Expectations, Stagnation and Fiscal Policy*

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Abstract

Some observers have argued that stagnation may become the new norm. We examine this possibility in a New Keynesian model with agents forming expectations using adaptive learning, and consider the role of fiscal policy in this context. We also impose inflation and consumption lower bounds, which can be relevant when agents are pessimistic. When the economy is near the targeted steady state output multipliers are front-loaded under learning compared to rational expectations. When initial expectations are pessimistic, the economy can sink into a stagnation trap. A large fiscal stimulus is then needed to avoid or emerge from stagnation.

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1 Introduction

We study the impact of government spending increases in a New Keynesian (NK) model when agents make forecasts using adaptive learning. Earlier

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work has shown that the fiscal policy effects under adaptive learning can sometimes be significantly different from those based on the assumption of rational expectations (RE). In undertaking this study it is clearly crucial to take into account both the monetary policy regime and the state of the economy. For example, under RE it has been shown that government spending multipliers are generally much larger when interest rates are fixed, as they are at the zero lower bound (ZLB). Furthermore, in practice the ZLB has arisen primarily in economies that have undergone severe recessions, as in the Great Recession.

There has also recently been a renewed interest in the possibility of the economy becoming stuck for long periods in a distinct stagnation state associated with the ZLB. If the stagnation state is due primarily to pessimism on the part of agents, this raises the question of whether policy can return the economy to a better steady state associated with output, inflation and interest rates at their normal levels. In particular, can fiscal policy prevent the economy from converging to stagnation, and if the economy has settled into stagnation, with deflation and interest rates at the ZLB, can fiscal policy dislodge the economy from stagnation and return it to the steady state targeted by monetary policy?

We are not the first to consider these issues. With interest rates close to zero and monetary policy seemingly proving ineffective to tackle the effects of the Great Recession, governments naturally turned their attention to fiscal measures to combat the severe recession. These measures in turn have led to renewed interest in fiscal policy and a fairly voluminous recent literature in which the discussion has often been conducted in terms of the magnitude of the multiplier; see for instance Hall (2009), Barro and Redlick (2011), Ramey (2011b), Ramey (2011a), Leeper, Traum, and Walker (2011), Coenen et al. (2012), and Ravn, Schmitt-Grohe, and Uribe (2012). There has also recently been interest in the possibility of the economy becoming stuck for long periods in a distinct stagnation state; see, for example, Summers (2013), Evans (2013), Benigno and Fornaro (2015) and Schaal and Taschereau-Dumouchel (2015).

A growing literature has reconsidered their effects owing to the relatively large fiscal stimuli adopted in various countries in the aftermath of the Great Recession. For example, Christiano, Eichenbaum, and Rebelo (2011), Corsetti, Kuester, Meier, and Muller (2010) and Woodford (2011) demonstrate the effectiveness of fiscal policy in models with monetary policy when the zero lower bound on nominal interest rate is reached. For a contrary

view see Mertens and Ravn (2014). Most of this literature explicitly makes the assumption of rational expectations (RE). The adaptive learning (AL) literature has shown that quite different results can arise both in NK and Real Business Cycle models; see Evans, Guse, and Honkapohja (2008), Benhabib, Evans, and Honkapohja (2014), Mitra, Evans, and Honkapohja (2013), Gasteiger and Zhang (2014) and Mitra, Evans, and Honkapohja (2015).

In Evans, Guse, and Honkapohja (2008) and Benhabib, Evans, and Honkapohja (2014) AL was introduced into the NK model with a ZLB creating two steady states: the targeted steady state and the unintended steady state arising from the ZLB. Those papers showed that while the unintended steady state is not locally stable under learning, it is on the edge of a deflation trap region in which inflation and output fall without bound. In the current paper we introduce lower bounds to inflation and consumption into a NK model. We think such bounds are both plausible and more consistent with observed data. Depending on the magnitude of the inflation lower bound there are then one or three steady states. The critical level is a deflation rate equal to the discount rate. If the inflation lower bound is higher than this critical rate then the deflation trap region cannot be reached and the targeted steady state is unique. However, if the inflation bound is below this critical rate, then there are three steady states. We will show that in this case there is a stagnation steady state, at the inflation lower bound, which is locally stable under learning. This stagnation or "trap" steady state can have very low output accompanied by moderate deflation.

Adaptive learning plays a central role in the economy when there are multiple steady states. Even in the absence of shocks, under RE the solution is indeterminate, as originally emphasized by Benhabib, Schmitt-Grohe, and Uribe (2001b). From the RE viewpoint this appears to suggest that convergence to the unintended low inflation steady state may be quite likely. Furthermore, as emphasized by Bullard (2010), the recent inflation and interest-rate data for Japan and the US appear to be consistent with convergence to the unintended deflation steady state. However, note that because the level of output is about the same between the targeted and the unintended steady state, it is difficult to use view convergence to the unintended steady state as consistent with deep recession or stagnation.

Adaptive learning resolves the indeterminacy issue in the sense that, given initial expectations and the learning rule, the time path of the economy is pinned down. AL explains how deep recessions accompanied by deflation can emerge and points to the possibility of deflationary stagnation. Consider

the case in which there are three steady states. We will show that the usual targeted steady state is locally stable under learning, i.e. there is convergence to the intended steady state under learning from nearby initial expectations, and indeed the basin of attraction is quite large. In contrast, the unintended steady state emphasized by Benhabib, Schmitt-Grohe, and Uribe (2001b) is not locally stable under learning: for nearby initial expectations there will either be convergence to the intended steady state or expectations will evolve toward lower expected inflation and output. Expectations then are driven down to the stagnation steady state, which is also locally stable under learning.[1]

Of course exogenous shocks also influence the economy and it is possible for these to drive the economy to the ZLB and to low output levels. Exogenous discount rate or credit-spread shocks have been emphasized in the RE literature on the ZLB and macroeconomic policy. See Eggertsson and Woodford (2003), Christiano, Eichenbaum, and Rebelo (2011), Corsetti, Kuester, Meier, and Muller (2010) and Woodford (2011) These shocks are often assumed to follow a two-state Markov process in which the credit-spread shock disappears each period with a fixed probability, with the aggregate output and inflation recovering as soon as the exogenous shock stops operating.

This approach, however, which is sometimes developed using a linear approximation around the intended steady state with the ZLB appended, does not do justice to the multiplicities issue raised by Benhabib, Schmitt-Grohe, and Uribe (2001b) and faced in the adaptive learning literature cited.[2] The key point of the latter literature is that the low output and inflation during the period of exogenous discount rate or credit shocks, may have made agents generally more pessimistic about the future, and that these pessimistic expectations may well continue for a time after the exogenous shocks have ceased. In effect expectations have been re-initialized by the severe recession and if these expectations are sufficiently pessimistic then they may have taken

[1] In his August 13, 2015 conference speech "Neo-Fisherianism" (available on the FRB St. Louis website) at the University of Oregon Conference on "Expectations in Dynamic Macroeconomic Models", James Bullard suggests that instability of the unintended low steady state under learning goes against the empirical evidence of major developed economies, in recent times, which are close to the ZLB and the unintended inflation steady state. In contrast we find that this instability can lead to convergence to a stagnation steady state at the ZLB with approximately the same inflation rate, but significantly lower output, which better matches the recent experience of major economies.

[2] See the beginning of Section 4 for further comparison of the RE and AL approaches.

the economy out of the basin of attraction of the targeted steady state. To emphasize this issue, when we take up fiscal policy at the ZLB in Section 5 we will abstract from credit-spread shocks on the assumption that any such shocks have already dissipated. Our focus will be on whether fiscal policy can then shift the pessimistic expectations sufficiently to avoid convergence to the stagnation steady state.

The AL approach used in the current paper is implemented as follows. Because we consider temporary changes in fiscal policy like the stimulus measures adopted in practice in recent recessions, we use the general infinite-horizon approach advocated by Preston (2005) and Eusepi and Preston (2010), but modified for policy changes as discussed in Evans, Honkapohja, and Mitra (2009) and Mitra, Evans, and Honkapohja (2013). Agents are assumed to incorporate the announced path of future government spending and taxes into their intertemporal budget constraint, and thus take into account the known direct impact of the policy. At the same time, agents are assumed not to know the general equilibrium effects of the temporary change in fiscal policy, and to use adaptive learning to forecast future values of output and inflation. Under AL agents update each period their estimates of the coefficients in their forecast model, and the evolution of these parameters over time modulates the impact of fiscal policy under learning vis-a-vis the impacts under rational expectations.

As mentioned above, we also explicitly impose inflation and consumption lower bounds, which can be relevant when expectations are pessimistic. The inflation lower bound is motivated by empirical experience that finds a smaller reduction in inflation rates at very low levels of output than would have been expected from the standard NK Phillips curve. See for example Ball and Mazumder (2011) and IMF (2013). We also introduce a consumption lower bound that would plausibly arise when consumption is substantially below the level corresponding to the usual steady state. Although in normal times the inflation and consumption lower bounds are not relevant, they play an important role during times of deep recession.

The structure of our paper is as follows. In Section 2 we present the Rotemberg adjustment-cost version of the NK model when the ZLB and other lower bounds are not applicable. In this setting we obtain the household and firm decision rules, the temporary equilibrium equations, and the updating rules for agents' forecast rules. In Section 3 we then compare fiscal policy under RE and AL in normal times. The main result is that the overall size of the output multipliers for government spending under AL and RE are about

the same, but the AL multipliers are front-loaded, i.e. the bulk of the impact on output occurs during the early part of the stimulus, whereas under RE the output effect is greater near the end of the stimulus.

In Section 4 we modify the model to include lower bounds for interest-rates, inflation and consumption. Because monetary policy is assumed transparent, household forecasts of future interest rates must allow for the possibility of the ZLB binding. This Section obtains the key existence and learning stability results for the different steady states, in the model with lower bounds, demonstrating in particular the possibility of a stagnation steady state, with low output and moderate deflation, which is locally stable under learning.

Section 5 turns to numerical results and fiscal policy when expectations are sufficiently pessimistic that they imply a high likelihood under unchanged policy of the economy converging to stagnation. We examine the impact of a fiscal stimulus, in which government purchases are increased for a temporary stated period of time, and we show that in this situation the impact of fiscal policy is nonlinear. For a given duration, a small stimulus can fail to prevent convergence to the stagnation state, while a sufficiently large temporary stimulus can be very effective in returning the economy to the targeted steady state. In either case the size of the multipliers is large compared to normal times when none of the lower bounds apply, but in these settings a large stimulus can have an extremely large cumulative output multiplier.

