Low Interest Rate And Productivity Growth*

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Abstract

What is the effect of very low interest rate on productivity growth? We write down an endogenous growth model to show that the relationship between interest rate and productivity growth follows an inverted-U. A further decline in interest rate beyond a threshhold reduces overall productivity growth due to the impact of low interest rate on strategic competition. Very low interest rate increases the equilibrium distance between leader and follower, thus reducing innovation incentives. Consistent with empirical evidence, our model predicts that very low interest rate leads to a slowdown in productivity growth, increase in market concentration and mark ups, a decline in creative destruction and firm entry, a widening productivity-gap between leader and follower within an industry, and an increasing wedge between market-q and investment.

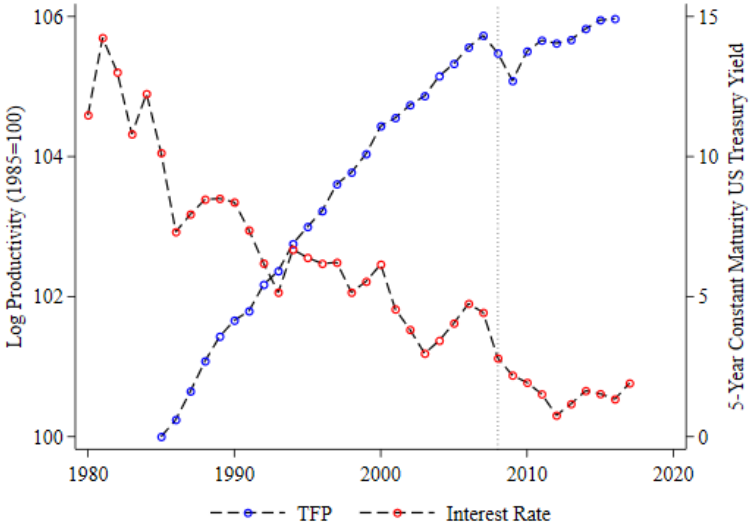
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1 Introduction

Long-run growth is foremost determined by growth in total factor productivity. A somewhat worrisome sign in recent decade has been the persistent slow-down in the rate of productivity growth. The slowdown in productivity growth is a global phenomena, not limited to a single country or region, suggesting that the factors behind the slowdown are likely to be common and global as well. One such potential factor is the very low level of global interest rate.

Figure 1 plots the log of average total factor productivity for 20 OECD countries[1] from 1985 onwards, and the 5-year constant maturity U.S. treasury rate as a measure of the global risk free rate. There is clear indication of a slowdown in productivity growth in recent years as reflected by the declining slope of the productivity graph. The slowdown in productivity growth starts before the Great Recession in 2008 (vertical dotted line in the figure), but the strongest decline occurs in the post-2008 period. The decline in global productivity growth is correlated with the decline in global interest rates, particularly the decline in global real rates to levels close to zero.

Figure 1: Productivity growth and interest rates



Is there a connection between the very low global real rate and the decline in rate of productivity growth? In particular, could the very low global rate be partly responsible for the slowdown in productivity growth? We formally analyze this question and describe an important mechanism that connects very low real rates with an endogenous slowdown in productivity growth. We also show that our highlighted mechanism is consistent with a number of empirical regularities, some quite unique to our specific hypothesis.

At first pass it might appear odd that low interest rates should lead to a slowdown in productivity growth. In fact the traditional mechanism linking investment with interest rate suggests that lower

[1] Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Ireland, Italy, Japan, Korea, Netherlands, New Zealand, Portugal, Spain, Sweden, Switzerland, U.K. and U.S.

interest rates should lead to *higher* productivity growth. For example, if productivity growth depends on the level of R&D investment in the economy, then a reduction in interest rates - all else equal - will increase the present value of R&D investment, making everyone invest more and boosting productivity growth in the process. However, as we show, while this positive effect of lower interest rates on productivity growth in always there, at very low interest rates a second negative effect of lower interest rates on productivity growth starts to dominate. In fact, we show that under reasonably general conditions, as real rates go towards zero, productivity growth is *guaranteed* to slow-down. There is thus an inverted-U relationship between growth and the level of interest rate.

We illustrate our mechanism by writing down an endogenous growth model in the spirit of Aghion et al. (2001) where growth is the outcome of a stochastic investment game between a leader and a follower in each industry. Specifically, the economy has a continuum of product lines, each of which is occupied by a high productivity leader and lower productivity follower. Both the leader and the follower invest and stochastically improve their respective productivities. The leader always has a stronger incentive to innovate than the follower because doing so enables the leader to prolong its market leadership. Leader's incentive to innovate also increases when the productivity gap between the two firms narrows.

The key question is what happens in this typical model if interest rate declines? A drop in interest rate causes both the leader and the follower to increase investment for any productivity gap. This is the traditional effect of lower interest rate already mentioned earlier. However, we show that it is also that case that the leader's increase in effort as a result of lower interest rate is always stronger than the follower's. The stronger incremental effect for the leader implies that equilibrium productivity gap between the leader and follower *increases*. An increases in equilibrium gap in turn reduces the incentives for both leader and follower to innovate: the average leader gets so far ahead of the follower that it behaves like a "lazy monopolist" and reduces effort to innvoate. The follower on the other hand gets discouraged from the wide gap and also reduces effort. Since effort of leader and follower are strategic complements, the decline in effort reinforces each other. This negative strategic effect of lower interest rates on R&D effort becomes the dominant force if interest rates are low enough. Contrary to conventional wisdom, very low interest rates can be contractionary.

Our model delivers a number of predictions that are consistent with recent changes in market structure and productivity growth, both in the time-series and cross-section. First, as interest rate declines beyond a threshhold, the endogenous productivity gap between leader and follower increases. Thus market concentration increases, markups increase and market power becomes more persistent. All of these are well-documented facts.

Second, our theory has precise predictions on the evolution of productivity growth differential between a leader and follower as interest rate declines. Beyond the threshhold where a further reduction in interest rate reduces productivity growth, the average gap in productivity growth between the leader and follower increases. As we discuss later in the paper, this is exactly what one find the firm-level OECD data. The slowdown in overall productivity growth is accompanied by a simultaneous increase in the productivity growth gap between leaders and followers within industries. Moreover, the model predicts that the overall fall in productivity growth will be stronger in sectors where the distance be-

tween the leader and the follower diverges more as interest rate falls. This prediction is again confirmed in the analysis by Andrews et al.

Third, consistent with empirical evidence, our theory predicts a slow down in the rate of creative destruction and startups at very low interest rate. Fourth, firms with higher market value have lower incentive to invest - creating a wedge between market "q" and investment. Our framework can thus rationalize the low levels of investment made by the corporate sector despite very high valuations.

Overall our model provides a unified and parsimoneous explanation for a wide range of recent salient trends in the U.S. and other developed economies. Our paper is also related to a number of recent papers on the question of "secular stagnation" and liquidity traps[2]. These papers typically focus on aggregate demand and show how demand-side factors such as aging population and household indebtedness can generate low interest rates and a demand-driven "Keynesian" decline in output. Our paper is one of the few that highlights a potential supply-side feedback mechanism. If interest rate declines due to contractionary demand-side forces, a sufficiently strong decline can independently unleash contractionary supply-side forces as well. The net result from these interacting demand and supply forces could be more persistent slowdown in productivity growth.

2 Model

The model has two blocks. We first provide a stylized model of dynamic duopoly competition to illustrate how low interest rate could lead to the emergence of large firms, thereby reducing innovation and growth. We then embed a generalized version of the stylized model into an endogenous growth model.

