Is Index Trading Benign?

Shmuel Baruch

Xiaodi Zhang*

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Abstract

We develop a conditional capital asset pricing model (CAPM) that maintains the rationale for index investment: In equilibrium, it is optimal for nonindex investors to index. The model demonstrates that index investment is not benign. As more nonindexers become indexers, the proportion of an asset's idiosyncratic risk to total risk increases; correlation in asset prices increases; correlation in returns decreases; and for any portfolio other than the market portfolio, the Sharpe ratio decreases and the conditional variance of payoff increases. As a limiting case of our model, we present an equilibrium in which all investors are indexers.

JEL: G11, D82

Keywords: CAPM, index investment, Sharpe ratio, partially revealing equilibrium

*Baruch: David Eccles School of Business, University of Utah, Salt Lake City, UT 84112; shmuel.baruch@business.utah.edu. Zhang: Department of Finance, University of Central Florida, Orlando, FL 32816; Xiaodi.Zhang@ucf.edu. We thank Kerry Back, Dan Bernhardt, Joel Hasbrouck, Davar Khoshnevisan, Mark Loewenstein, Matt Ringgenberg, Yajun Wang, and seminar participants at the SEC and Utah for their helpful comments.

1 Introduction

The separation theorem provides the intellectual underpinning of index investment, and the capital asset pricing model (CAPM) is the most important pricing implication of the separation theorem.[1] But the CAPM is silent about the impact of index investment on pricing. In this paper, we examine the implications of indexing in an extension of the mean–variance equilibrium analysis in which (*i*) the separation result holds and (*ii*) a conditional CAPM holds. Thus, the equilibrium we present in this paper maintains the rationale for index investment, and the results are framed in the standard CAPM terminology.

To build our model, we depart from the standard CAPM in two ways. First, we adopt a rational expectation setup (Grossman, 1976) in which investors combine their private information with the information contained in equilibrium prices. In a multiasset extension of his model, Grossman (1978) shows that a conditional CAPM emerges. But Grossman's conditional CAPM is silent about the impact of indexing in exactly the same way the standard CAPM is. We therefore deviate further from the standard model by explicitly dividing the investors into two groups, indexers and nonindexers. Index investors confine themselves to combinations of the risk-free asset and the market portfolio. Nonindex investors, by contrast, solve an unconstrained investment problem. The presence of index investors in our model is the second and last departure from the standard CAPM world.

We prove the existence of a partially revealing rational expectation equilibrium in which a separation result holds: Both indexers' and nonindexers' investment positions are located, in the volatility–return plane, on the capital market line. Moreover, similar to the fully

[1] In August 1976, when Vanguard started offering its index fund, Samuelson (1976) articulated the separation theorem in the popular press: "What each prudent investor must do is to decide what fraction of savings he can afford, in this age of inflation, to keep in equities and in other things. An unmanaged, low-turnover, low-fee index fund is merely an efficient way of holding that part deemed appropriate for equities." Lo (2016) writes that it was the academic research, specifically the CAPM and the efficient market hypothesis, that "provided the seeds from which the index fund business grew."

revealing equilibrium described by Grossman (1978), a conditional CAPM relation holds.[2] However, asset prices and betas depend on the specific partition of investors into indexers and nonindexers.

By means of comparative statics, we find that as more nonindex investors become index investors, the proportion of idiosyncratic risk to total risk increases.[3] This increase manifests in different ways. The statistical fit (measured by R^2) of the CAPM regression decreases. Correlation in returns decreases in the sense that, provided an asset is positively correlated with the portfolio of remaining assets, this correlation decreases. Correlation in asset prices increases in the sense that, provided an asset price is positively correlated with the price of the portfolio of remaining assets, this correlation increases.[4] For any portfolio other than the market portfolio, the portfolio's Sharpe ratio decreases, and the variance of the portfolio's payoff increases.[5] By contrast, both the market portfolio's Sharpe ratio and the variance of the market portfolio's payoff are unaffected. Finally, using numerical computations, we find that the distributions of betas become less dispersed.

Interestingly, Campbell, Lettau, Malkiel, and Xu (2001) find that from 1962 to 1997, market variance was stable while firms' variances more than doubled. They find that comovement in returns decreased, and the coefficient of determination also decreased. The authors provide several possible explanations for their findings, such as the breaking up of conglomerates. This paper provides a new explanation, namely the rise of index investment.

Whereas the results discussed thus far highlight the impact of index investment, we also

[2] A fully revealing equilibrium is an equilibrium in which, given the aggregate information in the economy, prices are jointly sufficient statistics for the payoff ("future prices") of the assets. Our model also possesses a fully revealing equilibrium.

[3] Idiosyncratic risk is the "unexplained variance" of the stock's return. The proportion of idiosyncratic risk to total risk is the fraction of unexplained variance, and it equals $1 - R^2$, where R^2 is the coefficient of determination of the CAPM regression.

[4] Like in other conditional CAPM models, prices and betas are realizations of random variables. It is therefore meaningful to study their statistical properties.

[5] In this paper, a portfolio with a return r has a Sharpe ratio $(Er - r_f)/\operatorname{sd}(r)$.

identify some market outcomes that do not depend on the specific partition of investors into indexers and nonindexers. Considering the complete set of signals (of both indexers and nonindexers) as the "data" and the payoff of the market portfolio as the "parameter," we show that the forward price of the market portfolio is a sufficient statistic for the payoff of the market portfolio.[6] In fact, we show that the forward price of the market portfolio is a *minimal* sufficient statistic. Furthermore, we show that the endogenous interest rate, the capital market line, and investors' welfare are all independent of the partition of investors.

Our model is relevant to a market with many layman investors who costlessly observe extremely noisy signals.[7] Even though the market is large, the law of large numbers is not applicable; even though each signal is very noisy, equilibrium prices are informative. The signals are a mathematical representation of information readily available in the investors' environments. Sources of the signals can be, for example, having experience with the customer services of firms, being enthusiastic about products of certain brands, or even noting how full retailers' parking lots are. Each individual signal can hardly be called informative; however, when all the signals in the economy are aggregated, valuable information emerges (Treynor 1987). This is our interpretation, but we do not impose this interpretation on our model.

Our model adds to the literature on partially revealing equilibria. To avoid the fully revealing outcome, this literature relies on noise trading (Kyle 1985), supply uncertainty (Hellwig 1980 and Admati 1985), extrinsic noise (DeMarzo and Skiadas 1998), or preference uncertainty (see Ausubel 1990 and the dynamic model of Detemple 2002). Our model relies on none of these. Instead, the equilibrium we compute is partially revealing because index investors solve a constrained optimization problem.

[6] We do not have a derivative securities market in our model, but nevertheless we can compute synthetic forward prices. Forward prices are prices divided by the price of the risk-free bond. In other words, these are the prices at which the bond acts as the numeraire.

[7] Evidence that the trade of layman investors conveys information is provided by Kaniel, Liu, Saar, and Titman (2012), Kelley and Tetlock (2013), and Boehmer, Jones, and Zhang (2016).

In addition, our model is related to those of Levy (1978) and Malkiel and Xu (2002). These authors also study markets in which some investors—much like the index investors in our model—do not solve a complete portfolio optimization problem. These two papers demonstrate how idiosyncratic risk can become relevant. By contrast, in our model, idiosyncratic risk is irrelevant because no investor (indexer or nonindexer) is exposed to it.

Our model also adds to the literature that examines comovements and exchange-traded funds. In Barberis and Shleifer (2003), rational traders take advantage of the extrapolative expectations of switchers who move their holding from one set of assets to another. Barberis, Shleifer, and Wurgler (2005) review additional theories of comovement that stem from market frictions or noise traders' sentiment. Comovement also shows up in market structure–type models. According to Bhattacharya and O'Hara (2016), the source of comovement is the inability to precisely tease out information relevant to individual assets from the exchange-traded funds. Also, Cong and Xu (2016) and Glosten, Nallareddy, and Zou (2016) show that the presence of a composite security, created to cater to factor investors, enhances comovement.

The remainder of this paper is organized as follows. In Section 2, we describe the model. In Section 3, we expand Grossman's (1978) notion of artificial economies to a world with index investors. In Section 4, we compute a partially revealing equilibrium. In Section 5, we present comparative statics. In Section 6, we present, as a limiting case of our model, an equilibrium in which all investors are indexers. In Section 7, we conclude.

2 The Model

We consider a two-period, single-good exchange economy with one financial (i.e., zero-net-supply) risk-free asset (a bond) and n risky real assets (firms). The prices of the assets are denominated in units of the time-zero consumption good, and the assets' payoffs are denom-

inated in units of the time-one consumption good. The consumption good is perishable, so the only way to transfer consumption between periods is through the capital market.

The payoff of the risk-free asset is one. The random payoff of the risky assets is denoted by $\mathbf{v} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}'$.[8] We assume $\mathbf{v} \sim \mathcal{N}(\boldsymbol{\mu}_{\mathbf{v}}, \boldsymbol{\Sigma}_{\mathbf{vv}})$, where $\boldsymbol{\Sigma}_{\mathbf{vv}}$ is a symmetric, positive definite matrix of rank n.[9] The implication of the assumption that $\boldsymbol{\Sigma}_{\mathbf{vv}}$ is of full rank is that the bond is a nonredundant asset. The price vector of the risky assets is denoted by $\mathbf{p} = \begin{bmatrix} p_1 & \dots & p_n \end{bmatrix}'$. The price of the bond is denoted by p_f. We define the risk-free interest rate as follows: $r_f = 1/p_f - 1$.

> **Comment 1:** Why do we deviate from the literature that assumes an exogenous price for the risk-free asset? The risk-free asset is nonredundant, and the equilibrium we study is not fully revealing. Thus, the risk-free asset may serve as a channel for information transmission, which is exactly what happens in the paper by Detemple (2002). Moreover, the risk-free rate may depend on the level of index investment in the economy. Ignoring this possibility may distort our comparative statics analysis. For these two reasons, we choose to deviate from the literature.
>
> In this paper, the price of the risk-free asset does not transmit novel information. Moreover, the price is independent of the level of index investment.

For each risky asset, we normalize the number of outstanding shares to one. A portfolio (of

[8] Notation: All vectors are column vectors. The transpose operation is denoted by a single quotation mark. Bold lowercase (Greek or upright Roman) letters are used for vectors. Bold uppercase (Greek or upright Roman) letters are used for matrices. We have no special notation to distinguish random variables from their realizations. The context should make our intention clear.

[9] Given two random vectors, $\mathbf{z} = \begin{bmatrix} z_1 & \dots & z_n \end{bmatrix}'$ and $\mathbf{y} = \begin{bmatrix} y_1 & \dots & y_m \end{bmatrix}'$, we interchangeably use the notations $\operatorname{cov}(\mathbf{z}, \mathbf{y})$ and $\Sigma_{\mathbf{zy}}$ to denote the $n \times m$ covariance matrix $\begin{bmatrix} \operatorname{cov}(z_i, y_j) \end{bmatrix}_{n \times m}$. Consequently, using submatrix notation, we have

$$\operatorname{cov}\left(\mathbf{z}, \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}\right) = \begin{bmatrix} \operatorname{cov}(\mathbf{x}, \mathbf{y}_1) & \operatorname{cov}(\mathbf{x}, \mathbf{y}_2) \end{bmatrix}$$

risky assets) is a vector $\mathbf{x} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}'$ with the interpretation that x_i is the number of shares of the ith risky asset.[10] The cost of the portfolio is $\mathbf{x}'\mathbf{p}$; the random payoff of the portfolio is $\mathbf{x}'\mathbf{v}$; the mean of the payoff is $\mathbf{x}'\boldsymbol{\mu}_{\mathbf{v}}$; and the variance of the payoff is $\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{x}$. Let $\mathbf{1} \in R^n$ denote the vector of all-ones, so $\mathbf{1}$ is the *market portfolio*.

Every portfolio with a nonzero cost has a return $\mathbf{x}'\mathbf{v}/(\mathbf{x}'\mathbf{p}) - 1$. Two portfolios have the same return if one is a strictly positive scalar multiplication of the other. This is an equivalence relation that is invariant under change of prices.[11] In the portfolio analysis literature, an equivalent class is identified with a vector of market-value weights, termed portfolio weights. (Often, the word "weights" is omitted.) But, under a different set of prices, the same weights represent a different equivalence class. (For example, the market-value weights of the market portfolio change as we change prices.) We therefore avoid market-value weights altogether. Whenever an investor holds a portfolio that is a strictly positive scalar multiplication of $\mathbf{1}$, we say that the investor holds the market.

There are m investors, labeled $k = 1, \dots, m$. Investors observe the realization of private signals centered around $\mathbf{v}$. The signals are

$$\forall k = 1 \dots m, \quad \mathbf{s}_k = \begin{bmatrix} s_{1k} \\ \vdots \\ s_{nk} \end{bmatrix} = \mathbf{v} + \boldsymbol{\epsilon}_k$$

where

$$\boldsymbol{\epsilon}_k \sim N\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}\boldsymbol{\epsilon}} \right)$$

The random vectors $\{\mathbf{v}, \boldsymbol{\epsilon}_1, \dots, \boldsymbol{\epsilon}_m\}$ are jointly normally distributed and mutually independent.

Let $\bar{c}_k, \bar{b}_k$, and $\bar{\mathbf{x}}_k$ denote the kth investor's endowment. The budget set of the kth investor

[10] In this paper, the word "portfolio" is a shorthand for portfolio of risky assets.

[11] In other words, if two portfolios have the same return under one set of prices, and we then change prices, then either both portfolios will not have a return or both portfolios will have the same return.

is

$$\mathcal{B}_k(p_f, \mathbf{p}) = \left\{c \in R, b \in R, \mathbf{x} \in R^n : \bar{c}_k - c + (\bar{b}_k - b)p_f + (\bar{\mathbf{x}}_k - \mathbf{x})'\mathbf{p} = 0\right\} \tag{1}$$

Subject to their budget constraint, the investors choose the number of time-zero consumption units, c; the number of bonds, b; and a portfolio of risky assets, $\mathbf{x}$, to maximize the expected value of the (same) utility function:

$$U(c, b, \mathbf{x}'\mathbf{v}) \equiv -e^{-\rho_0 c} - e^{-\rho_1(b + \mathbf{x}'\mathbf{v})}$$

Investors belong to one of two groups, indexers or nonindexers. Index investors confine themselves to combining the risk-free asset with the market portfolio. Nonindex investors solve a complete portfolio selection problem. We denote the set of indices of index investors by $\mathcal{I}$ and the set of indices of the nonindex investors by $\mathcal{NI}$. We then have $|\mathcal{I}| + |\mathcal{NI}| = m$. For expositional reasons, we assume both types of investors are present. In other words, $0 < |\mathcal{I}| < m$. We remove this assumption in Section 6.

