Wang (2013), and Ang, Ayala, and Goetzmann (2014)). Consistent with these studies, in unreported results we also find a positive relation between endowment fund alpha and unspanned volatility. Given this assumed alpha, the expected overall return on the alternative asset is $\mu_A = 0.02 + 0.04 + 0.6 \times (0.10 - 0.04) = 0.096 = 9.6\%$.

For voluntary liquidations, we assume the proportional transaction cost is $\theta_L = 10\%$, based on empirical findings and the following back-of-the-envelope calculation: For secondary market liquidations of private equity, Kleymenova, Talmor, and Vasvari (2012) and Nadauld, Sensoy, Vorkink, and Weisbach (2017) find average discounts of 25.2% and 13.8%, respectively. For secondary market liquidations of hedge funds, Ramadorai (2012) finds an average discount of 0.9%, which rises to 7.8% during the financial crisis. Therefore, we combine the aggregate endowment fund portfolio weights with liquidation costs of 20% for PE and VC, 1% for hedge funds, and 10% for private real estate and timberland, to obtain a proportional liquidation cost of 9.3% for the representative alternative asset. For acquisitions, we assume the proportional acquisition cost is $\theta_X = 2\%$, which is equal to the average placement agent fee reported by Rikato and Berk (2015) and Cain, McKeon, and Solomon (2016).

Calibrating the model also requires a payout parameter, which determines the amount of automatic liquidity generated for the portfolio through automatic liquidity events (e.g., liquidity from funds maturing and paying out capital to the limited partners). The payout rate depends on the number of alternative asset funds held by the investor. For example, given the target horizon of $H = 6$ years, an investor with a single alternative asset

fund would receive a large payout once every six years. In contrast, an investor with a very large number of funds would receive payouts that are much smaller but much more frequent. For any given number of funds, Appendix E shows how it is possible to use the previously described parameter values to impute the payout rate. For our baseline calibration, we use the case of $n \to \infty$. As shown in Appendix E, this implies a payout rate of $\delta_A = 4.0\%$. Appendix E also considers other cases such as $n = 1$, $n = 12$, etc.

Table 4 summarizes the symbols for the key variables in the model and the parameter values in the baseline model.

Table 4: Summary of Key Variables and Parameters

This table summarizes the symbols for the key variables in the model and baseline parameter values. For certain upper case variable names in the left column (Π, W, C, and P), we use its lower case to denote the ratio of this variable to the amount of the alternative investment, A. For completeness, the table also reports *implied parameters*; redundant parameters whose values are determined by other parameters.

Variable	Symbol	Parameter	Symbol	Value
Alternative asset	A	Coefficient of relative risk aversion	γ	2
Market portfolio allocation	Π	Subjective discount rate	ζ	4%
Liquid wealth	W	Risk-free rate	r	4%
Total net worth	N	Public equity expected return	μ_S	10%
Consumption (expenditure)	C	Volatility of market portfolio	σ_S	20%
Certainty equivalent wealth	P	Beta of the alternative asset	β_A	0.6
Scaled net certainty equivalent wealth	q	Alternative asset alpha	α	2%
Cumulative interim alternative asset liquidations	L	Alternative asset's expected return	μ_A	9.6%
Cumulative interim alternative asset acquisitions	X	Volatility of alternative asset	σ_A	19.2%
Value function	V	The alternative asset's target horizon	H	6
Brownian motion for public equity return	$\mathbb{B}^S$	Proportional cost of liquidation	θ_L	0.1
Brownian motion for alternative asset return	$\mathbb{B}^A$	Proportional cost of acquisition	θ_X	0.02
Acquisition boundary	$\overline{w}$			
Liquidation boundary	$\underline{w}$	*Implied parameters*		
Liquid wealth at $\to mT$	$\widehat{W}_{mT}$	Public equity Sharpe ratio	η_S	0.30
Alternative asset at $\to mT$	$\widehat{A}_{mT}$	Correlation between risky assets	ρ	0.625
		Unspanned volatility	ϵ	15%
		The payout rate	δ_A	4.00%

5. Results

In this section, we use the parameter values from the preceding section to analyze our model's implications. We begin with the limiting case as $n \to \infty$, and then explore the case of $n = 1$. Following this, we study the comparative statics of the model.

For the purposes of comparison, we also providing a benchmark in which the alternative asset is fully liquid. In this case, the alternative asset simply expands the investment opportunity set by providing a second liquid risky asset, and the optimization problem is again the Merton's model with two publicly traded liquid risky assets. Thus, the value function for this case is clearly higher than when the alternative asset is illiquid. Indeed, the case of full spanning serves as an upper bound for the investor's value function. See Appendix B.1 for a detailed discussion.

5.1. The case of $n \to \infty$

Net certainty equivalent wealth. Figure 2 plots the results for the case of $n \to \infty$, in which $\delta_A = 4\%$. In both panels, the x-axis displays the liquidity ratio w. Panel A shows the investor optimally sets the range of w to lie between $(\underline{w}, \overline{w}) = (0.54, 2.64)$. That is, if the liquidity ratio w falls to the endogenous liquidation boundary, $\underline{w} = 0.54$, the investor immediately sells just enough units of the illiquid alternative asset and invests the proceeds in liquid assets with the objective of keeping $w \geq 0.54$. If the liquidity ratio rises to the endogenous acquisition boundary, $\overline{w} = 2.64$, the investor acquires just enough units of the illiquid asset so that w falls back to 2.64. In sharp contrast, when the alternative asset is perfectly liquid, (i.e., the full spanning case), the optimal liquidity ratio is a singleton rather than a line: $w^* = 1.25$, which one can show by rearranging (B.3): $\frac{W_t}{A_t} = \frac{N_t - A_t}{A_t} \equiv \frac{\gamma \epsilon^2}{\alpha} - 1 = 1.25$.