Setup Time is continuous. Two risk-neutral firms compete in a product line. Let $s \in \mathbb{Z}_{\geq 0}$ be the state variable that captures the productivity gap between the two firms. When $s = 0$, the two firms are said to be neck-to-neck; when $s > 0$, one of the firm is a leader (L) while the other is a follower (F) . Firms compete in the product market: the leader earns flow profit π_s^L and the follower earns π_s^F. For simplicity, we assume the total flow profit is constant across states, with $\pi_s^L + \pi_s^F = \pi$, and that π_s^L is increasing in s while π_s^F is decreasing in s. In state $s = 0$, the two neck-to-neck firms each earns $\pi/2$.

Firms invest in order to improve their productivities, and hence the productivity gap s transitions endogenously. For simplicity, the investment decision is binary: a firm can choose to pay flow cost c in exchange for Poisson rate η that its productivity increases by one step. Furthermore, with exogenous flow rate κ, the follower closes its productivity gap with the leader by one step. Thus the state transition

[2]While there has been work in the past that links interest rate to the *level* of productivity (e.g. Caballero, Hoshi, and Kashyap; and Gopinath et al.), our paper is fundamentally different in that it focuses on the relationship between the level of interest rate and the *growth rate* of productivity.

follows

$$s(t+\Delta) = \begin{cases} s(t)+1 & \text{with probability } \Delta\left(\eta \cdot Pr\left(\text{leader invests in state } s(t)\right)\right) \\ s(t)-1 & \text{with probability } \Delta\left(\kappa + \eta \cdot Pr\left(\text{leader invests in state } s(t)\right)\right) \\ s(t) & \text{otherwise} \end{cases}$$

In later parts, we are going to generalize the model by 1) microfounding the profit-sharing rule with an explicit model of competition; 2) extend the investment decision to be a continuous variable with convex cost.

We study the stationary MPE of this game. Let v_s^L and v_s^F be the value function of the two firms; we abuse the notation and simply use $v_s \equiv v_s^L$, $v_{-s} \equiv v_s^F$, $\pi_s \equiv \pi_s^L$, $\pi_{-s} \equiv \pi_s^F$. We denote

$$\eta_s \equiv \eta \cdot Pr\left(\text{leader invests in state } s\right)$$

$$\eta_{-s} \equiv \eta \cdot Pr\left(\text{follower invests in state } s\right).$$

We show below that generically, the leader plays pure strategy and the follower plays pure strategy in all but one state.

Under interest rate r, the leader's HJB equation is

$$rv_s = \max_{0\leq\eta_s\leq\eta} \pi_s - c\frac{\eta_s}{\eta} + \eta_s\left(v_{s+1}-v_s\right) + \left(\kappa+\eta_{-s}\right)\left(v_{s-1}-v_s\right)$$

The follower's HJB equation is

$$rv_{-s} = \max_{0\leq\eta_{-s}\leq\eta} \pi_{-s} - c\frac{\eta_{-s}}{\eta} + \eta_s\left(v_{-(s+1)}-v_{-s}\right) + \left(\kappa+\eta_{-s}\right)\left(v_{-(s-1)}-v_{-s}\right)$$

A symmetric Markov-perfect equilibrium (SMPE) is the collection of $\{\eta_s, \eta_{-s}\}_{s=0}^{\infty}$ such that η_s and η_{-s} solves the HJB equations for both firms in all states.

Results

Proposition 1. *There exists a monotone SMPE, such that the investment decision is weakly decreasing in the state. Such equilibria can be characterized by $(n,k) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, with*

$$\eta_s = \begin{cases} \eta & \text{if } -\lfloor k \rfloor < s < \lfloor n \rfloor \\ (n-\lfloor n \rfloor)\,\eta & \text{if } s = \lfloor n \rfloor \\ (k-\lfloor k \rfloor)\,\eta & \text{if } s = \lfloor k \rfloor \\ 0 & \text{if } s \geq \lceil n \rceil \text{ or } s \leq -\lceil k \rceil. \end{cases}$$

That is, both players is indifferent between investing or not in at most one state. Furthermore,

1. *there exists $\bar{r}$ such that $n > k$ for all $r < \bar{r}$;*

2. *there exists* $\bar{\kappa}$ *such that for all* $\kappa < \bar{\kappa}$*, there exists* $\bar{\bar{r}}(\kappa)$ *such that for all* $r < \bar{\bar{r}}(\kappa)$*, all SMPE must be monotone.*

The proposition establishes that when r and κ are small, all SMPE exhibit the following properties:

1. Monotonicity: both players's investment is weakly decreasing in the productivity gap s.

2. Leader dominance: the leader has a stronger incentive to invest than the follower. In this stylized binary action model, $n \geq k$ thus the leader invests in a larger set of states. To understand this, note that the incentive to innovate originates not only from the higher flow profit associated with a higher productivity, but more importantly comes from the fact that an improved productivity is a necessary step to future productivity gains. The follower innovates to close the gap and eventually catches up with the leader, whereas the leader innovates to widen the productivity gap and prevents the follower from catching up. The result suggests that the leader's incentive is always stronger than the followers, as the change in value is always higher for the leader than for the follower: $(v_{s+1} - v_s) > \left(v_{-(s-1)} - v_{-s}\right)$. The intuition is as follows:

   (a) the leader's flow profit is higher than follower's;

   (b) any gain in productivity benefits the leader in maintaining its market leadership and prolong its market power;

   (c) any gain in productivity benefits the follower in closing the gap with the leader and eventually becomes the leader in the future;

   (d) the gain for the leader materializes immediately while the gain for the follower materializes in the future, thus the gain is more valued by the leader than by the follower.

3. Two important intuitions in underlying the equilibrium structure.

   (a) Discouragement effect: when the follower is too far behind, it ceases investment because the prospects of catching up are dim.

   (b) Lazy monopolist effect: when the leader is too far ahead of the follower, it ceases investment because there is little chance that the follower can catch up. Even if the gap closes (through the exogenous flow rate κ), there is enough distance $(n-k)$ for the leader to re-consolidate the gap before the follower starts to invest again. The marginal gain in widening the productivity gap further to $n+1$ is small.

Under certain parameter values, there could exist multiple monotone equilibria. We select the equilibrium with the maximum growth rate, or equivalently the smallest investment gap $(n-k)$.

Proposition 2. *The incentive to invest is decreasing in r in all states. Furthermore, leader's incentive to invest is more responsive to changes in the interest rate:*

$$\left.\frac{-\partial\left(v_{s+1}-v_s\right)}{\partial r}\right|_{(n,k)\text{ held fixed}} > \left.\frac{-\partial\left(v_{s'+1}-v_{s'}\right)}{\partial r}\right|_{(n,k)\text{ held fixed}} > 0.$$

Conjecture. *$k(r)$, $n(r)$, and $n(r) - k(r)$ are all weakly decreasing in r.*

The proposition shows that following a drop in the interest rate, the incentive to invest is higher for both the leader and the follower, but the change in incentive is larger for the leader. The conjecture further states that the "investment gap" between the two players widens: there are more states in which the leader invests and the follower does not.