We can write the investors' problems as follows:[12]

$$\begin{aligned} \forall k \in \mathcal{NI}, \quad & \max_{c,b,\mathbf{x}} E\left[U(c, b, \mathbf{x}'\mathbf{v}) \,|\mathbf{s}_k, \mathbf{p}, p_f\right] \\ & \text{s.t. } (c, b, \mathbf{x}) \in \mathcal{B}_k(p_f, \mathbf{p}) \\ \forall k \in \mathcal{I}, \quad & \max_{c,b,q} E\left[U(c, b, q\mathbf{1}'\mathbf{v}) \,|\mathbf{s}_k, \mathbf{p}, p_f\right] \\ & \text{s.t. } (c, b, q\mathbf{1}) \in \mathcal{B}_k(p_f, \mathbf{p}) \end{aligned}$$

Let c_k, b_k, and $\mathbf{x}_k$ denote the optimal solution of the above maximization problems, where $\mathbf{x}_k \equiv q_k\mathbf{1}$, whenever $k \in \mathcal{I}$. The solutions of the maximization problems of the investors

[12] Indexers can infer information from the prices of individual assets. Some readers may object to this modeling choice, preferring a model where an indexer can observe only the price of the market portfolio. However, unless her endowment is the market portfolio, to know what her budget is, an indexer has to know the prices of individual assets. If an indexer is endowed with the market portfolio, then, in the equilibrium presented in this paper, the objective of the indexer can be replaced with $E\left[U(c, b, q\mathbf{1}'\mathbf{v}) \,|\mathbf{s}_k, \mathbf{1}'\mathbf{p}, p_f\right]$ instead of $E\left[U(c, b, q\mathbf{1}'\mathbf{v}) \,|\mathbf{s}_k, \mathbf{p}, p_f\right]$, thus removing the objection some readers may have. We choose not to impose assumptions on initial endowments. Hence, our modeling choice is to allow the indexer to observe the equilibrium vector of prices.

depend on the realization of the prices and signals, but for brevity our notation does not show this dependency.

Denote by $\mathbf{s}$ the concatenation of all signals. A *rational expectation equilibrium* is a random pair $(\mathbf{p}, p_f)$ such that for each joint realization of $\mathbf{s}$ and $(\mathbf{p}, p_f)$, the market for the consumption good, the market for debt, and the market for risky assets clear:

$$\sum_{k=1}^{m} c_k = \sum_{k=1}^{m} \bar{c}_k, \qquad \sum_{k=1}^{m} b_k = \sum_{k=1}^{m} \bar{b}_k = 0, \qquad \sum_{k \in \mathcal{NI}} \mathbf{x}_k + \sum_{k \in \mathcal{I}} q_k \mathbf{1} = \sum_{k=1}^{m} \bar{\mathbf{x}}_k = \mathbf{1}$$

3 Artificial Economies

Grossman (1978) devises a heuristic for finding a rational expectation equilibrium. He considers an artificial economy in which each investor has access to all private information in the economy. He proves that if the equilibrium price vector in this artificial economy is a sufficient statistic for the mean of the investors' signals (which is the payoff vector, $\mathbf{v}$), then this price vector is also a rational expectation equilibrium in the actual economy.

The heuristic can be successfully applied to our model because in Grossman's *fully revealing equilibrium* everyone holds the market portfolio; therefore, in the fully revealing equilibrium, the additional constraint on index investors is nonbinding. This means that the fully revealing equilibrium is silent about the implications of index investment in exactly the same way that the standard CAPM is. We are searching for a different rational expectation equilibrium.

For the purpose of finding a new rational expectation equilibrium, we note that statistical sufficiency of prices is a requirement stronger than needed. Indeed, when the price vector is a sufficient statistic, *any decision maker* is indifferent between knowing all the information and knowing only the prices. But we should heed only what the decision makers in our

model prefer (in particular, those index investors who solve a constraint problem), not what every hypothetical decision maker prefers.

That said, as in Grossman's fully revealing equilibrium, we are searching for a rational expectation equilibrium in which the equilibrium prices reveal "information to each trader which is of 'higher quality' than his own information" (Grossman 1976):

$$\forall k \in \mathcal{NI}, \forall \mathbf{x} \in R^n \;\; E\left[U(c, b, \mathbf{x}'\mathbf{v}) \,|\mathbf{s}_k, \mathbf{p}, p_f\right] = E\left[U(c, b, \mathbf{x}'\mathbf{v}) \,|\mathbf{p}, p_f\right]$$

$$\forall k \in \mathcal{I}, \forall q \in R \;\; E\left[U(c, b, q\mathbf{1}'\mathbf{v}) \,|\mathbf{s}_k, \mathbf{p}, p_f\right] = E\left[U(c, b, q\mathbf{1}'\mathbf{v}) \,|\mathbf{p}, p_f\right]$$

3.1 The Artificial Economy $\mathcal{E}_{\mathbf{y}}$

We fix an arbitrary, nondegenerate, multivariate normal random vector $\mathbf{y}$, such that $\mathbf{y}$ and $\mathbf{v}$ are jointly normal. We do not specify the dimension of the vector $\mathbf{y}$.[13]

Let $\boldsymbol{\mu}_{\mathbf{y}}$ be the expected value of $\mathbf{y}$. Because $\mathbf{v}$ and $\mathbf{y}$ are jointly normally distributed, a standard result in probability theory is that $\mathbf{v}$ conditional on the realization of $\mathbf{y}$ is multivariate normal with a conditional mean and a conditional (deterministic) covariance matrix:

$$\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{v}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}) \tag{2}$$

$$\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} = \boldsymbol{\Sigma}_{\mathbf{vv}} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yv}} \tag{3}$$

We define the artificial economy $\mathcal{E}_{\mathbf{y}}$ as follows. Investors have the same initial endowment as in the actual economy. All investors are nonindexers, and investors do not observe realizations of private signals. Instead, they observe the realization of the random vector $\mathbf{y}$. An equilibrium in $\mathcal{E}_{\mathbf{y}}$ is a pair $(p_f, \mathbf{p})$ such that for each realization of $\mathbf{y}$, and for each $k = 1, \ldots, m$, $(c_k, b_k, \mathbf{x}_k)$ solves

[13] A multivariate normal random vector is nondegenerate if its covariance matrix is invertible.

$$\max_{c,b,\mathbf{x}} E\left[U(c,b,\mathbf{x}'\mathbf{v})\,|\mathbf{y}\right]$$
$$\text{s.t. } (c,b,\mathbf{x}) \in \mathcal{B}_k(p_f,\mathbf{p})$$

and the three markets clear:

$$\sum_{k=1}^{m} c_k = \sum_{k=1}^{m} \bar{c}_k, \qquad \sum_{k=1}^{m} b_k = \sum_{k=1}^{m} \bar{b}_k = 0, \qquad \sum_{k=1}^{m} \mathbf{x}_k = \sum_{k=1}^{m} \bar{\mathbf{x}}_k = \mathbf{1}$$

The following two results are standard.

Theorem 3.1. *The artificial economy $\mathcal{E}_{\mathbf{y}}$ has a unique equilibrium. The equilibrium asset prices are defined implicitly as follows. Let*

$$\mathbf{f} := \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_1}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} \tag{4}$$

The equilibrium price of the bond, p_f, is given by

$$\log(p_f) = \log(\rho_1) - \log(\rho_0) - \frac{\rho_1}{m}\left(\mathbf{1}'\mathbf{f} + \frac{\rho_1}{2m}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right) + \frac{\rho_0}{m}\sum_{k=1}^{m}\bar{c}_k \tag{5}$$

and the equilibrium price of the risky assets is

$$\mathbf{p} = p_f\mathbf{f} \tag{6}$$

In equilibrium, the portfolio holding of risky assets is identical for all investors: $\mathbf{x}_k = \frac{1}{m}\mathbf{1}$.

The proof of Theorem 3.1 is in Appendix A.

From Equation 6, we see that $\mathbf{f}$ is the vector of synthetic *forward prices*. In other words, the price of the risky assets denominated in units of the bond is $\mathbf{f}$. Next, we show that the CAPM risk–return relation holds in the equilibrium in the artificial economy.

Let $r_{\text{mkt}} = \mathbf{1}'\mathbf{v}/(\mathbf{1}'\mathbf{p}) - 1$, $r_i = v_i/p_i - 1$, and

$$\beta_i = \frac{\text{cov}(r_{\text{mkt}}, r_i|\mathbf{y})}{\text{var}(r_{\text{mkt}}|\mathbf{y})} \tag{7}$$

Theorem 3.2. *In the artificial economy $\mathcal{E}_{\mathbf{y}}$, the CAPM holds:*

$$E[r_i|\mathbf{y}] = r_f + \beta_i(E[r_{\text{mkt}}|\mathbf{y}] - r_f) \tag{8}$$

The proof of Theorem 3.2 is in Appendix A. So far, our choice of $\mathbf{y}$ has been arbitrary. We now introduce a strong assumption:

(GR) The random vector $\mathbf{y}$ is such that

$$\begin{cases} \forall k \in \mathcal{NI} & E[\mathbf{v}|\mathbf{s}_k, \mathbf{y}] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} & \text{var}(\mathbf{v}|\mathbf{s}_k, \mathbf{y}) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} \\ \forall k \in \mathcal{I} & E[\mathbf{1}'\mathbf{v}|\mathbf{s}_k, \mathbf{y}] = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} & \text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}_k, \mathbf{y}) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} \end{cases}$$

A random vector that trivially satisfies (GR) is $\mathbf{s}$, the concatenation of all signals. Therefore, (GR) is not vacuous.

Our next goal is to show that when $\mathbf{y}$ satisfies (GR), the equilibrium prices in the artificial economy $\mathcal{E}_{\mathbf{y}}$ are the equilibrium prices in the actual economy. To that end, we first need to establish that the prices in the artificial economy carry the same information as $\mathbf{y}$, so that the conditioning in (GR) on $\mathbf{y}$ can be replaced with conditioning on prices.

Because equilibrium prices are not normally distributed, we find it easier to replace the conditioning on random vectors by conditioning on the σ–algebras generated by the random vectors. We have

Lemma 3.3. *Assume $\mathbf{y}$ satisfies (GR), and let p_f and $\mathbf{p}$ be the equilibrium prices in the artificial economy $\mathcal{E}_{\mathbf{y}}$. Then, $\sigma(\mathbf{p}, p_f) = \sigma(\mathbf{f}) = \sigma(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}) \subseteq \sigma(\mathbf{y})$.*

The proof of Lemma 3.3 is in Appendix A. We also need the following simple result.

Lemma 3.4. *Let $\mathcal{G}$ and $\mathcal{F}$ be two sigma algebras such that $\mathcal{G} \subseteq \mathcal{F}$ ($\mathcal{F}$ contains more information than $\mathcal{G}$).*

1. *If $E[\mathbf{v}|\mathcal{F}] \in \mathcal{G}$, then $E[\mathbf{v}|\mathcal{G}] = E[\mathbf{v}|\mathcal{F}]$.*
2. *If $E[\mathbf{v}|\mathcal{F}] \in \mathcal{G}$ and $\text{var}(\mathbf{v}|\mathcal{F}) \in \mathcal{G}$, then $\text{var}(\mathbf{v}|\mathcal{G}) = \text{var}(\mathbf{v}|\mathcal{F})$.*

The proof of Lemma 3.4 is in Appendix A.

Corollary 3.5. *Assume $\mathbf{y}$ satisfies (GR). Let p_f and $\mathbf{p}$ be the equilibrium prices in the artificial economy $\mathcal{E}_{\mathbf{y}}$. We have*

1. *The distribution of $\mathbf{v}$ conditional on the realization of p_f and $\mathbf{p}$ is multivariate normal with the same mean and variance as the distribution of $\mathbf{v}$ conditional on $\mathbf{y}$:*

$$E\left[\mathbf{v}\middle|\mathbf{p}, p_f\right] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} \quad \text{var}\left(\mathbf{v}\middle|\mathbf{p}, p_f\right) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$$

2. *For all $k \in \mathcal{NI}$, the distribution of $\mathbf{v}$ conditional on the realization of p_f, $\mathbf{p}$, and $\mathbf{s}_k$ is multivariate normal with the same mean and variance as the distribution of $\mathbf{v}$ conditional on $\mathbf{y}$:*

$$E\left[\mathbf{v}\middle|\mathbf{s}_k, \mathbf{p}, p_f\right] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} \quad \text{var}\left(\mathbf{v}\middle|\mathbf{s}_k, \mathbf{p}, p_f\right) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}} \tag{9}$$

3. *For all $k \in \mathcal{I}$, the distribution of $\mathbf{1}'\mathbf{v}$ conditional on the realization of p_f, $\mathbf{p}$, and $\mathbf{s}_k$ is normal with the same mean and variance as the distribution of $\mathbf{1}'\mathbf{v}$ conditional on $\mathbf{y}$. In other words,*

$$E\left[\mathbf{1}'\mathbf{v}\middle|\mathbf{s}_k, \mathbf{p}, p_f\right] = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} \quad \text{var}\left(\mathbf{1}'\mathbf{v}\middle|\mathbf{s}_k, \mathbf{p}, p_f\right) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} \tag{10}$$

The proof of Corollary 3.5 is in Appendix A.

Corollary 3.6. *Assume that* $\mathbf{y}$ *satisfies (GR). The equilibrium prices in the artificial economy* $\mathcal{E}_{\mathbf{y}}$ *form a rational expectation equilibrium in the actual economy. The allocations are identical in both equilibria.*

Proof of Corollary 3.6. Assume that investors in the actual economy face prices that are the equilibrium prices in the artificial economy.

Let $k \in \mathcal{NI}$. The investor's objective is to maximize $E[U(c, b, \mathbf{x}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f]$. We have

$$E[U(c, b, \mathbf{x}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f] \underset{(9)}{=} -e^{-\rho_0 c} - e^{-\rho_1 \left(b + \mathbf{x}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_1}{2}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{x}\right)} = E[U(c, b, \mathbf{x}'\mathbf{v})|\mathbf{y}]$$

Because the budget constraints are the same in both the actual economy and the artificial economy, we conclude that the equilibrium allocation of a nonindex investor in the artificial economy is also optimal in the actual economy.

