Central Bank Balance Sheet Policies without Rational Expectations*

Luigi Iovino† Dmitriy Sergeyev‡

December 28, 2017

Abstract

We study the effects of central bank balance sheet policies—quantitative easing and foreign exchange interventions—in a model where agents form expectations through an iterative process known as level-k thinking. This process is consistent with the experimental evidence on the behavior of people who are confronted with novel strategic situations. This belief formation choice is motivated by the fact that some balance sheet policies are novel while others are still not well understood. We emphasize three main results. First, under a broad set of conditions, central bank interventions are effective under level-k thinking, while they are neutral in the rational expectations equilibrium. Second, we derive testable predictions that can be used to differentiate this channel from the portfolio balance and signaling channels of balance sheet policies. Third, balance sheet policies become neutral in the long run when agents become more sophisticated over time.

*We would like to thank Nicola Gennaioli, Yuriy Gorodnichenko, Pierre-Olivier Gourinchas, Matteo Maggiori, Tommaso Monacelli, Rosemarie Nagel, Esteban Rossi-Hansberg, and seminar and conference participants in Bocconi University, UC Berkeley, London Business School, Barcelona GSE Summer Forum 2017, SED 2017 Meeting for comments and helpful discussions.

†Bocconi University, Department of Economics, CEPR, IGIER, e-mail: luigi.iovino@unibocconi.it

‡Bocconi University, Department of Economics, CEPR, IGIER, e-mail: dmytro.sergeyev@unibocconi.it

1 Introduction

Balance sheets of central banks are among the most important and widely used stabilization policy tools (Bernanke, 2012; Draghi, 2015; Yellen, 2016). A recent example is the policy of quantitative easing (QE)—a purchase of long-term public and risky private assets financed with central bank liabilities. Several central banks in developed countries have recently used this policy to stimulate their economies when the conventional nominal interest rate tool reached its effective zero lower bound. Yet balance sheet policies are not confined to liquidity traps. Another example of such policies, which arguably has been used more often across countries and over time, is foreign exchange (FX) interventions—a purchase of foreign sovereign bonds denominated in foreign currency, usually financed by selling holdings of domestic sovereign bonds. Advanced economies had to routinely rely on this policy during periods of fixed exchange rate arrangements (e.g., the Gold Standard, the Bretton Woods, the European Exchange Rate Mechanism). What is more, during the recent financial crisis, some economies have again resorted to such interventions to tame speculative capital flows (e.g., in Switzerland and Israel) and to stimulate domestic production (e.g., in the Czech Republic). Finally, emerging economies have also been using FX interventions to limit exchange rate fluctuations and to accumulate buffers against future sudden stops.

Despite the popularity, central bank balance sheet policies are among the least understood. First, from an empirical perspective, it is challenging to identify a causal effect: Balance sheet policies are usually implemented in response to shocks that hit the economy creating an endogeneity problem. There is nonetheless some empirical evidence for QE and FX interventions effects on assets markets and the real economy.[1] Second, from a theoretical perspective, a wide class of standard macroeconomic models predicts that balance sheet policies have no effect on the economy. More precisely, Wallace (1981) showed that these policies can be irrelevant because investors can completely undo central bank

[1] Using high frequency financial data, Gagnon *et al.* (2011), Krishnamurthy and Vissing-Jorgensen (2011), Hancock and Passmore (2011) found that large-scale mortgage-backed securities (MBS) purchases by the Fed have affected mortgage market yields and have spread to other assets markets. At the same time, Stroebel and Taylor (2012) find no effects of MBS purchases in the first round of QE by the Fed. Di Maggio *et al.* (2016) and Chakraborty *et al.* (2016) found evidence of the effects of Fed's MBS purchases on mortgage lending. Fieldhouse *et al.* (2017) show that purchases of MBS by the government sponsored enterprises in the US, which can be interpreted as QE, affected not only mortgage rates and lending but also residential investments. Gorodnichenko and Ray (2017) present evidence that demand shocks at the primary market for U.S. Treasuries of specific maturities affect the yield curve, which can be interpreted as a non-neutrality of quantitative easing in which the central bank trades long-term government bonds.

Dominguez and Frankel (1990, 1993) estimate significant effects of foreign exchange interventions. Sarno and Taylor (2001) summarize the earlier literature presenting a more balanced view. Kearns and Rigobon (2005) study a "natural experiment" in which Japan and Australia "exogenously" changed their FX policies which resulted in significant change in their exchange rate. Blanchard *et al.* (2014) present evidence of the significant effects of currency interventions using more recent data.

interventions. There are two main steps in Wallace's argument, which represents an application of the classical Ricardian equivalence (Ricardo, 1821; Barro, 1974). First, when a central bank purchases private risky assets and issues safe liabilities (as in the case of QE), investors understand that gains or losses incurred on the central bank's portfolio will be directly transferred to the fiscal authority and, through taxes, they will indirectly return to investors. In turn, investors reduce their demand for risky assets to hedge against this tax risk. Second, investors accurately predict the magnitude of the tax risk in rational expectations equilibrium. Thus they reduce their demand for risky assets in a way that completely undoes the direct effect of the policy intervention. The same logic applies to FX interventions.[2]

Laboratory experiments repeatedly documented that people fail to play equilibrium strategies when confronted with novel strategic situations (Crawford *et al.*, 2013). Instead, many people behave consistent with forming their expectations about the behavior of other people according to the level-k thinking process (Nagel, 1995; Stahl and Wilson, 1995). This process can be particularly relevant in macroeconomic settings, especially when people observe new policies or policies—such as QE and FX interventions—whose effects are not precisely estimated or even understood. Without sufficient past experience, agents' ability to predict the effects of the new policies is hindered and their expectations are unlikely to be rational. In these cases, level-k thinking provides a plausible alternative as it does not require the knowledge of past policy effects.

In this paper, we introduce the level-k thinking process into a standard dynamic equilibrium model based on Wallace (1981), which we extend to an international setting along the lines of Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006).[3] In particular, we show that the assumption that investors forecast future endogenous variables through the level-k thinking process overcomes Wallace's irrelevance result and provides a new channel for central bank balance sheet policies.

Level-k thinking and the equilibrium notions associated with it work as follows (we adopt the formulation in García-Schmidt and Woodford 2015; Farhi and Werning 2016). Agents are perfectly aware of current balance sheet policies (assets traded by the government) as well as their own income and asset positions. However, they have to form expectations about the effects of the policy on future endogenous variables, such as taxes and asset prices. They form expectations according to the following iterative procedure.

[2]Below, we discuss other mechanisms through which balance sheet policies can have an effect on the economy.

[3]These two papers feature deviations from full information rational expectations equilibria. Jeanne and Rose (2002) introduce "noise traders," while Bacchetta and Van Wincoop (2006) add private signals to agents information sets. We instead incorporate level-k thinking and study central bank balance sheet policies.

"Level-1 thinking" assumes that agents keep expectations identical to those before the change in the policy. As a result, "level-1 thinkers" completely ignore the Wallace irrelevance result. These agents choose consumption and portfolio holdings given their naive expectations and markets clear on period-by-period basis. "Level-2 thinker" forecasts the future by computing equilibrium outcomes of the balance sheet policy conditional on believing that economy is populated by level-1 thinkers only. Thus, this more sophisticated thinker will expect future to depend on the policy intervention, however, these expectations may not coincide with rational expectations. Conditional on these updated expectations, these more sophisticated thinkers make consumption and portfolio decisions and markets clear on a period-by-period basis. Applying the same steps, these deduction rounds can be generalized to "level-k thinking" and carried over to infinity. Following García-Schmidt and Woodford (2015), we assume that the economy is populated by agents with all levels of thinking, with the mass of agents of each level of thinking given by an exogenous distribution.[4] This results in a notion of reflective equilibrium which we describe in Section 2.

Our first main result shows that balance sheet policies affect asset prices in the reflective equilibrium while they are neutral in the rational expectations equilibrium. Intuitively, agents do not hold rational expectations about future endogenous variables, and, in fact, they always underestimate the response of these variables to policy interventions. As a result, agents do not change their asset demand enough to undo the intervention and balance sheet policies become effective.

As the average level of sophistication in the economy increases, the reflective equilibrium converges to the rational expectations equilibrium. However, we show that this convergence may be non-monotonic when the policy intervention is expected to persist over time. Specifically, an increase in agents' sophistication has two opposing effects. On the one hand, more sophisticated agents can better foresee the effects of policies on future endogenous variables, bringing the policy closer to full neutrality. On the other hand, more sophisticated agents become endogenously more forward looking, thus, making persistent policies more effective.

We present numerical and comparative statics exercises that suggest that the effects of level-k thinking can be sizable. More specifically, we show that the price effect of quantitative easing is a fraction of the price effect of an equivalent increase in foreign demand, where the fraction is given by the inverse of the average sophistication of agents in the economy. For example, if the average level of sophistication of agents in the economy is $\overline{k} = 2$ (i.e., a typical agent thinks that all other agents do not change their expectations

[4] García-Schmidt and Woodford (2015) assume that the average expectations are updated in a continuous fashion following a first-order differential equation. This assumption is equivalent to assuming that beliefs are formed in a discrete way and that the economy is populated by all types of thinkers.

following a policy change) then QE is only twice weaker than foreign purchases of assets. In a very stylized numerical exercise, we evaluate a policy experiment that resembles the Fed's mortgage-backed securities (MBS) purchases in 2009-10. When the home government buys 10% of the overall supply of risky assets that command the excess return of 2%, then the price of these assets change by 10 basis points if policy is short-lived and by 20 basis points in case of the permanent policy.

Our second main result characterizes the behavior of forecast errors of asset prices after policy interventions. We show that individual and cross-sectional-average forecast errors are related to policy interventions. Importantly, this result can help differentiate the mechanism proposed in this paper from other theories of the effectiveness of balance sheet policies in the literature. First, forecast errors are absent in the standard models that assume limited market participation, but retain rational expectations. Second, in models with heterogeneous information (Lucas, 1972; Mankiw and Reis, 2002; Sims, 2003), predictable forecast errors would arise only if agents had incomplete information about policy interventions. If agents had access to all the relevant information regarding the policy, forecast errors would no longer be predictable. In our model, instead, agents are fully aware of the policy intervention, yet they make mistakes due to their inability to form rational expectations.[5]

The third main result of the paper shows that balance sheet policies are neutral in the long run if agents "learn to play" rational expectations equilibrium (equilibrium unraveling). The presence of forecast errors suggests that agents could learn by accumulating evidence on the effects of central bank interventions. We thus extend the model by introducing a simple unraveling mechanism. In particular, we let the average level of sophistication in the economy to grow over time. This assumption captures the idea that, as new evidence becomes available, each agent expects other agents to have a better understanding of the effects of the new policy. We show that over time the reflective equilibrium converges to the rational expectations equilibrium and the effects of balance sheet policies fade. Therefore, a central bank that wants to keep asset prices elevated must continuously expand its balance sheet. In addition, new policy rounds must be less effective if people use the knowledge accumulated in previous rounds to forecast future endogenous variables. For example, our model implies that the Fed's QE1 has been more effective than QE2, after controlling for the different size of these policies.

Related literature. Our paper is related to several strands of literature. First and foremost, we contribute to the theoretical literature that proposes mechanisms of the effects of the balance sheet policies. An important starting point is the Wallace (1981) irrelevance

[5] In ongoing work, we empirically test these predictions.

result, which we mentioned earlier. Backus and Kehoe (1989) reach an even stronger conclusion than the Wallace irrelevance result. They show that, following an intervention in foreign exchange markets, the portfolio of the government has exactly the same state-contingent payoffs as the original one, therefore, future taxes are not affected by the policy. To deviate from the irrelevance result, the literature augmented the Wallace's model with various frictions. The two main frictions are incomplete information and market segmentation. The former friction generates the so-called "signaling" channel and the letter generates the "portfolio balance" channel of the balance sheet policies.

According to the signaling channel view, changes in the composition of central bank's balance sheet does not have a direct effect on the economy but rather serve as a signal of the central bank private information about its objectives and economic fundamentals. Mussa (1981), Bhattacharya and Weller (1997), Popper and Montgomery (2001), Vitale (1999, 2003) applied this idea to FX interventions. Some authors considered a situation when the central bank cannot commit to a desired future monetary policy and use the costly balance sheet policy as a signal about future intentions. See Jeanne and Svensson (2007) and Bhattarai *et al.* (2015) for FX and QE interventions respectively.

The portfolio-balance channel posits that changes in the supplies of different assets affect asset prices because of asset markets segmentation due to fixed costs of entry or because of limited market participation due to impossibility to trade asset for yet-unborn people. Kouri (1976) and more recently Gabaix and Maggiori (2015), Fanelli and Straub (2016), Amador *et al.* (2017), Cavallino (2017) apply this idea to FX interventions, while Vayanos and Vila (2009), Curdia and Woodford (2011), Chen *et al.* (2012), Hamilton and Wu (2012), Silva (2016) apply this idea to quantitative easing, and Krishnamurthy and Vissing-Jorgensen (2011) summarizes the recent literature on quantitative easing. Reis (2017) proposes that quantitative easing is a powerful stabilization tool in times of fiscal crisis. Sterk and Tenreyro (2013) show that standard open market operations have sizable effect on real economy in the presence of durable goods and asset markets segmentation.