Why are these results interesting, and how do they relate to growth? Notice that the state variable, s, follows an ergodic Markov chain with endogenous transition probabilities that depends on the investment decisions of the duopoly. Thus, if there is measure 1 of product lines, each of which has two firms playing the investment game outlined above, then the equilibrium induces a stationary (steady-state) distribution of the state variable. In the generalized model below we provide a specific microfoundation of the duopolistic competition, in which case productivity growth of the leader and follower in each product line can be mapped to the general equilibrium economy-wide productivity growth. We now briefly sketch the intuitions. Loosely, aggregate growth can be written as a weighted average productivity growth in each product line. Let μ_s be the probability mass of product lines in state s in the stationary distribution. The stationary distribution satisfies $\sum_{s=0}^{n} \mu_s = 1$ as well as the following set of equations:

moving from state s to $s+1$		moving from state $s+1$ to s
$2\mu_0\eta_0$	$=$	$\mu_1(\eta_{-1} + \kappa)$
$\mu_1\eta_1$	$=$	$\mu_2(\eta_{-2} + \kappa)$
	$\vdots$	
$\mu_{n-1}\eta_{n-1}$	$=$	$\mu_n\kappa$

Suppose we want to track the growth rate of the frontier (leader) productivity of each product line, which is given by

$$\begin{aligned} g &= \sum_{s=0}^{n-1} \mu_s \eta_s \\ &= \eta(1 + \mu_0 - \mu_n) \end{aligned}$$

which is increasing in the mass of product lines in the neck-to-neck state and decreasing in the mass of lines in state n. Interpretation:

- Under neck-to-neck status, investment by either of the two firms can contribute to frontier growth; in state n, neither firm invests. Hence high μ_0 and low μ_n contributes to high growth.
- Why is the growth rate of the frontier not a function of k? It actually is: the mass of firms in each state under a stationary distribution endogenously depends on the investment decision by the two firms. Let $\alpha \equiv \frac{\kappa}{\eta}$ and $\beta \equiv \frac{\kappa+\eta}{\eta} = \frac{\alpha+1}{\alpha}$ (ignore mixed strategy for now and assume the follower does

not invest in state k, i.e. $p = 0$), we have

$$\begin{aligned}
\mu_{n-1} &= \mu_n \alpha \\
&\vdots \\
\mu_{k-1} &= \mu_n \alpha^{n-(k-1)} \\
\mu_{k-2} &= \mu_{k-1}\beta = \mu_n \alpha^{n-(k-1)}\beta \\
&\vdots \\
\mu_1 &= \mu_n \alpha^{n-(k-1)}\beta^{k-2} \\
\mu_0 &= \frac{1}{2}\mu_n \alpha^{n-(k-1)}\beta^{k-1}
\end{aligned}$$

Since $\sum \mu_i = 1$, we have

$$\mu_n \left(1 + \alpha + \cdots + \alpha^{n-(k-1)} \left(1 + \beta + \cdots + \beta^{k-2} + \frac{1}{2}\beta^{k-1}\right)\right) = 1$$

and

$$1 + \mu_0 - \mu_n = \mu_n \left(\alpha + \cdots + \alpha^{n-(k-1)} \left(1 + \beta + \cdots + \beta^{k-2} + \beta^{k-1}\right)\right).$$

- We can see that growth is increasing in k, decreasing in n, and decreasing in $(n - k)$.

A drop in r leads both firms in each product to increase investment. This is the neoclassical effect: lower r reduces cost of funds today and effectively raises the patience of firms. The discussion so far, however, shows that the strategic interaction between the leader and the follower can lead to lower growth in a low-rate environment through the change in the equilibrium distribution of productivity gap: when the equilibrium gap widens, more firms are in state n and fewer firms are in the neck-to-neck state, causing aggregate growth rate to drop. We summarize this as a proposition, and we elaborate on this point in the generalized model below.

Proposition. *Equilibrium growth could decrease following a drop in interest rate. This is despite that both the leader and the follower strictly increases investment under lower interest rate.*

3 A Model of Endogenous Growth with Strategic Investments

Consider an economy with a continuum of markets of measure 1, exogenous labor endowment L which we also normalize to one, and Cobb-Douglas final demand:

$$\ln Y = \int_0^1 \ln Y_i di.$$

Time is continuous. We are going to analyze a steady-state of the economy (balanced-growth path) in which output and consumptions grow at a constant rate; for this reason, I suppress time subscripts for notational convenience.

Each product line is occupied by two competing firms with linear functions

$$y_{i,1} = q_{i,1}\ell_{i,1}, \quad y_{i,2} = q_{i,2}\ell_{i,2}$$

where $q_{i,1}$ and $q_{i,2}$ are the productivities in market i. Productivities evolve endogenously in multiples of $\lambda > 1$; that is, $q_{i,j} = \lambda^m$ at any instance with $m \in \mathbb{Z}_{\geq 0}$. Without loss of generality, we refer to the firm with higher productivity as the "leader" and that with lower productivity as the "follower", with $q_{i,L} = \lambda^{m_L} \geq q_{i,F} = \lambda^{m_F}$. When the two firms have the same productivity, we say they are neck-to-neck. Let $s_i \equiv \frac{\ln(q_{i,L}/q_{i,F})}{\ln \lambda} = m_L - m_F$ denote the technological gap of variety s.

Production Side—Static Competition The two firms in each market take their productivities as given and engage in price competition. The output by the two firms may or may not be perfect substitutes. Assume the environment is such that we write the operating profit of both the leader and the follower as a function of the technological gap as the state variable, with $\pi^L(s)$ increasing in s and $\pi^F(s)$ decreasing in s.

A simple case is when the two firms produce perfect substitutes, in which case only the leader would produce and the price charged will be exactly the cost of production of the follower. The leader has profit

$$\begin{aligned} \frac{\Pi(s)}{Y} &= (p_{i,L} - c_{i,L}) y_{i,L}/Y \\ &= \left(1 - \frac{q_{i,F}}{q_{i,L}}\right) p_{i,L} y_{i,L}/Y \\ &= 1 - \lambda^{-s} \\ &\equiv \pi_s \end{aligned}$$

while the follower has profit zero.

This is the only case with a closed form solution of π_s; we have to rely on numerical methods to compute the π_s under more general elasticity of substitution $1 < \sigma < \infty$ between the two firms.

Production Side—Dynamic Investment Productivity growth is modeled as a controlled Poisson process. In a time interval dt, each firm can invest $c(\eta)\,dt$ units of output in exchange for a flow rate ηdt to improve its technology by one step, changing its productivity from λ^m to λ^{m+1}. Moreover, there's also an exogenous Poisson rate κdt that the technology gap closes by one step.

Let $V(s)$ be the value function for the firm that is s-steps ahead of the other firm ($s > 0$ when the firm is a leader and $s < 0$ when it is a follower). We normalize value functions by the total output and define $v_s \equiv V(s)/Y$. The HJB equation for v_s is:

$$r v_s = \begin{cases} \max_\eta \pi_s - c(\eta) + \eta(v_{s+1} - v_s) + (\eta_{-s} + \kappa)(v_{s-1} - v_s) & \text{if } s > 0 \\ \max_\eta \pi_s - c(\eta) + \eta(v_{s+1} - v_s) + (\eta_{-s})(v_{s-1} - v_s) & \text{if } s = 0 \\ \max_\eta \pi_s - c(\eta) + (\eta + \kappa)(v_{s+1} - v_s) + \eta_{-s}(v_{s-1} - v_s) & \text{if } s < 0 \end{cases}$$

Let μ_s be the mass of product lines with technological gap s, with $\sum_{s=0}^{\infty} \mu_s = 1$.

The total value of firms in the economy (relative to aggregate output) is

$$\bar{v} \equiv \sum_{s=0}^{\infty} \mu_s \left(v_s + v_{-s} \right)$$

The total investment in the economy is

$$I \equiv Y \sum_{s=0}^{\infty} \mu_s \left(c\left(\eta_s\right) + c\left(\eta_{-s}\right) \right).$$

Aggregate profits:

$$\Pi \equiv Y \sum_{s=0}^{\infty} \mu_s \left(\pi_s + \pi_{-s} \right).$$

We impose two transversality conditions on v's: $v_{-\infty} = 0$ and $v_{\infty} = \frac{1}{r}$. That is, the follower's value function converges to zero while the leader's value converges to $1/r$ as $s \to \infty$ (because the leader can charge an arbitrarily high price relative to its production cost and essentially generate sales revenue of Y with no production cost).