Let $k \in \mathcal{I}$. The investor's objective is to maximize $E[U(c, b, q\mathbf{1}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f]$. We have

$$\begin{aligned}
& \max_{c,b,q} E[U(c, b, q\mathbf{1}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f] && \text{s.t. } (c, b, q\mathbf{1}) \in \mathcal{B}_k(p_f, \mathbf{p}) \\
\underset{(10)}{=} & \max_{c,b,q} -e^{-\rho_0 c} - e^{-\rho_1 \left(b + q\mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_1}{2}q^2\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)} && \text{s.t. } (c, b, q\mathbf{1}) \in \mathcal{B}_k(p_f, \mathbf{p}) \\
= & \max_{c,b,q} E[U(c, b, q\mathbf{1}'\mathbf{v})|\mathbf{y}] && \text{s.t. } (c, b, q\mathbf{1}) \in \mathcal{B}_k(p_f, \mathbf{p}) \\
= & \max_{c,b,\mathbf{x}} E[U(c, b, \mathbf{x}'\mathbf{v})|\mathbf{y}] && \text{s.t. } (c, b, \mathbf{x}) \in \mathcal{B}_k(p_f, \mathbf{p})
\end{aligned}$$

where the last equality arises because in the equilibrium in the artificial economy it is optimal for investors to hold the market (see Theorem 3.1).

We have shown that the equilibrium allocations in the artificial economy are optimal in the actual economy. We are left to show that the three markets clear. Those allocations clear the three markets in the artificial economy; therefore, they also clear the markets in the actual economy.

■

4 A Partially Revealing Equilibrium

Guided by hindsight, we define the vector $\mathbf{g} \in R^n$ as follows:

$$\mathbf{g} := (m\boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}})^{-1} \boldsymbol{\Sigma}_{\mathbf{vv}} \mathbf{1} \tag{11}$$

and for the remainder of this paper, we set $\mathbf{y} \in R^{n+1}$ to

$$\mathbf{y} := \begin{bmatrix} \dfrac{1}{|\mathcal{NI}|} \displaystyle\sum_{k \in \mathcal{NI}} \mathbf{s}_k \\ \\ \mathbf{g}' \displaystyle\sum_{k=1}^{m} \mathbf{s}_k \end{bmatrix}_{(n+1)\times 1} \tag{12}$$

Theorem 4.1. *The vector* $\mathbf{y}$ *satisfies (GR).*

The proof of Theorem 4.1 is in Appendix B.

According to Corollary 3.6 and Theorem 4.1, there is a rational expectation equilibrium with prices and allocations identical to those in the artificial economy $\mathcal{E}_{\mathbf{y}}$. In particular, every investor, whether indexer or nonindexer, holds the market portfolio, and a conditional CAPM holds. Importantly, because the first n coordinates of $\mathbf{y}$ depend on the set $\mathcal{NI}$, the equilibrium prices depend on the specific partition of investors into indexers and nonindexers. The next corollary points out that the equilibrium prices are informationally equivalent to $\mathbf{y}$.

Corollary 4.2. *We have* $\sigma(\mathbf{p}, p_f) = \sigma(\mathbf{f}) = \sigma(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}) = \sigma(\mathbf{y})$.

The proof of Corollary 4.2 is in Appendix C. For notational consistency and brevity, we

continue to condition on $\mathbf{y}$. Corollary 4.2 implies that doing so is equivalent to conditioning on equilibrium prices.

We have constructed a rational expectation equilibrium. But what is the information content of $\mathbf{y}$ (or, equivalently, the equilibrium prices $(\mathbf{p}, p_f)$)? The first n coordinates of y are the average of the nonindexers' signals. It is a well-known result in statistics that, in the case of normal distribution with a known variance, the sample mean is a sufficient statistic for the mean. Here, the mean is the asset payoff vector, $\mathbf{v}$, and the variance is known. Thus, the first n coordinates of $\mathbf{y}$ contain all the information about $\mathbf{v}$ that there is in the entire pool of nonindexers' private information.

The next theorem articulates the informational content of the $(n+1)$th coordinate of $\mathbf{y}$, $y_{n+1} = \mathbf{g}' \sum_{k=1}^{m} \mathbf{s}_k$.

Theorem 4.3. *We have*

1. $E[\mathbf{1}'\mathbf{v}|\mathbf{s}] = E[\mathbf{1}'\mathbf{v}|y_{n+1}]$, *and* $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}) = \text{var}(\mathbf{1}'\mathbf{v}|y_{n+1})$.
2. *Given* $\mathbf{s}$, y_{n+1} *is a minimal sufficient statistic for* $\mathbf{1}'\mathbf{v}$.

In part 2 of the Theorem, $\mathbf{s}$ plays the role of the data, y_{n+1} the role of the statistic, and $\mathbf{1}'\mathbf{v}$ the role of the parameter.

The proof of Theorem 4.3 is in Appendix C.

The next theorem shows that the forward price of the market portfolio, $\mathbf{1}'\mathbf{f}$, contains *all* the information in the economy about the future payoff of the market portfolio. The proof is based on the observation that $\mathbf{1}'\mathbf{f}$ is informationally equivalent to y_{n+1}.

Theorem 4.4 (Sufficient Statistics). *We have*

1. $E[\mathbf{1}'\mathbf{v}|\mathbf{s}] = E[\mathbf{1}'\mathbf{v}|\mathbf{1}'\mathbf{f}]$, *and* $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}) = \text{var}(\mathbf{1}'\mathbf{v}|\mathbf{1}'\mathbf{f})$.

2. *Given* $\mathbf{s}$, $\mathbf{1}'\mathbf{f}$ *is a minimal sufficient statistic for* $\mathbf{1}'\mathbf{v}$.

The proof of Theorem 4.4 is in Appendix C.

We conclude this section with a paradox: Nonindexers' equilibrium investment positions are points on the capital market line, whereas the constraint on indexers to stay on the capital market line is binding. In other words, if the constraint is removed, it is suboptimal for indexers to hold the market portfolio.

Theorem 4.5 (Index Investment Paradox). *Let* $(\mathbf{p}, p_f)$ *be the equilibrium prices. Let*

$$(c_k, b_k, \mathbf{x}_k) = \underset{c,b,\mathbf{x}}{\operatorname{argmax}} \; E[U(c, b, \mathbf{x}'\mathbf{v})|\mathbf{s}_k, \mathbf{p}, p_f]$$

$$\text{s.t. } (c, b, \mathbf{x}) \in \mathcal{B}_k(p_f, \mathbf{p})$$

1. *If* $k \in \mathcal{NI}$, *then there is a nonzero scalar* q *such that* $\mathbf{x}_k = q\mathbf{1}$.

2. *If* $k \in \mathcal{I}$, *then almost surely there is no scalar* q *such that* $\mathbf{x}_k = q\mathbf{1}$.

The proof of Theorem 4.5 is in Appendix C.

5 Comparative Statics

So far, the sets $\mathcal{NI}$ and $\mathcal{I}$ have been fixed. In this section, we study how the equilibrium outcomes depend on the level of index investment. The next result is based on the observation that the $(n+1)$th element of $\mathbf{y}$, $y_{n+1} = \mathbf{g}' \sum_{k=1}^{m} \mathbf{s}_k$, is independent of the the partition of the group of investors into indexers and nonindexers.

Theorem 5.1. *Fix a joint realization of* $\mathbf{v}$ *and* $\mathbf{s}$. *Then,* p_f, r_f, *the price of the market portfolio, the return on the market portfolio, the capital market line, and the investors' welfare do not depend on the specific partition of investors into indexers and nonindexers.*

The proof of Theorem 5.1 is in Appendix C.

The above result sharpens the Index Investment Paradox. Theorem 4.5 shows that (i) the indexers' constraint to stay on the capital market line is binding, and (ii) if an indexer decides to become a nonindexer and we recompute equilibrium prices, the "new" position is still on the capital market line. Theorem 5.1 adds that the new position is the original position.

Figure 1 shows an example in which the realization of the signals is fixed. The figure depicts, in the volatility–return plane, the capital market line and the efficient frontiers for two different partitions of the set of investors. As stated in Theorem 5.1, the capital market line is the same in both examples. The figure shows the most common situation we find in the many simulations we tried: The efficient frontier that corresponds to a partition of the investors with a large set of indexers is nested in the efficient frontier that corresponds to a partition with a smaller set of indexers.

In Theorem 5.1, the realizations of the signals and payoffs were fixed. In contrast, in the remainder of this section, we study outcomes that depend solely on conditional covariance matrices, which are matrices of scalars. Even the conditional Sharpe ratio (Theorem 5.3), after algebraic simplifications, can be expressed in terms of scalars taken from covariance matrices. Thus, in the rest of this section, we need not assume that the realization of $\mathbf{v}$ or the realization of $\mathbf{s}$ is fixed. In particular, when we compare two partitions of the set of investors, one with $\mathcal{I}_1$ and the other with $\mathcal{I}_2$ such that $|\mathcal{I}_1| < |\mathcal{I}_2|$, we do not assume that $\mathcal{I}_1 \subset \mathcal{I}_2$ (the two sets may even be disjoint).

Theorem 5.2 (Conditional Variance of the Portfolio's Payoff). *Let* $\mathbf{x} \in R^n$ *be a portfolio, and consider changes as we increase* $|\mathcal{I}|$. *If* $\mathbf{x}$ *is a scalar multiplication of the market portfolio, then the conditional variance* $\operatorname{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})$ *does not change. For all other portfolios,* $\operatorname{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})$ *strictly increases.*

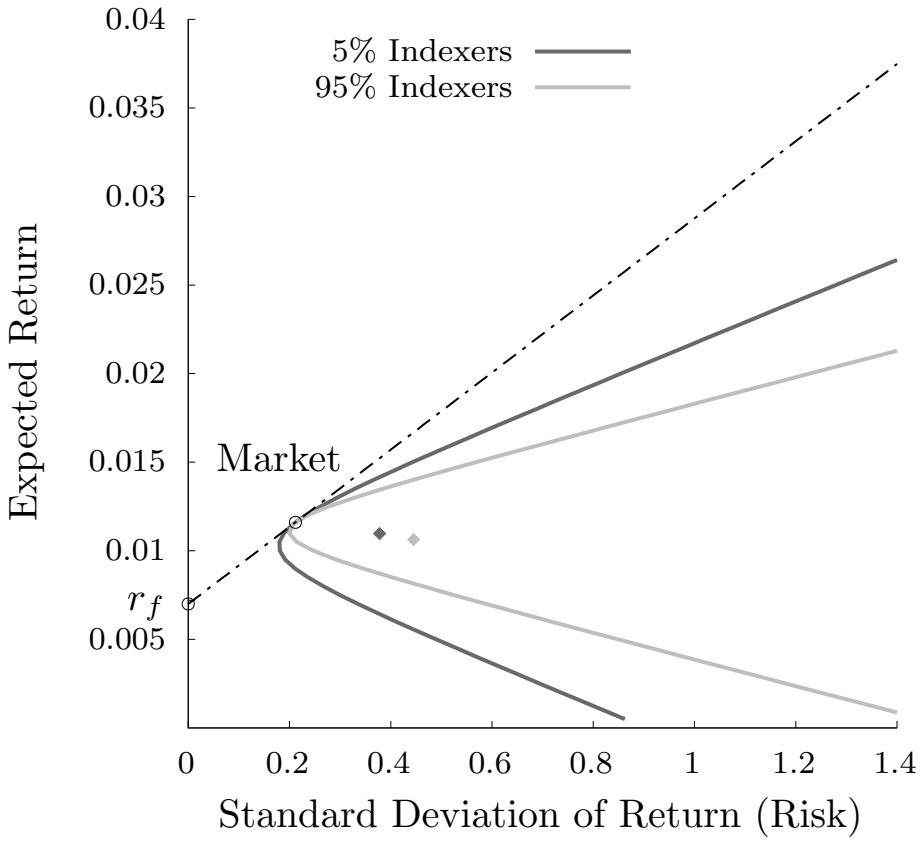


Figure 1: The efficient frontiers and the capital market line in the volatility–return plane. These are the realized efficient frontiers in an example with six risky assets and 10,000 investors. The capital market line and the position of the market portfolio (*tangency point*) are the same in both cases. The two points in the center of the frontiers stand for the same risky asset. The conditional Sharpe ratios are the slopes of the straight lines joining these points and the risk-free asset (*denoted by* r_f). These lines are not drawn, but it is apparent the Sharpe ratio is lower (the slope of the invisible line is gradual) when 95% of the investors are indexers.

The proof of Theorem 5.2 is in Appendix C.

For any portfolio with a return (i.e., a portfolio with a nonzero cost), we define the conditional Sharpe ratio to be

$$\frac{E\left[\frac{\mathbf{x}'\mathbf{v}}{\mathbf{x}'\mathbf{p}} - 1 \middle| \mathbf{y}\right] - r_f}{\sqrt{\text{var}\left(\frac{\mathbf{x}'\mathbf{v}}{\mathbf{x}'\mathbf{p}} - 1 \middle| \mathbf{y}\right)}}$$

Theorem 5.3 (Conditional Sharpe Ratio). *Let $\mathbf{x} \in R^n$ be a portfolio with a positive Sharpe ratio, and consider changes as we increase $|\mathcal{I}|$. If $\mathbf{x}$ is a scalar multiplication of the market portfolio, then the Sharpe ratio does not change. For all other portfolios, the Sharpe ratio strictly decreases.*

The proof of Theorem 5.3 is in Appendix C.

We can always write

$$r_i = r_f + \beta_i(r_{\text{mkt}} - r_f) + \epsilon \tag{13}$$

where

$$\epsilon \equiv r_i - r_f - \beta_i(r_{\text{mkt}} - r_f)$$

Because the conditional CAPM holds, we know that $E[\epsilon|\mathbf{y}] = 0$. From the definition of β_i (see Equation 7), it is immediately apparent that $E\left[(r_{\text{mkt}} - r_f)\epsilon \,|\, \mathbf{y}\right] = 0$. Thus, the theory of linear regression is applicable, and it is meaningful to use R^2 as a measure of the strength of the CAPM regression (Equation 13). In practice, one computes the sample R^2. We compute the population R^2. We now ask how R^2 depends on $|\mathcal{I}|$.

Theorem 5.4 (Conditional R^2). *For every asset i, R^2 of the CAPM relation decreases with $|\mathcal{I}|$.*

The proof of Theorem 5.4 is in Appendix C.

Fix i. Let $\mathbf{1}-\mathbf{e}_i$ be the portfolio that includes all assets except for asset i. Denote the payoff and return by v_{-i} and r_{-i}, respectively.

Theorem 5.5 (**Conditional Correlation in Returns**). *Assume* $\operatorname{corr}(r_i, r_{-i}|\mathbf{y}) > 0$. *Then, this correlation decreases with* $|\mathcal{I}|$.

The proof of Theorem 5.5 is in Appendix C.

Theorem 5.6 (**Unconditional Variance of Portfolio's Price**). *Let* $\mathbf{x} \in R^n$ *be a portfolio, and consider changes as we increase* $|\mathcal{I}|$. *If* $\mathbf{x}$ *is a scalar multiplication of the market portfolio, then* $\operatorname{var}(\mathbf{x}'\mathbf{p})$ *does not change. For all other portfolios,* $\operatorname{var}(\mathbf{x}'\mathbf{p})$ *strictly decreases.*

The proof of Theorem 5.6 is in Appendix C.