Our paper relies on the assumption that people form their expectations according to the level-k thinking process. This assumption has been widely used in behavioral game theory to rationalize a non-equilibrium behavior of subjects in various laboratory and field experiments on games of full information (Nagel, 1995; Stahl and Wilson, 1995; Bosch-Domenech *et al.*, 2002; Crawford *et al.*, 2013). Camerer *et al.* (2004) proposes a related "cognitive hierarchy" model in which level-k thinkers assume that the other players are not only level-$(k-1)$ but also level-$(k-2)$ and so on. This alternative retains most of tractability of level-k thinking but outperforms it in some applications. Deviations from Nash equilibrium behavior in many simple games are most stark on the first round of play, when agents do not have prior experience. In these games, subjects usually do not

exhibit levels of thinking higher than 3.

Our paper contributes to the literature that incorporates deviations from rational expectations in macroeconomics models. Woodford (2013) provides a summary of recent advances in the literature. The level-k thinking process of expectations formation was first used in macro by Evans and Ramey (1992, 1998) to study conventional monetary policy. García-Schmidt and Woodford (2015) and Farhi and Werning (2016) analyze forward guidance policy under level-k beliefs formation, and level-k belief formation and incomplete asset markets respectively. Our paper instead focuses on domestic and international balance sheet policies in an environment with aggregate risk. Gabaix (2016) augments a New Keynesian model with agents' inattention to future policy instruments which help to resolve a number of puzzles in the New Keynesian literature. In contrast, in our environment, agents perfectly understand announced current and future policy changes, however, they do not adjust their expectations about endogenous variables to achieve rational expectations after the policy changes. Bordalo *et al.* (2016) incorporate "diagnostic expectations" (agents over-weigh future likelihood of events that occurred in the recent past) into a dynamic equilibrium model and show that credit spreads are excessively volatile and exhibit predictable reversals. In the current paper, we assume that agents expectations about future shocks coincide with their realizations, the agents, however, do not form rational expectations about future endogenous variables.

The rest of the paper is organized as follows. Section 2 presents the model and introduces equilibrium concepts. Section 3 analyzes the effects of balance sheet policies. Section 4 discusses the testable implications. Section 5 considers the learning process. Section 6 concludes.

2 A Model of the World Economy

We now present our baseline model, which we use to investigate the effects of domestic (e.g., quantitative easing) and international (e.g., foreign exchange interventions) balance-sheet policies. The model features a nominal friction in the form of demand for money and an international dimension so that we can analyze both closed- and international-economy interventions involving nominal variables in a single environment.

We build on Wallace (1981) by adding an international dimension following the open-economy models of Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006).

2.1 Agents, Assets, and Expectations

Countries. There are two countries: home and foreign. Foreign country variables will bear an asterisk. Both countries produce the same good, which is traded freely across borders. As a result, the law of one price applies and we have $P_t = E_t P_t^*$, where P_t and P_t^* are the nominal price levels in the home and foreign countries, respectively, and E_t is the nominal exchange rate. The exchange rate is defined as the quantity of home currency bought by one unit of foreign currency. Consequently, an increase in E_t corresponds to depreciation of home currency. For convenience, we let $e_t \equiv \log E_t$ and $p_t \equiv \log P_t$.

Time. Time is discrete, infinite, and indexed by $t = 0, 1, 2, \ldots$.

Assets. There are several assets in the world. Households in the home country can hold money issued by their own country,[6] nominal bonds issued by both countries—which pay, respectively, interest rates i_t and i_t^*—a riskless technology asset, available in perfectly elastic supply, that pays off a real continuously compounded return $r \equiv \log R$, and a home country risky asset that pays off a return r_{t+1}^x in units of consumption good in the following period. The total supply of the risky asset is $\overline{X}$ and they trade at price q_t. Similarly, households in the foreign country can hold money issued by their own country, nominal bonds issued by both countries, and the riskless technology asset. The assumption that foreign-country households cannot invest in this risky technology is made to simplify the analysis, but has no substantial consequence for the results.[7]

Risk. There are three sources of risk in the economy. First, returns on the risky asset satisfy $r_t^x = r^x + \epsilon_t^x$. Second, both home and foreign-country money supplies follow stochastic processes given by $\log M_{t+1} = \log \overline{M} + \epsilon_t^h$ and $\log M_{t+1}^* = \log \overline{M}^* + \epsilon_t^f$. The disturbances ϵ_t^x, ϵ_t^h, and ϵ_t^f are assumed to be independent from each other, iid over time, and Normally distributed with mean 0 and standard deviation σ_x, σ_h, and σ_f, respectively.

Households. We explicitly consider home-country households, foreign-country households are symmetric. Households live for two periods. In the first period of their life, they receive real endowment w and can buy four types of assets: home currency, home and foreign nominal bonds, and the riskless asset. In the second period of their lives, they get a return on their portfolio, pay taxes, and consume. Each period there is a mass of ω

[6] We follow the international economics literature and make the simplifying assumption that households in a country can only hold the money of the country they live in.

[7] Note that this assumption does not explicitly prohibit the government of the foreign country to invest in risky asset.

of "young" households and a mass of ω of "old" households, where ω represents the size of home country. The size of the foreign country is $1-\omega$.

To solve their problem, households need to form expectations about future variables, both exogenous and endogenous ones. We describe expectations in detail below, for now we use a tilde on the expectation operator to emphasize the fact that households may use a probability distribution over future variables that is not necessarily consistent with equilibrium outcomes.

Specifically, given beliefs and prices, households choose consumption, investment in the safe and risky technology, real money holdings, and saving in home and foreign bonds, so as to solve the following problem in period t:

$$\max_{\substack{s_{t+1},x_{t+1},b_{H,t+1},b_{F,t+1},\\ m_{t+1},c_{t+1}}} -\frac{1}{\gamma}\widetilde{\mathbb{E}}_t \exp\left[-\gamma\left(c_{t+1}-\frac{m_{t+1}\left[\log\left(m_{t+1}/\overline{m}\right)-1\right]}{v}\right)\right]$$

subject to the current budget constraint

$$P_t\left(s_{t+1}+q_t x_{t+1}+b_{H,t+1}+m_{t+1}\right)+E_t P_t^* b_{F,t+1}\leq P_t w, \tag{1}$$

and the future budget constraint

$$\begin{aligned} P_{t+1}\left(c_{t+1}+T_{t+1}\right)\leq & P_{t+1}\left[e^r s_{t+1}+\left(r_{t+1}^x+q_{t+1}\right)x_{t+1}\right], \\ & +P_t\left(e^{i_t}b_{H,t+1}+m_{t+1}\right)+E_{t+1}P_t^* e^{i_t^*}b_{F,t+1} \end{aligned} \tag{2}$$

where s_{t+1} is investment in the riskless technology, x_{t+1} is investment in the risky technology, $b_{H,t+1}$ and $b_{F,t+1}$ are purchases of home and foreign bonds, respectively, and m_{t+1} denotes a choice of period-t home real money balances expressed in units of period t consumption. In the budget constraints, all quantity variables are expressed in units of contemporaneous consumption good and converted into units of domestic currency.

Preferences are assumed to depend also on real home money balances. Money in the utility function is a standard approach to introduce demand for money in macroeconomics and international finance. In addition, the particular functional form assumed here simplifies the analysis by making money demand independent of the consumption choice. Note that utility is increasing in m_{t+1} for $m_{t+1}\leq\overline{m}$ and decreasing for $m_{t+1}>\overline{m}$. We thus assume that $m_{t+1}\leq\overline{m}$.

It is worth explaining our modeling choice of the overlapping generations framework. In such an environment, unborn generations are excluded from participating in current asset markets, thus, government asset purchases can affect the economy even under rational expectations. The overlapping generations model is therefore an example of models

with limited participation where the Wallace irrelevance result does not apply. In this sense, the level-k thinking of expectations formation—the focus on this paper—is not necessary to make asset purchases effective.

There are two main reasons that lead us to choose this particular environment. First, maximization of the CARA preferences with Gaussian shocks is equivalent to maximization of the mean-variance preferences. This property makes the analysis extremely tractable when incorporating the effects of uncertainty. In fact, such an environment has been a workhorse model in finance literature starting from seminal contribution by De Long *et al.* (1990). Second, we can obtain the Wallace irrelevance result even in this environment. In fact, Wallace (1981) also uses a two-period overlapping generations model to derive his irrelevance result. To derive the irrelevance result, we assume that agents are only taxed when they are old, as can be seen from budget constraints (1) and (2). In addition, the government imposes its gains or losses from the portfolio choice in period t on the households in period $t+1$ (see the formal assumption on the government behavior below). Because of these two assumptions, the agents who actively participate on the risky assets market are those agents who will be exposed to future taxation risk. With similar assumptions on fiscal variables, limited participation models and, in particular, OLG models reproduce the irrelevance result. Once we guarantee that in the baseline model asset purchases are irrelevant when expectations are rational, we focus on the effects of deviating from such expectations while, at the same time, keep the tractability of the OLG framework.

Expectations. We let households hold beliefs that may in principle differ from rational expectations. More precisely, consistently with the idea that households understand policy announcements but may be unable to solve for the equilibrium of the economy, we make the following assumptions. First, we assume that, when it comes to future *exogenous* variables, that is, $\{M_{t+1}\}$, $\{M^*_{t+1}\}$, and $\{r^x_{t+1}\}$, households form expectations using the true distribution of such variables. Second, letting Z_t denote the vector of *endogenous* variables $(q_t, p_t, i_t, T_t, p^*_t, i^*_t, T^*_t)$, we assume that at time $s < t$ households expect Z_t to be distributed according to some cdf $\widetilde{\Phi}_t$, which can potentially differ from the distribution Φ_t implied in equilibrium.

In general, we could let $\widetilde{\Phi}_t$ be any distribution. However, to preserve tractability, we also assume that households believe that each component of Z_t is a linear function of the exogenous shocks at time t, that is, under the distribution $\widetilde{\Phi}_t$ each component of Z_t satisfies

$$Z_{i,t} = \alpha_{i,t} + \beta^x_{i,t}\epsilon^x_t + \beta^h_{i,t}\epsilon^h_t + \beta^f_{i,t}\epsilon^f_t, \tag{3}$$

for some scalars $\alpha_{i,t}$, $\beta^x_{i,t}$, $\beta^h_{i,t}$, and $\beta^f_{i,t}$. Here, $\alpha_{i,t}$ represents the expected average of $Z_{i,t}$,

while $\beta_{i,t}^{j}$, $j = x, h, f$, capture the expected sensitivity of $Z_{i,t}$ to the aggregate shocks.

Importantly, while assumption (3) may seem restrictive, it will have a bite only with the temporary equilibrium of our economy (see the definition below). In all the other equilibrium concepts we will consider—i.e., rational expectations, level-k, and reflective— assumption (3) will be redundant. For example, in the rational expectations equilibrium benchmark, expectations take the form of (3) even if we do not impose any restrictions on $\widetilde{\Phi}_t$.

In the equilibria we consider below—i.e., rational expectations, level-k, and reflective—household expectations will satisfy three important properties. First, households do not revise their expectations of endogenous variables as time progresses. The reason is that our simple model does not feature any endogenous propagation mechanisms.[8] The only time-dependance that we consider is the persistence of balance sheet policies, therefore, once such policies are announced and agents have incorporated them into their beliefs, there will be no further reasons for belief revision. Second, households expect endogenous variables to depend only on contemporaneous shocks. Since both of these properties are satisfied in the equilibria we consider, we simplify notation and remove dependence on time and on non-contemporaneous shocks right from the start. Finally, although non explicit in the notation, in the equilibria we consider households will expect the law of one price to hold in any future period. Formally, $\widetilde{\Phi}_t$ is such that

$$p_t = p_t^* + e_t,$$

for all t.

Government. We explicitly specify the behavior of government in both countries. We use capital letters to denote government choices. The government of the home country controls real per capita taxes $\{T_{t+1}\}$, nominal money supply $\{M_{t+1}\}$, the real purchases of private risky assets $\{X_{t+1}\}$, the *real* amount of home currency nominal bonds $\{B_{t+1}^h\}$, and the *real* amount of foreign-currency purchases $\{B_{t+1}^f\}$. We let $\Pi_t = \{T_{t+1}, M_{t+1}, X_{t+1}, B_{t+1}^h, B_{t+1}^f\}$. At time 0 the government announces the process for money supply and balance-sheet policies $\{M_{t+1}, X_{t+1}, B_{t+1}^h, B_{t+1}^f\}$, which becomes common knowledge in both countries. The consolidated government budget constraint is

$$\begin{aligned} P_t q_t X_{t+1} + P_t B_{t+1}^h + E_t P_t^* B_{t+1}^f + M_t & \\ = P_t \left(r_t^x + q_t\right) X_t + e^{i_{t-1}} P_{t-1} B_t^h + E_t e^{i_{t-1}^*} P_{t-1}^* B_t^f + \omega P_t T_t + M_{t+1}. & \end{aligned} \tag{4}$$

[8] This property does not hold if agents can update their beliefs over time, for example, through a process of equilibrium unraveling. We investigate this extension in Section 5.

The left-hand side represents government's nominal outlays, consisting of purchases of home-country risky asserts $P_t q_t X_{t+1}$, purchases of home-country nominal bonds $P_t B^h_{t+1}$, purchases of foreign-country nominal bonds $E_t P^*_t B^f_{t+1}$, and repayment of money liabilities. The right-hand side is government income.