Consumption Side Each worker receives flow wage W and profits Π, consumes C, and makes investment I, subject to budget constraint

$$C + I = W + \Pi.$$

The consumer makes consumption-investment tradeoff following an Euler equation

$$g_C = \frac{1}{\theta} \left(r - \rho + \alpha \right).$$

The term $\alpha \geq 0$ represents potential consumer-side market imperfections (such as incomplete market with idiosyncratic income shocks).

Market clearing condition implies $C + I = Y$.

Steady-State In a steady-state, the mass of product lines leaving any state over an infinitesimal time period is equal to the mass of firms entering into the state. We have

State	moving from state s to $s+1$		moving from state $s+1$ to s
0	$2\mu_0\eta_0$	$=$	$\mu_1(\eta_{-1}+\kappa)$
1	$\mu_1\eta_1$	$=$	$\mu_2(\eta_{-2}+\kappa)$
2	$\mu_2\eta_2$	$=$	$\mu_3(\eta_{-3}+\kappa)$
		$\vdots$	
s	$\mu_s\eta_s$	$=$	$\mu_{s+1}\left(\eta_{-(s+1)}+\kappa\right)$
		$\vdots$	

In the steady-state, the growth rate g_s in output of a variety with productivity gap s is increasing in η_s and η_{-s}, and the growth rate of aggregate output is a weighted average of g_s:

$$g = \ln\lambda \cdot \left(\sum_{s=0}^{\infty} \mu_s g_s\right).$$

When the two firms produce perfect substitutes, only the leader produces thus output growth is driven by the improvement in leader's productivity, with

$$g = \ln\lambda \cdot \left(2\mu_0\eta_0 + \sum_{s=1}^{\infty} \mu_s\eta_s\right).$$

Definition 1. A stationary equilibrium is defined by $\{C, Y, W, \Pi, V, I, g, r\}$ and $\{\pi_s, \pi_{-s}, \eta_s, \eta_{-s}, v_s, v_{-s}, \mu_s\}_{s=0}^{\infty}$ such that π_s and π_{-s} are the equilibrium profits in the (static) production games taking productivity gap s as given; η's and v's solve firms' profit max problems; μ_s is mass of varieties in with productivity gap s in the stationary distribution; g is the growth rate of the aggregate output, consumption, wages, profits, and value of the stock market in the economy; consumers choose investment / consumption following the potentially distorted Euler equation.

4 Equilibrium

I solve the model numerically under the assumption that two firms produce perfect substitutes within each variety, and the investment cost is a power function $c(\eta) = c \cdot \frac{\eta^\alpha}{\alpha}$. Power investment cost implies

$$c \cdot \eta_s^{\alpha-1} = v_{s+1} - v_s$$

and the HJB can be written as

$$rv_s = \pi_s + \frac{\alpha-1}{\alpha} c\eta_s^\alpha - (\eta_{-s}+\kappa)\, c\eta_{s-1}^{\alpha-1}$$

$$rv_{-s} = c\left(\frac{\alpha-1}{\alpha}\eta_{-s}^\alpha - \eta_s\eta_{-(s+1)}^{\alpha-1} + \kappa\eta_{-s}^{\alpha-1}\right)$$

or

$$0 = \pi_s - \pi_{s-1} + c\left(\frac{\alpha-1}{\alpha}\left(\eta_s^{\alpha} - \eta_{s-1}^{\alpha}\right) - \left(\eta_{-s} + \kappa + r\right)\eta_{s-1}^{\alpha-1} + \left(\eta_{-(s-1)} + \kappa\right)\eta_{s-2}^{\alpha-1}\right)$$

$$0 = c\left(\frac{\alpha-1}{\alpha}\left(\eta_{-s}^{\alpha} - \eta_{-(s+1)}^{\alpha}\right) - \eta_{-(s+1)}^{\alpha-1}\left(\eta_s + \kappa + r\right) + \kappa\eta_{-s}^{\alpha-1} + \eta_{s+1}\eta_{-(s+2)}^{\alpha-1}\right)$$

I now numerically solve for the equilibrium, examine the equilibrium structure, and discuss implications of the model. Note that the model is not yet calibrated. I first explore the supply-side of the economy, treating interest rate as exogenous and examine firms' incentive to invest and the resulting equilibrium growth rate, $g(r)$. The general equilibrium can be closed by the Euler equation, which is the demand side of the economy and generates another relationship between g and r.

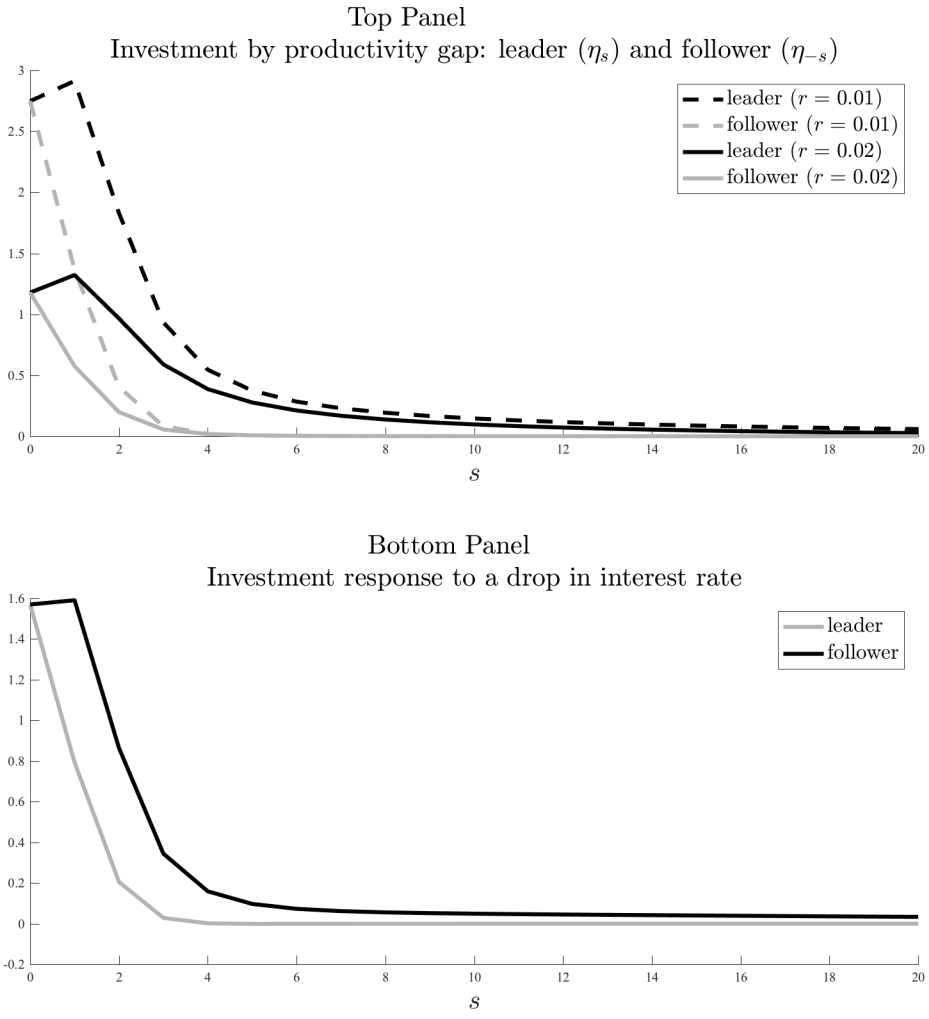


Figure 2: Investment by state

The top panel of Figure 1 plots investment η_s, η_{-s} as a function of the state, for two levels of interest

rates ($r_{high} = 0.03$ and $r_{low} = 0.01$). The bottom panel plots $\left(\eta_s\left(r_{low}\right) - \eta_s\left(r_{high}\right)\right)$ for the leader and follower. Key observations:

1. Leader dominance: leader always invests more than the follower;
2. Lazy monopolist effect: leader's investment decreases in his distance with the follower. This is because when the leader is far ahead, the gain from widening the gap further is low because there is little chance the follower can catch up.
3. Leader respond to the drop in interest rate more than the follower does:

$$\eta_s\left(r_{low}\right) - \eta_s\left(r_{high}\right) > \eta_{-s}\left(r_{low}\right) - \eta_{-s}\left(r_{high}\right).$$

4. Standard economic reasoning dictates that both players should invest more in every state as interest rate drops. This is the neoclassical force: low interest rate means low opportunity cost of current funds and low cost of investment.