Theorem 5.7 (**Unconditional Correlation in Asset Prices**). *For every* i, *assume* $\operatorname{corr}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p}) > 0$. *Then, this correlation strictly increases with* $|\mathcal{I}|$.

Proof of Theorem 5.7. We have

$$\operatorname{corr}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p}) = \frac{\operatorname{cov}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p})}{\sqrt{\operatorname{var}(\mathbf{e}_i'\mathbf{p})}\sqrt{\operatorname{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{p})}}$$

To see that the denominator decreases and the numerator increases, we write

$$\operatorname{var}(p_{\text{mkt}}) = \operatorname{var}(\mathbf{e}_i'\mathbf{p}) + \operatorname{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{p}) + 2\operatorname{cov}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p}))$$

From Theorem 5.6, $\operatorname{var}(p_{\text{mkt}})$ does not change, whereas both $\operatorname{var}(\mathbf{e}_i'\mathbf{p})$ and $\operatorname{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{p})$ decrease. Hence, $\operatorname{cov}(\mathbf{e}_i'\mathbf{p}, (\mathbf{1}-\mathbf{e}_i)'\mathbf{p})$ increases.

■

Like in other conditional CAPM models, betas are realizations of random variables. In our model, the distributions of betas are ratios of Gaussian random variables, and therefore their moments do not exist. However, we can draw their densities. Figure 2 depicts a typical

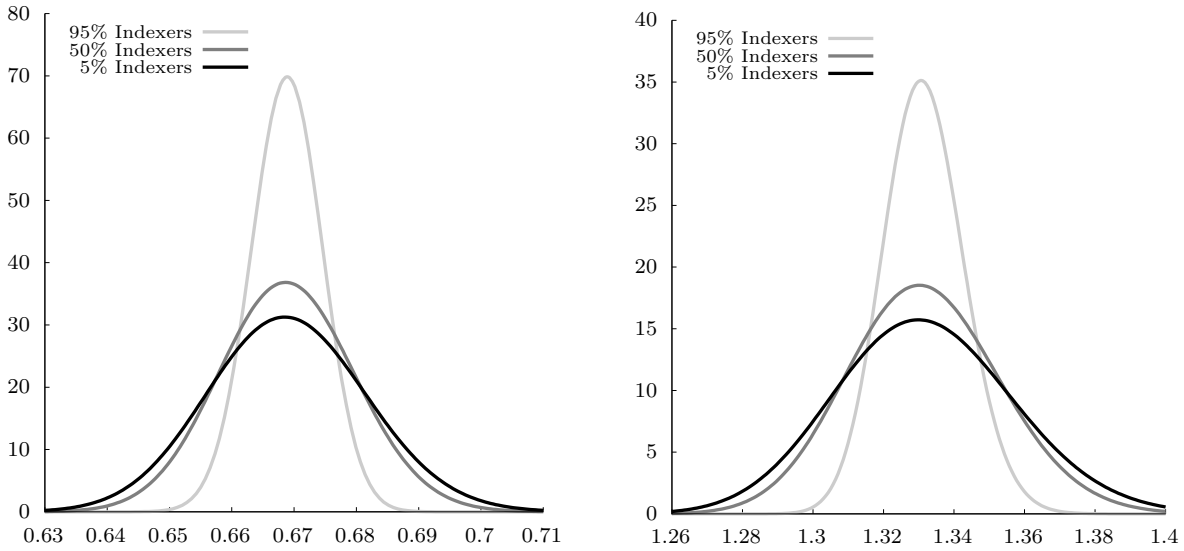


Figure 2: Densities of betas. These are the densities of betas in an example with two risky assets. The densities are shown for three different levels of index investment. The parameters in this example are $n = 2$, $m = 10,000$, an arbitrary ρ_0, $\rho_1 = 2$, $\boldsymbol{\mu}_{\mathbf{v}} = \left(\begin{smallmatrix}1000\\1000\end{smallmatrix}\right)$, $\boldsymbol{\Sigma}_{\mathbf{vv}} = \left(\begin{smallmatrix}1000 & 10\\10 & 2000\end{smallmatrix}\right)$, and $\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} = 1,000 \times \boldsymbol{\Sigma}_{\mathbf{vv}}$.

example. We see that (i) the dispersion of a beta decreases with $\mathcal{I}$, while (ii) the abscissa of the global maximum of its density is unaffected.

6 The Limiting Case

We have already pointed out that, regardless of what the set of index investors is, the concatenation of all signals, $\mathbf{s}$, satisfies (GR). Therefore, regardless of what the partition of investors into indexers and nonindexers is, there is always a fully revealing equilibrium. We think that in the presence of indexers, the fully revealing equilibrium is unappealing.

That said, the fully revealing equilibrium is the equilibrium that corresponds to the limiting case of the partially revealing equilibrium, in which all investors are nonindexers; that is, $|\mathcal{NI}| = m$. When all investors are nonindexers, the $(n+1)$th coordinate of $\mathbf{y}$ is a linear function of the first n coordinates. Therefore, $\mathbf{y}$ is degenerate, and its covariance matrix

is noninvertible. We can still use the artificial economy apparatus. We simply remove the redundant $(n+1)$th coordinate, and we are left with the nondegenerate n-dimensional vector, $\frac{1}{m}\sum_{k=1}^{m}\mathbf{s}_k$. This vector satisfies (GR) because a sample mean of a multivariate normal random vector with a known covariance matrix is a sufficient statistic for its mean. Moreover, this is a fully revealing equilibrium.[14] Therefore, the limiting case in which all investors are nonindexers is a fully revealing equilibrium.

When all investors are indexers, it is not clear how individual assets are priced. It is conceivable that there are infinitely many equilibria, all of which agree on the price of the market portfolio but disagree on the prices of individual assets. Our model can be used to pick one of these equilibria, the equilibrium that corresponds to the limiting case of our model in which all investors are indexers.

When all investors are indexers, the random vector $\mathbf{y}$ is not defined (because of the division by zero of the first n coordinates). It is natural to remove the first n coordinates altogether, and use only the $(n+1)$th coordinate. Indeed, $y_{n+1} = \mathbf{g}' \sum_{k=1}^{m}$ satisfies (GR) because, according to Theorem 4.3, this random variable is a sufficient statistic for $\mathbf{1}'\mathbf{v}$. Hence, the conditions in (GR) with regard to the set of indexers are satisfied. Moreover, the conditions in (GR) with regard to the set of nonindexers are trivially satisfied because the set of nonindexers is empty. Hence, the equilibrium in the artificial economy in which everyone observes only y_{n+1} is also a rational expectation equilibrium when all investors are indexers.

Figure 3 repeats the same example shown in Figure 1, except that we add the middle case (50% indexers) and the two extreme cases. It is apparent from the figures that even when 100% of the investors are indexers, the efficient frontier is not degenerated.

[14] The artificial economy in which everyone observes the sufficient statistic has the same outcomes as the artificial economy in which everyone observes $\mathbf{s}$ (Grossman, 1978).

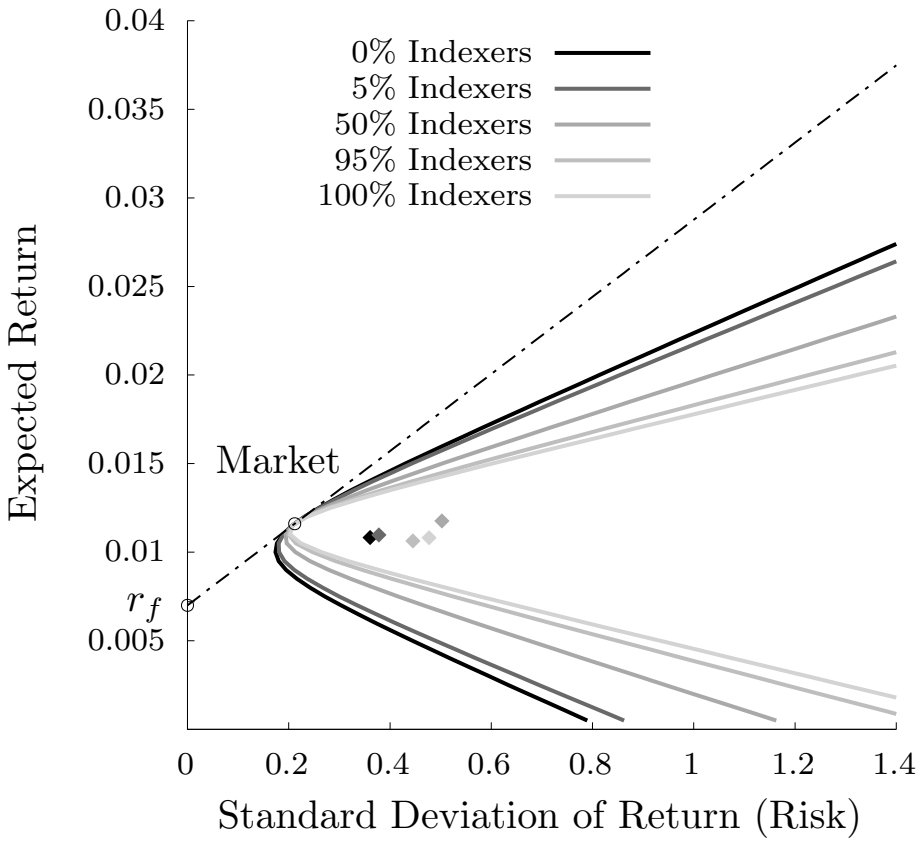


Figure 3: The efficient frontiers and the capital market line. This is a continuation of the example shown in Figure 1. Here, there are three more partitions of investors into indexers and nonindexers: 0% indexers, 50% indexers, and 100% indexers. These are the realized efficient frontiers in an example with six risky assets and 10,000 investors. The capital market line and the position of the market portfolio (*tangency point*) are the same in all cases. The five points in the center of the frontiers stand for the same risky asset. The conditional Sharpe ratios are the slopes of the straight lines joining these points and the risk-free asset (*denoted by* r_f). These lines are not drawn, but it is apparent the Sharpe ratio is lower (the slope of the invisible line is gradual) when there are more indexers.

7 Concluding Remarks

Markowitz (1952) studies mean-variance portfolio selection and discovers the efficient frontier. Tobin (1958) adds the risk free asset, discovers the separation theorem (Tobin 1958, page 84), and explains that "Markowitz's main interest is prescription of rules of rational behaviour for investors; [while] the main concern of [my] paper is the implications for economic theory, mainly comparative statics, that can be derived from assuming that investors do in fact follow [Markowitz's] rules" (Tobin 1958, page 85). We add costless private signals, and the main concern of our paper is the implications, mainly comparative statics, that can be derived from assuming that some investors follow Tobin's rule. In the same manner that Markowitz's rules are compatible with Tobin's extension, Tobin's rule is compatible with our extension: In our model, it is optimal for nonindex investors to index.

The model shows that index investment is not benign. As more non-index investors become index investors, the proportion of idiosyncratic risk to total risk increases, the R^2 of the CAPM regression decreases, comovement in returns decreases, comovement in asset prices increases. For any portfolio other than the market portfolio, the portfolio's Sharpe ratio decreases, and the variance of the portfolio's payoff increases. The following examples illustrate some of the implications of our model.

There is a known link between corporate underinvestment in real projects and total risk. Panousi and Papanikolaou (2012) find that the link is stronger when managers hold a large equity stake in the firm, and Deng, Chen, and Kong (2014) go even further and document that the link tends to be insignificant when managerial ownership is very low. Those empirical papers support the notion that when managers are exposed to firm's total risk, they are reluctant to invest in risky real projects. Our model shows that the larger the set of index investors is, the greater is the uncertainty about future value of individual assets. Thus, our model suggests a possible link between index investment and corporate underinvestment.

Similarly, the cost of financial hedging depends on the volatility of future payoff of the asset. In particular, our model suggests the corporate practice of awarding options to management is costlier in the presence of index investment.

Finally, a firm's ability to raise financing (bank loans, or bonds) is likely to depend on its stock price. Our model shows that the larger the set of index investors is, the greater is the comovement in pricing. It is therefore conceivable that in the presence of index investors, a large negative shock to one firm may make it difficult for another firm to raise the capital needed to invest in real projects or repay old debt. The former implies corporate underinvestment; the latter implies financial contagion.

Appendices

A Artificial Economies: Proofs

Proof of Theorem 3.1. As stated in the theorem, $\mathbf{f}$ is given in Equation 4; the price of the bond, p_f, is given in Equation 5; and $\mathbf{p}$ is given in Equation 6. Our goal is to show not only that these prices clear the markets but also that they are the only prices that can clear the markets.

In the artificial economy, the problem of the kth investor is

$$\begin{aligned}
&\max_{c,b,\mathbf{x}} E\left[U(c,b,\mathbf{x}'\mathbf{v})\,|\mathbf{y}\right], \text{ subject to } (c,b,\mathbf{x}) \in \mathcal{B}_k(p_f,\mathbf{p}) \\
&= \max_{c\in R,\mathbf{x}\in R^n} E\left[U\left(c,(\bar{c}_k-c)\frac{1}{p_f}+\bar{b}_k+(\bar{\mathbf{x}}_k-\mathbf{x})'\frac{1}{p_f}\mathbf{p},\mathbf{x}'\mathbf{v}\right)\middle|\,\mathbf{y}\right] \\
&= \max_{c\in R,\mathbf{x}\in R^n} -e^{-\rho_0 c} - e^{-\rho_1\left((\bar{c}_k-c)\frac{1}{p_f}+\bar{b}_k+(\bar{\mathbf{x}}_k-\mathbf{x})'\frac{1}{p_f}\mathbf{p}+\mathbf{x}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}-\frac{\rho_1}{2}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{x}\right)} \\
&= \max_{c\in R} -e^{-\rho_0 c} + e^{-\rho_1\left((\bar{c}_k-c)\frac{1}{p_f}+\bar{b}_k+\bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p}\right)} \times \max_{\mathbf{x}\in R^n} -e^{-\rho_1\left(\mathbf{x}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}-\frac{1}{p_f}\mathbf{p}\right)-\frac{\rho_1}{2}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{x}\right)}
\end{aligned} \tag{A.1}$$

We solve the maximization problem "backward." The first-order condition with respect to $\mathbf{x}$ is

$$-\frac{1}{p_f}\mathbf{p}+\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}-\frac{\rho_1}{2}\left(\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}+\boldsymbol{\Sigma}'_{\mathbf{vv}|\mathbf{y}}\right)\mathbf{x}=0$$

We replace $\mathbf{x}$ with $\mathbf{x}_k$ to emphasize that this is the optimal portfolio of the kth investor, and using the symmetry of the covariance matrix, we rearrange and obtain

$$\mathbf{x}_k=\frac{1}{\rho_1}\boldsymbol{\Sigma}^{-1}_{\mathbf{vv}|\mathbf{y}}\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}-\frac{1}{p_f}\mathbf{p}\right)$$