We assume that only the home-country government conducts balance-sheet policies. More specifically, for simplicity we assume that the foreign government sets money supply and taxes so as to keet a constant level of real bonds. Formally, the foreign government chooses $\Pi^*_t = \{T^*_{t+1}, M^*_{t+1}, B^*\}$, where B^* is the constant level of real foreign-country bonds, that satisfy the budget constraint

$$e^{i^*_{t-1}} P^*_{t-1} B^* + M^*_t = P^*_t B^* + (1-\omega) P^*_t T^*_t + M^*_{t+1}. \tag{5}$$

We call QE a policy of risky-asset purchases financed with the issuance of home-country bonds. Similarly, a policy of foreign-bond purchases financed with the issuance of home-country bonds will be referred to as FX intervention.

2.2 Equilibrium Concepts

Temporary Equilibrium. Our goal is to investigate the equilibrium implications of letting agents' expectations deviate from standard rational expectations. In rational expectations equilibrium (REE), expectations are required to be consistent with equilibrium objects. This requirement is restrictive for our purpose, therefore, we adopt a more general notion of equilibrium known as temporary equilibrium (TE). Intuitively, such a concept generalizes the standard REE insofar as it does not restrict beliefs about endogenous variables, which are free to deviate from equilibrium outcomes. More specifically, a TE takes as given household beliefs about future endogenous variables and requires only that *(i)* households optimize given these beliefs and that *(ii)* markets clear in every period.

Definition (Temporary Equilibrium)**.** Given beliefs $\{\widetilde{\Phi}_t\}$ that satisfy (3), a temporary equilibrium is a collection of household choices $\{c_t, x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}, s_{t+1}\}$ and $\{c^*_t, b^*_{H,t+1}, b^*_{F,t+1}, m^*_{t+1}, s^*_{t+1}\}$, government policies $\{\Pi_t, \Pi^*_t\}$, and prices $\{q_t, i_t, p_t, i^*_t, p^*_t, e_t\}$ such that

1. Given beliefs and prices, households make consumption, storage, and portfolio choices optimally;
2. Risky-asset, bonds, and money markets clear;
3. Government budget constraints (4) and (5) are satisfied for all t.

A TE induces a sequence of distributions $\{\Phi_t\}$ over $\{Z_t\}$. In general, this sequence can differ from the original sequence of household beliefs $\{\widetilde{\Phi}_t\}$, except in one special case.

Rational Expectations Equilibrium. A REE is a TE with the extra requirement that expectations must be consistent with equilibrium distributions.

Definition (Rational Expectations Equilibrium)**.** A REE is a TE which satisfies

$$\widetilde{\Phi}_t = \Phi_t, \text{ for all } t.$$

Level-k Equilibrium. We now consider an alternative process of belief formation. This process captures the idea that, when confronted with a new policy in a complex environment, households do not form expectations rationally, either because they find it hard to deduce the equilibrium consequences of the new policy or because they believe that other agents may hold non-rational expectations.

This alternative process of belief formation is known as level-k thinking, where k denotes the level of sophistication of an agent (which we define formally below). Intuitively, agents are assumed to have the correct model of the economy, but they hold incorrect beliefs about the sophistication of other agents. Suppose, for example, that a new policy is announced. Agents correctly understand the policy, but they fail to predict the behavior of the other agents. What is more, since endogenous variables result from the aggregation of individual actions, level-k agents incorrectly predict such variables.

Remember that a TE induces a mapping from sequences of beliefs and policies into equilibrium distributions, which we represent compactly as

$$\Phi_t = \Psi(\{\widetilde{\Phi}_{t+s}\}, \{X_{t+s+1}, B^h_{t+s+1}, B^f_{t+s+1}\}), \tag{6}$$

for all t. Moreover, by definition, in a REE beliefs and equilibrium distributions coincide, that is, $\{\Phi^{REE}_t\}$ is a fixed point of (6).

We start with $k = 1$, the lowest level of sophistication. We assume that level-1 agents believe that all other agents do not respond to policy announcements. Following a policy announcement, therefore, they will hold incorrect beliefs that asset prices and future taxes will coincide with those in the REE *before* government intervention. Formally, we assume households have beliefs $\{\widetilde{\Phi}^1_t\} = \{\Phi^{REE}_t\}$. The additional superscript denotes "level-1" beliefs. We define a level-1 equilibrium as a TE where agents' beliefs are given by $\{\widetilde{\Phi}^1_t\}$.

Starting from level-1 agents, we define level-k agents and level-k equilibria recursively. Let $\{\widetilde{\Phi}^k_t\}$ be the beliefs of a level-k agent, $k \geq 1$, and define a level-k equilibrium accordingly. From (6), we obtain the distribution of asset prices and taxes in a level-k equilib-

rium. We then assume that these distributions coincide with the beliefs of level-$k+1$ agents. The entire process entire process of belief formation with the following recursion:

$$\widetilde{\Phi}_t^{k+1} = \Psi(\{\widetilde{\Phi}_{t+s}^k\}, \{X_{t+s+1}, B_{t+s+1}^h, B_{t+s+1}^f\}), \tag{7}$$

for all k and t.

Definition (Level-k Equilibrium)**.** Given a REE sequence of distributions $\{\Phi_t^{REE}\}$ and sequences of government purchases $\{X_{t+1}, B_{t+1}^h, B_{t+1}^f\}$, a level-$k$ equilibrium is a TE where beliefs are obtained recursively from the mapping (7) with initial condition $\{\widetilde{\Phi}_t^1\} = \{\Phi_t^{REE}\}$.

Reflective Equilibrium. All the equilibrium concepts so far assumed that agents in the economy were homogeneous, in particular, they had the same beliefs. We now consider an economy populated by households who are heterogeneous in their beliefs. In doing so, we follow García-Schmidt and Woodford (2015). In particular, the population is split into different groups depending on their beliefs. Each group contains households with the same level of sophistication k and has mass given by exogenous probability density function $f(k)$. One advantage of this approach is that the economy is not indexed by a discrete level of sophistication of agents but rather by a continuous level of average sophistication of agents. This allows us to do comparative statics in continuous manner. We now formally introduce the equilibrium notion for this economy.

Definition (Reflective Equilibrium)**.** Given beliefs $\{\widetilde{\Phi}_t^k\}_{t,k}$, with $\{\widetilde{\Phi}_t^1\} = \{\Phi_t^{REE}\}$, a Reflective Equilibrium (RE) is a collection of household choices $\{c_t^k, x_{t+1}^k, b_{H,t+1}^k, b_{F,t+1}^k, m_{t+1}^k, s_{t+1}^k\}$ and $\{c_t^{*,k}, b_{H,t+1}^{*,k}, b_{F,t+1}^{*,k}, m_{t+1}^{*,k}, s_{t+1}^{*,k}\}$, government policies $\{\Pi_t, \Pi_t^*\}$, and prices $\{q_t, i_t, p_t, i_t^*, p_t^*, e_t\}$ such that

1. Given beliefs and prices, households make consumption, storage, and portfolio choices optimally;
2. Risky-asset, bonds, and money markets clear;
3. Government budget constraints (4) and (5) are satisfied for all t;
4. Beliefs are generated recursively with the mapping (7).

Cashless economy. To streamline our analysis, we assume that that the demand for and supply of money is negligibly small after we solve for equilibrium outcomes. This cashless limit is a standard assumption employed in, for example, New Keynesian Open

Economy literature to eliminate the real effects of nominal money supply above and beyond its effects on inflation and nominal interest rate. We can thus abstract from money holdings when computing equilibria.[9]

Definition (Cashless limit)**.** A cashless limit of the economy described above is a limit of TE such that home-country parameters $\overline{m} \to 0$, $\overline{M} \to 0$, and $\omega\overline{m}/\overline{M} \to 1$, and foreign-country parameters $\overline{m} \to 0$, $\overline{M}^* \to 0$, and $(1-\omega)\,\overline{m}/\overline{M}^* \to 1$.[10]

3 Equilibrium Effects of Balance Sheet Policies

Household behavior. We begin with the household problem and then derive the TE for general sequences of balance sheet policies.

Let $\mathcal{R}_{t+1} = \left(r^x_{t+1} + q_{t+1} - Rq_t, i_t - \pi_{t+1} - r, i^*_t - \pi^*_{t+1} - r, -r - \pi_{t+1}\right)'$ be the vector of realized real excess returns to the risky asset and the two bonds. Let $\widetilde{\Sigma}_t$ denote the variance-covariance matrix of $\mathcal{R}_{t+1}$ under the distribution $\widetilde{\Phi}_{t+1}$ and *conditional* on all the variables realized at time t. The other moments are defined analogously.

To derive closed-form solutions, after combining the budget constraints (1) and (2) into a single intertemporal budget constraint, we take a first-order approximation, and treat the resulting budget constraint as exact:[11]

$$c_{t+1} = Rw + \left(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}\right) \cdot \mathcal{R}_{t+1} - \left(r + \pi_{t+1}\right) m_{t+1} - T_{t+1},$$

where the prime next denotes the transpose.

When preferences are exponential and shocks are normally distributed, the household problem becomes a standard mean-variance portfolio optimization problem, which has a simple closed-form solution.

Lemma 1. *Given beliefs expressed in equation* (3)*, household asset demand satisfy*

$$\widetilde{\Sigma}_t \begin{pmatrix} x_{t+1} \\ b_{H,t+1} \\ b_{F,t+1} \\ m_{t+1} \end{pmatrix} = \frac{1}{\gamma}\widetilde{\mathbb{E}}_t(\mathcal{R}_{t+1}) + \frac{1}{\gamma}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{v}\log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + \widetilde{Cov}_t\left(\mathcal{R}_{t+1}, T_{t+1}\right), \tag{8}$$

[9] Alternatively, we could introduce money-in-production-function, similar to the assumption in Bacchetta and Van Wincoop (2006), which does not lead to distortion of the Euler equations.

[10] The fact that $\omega\overline{m}/\overline{M}$ and $(1-\omega)\,\overline{m}/\overline{M}^*$ approach one is not consequential for any of the results. It avoids writing unimportant constants.

[11] Jeanne and Rose (2002) and Bacchetta and Van Wincoop (2006) make the same simplifying assumption. Alternatively, we could assume that households hold the mean-variance preferences instead of exponential. This would allow us to avoid making linearization because we could express mean and variance of log-normally distributed random variables in closed forms.

with analogous equations for $(b^*_{H,t+1}, b^*_{F,t+1}, m^*_{t+1})$.

The first term in equation (8) captures the excess return of each asset divided by the coefficient of absolute risk aversion γ. The second term captures a hedging motive coming from the fact that, when the government conducts balance sheet policies, households expect future taxes to correlate with asset returns. For example, suppose agents expect future taxes to be negatively related to the return on the risky asset, that is, $\widetilde{Cov}_t(r^x_{t+1} + q_{t+1} - (1+r)\, q_t, T_{t+1}) < 0$. The risky asset is then a bad hedge against future tax risk, thus, the household scales down her demand for this asset. The same intuition applies to the other assets. The final term represents non-pecuniary returns from holding assets. Specifically, money is the only asset in the economy that yields non-zero pecuniary return.

A convenient property of the demand function (8) is that we can impose market-clearing and, thus, solve for endogenous prices, without solving for the optimal choice of consumption and storage. Moreover, asset demands (8) suggest that, by affecting beliefs of future taxes, balance sheet policies can potentially influence investor demands and, by market clearing, equilibrium asset prices. Equilibrium asset prices in turn feed back into asset demands. The latter intuition is, however, incorrect and balance sheet policies are completely irrelevant in an important benchmark.

3.1 Neutrality under Rational Expectations

In this section, we study the response of the economy to balance-sheet policies in rational expectations equilibrium. Using the demand for the assets in (8) and the consistency of beliefs and outcomes under rational expectations, we derive the irrelevance result and summarize it the following proposition.

Proposition 1. *For any sequences of balance sheet policies* $\{X_{t+1}, B^h_{t+1}, B^f_{t+1}\}$*, there is a unique no-bubble REE. In the cashless limit of this equilibrium, the risky asset price is constant:*

$$q^{REE}_t = q^{REE} \equiv \frac{1}{R-1}\left(r^x - \frac{1}{\omega}\gamma\sigma^2_x\overline{X}\right);$$

the home-country nominal interest rate and price level are:

$$i^{REE}_t = r - \frac{1}{1+v}\epsilon^h_t,$$
$$p^{REE}_t = vr + \frac{1}{1+v}\epsilon^h_t;$$

with analogous expressions for the foreign country. Finally, the exchange rate satisfies

$$e_t^{REE} = \frac{1}{1+v}\left(\epsilon_t^h - \epsilon_t^f\right).$$

In particular, balance-sheet policies are irrelevant.

Proposition 1 states that, when agents anticipate future taxes rationally, government intervention is irrelevant. This is the celebrated result that in an economy where the Ricardian equivalence holds asset purchases by the government—or, equivalently, by the central bank—are irrelevant. The reason is that, while assets are removed from households' budgets, they are still present indirectly through future taxes. In a REE, households correctly anticipate that future taxes will depend on government purchases and react by adjusting their demand for risky assets. In the end, equilibrium allocations and prices are unaffected. Crucially, this reasoning requires households to hold rational expectations about future endogenous variables such as taxes.