In this economy, growth is generated by the productivity improvements in the frontier (leader's productivity). Thus one might think that because leader invests more in every state under low interest rates, aggregate growth has to increase. This line of reasoning is incorrect: low-rate not only changes the leader's incentive to invest, but also shifts the equilibrium distribution of productivity gaps. Indeed, because low interest rate widens the investment gap between leaders and followers, in the steady state there are more markets with large productivity gaps (as shown in Figure 2). Because leaders tend to invest less when productivity gap is large (lazy monopolist effect), aggregate growth could decrease under low-rate.

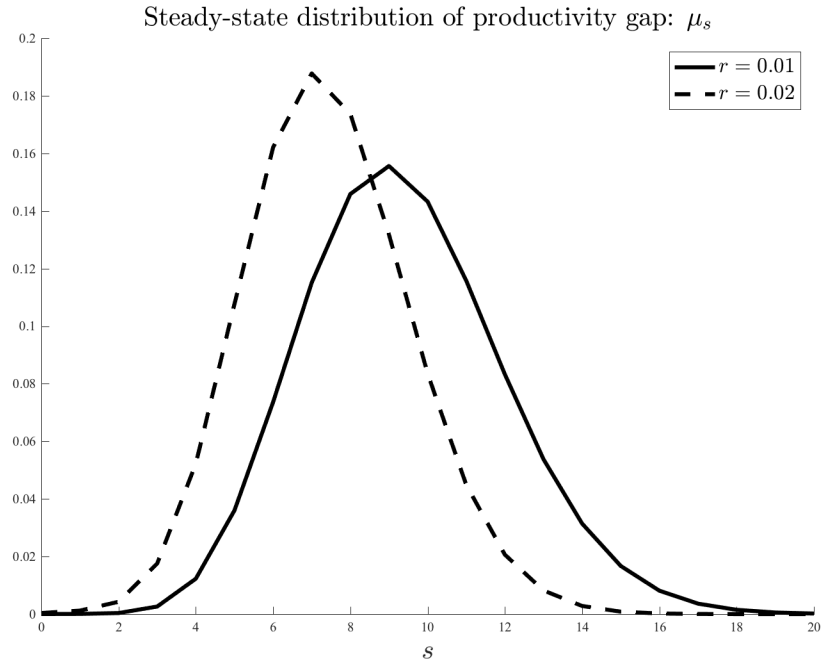


Figure 3: Steady-state distribution of productivity gap

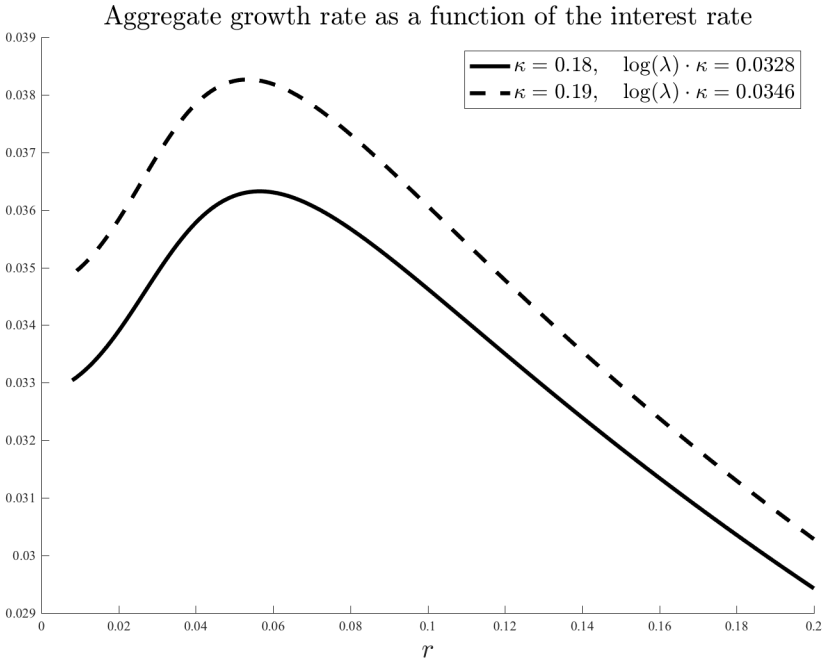


Figure 4: Productivity growth rate (supply-side) as a function of interest rate

Figure 3 plots the aggregate growth rate of the economy as a function of interest rate, for different levels of κ (the rate at which productivity gap exogenously closes; this is interpreted as "patent protection"

in the growth literature, but we can interpret it more broadly). Overall, growth rate generated by investments in the supply-side of the economy has an inverted-U relationship with the risk-free interest rate. For high levels of interest rate, neoclassical force dominates and growth is decreasing in the interest rate. On the other hand, when interest rate is low and close to zero, the strategic force dominates and growth becomes increasing in the interest rate. This is consistent with the VAR evidence found by Fernald.

A technical note: as interest rate goes to zero, the investment gap between leader and follower becomes arbitrarily large, and growth converges to $\ln \lambda \cdot \kappa$ and becomes purely driven by the exogenous parameters.

Demand side: Euler equation equation represents an upward-sloping relationship between g and r; it could have multiple interceptions with the supply-side $g(r)$ curve, generating multiple steady-states in the economy: one with high rate and high growth, another with low rate and low growth.

5 Implications

We think the model fits a broad range of aggregate economic trends over the last decade or so in the U.S. and Europe.

Fact 1. Secular stagnation (a large literature on this) We are experiencing long-term low productivity growth, coupled with low interest rates. Existing literature takes productivity growth as exogenous (i.e. does not slow down) and uses Keynesian / demand-side frictions to generate low output ("destroy output" despite productivity growth) and low interest rate. An easy rebuttal of the existing explanation is to simply show that productivity growth has slowed down.

- A related "zombie" idea is that in a low-rate environment, incumbents have access to better financing (e.g. Gopinath et al.). But such explanations can only explain "level" drops in aggregate productivity, and cannot explain changes in "growth rate" of productivity.

Fact 2. Rising market power (large literature, de Loecker and Eeckout, Autor et al.) When interest rate is low, the stationary distribution has more large firms with high markups and profit shares. Figure 4 plots the aggregate profit share output (recall $Y = \Pi + W$) as a function of interest rate.