The right-hand side is independent of k; thus, all m investors hold the same portfolio. Therefore, the only way the market of risky assets clears is if $\mathbf{x}_k=\frac{1}{m}\mathbf{1}$. By plugging $\mathbf{x}_k=\frac{1}{m}\mathbf{1}$ back into the first-order condition, we conclude that whatever $\mathbf{p}$ and p_f are, their

ratio in equilibrium is uniquely defined:

$$\frac{1}{p_f}\mathbf{p} = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_1}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} \tag{A.2}$$

Inserting $\mathbf{x}_k = \frac{1}{m}\mathbf{1}$ back into the investors' problem (Equation A.1), we can write the problem of the kth investor as

$$\max_{c\in R} \; -e^{-\rho_0 c} - e^{-\rho_1\left((\bar{c}_k - c)\frac{1}{p_f} + \bar{b}_k + \bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p} + \frac{1}{m}\mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\rho_1}{2m^2}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)}$$

The first-order condition with respect to c is

$$\rho_0 e^{-\rho_0 c} = \frac{\rho_1}{p_f} e^{-\rho_1\left((\bar{c}_k - c)\frac{1}{p_f} + \bar{b}_k + \bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p} + \frac{1}{m}\mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\rho_1}{2m^2}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)}$$

We then take log on both sides, and replace c with c_k to emphasize that this is the optimal consumption of the kth investor:

$$\begin{aligned}\log(\rho_0) - \rho_0 c_k = {} & \log(\rho_1) - \log(p_f) \\ & - \rho_1\left((\bar{c}_k - c_k)\frac{1}{p_f} + \bar{b}_k + \bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p} + \frac{1}{m}\mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\rho_1}{2m^2}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)\end{aligned}$$

Adding up the m first-order conditions of all investors yields

$$\begin{aligned}m\log(\rho_0) - \rho_0\sum_{k=1}^{m} c_k = {} & m\log(\rho_1) - m\log(p_f) \\ & - \rho_1\left(\sum_{k=1}^{m}(\bar{c}_k - c_k)\frac{1}{p_f} + \sum_{k=1}^{m}\bar{b}_k + \sum_{k=1}^{m}\bar{\mathbf{x}}_k'\frac{1}{p_f}\mathbf{p} + \mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{1}{p_f}\mathbf{p}\right) - \frac{\rho_1}{2m}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)\end{aligned}$$

The aggregate endowment of bonds is zero, and the aggregate endowment of risky assets is the market portfolio; in other words, $\sum_{k=1}^{m}\bar{b}_k = 0$, and $\sum_{k=1}^{m}\bar{\mathbf{x}}_k = \mathbf{1}$. We use these two identities and Equation A.2 to conclude that in every equilibrium we must have

$$\begin{aligned}m\log(\rho_0) - \rho_0\sum_{k=1}^{m} c_k = {} & m\log(\rho_1) - m\log(p_f) \\ & - \rho_1\left(\sum_{k=1}^{m}(\bar{c}_k - c_k)\frac{1}{p_f} + \bar{\mathbf{1}}'\left(\boldsymbol{\mu}_{\boldsymbol{v}|\boldsymbol{y}} - \frac{\rho_1}{2m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)\right)\end{aligned} \tag{A.3}$$

We have two immediate goals. Our first goal is to show that there is a bond price that clears the market for the consumption good, and our second goal is to show that this is the only bond price that clears the market.

Let the bond price be given via

$$m\log(p_f) = m\log(\rho_1) - m\log(\rho_0) - \rho_1 \mathbf{1}' \left(\boldsymbol{\mu}_{\boldsymbol{v}|\boldsymbol{y}} - \frac{\rho_1}{2m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right) + \rho_0 \sum_{k=1}^{m} \bar{c}_k \tag{A.4}$$

Inserting this bond price into Equation A.3, we find that the sum of the m first-order conditions is

$$\rho_0 \sum_{k=1}^{m} \bar{c}_k - \rho_0 \sum_{k=1}^{m} c_k = -\rho_1 \sum_{k=1}^{m} (\bar{c}_k - c_k)\frac{1}{p_f}$$

Or

$$(\rho_1 + p_f \rho_0) \sum_{k=1}^{m} (\bar{c}_k - c_k) = 0$$

From Equation A.4 we know $(\rho_1 + p_f\rho_0) \neq 0$, so we conclude that $\sum_{k=1}^{m}(\bar{c}_k - c_k) = 0$. In other words, we conclude that the bond price given in Equation A.4 clears the market for the consumption good.

To see that this is the only bond price that clears the market, let us say there is another bond price, p_f', that clears the market; that is, $\sum c_k = \sum \bar{c}_k$. By inserting the clearing condition back into Equation A.3 and replacing p_f with p_f' to emphasize that this is the "other" equilibrium price, we obtain

$$m\log(\rho_0) - \rho_0 \sum_{k=1}^{m} \bar{c}_k = m\log(\rho_1) - m\log(p_f') - \rho_1 \mathbf{1}' \left(\boldsymbol{\mu}_{\boldsymbol{v}|\boldsymbol{y}} - \frac{\rho_1}{2m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)$$

But by comparing this identity to the definition of the equilibrium price in Equation A.4 we conclude that $p_f' = p_f$; thus, the equilibrium price is unique. To sum up, when p_f is given as in Equation A.4 and the ratio $\frac{1}{p_f}\mathbf{p}$ is given as in Equation A.2, then the market for the consumption good and the market for the risky assets clear. The budget constraint (Equation 1) implies that the bond market also clears (Walras's law).

To conclude the proof, we note that the equilibrium prices stated in the theorem (Equations 4, 5, and 6) are the same at the equilibrium identities we computed in the proof (Equations A.2 and A.4).

■

Proof of Theorem 3.2. Fix an investor. Because we fix the investor, we omit the subscript k.

Let c^*, b^*, and x^* be the investor's decisions in equilibrium. From Theorem 3.1, we know that the investor holds the market. In other words, there is a scalar q^* such that $x^* = q^*\mathbf{1}$.

Define $w^* = b^* p_f + q^* \mathbf{1}'\mathbf{p}$, where p_f and $\mathbf{p}$ are the equilibrium prices in the artificial economy. We invoke a calculus of variation-type argument. Instead of looking at the full-blown problem, we restrict our attention to a subclass of feasible allocations that includes the optimal one. Specifically, let us say that the investor contemplates consuming the optimal c^* and investing a fraction α of w^* in the market portfolio, a fraction κ of w^* in asset i, and the remaining $(1-\alpha-\kappa)w^*$ in bonds. In other words, the allocation the investor contemplates is to buy $w^*(1-\alpha-\kappa)/p_f$ bonds, a fraction $\frac{w^*\alpha}{\mathbf{1}'\mathbf{p}}$ of the market portfolio $\mathbf{1}$, and an additional $\frac{w^*\kappa}{p_i}$ shares of asset i. The optimal fraction invested in asset i must satisfy $\kappa = 0$.

We can write the investor's problem as follows:

$$
\begin{aligned}
&\max_{c,b,\mathbf{x}} E\left[U(c,b,\mathbf{x}'\mathbf{v})\,|\mathbf{y}\right], \text{ subject to } (c,b,\mathbf{x}) \in \mathcal{B}(p_f,\mathbf{p}) \\
=&\max_{b,\mathbf{x}\in R^n} E\left[U\left(c^*,b,\mathbf{x}'\mathbf{v}\right)|\,\mathbf{y}\right], \text{ subject to } bp_f + \mathbf{x}'\mathbf{p} = w^* \\
=&-e^{-\rho_0 c^*} + \max_{b,\mathbf{x}\in R^n} -E\left[e^{-\rho_1\left(b+\mathbf{x}'\mathbf{v}\right)}\middle|\,\mathbf{y}\right], \text{ subject to } bp_f + \mathbf{x}'\mathbf{p} = w^* \\
=&-e^{-\rho_0 c^*} + \max_{\alpha,\kappa} -E\left[\exp\left(-\rho_1 w^*\left(\frac{1-\alpha-\kappa}{p_f} + \frac{\alpha}{\mathbf{1}'\mathbf{p}}\mathbf{1}'\mathbf{v} + \frac{\kappa}{p_i}v_i\right)\right)\middle|\,\mathbf{y}\right]
\end{aligned}
$$

Thus, the maximization problem is equivalent to

$$\max_{\alpha,\kappa}(1-\alpha-\kappa)(1+r_f)+\alpha(1+E[r_{\text{mkt}}|\mathbf{y}])+\kappa(1+E[r_i|\mathbf{y}]) \\ -\frac{\rho_1 w^*}{2}\left(\alpha^2 \operatorname{var}(r_{\text{mkt}}|\,\mathbf{y})+2\alpha\kappa \operatorname{cov}(r_{\text{mkt}},r_i|\,\mathbf{y})+\kappa^2 \operatorname{var}(r_i|\,\mathbf{y})\right)$$

Taking the first-order condition with respect to α, and evaluating at $\kappa = 0$, yields

$$E[r_{\text{mkt}}|\mathbf{y}]-r_f-\alpha\rho_1 w^* \operatorname{var}(r_{\text{mkt}}|\,\mathbf{y})=0 \quad\longrightarrow\quad \alpha\rho_1 w^* = \frac{E[r_{\text{mkt}}|\mathbf{y}]-r_f}{\operatorname{var}(r_{\text{mkt}}|\,\mathbf{y})}$$

Taking the first-order condition with respect to κ, and evaluating at $\kappa = 0$, yields

$$E[r_i|\mathbf{y}]-r_f-\alpha\rho_1 w^* \operatorname{cov}(r_{\text{mkt}},r_i|\,\mathbf{y})=0$$

Combining both conditions, we obtain

$$E[r_i|\mathbf{y}]-r_f-\beta_i(E[r_{\text{mkt}}|\mathbf{y}]-r_f)=0$$

■

Proof of Lemma 3.3. In this proof, we repeatedly use the Doob–Dynkin lemma. The pair $(\mathbf{p}, p_f)$ is defined as a measurable mapping of $\mathbf{f}$. Indeed, p_f, given in Equation 5, is a measurable function of $\mathbf{f}$. So $\mathbf{p}$, given in Equation 6, is also a function of $\mathbf{f}$. Therefore, $\sigma(\mathbf{p}, p_f) \subseteq \sigma(\mathbf{f})$. The reverse is also true: Given the pair $(\mathbf{p}, p_f)$, we have $\mathbf{f} = \frac{1}{p_f}\mathbf{p}$. Therefore, $\sigma(\mathbf{p}, p_f) \supseteq \sigma(\mathbf{f})$, and we conclude that $\sigma(\mathbf{p}, p_f) = \sigma(\mathbf{f})$. Next, Equation 4 implies $\sigma(\mathbf{f}) = \sigma(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}})$. Finally, from the definition of conditional expectation, we know that $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ is a measurable function of $\mathbf{y}$, and hence $\sigma(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}) \subseteq \sigma(\mathbf{y})$.

■

Proof of Lemma 3.4. For part 1, we have

$$E[\mathbf{v}|\mathcal{G}] \underset{\mathcal{G}\subseteq\mathcal{F}}{=} E[E[\mathbf{v}|\mathcal{F}]|\mathcal{G}] \underset{E[\mathbf{v}|\mathcal{F}]\in\mathcal{G}}{=} E[\mathbf{v}|\mathcal{F}]$$

For part 2, we use the result from part 1:

$$\begin{aligned}\operatorname{var}(\mathbf{v}|\mathcal{G}) &\underset{\text{def.}}{=} E\big[\,(\mathbf{v}-E[\mathbf{v}|\mathcal{G}])\,(\mathbf{v}-E[\mathbf{v}|\mathcal{G}])'\,\big|\mathcal{G}\big]\\ &\underset{\text{part 1}}{=} E\big[\,(\mathbf{v}-E[\mathbf{v}|\mathcal{F}])\,(\mathbf{v}-E[\mathbf{v}|\mathcal{F}])'\,\big|\mathcal{G}\big]\\ &\underset{\mathcal{G}\subseteq\mathcal{F}}{=} E\Big[E\big[\,(\mathbf{v}-E[\mathbf{v}|\mathcal{F}])\,(\mathbf{v}-E[\mathbf{v}|\mathcal{F}])'\,\big|\mathcal{F}\big]\Big|\mathcal{G}\Big]\\ &\underset{\text{def.}}{=} E\big[\operatorname{var}(\mathbf{v}|\mathcal{F})\big|\mathcal{G}\big] \underset{\operatorname{var}(\mathbf{v}|\mathcal{F})\in\mathcal{G}}{=} \operatorname{var}(\mathbf{v}|\mathcal{F})\end{aligned}$$

■

Proof of Corollary 3.5. We study conditional distributions, where the conditioning is on prices (and, in parts 2 and 3, also on signals). Lemma 3.3 implies that whenever we condition on prices, we can condition instead on $\mathbf{f}$, which is Gaussian. Thus, all the conditional distributions stated in the corollary are indeed Gaussian. We now verify that the means and the covariance matrices are as stated in each part of the corollary.

For part 1, we have $\sigma(p_f,\mathbf{p})\subseteq\sigma(\mathbf{y})$ and $E[\mathbf{v}|\mathbf{y}]=\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}\in\sigma(p_f,\mathbf{p})$. Thus, we can apply part 1 of Lemma 3.4 to conclude that $E[\mathbf{v}|p_f,\mathbf{p}]=\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$. Next, we have $\operatorname{var}(\mathbf{v}|\mathbf{y})=\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$, which is nonrandom. Thus, we can apply part 2 of Lemma 3.4 to conclude that $\operatorname{var}(\mathbf{v}|p_f,\mathbf{p})=\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$. This concludes the proof of part 1.

For part 2, let $k\in\mathcal{NI}$. We have $\sigma(\mathbf{s}_k,p_f,\mathbf{p})\subseteq\sigma(\mathbf{s}_k,\mathbf{y})$. According to (GR), $E[\mathbf{v}|\mathbf{s}_k,\mathbf{y}]=\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$, which is measurable with respect to $\sigma(\mathbf{s}_k,p_f,\mathbf{p})$. Thus, we can apply part 1 of Lemma 3.4 to conclude that $E[\mathbf{v}|\mathbf{s}_k,p_f,\mathbf{p}]=\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$.

Next, according to (GR), $\operatorname{var}(\mathbf{v}|\mathbf{s}_k,\mathbf{y})=\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$, which is nonrandom. Thus, we can apply part 2 of Lemma 3.4 to conclude that $\operatorname{var}(\mathbf{v}|\mathbf{s}_k,p_f,\mathbf{p})=\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$. This concludes the proof of Equation 9.