Interestingly, all the variables in Proposition 1 take a particularly simple form. First, the risky-asset price equals the average return minus a term capturing the risk premium demanded by the risk-averse investors. Secondly, interest rates are given by the risk-free real interest rate minus a shock to money supply. Intuitively, to stimulate agents to hold more money, the opportunity cost of holding money, represented by the nominal interest rate i_t, must go down. Third, the nominal interest rate i_t equals the constant real interest rate r plus expected inflation $\mathbb{E}_t p_{t+1} - p_t$ in this economy.[12] As a result, the expected inflation must decline with a shock to money supply. Because the economy is stationary, and future expected price is constant, hence, the drop in expected inflation is achieved through an increase in the current price level p_t. Finally, the law of one price requires that nominal exchange rate of home currency depreciates, i.e., e_t goes up, after a positive shock to home money supply.

3.2 Non-neutrality under Level-k Thinking

We now depart from rational expectations and assume that, following an announcement of intervention, households in both countries form expectations following the level-k process described in Section 2.2. As a starting point, we further assume that, before the announcement, the world economy is in its REE, thus, households in both countries correctly forecast the behavior of future taxes in each country. In particular, households correctly understand that the home-country government has zero holdings of government

[12] Inflation risk reduces the demand for nominal government bonds, but, at the same time, this risk makes future real taxes positively correlated with real bond returns. The last effect increases the demand for nominal bonds. In REE, the two effects cancel each other out.

bonds, thus, future taxes are expected to be zero. In addition, households understand that the foreign-country government has a real value B^* of outstanding bonds, thus, future taxes in the foreign country are expected to be a function of the future foreign money supply. One can justify such an assumption, for example, through learning: If governments kept their outstanding supply of debt at constant (real) levels for a long enough time, households would have time to learn the stochastic process of taxes. We refer to the REE *before* the foreign government intervention as the "status quo" and we use the superscript "REE" to denote status-quo variables. By assumption, the status quo is the REE in the absence of asset purchases, that is, status-quo beliefs $\widetilde{\Phi}_t^{REE}$ are a fixed point of (6) with $X_{t+1} = B_{t+1}^h = B_{t+1}^f = 0$, for all t.

We begin with the TE under the assumption that households are level-1, that is, we assume that households do not change their status-quo beliefs after new balance-sheet policies are announced. From (6), we then obtain the equilibrium variables Z_t that are compatible with level-1 beliefs and the new policy. In turn, these values coincide with the beliefs of level-2 agents. In general, using (7), we can obtain the beliefs and the equilibrium outcomes in any level-k equilibrium recursively.

As an example, it is instructive to consider the recursion for the risky-asset price in a level-k equilibrium:

$$q_t^k = \frac{R-1}{R} q^{REE} + \frac{1}{R} q_{t+1}^{k-1}. \tag{9}$$

Equation (9) can be iterated forward until we reach level-1 equilibrium:

$$q_t^1 = q^{REE} + \frac{1}{R} \frac{\gamma \sigma_x^2 X_{t+1}}{\omega};$$

The iterative procedure implied by (9) is depicted in Figure 1, where the horizontal axis represents time and the vertical axis plots the level k. Every bold dot is the equilibrium price q_t^k of a particular level-k equilibrium. This figure visually shows that if one wants to compute, for example, the asset price at time 0 of a level-5 equilibrium by iterating equation (9) forward, one has to move diagonally and compute the asset price at time-1 in a level-4 equilibrium, q_1^4, the asset price at time-2 in a level-3 equilibrium, q_2^3, and so on.

Importantly, these iterations always stop when level-1 equilibrium is reached because level-1 agents do not change their beliefs when the new policy is announced. The reason is that, households in a level-1 equilibrium do not understand the connection between future taxes and government purchases, thus, they do not react by varying their demand for the risky asset, as it was the case in the REE. In turn, since level-1 households do not change their beliefs, government purchases of risky assets affect the asset price in a level-1 equilibrium and, through (9), in any k-level equilibrium with $k > 1$. What is more, if

we start from a higher k, equation (9) shows that the weight on q^{REE} increases, thus, asset purchases become less effective when sophistication increases. In the example of a level-5 equilibrium, it is as if agents were fully discounting any changes happening after period 4.

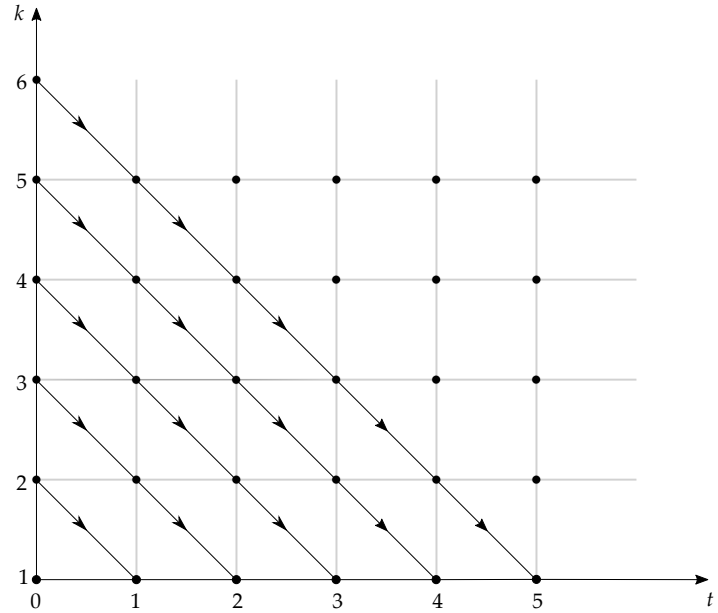


Figure 1: Level-k price equilibrium solution.

Having characterized beliefs for any level of sophistication, we turn to our last equilibrium concept, the RE, which allows heterogeneous agents to coexist in the economy.

The following proposition contains the main result of this section.

Proposition 2. *For sequences of balance sheet policies* $\{X_{t+1}, B^h_{t+1}, B^f_{t+1}\}$*, in the cashless limit of the RE the asset price is*

$$q_t = q^{REE} + \frac{\gamma\sigma_x^2}{\omega}\sum_{k=1}^{\infty} f(k)\frac{X_{t+k}}{R^k};$$

the home-country nominal interest rate and price level are

$$i_t = i_t^{REE} + \gamma\sigma_h^2\frac{1}{v}\left(\frac{1}{1+v}\right)^2\sum_{k=1}^{\infty} f(k)\left(\frac{v}{1+v}\right)^k\left(-B^h_{t+k}\right),$$

$$p_t = p_t^{REE} + \gamma\sigma_h^2\left(\frac{1}{1+v}\right)^2\sum_{k=1}^{\infty} f(k)\left(\frac{v}{1+v}\right)^k\left(-B^h_{t+k}\right);$$

with analogous expressions for the foreign country. Finally, the exchange rate satisfies

$$e_t = e_t^{REE} + \gamma\left(\frac{1}{1+v}\right)^2\sum_{k=1}^{\infty} f(k)\left(\frac{v}{1+v}\right)^k\left(\sigma_f^2 B^f_{t+k} - \sigma_h^2 B^h_{t+k}\right).$$

In particular, equilibrium variables depend on balance-sheet policies.

Proposition 2 shows that, in a RE, balance sheet-policies are effective tools in controlling asset prices. In particular, by purchasing risky assets, the government–or, equivalently, the central bank–can increase the price of this type of asset. Moreover, since the risk-free real return is fixed by the storage technology, we can relate the inverse of the risk-asset price to the risk-premium required by investors. Asset purchases, therefore, affect asset prices by lowering the equilibrium risk-premium. Similarly, Proposition 2 states that the nominal interest rate and the price level–and, therefore, the exchange rate–are now functions of the entire path of nominal-bond purchases or issuances (remember that an issuance is a negative value for B^h_{t+1}). Take for example, the nominal bond in the home country. Suppose, for simplicity, that the home-country government issues some of these bonds at time t, i.e., $B^h_{t+1} < 0$. In a RE, households fail to anticipate that, since bonds promise a risk-free *nominal* payment, future taxes will now depend on the fluctuations of the future price level. They will thus fail to hedge the tax risk by increasing their demand for nominal bonds. To induce households to hold the additional bonds and clear the market, therefore, the interest rate and the current price level have to increase.

We now consider several special cases to highlight important properties of government intervention effects summarized in Proposition 2.

Absence of level-1 households. First, we assume that there are no level-1 agents in the economy. Formally, this assumption corresponds to $f(1) = 0$. In this case, the infinite sums in Proposition 2 start from $k = 2$, implying that the current (i.e., period t) government purchases do not affect the equilibrium prices. The future purchases have the same effect as before. Note that it is nevertheless crucial that level-k households, with $k \geq 2$, form their beliefs starting from level-1 agents beliefs, despite the fact that there are no level-1 households in the economy.

Opposing effects of level-k thinking on effects of future interventions. There are two effects of level-k thinking on the strength of, for example, quantitative easing. First, in the current model, the further away the agents are from rational expectations (i.e., the lower is the average level-k in the economy), the stronger is the direct effect of QE. Second, the further away the agents are from rational expectations, the stronger they discount the effects of future QE. Recall that level-k households "discount" the future completely after k periods. They thus do not take into account any part of policy occurring more than k periods ahead.

To illustrate this point formally, we compute the risky-asset price after the announcement by the government to make a one-time purchase of the risky assets in the following

period. For concreteness assume that $f(k)$ is represented by exponential distribution, i.e., $f(k) = (1-\lambda)\lambda^{k-1}$, $\lambda \in [0,1)$. With the exponential distribution, the average level of sophistication in the economy $\bar{k} \equiv 1/(1-\lambda)$. The risky asset price in the period of policy announcement equals

$$q_t - q^{REE} = \frac{\gamma\sigma_x^2}{\omega R^2}\left(\frac{1}{\bar{k}} - \frac{1}{\bar{k}^2}\right) X_{t+2}.$$

The nonlinear effect of $\bar{k}$ on the price is clear from this formula. If $\bar{k} < 2$, then a small increase in the average level of sophistication $\bar{k}$ increases the effect of the policy. The policy strength peaks at $\bar{k} = 2$, and then it declines to zero as $\bar{k}$ approaches infinity, as depicted in Figure 2.

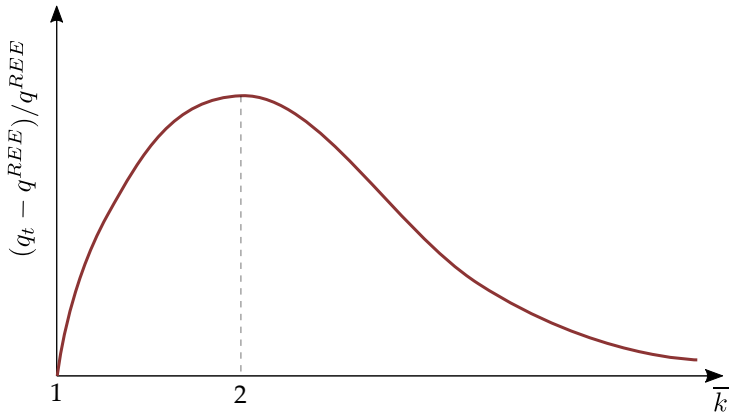


Figure 2: Price effect of the risky asset purchases as a function of the average sophistication in the economy.

Persistence. Suppose that the path of government asset purchases follows an exponentially decaying process $X_{t+1+s} = X_{t+1}\mu^s$, for $s \geq 0$, and $\mu \in [0,1]$. A one-time purchase in the current period is a special case of this process with $\mu = 0$. A permanent increase in asset purchases corresponds to $\mu = 1$. When the distribution of households $f(k)$ is exponential, we have

$$q_t - q^{REE} = \frac{\gamma\sigma_x^2}{\omega} \cdot \frac{1}{\bar{k}(R-\mu)+\mu} X_{t+1}.$$

There are two important observations. First, the higher the persistence of QE (higher μ), the higher the effect on the price. Second, in line with our discussion about sophistication and discounting of the future, there are still two effects of level-k thinking on the effectiveness of QE. However, in this example, the discounting effect is weaker and, as a result, a higher level of sophistication leads to lower prices, i.e., $dq_t/d\bar{k} < 0$.

3.3 Quantitative Illustration

In this section, we present the results of several numerical exercises. Our aim is to numerically investigate the sensitivity of results to changes in various parameters and to illustrate the possible magnitudes of the effects of the level-k thinking process of expectations formation. We stress that our numerical exercises are not a substitute for a serious calibration of a more realistic model.

Quantitative easing. We consider a quantitative easing policy in home country. We assume that the government purchases of risky assets and issuance of riskless nominal bonds start from some initial values and decay exponentially. Specifically, $\{X_{t+1}, B^h_{t+1}\}$ follow $X_{t+1} = X_1\mu^t$ and $B^h_{t+1} = B^h_1\mu^t$ for all $t \geq 0$, with positive initial values of X_1 and B^h_1, and $\mu \in [0,1]$. Note that it is not necessary that B^h_1 and X_1, and their subsequent values, are related to each other because the government can close any gap in financing through taxes. For example, if in period 0 the amount of issued government bonds is not enough to cover the purchases of private risky assets, the government taxes young households and closes the fiscal gap.