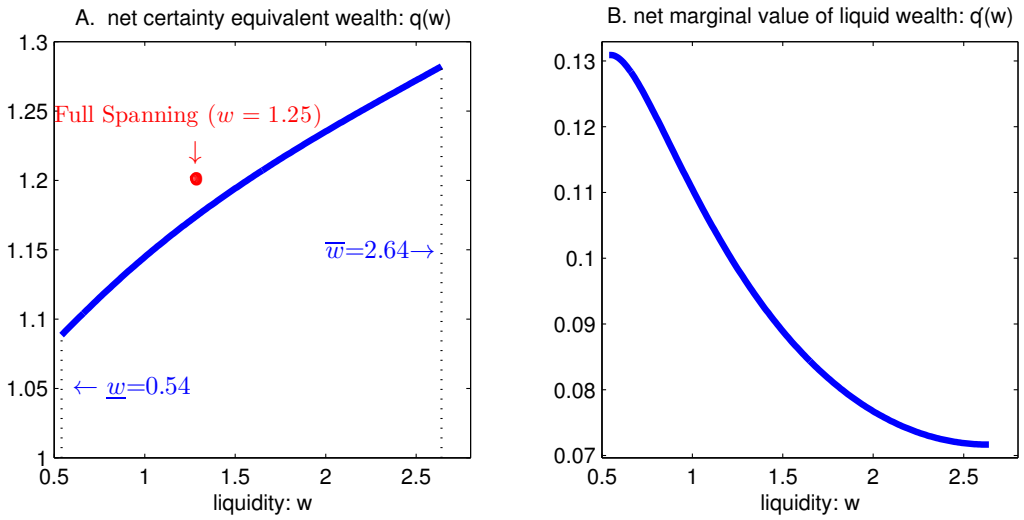


Figure 2: Panels A and B plot investors' net certainty equivalent wealth $q(w) = p(w) - w$ and net marginal value of liquid wealth $q'(w)$ as functions of the liquidity ratio w, respectively. The input parameter values are given in Table 4. For this figure, we set $n = \infty$ and $\delta_A = 4\%$. The blue solid lines show results for the general case within the optimal rebalancing boundaries $\underline{w} = 0.54$ and $\overline{w} = 2.64$. The red dot shows the solution under Full Spanning, which involves a static optimal $w^* = 1.25$ and $q^* = 1.20$.

Panel A of Figure 2 plots the liquidity ratio, w, on the x-axis and net certainty equivalent wealth

$$q(w) = p(w) - w \tag{36}$$

on the y-axis. Naturally, $q(w)$ conveys the same information as $p(w)$ but is easier to read graphically. As Panel A shows, the net certainty equivalent wealth $q(w)$ is increasing and concave in w. Intuitively, the curve is upward sloping because, as the illiquid asset becomes a smaller fraction of the portfolio, the investor incurs less and less disutility from the asset's illiquidity and unspanned volatility. Under full spanning, the net certainty equivalent wealth is higher than the for the same w under the general case. The vertical difference between ull spanning and the general case can be interpreted as the utility cost of illiquidity (measured in dollars per unit of contemporaneous A). For illustration, consider a simple example in which the investor has total net worth $N_t = \$225M$ with $A_t = \$100M$ invested in the alternative asset and the remaining $W_t = \$125M$ invested in liquid assets, implying $w_t = 1.25$. At this point, $q(w_t) = 1.17$ implying that, to avoid loss of utility, the investor would require $0.17 \times \$100M = \$17M$ in compensation to permanently give up the opportunity of investing in the alternative asset and invest only in the risk-free asset and public equity. Additionally, when the alternative asset is fully liquid, $q(w) = 1.20$ meaning that the additional value from full spanning is $(1.20-1.17) \times \$100M = \$3M$. For this numerical example, the vast majority (85%=\$17M/\$20M) of the alternative asset's value-added is due to the excess return. The additional value-add due to complete spanning accounts for only 15% of \$20M.

Panel B plots the net marginal value of liquid wealth $q'(w)$, which is always positive, reflecting the disutility from the alternative asset's illiquidity and unspanned volatility. The net marginal value of W, $P_W - 1 = q'(w_t)$ declines from 13 cents (per dollar of A)

to seven cents as w increases from $\underline{w} = 0.54$ to $\overline{w} = 2.64$, indicating that the marginal benefits of additional liquid wealth decrease as the fraction of the portfolio held in liquid wealth increases.

Determining the optimal portfolio target $\widehat{w}$. Figure 3 shows the ratio of the portfolio's certainty equivalent wealth to its book value, plotted for all values of w within the endogenous rebalancing boundaries of w, $(\underline{w}, \overline{w}) = (0.54, 2.64)$. Although all values of w within these rebalancing boundaries are possible, given the transaction costs, not all values of w give the investor the same utility. Hypothetically, if the investor could costlessly choose the split between W and A for a given $N = W + A$, he would choose $\widehat{w} = 1.9$. At this point, the investor's certainty equivalent wealth from the portfolio is 7.8% higher than the investor's net worth. The curve is noticeably asymmetric and declines more rapidly to the left of the maximum, as the investor approaches the voluntary liquidation boundary. This is caused by the asymmetry of the rebalancing costs, with the proportional liquidation cost ($\theta_L = 10\%$) larger than the proportional acquisition cost ($\theta_X = 2\%$).

5.2. The case of $n = 1$

We have just analyzed the case of $n \to \infty$, in which the alternative asset recurrently provides liquidity at the constant dividend yield δ_A to replenish the liquid asset holdings. Now, we consider the other extreme possibility of an investor with $n = 1$. In this case, the alternative asset has an automatic liquidity event every six years, at which point the investor reinvests 78.66% of the proceeds back into the alternative asset and keeps 21.34% of the proceeds in liquid assets. See Appendix E for more details.