For part 3, let $k\in\mathcal{I}$. We have $\sigma(\mathbf{s}_k,p_f,\mathbf{p})\subseteq\sigma(\mathbf{s}_k,\mathbf{y})$. According to (GR), $E[\mathbf{1}'\mathbf{v}|\mathbf{s}_k,\mathbf{y}]=\mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$, which is measurable with respect to $\sigma(\mathbf{s}_k,p_f,\mathbf{p})$. Thus, we can apply part 1 of Lemma

3.4 to conclude that $E[\mathbf{1}'\mathbf{v}|\mathbf{s}_k, p_f, \mathbf{p}] = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$.

Finally, according to (GR), $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}_k, \mathbf{y}) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$, which is nonrandom. Thus, we can apply part 2 of Lemma 3.4 to conclude that $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}_k, p_f, \mathbf{p}) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$. This concludes the proof of Equation 10.

■

B Proof of Theorem 4.1

Before we prove the theorem, we need some preliminary results.

Define the auxiliary matrix as follows:

$$\mathbf{M} := \begin{bmatrix} \mathbf{I}_{n\times n} & |\mathcal{NI}|\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}}^{-1}\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g}) \\ \mathbf{0}_{n\times n} & -|\mathcal{NI}|\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}}^{-1}\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g}) + m\mathbf{g} \end{bmatrix}_{2n\times(n+1)} \tag{B.1}$$

We can now write

$$\boldsymbol{\Sigma}_{\mathbf{vy}} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} & m\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{g} \end{bmatrix} \tag{B.2}$$

$$= \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix} \mathbf{M} \tag{B.3}$$

Also,

$$\forall k \in \mathcal{NI}, \quad \text{cov}\left(\mathbf{y}, \mathbf{s}_k\right) = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} + \dfrac{1}{|\mathcal{NI}|}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} \\ \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix}_{(n+1)\times n} \tag{B.4}$$

$$\forall k \in \mathcal{I} \quad \text{cov}\left(\mathbf{y}, \mathbf{s}_k\right) = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} \\ \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix}_{(n+1)\times n} \tag{B.5}$$

So, for any $nind \in \mathcal{NI}$ and $ind \in \mathcal{I}$, we have

$$\mathbf{1}' \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{1\times n} & 1_{1\times 1} \end{bmatrix} \begin{bmatrix} \text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right) \end{bmatrix} \tag{B.6}$$

and

$$\begin{bmatrix}\text{cov}(\mathbf{y}, \mathbf{s}_{nind}) & \text{cov}(\mathbf{y}, \mathbf{s}_{ind})\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{vv}} + \frac{1}{|\mathcal{NI}|}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \boldsymbol{\Sigma}_{\mathbf{vv}} \\ \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} & \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}\end{bmatrix}_{(n+1)\times 2n}$$

The latter implies that

$$\boldsymbol{\Sigma}_{\mathbf{yy}} = \begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{vv}} + \frac{1}{|\mathcal{NI}|}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1} \\ \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} & m\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{g}\end{bmatrix} = \begin{bmatrix}\text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right)\end{bmatrix}\mathbf{M} \tag{B.7}$$

Lemma B.1. *Let* $\mathbf{x} \in R^n$ *be an arbitrary portfolio. By means of matching terms, define* $\mathbf{q} \in R^n$ *and the scalar* q *to be*

$$\begin{bmatrix}\mathbf{q}' & q\end{bmatrix}_{1\times(n+1)} := \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1} \tag{B.8}$$

Then, $\mathbf{q} = \mathbf{0}_{n\times 1}$ *if and only if* $\mathbf{x}$ *is a scalar multiplier of* $\mathbf{1}$ *with* $\mathbf{x} = q\mathbf{1}$.

Proof of Lemma B.1. Let $\mathbf{x}$ be arbitrary, and multiply both sides of Equation B.8 by $\boldsymbol{\Sigma}_{\mathbf{yy}}$ on the right:

$$\begin{bmatrix}\mathbf{q}' & q\end{bmatrix}\boldsymbol{\Sigma}_{\mathbf{yy}} = \mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vy}}$$

We use Equations B.2 and B.7 to obtain

$$\begin{bmatrix}\mathbf{q}'\left(\boldsymbol{\Sigma}_{\mathbf{vv}} + \frac{1}{|\mathcal{NI}|}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\right) + q\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}} & \mathbf{q}'|\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1} + mq\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{g}\end{bmatrix}_{1\times(n+1)} = \begin{bmatrix}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}} & m\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{g}\end{bmatrix}_{1\times(n+1)}$$

Matching terms makes it clear that $\mathbf{q} = \mathbf{0}$ implies $\mathbf{x} = q\mathbf{1}$, and hence $\mathbf{x}$ is a scalar multiplier of $\mathbf{1}$. We need to show that the opposite is also true. Let us say that $\mathbf{x} = q\mathbf{1}$, and assume by means of contradiction that $\mathbf{q} \neq \mathbf{0}$. Matching terms must yield

$$\mathbf{q}'\left(\boldsymbol{\Sigma}_{\mathbf{vv}} + \frac{1}{|\mathcal{NI}|}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\right) = \mathbf{0}_{1\times n}$$

Multiplying on the right by $\mathbf{q}$ and noticing that $\left(\mathbf{\Sigma_{vv}} + \frac{1}{|\mathcal{NI}|}\mathbf{\Sigma_{\epsilon\epsilon}}\right)$ is a positive definite matrix, we obtain a contradiction:

$$0 < \mathbf{q}'\left(\mathbf{\Sigma_{vv}} + \frac{1}{|\mathcal{NI}|}\mathbf{\Sigma_{\epsilon\epsilon}}\right)\mathbf{q} = \mathbf{0}_{1\times n}\mathbf{q} = 0$$

■

Applying Lemma B.1 to $\mathbf{x} = \mathbf{1}$, we obtain

$$\begin{bmatrix}\mathbf{0}_{1\times n} & 1\end{bmatrix} = \mathbf{1}'\mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}} \tag{B.9}$$

We define the n-dimensional random vector as follows:

$$\mathbf{z} := \mathbf{v} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{y} \tag{B.10}$$

Lemma B.2. *We have*

$$E\mathbf{z} = \boldsymbol{\mu}_\mathbf{v} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\boldsymbol{\mu}_\mathbf{y} \tag{B.11}$$

$$\text{var}(\mathbf{z}) = \mathbf{\Sigma_{vv|y}} \tag{B.12}$$

Proof of Lemma B.2. Equation B.11 follows from the definition of $\mathbf{z}$. As for Equation B.12, we have

$$\begin{aligned}
\text{var}(\mathbf{z}) &= \text{var}(\mathbf{v} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{y}) \\
&= \mathbf{\Sigma_{vv}} + \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{\Sigma_{yy}}\Sigma_{\mathbf{yy}}^{-1}\mathbf{\Sigma_{yv}} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{\Sigma_{yv}} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{\Sigma_{yv}} \\
&= \mathbf{\Sigma_{vv}} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{\Sigma_{yv}} \\
&\underset{(3)}{=} \mathbf{\Sigma_{vv|y}}
\end{aligned}$$

■

Lemma B.3. *We have*

1. $\text{cov}(\mathbf{z}, \mathbf{y}) = \mathbf{0}_{n\times(n+1)}$.

2. $\forall k \in \mathcal{NI}$, $\text{cov}(\mathbf{z}, \mathbf{s}_k) = \mathbf{0}_{n\times n}$.

3. $\forall k \in \mathcal{I}$, $\text{cov}(\mathbf{1}'\mathbf{z}, \mathbf{s}_k) = \mathbf{0}_{1\times n}$.

Proof of Lemma B.3. For part 1 of the lemma,

$$\text{cov}(\mathbf{z}, \mathbf{y}) = \text{cov}(\mathbf{v} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\mathbf{y}, \mathbf{y}) = \mathbf{\Sigma_{vy}} - \mathbf{\Sigma_{vy}} = 0$$

For part 2 of the lemma, it is convenient to compute an $n \times 2n$ covariance matrix. For any $nind \in \mathcal{NI}$ and any $ind \in \mathcal{I}$, we have

$$\begin{bmatrix}\text{cov}\left(\mathbf{z}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{z}, \mathbf{s}_{ind}\right)\end{bmatrix} = \begin{bmatrix}\text{cov}\left(\mathbf{v}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{v}, \mathbf{s}_{ind}\right)\end{bmatrix} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\begin{bmatrix}\text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right)\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{\Sigma_{vv}} & \mathbf{\Sigma_{vv}}\end{bmatrix} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\begin{bmatrix}\text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right)\end{bmatrix}$$

Multiply both sides of the equations on the right by $\mathbf{M}$:

$$\begin{aligned}\begin{bmatrix}\text{cov}\left(\mathbf{z}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{z}, \mathbf{s}_{ind}\right)\end{bmatrix}\mathbf{M} &= \underbrace{\begin{bmatrix}\mathbf{\Sigma_{vv}} & \mathbf{\Sigma_{vv}}\end{bmatrix}\mathbf{M}}_{\underset{(B.3)}{=}\mathbf{\Sigma_{vy}}} - \mathbf{\Sigma_{vy}}\mathbf{\Sigma_{yy}^{-1}}\underbrace{\begin{bmatrix}\text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) & \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right)\end{bmatrix})\mathbf{M}}_{\underset{(B.7)}{=}\mathbf{\Sigma_{yy}}} \\ &= \mathbf{\Sigma_{vy}} - \mathbf{\Sigma_{vy}} \\ &= \mathbf{0}_{n\times(n+1)}\end{aligned}$$

In submatrix notation, we can write the above, using the definition of $\mathbf{M}$ from Equation B.1, as

$$\begin{bmatrix}\text{cov}(\mathbf{z}, \mathbf{s}_{nind}) & \mathbf{t}\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1}\end{bmatrix}$$

where $\mathbf{t}$ is some vector.[15] This proves that $\text{cov}(\mathbf{z}, \mathbf{s}_{nind}) = \mathbf{0}$.[16]

[15] Specifically,

$$\mathbf{t} = \left(\text{cov}(\mathbf{z}, \mathbf{s}_{nind}) - \text{cov}(\mathbf{z}, \mathbf{s}_{ind})\right)|\mathcal{NI}|\mathbf{\Sigma_{\epsilon\epsilon}^{-1}}\mathbf{\Sigma_{vv}}(\mathbf{1} - m\mathbf{g}) + m\,\text{cov}(\mathbf{z}, \mathbf{s}_{ind})\mathbf{g}$$

[16] Although it does not prove, and it is not true, that $\text{cov}(\mathbf{z}, \mathbf{s}_{ind}) = \mathbf{0}$.

Next, for part 3 of the lemma, we need to prove that $\text{cov}(\mathbf{1}'\mathbf{z}, \mathbf{s}_{ind}) = \mathbf{0}_{1\times n}$. Again, to facilitate linear algebra manipulations, it is convenient to compute a vector of dimension $2n$:

$$\begin{aligned}
&\left[\text{cov}\left(\mathbf{1}'\mathbf{z}, \mathbf{s}_{nind}\right) \quad \text{cov}\left(\mathbf{1}'\mathbf{z}, \mathbf{s}_{ind}\right)\right] = \mathbf{1}'\left[\text{cov}\left(\mathbf{z}, \mathbf{s}_{nind}\right) \quad \text{cov}\left(\mathbf{z}, \mathbf{s}_{ind}\right)\right] \\
&= \mathbf{1}'\left[\text{cov}\left(\mathbf{v}, \mathbf{s}_{nind}\right) \quad \text{cov}\left(\mathbf{v}, \mathbf{s}_{ind}\right)\right] - \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\left[\text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) \quad \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right)\right] \\
&= \mathbf{1}'\left[\boldsymbol{\Sigma}_{\mathbf{vv}} \quad \boldsymbol{\Sigma}_{\mathbf{vv}}\right] - \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\left[\text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) \quad \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right)\right] \\
&\underset{\text{(B.9)}}{=} \mathbf{1}'\left[\boldsymbol{\Sigma}_{\mathbf{vv}} \quad \boldsymbol{\Sigma}_{\mathbf{vv}}\right] - \left[\mathbf{0}_{1\times n} \quad 1_{1\times 1}\right]\left[\text{cov}\left(\mathbf{y}, \mathbf{s}_{nind}\right) \quad \text{cov}\left(\mathbf{y}, \mathbf{s}_{ind}\right)\right] \\
&\underset{\text{(B.6)}}{=} \mathbf{0}_{1\times 2n}
\end{aligned}$$

■

Lemma B.4. *For all $k \in \mathcal{NI}$, we have*

$$E\left[\mathbf{v}\left|\mathbf{y}, \mathbf{s}_k\right.\right] = E[\mathbf{v}|\mathbf{y}]$$
$$\text{var}\left(\mathbf{v}\middle|\mathbf{y}, \mathbf{s}_k\right) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$$

Proof of Lemma B.4. Let $k \in \mathcal{NI}$. Part 1 of Lemma B.3 states that $\text{cov}(\mathbf{z}, \mathbf{y}) = \mathbf{0}$, and part 2 states that $\text{cov}(\mathbf{z}, \mathbf{s}_k) = \mathbf{0}$. Therefore, $\text{cov}\left(\mathbf{z}, \begin{bmatrix}\mathbf{y} \\ \mathbf{s}_k\end{bmatrix}\right) = \mathbf{0}_{n\times(2n+1)}$.

Thus,

$$E\left[\mathbf{z}\left|\mathbf{y}, \mathbf{s}_k\right.\right] = E\mathbf{z} \tag{B.13}$$
$$\text{var}\left(\mathbf{z}\middle|\mathbf{y}, \mathbf{s}_k\right) = \text{var}(\mathbf{z}) \tag{B.14}$$

We have

$$E\left[\mathbf{v}\left|\mathbf{y}, \mathbf{s}_k\right.\right] \underset{\text{(B.10)}}{=} E\left[\mathbf{z} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y}\left|\mathbf{y}, \mathbf{s}_k\right.\right] \underset{\text{(B.13)}}{=} E\mathbf{z} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y} \underset{\text{(B.11)}}{=} \boldsymbol{\mu}_{\mathbf{v}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}) \underset{(2)}{=} E[\mathbf{v}|\mathbf{y}] \tag{B.15}$$

Next,

$$\text{var}\left(\mathbf{v}\left|\mathbf{y}, \mathbf{s}_k\right.\right) \underset{\text{(B.10)}}{=} \text{var}\left(\mathbf{z} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y}\left|\mathbf{y}, \mathbf{s}_k\right.\right) = \text{var}\left(\mathbf{z}\left|\mathbf{y}, \mathbf{s}_k\right.\right) \underset{\text{(B.14)}}{=} \text{var}(\mathbf{z}) \underset{\text{(B.12)}}{=} \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$$

■

Lemma B.5. *For all $k \in \mathcal{I}$, we have*

$$E\left[\mathbf{1}'\mathbf{v}\,|\mathbf{y}, \mathbf{s}_k\right] = \mathbf{1}'E[\mathbf{v}|\mathbf{y}]$$
$$\text{var}\left(\mathbf{1}'\mathbf{v}\middle|\mathbf{y}, \mathbf{s}_k\right) = \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$$

Proof of Lemma B.5. Let $k \in \mathcal{I}$. Part 1 of Lemma B.3 implies that $\text{cov}(\mathbf{1}'\mathbf{z}, \mathbf{y}) = \mathbf{0}$, and part 3 states that $\text{cov}(\mathbf{1}'\mathbf{z}, \mathbf{s}_k) = \mathbf{0}$. Therefore, $\text{cov}\left(\mathbf{1}'\mathbf{z}, \begin{bmatrix}\mathbf{y}\\ \mathbf{s}_k\end{bmatrix}\right) = \mathbf{0}_{1\times(2n+1)}$.