These results imply that multipliers are both state-dependent and nonlinear. They are also stochastic. For a given initial state, in which expectations are pessimistic, and a given announced fiscal stimulus, the convergence to the targeted steady state will depend on the sequence of stochastic shocks. We give numerical results for the proportion of times stagnation is avoided for different magnitudes and horizons of fiscal stimulus.

Section 6 considers some extensions. Fiscal policy is most effective when it is implemented early. We also consider worst-case situations in which the economy has converged to and fully adapted to a stagnation steady state. We show numerically that even in this case there are fiscal policies that in the majority of cases will dislodge the economy from the stagnation state and return it over time to the targeted steady state.

Finally, we discuss the connection between the discount factor and the magnitude of the deflation rate in the stagnation state. While in our benchmark simulations we use a standard calibration of the discount factor, there are reasons for thinking the relevant discount rate is much lower. This would

imply a smaller critical deflation rate and thus increase the likelihood of experiencing deep recessions or depressions with mild deflation.

2 New Keynesian Model

We use a NK model with households and firms following the method developed in Eusepi and Preston (2010) and Evans and Honkapohja (2010). Our version of the NK model uses the Rotemberg adjustment cost version of the pricing friction, which we adopt because of its analytical convenience in looking at global dynamics. We index households by i and firms by j, but in the temporary equilibrium dynamics that we study all households and firms will make identical decisions. We start with the households. We assume a cashless limit and that households are Ricardian. Throughout Sections 2 and 3 we assume that the zero lower bound (ZLB) on interest rates is never binding.

2.1 Households

The objective for agent i is to maximize expected, discounted utility subject to a standard flow budget constraint:

$$Max\ \hat{E}_{0,i}\sum_{t=0}^{\infty}\beta^t U_{t,i}\left(C_{t,i},h_{t,i}\right) \tag{1}$$

$$\text{s.t. } C_{t,i}+b_{t,i}+\Upsilon_{t,i}=R_{t-1}\pi_t^{-1}b_{t-1,i}+Y_{t,i}, \tag{2}$$

where $C_{t,i}$ is the Dixit-Stiglitz consumption aggregator, $h_{t,i}$ is the labour input into production, $b_{t,i}$ denotes the real quantity of risk-free one-period nominal bonds held by the agent at the end of period t, $\Upsilon_{t,i}$ is the lump-sum tax collected by the government, R_{t-1} is the nominal interest rate factor between periods $t-1$ and t, P_t is the aggregate price level and the inflation rate is $\pi_t = P_t/P_{t-1}$. Consumers' income is denoted by $Y_{t,i}$ where

$$Y_{t,i}=\frac{W_t}{P_t}h_{t,i}+\Omega_t^i.$$

W_t is the nominal wage and Ω_t^i denotes profits from holding shares in equal part of each firm. The subjective discount factor is denoted by β. The utility

function has the parametric form

$$U_{t,i} = \log C_{t,i} - \gamma \frac{h_{t,i}^{1+\varepsilon}}{1+\varepsilon},$$

where $\varepsilon > 0$. The household decision problem is also subject to the usual 'no Ponzi game' condition.

There is a static FOC for the household concerning labor-leisure choice, which is

$$\frac{W_t}{P_t} = \gamma h_{t,i}^{\varepsilon} C_{t,i}. \tag{3}$$

To derive the linearized consumption function, we first linearize the Euler equation

$$C_{t,i}^{-1} = \beta R_t \hat{E}_{t,i} \left(\pi_{t+1}^{-1} C_{t+1,i}^{-1} \right) \tag{4}$$

to get

$$\tilde{C}_{t,i} = \hat{E}_{t,i} \tilde{C}_{t+1,i} - \beta \bar{C} \hat{E}_{t,i} \tilde{r}_{t+1}, \tag{5}$$

where tilde indicates deviation from the steady state, e.g. $\tilde{C}_{t,i} = C_{t,i} - \bar{C}$, and the bar denoting the deterministic steady state. Here

$$r_{t+1} \equiv \frac{R_t}{\pi_{t+1}}.$$

As shown in Appendix 1, the linearized lifetime budget constraint of the household is

$$\sum_{s=0}^{\infty} \beta^s \tilde{C}_{t+s,i} = \sum_{s=0}^{\infty} \beta^s (\tilde{Y}_{t+s,i} - \tilde{G}_{t+s}).$$

Here G_t is the level of government purchases, assumed exogenous, and we are assuming Ricardian households with identical taxes so that for each agent we may set $\Upsilon_t = G_t$.

Iterating the Euler equation gives

$$\hat{C}_{t,i} = \hat{E}_{t,i} \hat{C}_{t+s,i} - \hat{E}_{t,i} \sum_{i=1}^{s} \hat{r}_{t+i},$$

where we have used $\beta^{-1} = \bar{r}$ and a hatted variable indicates proportional deviations from its mean. e.g.

$$\hat{C}_{t,i} = \frac{C_{t,i} - \bar{C}}{\bar{C}} \text{ and } \hat{r}_{t+1} = \frac{r_{t+1} - \bar{r}}{\bar{r}}.$$

Combining the lifetime budget constraint and the iterated Euler equation and using the representative agent assumption $\hat{C}_{t,i} = \hat{C}_t$, $\hat{Y}_{t,i} = \hat{Y}_t$ and $\hat{E}_{t,i} = \hat{E}_t$ yields

$$\begin{aligned}\hat{C}_t &= (1-\beta)\left[\frac{\hat{Y}_t}{(\bar{C}/\bar{Y})} - \frac{\hat{G}_t}{(\bar{C}/\bar{G})} + \sum_{s=1}^{\infty}\beta^s \hat{E}_t\left(\frac{\hat{Y}_{t+s}}{(\bar{C}/\bar{Y})} - \frac{\hat{G}_{t+s}}{(\bar{C}/\bar{G})}\right)\right] \\ &\quad -\hat{E}_t\sum_{s=1}^{\infty}\beta^s \hat{r}_{t+s}. \qquad (6)\end{aligned}$$

2.2 Firms

The production function for each firm, producing good j, is given by

$$Y_{t,j} = A_t h_{t,j}^{\alpha}$$

where $0 < \alpha \leq 1$ and A_t is a random productivity shock with mean $\bar{A}$. Output is differentiated and firms operate under monopolistic competition. Each firm faces a downward-sloping demand curve given by

$$P_{t,j} = \left(\frac{Y_{t,j}}{Y_t}\right)^{-1/\theta_t} P_t. \qquad (7)$$

Here $P_{t,j}$ is the profit maximizing price set by firm j consistent with its production $Y_{t,j}$ (this optimization will be done below). The parameter θ_t is the elasticity of substitution between two goods and is assumed to be greater than one. θ_t is assumed to be a random stationary process with mean θ. Y_t is aggregate output, which is exogenous to the firm. The firms' problem is

$$Max\ \hat{E}_{T,j}\sum_{T=t}^{\infty} Q_{t,T} P_T \Omega_{T,j},$$

where, due to log utility, $Q_{t,T} = \beta^{T-t}\frac{P_t C_t}{P_T C_T}$, for $T \geq t$, where

$$\Omega_{t,j} = (1-\tau)\frac{P_{t,j}}{P_t}Y_{t,j} - \frac{W_t}{P_t}h_{t,j} - \frac{\psi}{2}(\frac{P_{t,j}}{P_{t-1,j}} - \pi^*)^2,$$

and where τ is the revenue tax rate to eliminate the steady state distortion in output caused by monopolistic competition. Here π^* is the (gross) rate

of inflation $\pi_t = P_t/P_{t-1}$ that is targeted by policymakers. Thus firms view it as costly to change prices by an amount that differs from the monetary policymaker target π^*. We interpret the quadratic term as reflecting the costs of justifying to consumers price increases at a rate higher than the target rate and the additional marketing costs of making customers aware of price increases below the target rate.[3]

The first-order condition for the firm's choice of $P_{t,j}$ is given by

$$\begin{aligned} 0 &= (1-\tau)(1-\theta_t)\left(\frac{P_{t,j}}{P_t}\right)^{-\theta_t} Y_t + S_t Y_t \theta_t \left(\frac{P_{t,j}}{P_t}\right)^{(-\theta_t/\alpha)-1} \\ &\quad -\psi \frac{P_t}{P_{t-1,j}}\left(\frac{P_{t,j}}{P_{t-1,j}} - \pi^*\right) + \hat{E}_{t,j} Q_{t,t+1} \frac{P_{t+1}}{P_{t,j}} \psi \frac{P_{t+1,j}}{P_{t,j}}\left(\frac{P_{t+1,j}}{P_{t,j}} - \pi^*\right). \end{aligned} \tag{8}$$

Here we use $Q_{t,t} = 1$ and

$$S_t = \frac{W_t/P_t}{\partial Y_{t,j}/\partial h_{t,j}} = \frac{W_t/P_t}{\alpha A_t h_t^{\alpha-1}}. \tag{9}$$

is the real marginal cost. It's useful to define the mark-up μ by

$$\mu_t = \frac{\theta_t}{\theta_t - 1}. \tag{10}$$

The steady state at π^* satisfies

$$(1-\tau)(1-\theta) + \bar{S}\theta = 0.$$

In the steady state, of course, $\mu = \theta(\theta-1)^{-1}$. From above steady state real marginal cost is

$$\bar{S} = \frac{(\theta-1)(1-\tau)}{\theta} = (1-\tau)\mu^{-1}. \tag{11}$$

We make the assumption that the target inflation rate is $P_t/P_{t-1} = \pi^* \geq 1$, i.e. the net inflation rate may be positive. As discussed above, we are making the assumption that price adjustment costs are quadratic in terms of the deviation from the target inflation rate and this is also analytically convenient. The market clearing condition is

$$Y_t = C_t + G_t + \frac{1}{2}\psi(\pi_t - \pi^*)^2. \tag{12}$$

[3]Benhabib, Schmitt-Grohe, and Uribe (2001b) and Benhabib and Eusepi (2005) use this formulation of price adjustment costs, though they do so in the context of the utility loss of household firms. Eusepi and Preston (2010) also use this formulation but set $\pi^* = 1$.