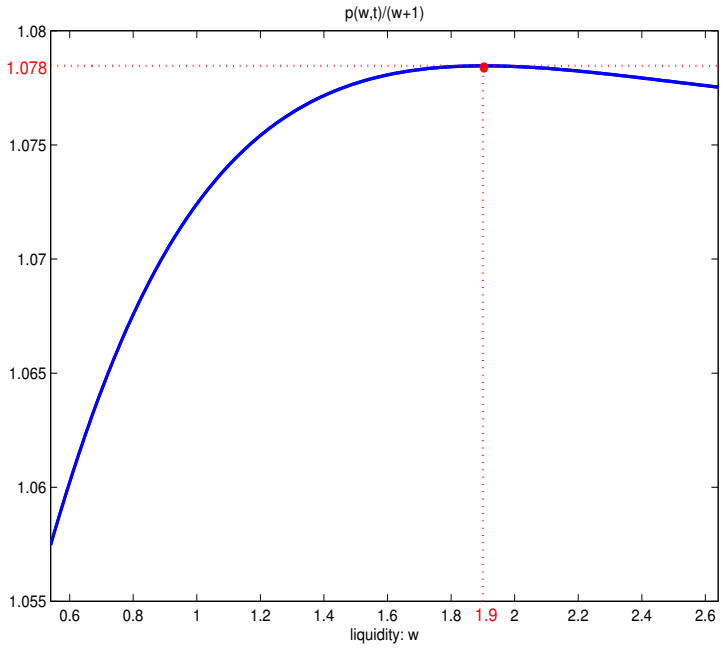


Figure 3: This figure plots $\frac{p(w)}{(w+1)}$, the ratio of the certainty equivalent wealth $p(w)A$ and net worth $N = W + A$ on the y-axis and the liquidity ratio w on the x-axis over the support range $(\underline{w}, \overline{w}) = (0.54, 2.64)$ for the case with with $n = \infty$ and $\delta_A = 4\%$. Other parameter values are given in Table 4. Note that $\widehat{w} = 1.9$ and the maximand is $\frac{p(\widehat{w})}{(\widehat{w}+1)} = 1.078$, highlighted in the figure.

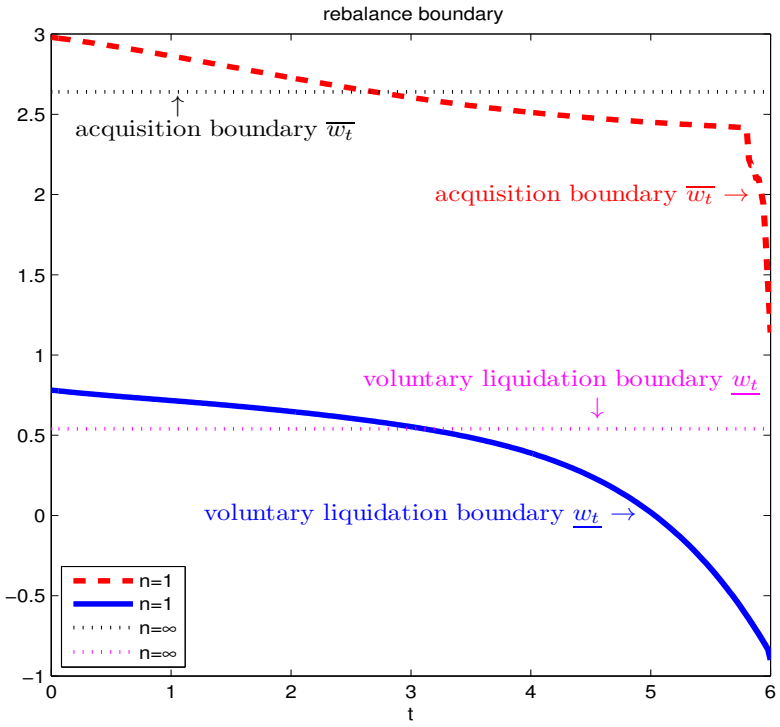


Figure 4: This figure shows the optimal (lower) liquidation boundary $\underline{w_t}$ and (upper) acquisition boundary $\overline{w_t}$ over time. The input parameter values are given in Table 4. For this figure, $n = 1$ and the implied payout $\delta_T = 21.34\%$ every six years, i.e., $H = 6$.

Rebalancing boundaries. Figure 4 shows the rebalancing boundaries over the period $0 < t < T$. The blue solid lines show the voluntary liquidation boundary, $\underline{w_t}$, and red dashed lines show the acquisition boundary $\overline{w_t}$. As a basis of comparison, the horizontal dotted lines show results for the case with $n \to \infty$.

Both boundaries decrease as $t \to T$. This means that the investor is less willing to liquidate alternative assets and more willing to voluntarily acquire alternative assets as the anticipated liquidity event dates $t = mT$ approaches. This is intuitive as the investor can simply wait until the automatic liquidity event rather than incurring the liquidation cost to reach the "target" steady state. Also, anticipating the cash inflows from the automatic liquidity event, the investor rationally becomes more willing to acquire alternative

assets as t gets closer to mT. The decrease for the voluntary liquidation boundary $\underline{w}_t$ is particularly pronounced, because of the liquidation cost $\theta_L = 10\%$. Indeed, the liquidation boundary $\underline{w}_t$ eventually becomes negative, indicating that when t is sufficiently near mT the investor prefers borrowing against the value of the alternative asset rather than incurring the liquidation cost.

Net certainty equivalent wealth Panels A and B of Figure 5 plot net certainty equivalent wealth $q(w)$ and net marginal value of liquid wealth $q'(w)$, respectively, for the case of $n = 1$ at time $t = 0$. Compared with the case of $n \to \infty$ shown in Figure 2, the rebalancing boundaries are shifted to the right as the investor is less comfortable holding the illiquid alternative asset at time $t = 0$. Recall that $0.54 \leq w \leq 2.64$ for $n = \infty$ as compared to $0.78 \leq w \leq 2.98$ for $n = 1$. This is because the cost of illiquidity is greater and hence the demand for the liquid asset increases. Aside from the changes in the rebalancing boundaries, the results in Figure 5 are generally similar to those for the $n \to \infty$ case shown in Figure 2.