Thus,

$$E\left[\mathbf{1}'\mathbf{z}\,|\mathbf{y}, \mathbf{s}_k\right] = E\mathbf{1}'\mathbf{z} \tag{B.16}$$
$$\text{var}\left(\mathbf{1}'\mathbf{z}\middle|\mathbf{y}, \mathbf{s}_k\right) = \text{var}(\mathbf{1}'\mathbf{z}) \tag{B.17}$$

Now,

$$\begin{aligned} E\left[\mathbf{1}'\mathbf{v}\,|\mathbf{y}, \mathbf{s}_k\right] &\underset{(B.10)}{=} E\left[\mathbf{1}'\mathbf{z}\,|\mathbf{y}, \mathbf{s}_k\right] + \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y} \underset{(B.16)}{=} E\mathbf{1}'\mathbf{z} + \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y} \\ &= \mathbf{1}'\left(\boldsymbol{\mu}_{\mathbf{v}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}})\right) \underset{(2)}{=} \mathbf{1}'E[\mathbf{v}|\mathbf{y}] \end{aligned} \tag{B.18}$$

Next,

$$\text{var}\left(\mathbf{1}'\mathbf{v}\middle|\mathbf{y}, \mathbf{s}_k\right) \underset{(B.10)}{=} \text{var}\left(\mathbf{1}'\mathbf{z}+\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{y}\middle|\mathbf{y}, \mathbf{s}_k\right) = \text{var}\left(\mathbf{1}'\mathbf{z}\middle|\mathbf{y}, \mathbf{s}_k\right) \underset{(B.17)}{=} \text{var}(\mathbf{1}'\mathbf{z}) \underset{(B.12)}{=} \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$$

■

Proof of Theorem 4.1. The proof follows from the fact that Lemmas B.5 and B.4 show that the four conditions in (GR) are satisfied.

■

C Additional Lemmas and Proofs

Lemma C.1. *We have*

$$\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} = \boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g}) \tag{C.1}$$

$$\mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k - \boldsymbol{\mu}_{\mathbf{v}}) \tag{C.2}$$

$$\mathbf{1}'\mathbf{f} = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k - \boldsymbol{\mu}_{\mathbf{v}}) - \frac{\rho_1}{m}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g}) \tag{C.3}$$

Proof of Lemma C.1. To prove Equation C.1, we have

$$\begin{aligned}
\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} &\underset{(3)}{=} \boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1} - \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\boldsymbol{\Sigma}_{\mathbf{yv}}\mathbf{1} \\
&\underset{(B.9)}{=} \boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1} - \boldsymbol{\Sigma}_{\mathbf{vy}}\begin{bmatrix}\mathbf{0}_{n\times 1}\\ 1_{1\times 1}\end{bmatrix} \\
&\underset{(B.2)}{=} \boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{1} - m\boldsymbol{\Sigma}_{\mathbf{vv}}\mathbf{g} \\
&= \boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g})
\end{aligned}$$

This proves Equation C.1. To prove Equation C.2, we have

$$\begin{aligned}
\mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} &\underset{(2)}{=} \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}) \\
&\underset{(B.9)}{=} \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \begin{bmatrix}\mathbf{0}_{1\times n} & 1_{1\times 1}\end{bmatrix}(\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}}) \\
&\underset{(12)}{=} \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k - \boldsymbol{\mu}_{\mathbf{v}})
\end{aligned}$$

This proves Equation C.2.

Now, we use Equation C.1 to re-express Equation 4, which yields the following equation.

$$\mathbf{f} = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_1}{m}\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g})$$

We multiply the above by $\mathbf{1}'$ from the left, and we use Equation C.2 to obtain Equation C.3.

■

Proof of Corollary 4.2. This corollary adds only one additional property to Lemma 3.3: The last set inequality in Lemma 3.3 is an equality. Therefore, to prove the corollary, it is sufficient to show that $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ reveals $\mathbf{y}$.

Equation C.2 implies that $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ reveals the $(n+1)$th coordinate of $\mathbf{y}$. Indeed, rearrenging Equation C.2, we have $y_{n+1} = \mathbf{1}'(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \boldsymbol{\mu}_{\mathbf{v}}) + m\mathbf{g}'\boldsymbol{\mu}_{\mathbf{v}}$.

Taking advantage of the fact that $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ reveals the $(n+1)$th coordinate of $\mathbf{y}$, we rewrite Equation 2:

$$\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} = \boldsymbol{\mu}_{\mathbf{v}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\begin{bmatrix} y_1 - Ey_1 \\ \vdots \\ y_n - Ey_n \\ y_{n+1} - Ey_{n+1} \end{bmatrix} = \boldsymbol{\mu}_{\mathbf{v}} + \boldsymbol{\Sigma}_{\mathbf{vy}}\boldsymbol{\Sigma}_{\mathbf{yy}}^{-1}\begin{bmatrix} y_1 - Ev_1 \\ \vdots \\ y_n - Ev_n \\ \mathbf{1}'(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \boldsymbol{\mu}_{\mathbf{v}}) \end{bmatrix}$$

To obtain the first n coordinates of $\mathbf{y}$, we solve the above system of n linear equations; the unknowns are the first n coordinates of $\mathbf{y}$. Thus, $\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$ reveals all $n+1$ coordinates of $\mathbf{y}$.

■

Proof of Theorem 4.3. Because $\mathbf{s}$ and $\mathbf{1}'\mathbf{v}$ are jointly normally distributed, the conditional distribution of $\mathbf{1}'\mathbf{v}$, given $\mathbf{s}$, is normal with mean $E[\mathbf{1}'\mathbf{v}|\mathbf{s}]$ and variance $\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s})$.

We have

$$\boldsymbol{\Sigma}_{\mathbf{ss}} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \boldsymbol{\Sigma}_{\mathbf{vv}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}} \\ \boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}} \\ \vdots & & \ddots & \vdots \\ \boldsymbol{\Sigma}_{\mathbf{vv}} & \boldsymbol{\Sigma}_{\mathbf{vv}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}} \end{bmatrix}_{nm\times nm}$$

Thus,

$$\begin{bmatrix} \mathbf{I}_{n\times n} & \cdots & \mathbf{I}_{n\times n} \end{bmatrix}_{n\times nm} = \begin{bmatrix} (m\boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}})^{-1} & \cdots & (m\boldsymbol{\Sigma}_{\mathbf{vv}} + \boldsymbol{\Sigma}_{\boldsymbol{\epsilon\epsilon}})^{-1} \end{bmatrix}_{n\times nm} \boldsymbol{\Sigma}_{\mathbf{ss}} \tag{C.4}$$

We also have

$$\boldsymbol{\Sigma}_{\mathbf{vs}} = \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{vv}} & \cdots & \boldsymbol{\Sigma}_{\mathbf{vv}} \end{bmatrix}$$

Therefore, by multiplying each side of Equation C.4 on the right by $\mathbf{\Sigma}_{\mathbf{ss}}^{-1}$ and on the left by $\mathbf{\Sigma}_{\mathbf{vv}}$, we obtain

$$\mathbf{\Sigma}_{\mathbf{vs}}\mathbf{\Sigma}_{\mathbf{ss}}^{-1} = \begin{bmatrix}\mathbf{\Sigma}_{\mathbf{vv}}\left(m\mathbf{\Sigma}_{\mathbf{vv}} + \mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\right)^{-1} & \cdots & \mathbf{\Sigma}_{\mathbf{vv}}\left(m\mathbf{\Sigma}_{\mathbf{vv}} + \mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\right)^{-1}\end{bmatrix}_{n\times nm}$$

Multiplying both sides of the above equation on the left by $\mathbf{1}'$, and recalling the definition of $\mathbf{g}$ (see Equation 11), we conclude that

$$\mathbf{1}'\mathbf{\Sigma}_{\mathbf{vs}}\mathbf{\Sigma}_{\mathbf{ss}}^{-1} = \begin{bmatrix}\mathbf{g}' & \cdots & \mathbf{g}'\end{bmatrix}_{1\times nm}$$

Thus,

$$\begin{aligned} E[\mathbf{1}'\mathbf{v}|\mathbf{s}] &= \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{1}'\mathbf{\Sigma}_{\mathbf{vs}}\mathbf{\Sigma}_{\mathbf{ss}}^{-1}(\mathbf{s} - E\mathbf{s}) = \mathbf{1}'\boldsymbol{\mu}_{\mathbf{v}} + \mathbf{g}'\sum_{k=1}^{m}(\mathbf{s}_k - \boldsymbol{\mu}_{\mathbf{v}}) \\ &= (\mathbf{1}' - m\mathbf{g}')\boldsymbol{\mu}_{\mathbf{v}} + y_{n+1} \end{aligned} \tag{C.5}$$

depends on $\mathbf{s}$ only through y_{n+1}. Therefore, applying part 1 of Lemma 3.4, we conclude that

$$E[\mathbf{1}'\mathbf{v}|\mathbf{s}] = E[\mathbf{1}'\mathbf{v}|y_{n+1}]$$

Next, we note that

$$\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s})$$

is merely a scalar;[17] therefore, we can apply part 2 of Lemma 3.4 to conclude that

$$\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}) = \text{var}(\mathbf{1}'\mathbf{v}|y_{n+1})$$

To prove that y_{n+1} is a sufficient statistic, we note that the density of a normal random variable depends only on the mean and variance. Therefore, from part 1 of the lemma, we conclude that the conditional density satisfies $f(\mathbf{1}'\mathbf{v}|\mathbf{s}) = f(\mathbf{1}'\mathbf{v}|y_{n+1})$. In our model, the "data," $\mathbf{s}$; the "statistic," y_{n+1}; and the "parameter," $\mathbf{1}'\mathbf{v}$, have a known joint distribution. Thus, we use Bayes's rule (see Theorem 2.2.1 of Geweke 2005) to conclude that y_{n+1} is a sufficient statistic for $\mathbf{1}'\mathbf{v}$.

[17] Specifically,

$$\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{s}) = \mathbf{1}'\mathbf{\Sigma}_{\mathbf{vv}}\mathbf{1} - \mathbf{1}'\mathbf{\Sigma}_{\mathbf{vs}}\mathbf{\Sigma}_{\mathbf{ss}}^{-1}\mathbf{\Sigma}_{\mathbf{sv}}\mathbf{1} = \mathbf{1}'\mathbf{\Sigma}_{\mathbf{vv}}\mathbf{1} - m\mathbf{g}'\mathbf{\Sigma}_{\mathbf{vv}}\mathbf{1}$$

Now, let $t(\mathbf{s})$ be another arbitrary sufficient statistic. To show that y_{n+1} is minimal, we need to show that y_{n+1} is a measurable function of $t(\mathbf{s})$.[18] From Bayes's rule (again, see Theorem 2.2.1 of Geweke 2005), it follows that $f(\mathbf{1}'\mathbf{v}|\mathbf{s}) = f(\mathbf{1}'\mathbf{v}|t(\mathbf{s}))$. Thus,

$$E[\mathbf{1}'\mathbf{v}|t(\mathbf{s})] = E[\mathbf{1}'\mathbf{v}|\mathbf{s}] \underset{(C.5)}{=} (\mathbf{1}' - m\mathbf{g}')\boldsymbol{\mu}_{\mathbf{v}} + y_{n+1}$$

In other words,

$$y_{n+1} = E[\mathbf{1}'\mathbf{v}|t(\mathbf{s})] - (\mathbf{1}' - m\mathbf{g}')\boldsymbol{\mu}_{\mathbf{v}}$$

which shows that y_{n+1} is a measurable function of $t(\mathbf{s})$, and hence minimal.

■

Proof of Theorem 4.4. From Equation C.3, we have

$$\mathbf{1}'\mathbf{f} = y_{n+1} + (\mathbf{1}' - m\mathbf{g}')\boldsymbol{\mu}_{\mathbf{v}} - \frac{\rho_1}{m}\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1} - m\mathbf{g})$$

Thus, there is a one-to-one mapping between $\mathbf{1}'\mathbf{f}$ and y_{n+1}. As a result, $\sigma(\mathbf{1}'\mathbf{f}) = \sigma(y_{n+1})$. The proof follows from Theorem 4.3.

■

Proof of Theorem 4.5. Note that in Corollary 4.2 we show that $\sigma(\mathbf{p}, p_f) = \sigma(\mathbf{y})$, so $\sigma(\mathbf{p}, p_f, \mathbf{s}_k) = \sigma(\mathbf{y}, \mathbf{s}_k)$. The first-order condition implies

$$\begin{aligned}\mathbf{x}_k &= \frac{1}{\rho_1}\operatorname{var}^{-1}(\mathbf{v}|\mathbf{s}_k, \mathbf{y})(E[\mathbf{v}|\mathbf{s}_k, \mathbf{y}] - \mathbf{f}) \\ &= \frac{1}{m}\operatorname{var}^{-1}(\mathbf{v}|\mathbf{s}_k, \mathbf{y})\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} + \frac{1}{\rho_1}\operatorname{var}^{-1}(\mathbf{v}|\mathbf{s}_k, \mathbf{y})(E[\mathbf{v}|\mathbf{s}_k, \mathbf{y}] - \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}})\end{aligned}$$

Consider $k \in \mathcal{NI}$. Because $\boldsymbol{y}$ satisfies (GR), we know that $\operatorname{var}(\mathbf{v}|\mathbf{s}_k, \mathbf{y}) = \boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}$ and $E[\mathbf{v}|\mathbf{s}_k, \mathbf{y}] = \boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}$. Thus, we can insert those into the first-order condition to conclude that $\mathbf{x}_k = \frac{1}{m}\mathbf{1}$. Therefore, the nonzero scalar in part 1 is $1/m$.[19]

[18] The intuition that underlies the formal definition of the minimality of a sufficient statistic is as follows. If y_{n+1} is a measurable function of $t(\mathbf{s})$, then $y_{n+1} \in \sigma(t(\mathbf{s}))$. Therefore, $\sigma(y_{n+1}) \subseteq \sigma(t(\mathbf{s}))$, so y_{n+1} contains less (not necessarily in a strict sense) information than $t(\mathbf{s})$.