We need to linearize around the steady state $\pi^*, \bar{Y}, \bar{S}, \bar{C}, \bar{Q}, \bar{h}$. Clearly $\bar{Q} = \beta/\pi^*$ is the steady state value of $Q_{t,t+1}$ and $\bar{S}$ is given above. From (12) with $\pi_t = \pi^*$ we have $\bar{Y} = \bar{C} + \bar{G}$. Finally, in a steady state (9) and (3) can be combined to give

$$\bar{S} = \gamma \alpha^{-1} A^{-1} \bar{h}^{1+\varepsilon-\alpha} \bar{C}. \tag{13}$$

Equation (13) together with the steady-state production function $\bar{Y} = A\bar{h}^{\alpha}$, market-clearing $\bar{Y} = \bar{C} + \bar{G}$ and (11) determines steady values of $\bar{Y}, \bar{S}, \bar{C}, \bar{h}$ at the targeted steady state π^*.

We assume that firms use a decision-rule for price setting based on a linearization around the targeted steady state. Appendix 1 shows how to obtain the infinite horizon linearized New Keynesian Phillips curve

$$\begin{aligned}(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t &= a_1 \sum_{s=1}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{\pi}_{t+s} + a_2 \sum_{s=1}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{Y}_{t+s} \\ &\quad - a_3 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{A}_{t+s} - a_4 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{G}_{t+s} \\ &\quad + a_5 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{\mu}_{t+s}, \end{aligned} \tag{14}$$

where the coefficients $a_1, ..., a_5$ and γ_1 are defined in the Appendix.

The interpretation of equation (14) is as follows. As is standard higher expected future inflation and higher current and expected future aggregate output lead to higher current inflation. Current inflation is also increased when future monopoly power is expected to be higher. The remaining terms reflect the impact of productivity and government spending on real marginal cost. When expected future productivity is high this lowers expected future marginal costs and hence reduces inflation. Finally, conditional on expected future output, higher current and expected future government spending is associated with lower consumption, higher labor supply (conditional on real wages) and hence lower real wages, which leads to lower inflation.

2.3 Temporary equilibrium

We can now define the temporary equilibrium which is given by the Phillips curve (14), the NK IS curve and the interest rate rule. To get the IS curve, we aggregate (6) and use the representative agent assumption and combine

it with the market clearing condition $\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{G}}{\bar{Y}}\hat{G}_t$, or

$$\hat{Y}_t = (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t, \tag{15}$$

where $\bar{g} = \frac{\bar{G}}{\bar{Y}}$.[4] This yields

$$\hat{Y}_t = \bar{g}\hat{G}_t + (1-\beta)\left[\hat{Y}_t - \bar{g}\hat{G}_t + \sum_{s=1}^{\infty}\beta^s \hat{E}_t\left(\hat{Y}_{t+s} - \bar{g}\hat{G}_{t+s}\right)\right] - (1-\bar{g})\hat{E}_t\sum_{s=1}^{\infty}\beta^s\hat{r}_{t+s}.$$

Note that from $r_{t+1} \equiv \frac{R_t}{\pi_{t+1}}$ we have

$$\hat{r}_{t+1} = \hat{R}_t - \hat{\pi}_{t+1}. \tag{16}$$

The equation for $\hat{Y}_t$ can be rewritten as

$$\begin{aligned}\hat{Y}_t &= \hat{G}_t + (1-\beta)\beta^{-1}\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{Y}_{t+s} - (1-\beta)\bar{g}\beta^{-1}\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s} \\ &\quad -(1-\bar{g})(\hat{R}_t - \hat{E}_t\hat{\pi}_{t+1}) - (1-\bar{g})\sum_{s=2}^{\infty}\beta^{s-1}\hat{E}_t\hat{r}_{t+s}. \end{aligned}\tag{17}$$

To complete the model one must specify an interest rate rule, for example,

$$R_t = \beta^{-1}\left(\pi^* + \phi_\pi(\pi_t - \pi^*) + \tilde{\phi}_y(Y_t - \bar{Y})\right),$$

which in log-linearized form becomes

$$\hat{R}_t = \phi_\pi\hat{\pi}_t + \phi_y\hat{Y}_t, \tag{18}$$

where $\hat{R}_t = (R_t - \bar{R})/\bar{R}$, and $\phi_y = \frac{\tilde{\phi}_y\bar{Y}}{\pi^*}$. We also assume a government fiscal policy in which government spending is financed by lump-sum taxes. Here we are assuming that the zero lower bound (ZLB) is not violated. Later in the paper we allow for cases in which the ZLB is binding.

Finally, the shocks to A_t and μ_t are assumed to follow exogenous AR(1) processes given by

$$\hat{A}_t = \rho_A\hat{A}_{t-1} + \upsilon_{At}, \tag{19}$$

$$\hat{\mu}_t = \rho_\mu\hat{\mu}_{t-1} + \upsilon_{\mu t}, \tag{20}$$

where $0 < \rho_A, \rho_\mu < 1$ and the shocks υ_{At} and $\upsilon_{\mu t}$ are iid normal variables with zero mean and constant variances σ_A^2 and σ_μ^2.

This completes the description of the model apart from a specification of how expectations are formed.

[4] As in the Appendix to Eusepi and Preston (2010), the adjustment costs drop out from the log-linearized market clearing equation.

2.4 Expectations, learning and fiscal policy

Under rational expectations (RE) the solution technique is standard. See Appendix 1 for details. Under adaptive learning, agents need to form forecasts of future inflation, output and, when a fiscal policy change occurs, of government spending and taxes. Until Section 4 we assume that agents know the interest rate rule and that the ZLB is never binding.

Agents are assumed to have perceived laws of motion (PLMs) of the same form as the RE solution of the economy under standard policy. To allow for the indirect general-equilibrium impact of a policy change on future output and inflation, agents use constant-gain learning, as discussed below. There is a stochastic steady state of the form

$$\hat{\pi}_t = f_\pi + d_{\pi A}\hat{A}_t + d_{\pi\mu}\hat{\mu}_t \quad (21)$$

$$\hat{Y}_t = f_Y + d_{YA}\hat{A}_t + d_{Y\mu}\hat{\mu}_t \quad (22)$$

where $\hat{A}_t, \hat{\mu}_t$ are observable processes (with known coefficients) given by (19) and (20). Under adaptive learning agents estimate the coefficients of (21)-(22). Given their time t estimates of the coefficients f_π, $d_{\pi A}$, $d_{\pi\mu}$, f_Y, d_{YA}, $d_{Y\mu}$ forecasts $\hat{E}_t\pi_{t+s}$ and $\hat{E}_t\hat{Y}_{t+s}$ are given by

$$\begin{aligned}\hat{E}_t\hat{\pi}_{t+s} &= f_\pi + d_{\pi A}\rho_A^s\hat{A}_t + d_{\pi\mu}\rho_\mu^s\hat{\mu}_t,\\ \hat{E}_t\hat{Y}_{t+s} &= f_Y + d_{YA}\rho_A^s\hat{A}_t + d_{Y\mu}\rho_\mu^s\hat{\mu}_t.\end{aligned}$$

These forecasts can then be inserted into the model (14)-(17), and the infinite series summed, to determine the temporary equilibrium at time t. When there is no fiscal policy, government spending is constant and $\hat{G}_t = \hat{E}_t\hat{G}_{t+s} = 0$ and the corresponding terms in (14)-(17) are zero.

When there is a change in fiscal policy, agents will take account of the announced path of policy, which takes the form of a temporary increase in government spending. We assume that initially, at $t = 0$, we are in the stochastic steady state corresponding to $G = \bar{G}$, and that at $t = 1$ the government announces an increase in government spending for T periods, i.e.

$$G_t = \tau_t = \begin{cases} \bar{G}', t = 1, ..., T \\ \bar{G}, t \geq T + 1. \end{cases}$$

Thus government spending and taxes are changed in period $t = 1$ and this change is reversed at a later period $T+1$. We assume that the announcement

is fully credible and actually implemented. These assumptions could, of course, be relaxed.

Denoting the change in government spending by ΔG ($= \bar{G}' - \bar{G}$) we have

$$\hat{G}_t = \begin{cases} \frac{\Delta G}{\bar{G}}, \, t = 1, ..., T \\ 0, \, t \geq T + 1. \end{cases}$$

It is straightforward to compute $\sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{G}_{t+s}$ and $\sum_{s=0}^{\infty} \beta^s \hat{E}_t \hat{G}_{t+s}$, which will depend on calendar time, and include these terms in (14)-(17) when determining the temporary equilibrium.

Finally we describe the least-squares updating rule for the forecast rule coefficients of $\hat{\pi}_t$ and $\hat{Y}_t$. Agents are assumed to use constant gain recursive least squares (RLS). The parameter estimates based on data through time t are

$$\phi_{\pi,t} = \begin{pmatrix} f_{\pi,t} \\ d_{\pi A,t} \\ d_{\pi\mu,t} \end{pmatrix}, \phi_{Y,t} = \begin{pmatrix} f_{Y,t} \\ d_{YA,t} \\ d_{Y\mu,t} \end{pmatrix}, z_t = \begin{pmatrix} 1 \\ \hat{A}_t \\ \hat{\mu}_t \end{pmatrix}.$$

The RLS formulae corresponding to estimates of equations (21)-(22) are

$$\begin{aligned} \phi_{\pi,t} &= \phi_{\pi,t-1} + \kappa \mathcal{R}_t^{-1} z_t (\pi_t - \phi_{\pi,t-1} z_t), \\ \phi_{Y,t} &= \phi_{Y,t-1} + \kappa \mathcal{R}_t^{-1} z_t (Y_t - \phi_{Y,t-1} z_t), \\ \mathcal{R}_t &= \mathcal{R}_{t-1} + \kappa (z_t z_t' - \mathcal{R}_{t-1}). \end{aligned}$$

Here $0 < \kappa < 1$ is the "gain" parameter that discounts old data at rate $1 - \kappa$ per period (taken to be one quarter), to allow for adaptation of parameters to structural changes like policy changes. We assume that parameter estimates under learning are updated at the end of the period. Thus in time t, when expectations are formed, agents observe the current value of the exogenous variables $\hat{A}_t$ and $\hat{\mu}_t$ but use estimates $\phi_{\pi,t-1}$, $\phi_{y,t-1}$ in making forecasts. The initial values of all parameter estimates ϕ and $\mathcal{R}$ are set to the initial steady state values under RE.