Using the risky asset price derived in Proposition 2, we can express the deviation of the price from its REE value as

$$\frac{q_t - q^{REE}}{q^{REE}} = \mathbb{E}_t\left(\mathcal{R}_{1,t+1}\right) \cdot \frac{X^G_t}{\overline{X}} \cdot \frac{1}{\bar{k}\left(R-\mu\right)+\mu}, \tag{10}$$

where $\mathcal{R}_{1,t+1}$ is the first component of $\mathcal{R}_{t+1}$, that is, the expected excess return on the risky asset. In the REE, we have

$$\mathbb{E}_t\left(\mathcal{R}_{1,t+1}\right) = \frac{r^x + q^{REE}}{q^{REE}} - R = \frac{\gamma\sigma_x^2\overline{X}}{\omega q^{REE}}.$$

We can quantify equation (10) as follows. We set the excess return on risky asset to 2%, which roughly corresponds to excess return on the Bloomberg Barclays U.S. MBS Index in the period of 2005-15. The government initial purchases of risky assets are 10% of the overall supply of the risky assets, i.e., $X_0/\overline{X} = 0.1$. This amount roughly corresponds to $1 trillion of mortgage-backed securities purchased by the Fed from January 2009 and June 2010 relative to overall value of the MBS market of about $10 trillion in that period. Finally, we set the safe rate of return is $R = 1.01$.

The risky assets price in equation (10) is positively related to the persistence of the quantitative easing policy. We now present two polar cases—permanent and one-period long intervention—to assess the magnitude of the intervention. If the government pur-

chases are permanent, i.e., $\mu = 1$, the risky asset price is

$$\frac{q_t - q^{REE}}{q^{REE}} = \frac{0.2\%}{0.01 \cdot \bar{k} + 1}.$$

Note that the magnitude is not sensitive to changes in the distribution of sophistication in the economy, governed by parameter $\bar{k}$, when $\bar{k}$ is not too large. As a result, the magnitude of the effect close to 0.2% obtains under the wide range of the average levels of sophistication of agents in the economy.

If the government intervention continues for only one period, i.e., $\mu = 0$, then, according to price equation (10), the price effect is inversely proportional to the average level of households sophistication $\bar{k}$. If, for example, $\bar{k} = 2$, i.e., a typical household thinks that other households do not change their expectations after the policy intervention, the increase in the risky asset price is 0.1%. This result suggests that increase the duration of quantitative easing from one period to very long horizon increases the effect of the policy on current price by only a factor two. This is a small effect relative to, for example, an infinite sum of 0.1% increase in every period discounted by R that would yield a price increase of 10%. The reason why the strength of QE does not explode with the persistence of the policy has to do with endogenous discounting under level-k thinking that we discussed in Section 3.2.

Decline in the supply of risky assets. The overall effect of quantitative easing on the price of risky assets consists of two main forces: (i) limited understanding of the effect on future taxes, and (ii) limited understanding of the effect on future prices. The first effect makes the police relevant. The second effect limits the strength of the policy. In the analysis so far, both of the effects were present together. Next, we separate the tax and price effects. To do this, we consider a change in the supply of risky assets similar to that under QE, i.e., $X_{t+1} = X_1 \mu^t$, however, without any consequences for current and future taxes. One interpretation of this experiment is the purchase of risky assets by the government of foreign country.[13]

It is straightforward to show that the change in the price of risky assets must equal

$$\frac{q_t - q^{REE}}{q^{REE}} = \frac{\gamma \sigma_x^2 \overline{X}}{\omega q^{REE}} \cdot \frac{\bar{k}}{\bar{k}(R - \mu) + \mu} \cdot \frac{X_{t+1}}{\overline{X}}. \tag{11}$$

The comparison of (10) and (11) reveals that the two formulas only differ by the term $1/\bar{k}$.

[13]Note that in the model we assumed that households abroad do not have access to home risky assets market. In this interpretation of the experiment, one can assume that the government of foreign country does not face the same information and trading costs as foreign households.

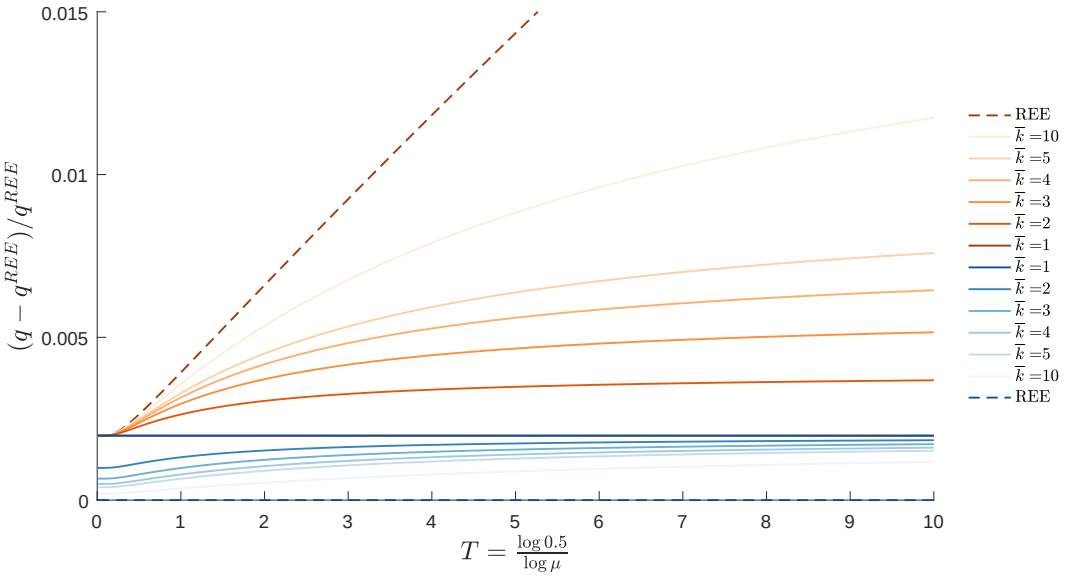


Figure 3: Price effects of risky assets purchases. Percentage change in the price of risky assets is on the vertical axis, the half life of the persistence of the policy is on horizontal axis. The orange lines represent the effects of an increase in foreign demand, while the blue lines show the effect of central bank purchases of risky assets. Various lines correspond to different average level of sophistication of agents in the economy $\overline{k}$.

As a result, the ratio of the price effect of quantitative easing over the price effect of a change in the supply of risky assets is constant.

We represent the price effects of the two policies in Figure 3. This figure plots the change in price of risky asset after a decline in net supply of risky assets and central bank assets purchases of same magnitude and duration as a function of the persistence of this purchase (measured in terms of half life). There are several things to note in this figure. First, when the average level of sophistication of agents in the economy equals 1 (all agents are level-1 thinkers), the effects of two policies coincide. This is represented by the horizontal line on the figure. Intuitively, when agents are level-1 thinkers they keep their expectations of future prices and taxes fixed after the changes in the economy. As a result, it does not matter if in reality the change in the net supply of risky assets is accompanied by future tax changes or not.

Second, as the level of sophistication of the agents in the economy increases, quantitative easing becomes less powerful: the set of blue curves move down from dark blue lines to light blue lines and, eventually, to dark blue dashed line, which represents the absence of quantitative easing effect on the price in rational expectations equilibrium. The more persistent the policy is the larger is the effect of QE policy.

Third, with more sophisticated agents, the effects of a decline in net supply of risky assets (e.g., an increase in foreign demand for these assets) becomes stronger because

agents realize the future equilibrium effects on prices more. This is represented by a set of orange curves in the plot that move up from horizontal line to lighter orange lines, and, eventually to dashed orange line. The difference between the orange and blue lines corresponding to the same level of average sophistication of households reflects the negative tax effect on the strength of QE. Interestingly, the ratio of the QE policy price effect over foreign purchases price effect does not depend on the persistence of the two policies and equals the inverse of the average level of sophistication in the economy. For example, in the case of $\bar{k} = 2$, the quantitative easing price effect is a half of the effect of foreign purchases.

FX intervention. We now turn to foreign exchange interventions by the home-country central bank. In period 0, The bank announces a path of real holdings of foreign country nominal bonds that follow $B^f_{t+1} = B_1 \mu^t$ for $t \geq 0$. At the same time the central bank issues the same real amount of home country nominal bonds, i.e., $-B^h_{t+1} = B^f_{t+1}$ for $t \geq 0$. Note that $-B^h_{t+1}$ denotes the issuance of home country nominal bonds expressed in real terms. The nominal exchange rate can be computed using the expression in Proposition 2. Specifically, we obtain

$$e_t - e_t^{REE} = \gamma \mathbb{V}_t \left(e_{t+1}^{REE} \right) \frac{1}{\bar{k} \left(1 - \mu + \frac{1}{v} \right) + \mu} B^f_{t+1},$$

where we used $\mathbb{V}_t e_{t+1}^{REE} = (\sigma_f^2 + \sigma_h^2) / (1+v)^2$ is the volatility of the nominal exchange rate in rational expectations equilibrium. To illustrate this relation numerically, we set $\nu = 0.3$, $\mathbb{V}_t \left(e_{t+1}^{REE} \right) = (0.03)^2$, which roughly correspond to the elasticity of money demand in the US and the volatility of dollar-yen exchange rate. We set the productγB_1^f to 100. Figure 4 illustrates comparative statics of the exchange rate response to FX intervention with respect to average level of thinking $\bar{k}$ and the persistence of the FX intervention, expressed in terms of half life. One can see that the response of the exchange rate is not very sensitive to the persistence of the policy. As in the case of quantitative easing, this is because level-k thinkers endogenously discount future effect of the intervention. In the extreme case when the economy is populated by only level-1 thinkers, the exchange rate effect is independent of the persistence of the policy. As the average level of sophistication in the economy increases, the effect of the policy becomes smaller completely disappearing the rational expectations equilibrium.

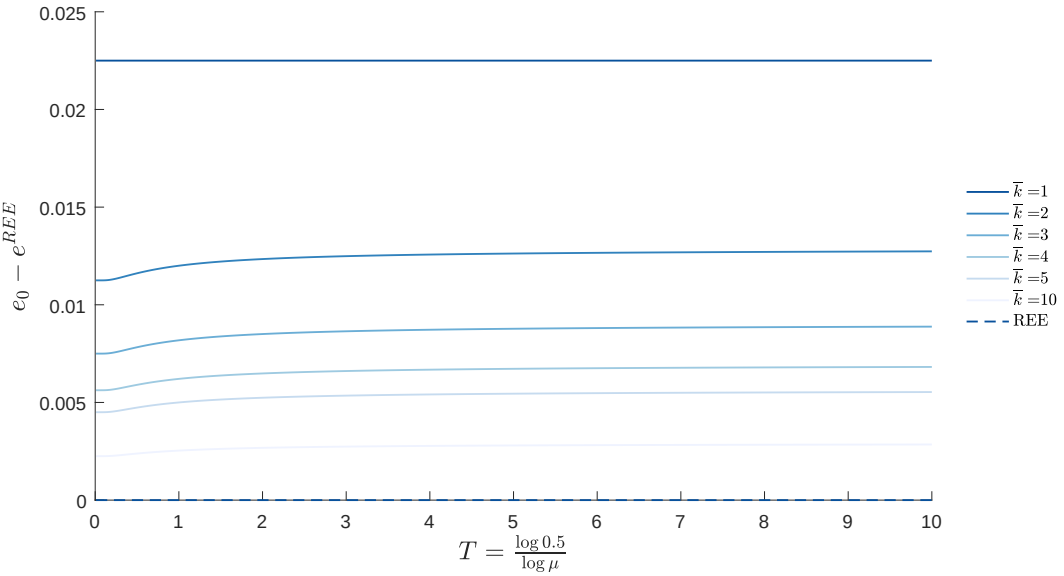


Figure 4: Exchange rate reaction to FX intervention as function of the persistence of intervention.

4 Forecast Errors and Balance Sheet Policies

A central element of the model with level-k thinking is its implications for endogenous variables forecasts. Specifically, because agents do not form expectations rationally, they make systematic mistakes. We can use the model to compute the errors that agents make following central bank balance sheet interventions. These predictions can help differentiate the mechanism in this paper from the mechanisms relying on *rational expectations* together with either market imperfections in the form of markets segmentation (i.e., the portfolio balance channel) or asymmetric information between the government and private agents (i.e., the signaling channel).

We next derive the predictions about the forecast errors that agents make in reflective equilibrium. We focus on the endogenous price of risky assets. The implications for the other prices can be obtained analogously. A level-k thinker who forms expectations about q_t by computing price q_t^{k-1}, which is obtained under the assumption that the economy is populated by level-$(k-1)$ thinkers only. We denote forecast error as $u_t^k \equiv q_t - q_t^{k-1}$. Using the expression for price q_t from Proposition 2 and the price q_t^{k-1} obtained from recursive equation (9), by iterating it $k-1$ times forward, we express the average forecast error as

$$\overline{u}_t = \sum_{k=1}^{\infty} f(k) u_t^k = \frac{\gamma \sigma_x^2}{\omega} \left(\sum_{k=1}^{\infty} f(k) \frac{X_{t+k}}{R^k} - \sum_{k=2}^{\infty} f(k) \frac{X_{t+k-1}}{R^{k-1}} \right) = \frac{\gamma \sigma_x^2}{\omega} \cdot \frac{1}{\overline{k}} \cdot \frac{1}{\overline{k}(R-\mu)+\mu} X_{t+1},$$

where the second equality presents the result of the special case when the types of think-

ing distributed according to exponential distribution across the households and government asset purchases decay exponentially at rate μ. The last expression underscores that the average forecast error of individual agents are related to the size of government intervention. Specifically, level-k thinkers necessarily make forecast errors that are positively related to the size of QE: agents always underestimate the power of the government intervention. This prediction of the model contrasts it from the predictions of the models with rational expectations and symmetric information across agents. In these models, agents do not make systematic forecast errors.