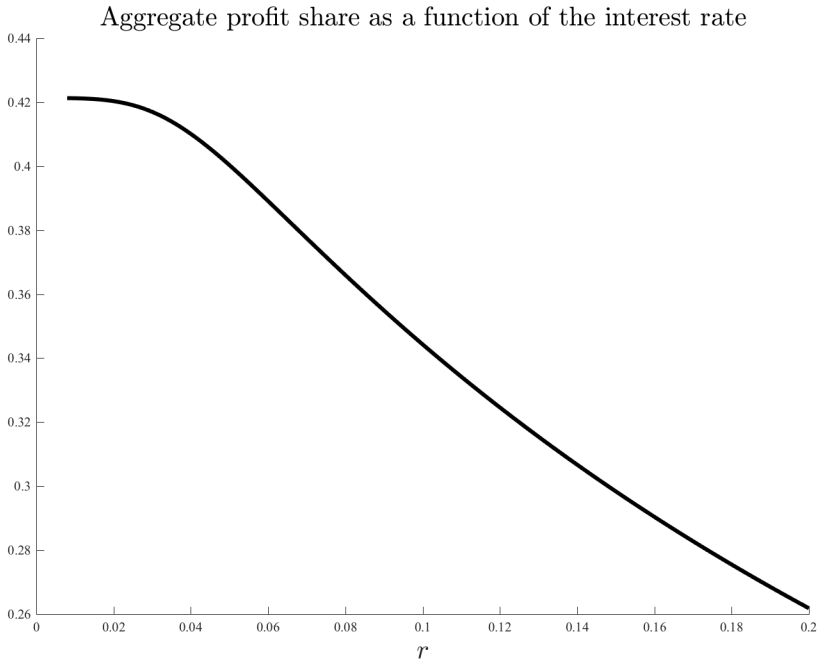


Figure 5: Aggregate profit share and interest rate

Fact 3. High stock market value The solid line in Figure 5 plots the value of all firms (relative to aggregate output) as a function of the interest rate. Aggregate market valuation can be written as a weighted average of firm valuation:

$$\bar{v} = \sum_{s=0}^{\infty} \mu_s \left(v_s + v_{-s} \right).$$

Let $f_s = \pi_s - c\left(\eta_s\right)$ be the flow payoff in state s. The firm value of being in a given state, v_s, can be written as a weighted average of the permanent-value of flow payoffs:

$$v_s = \sum_{s'} w_{s,s'} \cdot \frac{f_{s'}}{r}$$

with weights $\sum_{s'} w_{s,s'} = 1$ for all s. These weights can be written as the present discount fraction of time spent in s' if a firm starts in state s. Aggregate market valuation is affected by four factors.

1. Accounting effect. When interest rate drops, value of all firms increase proportionally to $\frac{1}{r}$.

2. Composition effect. The value of each market $(v_s + v_{-s})$ is increasing in the state s. When interest rate drops, the distribution of productivity gap $\{\mu_s\}$ undergoes a first-order-stochastic-dominant shift, resulting in larger productivity gaps in the economy. As a result, the value of stock market $\bar{v}$ increases as more weights (μ_s) are placed on high-valued markets (high s).

3. Rising investment cost. The flow value f_s represents the sum of flow-profit as well as invest cost, and it decreases for all $s > 0$ under low interest rate because leader invests more.

4. Persistent monopoly power. As the investment gap between leader and follower widens, monopoly power becomes more persistent and market leadership becomes harder to erode; as a result, there is reduced churning in the economy. This effect can be represented by changes in weights $w_{s,s'}$: when monopoly power is high (high s), the market leader's value places more weights on the flow-payoffs in higher states. Loosely, the persistence of monopoly power is reflected by the fact that $\left[w_{s,s'}\left(r^{low}\right) - w_{s,s'}\left(r^{high}\right)\right]$ exhibits increasing differences in (s, s').

The dashed line in Figure 5 isolates the accounting effect of interest rate on the value of stock market and plots $\left(\bar{v}\left(r\right)\Big|_{r=0.1}\right) \cdot \frac{0.1}{r}$. That is, it scales up the market value at $r = 10\%$ by $\frac{0.1}{r}$. The difference between solid and dashed lines correspond to the other 3 effects. In future analysis we can explore the decomposition further. For example, we can directly isolate the composition effect; we can also compute the "expected time before a leader falls behind" as a measure of churning. I am confident that we will see churning decreases as interest rate drops.

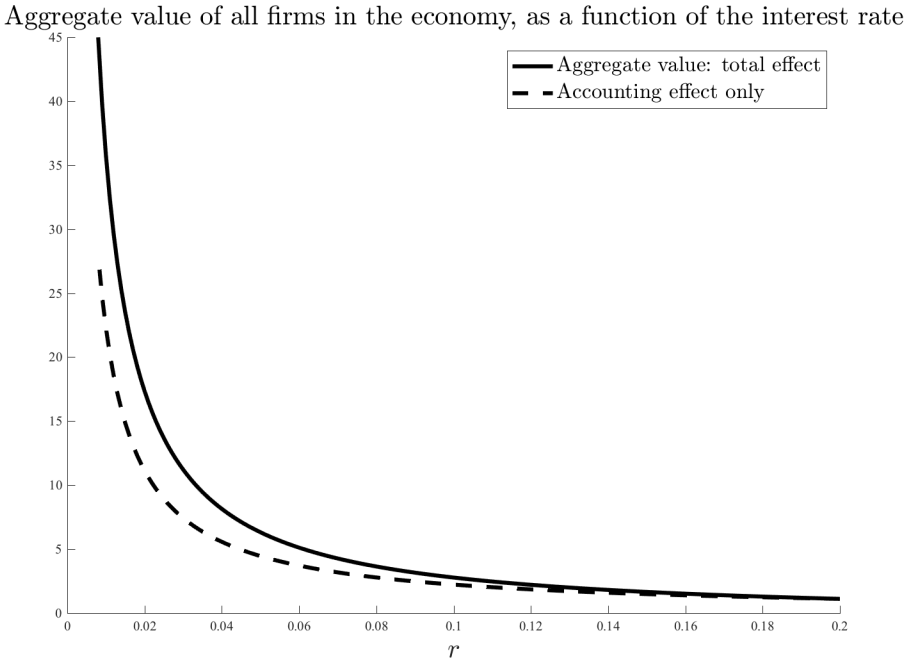


Figure 6: Value of stock market as a fraction of GDP

Fact 4. De-coupling of firm value and investment This can be seen from the top panel of Figure 1: high-valued firms invest less. Jones and Philippon "secular stagnation of investment" argue that a secular decline in competition in the goods market explains how low rates and weak corporate investment are correlated. Gutiï£¡rrez and Philippon "investment-less growth" shows that investment is not correlated with market valuation / profitability after 2000s. Stulz also presented a paper at NBER on this.

Fact 5. Widening productivity gap The investment gap between leader and follower is widening, as can be seen through the bottom panel of Figure 1 combined with Figure 2. Below is a figure produced from the OECD data by (paper by Andrews, Criscuolo, and Gal).