3 Fiscal Policy in Normal Times

Using this set-up we compute numerically the government spending multipliers during normal times, when the ZLB does not bind, and there is a temporary increase in G. We show both the multiplier viewed as a distributed lag response and the cumulative multiplier over time. For each graph

the RE and learning responses are both shown. The cumulative multipliers are computed as a discounted sum using the discount factor β. Specifically, we compute

$$ym_t = \frac{Y_t - \bar{Y}}{\bar{G}' - \bar{G}} \text{ and } ycm_t = \frac{\sum_{i=1}^{t} \beta^{i-1}(Y_i - \bar{Y})}{\left(\bar{G}' - \bar{G}\right)\sum_{i=1}^{T-1} \beta^{i-1}}, \text{ for } t = 1, 2, 3, \ldots,$$

Because of discounting the cumulative multiplier will be finite even in those cases considered below in which policy leads to a permanent change in the level of output.

In this section we consider the temporary policy change discussed above with $T = 10$. The baseline parameters used in the simulations in both this and the following section are

$$\begin{aligned} \alpha &= 0.66, \beta = 0.99, \theta = 7.67, \tau = 0, \epsilon = 2, \gamma = 1, \psi = 128.8, \\ \bar{G} &= 0.2, \rho_A = \rho_\mu = 0.95, \sigma_A = \sigma_\mu = 0.0007. \end{aligned}$$

$A = 1.085$ is chosen so that output is approximately one; precisely $Y = 1.0081$. We interpret the parameters as corresponding to a quarterly calibration. The gain parameter κ of agents in this section is set equal to 0.04. For the inflation target we set $\pi^* = 1.005$. For quarterly data this corresponds to an annual rate of inflation of 2% which is a frequently used target for monetary policymakers. The value for ψ is based on a 15% markup of prices over marginal cost suggested in Leeper, Traum, and Walker (2011) (see their Table 2) and the price adjustment costs estimated from the average frequency of price reoptimization at intervals of 15 months (see Table 1 in Keen and Wang (2007)). We remark that the numerical simulations in this section use the linearized system of equations given in the previous sections. It would be possible to combine the linearized decision rules for consumption and price setting with the non-linear equations for market clearing, production, factor prices and labor supply, but this adds considerable computational complexity.[5]

In the examples we set $\phi_Y = 0$ to prevent monetary policy from directly acting against the output effects of fiscal policy. However we set $\phi_\pi > 1$ in

[5]In Mitra, Evans, and Honkapohja (2015) we found in an RBC model that including nonlinear temporary equilibrium equations made little difference, even when the shocks were large or we were changing steady states. However, the computations were 150 times slower.

line with the Taylor principle, in order to ensure both that the economy is determinate and that it is stable under least-squares learning.

Figure 1 shows the output and inflation paths under learning (solid line) and RE (dotted line) and the output multipliers (impact and cumulative) for a surprise temporary policy change with $T = 10$. Initial beliefs of agents and the values of the exogenous variables are at the steady state. For this example we set $\phi_\pi = 1.5$ and $\phi_Y = 0$ and consider an increase in G of 5%. The Figure shows the mean values of percent deviations of inflation and output from the steady state over 2000 simulations. For this setting the ZLB is never violated.

The most notable results are that the output and multiplier effects are larger under learning in early periods of policy, compared to RE. Under learning the maximum positive output effect is at the beginning of the policy, while under RE the maximum effect is in the last period of policy. Once the policy ends, the output effects are reversed under learning, with negative deviations for several periods after the stimulus ends. This contraction is the result of the higher expected inflation of agents, developed during the policy implementation, which leads agents to anticipate higher future real interest rates in accordance with the active Taylor rule.

To understand these results, we first examine the path under RE, which is fairly complex, and best analyzed starting from the last period of the policy. From $t = T + 1 = 11$, because there are no endogenous predetermined state variables in the NK model, the economy will return to the initial RE stochastic steady state. Consider next the economy at $t = T = 10$. The extra government spending ΔG at $T = 10$ has an impact on aggregate demand that is much larger than the small reduction in consumption resulting from the corresponding one-period tax increase. Because of consumption smoothing the reduction in consumption at $T = 10$ turns out to be relatively small. The high level of output and employment at $t = 10$ leads to higher real wages, and thus higher marginal costs and higher inflation through the Phillips curve. This in turn leads to high nominal and real interest rates through the Taylor rule. Now consider the economy at earlier dates $t < 10$. The reduction in consumption is greater in earlier periods and largest at $t = 1$. This is because households anticipate both a longer period of higher taxes and a longer period of higher real interest rates. It follows that under RE the increase in output is smallest at $t = 1$, due to the crowding out, and also that the impact on inflation is low in early periods. Under RE the largest impact of fiscal policy is at the *end* of the period of increases government spending.

Consider in contrast the path under learning. This path is best understood beginning with the impact effect at $t = 1$. Households reduce consumption because of the foreseen period of temporary tax increases, but they do not foresee the sustained period of high real interest rates. The reduction in consumption is thus much smaller than under RE and there is a large increase in output and employment due to the additional government spending. Through the Phillips curve there is also an increase in inflation and interest rates. At $t = 2$ expectations of future output and inflation will both be revised upward. Because higher expected inflation translates, into higher expected future nominal and real interest rates (since $\phi_\pi > 1$ in the Taylor rule), consumption falls. For later periods with $t < T$ increasing expected inflation and real interest rates leads to further reductions in consumption and output, with continued moderate inflation. Finally, at $t = 11$, when the policy ends, there is a substantial drop in output because the reduction in government spending is not offset by an increase in consumption, which remains low due to continued high expected inflation and real interest rates under adaptive learning. The low output levels for $t > 10$ continue for a period of time until inflation expectations, in response to observed low inflation rates, return to the steady state level. Thus under learning the largest impact of fiscal policy is at the *start* of the policy, and is partially offset following the end of the policy.

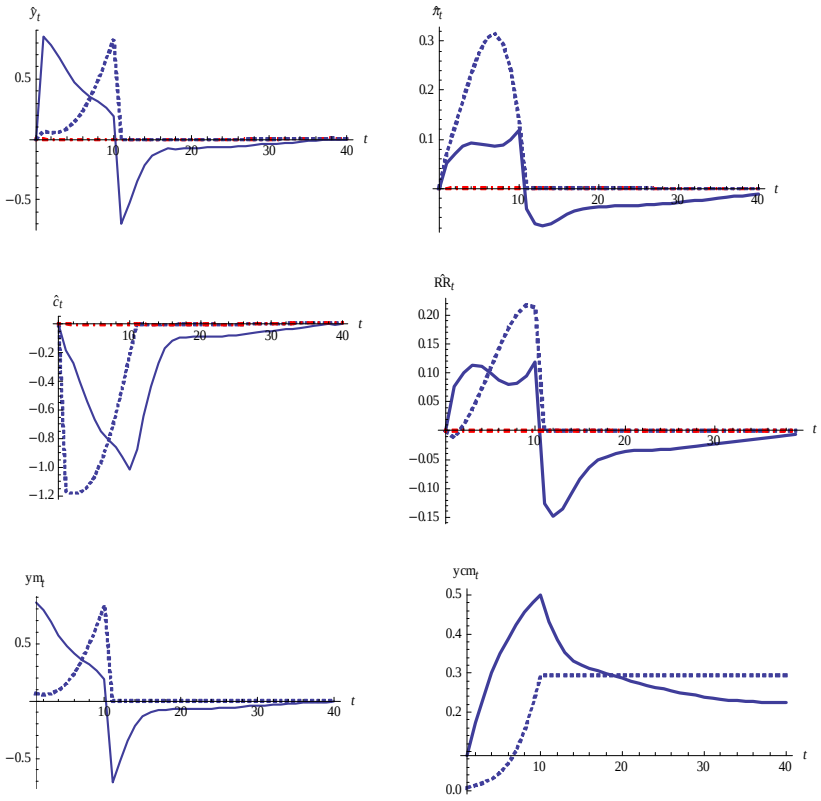


Figure 1: The upper panel shows the paths of output and inflation under RE (dotted line), learning (solid line) for a temporary policy change with $T = 10$. The middle panel shows the paths of the corresponding consumption and *ex ante* one period real rate of interest $(\hat{R}_t - \hat{E}_t\pi_{t+1})$. The lower panel shows the distributed lag and cumulative output multipliers for the policy change. Here and in subsequent figures $\hat{y}_t$ is used for $\hat{Y}_t$.

In Figure 2 we continue to assume that the exogenous variables are initially at their steady state levels, but now assume that initial beliefs of inflation and output are lower than the steady state values. In particular, we set $f_\pi = f_Y = -0.0008$ i.e. initial expectations of inflation and output are low. The gain parameter is the same as before. For this setting the ZLB is never violated. For this example we set $\phi_\pi = 1.1$ and $\phi_y = 0$ and again consider an increase in G of 5% for ten periods. The figure shows the mean values over 2000 simulations.

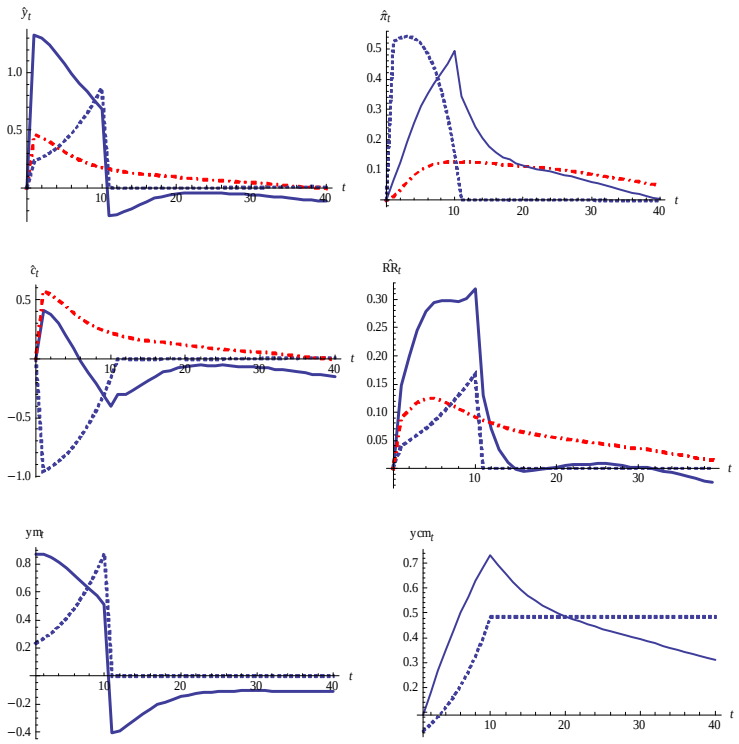


Figure 2: The upper panel shows the paths of output and inflation under RE with policy change (dotted line), learning with policy change (solid line) and learning without policy change (dot-dashed line) for a temporary policy change with $T = 10$. The middle panel shows the paths of the corresponding consumption and *ex ante* one period real rate of interest $(\hat{R}_t - \hat{E}_t \pi_{t+1})$. The lower panel shows the distributed lag and cumulative output multipliers for the policy change.