Heterogeneous information models. One can object, however, that a class of models in which agents form expectations rationally but perhaps posses heterogeneous information can generate non-neutrality of balance sheet policies and predictability of forecast errors (both average across agents and individual). For example, if some agents do not have accurate information about government intervention due to, for example, noisy information (Lucas, 1972; Woodford, 2001; Sims, 2003; Maćkowiak and Wiederholt, 2009; Angeletos and La'O, 2010; Angeletos and Lian, 2016), sticky information (Mankiw and Reis, 2002; Reis, 2006a,b), or rational inattention (Sims, 2003), they will make predictable forecast errors from a view point of an econometrician who is perfectly aware of the policy implementation.

A possible way to differentiate the predictions of the model with level-k thinking from the predictions of the models with heterogeneous information is to use survey data on joint behavior of forecasts of future endogenous variables and beliefs about government interventions. Specifically, individual forecast errors should not be related to policy interventions after controlling for the discrepancy between actual policy intervention and individual beliefs about the policy intervention. For example, in the sticky information model, in which some agents are completely unaware of policy interventions, the agents make forecasts errors because they are not aware of policy intervention at all. Hence, after controlling for the discrepancy in beliefs about policy and actual policy, forecast errors should not be predictable in models with sticky information. At the same time, level-k thinking belief formation process implies that agents make predictable forecast errors even if they perfectly know about policy intervention.

5 Equilibrium Unraveling and Long-Run Neutrality

Laboratory and field experiments provide some evidence that people "learn" to play Nash equilibrium after several repetition of one-shot games (Nagel, 1995). In the previous two sections, we assumed that in every period households initiate their iterations

from the status quo that corresponds to rational expectations equilibrium without policy actions. As a result, if not for policy dynamics, the model produces stationary outcomes. The reflective equilibrium does not converge to rational expectations equilibrium over time.

In this section, we modify our baseline setup and allow for dynamic equilibrium unraveling. To do this, we assume that the level of sophistication of agents changes over time. We introduce this assumption in the simplest possible way to highlight a number of qualitative results. Specifically, we assume that current level-k thinker becomes level-$(k+h)$ in the subsequent period, where h is a constant non-negative integer number. One interpretation of this assumption is as follows. In the first moment after a change in the policy an agent can compute k deductive iterations to form the expectations about all future endogenous variables. In every subsequent period, the agent uses already computed forecasts and performs h deductive iterations more. For example, if an agent is level-1 in period 0, when the policy is started, she turns level-3 thinker in period 1 if $h = 2$, and level-5 thinker in period 2.

The distribution of levels of sophistication of agents changes over time according to

$$f_t(k) = \begin{cases} f(k-ht), & k \geq 1+ht, \\ 0, & k < 1+ht, \end{cases} \tag{12}$$

where $f(k)$ is the distribution of levels of thinking in the moment of policy announcement.

Permanent quantitative easing. We can compute the price effect of QE by using the results in Proposition 2 evaluated for the distribution (12). Specifically, for a permanent QE of size X, we obtain

$$q_t = q^{REE} + \frac{\gamma\sigma_x^2}{\omega} \cdot \frac{X}{\bar{k}(R-1)+1} R^{-ht}. \tag{13}$$

Equation (13) shows that, over time, the price approaches q^{REE} at rate $1/R^h$, thus, h determines the speed of convergence. The key implication of equation (13) is that a central bank cannot stimulate the economy forever by keeping the size of its balance sheet at a constant, however high, level.

To counteract the dampening forces coming from equilibrium unraveling and to sustain low risk premium, it is crucial that asset purchases increase over time. Formally, we can show that if government asset purchases can increase at exponential rate, then the central bank needs to increase them at rate $\mu = R^{h/(1+h)} > 1$ to keep the price q_t elevated at a constant level.

New policy rounds. Another implication of equilibrium unraveling is the prediction of the price effects of new rounds of balance sheet policies. Specifically, if knowledge accumulated while observing the first occurrence of a policy is used to predict the reaction of endogenous variables during the subsequent rounds of this policy, then later occurrences of the policy will be less and less effective. For example, this logic predicts that, after controlling for the size of QE round, the first round of quantitative easing policy by the Federal Reserve implemented in 2009 should have higher asset price effect than the second round implemented in 2010 .

6 Conclusion

We showed that the central bank's balance sheet policies become effective in a model where agents form expectations according to level-k thinking process even if these policies are neutral in rational expectations equilibrium. These policies become neutral in the long run if agents "learn to play" equilibrium strategies over time. We derive testable implications of this channel of balance sheet policies. In ongoing work, we test these predictions empirically.

References

AMADOR, M., BIANCHI, J., BOCOLA, L. and PERRI, F. (2017). *Exchange rate policies at the zero lower bound*. Tech. rep., National Bureau of Economic Research.

ANGELETOS, G.-M. and LA'O, J. (2010). Noisy business cycles. *NBER Macroeconomics Annual*, **24** (1), 319–378.

— and LIAN, C. (2016). *Forward Guidance without Common Knowledge*. Tech. rep., National Bureau of Economic Research.

BACCHETTA, P. and VAN WINCOOP, E. (2006). Can information heterogeneity explain the exchange rate determination puzzle? *The American Economic Review*, **96** (3), 552–576.

BACKUS, D. K. and KEHOE, P. J. (1989). On the denomination of government debt: a critique of the portfolio balance approach. *Journal of Monetary Economics*, **23** (3), 359–376.

BARRO, R. J. (1974). Are government bonds net wealth? *Journal of political economy*, **82** (6), 1095–1117.

BERNANKE, B. S. (2012). "monetary policy since the onset of the crisis" speech at the federal reserve bank of kansas city economic symposium. *Jackson Hole, Wyoming*.

BHATTACHARYA, U. and WELLER, P. (1997). The advantage to hiding one's hand: Speculation and central bank intervention in the foreign exchange market. *Journal of Monetary Economics*, **39** (2), 251–277.

BHATTARAI, S., EGGERTSSON, G. B. and GAFAROV, B. (2015). *Time consistency and the duration of government debt: A signalling theory of quantitative easing*. Tech. rep., National Bureau of Economic Research.

BLANCHARD, O., DE CARVALHO FILHO, I. and ADLER, G. (2014). Can sterilized foreign exchange intervention stem exchange rate pressures from the global financial cycle? *Unpublished manuscript*.

BORDALO, P., GENNAIOLI, N. and SHLEIFER, A. (2016). *Diagnostic Expectations and Credit Cycles*. Tech. rep., National Bureau of Economic Research.

BOSCH-DOMENECH, A., MONTALVO, J. G., NAGEL, R. and SATORRA, A. (2002). One, two,(three), infinity,...: Newspaper and lab beauty-contest experiments. *The American Economic Review*, **92** (5), 1687–1701.

CAMERER, C. F., HO, T.-H. and CHONG, J.-K. (2004). A cognitive hierarchy model of games. *The Quarterly Journal of Economics*, **119** (3), 861–898.

CAVALLINO, P. (2017). Capital flows and foreign exchange intervention. *working paper*.

CHAKRABORTY, I., GOLDSTEIN, I. and MACKINLAY, A. (2016). Monetary stimulus and bank lending.

CHEN, H., CÚRDIA, V. and FERRERO, A. (2012). The macroeconomic effects of large-scale asset purchase programmes. *The economic journal*, **122** (564).

CRAWFORD, V. P., COSTA-GOMES, M. A. and IRIBERRI, N. (2013). Structural models of nonequilibrium strategic thinking: Theory, evidence, and applications. *Journal of Economic Literature*, **51** (1), 5–62.

CURDIA, V. and WOODFORD, M. (2011). The central-bank balance sheet as an instrument of monetarypolicy. *Journal of Monetary Economics*, **58** (1), 54–79.

DE LONG, J. B., SHLEIFER, A., SUMMERS, L. H. and WALDMANN, R. J. (1990). Noise trader risk in financial markets. *Journal of political Economy*, **98** (4), 703–738.

DI MAGGIO, M., KERMANI, A. and PALMER, C. (2016). Unconventional monetary policy and the allocation of credit. *Columbia Business School Research Paper*, (16-1).

DOMINGUEZ, K. and FRANKEL, J. A. (1990). Does foreign exchange intervention work? *Peterson Institute Press: All Books*.

DOMINGUEZ, K. M. and FRANKEL, J. A. (1993). Does foreign-exchange intervention matter? the portfolio effect. *The American Economic Review*, **83** (5), 1356–1369.

DRAGHI, M. (2015). Introductory statement to the press conference (with q&a). *European Central Bank, Frankfurt am Main*.

EVANS, G. W. and RAMEY, G. (1992). Expectation calculation and macroeconomic dynamics. *The American Economic Review*, **82** (1), 207–224.

— and — (1998). Calculation, adaptation and rational expectations. *Macroeconomic Dynamics*, **2** (2), 156–182.

FANELLI, S. and STRAUB, L. (2016). A theory of foreign exchange interventions.

FARHI, E. and WERNING, I. (2016). The power of forward guidance with bounded rationality and incomplete markets.

FIELDHOUSE, A., MERTENS, K., RAVN, M. *et al.* (2017). *The Macroeconomic Effects of Government Asset Purchases: Evidence from Postwar US Housing Credit Policy*. Tech. rep.

GABAIX, X. (2016). A behavioral new keynesian model. *preliminary New York University working paper*.

— and MAGGIORI, M. (2015). International liquidity and exchange rate dynamics. *The Quarterly Journal of Economics*, **130** (3), 1369–1420.

GAGNON, J., RASKIN, M., REMACHE, J., SACK, B. *et al.* (2011). The financial market effects of the federal reserve's large-scale asset purchases. *international Journal of central Banking*, **7** (1), 3–43.

GARCÍA-SCHMIDT, M. and WOODFORD, M. (2015). *Are Low Interest Rates Deflationary? A Paradox of Perfect-Foresight Analysis*. Working Paper 21614, National Bureau of Economic Research.

GORODNICHENKO, Y. and RAY, W. (2017). The effects of quantitative easing: Taking a cue from treasury auctions. *wp*.

HAMILTON, J. D. and WU, J. C. (2012). The effectiveness of alternative monetary policy tools in a zero lower bound environment. *Journal of Money, Credit and Banking*, **44** (s1), 3–46.

HANCOCK, D. and PASSMORE, W. (2011). Did the federal reserve's mbs purchase program lower mortgage rates? *Journal of Monetary Economics*, **58** (5), 498–514.

JEANNE, O. and ROSE, A. K. (2002). Noise trading and exchange rate regimes. *The Quarterly Journal of Economics*, **117** (2), 537–569.

— and SVENSSON, L. E. O. (2007). Credible commitment to optimal escape from a liquidity trap: The role of the balance sheet of an independent central bank. *The American Economic Review*, **97** (1), 474–490.

KEARNS, J. and RIGOBON, R. (2005). Identifying the efficacy of central bank interventions: evidence from australia and japan. *Journal of International Economics*, **66** (1), 31–48.

KOURI, P. J. (1976). The exchange rate and the balance of payments in the short run and in the long run: A monetary approach. *The Scandinavian Journal of Economics*, pp. 280–304.

KRISHNAMURTHY, A. and VISSING-JORGENSEN, A. (2011). *The effects of quantitative easing on interest rates: channels and implications for policy*. Tech. rep., National Bureau of Economic Research.

LUCAS, R. E. (1972). Expectations and the neutrality of money. *Journal of economic theory*, **4** (2), 103–124.

MAĆKOWIAK, B. and WIEDERHOLT, M. (2009). Optimal sticky prices under rational inattention. *The American Economic Review*, **99** (3), 769–803.

MANKIW, N. G. and REIS, R. (2002). Sticky information versus sticky prices: A proposal to replace the new keynesian phillips curve*. *The Quarterly Journal of Economics*, **117** (4), 1295–1328.

MUSSA, M. (1981). *The role of official intervention*. Group of Thirty New York.

NAGEL, R. (1995). Unraveling in guessing games: An experimental study. *The American Economic Review*, **85** (5), 1313–1326.

POPPER, H. and MONTGOMERY, J. D. (2001). Information sharing and central bank intervention in the foreign exchange market. *Journal of International Economics*, **55** (2), 295–316.

REIS, R. (2006a). Inattentive consumers. *Journal of monetary Economics*, **53** (8), 1761–1800.

— (2006b). Inattentive producers. *The Review of Economic Studies*, **73** (3), 793–821.

— (2017). Qe in the future: the central bank's balance sheet in a fiscal crisis. *IMF Economic Review*, **65** (1), 71–112.

RICARDO, D. (1821). *On the Principles of Political Economy, and Taxation*. John Murray.

SARNO, L. and TAYLOR, M. P. (2001). Official intervention in the foreign exchange market: is it effective and, if so, how does it work? *journal of Economic Literature*, **39** (3), 839–868.

SILVA, D. H. (2016). *The Risk Channel of Unconventional Monetary Policy*. Tech. rep., Working Paper, MIT.

SIMS, C. A. (2003). Implications of rational inattention. *Journal of monetary Economics*, **50** (3), 665–690.

STAHL, D. O. and WILSON, P. W. (1995). On players' models of other players: Theory and experimental evidence. *Games and Economic Behavior*, **10** (1), 218–254.