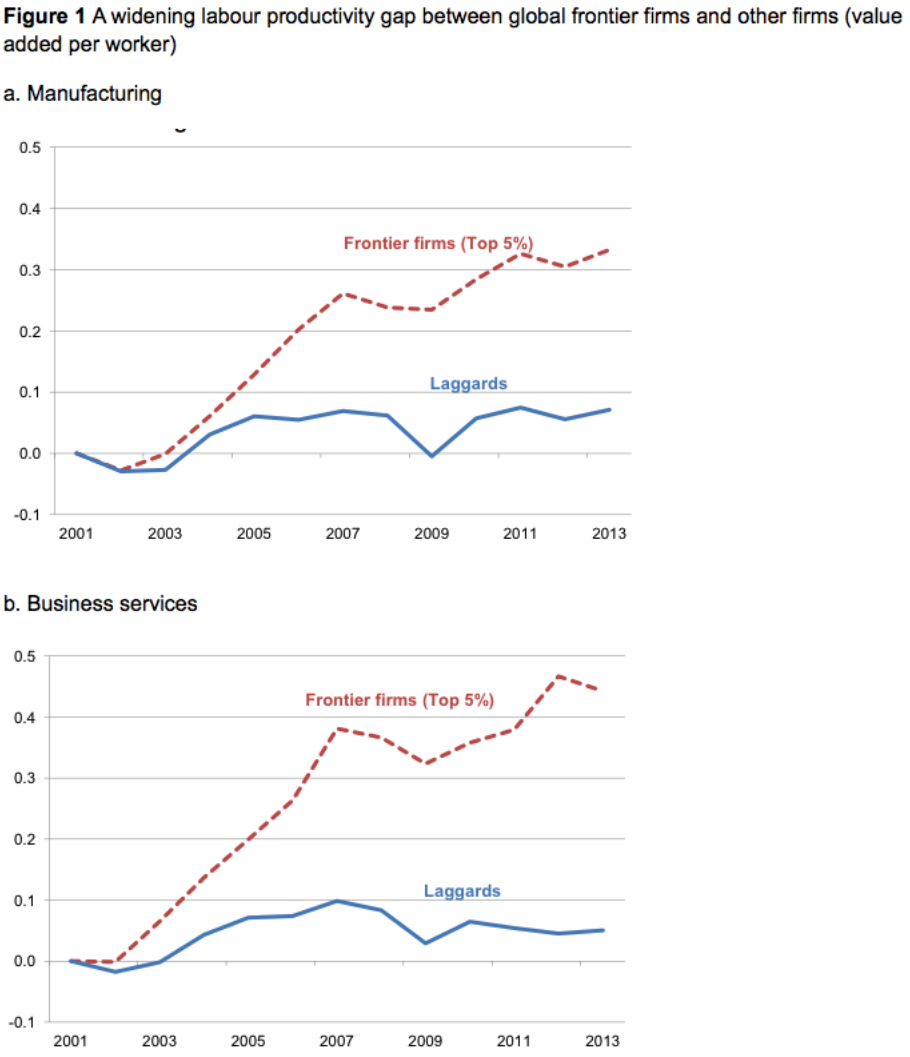


Figure 7: Widening productivity gap between leaders and followers

6 Appendix: Proof sketch

Basics

Setup Assume π_s, $(\pi_s + \pi_{-s})$, and $-\pi_{-s}$ are all weakly increasing in s, and that $\lim_{s\to\infty} \pi_s + \pi_{-s} = \pi$.

Monotone Equilibrium Label the players 1 and 2. In the model, an equilibrium is defined as the collection of investment decisions $\{\eta_s^1, \eta_s^2\}_{s=-\infty}^{\infty}$, such that these decisions induce value functions $\{v_s^1, v_s^2\}_{s=-\infty}^{\infty}$

and both players' investment decisions are consistent with maximizing his own value when taking the other player's investment decision as given: $\{\eta_s^1\}_{s=-\infty}^{\infty}$ is a best-response to $\{\eta_s^2\}_{s=-\infty}^{\infty}$ and vice-versa. The best-response is uniquely-valued for both players.

We look for symmetric monotone equilibria: "symmetric" in that player's investment decisions are functions of the state and not functions of the player's identity ($\eta_s^1 = \eta_s^2$ for all s); "monotone" in that investment decisions are weakly decreasing in the state. It can be shown that in any monotone equilibrium, both players can be indifferent between investing or not in at most one state. Thus a monotone equilibrium can be characterized by $(n, k) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ such that

- the leader
  - strictly invests in states 0 through $\lfloor n \rfloor$,
  - invest with probability $n - \lfloor n \rfloor$ in state $n + 1$,
  - does not invest in states $s \geq n + 2$;
- the follower
  - strictly invests in states 0 through $\lfloor k \rfloor$,
  - invest with probability $k - \lfloor k \rfloor$ in state $k + 1$,
  - does not invest in states $s \geq k + 2$.
- In what follows, let $\underline{n} \equiv \lfloor n \rfloor$, $\underline{k} \equiv \lfloor k \rfloor$, $\bar{n} \equiv \lceil n \rceil = \underline{n} + 1$, $\bar{k} \equiv \lceil k \rceil = \underline{k} + 1$.

If follower at least weakly invests in state $k + 1$, then leader must strictly invest in state k The statement can be equivalently written as $\underline{n} \geq \underline{k}$. To prove it, let $w_s \equiv v_s + v_{-s}$ be the joint value of leader and follower in state s. Then

$$w_s = \frac{\pi_s + \pi_{-s} - c\left(\eta_s + \eta_{-s}\right)/\eta + \eta_s w_{s+1} + \left(\eta_{-s} + \kappa\right)\eta_{s-1}}{r + \eta_s + \eta_{-s} + \kappa}.$$

In a monotone equilibrium, the flow payoffs $\{f_{s'}\}_{s'=0}^{\infty} \equiv \{\pi_{s'} + \pi_{-s'} - c\left(\eta_{s'} + \eta_{-s'}\right)/\eta\}_{s'=0}^{\infty}$ form a weakly increasing sequence. We can write $w_s = \sum_{s'=0}^{\infty} \omega_{s,s'} \frac{f_{s'}}{r}$ as a weighted average of the flow payoffs (with $\sum_{s=0}^{\infty} \omega_{s,s'} = 1$), with the highest weight on $s' = s$ and the weight strictly decays as s' moves away from s. Furthermore, the weights $\{\omega_{s,s'}\}$ first-order stochastically dominate $\{\omega_{k,s'}\}$ iff $k \geq s$. Thus the joint surplus w_s must be strictly increasing.

Now suppose the leader does not strictly invests in k while follower weakly invests in $k + 1$. This implies

$$v_{k+1} - v_k \leq c/\eta < v_{-k} - v_{-(k+1)}$$

$$\iff w_{k+1} < w_k,$$

a contradiction.

For small r, leader strictly invests more than the follower in any monotone equilibria The previous result shows that leader's invest cannot fall behind that of follower's by more than one state in a monotone equilibrium, but it does not rule out the possibility that $\underline{n} = \underline{k}$ and $n < k$. It turns out that n must strictly exceed k for $r < \bar{r}(\eta, \kappa)$. Computing $\bar{r}(\eta, \kappa)$ requires some algebra, but it is indeed a very weak requirement; in particular, $\bar{r}(\eta, \kappa)$ does not have to be vanishingly small r (I have to check the expressions, but I think the inequality is satisfied as long as firms choose to invest in state zero, thus a very weak assumption).

The proof strategy is as follows. Suppose $n < k$. The fact that leader mixes in the state implies

$$v_{\underline{n}} = \pi_{\bar{n}} - \frac{r+\kappa}{\eta} c$$

To see this, note that the leader is indifferent iff

$$v_{\bar{n}} - v_{\underline{n}} = c/\eta$$

$$\iff \frac{\pi + \kappa v_{\underline{n}}}{r+\kappa} - v_{\underline{n}} = c/\eta$$

$$\iff r v_{\underline{n}} \geq \pi_{\bar{n}} - \frac{r+\kappa}{\eta} c$$

and that the follower mixes implies a lower bound on $v_{-\underline{n}}$. It can be shown that for $r < \bar{r}(\eta, \kappa)$, the lower bound on joint value $w_{\underline{n}}$ can never be achieved, thus providing a contradiction.

A monotone equilibrium exists Starting with any $(n_0, k_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ such that $n_0 \geq k_0$, we can compute the value functions $\{v_s, v_{-s}\}_{s=0}^{\infty}$ induced by these investment decisions, and then compute optimal investment decisions based on the value functions. Under the same assumption that $r < \bar{r}(\eta, \kappa)$, the investment decisions will also be of the form (n, k) with $n \geq k$. The mapping from $\left\{ n, k \middle| (n,k) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \text{ and } n \geq k \right\}$ onto itself has at least one fixed point, which corresponds to a monotone equilibrium.