Intermediate cases with $n = 3, 6,$ and 12 Figure A1 extends the results in Figure 5 to cases with $n = 3, 6,$ and 12. As n increases, the rebalancing boundaries shift to the left, indicating that the investor becomes more comfortable holding the illiquid asset as the time span between automatic liquidity events decreases. Aside from the rebalancing boundaries, the lines in all four panels are essentially identical, by the continuity argument in n and the fact that the solutions for $n = 1$ and $n = \infty$ are close. Given this similarity, for the remainder of the paper we report results using δ_A instead of δ_T.

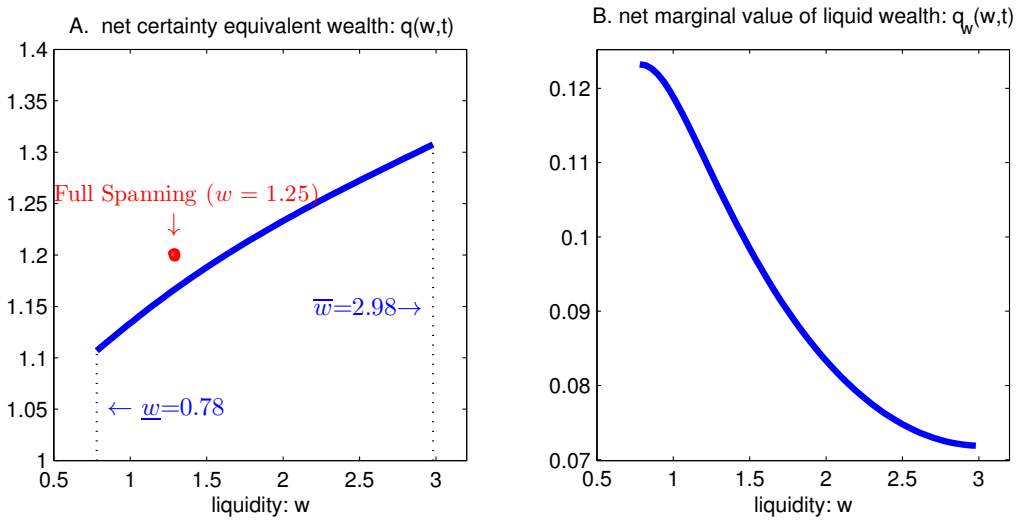


Figure 5: Panels A and B plot investors' net certainty equivalent wealth $q(w) = p(w) - w$ and net marginal value of liquid wealth $q'(w)$ as functions of the liquidity ratio w, respectively. The input parameter values are given in Table 4. For this figure, we set $n = 1$, $T = 6$, and $\delta_T = 21.34\%$. The blue solid lines show results for the general case within the optimal rebalancing boundaries $\underline{w} = 0.78$ and $\overline{w} = 2.98$. The red dot shows the solution under full spanning, which involves a static optimal $w^* = 1.25$ and $q^* = 1.20$.

5.3. Comparative statics

Table 5 reports the calibrated results. As shown earlier, given the transaction costs the model generates distinct feasible output values along the entire range of ($\underline{w}$, $\overline{w}$). For ease of interpretation, we do not report a range of possible outputs. Instead, we report comparative static results for only the case of $\widehat{w}$ (as shown in Figure 3), as it is easier to report a single number and moreover, this number gives the highest possible utility for the investor. The row in bold font shows results using the baseline parameter values summarized in Table 4. Panel A shows results for the main case, in which the alternative asset is illiquid. For comparison, Panel B shows results for the case of full spanning.

As the bold row in Panel A shows, using the baseline parameters the model implies allocations of 53.93% of the investors' net worth (N) to public equity, 34.48% to alternative assets, and the remaining 11.59% to bonds. These values are similar to the equal weighted average endowment fund portfolio allocations found in the data (see Table 1), which has allocations of 50.7% to public equity, 28.3% to alternative assets, and 21.0% to bonds. In the full-spanning case, the model implies approximately a 10 percentage point higher allocation to alternative assets, with reductions in public equity and bonds. Additionally, the full-spanning case will imply a much more volatile portfolio rebalancing strategy due to perfect liquidity and no transaction costs. The spending rate is slightly higher in the full-spanning case due to the greater liquidity.

5.3.1. The effect of α

Table 5 reports the comparative static effect of varying the α of the alternative asset. As α increases the implied allocations to alternative assets and the spending rate both rise as well. Intuitively, as α increases the economic value of the portfolio is greater and thus the spending rate rises due to the wealth effect.

Table 5: The Effect of α on Asset Allocation and Spending Rates

This table reports the asset allocation out of total net worth $N = W + A$. The three columns, Public Equity, Alternative Assets (Alternative), and Bonds report Π/N, A/N, and $(W - \Pi)/N$, respectively, evaluated at $\widehat{w}$, highlighted in Figure 3. Summing up across these three columns for each row equals 100%. The Spending column reports C/N. Panel A reports results for the case with illiquidity. Panel B reports results for the case of full spanning. The baseline parameter values are: $r = \zeta = 4\%$, $\mu_S = 10\%$, $\sigma_S = 20\%$, $\gamma = 2$, $\alpha = 2\%$, $\epsilon = 15\%$, $\beta_A = 0.6$, $\theta_L = 0.1$, $\theta_X = 0.02$, $\delta_A = 0.04$, $n \to \infty$, which implies $\rho = 0.625$, $\mu_A = 9.6\%$, and $\sigma_A = 19.2\%$. All results are presented in per cent (%), which are omitted for simplicity.