The RE results are similar to those in Figure 1 since the only change is that the Taylor-rule coefficient is lower. The results under learning are again quite different from those under RE, and because initial expectations are no longer at the steady state values we need to consider the paths of variables both without and with the fiscal policy. The paths without fiscal policy are shown by the dot-dashed red line. The low expected inflation leads to low expected future real interest rates under the assumption $\phi_\pi > 1$. This more than offsets expectations of low future incomes, raising consumption

and hence output $\hat{y}_t$. Through the Phillips curve this leads to above target inflation (and hence high *current* real interest rates), though the effect on inflation is initially small due to the initial low inflation expectations. Over time under learning the economy converges back to the steady state.

Under the fiscal policy with learning, there are significant increases in output. These increases are well in excess of those that would obtain under RE. The mechanisms are quite similar to those described in connection with Figure 1: under learning agents initially do not foresee the sustained period of high real interest rates that will be coming during the policy; consequently there is little crowding out of consumption during the policy, output is high, especially during the earlier stages of policy, and inflation and expected inflation build up. When the policy ends, there is a substantial drop in output because of the reduction in government spending, which is not offset by households because of consumption smoothing. Post policy there are low output levels for a period of time until inflation expectations adjust, in response to observed low inflation rates, and the economy returns to steady state levels.

In summary, the cumulative output multipliers are higher under AL than under RE during the policy implementation period, with the impact, relative to RE, concentrated in the early part of the policy. Again, we see that the maximum output effect of the fiscal policy under learning is in the early part of the policy, while the maximum output effect under RE occurs as the policy ends. The additional output increase under learning during the policy period is offset by lower levels of output after the policy ends.

4 Model with Lower Bounds

As extensively discussed in the recent literature on fiscal policy in New Keynesian models, the size of the multipliers can be very sensitive to the response of monetary policy: fiscal multipliers are smaller when the induced changes in inflation and output lead to increases in interest rates through the monetary policy rule. A particular case in which the multiplier can be expected to be large is when interest rates are at or near the zero-lower bound (ZLB), so that they are unresponsive to fiscal policy changes.

We now extend the temporary equilibrium framework of the model under learning to allow for the ZLB. In this section our focus is on AL when the initial expectations are sufficiently pessimistic that the ZLB is binding or is

expected to be binding. We remark that in contrast to much of the literature on the liquidity trap, and in particular most of the literature on fiscal multipliers at the ZLB assuming RE, in our framework the ZLB is primarily driven by a pessimistic expectations shock rather than by fundamental exogenous shocks to preferences (or natural interest rate shocks). Following the seminal paper of Eggertsson and Woodford (2003), much of the literature has assumed that low inflation and output at the ZLB are triggered by an exogenous preference shock that shifts the targeted RE equilibrium in such a way that the ZLB becomes a constraint for that equilibrium.[6] The shock is assumed to vanish according to a Markov process with known transition probability and an absorbing state, leading to a return to the intended steady state. Under RE the path of the economy with and without fiscal policy is largely determined by these exogenous preference shocks.

In contrast, the approach followed here focuses directly on a pessimistic shock to expectations. Although in our numerical analysis we do allow for exogenous shocks, as in the first part of the paper, we do not need to introduce an exogenous Markov preference shock or beliefs influenced by a sunspot that drive the recession and its recovery. Instead we assume an initial pessimistic expectations shock that, under learning, has the capacity to drive the economy to low levels of output and inflation and become self-sustaining.

It is known from earlier work on AL in the New Keynesian model that there is the possibility of deflation traps that cannot be overcome by monetary policy due to the ZLB and which push the economy along divergent trajectories.[7] We think that in these circumstances other bounds may also be important, which will act to stabilize the economy along an otherwise divergent trajectory. We begin with a discussion of these bounds and their implications for the possible steady states in the model. After presenting this analysis, and examining their stability properties under learning, we return in Section 5 to the role of fiscal policy and the size of government spending multipliers when their are large pessimistic shocks.

[6] In this approach global indeterminacy is ignored even though models describe monetary policy in terms of a Taylor rule subject to the ZLB. For example, see Christiano, Eichenbaum, and Rebelo (2011) and Woodford (2011). Aruoba, Cuba-Borda, and Schorfheide (2014) and Mertens and Ravn (2014) focus attention on sunspot solutions that are constructed using the indeterminacy.

[7] See Evans, Guse, and Honkapohja (2008), Evans and Honkapohja (2010) and Benhabib, Evans, and Honkapohja (2014). An earlier discussion of deflation traps in a backward-looking model was provided by Reifschneider and Williams (2000).

4.1 Lower bounds on R, π and C

A zero lower bound on net nominal interest rates correspond to a bound on the gross nominal one-period interest rate $R_t \geq 1$. In practice central banks prefer not to reduce net interest rates below a small positive number $\eta > 0$ and we thus impose the lower bound $R_t \geq 1 + \eta$.[8] At the global level we also now introduce two other lower bounds that will plausibly arise in extreme circumstances: an inflation lower bound $\underline{\pi}$ and a consumption lower bound $\underline{C}$. An inflation lower bound was discussed in Benhabib, Evans, and Honkapohja (2014) and Evans (2013). It is empirically appealing because the extent of deflation appears bounded even at very low levels of aggregate output. See for example Ball and Mazumder (2011), IMF (2013), and Coibion and Gorodnichenko (2015). Possible explanations include downward wage rigidity or money illusion, as discussed in Akerlof, Dickens, and Perry (1996), Akerlof, Dickens, and Perry (2000), Akerlof and Dickens (2007) and Akerlof and Shiller (2009)). We capture these factors through the simple device of an inflation lower bound $\underline{\pi}$, which we usually take to correspond to a modest rate of deflation.[9] The value of $\underline{\pi}$ may vary over time and across countries.

We assume $\underline{\pi} < \pi^*$, where π^* is the inflation rate targeted by monetary policy. A consumption lower bound would plausibly arise when consumption approaches the (perhaps socially determined) subsistence level. Below we assume that the bound $\underline{C}$ is significantly below the targeted steady state. The spirit of this bound is similar to the subsistence level parameter used in Stone-Geary preferences; see, for example King and Rebelo (1993) and Ravn, Schmitt-Grohe, and Uribe (2008).[10] Although in normal times these bounds would not be apparent, they can play a role in stabilizing the economy at low levels of output at the ZLB.

We begin with a discussion of the steady states that may arise when these lower bounds may be binding. In this section it is convenient to simplify the monetary policy rule, so that the Taylor-type rules responds only to inflation.

[8] This is also convenient theoretically because it ensures money demand is finite at the lower bound.

[9] As shown in Benhabib, Evans, and Honkapohja (2014), one can justify $\underline{\pi}$ formally by introducing an asymmetry into the inflation adjustment cost term.

[10] Our procedure for incorporating the consumption lower bound differs somewhat from using Stone-Geary preferences, but is convenient given our treatment of the two other lower bounds. Changing to Stone-Geary preferences would give the same qualitative results.

Together with the interest-rate lower bound we have

$$\begin{aligned} R_t^* &= \beta^{-1}\phi_\pi(\pi_t - \pi^*) + \beta^{-1}\pi^*, \text{ with } \phi_\pi > 1, \text{ and} \\ R_t &= \max(R_t^*, 1+\eta). \end{aligned}$$

Here the parameterization is consistent with our earlier log linearization $\hat{R}_t = \phi_\pi \hat{\pi}_t$ at the intended steady state. Throughout this Section it is convenient to abstract from the intrinsic random productivity and mark-up shocks.

To analyze the possible non-stochastic steady states we can focus attention on the Euler equations. Setting $C_t = C_{t+1} = C$ and $\pi_t = \pi$, it follows from (4) that the Fisher equation holds:

$$R/\pi = \beta^{-1}.$$

Figure 3, which shows this relationship together with the steady state interest rate rule $R = \max\left(\beta^{-1}\phi_\pi(\pi - \pi^*) + \beta^{-1}\pi^*, 1+\eta\right)$, illustrates the usual indeterminacy result that in addition to the intended steady state at $\pi = \pi^*$ there is an unintended steady state at $\pi = \pi_L \equiv \beta(1+\eta)$.

We assume throughout that $0 < \eta < \beta^{-1} - 1$ so that the interest rate lower bound below the level implied by the Fisher equation at π^*, and then $\phi_\pi > 1$ implies the existence of the unintended deflation steady state at $\pi_L < 1$. This multiplicity issue was analyzed in detail, under the RE assumption, in Benhabib, Schmitt-Grohe, and Uribe (2001b) and Benhabib, Schmitt-Grohe, and Uribe (2001a). Bullard (2010) gave a forceful argument that the pattern of inflation and interest rates in Japan and the US was cause for concern that the US experience might converge to a Japanese style stagnation with steady mild deflation.

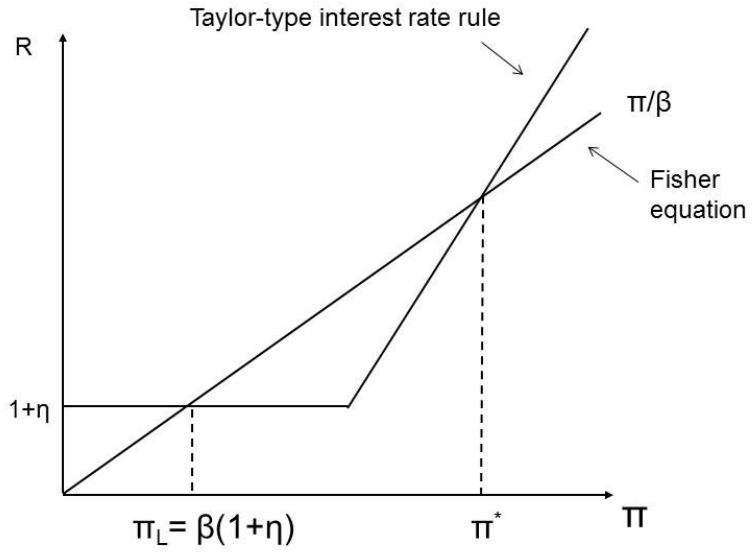


Figure 3: Existence of multiple steady states.