Pseudo-best-response I define pseudo-best-response as follows. Fix follower's investment decision k. For what values of n, such that (n, k) would induce value functions for which leader's investment decision is consistent with n? Such levels of $n^*(k)$ are refered to as the pseudo-best-response for the leader. We can similarly define the pseudo-best-response for the follower, $k^*(n)$. If $n = n^*(k)$ and $k = k^*(n)$, then (n, k) is an equilibrium.

We can also define $k(n-k)$ as the level of investment for the follower that is consistent with the investment gap $n - k$. It can be shown that $k(n-k)$ is unique. That is, there's a unique level of follower investment k that is consistent with any investment gap $n - k$. The reason is that holding $n - k$ fixed, raising k lowers the incentive to invest, and vice versa; thus there exists a unique level of k that is consistent with $n - k$.

Lastly, holding k fixed, consider a planner that makes investment decisions on behalf of the leader in

order to maximize the joint value of leader and follower in state $\underline{k}$. We define the investment decision of this fictitious planner as $n^{**}(k)$. It can be shown that $n^{**}(k)$ is uniquely valued, and that $n^{**}(k) - k$ is increasing in k. In words, this is because for high k (follower invests in more states), the joint value is lower in state $\underline{k}$, and thus the fictitious planner has to invest in more states (thus higher $(n^{**}(k) - k)$) in order for the joint value to reach $\pi_s - \frac{r+\kappa}{\eta}c$, the condition for stopping investment.

To see this, note that the planner stops investment iff

$$w_{s+1} - w_s < c/\eta$$

$$\iff \frac{\pi + \kappa w_s}{r+\kappa} - w_s < c/\eta$$

$$\iff rw_s \geq \pi - \frac{r+\kappa}{\eta}c$$

Why do we compute the decision for the fictitious planner? This is useful because holding k constant, $n^{**}(k)$ maximizes $w_{\underline{n}}$ conditioning on $n^{**} \geq k$; in other words, for any $n > n^{**}(k)$, $w_{\underline{n}}$ is decreasing in n. Since $v_0 = \frac{1}{2}w_0$ is an increasing function of $w_{\underline{n}}$, we know that raising n beyond n^{**} lowers the value of reaching state 0.

Consider figure 7, which plots the $k(n-k)$. It is inverted-U because starting with low levels of $(n-k)$, a small increase in $n-k$ while holding k constant raises the joint value in state 0, increasing the incentive for the follower to invest and catch up. On the other hand, when $n - k > n^{**}(k) - k$, joint welfare is strictly decreasing in $n-k$, thus raising $n-k$ lowers the incentive to invest. Because $w_s - w_{s-1} < v_s - v_{s-1}$, we know $n^*(k) \geq \lceil n^{**}(k) \rceil$: leader invests more than the fictitious planner, thus in any equilibrium, $(n-k, k)$ must be on the decreasing portion of the figure. We can also show that the slope of the decreasing portion cannot be less than -1: raising $n-k$ by ϵ cannot lower k by more than ϵ. This is because if k decreases by more than ϵ, n must decrease. Lowering n (moving n closer to n^{**}) and raising $n-k$ both raise the incentive for the follower to invest and is inconsistent with the fact that k decreases.

We can now translate the pseudo-best-response onto (n, k) space. The bold curve in figure 8 plots n against $k(n)$. The curve must be backward-bending (because $(k, n-k)$ is inverted-U), and any equilibrium must be on the downward-sloping portion of the curve.

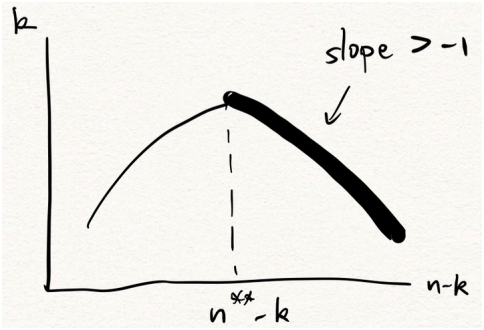


Figure 8:

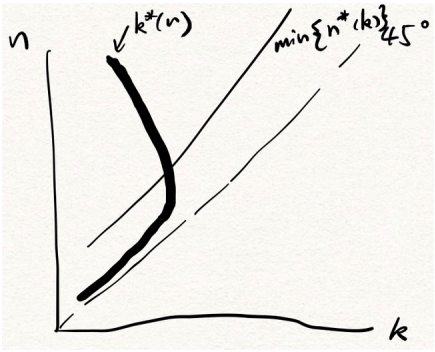


Figure 9:

Multiplicity Though I have not found any numerical example of non-monotone equilibria, the theoretical possibility cannot be ruled out unless we impose conditions on (κ, η). In particular, theoretically I can prove that for $\kappa < \bar{\kappa}(\eta)$, then there exists $\bar{r}(\kappa, \eta)$ such that all equilibria must be monotone for $r < \bar{r}(\kappa, \eta)$.

There could also exist more than one monotone equilibria under certain parameter values. The source of multiplicity is that $n^*(k)$ is set-valued (holding k fixed, there can be multiple levels of $n^*(k)$ consistent with maximization). The intuition is as follows. Holding k fixed, increasing n beyond $n^{**}(k)$ lowers the value of being in state zero (v_0), thereby lowering the value of leader in any given state and raising his incentive to invest (the leader stops in state s if and only if $v_s \geq \pi_{s+1} - \frac{r+\kappa}{\eta}c$).

Figure 2 plots $\min\{n^*(k)\}$ as a function of k. Because $k^*(n)$ is uniquely-valued and is downward-sloping around any equilibrium level of (n, k), the intersection between the two curves represents the monotone equilibrium with the minimum investment gap $(n - k)$. We focus on this equilibrium. When there are multiple equilibria, this equilibrium corresponds to the one in which growth is maximized.

Comparative Statics

To examine how investment incentives change in response to interest rate, we use a trick called "uniformization", i.e. adding a fictitious transition from one state onto itself. Formally, let $\xi \equiv 2\eta + \kappa$, and we can write the value functions as

$$\begin{aligned} v_s &= \frac{\pi_s - c\eta_s/\eta + \eta_s v_{s+1} + (\eta_{-s} + \kappa) v_{s-1}}{r + \eta_s + \eta_{-s} + \kappa} \\ &= \frac{\pi_s}{r+\xi} + \frac{\xi}{r+\xi}\left\{ \frac{2\eta}{\xi} v_s + \frac{\eta_{-s} + \kappa}{\xi}(v_{s-1} - v_s) + \frac{\eta_s (v_{s+1} - v_s - c/\eta)}{\xi} \right\} \end{aligned}$$

$$v_{-s} = \frac{\pi_{-s}}{r+\xi} + \frac{\xi}{r+\xi}\left\{ \frac{2\eta}{\xi} v_{-s} + \frac{\kappa}{\xi} v_{-(s-1)} + \frac{\eta_s}{\xi}\left(v_{-(s+1)} - v_{-s}\right) + \frac{\eta_{-s}\left(v_{-(s-1)} - v_s - c/\eta\right)}{\xi} \right\}.$$

Lowering r raises $\frac{\pi_s}{r+\xi}$ and $\frac{\xi}{r+\xi}$. Because π_s is increasing and that the weights in curly brackets has the monotone-hierarchy property (higher states have higher weights on high values), it can be shown that

$$\left.\frac{-\partial (v_{s+1} - v_s)}{\partial r}\right|_{(n,k) \text{ held fixed}} > \left.\frac{-\partial (v_{s'+1} - v_{s'})}{\partial r}\right|_{(n,k) \text{ held fixed}}$$

for all $s > s'$, thereby establishing that the change in leader's incentive to invest is larger than the change in the follower's incentive.

In progress to show $n(r) - k(r) > n(r') - k(r')$ for $r < r'$, i.e. show leader respond more than the follower does for discrete changes in interest rate.