[19] This is, of course, simply reiterating a result we already proved in Theorem 3.1, namely that in the artificial economy everyone's equilibrium portfolio is $\frac{1}{m}\mathbf{1}$.

Consider $k \in \mathcal{I}$. We know that $E[\mathbf{v}|\mathbf{y}, \mathbf{s}_k]$ depends on $\boldsymbol{s}_k$.[20] This means that $\mathbf{x}_k$ also depends on the realization of $\mathbf{s}_k$; thus, with probability one, $\mathbf{x}_k$ is not a scalar multiplier of $\mathbf{1}$.

■

Proof of Theorem 5.1. The price of the bond, $p_f = 1/(1 + r_f)$, is given in Equation 5. Two terms in Equation 5 can cause p_f to depend on how we divide the investors into index and nonindex investors. The first term is $\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$. Equation C.1 shows that this term depends on $\boldsymbol{\Sigma}_{\mathbf{vv}}$ and $\mathbf{g}$, which do not depend on the specific partition. The second term is $\mathbf{1}'\mathbf{f}$. According to Equation C.3, $\mathbf{1}'\mathbf{f}$ does not depend on the specific partition, so we conclude that p_f and r_f also do not depend on the specific partition.

The price of the market portfolio is $\mathbf{1}'\mathbf{p} \underset{(6)}{=} \dfrac{\mathbf{1}'\mathbf{f}}{1 + r_f}$. Therefore, the price of the market portfolio is also independent of the specific partition. Because payoffs are exogenous, and the price is independent, we conclude that the return on the market portfolio is independent of the specific partition. In particular, the conditional expected return on the market portfolio and the conditional variance of return on the market portfolio must be independent of the specific partition, so the capital market line remains unchanged as we change $\mathcal{I}$. But now, because investors have mean–variance preferences, the welfare of each investor is also independent of the specific partition.

■

Proof of Theorem 5.2. We use the following standard notation. If $\mathbf{A} = \left[a_{ij}\right]$ is an arbitrary matrix with elements that depend on a parameter π, then $\dfrac{\partial \mathbf{A}}{\partial \pi}$ is the matrix $\left[\dfrac{\partial a_{ij}}{\partial \pi}\right]$. If $\mathbf{B}$ is an arbitrary matrix with elements that do not depend on the parameter π, then it is well known that

[20] The point that the conditional expectation, whenever $k \in \mathcal{I}$, depends on $\mathbf{s}_k$ is so obvious that we decided not to mobilize matrix algebra to explicitly compute this conditional expectation.

$$\frac{\partial \mathbf{B}\mathbf{A}^{-1}\mathbf{B}'}{\partial \pi} = -\mathbf{B}\mathbf{A}^{-1}\frac{\partial \mathbf{A}}{\partial \pi}\mathbf{A}^{-1}\mathbf{B}' \tag{C.6}$$

For a qualitative comparative statics, any monotonic transformation of $|\mathcal{I}|$ is a measure of the level of index investment. It is convenient to use as a measure of index investment

$$\pi \equiv \frac{1}{m - |\mathcal{I}|} = \frac{1}{|\mathcal{NI}|}$$

Although π takes values only on a discrete set of numbers, we can nevertheless differentiate $\mathbf{x}'\mathbf{\Sigma}_{\mathbf{vv|y}}\mathbf{x}$ with respect to π.

From Equation 3, we have

$$\mathbf{\Sigma}_{\mathbf{vv|y}} = \mathbf{\Sigma}_{\mathbf{vv}} - \mathbf{\Sigma}_{\mathbf{vy}}\mathbf{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{\Sigma}_{\mathbf{yv}}$$

where $\mathbf{\Sigma}_{\mathbf{vv}}$ and $\mathbf{\Sigma}_{\mathbf{vy}} = \begin{bmatrix}\mathbf{\Sigma}_{\mathbf{vv}} & m\mathbf{\Sigma}_{\mathbf{vv}}\mathbf{g}\end{bmatrix}$ are independent of π. We use Equation C.6 with $A = \mathbf{\Sigma}_{\mathbf{yy}}$ and $B = \mathbf{\Sigma}_{\mathbf{vy}}$.

Equation B.7 shows that

$$\frac{\partial \mathbf{\Sigma}_{\mathbf{yy}}}{\partial \pi} = \begin{bmatrix}\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}} & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0_{1\times 1}\end{bmatrix} \tag{C.7}$$

Finally, by matching terms, we define the vector $\mathbf{q}$ and the scalar q such that $\begin{bmatrix}\mathbf{q}', q\end{bmatrix} = \mathbf{x}'\mathbf{\Sigma}_{\mathbf{vy}}\mathbf{\Sigma}_{\mathbf{yy}}^{-1}$.

We have

$$\frac{\partial \operatorname{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})}{\partial \pi} = \frac{\partial \mathbf{x}'\mathbf{\Sigma}_{\mathbf{vv|y}}\mathbf{x}}{\partial \pi} = \mathbf{x}'\mathbf{\Sigma}_{\mathbf{vy}}\mathbf{\Sigma}_{\mathbf{yy}}^{-1}\frac{\partial \mathbf{\Sigma}_{\mathbf{yy}}}{\partial \pi}\mathbf{\Sigma}_{\mathbf{yy}}^{-1}\mathbf{\Sigma}_{\mathbf{yv}}\mathbf{x} = \mathbf{q}'\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}\mathbf{q} \geq 0$$

where the inequality arises because $\mathbf{\Sigma}_{\boldsymbol{\epsilon\epsilon}}$ is positive definite. The inequality is strict whenever $\mathbf{q} \neq \mathbf{0}$. According to Lemma B.1, $\mathbf{q} = \mathbf{0}$ if and only if $\mathbf{x}$ is a scalar multiplication of $\mathbf{1}$.

■

Proof of Theorem 5.3. The numerator of the Sharpe ratio is

$$E\left[\frac{\mathbf{x}'\mathbf{v}}{\mathbf{x}'\mathbf{p}}-1\middle|\mathbf{y}\right]-r_f=\frac{1}{\mathbf{x}'\mathbf{p}}\left(\mathbf{x}'\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}}-\mathbf{x}'\mathbf{p}(1+r_f)\right)\underset{(6)}{=}\frac{1}{\mathbf{x}'\mathbf{p}}\frac{\rho_1}{m}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\underset{(C.1)}{=}\frac{1}{\mathbf{x}'\mathbf{p}}\frac{\rho_1}{m}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1}-m\mathbf{g})$$

The denominator of the Sharpe ratio is

$$\sqrt{\text{var}\left(\frac{\mathbf{x}'\mathbf{v}}{\mathbf{x}'\mathbf{p}}-1\middle|\mathbf{y}\right)}=\frac{1}{\mathbf{x}'\mathbf{p}}\sqrt{\text{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})}$$

Therefore, the Sharpe ratio equals

$$\frac{\frac{\rho_1}{m}\mathbf{x}'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1}-m\mathbf{g})}{\sqrt{\text{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})}}$$

and its numerator is a scalar that is independent of the sets $\mathcal{NI}$ and $\mathcal{I}$. The numerator must be positive because we assume the Sharpe ratio is positive. Theorem 5.2 states that as long as $\mathbf{x}$ is not a scalar multiplication of the market portfolio, the denominator increases as $|\mathcal{I}|$ increases.

■

Proof of Theorem 5.4. We have

$$\begin{aligned}R^2&=\text{corr}^2(r_i,r_f+\beta_i(r_{\text{mkt}}-r_f)|\mathbf{y})=\frac{\text{cov}^2(r_i,\beta_i r_{\text{mkt}}|\mathbf{y})}{\text{var}(r_i|\mathbf{y})\times\text{var}(\beta_i r_{\text{mkt}}|\mathbf{y})}\\&=\frac{\text{cov}^2(v_i,\mathbf{1}'\mathbf{v}|\mathbf{y})}{\text{var}(v_i|\mathbf{y})\times\text{var}(\mathbf{1}'\mathbf{v}|\mathbf{y})}=\frac{\left(\mathbf{e}_i'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}\right)^2}{\text{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})\times\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}}\\&\underset{(C.1)}{=}\frac{\left(\mathbf{e}_i'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1}-m\mathbf{g})\right)^2}{\text{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})\times\mathbf{1}'\boldsymbol{\Sigma}_{\mathbf{vv}}(\mathbf{1}-m\mathbf{g})}\end{aligned}$$

where $\mathbf{e}_i$ is the vector with one in the ith coordinate and zero elsewhere.

Because R^2 is positive and the only term that depends on the level of index investment is $\text{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})$, whether or not R^2 decreases depends on whether or not $\text{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})$ increases. We apply Theorem 5.2 with $\mathbf{x}=\mathbf{e}_i$ to conclude that this conditional variance increases with $|\mathcal{I}|$. Therefore, R^2 of the CAPM regression decreases with $|\mathcal{I}|$.

■

Proof of Theorem 5.5. We have

$$\operatorname{corr}(r_i, r_{-i}|\mathbf{y}) = \frac{\operatorname{cov}(r_i, r_{-i}|\mathbf{y})}{\sqrt{\operatorname{var}(r_i|\mathbf{y})}\sqrt{\operatorname{var}(r_{-i}|\mathbf{y})}} = \frac{\operatorname{cov}(v_i, v_{-i}|\mathbf{y})}{\sqrt{\operatorname{var}(v_i|\mathbf{y})}\sqrt{\operatorname{var}(v_{-i}|\mathbf{y})}}$$

$$= \frac{\mathbf{e}_i'\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1} - \operatorname{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})}{\sqrt{\operatorname{var}(\mathbf{e}_i'\mathbf{v}|\mathbf{y})}\sqrt{\operatorname{var}((\mathbf{1}-\mathbf{e}_i)'\mathbf{v}|\mathbf{y})}}$$

According to Equation C.1, the first term in the numerator is independent of the level of index investment. From Theorem 5.2, applied to the portfolio $\mathbf{e}_i$, we conclude that the numerator decreases as we increase $|\mathcal{I}|$. We also know that the denominator is positive, and Theorem 5.2 (applied to $\mathbf{e}_i$, then separately applied to $\mathbf{1}-\mathbf{e}_i$) implies that the denominator is increasing. In summary, we can formally write the correlation as

$$\frac{N(\mathcal{I})}{D(\mathcal{I})}$$

where we have shown that $N(\mathcal{I})$ decreases with $|\mathcal{I}|$ and that $D(\mathcal{I})$ is positive and increases with $|\mathcal{I}|$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be such that $|\mathcal{I}_1| < |\mathcal{I}_2|$. We have

$$\frac{N(\mathcal{I}_1)}{D(\mathcal{I}_1)} \underset{\substack{\uparrow \\ 0 < D(\mathcal{I}_1) < D(\mathcal{I}_2) \\ 0 < N(\mathcal{I}_1)}}{>} \frac{N(\mathcal{I}_1)}{D(\mathcal{I}_2)} \underset{\substack{\uparrow \\ N(\mathcal{I}_1) > N(\mathcal{I}_2)}}{>} \frac{N(\mathcal{I}_2)}{D(\mathcal{I}_2)}$$

which shows that the correlation decreases.

■

Proof of Theorem 5.6. Let $\mathbf{x} \in R^n$. Our goal is to examine $\operatorname{var}(\mathbf{x}'\mathbf{p})$.

We have

$$\begin{aligned} \operatorname{var}(\mathbf{x}'\mathbf{p}) &\underset{(6)}{=} \operatorname{var}(p_f\mathbf{x}'\mathbf{f}) = \operatorname{var}(p_f\mathbf{x}'(\boldsymbol{\mu}_{\mathbf{v}|\mathbf{y}} - \frac{\rho_1}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})) \\ &= \operatorname{var}\left(E[p_f\mathbf{x}'(\mathbf{v} - \frac{\rho_1}{m}\boldsymbol{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})|\mathbf{y}]\right) \end{aligned} \tag{C.8}$$

The law of total variance, applied to the random variable $p_f \mathbf{x}'(\mathbf{v} - \frac{\rho_1}{m}\mathbf{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})$, states that

$$\begin{aligned}
\operatorname{var}\left(p_f \mathbf{x}'(\mathbf{v} - \frac{\rho_1}{m}\mathbf{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})\right) &= E \operatorname{var}\left(p_f \mathbf{x}'(\mathbf{v} - \frac{\rho_1}{m}\mathbf{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})\middle|\, \mathbf{y}\right) + \operatorname{var}\left(E[p_f \mathbf{x}'(\mathbf{v} - \frac{\rho_1}{m}\mathbf{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})|\mathbf{y}]\right) \\
&\underset{(C.8)}{=} E \operatorname{var}\left(p_f \mathbf{x}'(\mathbf{v} - \frac{\rho_1}{m}\mathbf{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})\middle|\, \mathbf{y}\right) + \operatorname{var}(\mathbf{x}'\mathbf{p}) \\
&= E p_f^2 \operatorname{var}\left(\mathbf{x}'(\mathbf{v} - \frac{\rho_1}{m}\mathbf{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1})\middle|\, \mathbf{y}\right) + \operatorname{var}(\mathbf{x}'\mathbf{p}) \\
&= E p_f^2 \operatorname{var}\left(\mathbf{x}'\mathbf{v}|\, \mathbf{y}\right) + \operatorname{var}(\mathbf{x}'\mathbf{p}) \\
&= \operatorname{var}\left(\mathbf{x}'\mathbf{v}|\, \mathbf{y}\right) E p_f^2 + \operatorname{var}(\mathbf{x}'\mathbf{p})
\end{aligned}$$

where the last equality arises because $\operatorname{var}\left(\mathbf{x}'\mathbf{v}|\, \mathbf{y}\right)$ is a scalar.

Now, the left-hand side, $\operatorname{var}(p_f \mathbf{x}'(\mathbf{v} - \frac{\rho_1}{m}\mathbf{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}))$, is independent of the specific partition of investors into indexers and nonindexers because each element inside the variance operator is independent. Indeed, p_f is independent (see Theorem 5.1), $\mathbf{x}$ is an arbitrary vector of scalars, $\mathbf{v}$ is exogenous and hence independent, and finally $\mathbf{\Sigma}_{\mathbf{vv}|\mathbf{y}}\mathbf{1}$ is independent because the right-hand side of Equation C.1 is independent of the specific partition.

Also, $E p_f^2$ is positive and independent of the level of index investment. Therefore, $\operatorname{var}(\mathbf{x}'\mathbf{p})$ decreases as we increase $|\mathcal{I}|$ if and only if $\operatorname{var}(\mathbf{x}'\mathbf{v}|\mathbf{y})$ increases as we increase $|\mathcal{I}|$. Thus, the proof follows from Theorem 5.2.

■

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