The remaining steady state equation is obtained from the NK Phillips relationship (8), setting $P_{t,j} = P_t$, $S_t = S$, $Y_t = Y$, $h_t = h, \theta_t = \theta, P_{t+1}/P_t = \pi, A_t = A$ and $Q_{t,t+1} = \beta/\pi$. This gives

$$0 = (1-\tau)(1-\theta)Ah^{\alpha} + SAh^{\alpha}\theta - \psi\pi(\pi-\pi^*) + \beta\psi\pi(\pi-\pi^*)$$

Using (3) and (9) gives $S = \alpha^{-1}\gamma A^{-1}h^{1+\varepsilon-\alpha}C$, which leads to[11]

$$(\pi-\pi^*)\pi(1-\beta) = \frac{\theta}{\alpha\psi}\left[C\gamma h^{1+\varepsilon} - \alpha(1-\tau)(1-\theta^{-1})Ah^{\alpha}\right]. \qquad (23)$$

This is the steady-state NK Phillips curve equation. We will also need the GDP steady state accounting identity

$$Ah^{\alpha} = C + G + \frac{\psi}{2}(\pi-\pi^*)^2. \qquad (24)$$

[11] The steady-state Phillips curve equation here differs from the one in Evans, Guse, and Honkapohja (2008). The latter paper uses a representative household-firm in which the price-adjustment costs are quadratic in utility. In the current set-up households and firms are distinct. With utility $\log(C)$ this leads to a multiplicative factor C on the right-hand side of (23) not present in Evans, Guse, and Honkapohja (2008).

However, the above Phillips curve and Fisher equations do not take into account the inflation and consumption lower bounds. The inflation lower bound $\underline{\pi}$ holds if (8) would otherwise lead to an inflation rate lower than this bound, and similarly the consumption lower bound holds if otherwise we would have $\beta R > \pi$. Taking into account these bounds, the Euler equations thus lead to the inequalities

$$R/\pi \geq \beta^{-1} \text{ and } C \geq \underline{C}, \text{ with c.s.,} \tag{25}$$

which one could call the Fisher inequality, and the Phillips curve inequality

$$\begin{aligned} (\pi - \pi^*)\pi(1-\beta) &\geq \frac{\theta}{\alpha\psi}\left[C\gamma h^{1+\varepsilon} - \alpha(1-\tau)(1-\theta^{-1})Ah^{\alpha}\right] \\ \text{and } \pi &\geq \underline{\pi}, \text{ with c.s.} \end{aligned} \tag{26}$$

Here c.s. denotes that these inequalities hold with complementary slackness, i.e. if either inequality holds strictly then the other holds with equality. We can also write the interest-rate rule subject to its lower bound as

$$R \geq \beta^{-1}\phi_{\pi}(\pi - \pi^*) + \beta^{-1}\pi^* \text{ and } R \geq 1 + \eta, \text{ with c.s.} \tag{27}$$

Using the three inequalities (25), (26), (27) we can examine the possible steady states. We assume throughout that $G > 0$ and it is convenient to strengthen this slightly and assume that $G > \underline{G} > 0$ where $\underline{G}$ is specified below. In addition we assume that the consumption lower bound $\underline{C}$ is not too large, as further specified below.

4.2 Steady states

The number of steady states in the economy will depend critically on the inflation lower bound $\underline{\pi} < 1$, specifically on whether $\underline{\pi} < \pi_L, \underline{\pi} = \pi_L$ or $\underline{\pi} > \pi_L$. Full analytical results are available for cases in which price adjustment costs are small. The steady state results are given in the following proposition:

Proposition 1 *Suppose that $\underline{\pi} < \pi_L$. Then for $\psi > 0$ sufficiently small, there are exactly three steady states:*
(i) $\pi = \pi^*$, with $R = \beta^{-1}\pi^*$ and C uniquely determined by (23) and (24),
(ii) $\pi = \pi_L$, with $R = 1 + \eta$ and C uniquely determined by (23) and (24),
(iii) $\pi = \underline{\pi}$, with $R = 1 + \eta$ and $C = \underline{C}$.

If $\pi_L < \underline{\pi} < 1$ then there is a unique steady state at $\pi = \pi^*$, with $R = \beta^{-1}\pi^*$ and C uniquely determined by (23)-(24).
If $\underline{\pi} = \pi_L$ then for $\psi > 0$ sufficiently small there is a steady state at $\pi = \pi^*$, with $R = \beta^{-1}\pi^*$ and C uniquely determined by (23)-(24) and a continuum of steady states at $\pi = \pi_L$, with $R = 1 + \eta$ and with C satisfying $\underline{C} \leq C \leq C_L$, where C_L is uniquely determined by (23)-(24).

The proofs of all propositions are given in Appendix 2.

4.3 Local stability of steady states under learning

We now consider the stability under AL of the steady states just described. As is well known, the local stability of an RE solution under least-squares learning, of the type outlined in Section 2.4 is determined by expectational stability, or "E-stability," conditions, as discussed, for example in Evans and Honkapohja (2001). Although one could allow for the inclusion of exogenous productivity and mark-up shocks in this analysis, local stability in the current setting is governed by the intercepts of the forecast rules. We therefore simplify the theoretical stability results by assuming that the PLM for both output and inflation takes the form of an unknown constant plus a perceived white noise disturbance. Furthermore, for theoretical convenience in this section, we assume a forward-looking interest-rate rule of the form $\hat{R}_t = \phi_\pi \hat{E}_t \hat{\pi}_{t+1}$.

The local stability results are given by the following proposition.

Proposition 2 *If $\underline{\pi} < \pi_L$ then the steady state at π^* is locally E-stable and the steady state at $\underline{\pi}$ is locally E-stable, while the steady state at π_L is locally E-unstable for ψ sufficiently small. If $\underline{\pi} > \pi_L$ then the (unique) steady state at π^* is locally E-stable.*

The following Figure shows the E-stability dynamics that give the mean dynamics under constant gain learning with small constant gain, based on linearized decision rules subject to the lower-bound constraints, but incorporating the nonlinear market clearing condition (24).[12] In this figure we use the standard calibrated values used earlier for the structural parameters.

[12] We also impose an upper bound to inflation to ensure existence of a temporary equilibrium. This is not needed in the linearized model with market clearing linearized around the targeted steady state.

Here $\phi_\pi = 1.5, \eta = 0.0001$ and for convenience we set $\pi^* = 1$. We set the lower bound for consumption at 10% below the intended steady state and the lower bound for (net) inflation at -1.3%,i.e. $\underline{\pi} = -0.013$.

The origin of Figure 4 represents the targeted steady state $\hat{y} = \hat{\pi} = 0$, and y, π are in proportional deviation form. In addition to the targeted steady state it can be seen that the unintended low steady state has a corresponding output level very close to the targeted steady state; specifically, it is only -0.0397% below the value of output at the targeted steady state. The corresponding (net) inflation rate at the unintended steady state is -0.9901% i.e. $\pi_L = -0.009901$. Finally the trap steady state, corresponding to $\underline{\pi} = -0.013$, has an output level equal to 6.92% below the value of output at the targeted steady state.

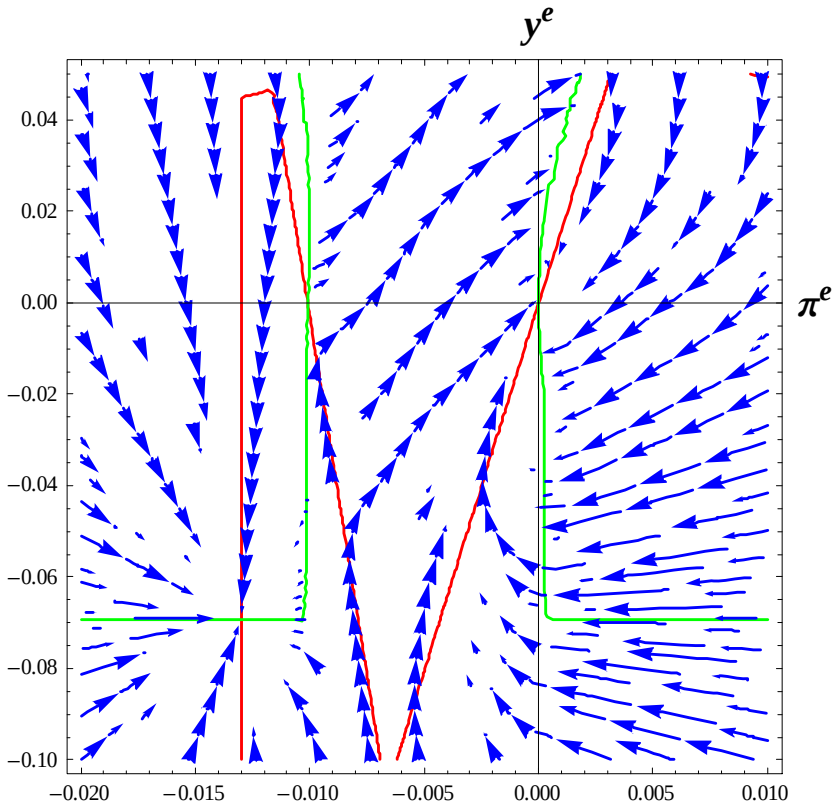


Figure 4: E-stability dynamics with forward looking Taylor rule in the case of three steady states. Here y^e and π^e denote expectations as proportional deviations from the targeted steady state, i.e. $\hat{Y}^e$ and $\hat{\pi}^e$.

It can be seen that the intended steady state at $\hat{\pi} = \hat{c} = 0$ is locally stable under learning (with the dynamics locally cyclical). The unintended steady state created by the ZLB is locally unstable (the dynamics are a saddle) and the trap steady state is locally stable. It can be seen that if π^e is sufficiently pessimistic then under learning the economy converges to the trap steady state with low output and mild deflation.

There has been considerable concern among US and European policymakers about deflation and the possibility of their economies, following the financial crisis of 2007-9, becoming enmeshed in a long period of stagnation with mild deflation, similar to that experienced by Japan since the mid $1990s$. This concern, which was forcefully stated in Bullard (2010), has been a large part of the motivation for setting and keeping policy interest rates near zero, and for innovative monetary policies like "quantitative easing" and "forward guidance." The above analysis shows that under adaptive learning this concern is acute if the inflation lower bound $\underline{\pi}$ is below the unintended steady state inflation rate π_L. There is then a stable deflation-trap steady state at $\pi = \underline{\pi}$ and a low level of output underpinned by the consumption lower bound. Because in the deflation-trap steady state interest rates are at the ZLB, conventional monetary policy cannot move the economy back to the targeted steady state. The effectiveness of fiscal policy in this setting is then of particular interest.