A. The case when the alternative asset is illiquid

	Public Equity	Alternative	Bonds	Spending
$\alpha = 0\%$	75.00	0.00	25.00	5.13
$\alpha = 1\%$	67.29	12.69	20.02	5.16
$\alpha = \mathbf{2\%}$	**53.93**	**34.48**	**11.59**	**5.32**
$\alpha = \textit{3\%}$	*37.91*	*60.24*	*1.87*	*5.60*
$\alpha = 4\%$	20.41	87.72	−8.12	6.00

B. The case of full spanning

	Public Equity	Alternative	Bonds	Spending
$\alpha = 0\%$	75.00	0.00	25.00	5.13
$\alpha = 1\%$	61.67	22.22	16.11	5.18
$\alpha = \mathbf{2\%}$	**48.33**	**44.44**	**7.22**	**5.35**
$\alpha = \textit{3\%}$	*35.00*	*66.67*	*-1.67*	*5.63*
$\alpha = 4\%$	21.67	88.89	−10.56	6.01

Table 5 shows that asset allocations are quite sensitive to changes in α. For example, increasing α from 2% to 3% increases the alternative asset allocation from 34.5% to 60.2%. As the investor allocates more to the alternative asset, they reduce allocations to public equity from 54% to 38% to manage the overall portfolio β and because of the additional liquidity risk. With an even higher α of 4%, the investor will optimally borrow 8.1% of net worth to invest in the alternative asset. It is worth noting that some endowment funds, such as Harvard, have occasionally taken on debt to invest in public and private equities. For example, during the financial crisis period, Harvard chose not to liquidate its endowment but rather to issue bonds (see Ang (2012)).

The sensitivity of the implied portfolio allocations to changes in α is consistent with the large empirically observed cross-sectional dispersion in endowment fund's allocations to alternative assets, shown in Table 2. An α of 0% can explain non-participation in alternative assets, while an α of 3% implies allocations that are broadly consistent with those of large endowments such as Yale and Stanford. Thus, with reasonable parameter values, our model is consistent with not only the average allocations but also the cross-sectional dispersion of allocations to alternative assets.

The sensitivity of allocations to α is also consistent with the empirically observed strong relation between endowment fund size and allocations to alternative assets. Lerner, Schoar, and Wang (2008), Brown, Garlappi, and Tiu (2010), Barber and Wang (2013), and Ang, Ayala, and Goetzmann (2014) empirically find that large endowment funds persistently earn significant alpha, which they attribute to superior alternative asset investments, while small endowments do not earn significant alphas. Lerner, Schoar, and Wang (2008) discuss how large endowments typically have better investment committees, better access to elite managers, and that there may be economies of scale in selecting alternative assets.

5.3.2. The effects of varying other parameters

The effect of ϵ. Panel A of Table 6 shows that the unspanned volatility of the alternative asset has a large effect on allocations. If $\epsilon = 5\%$ the investor takes unrealistic, large short positions in both public equity and bonds (65% and 62%, respectively), and allocates 227% of the portfolio to the alternative asset, because a 2% alpha is high compensation for the 5% unspanned volatility. However, when ϵ increases to 15% the investor allocates only 34.48% of the portfolio to the alternative asset and 54% to public equity. That is, just like α, unspanned volatility has very large effects on allocations to alternatives.

The effect of β_A. Panel B of Table 6 shows that as $beta_A$ increases the investor allocates more of the portfolio to alternative assets and less to public equities and bonds, holding the overall portfolio β approximately constant.

The effect of γ. Panel C of Table 6 shows that the coefficient of relative risk aversion has a large implied effect on asset allocation and spending. As γ increases, allocations to both public equity and alternative assets decrease. With higher risk aversion, the investor reduces exposure to systematic risk, unspanned volatility, and liquidity risk.

6. Conclusion

We analyze dynamic consumption and asset allocation decisions for a long-term institutional investor, such as University Endowments, by incorporating an illiquid investment opportunity into an otherwise standard Modern Portfolio Theory framework. Our model captures realistic aspects of investing in illiquid alternative assets by a large institutional investor, including: (1) the alternative asset has a target finite duration, at the end of

Table 6: The Effect of ϵ, β_A, and γ on Asset Allocation and Spending Rates

This table reports the asset allocation out of total net worth $N = W + A$. The three columns, Public Equity, Alternative Assets (Alternative), and Bonds report Π/N, A/N, and $(W - \Pi)/N$, respectively, evaluated at $\widehat{w}$, highlighted in Figure 3. Summing up across these three columns for each row equals 100%. The Spending column reports C/N. The baseline parameter values are: $r = \zeta = 4\%$, $\mu_S = 10\%$, $\sigma_S = 20\%$, $\gamma = 2$, $\alpha = 2\%$, $\epsilon = 15\%$, $\beta_A = 0.6$, $\theta_L = 0.1$, $\theta_X = 0.02$, $\delta_A = 0.04$, $n \to \infty$, which implies $\rho = 0.625$, $\mu_A = 9.6\%$, and $\sigma_A = 19.2\%$. All results are presented in per cent (%), which are omitted for simplicity.