STERK, V. and TENREYRO, S. (2013). The transmission of monetary policy operations through redistributions and durable purchases.

STROEBEL, J. and TAYLOR, J. B. (2012). Estimated impact of the federal reserve's mortgage-backed securities purchase program. *International Journal of Central Banking*.

VAYANOS, D. and VILA, J.-L. (2009). *A preferred-habitat model of the term structure of interest rates*. Tech. rep., National Bureau of Economic Research.

VITALE, P. (1999). Sterilised central bank intervention in the foreign exchange market. *Journal of International Economics*, **49** (2), 245–267.

— (2003). Foreign exchange intervention: how to signal policy objectives and stabilise the economy. *Journal of Monetary Economics*, **50** (4), 841–870.

WALLACE, N. (1981). A modigliani-miller theorem for open-market operations. *The American Economic Review*, **71** (3), pp. 267–274.

WOODFORD, M. (2001). *Imperfect common knowledge and the effects of monetary policy*. Tech. rep., National Bureau of Economic Research.

— (2013). Macroeconomic analysis without the rational expectations hypothesis. *Annu. Rev. Econ.*, **5** (1), 303–346.

YELLEN, J. L. (2016). The federal reserve's monetary policy toolkit: Past, present, and future. In *speech delivered at "Designing Resilient Monetary Policy Frameworks for the Future," a symposium sponsored by the Federal Reserve Bank of Kansas City, held in Jackson Hole, Wyoming, August*, vol. 26.

A Appendix

A.1 Proof of Lemma 1

We focus on the household problem in the home country, the problem in the foreign country is analogous. If we combine the budget constraints (1) and (2) and approximate the resulting intertemporal budget constraint around $i_t = \pi_{t+1} = r = 0$, we obtain

$$c_{t+1} = Rw + (x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1} - T_{t+1}. \tag{A.1}$$

where $\mathcal{R}_{t+1} = \left(r^x_{t+1} + q_{t+1} - Rq_t, i_t - \pi_{t+1} - r, i^*_t - \pi^*_{t+1} - r, -r - \pi_{t+1}\right)'$Equation (A.1) is a linear transformation of jointly normal variables, thus, standard properties of CARA preferences imply that the household maximization problem can be equivalently rewritten as

$$\max_{\substack{x_{t+1}, b_{H,t+1}, b_{F,t+1}, \\ m_{t+1}, c_{t+1}}} \widetilde{\mathbb{E}}_t c_{t+1} - \frac{\gamma}{2}\widetilde{\mathbb{V}}_t c_{t+1} - \frac{m_{t+1}\left[\log\left(m_{t+1}/\overline{m}\right) - 1\right]}{v},$$

subject to (A.1), where the tilde emphasizes that the household uses the distribution $\widetilde{\Phi}_{t+1}$ to predict endogenous variables at $t+1$. In particular, we can use (A.1) to rewrite the first two terms explicitly:

$$\widetilde{\mathbb{E}}_t c_{t+1} = Rw + \widetilde{\mathbb{E}}_t\left[(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1}\right] - \widetilde{\mathbb{E}}_t T_{t+1},$$

and

$$\widetilde{\mathbb{V}}_t c_{t+1} = \widetilde{\mathbb{V}}_t\left[(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1}\right] + \widetilde{\mathbb{V}}_t\left(T_{t+1}\right) - 2\left(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}\right) \cdot \widetilde{Cov}_t\left(\mathcal{R}_{t+1}, T_{t+1}\right).$$

We can then use linearity of beliefs (3) to rewrite the first four terms as follows:

$$\widetilde{\mathbb{V}}_t\left[(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}) \cdot \mathcal{R}_{t+1}\right] = \left(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}\right)\widetilde{\Sigma}_t\left(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1}\right)',$$

$$\widetilde{\mathbb{V}}_t\left(T_{t+1}\right) = \left(\beta^x_{T,t+1}\right)^2\sigma_x^2 + \left(\beta^h_{T,t+1}\right)^2\sigma_h^2 + \left(\beta^f_{T,t+1}\right)^2\sigma_f^2,$$

$$\begin{aligned}\widetilde{Cov}_t\left(\mathcal{R}_{t+1}, T_{t+1}\right) =& \widetilde{Cov}_t\left(\left(r^x_{t+1} + q_{t+1} - Rq_t, i_t - \pi_{t+1} - r, i^*_t - \pi^*_{t+1} - r, -\pi_{t+1} - r\right)', T_{t+1}\right) \\ =& \begin{pmatrix} \left(1 + \beta^x_{q,t+1}\right)\beta^x_{T,t+1}\sigma_x^2 + \beta^h_{q,t+1}\beta^h_{T,t+1}\sigma_h^2 + \beta^f_{q,t+1}\beta^f_{T,t+1}\sigma_f^2 \\ -\beta^x_{p,t+1}\beta^x_{T,t+1}\sigma_x^2 - \beta^h_{p,t+1}\beta^h_{T,t+1}\sigma_h^2 - \beta^f_{p,t+1}\beta^f_{T,t+1}\sigma_f^2 \\ -\beta^x_{p^*,t+1}\beta^x_{T,t+1}\sigma_x^2 - \beta^h_{p^*,t+1}\beta^h_{T,t+1}\sigma_h^2 - \beta^f_{p^*,t+1}\beta^f_{T,t+1}\sigma_f^2 \\ -\beta^x_{p,t+1}\beta^x_{T,t+1}\sigma_x^2 - \beta^h_{p,t+1}\beta^h_{T,t+1}\sigma_h^2 - \beta^f_{p,t+1}\beta^f_{T,t+1}\sigma_f^2 \end{pmatrix}.\end{aligned}$$

Finally, the first order conditions with respect to $(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1})$ give

$$\widetilde{\Sigma}_t\begin{pmatrix} x_{t+1} \\ b_{H,t+1} \\ b_{F,t+1} \\ m_{t+1} \end{pmatrix} = \frac{1}{\gamma}\widetilde{\mathbb{E}}_t\mathcal{R}_{t+1} + \frac{1}{\gamma}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{v}\log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + \widetilde{Cov}_t\left(\mathcal{R}_{t+1}, T_{t+1}\right).$$

A.2 Proof of Proposition 1

In a REE, beliefs are consistent with equilibrium variables, therefore, taxes can be computed from the government budgets (4) and (5). In particular, the latter imposes the following restrictions on the coefficients of taxes:

$$\begin{aligned}
\omega\beta_{T,t+1}^{x,REE} &= \beta_{q,t+1}^{x,REE} X_{t+2} - \left(1+\beta_{q,t+1}^{x,REE}\right) X_{t+1} + \beta_{p,t+1}^{x,REE} B_{t+1}^{h} + \beta_{p^*,t+1}^{x,REE} B_{t+1}^{f},\\
\omega\beta_{T,t+1}^{h,REE} &= \beta_{q,t+1}^{h,REE} X_{t+2} - \beta_{q,t+1}^{h,REE} X_{t+1} + \beta_{p,t+1}^{h,REE} B_{t+1}^{h} + \beta_{p^*,t+1}^{h,REE} B_{t+1}^{f} - \overline{M},\\
\omega\beta_{T,t+1}^{f,REE} &= \beta_{q,t+1}^{f,REE} X_{t+2} - \beta_{q,t+1}^{f,REE} X_{t+1} + \beta_{p,t+1}^{f,REE} B_{t+1}^{h} + \beta_{p^*,t+1}^{f,REE} B_{t+1}^{f},
\end{aligned}$$

for the home country, and

$$\begin{aligned}
(1-\omega)\,\beta_{T^*,t+1}^{x,REE} &= -B^*\beta_{p^*t+1}^{x,REE},\\
(1-\omega)\,\beta_{T^*,t+1}^{h,REE} &= -B^*\beta_{p^*t+1}^{h,REE} - \overline{M}^*,\\
(1-\omega)\,\beta_{T^*,t+1}^{f,REE} &= -B^*\beta_{p^*t+1}^{f,REE},
\end{aligned}$$

for the foreign one. In equilibrium, markets have to clear, that is,

$$\omega x_{t+1} = \overline{X} - X_{t+1}, \tag{A.2}$$

$$\omega b_{H,t+1} + (1-\omega) b_{F,t+1}^{*} = -B_{t+1}^{h}, \tag{A.3}$$

$$\omega b_{F,t+1} + (1-\omega) b_{H,t+1}^{*} = B^* - B_{t+1}^{f}, \tag{A.4}$$

$$\omega m_{t+1}^{h} = \frac{M_{t+1}}{P_t}, \tag{A.5}$$

$$(1-\omega)\, m_{t+1}^{f} = \frac{M_{t+1}^{*}}{P_t^{*}}. \tag{A.6}$$

where optimal choices $(x_{t+1}, b_{H,t+1}, b_{F,t+1}, m_{t+1})$ are obtained from (8) together with the restrictions on expectations of taxes above. From Lemma 1, home-country asset demands satisfy

$$\Sigma_t \begin{pmatrix} x_{t+1} \\ b_{H,t+1} \\ b_{F,t+1} \\ m_{t+1} \end{pmatrix} = \frac{1}{\gamma} \begin{pmatrix} r^x + \mathbb{E}_t q_{t+1}^{REE} - R q_t^{REE} \\ i_t - \mathbb{E}_t p_{t+1}^{REE} + p_t^{REE} - r \\ i_t^* - \mathbb{E}_t p_{t+1}^{*,REE} + p_t^{*,REE} - r \\ -\mathbb{E}_t p_{t+1}^{REE} + p_t^{REE} - r - \frac{1}{v}\log\left(\frac{m_{t+1}}{\overline{m}}\right) \end{pmatrix} + Cov_t \left(\begin{pmatrix} q_{t+1}^{REE} \\ -p_{t+1}^{REE} \\ -p_{t+1}^{*,REE} \\ -p_{t+1}^{REE} \end{pmatrix}, T_{t+1} \right), \tag{A.7}$$

where we dropped the tilde to emphasize that expectations are rational. A similar expressions holds for the foreign country.

We conjecture and later verify that $\left(q_t^{REE}, p_t^{REE}, p_t^{*,REE}\right)$ are linear functions of the underlying shocks. Standard properties of Normal distributions then imply that the conditional first and second moments are functions of time only. In addition, since balance sheet policies are assumed to be only functions of time, equations (A.2)-(A.4) imply that q_t^{REE} will be a deterministic function of time, while p_t^{REE} and $p_t^{*,REE}$ will depend only on time and on the monetary shocks ϵ_t^h and ϵ_t^f, respectively. Formally, $\beta_{q,t+1}^{x,REE} = \beta_{p,t+1}^{x,REE} =$

$\beta^{x,REE}_{p^*,t+1} = 0$, $\beta^{h,REE}_{q,t+1} = \beta^{h,REE}_{p^*,t+1} = 0$, and $\beta^{f,REE}_{q,t+1} = \beta^{f,REE}_{p,t+1} = 0$ and, in particular,

$$\Sigma_t = \begin{pmatrix} \sigma_x^2 & 0 & 0 & 0 \\ 0 & \left(\beta^{h,REE}_{p,t+1}\right)^2 \sigma_h^2 & 0 & \left(\beta^{h,REE}_{p,t+1}\right)^2 \sigma_h^2 \\ 0 & 0 & \left(\beta^{f,REE}_{p^*,t+1}\right)^2 \sigma_f^2 & 0 \\ 0 & \left(\beta^{h,REE}_{p,t+1}\right)^2 \sigma_h^2 & 0 & \left(\beta^{h,REE}_{p,t+1}\right)^2 \sigma_h^2 \end{pmatrix}. \tag{A.8}$$

Note that the second and the fourth rows are identical, implying that matrix Σ_t is not invertible.

We focus on the risky-asset market and derive q_t^{REE}, analogous arguments lead to p_t^{REE} and $p_t^{*,REE}$. Combining the first-order conditions for m_{t+1} and $b_{H,t+1}$ yields

$$\log\left(\frac{m_{t+1}}{\overline{m}}\right) = -vi_t. \tag{A.9}$$

Market clearing in the money market in the home country (A.5) requires

$$\log\left(\frac{\overline{M}}{\overline{m}\omega}\right) - p_t + \epsilon_t^h = -vi_t.$$

In the cashless limit, we get

$$p_t = vi_t + \epsilon_t^h, \tag{A.10}$$

with an analogous expression for the foreign country.

Combining the first row of (A.7) with (A.8) and imposing market clearing (A.2), we obtain

$$\overline{X} - X_{t+1} = \omega \frac{1}{\gamma\sigma_x^2}\left(r^x + q_{t+1}^{REE} - Rq_t^{REE}\right) + \omega\beta^{x,REE}_{T,t+1}.$$

Also, plugging $\beta^{x,REE}_{q,t+1} = \beta^{x,REE}_{p,t+1} = \beta^{x,REE}_{p^*,t+1} = 0$ into the restrictions on expectations of taxes imposed by the government budget constraint, gives

$$\begin{aligned} \omega\beta^{x,REE}_{T,t+1} &= \beta^{x,REE}_{q,t+1} X_{t+2} - \left(1 + \beta^{x,REE}_{q,t+1}\right) X_{t+1} + \beta^{x,REE}_{p,t+1} B^h_{t+1} + \beta^{x,REE}_{p^*,t+1} B^f_{t+1} \\ &= -X_{t+1} \end{aligned}$$

and, therefore,

$$\overline{X} = \omega \frac{1}{\gamma\sigma_x^2}\left(r^x + q_{t+1}^{REE} - Rq_t^{REE}\right). \tag{A.11}$$

Iterating (A.11) forward and looking for the non-explosive path satisfying $R^{-t}q_t^{REE} \to 0$, gives

$$q^{REE} = \frac{1}{R-1}\left(r^x - \frac{\gamma\sigma_x^2}{\omega}\overline{X}\right),$$

which is independent of time, thus, we omitted the subscript t from the notation.