5 Fiscal Policy When the ZLB may be Binding

5.1 Preliminary considerations

A zero lower bound on net nominal interest rates correspond to a bound on the gross nominal one-period interest rate $R_t \geq 1$. Recall that the steady state real interest rate factor is $\bar{r} = \beta^{-1}$. When the target inflation rate is $\pi^* \geq 1$, the steady state nominal interest-rate factor is $\bar{R} = \beta^{-1}\pi^*$. Because $\hat{R}_t = (R_t - \bar{R})/\bar{R} = (\beta R_t)/\pi^* - 1$, it follows that at the ZLB we have $\hat{R}_t = \beta/\pi^* - 1$.

In practice, in our numerical simulations, much like the actual monetary policy followed in the US and the UK in the 2008 - 2014 period, we will assume net interest rates are bounded by some small value $\eta > 0$. Thus the

ZLB is defined by the bound

$$\hat{R}_t \geq \frac{\beta(1+\eta)}{\pi^*} - 1.$$

We continue to assume that in normal times the interest-rate policy is given by the Taylor rule. With the ZLB added policy takes the form

$$\begin{aligned} \hat{R}_t &= \max\left\{\hat{R}_t^*, \frac{\beta(1+\eta)}{\pi^*} - 1\right\}, \text{ where} \\ \hat{R}_t^* &= \phi_\pi \hat{\pi}_t + \phi_y \hat{Y}_t. \end{aligned}$$

We first note that the New Keynesian Phillips curve (14) is unaffected because it does not depend on the interest rate. However, the New Keynesian IS equation (17) is altered because both expected future real interest rates and the current interest rate may be subject to the ZLB.

Whether agents expect the ZLB to bind in the future depends both on the fundamental shocks and the beliefs of agents as measured by their estimates of the parameters of the PLM. There are actually four cases to consider depending on whether the ZLB is expected to bind in all future periods, no future periods, after a finite number of periods or up to some date. Case 1 corresponds to normal times in which the ZLB is not expected to hold in the future. Case 2 corresponds to very pessimistic inflation expectations, in which the ZLB is expected to hold throughout the future. In Case 3 expectations are pessimistic but offset for an initial period because of current A_t, μ_t shocks, and in Case 4 expectations are not too pessimistic but the ZLB is expected to hold for an initial period because of unfavorable current A_t, μ_t shocks.[13]

These cases are discussed in detail in Appendix 3. The condition determining the applicable case depends in part on the parameters ρ_A and ρ_μ, and for analytical convenience we restrict attention to the case $\rho_A = \rho_\mu$. For each case we must also allow for the possibility of the ZLB binding in the temporary equilibrium. Note that the model of Section 2 corresponds to normal times in which the ZLB never holds either in expectation or in the temporary equilibrium.

In this section we make the additional assumptions that inflation and consumption are also subject to lower bounds, as described in Section 4.1. We

[13]Related ideas have been discussed in Williams (2010) in the context of price-level targeting under imperfect knowledge and learning.

normally take $\underline{\pi}$ to correspond a modest rate of deflation. The lower bounds $\underline{c}$ and $\underline{\pi}$ come into play when expectations are very pessimistic and can become binding in the "deflation trap" regions, emphasized by Benhabib, Evans, and Honkapohja (2014), in which adaptive learning can produce otherwise divergent paths with falling inflation and output. In Section 2 we implicitly assumed paths in which the consumption and inflation lower bounds were never violated.

5.2 Simulations

The impact of fiscal policy will depend sensitively on the values of $\underline{\pi}$ and $\underline{c}$ and the nonstochastic component of the initial expectations $\hat{\pi}^e(0) = f_\pi(0)$ and $\hat{Y}^e(0) = f_Y(0)$. There are cases in which without policy the economy will converge to a stagnation steady state rather than to the targeted steady state. If initial expectations are close to the edge of the deflation trap region, fiscal policy may be able to shift the path to the targeted steady state. In cases involving possible convergence to the stagnation regime, the impact of fiscal policy may depend critically on the size and length of fiscal policy.

Before turning to simulations, we note that there are three possible steady states when the inflation lower bound $\underline{\pi}$ is below $\pi_L = \beta(1+\eta)$. In proportional deviation form this corresponds to $\underline{\hat{\pi}} < \hat{\pi}_L$ where $\underline{\hat{\pi}} = \underline{\pi}/\pi^* - 1$ and $\hat{\pi}_L = \beta(1+\eta)/\pi^* - 1$. The first steady state is the targeted steady state at $\pi = \pi^*$ i.e. at $\hat{\pi} = 0$. There is a second steady state at $\hat{\pi} = \hat{\pi}_L$ with $\hat{c}_L > \underline{\hat{c}}$. This steady state, however, is unstable under learning. Finally, there is the deflation trap steady state at $\hat{\pi} = \underline{\hat{\pi}}$ and $\hat{c} = \underline{\hat{c}}$.

If instead $\underline{\hat{\pi}} > \hat{\pi}_L$ then the usual targeted steady state is the unique steady state, and if $\underline{\hat{\pi}} = \hat{\pi}_L$ there will also be a continuum of steady states at $\hat{\pi} = \underline{\hat{\pi}}$ with $\hat{c} > \underline{\hat{c}}$.

For our parameterization with $\beta = 0.99$ and $\eta = 0.0001$, the critical value π_L for the inflation bound is approximately -0.0099. This is a deflation rate of 0.99% per period, which we take to be a quarter, i.e. just under 4% per annum. In our simulations below we set $\underline{\hat{\pi}} = -0.017$ or $\underline{\hat{\pi}} = -0.01475$. The inflation lower bound $\underline{\hat{\pi}} = -0.017$, which corresponds to about -1.21% per quarter, leads to three steady states, while the lower bound $\underline{\hat{\pi}} = -0.01475$, which corresponds to about -0.98% per quarter, leads to a unique steady state. For the consumption lower bound, we set $\underline{\hat{c}} = -0.3$ which corresponds to a 30% reduction in consumption from the targeted steady state which in turn corresponds to a drop of roughly 24% in aggregate output in our model.

Thus the level of output in the stagnation state corresponds roughly to the output drop in the Great Depression in the United States in the 1930s.[14]

In this Section we set the benchmark gain to 0.10 because we consider expectations that are sometimes far from rational values. In these circumstances agents have an incentive to adjust expectations relatively quickly to eliminate systematic forecast errors.[15]

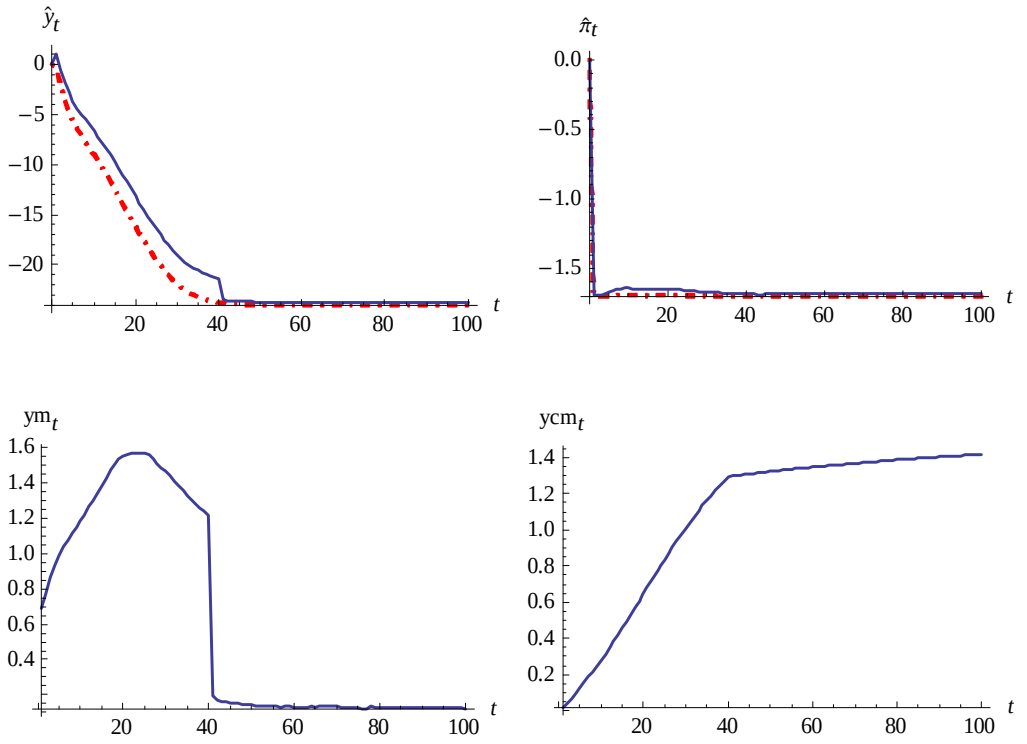


Figure 5: Small policy change. The upper panel shows the paths of output and inflation under learning with policy change (solid line) and learning without policy change (dashed line) for a temporary policy change with $T = 40$. The lower panel shows the distributed lag and cumulative output multipliers for the policy change.

[14] In our simulations we continue to use the linearized model, for the reasons given earlier, but now the relevant equations are subject to the lower bounds on the interest rate, inflation and consumption.

[15] The qualitative features are fairly robust to the value of the learning gain parameter κ, but quantitative results may be affected by κ.