A. Unspanned volatility (holding β_A fixed)				
	Public Equity	Alternative	Bonds	Spending
$\epsilon = 5\%$	−64.96	227.27	−62.31	6.66
$\epsilon = \mathbf{15\%}$	**53.93**	**34.48**	**11.59**	**5.32**
$\epsilon = 25\%$	68.10	11.31	20.58	5.18
B. β_A (holding ϵ fixed)				
	Public Equity	Alternative	Bonds	Spending
$\beta_A = 0.3$	66.34	26.32	7.35	5.29
$\beta_A = \mathbf{0.6}$	**53.93**	**34.48**	**11.59**	**5.32**
$\beta_A = 1.2$	21.53	45.45	33.01	5.33
C. Risk aversion				
	Public Equity	Alternative	Bonds	Spending
$\gamma = 1$	104.29	64.94	−69.23	4.03
$\gamma = \mathbf{2}$	**53.93**	**34.48**	**11.59**	**5.32**
$\gamma = 4$	27.25	16.84	55.64	4.99

which it is liquidated, (2) the alternative asset is not fully spanned by public equity, (3) the investor can voluntarily liquidate a part of the alternative asset holding prior to maturity by incurring a proportional transaction cost, (4) the investor can increase alternative asset holdings at any time by paying a proportional acquisition cost, and (5) the investor can create *liquidity diversification* by staggering the maturity dates of alternative asset holdings. We calibrate the model and show that our model's quantitative implications are consistent with both the level and cross-sectional variation in alternative asset allocations by university endowment funds.

The model provides clear guidelines for an institutional investor making spending and asset allocation decisions. Our theory justifies the seemingly aggressive allocations to alternative assets, as advocated by Swensen's Endowment model, if the alternative asset can generate an expected risk-adjusted excess return of 2-3% (with public equity as the benchmark), with moderate level of unspanned volatility, and standard levels of risk aversion for investors (e.g., around two and three). However, investors with limited access to sufficiently high (net-of-fees) alphas should hold conventional portfolios that follow the guidelines of MPT.

Due to space considerations we have not included several quantitatively interesting generalizations. For example, stochastic calls and distributions are potentially important as they drive much of uncertainty about the precise timing of cash flows. Extending the model to allow for time-variation in the liquidation cost is also an interesting direction for future research, particularly given the rise in illiquidity observed during the financial crisis.

Appendix

A Public Equity and Bonds with No Alternative Asset

First, we summarize the Markowitz-Merton MPT with no illiquid alternative assets. In this classic framework, investors have the standard investment opportunities defined by the public equity's risky return process given by (1) and a risk-free bond that pays a constant rate of interest r. Investors dynamically adjust their consumption/spending and frictionlessly rebalance their portfolios. The following proposition summarizes the main results in Merton (1971).

Proposition 2 *The optimal spending C_t is proportional to wealth W_t: $C_t = \phi_1 W_t$ where*

$$\phi_1 = \zeta + (1 - \gamma^{-1}) \left(r - \zeta + \frac{\eta_S^2}{2\gamma} \right) \tag{A.1}$$

is the constant marginal propensity to consume (MPC). Investors allocate a constant fraction, denoted by π, of their wealth W_t to public equity, i.e., the total investment amount in public equity is $\Pi = \pi W$ where $\pi = \eta_S/(\gamma\sigma_S) = (\mu_S - r)/(\gamma\sigma_S^2)$. The investor's value function $J(W)$ is given by

$$J(W) = \frac{(b_1 W)^{1-\gamma}}{1-\gamma}, \tag{A.2}$$

where b_1 is a constant given by:

$$b_1 = \phi_1^{\frac{1}{1-\gamma^{-1}}}. \tag{A.3}$$

Note that the asset allocation rule is the standard Merton mean-variance result in that the fraction of investment in public equity increases in the equity risk premium $(\mu_S - r)$ and decreases in variance σ_S^2 and risk aversion γ. Next, we analyze the general case where investors can also invest in illiquid alternative assets in addition to public equity and bonds.

B Proof for Proposition 1

Using the value-matching condition and taking the limit, we have

$$\begin{aligned}
&\lim_{\Delta \to 0} \frac{1-\theta_L}{(1-\theta_L)\Delta} \left(P(W_t + (1-\theta_L)\Delta, A_t, t) - P(W_t, A_t, t) \right) \\
&= \lim_{\Delta \to 0} \frac{1}{\Delta} \left(P(W_t + (1-\theta_L)\Delta, A_t, t) - P(W_t + (1-\theta_L)\Delta, A_t - \Delta, t) \right).
\end{aligned}$$

Using differentiability, we have

$$(1-\theta_L) P_W(W_t, A_t, t) = P_A(W_t, A_t, t).$$

By substituting $P(W_t, A_t, t) = p(w_t, t) A_t$ into the equation above, we have

$$(1-\theta_L) p_w(w_t, t) = p(w_t, t) - p_w(w_t, t) w_t.$$

Upper boundary. We use essentially the same argument as that for the lower boundary. Let (W_t, A_t) denote the pre-acquisition position in public equity and the alternative asset at time t. As the acquisition cost for the alternative asset is θ_X per unit of the asset, the investor's post-acquisition position is then (W'_t, A'_t), where $W'_t = W_t - (1+\theta_X)\Delta$, $A'_t = A_t + \Delta$, and $\Delta > 0$ denotes the unit of the alternative asset that the investor considers to acquire voluntarily. The continuity of the value function implies $P(W'_t, A'_t, t) = P(W_t, A_t, t)$, which implicitly defines the upper boundary $\overline{W_t}$ as a function of A_t at time t. Similarly, in the differential form, we have

$$(1 + \theta_X)P_W(W_t, A_t, t) = P_A(W_t, A_t, t)\,.$$

By substituting $P(W_t, A_t, t) = p(w_t, t)A_t$ into the equation above, we have

$$(1 + \theta_X)p_w(w_t, t) = p(w_t, t) - p_w(w_t, t)w_t,$$

which means (27) holds.

B.1 Full Spanning with the Liquid Alternative Asset

This appendix section summarizes the case of full spanning, in which the alternative asset is fully liquid. In this case, the alternative asset simply expands the investors' investment opportunity set by adding a second liquid risky asset. Therefore, the optimization problem is essentially Merton's model with two publicly traded risky assets. As this new risky asset provides diversification and incurs no costs, the value function for this case is clearly higher than in the general case. Indeed, it serves as an upper bound for the investor's value function. Below, we summarize the main results for this frictionless benchmark.