Finally, the equilibrium exchange rate follows from the expressions for the price levels and the law of one price: $e_t^{REE} = p_t^{REE} - p_t^{*,REE}$.

A.3 Proofs for Level-k Thinking Equilibrium

We start with level-1 equilibrium. By assumption, after the announcement of the new policy, level-1 household expectations coincide with REE variables before the announcement. In turn, if expectations do not change, asset demands in Lemma 1 coincide with their REE counterparts (A.7). Specifically, from Proposition 1, expectations of the risky-asset price are pinned down by

$$\begin{pmatrix} \alpha^1_{q,t+1} \\ \beta^{x,1}_{q,t+1} \\ \beta^{h,1}_{q,t+1} \\ \beta^{f,1}_{q,t+1} \end{pmatrix} = \begin{pmatrix} q^{REE} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{A.12}$$

expectations of the home-country price level are pinned down by

$$\begin{pmatrix} \alpha^1_{p,t+1} \\ \beta^{x,1}_{p,t+1} \\ \beta^{h,1}_{p,t+1} \\ \beta^{f,1}_{p,t+1} \end{pmatrix} = \begin{pmatrix} vr \\ 0 \\ \frac{1}{1+v} \\ 0 \end{pmatrix}, \tag{A.13}$$

and expectations of the foreign-country price level are pinned down by

$$\begin{pmatrix} \alpha^1_{p^*,t+1} \\ \beta^{x,1}_{p^*,t+1} \\ \beta^{h,1}_{p^*,t+1} \\ \beta^{f,1}_{p^*,t+1} \end{pmatrix} = \begin{pmatrix} vr \\ 0 \\ 0 \\ \frac{1}{1+v} \end{pmatrix}. \tag{A.14}$$

Finally, since we assume that, before the announcement, the home-country government does not conduct any intervention nor it has outstanding liabilities, taxes are expected to be 0, i.e.,

$$\begin{pmatrix} \alpha^1_{T,t+1} \\ \beta^{x,1}_{T,t+1} \\ \beta^{h,1}_{T,t+1} \\ \beta^{f,1}_{T,t+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{A.15}$$

In the foreign country, instead, before (and after) the announcement, the government has a real value B^* of outstanding nominal bonds, therefore, from (5),

$$\begin{pmatrix} \alpha^1_{T^*,t+1} \\ \beta^{x,1}_{T^*,t+1} \\ \beta^{h,1}_{T^*,t+1} \\ \beta^{f,1}_{T^*,t+1} \end{pmatrix} = \begin{pmatrix} rB^* \\ 0 \\ 0 \\ -\frac{1}{1+v}B^* \end{pmatrix}. \tag{A.16}$$

In equilibrium, market-clearing equations (A.2)-(A.4) must be satisfied. Importantly, as it was also true in the REE, standard properties of Normal distributions imply that the market-clearing condition (A.2) is satisfied by some price q^1_t which is only a function of time. Similarly, as it was also true in the REE, equations (A.3), (A.4), and (A.10) imply that p^1_t and $p^{*,1}_t$ are functions only of time and of the monetary shock at time

t in their respective country.

We now solve for equilibrium prices explicitly. Consider first market-clearing condition (A.2) and combine it with (A.8) and with the risky-asset demand given by the first row of (A.7):

$$\begin{aligned}\overline{X} - X_{t+1} =& \omega \frac{1}{\gamma \sigma_x^2} \left(r^x + \widetilde{\mathbb{E}}_t \left[q_{t+1} \right] - R q_t^1 \right) + \omega \frac{1}{\sigma_x^2} \widetilde{Cov}_t \left(r_{t+1}^x + q_{t+1}, T_{t+1} \right) \\ =& \omega \frac{1}{\gamma \sigma_x^2} \left(r^x + q^{REE} - R q_t^1 \right),\end{aligned}$$

where we used (A.12) and (A.15).[14] In particular, note that agents expect future prices and taxes to be those in the status-quo REE, however, due to unexpected government purchases X_{t+1}, the equilibrium price q_t^1 will be different from the status-quo price q^{REE}. Solving for the equilibrium price yields

$$q_t^1 = q^{REE} + \frac{\gamma \sigma_x^2}{\omega R} X_{t+1}. \tag{A.17}$$

Let's now compute p_t^1. From (A.15) and (A.16), $\widetilde{Cov}_t \left(p_{t+1}^1, T_{t+1} \right) = \widetilde{Cov}_t \left(p_{t+1}^1, T_{t+1}^* \right) = 0$ and the market-clearing condition (A.3), together with (A.10), gives

$$\frac{\frac{1}{v} \left(p_t^1 - \epsilon_t^h \right) - \left(\widetilde{\mathbb{E}}_t \left[p_{t+1} \right] - p_t^1 \right) - r}{\gamma \sigma_h^2 \left(\beta_{p,t+1}^{h,1} \right)^2} = -B_{t+1}^h.$$

Using (A.13) and solving for the equilibrium price,

$$p_t^1 = vr - \frac{v}{1+v} \gamma \sigma_h^2 \left(\frac{1}{1+v} \right)^2 B_{t+1}^h + \frac{1}{1+v} \epsilon_t^h. \tag{A.18}$$

Finally, analogous steps together with (A.14) and (A.16) yield the foreign-country price level in the level-1 equilibrium:

$$p_t^{*,1} = vr - \frac{v}{1+v} \gamma \sigma_f^2 \left(\frac{1}{1+v} \right)^2 B_{t+1}^f + \frac{1}{1+v} \epsilon_t^f. \tag{A.19}$$

The level-1 equilibrium determines the expectations of level-2 households. In particular, from (A.17),

$$\begin{pmatrix} \alpha_{q,t+1}^2 \\ \beta_{q,t+1}^{x,2} \\ \beta_{q,t+1}^{h,2} \\ \beta_{q,t+1}^{f,2} \end{pmatrix} = \begin{pmatrix} q^{REE} + \frac{\gamma \sigma_x^2}{\omega R} X_{t+1} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

from (A.18),

$$\begin{pmatrix} \alpha_{p,t+1}^2 \\ \beta_{p,t+1}^{x,2} \\ \beta_{p,t+1}^{h,2} \\ \beta_{p,t+1}^{f,2} \end{pmatrix} = \begin{pmatrix} vr - \frac{v}{1+v} \gamma \sigma_h^2 \left(\frac{1}{1+v} \right)^2 B_{t+1}^h \\ 0 \\ \frac{1}{1+v} \\ 0 \end{pmatrix},$$

[14] We use a tilde to emphasize that expectations are no longer rational.

and, from (A.19),

$$\begin{pmatrix} \alpha^2_{p^*,t+1} \\ \beta^{x,2}_{p^*,t+1} \\ \beta^{h,2}_{p^*,t+1} \\ \beta^{f,2}_{p^*,t+1} \end{pmatrix} = \begin{pmatrix} vr - \frac{v}{1+v}\gamma\sigma_f^2\left(\frac{1}{1+v}\right)^2 B^f_{t+1} \\ 0 \\ 0 \\ \frac{1}{1+v} \end{pmatrix}.$$

Finally, letting $\Gamma^h_{t+1} \equiv \gamma\sigma_h^2\left(\frac{1}{1+v}\right)^2\left(\frac{v}{1+v}B^f_{t+2} - B^f_{t+1}\right)$ and $\Gamma^f_{t+1} \equiv \gamma\sigma_f^2\left(\frac{1}{1+v}\right)^2\left(\frac{v}{1+v}B^f_{t+2} - B^f_{t+1}\right)$ and combining the home and foreign-government budget constraints (4) and (5) with the equilibrium asset prices, we can pin down the beliefs of future taxes:

$$\omega\begin{pmatrix} \alpha^2_{T,t+1} \\ \beta^{x,2}_{T,t+1} \\ \beta^{h,2}_{T,t+1} \\ \beta^{f,2}_{T,t+1} \end{pmatrix} = \begin{pmatrix} -r^x X_{t+2} + q^1_{t+1}\left(X_{t+2} - X_{t+1}\right) + B^h_{t+2} - \left(R + \Gamma^h_{t+1}\right)B^h_{t+1} + B^f_{t+2} - \left(R + \Gamma^f_{t+1}\right)B^f_{t+1} \\ -X_{t+1} \\ \frac{1}{1+v}B^h_{t+1} \\ \frac{1}{1+v}B^f_{t+1} \end{pmatrix},$$

for the home country, and

$$(1-\omega)\begin{pmatrix} \alpha^2_{T^*,t+1} \\ \beta^{x,2}_{T^*,t+1} \\ \beta^{h,2}_{T^*,t+1} \\ \beta^{f,2}_{T^*,t+1} \end{pmatrix} = \begin{pmatrix} R - 1 + \Gamma^f_{t+1}B^* \\ 0 \\ 0 \\ -\frac{1}{1+v}B^* \end{pmatrix},$$

for the foreign one.

Starting from level-2 beliefs, we can solve for level-3 equilibrium variables. In general, we can proceed recursively and derive q^k_t, p^k_t, and $p^{*,k}_t$, for any level k:

$$q^k_t = q^{REE} + \frac{\gamma\sigma_x^2 X_{t+k}}{\omega R^k}, \tag{A.20}$$

$$p^k_t = vr - \left(\frac{v}{1+v}\right)^k \gamma\sigma_h^2\left(\frac{1}{1+v}\right)^2 B^h_{t+k} + \frac{1}{1+v}\epsilon^h_t, \tag{A.21}$$

$$p^{*,k}_t = vr - \left(\frac{v}{1+v}\right)^k \gamma\sigma_f^2\left(\frac{1}{1+v}\right)^2 B^f_{t+k} + \frac{1}{1+v}\epsilon^f_t. \tag{A.22}$$

A.4 Proof of Proposition 2

By definition, in a RE there is a mass $f(k)$ of level-k households for each k, where level-$k+1$-household beliefs are generated from level-k equilibrium variables (A.20)-(A.22).

Let's start with the risky-asset market. Using the asset demand in Lemma 8 with market clearing (A.2), we have that the equilibrium price q_t must satisfy

$$\omega\left(\sum_{k=1}^{\infty} f(k)\frac{r^x + \alpha^k_{q,t+1} - Rq_t}{\gamma\sigma_x^2} + \beta^{x,k}_{T,t+1}\right) = \overline{X} - X_{t+1},$$

which yields

$$q_t = q^{REE} + \sum_{k=1}^{\infty} f(k)\frac{\gamma\sigma_x^2 X_{t+k}}{\omega R^k}.$$

Similarly, the equilibrium in the home-bond market is now

$$\sum_{k=1}^{\infty} f(k) \left(\frac{i_t - \left(\alpha^k_{p,t+1} - p_t\right) - r}{\gamma \sigma_h^2 \left(\beta^{h,k}_{p,t+1}\right)^2} - \omega \frac{\beta^{h,k}_{T,t+1}}{\beta^{h,k}_{p,t+1}} - (1-\omega) \frac{\beta^{f,k}_{T^*,t+1}}{\beta^{h,k}_{p,t+1}} \right) = -B^h_{t+1}$$

and, therefore,

$$p_t = vr - \gamma \sigma_h^2 \left(\frac{1}{1+v}\right)^2 \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B^h_{t+k} + \frac{1}{1+v} \epsilon^h_t.$$

Analogously,

$$p^*_t = vr - \gamma \sigma_f^2 \left(\frac{1}{1+v}\right)^2 \sum_{k=1}^{\infty} f(k) \left(\frac{v}{1+v}\right)^k B^f_{t+k} + \frac{1}{1+v} \epsilon^f_t.$$

Finally, as usual, the exchange rate follows directly from the law of one price:

$$e_t = p_t - p^*_t.$$

A.5 Proofs of Section 5

Permanent QE. From Proposition (2) and $X_{t+1} = X$, for all t,

$$q_t = q^{REE} + \frac{\gamma \sigma_x^2}{\omega} X \sum_{k=1}^{\infty} f_t(k) \frac{1}{R^k},$$

where now the distribution of level-k agents changes over time. In particular, using (12)

$$\begin{aligned} q_t &= q^{REE} + \frac{\gamma \sigma_x^2}{\omega} X \sum_{k=1+ht}^{\infty} f(k-ht) \frac{1}{R^k} \\ &= q^{REE} + \frac{\gamma \sigma_x^2}{\omega} X \frac{1}{R^{ht}} \frac{1-\lambda}{R-\lambda}. \end{aligned}$$

Equation (13) follows from the definition of $\bar{k}$.

Expanding QE. Assuming that $X_{t+1} = \mu^t X_1$, where $\mu < R/\mu$ otherwise the price will explode, we get

$$q_t - q^{REE} = \frac{\gamma \sigma_x^2}{\omega} \cdot \frac{1}{\bar{k}(R-\mu) + \mu} X_1 \left[\frac{\mu^{1+h}}{R^h}\right]^t.$$

If $\mu^{1+h}/R^h = 1$, then q_t does not depend on time.