The optimal consumption C is proportional to the net worth, N: $C = \phi_2 N$ where the MPC, denoted by ϕ_2, is given by

$$\phi_2 = \zeta + (1 - \gamma^{-1}) \left[r - \zeta + \frac{\eta_S^2 - 2\rho\eta_S\eta_A + \eta_A^2}{2\gamma(1-\rho^2)} \right] . \tag{B.1}$$

By comparing ϕ_2 given in (B.1) and ϕ_1 given in (A.1), we see that diversification ($|\rho| < 1$) and an additional risk premium $\eta_A > 0$ both influence the MPC making the investor better off. At each time t, the investor continuously rebalances the portfolio so the investment in public equity, Π, and in the alternative asset, A, are proportional to total net worth N, i.e.

$$\Pi = \frac{\eta_S - \rho\eta_A}{\sigma_S\gamma(1-\rho^2)} N , \tag{B.2}$$

$$A = \frac{\alpha}{\gamma\epsilon^2} N , \tag{B.3}$$

The remaining wealth, $N - (\Pi + A)$, is allocated to the risk-free bond.

The investor's value function $\overline{V}(N)$ is given by

$$\overline{V}(N) = \frac{(b_2 N)^{1-\gamma}}{1-\gamma} , \tag{B.4}$$

where b_2 is a constant given by

$$b_2 = \phi_2^{\frac{1}{1-\gamma^{-1}}} . \tag{B.5}$$

To sharpen our intuition, we first analyze the value-add of the alternative asset if it were fully liquid by comparing the solutions for the two frictionless benchmarks.

Because the case of full spanning has the additional alternative asset and the classic Merton case does not, the investor is better off in that $b_2 > b_1$ by comparing the value functions for the two cases. Let $\overline{q} = b_2/b_1 > 1$ denote the ratio. Intuitively speaking, $\overline{q} - 1$ is the fraction of wealth that the investor would need as compensation to permanently give up the opportunity to invest in the liquid alternative asset and instead live under the Merton case.

C Proof for the Case of Full Spanning

Using the standard dynamic programming method, we have

$$\begin{aligned}\zeta\overline{V} &= \max_{C,\Pi,A} U(C) + [rN + (\mu_S - r)\Pi + (\mu_A - r)A - C]\overline{V}_N \\ &\quad + \frac{(\Pi\sigma_S)^2 + 2\rho\Pi\sigma_S A\sigma_A + (A\sigma_A)^2}{2}\overline{V}_{NN}\,, \end{aligned} \tag{C.1}$$

and using the FOCs for Π, A and C, we have

$$U'(C) = V_N\,, \tag{C.2}$$

$$\Pi = -\frac{\eta_S}{\sigma_S}\frac{V_N}{V_{NN}} - \frac{\rho\sigma_A}{\sigma_S}A\,, \tag{C.3}$$

$$A = -\frac{\eta_A}{\sigma_A}\frac{V_N}{V_{NN}} - \frac{\rho\sigma_S}{\sigma_A}\Pi\,. \tag{C.4}$$

Conjecture the value function takes the following form

$$\overline{V}(N) = \frac{(b_2 N)^{1-\gamma}}{1-\gamma}\,. \tag{C.5}$$

Substituting it into the FOCs, we have $C = b_2^{1-\gamma^{-1}}N$, (B.2) and (B.3). Then substituting them into the HJB equation (C.1) and simplifying, we obtain (B.1).

D Sub-Asset Class Inputs for Calculating Unspanned Volatility

Table A1: Summary of Asset Class Risk and Correlations

This table shows summary statistics for the sub-asset classes within the illiquid asset class, which are used to calculate the unspanned volatility of the endowment fund portfolios. Panel A shows β_a, R_a^2, and σ_a for each alternative asset class a. Panel B shows the pairwise correlations between these sub-asset classes.

Panel A

	β_a	R_a^2	σ_a
Hedge Funds (HF)	0.54	0.32	19.1%
Private Equity (PrivEqu)	0.72	0.32	25.4%
Venture Capital (VC)	1.23	0.30	45.1%
Private Real Estate (PrivRE)	0.50	0.49	16.0%
Natural Resources (NatRes)	0.20	0.07	17.0%

Panel B

	FixedInc	*PubEqu*	*HF*	*PrivEqu*	*VC*	*PrivRE*	*NatRes*
FixedInc	1						
PubEqu	0.02	1					
HF	0.16	0.64	1				
PrivEqu	-0.23	0.78	0.73	1			
VC	-0.18	0.46	0.52	0.66	1		
PrivRE	-0.13	0.35	0.31	0.51	0.17	1	
NatRes	0.04	0.87	0.67	0.70	0.46	0.44	1

E Calibration of δ_A and δ_T

We focus on the steady state in which the investor always has n funds at any time t. This is feasible provided the investor immediately replaces each fund that exits.

To simplify exposition, assume that each fund's payoff structure involves only one contribution at its inception and one distribution upon its exit, and the horizon (or equivalently the lockup period) of each fund is H. At the steady state, n/H funds mature each year, which means that there is one liquidity event every $T = H/n$ years. For example, if the lockup period for each fund is $H = 6$ and there are three funds at the steady state ($n = 3$), then every two years ($T = 6/3 = 2$) an automatic liquidity event occurs. To ensure that the investor has three funds at the steady state, the investor immediately replaces the exited fund by investing in a new fund with 6-year lock-up.

To ensure growth stationarity, we assume that both the growth rate of each fund, g_A, and the growth rate of the inception size for each fund (vintage), g_I, are constant. Consider a vintage-t fund, which refers to the fund that enters the portfolio at time t. Let IS_t denote the fund's initial size (IS) at inception. Its size at $(t + iT)$ is then $e^{g_A iT} IS_t$ where $i = 1, 2, \cdots n$ and hence the fund's size when exiting at time $t + H$ is $e^{g_A H} IS_t$.

At time $(t+H)$ the investor holds a total of n illiquid alternative funds ranging from vintage-t to vintage-$(t + (n - 1)T)$. Note that the value of the vintage-$(t + i)$ fund is $e^{g_A(n-i+1)T} \times (IS_t e^{g_I(i-1)T})$ as its inception size is $IS_t e^{g_I(i-1)T}$ and has grown at the rate of g_A per year for $(n - i + 1)T$ years. Summing across all vintages, we obtain

$$\sum_{i=1}^{n} e^{g_A(n-i+1)T} \times (IS_t e^{g_I(i-1)T}) = e^{g_I H} IS_t \times \sum_{i=1}^{n} e^{(g_A - g_I)iT} .$$

The net payout at time $(t + H)$ is given by the difference between $e^{g_A H} IS_t$, the size of the exiting vintage-t fund, and $e^{g_I H} IS_t$, the size of the new vintage-$(t + H)$ fund. As the payout occurs once every T years, the annualized net payout rate is then

$$\frac{1}{T} \frac{e^{g_A H} IS_t - e^{g_I H} IS_t}{e^{g_I H} IS_t \times \sum_{i=1}^{n} e^{(g_A - g_I)iT}} = \frac{1}{T} \frac{e^{(g_A - g_I)H} - 1}{\sum_{i=1}^{n} e^{(g_A - g_I)iT}} = \frac{1}{T} \left(1 - e^{-(g_A - g_I)T}\right) . \tag{E.1}$$

Next, we use this annualized net payout rate to calibrate δ_A and δ_T. Although, for the sake of generality, the model includes both δ_A and δ_T, in any single calibration we use only one of either δ_A or δ_T. For the illustration, we provide three examples below.

First, consider the limiting case when $n \to \infty$, and with fixed finite holding period H for each fund, $T \equiv H/n \to 0$. Therefore, the investor continuously receives payout at a constant rate. This maps to the parameter δ_A in our model. As one may expect, the net payout rate is simply the difference between the value of the incumbent fund growth rate g_A and the growth of the new fund's initial size g_I:

$$\delta_A = g_A - g_I . \tag{E.2}$$

For the calibration, we set $\mu_A = g_A = 9.6\%$ and $g_I = 5.6\%$ (approximately equal to the average endowment fund growth rate over the past 20 years) resulting in $\delta_A = 4\%$.

Second, consider the case when the investor has only one fund outstanding at each point in time. Then, $T = H$, the payout occurs once every H years, and we can use δ_T to capture the payout. That is, when T is relatively large, δ_T is given by

$$\delta_T = 1 - e^{-(g_A - g_I)T}. \tag{E.3}$$

Note that δ_T as defined in the model is not annualized. Thus, with the values of g_A and g_I given above, for a single fund ($n = 1$) in the portfolio and $H = 6$, $\delta_T = 21.34\%$.

Third, consider an intermediate case when the investor has six funds at each point in time. Then, we have $T = H/n = 6/6 = 1$, and we could use $\delta_T = \delta_1 = 1 - e^{-0.04} = 3.92\%$ to capture the payout. Alternatively, we could approximate with a continuous constant dividend yield by annualizing δ_T and using this annualized value as δ_A in the calibration. In this case, we would have $\delta_A \approx (1 + \delta_T)^{1/T} - 1 = \delta_T = \delta_1 = 3.92\%$ when $T = 1$.

F Cases with $n = 3, 6,$ and 12

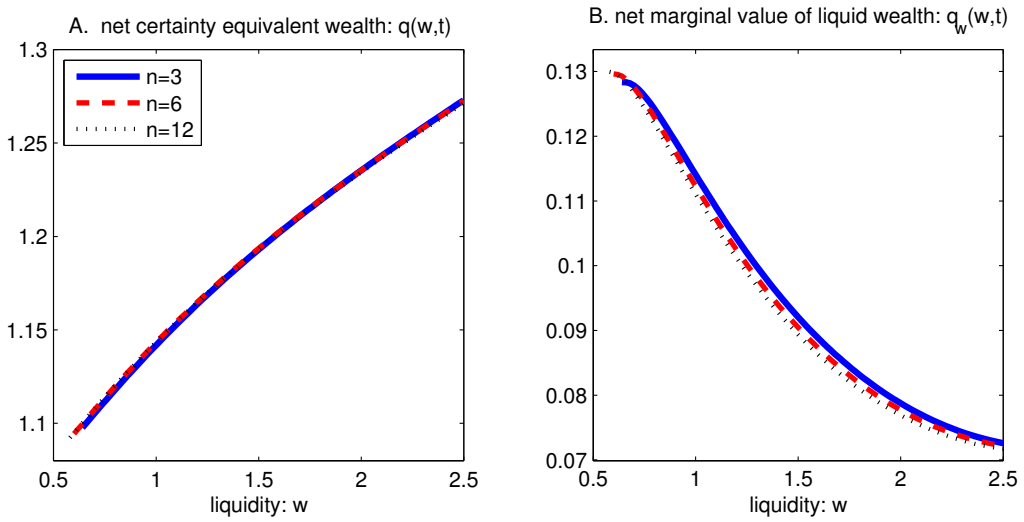


Figure A1: Panels A, B, C, and D plot investors' net certainty equivalent wealth $q(w) = p(w) - w$, net marginal value of liquid wealth $q'(w)$, scaled public equity allocation $\pi(w)$, and spending $c(w)$ as functions of the liquidity ratio w, respectively. The input parameter values are given in Table 4. The blue line shows results for $n = 3$, $T = 2$, and $\delta_T = 7.69\%$. The red dashed line shows results for $n = 6$, $T = 1$, and $\delta_T = 3.92\%$. The black dotted line shows results for $n = 12$, $T = 0.5$, and $\delta_T = 1.98\%$.

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