Figure 5 sets $\underline{\hat{\pi}} = -0.017$ and shows the results for initial $f_\pi = -0.01485$ and $f_Y = -0.009$. These values for f_π and f_Y correspond to initial inflation expectations of close to -1.0% per quarter and output expectations 0.9% below the level of the targeted steady state (assuming the exogenous shocks are at their mean values). This setting can be thought as follows. At the beginning of time $t = 1$ the economy suffers a pessimistic expectational shock, which resets mean expectations to levels below the targeted steady state, specifically $\hat{\pi}^e = -0.01485$ and $\hat{Y}^e = -0.009$. We then contrast the evolution of the economy under learning when fiscal policy is unchanged with the evolution of the economy under learning when at $t = 1$ a temporary fiscal stimulus is initiated of known duration.[16]

Without fiscal policy these initial expectations are sufficiently pessimistic so that inflation at $t = 1$ falls immediately to the lower bound $\underline{\hat{\pi}} = -0.017$. This is accompanied by small reductions in consumption and output, and the interest rate falls to the ZLB. Because of the ZLB, and with inflation at its (negative) lower bound, current and expected future real interest rates are positive and approximately equal to the deflation rate. Using the temporary consumption function and market clearing equations, with inflation and expected inflation at the lower bound, it is shown at the end of Appendix 3 that

$$\hat{Y}_t = f_{Y,t} - \Delta, \text{ where } \Delta = \frac{(1-\bar{g})\,(\pi_L - \underline{\pi})}{(1-\beta)\pi^*} > 0,$$

and $\hat{C}_t = (1-\bar{g})^{-1}\hat{Y}_t$. Thus at each time t output is lower than expected output. This results in expected output falling over time. More specifically, it can be shown that the RLS updating equation for f_Y in Section 2.4 can be approximated by $f_{Y,t+1} = f_{Y,t} + \kappa\left(\hat{Y}_t - f_{Y,t}\right) = f_{Y,t} - \kappa\Delta$, with also $d_{YA,t}, d_{Y\mu,t} \to 0$. It follows that $\hat{E}_t Y_{t+s} = f_{Y,t}, \hat{Y}_t$ and $\hat{C}_t$ will steadily fall over time until $\hat{C}_t$ is constrained by the consumption lower bound, at which point the economy reaches and stays in the stagnation steady state. The paths of key variables are illustrated in Figure 5 by the dot-dashed lines. With no fiscal policy change the initial pessimistic expectations lead the economy

[16] Thus on this interpretation in Figures 5 (and 6) we set the $t = 0$ variables at their steady state values. An alternative possible interpretation of these simulations is that due to past (unmodeled) shocks and the recent path of the economy, expectations have evolved to the pessimistic values given, and we study the impact of fiscal policy introduced at $t = 1$ to counteract this. On this alternative interpretation the $t = 0$ variables would differ from steady state values.

into a deflation trap at $\underline{\pi}$ with output approximately 24% below the targeted steady state.

Consider now the effectiveness fiscal policy. In Figure 5 we start by looking at a small fiscal stimulus with a duration of $T = 40$ periods. Under a fiscal policy that increases G by 10%, from $G = 0.2$ to $G = 0.22$, there are positive multipliers during the policy period, with a cumulative multiplier of around 1.4, after 100 periods, which is mostly reached by period 40. However, the economy sinks back into the deflation trap around period 40.

In contrast, when G is changed by a sufficiently large amount the economy can be shifted to the targeted steady state. In Figure 6 we start with the same initial beliefs as in Figure 5, but we consider a policy that increases government spending from $G = 0.2$ to $G = 0.28$ for a period of $T = 5$ periods. In approximately 91% of the simulations there is convergence to the intended steady state under this fiscal policy. Figure 6 illustrates the results based on $10,000$ simulations. The top panel shows the mean paths of output and inflation for the paths for which fiscal policy is effective in the sense that it leads to convergence to the intended steady state.[17] The middle panel shows that in the approximately 9% of the simulations in which the economy fails to escape the deflation trap, fiscal policy still has substantial positive effects on output. The bottom panel shows the distributed lag and cumulative output multipliers averaged over all $10,000$ simulations. Output and inflation increase monotonically during the period of the stimulus, $t = 1, \ldots, 5$.

[17] Recall that in Figures 5 and 6 the $t = 0$ values shown are for the steady state.

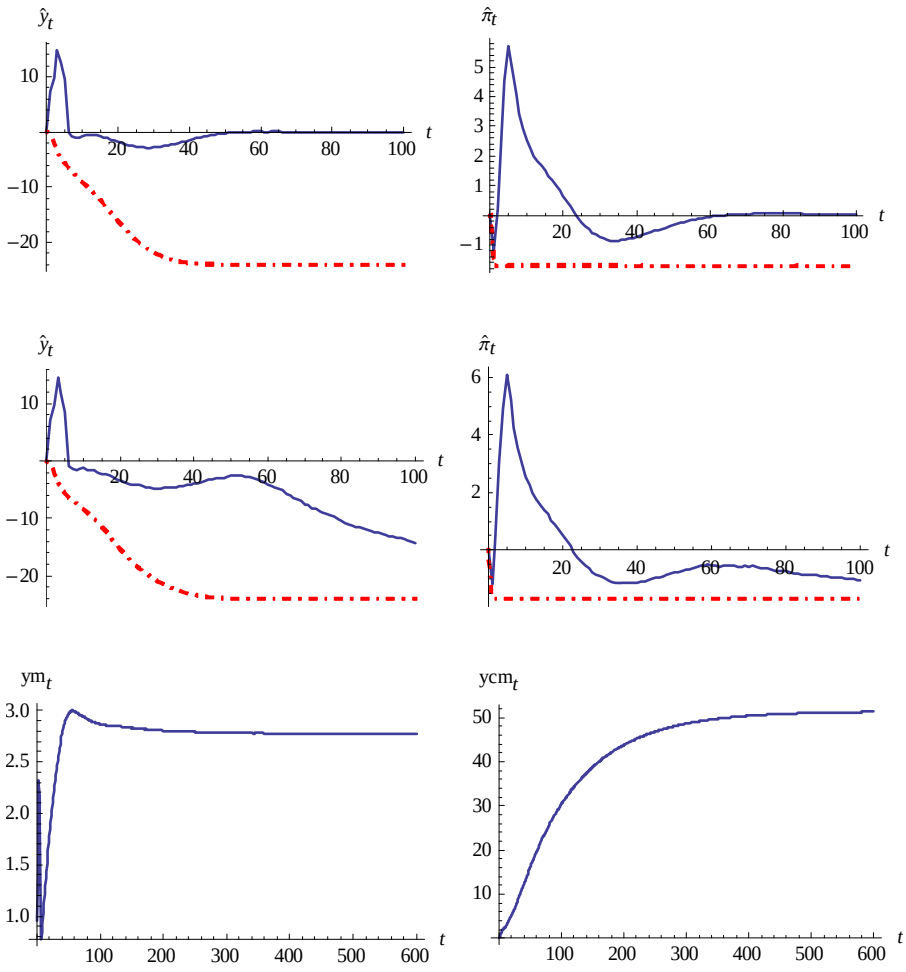


Figure 6: Large policy change. The top two panels show the paths of output and inflation under learning with policy change (solid line) and without policy change (dashed line) for $T = 5$. Top panel: means of paths with convergence to targeted steady state under policy. Middle panel: means of paths with convergence to deflation trap despite policy. Bottom panel: distributed lag and cumulative output multipliers across all paths.

The intuition for the results in which there is convergence to the intended steady state under policy is as follows. In period $t = 1$ pessimistic expectations for inflation and output are predetermined. The increase in G has

a large effect on output because there is only a small crowding out effect on consumption. Although there is deflation and interest rates are near the ZLB, the high level of output in period $t = 1$ increases inflation above its expected level. Consequently in the following period inflation and output expectations are both revised upward. The higher inflation and output expectations and continued high G lead in period $t = 2$ to even higher output and to inflation close to the target level. Beginning in period $t = 3$, inflation has risen above target. The high output and inflation levels in period $t = 3, 4, 5$ result in inflation and output expectations being large enough so that when the government stimulus is removed in $t = 6$, output falls back close to normal levels, though inflation remains above target for a sustained period of time. Because expectations of inflation are above target levels, it still takes significant time for the economy to converge to the targeted steady state, but expectations are now within the stable basin of attraction and there is asymptotic convergence to the targeted steady state.

In summary, temporary increases in G are effective in raising output. Small temporary increases in G lead only to temporary increases in y, but large temporary increases in G can shift the economy back to the targeted steady state resulting in a permanent increase in output. It is important to note that whether or not the fiscal policy is successful in pushing the economy back to the targeted steady state depends in part on the sequence of stochastic shocks hitting the economy over time. Table 1 illustrates these results, and Table 2 shows the cumulative multipliers as of $t = 40$ i.e. 10 years after the policy has been implemented. These results are based on 100 simulations.

In Figure 7 we consider the case in which the inflation lower bound is high enough so that there is a unique steady state. In this case the low level trap does not exist and the targeted steady state is unique. Initial pessimistic expectations can still lead to a very long transition to the targeted steady state and effectiveness of fiscal policy is of interest.

$G\backslash T$	1	2	3	5	10	20	40
0.21	4	4	5	1	0	0	0
0.22	31	37	34	19	7	4	0
0.25	92	94	93	91	90	47	7
0.27	92	91	94	91	91	89	23
0.28	91	92	97	93	90	85	43
0.30	94	97	95	93	87	86	75
0.35	96	95	96	86	84	80	80
0.40	93	95	92	85	80	67	75
0.50	93	91	87	83	68	62	45
0.60	94	88	84	71	44	23	18

Table 1: Percentage of simulations in which fiscal policy successfully results in convergence to the targeted steady state starting from pessimistic expectations. Based on 100 simulations.

$G\backslash T$	1	2	3	5	10	20	40
0.21	27.3	17.3	11	3.6	2.3	1.7	1.2
0.22	88.8	52.5	30.6	12.6	4.0	2.2	1.2
0.25	99.8	50.1	33.5	20.1	10.4	3.8	1.4
0.27	71.2	35.8	24	14.3	7.3	4	1.6
0.28	62.3	31.3	21	12.5	6.3	3.5	1.7
0.30	49.9	25.1	16.7	9.9	4.9	2.8	1.8
0.35	33.3	16.7	11	6.4	3.1	1.8	1.3
0.40	25	12.5	8.1	4.5	2.1	1.3	1.1
0.50	16.7	8.1	5.1	2.6	1.2	0.9	0.9
0.60	12.5	5.9	3.5	1.7	0.9	0.8	0.9

Table 2: Cumulative multipliers through $t = 40$ for fiscal policies starting from pessimistic expectations. Based on 100 simulations.

Figure 7 shows the impact on output and inflation of an increase in G from 0.20 to 0.22 for $T = 8$ periods. For this policy experiment agents are assumed to use a gain parameter $1/15 = 6.67\%$. For these simulations we set $\hat{\underline{\pi}} = -0.01475$, i.e. just above the level needed to avoid the low-level trap. As before, initial expectations following a large pessimistic shock are

set at $\hat{\pi}^e = -0.0165$ and $\hat{y}^e = -0.01$. The cumulative multipliers here appear to be smaller than those found in Table 2. However, they are substantially larger than those seen in Figures 1 and 2. This is because, although there is a unique steady state, the economy is initially in a liquidity trap. Consequently the multipliers are higher than in the first section of the paper in which fiscal policy was conducted in normal times.

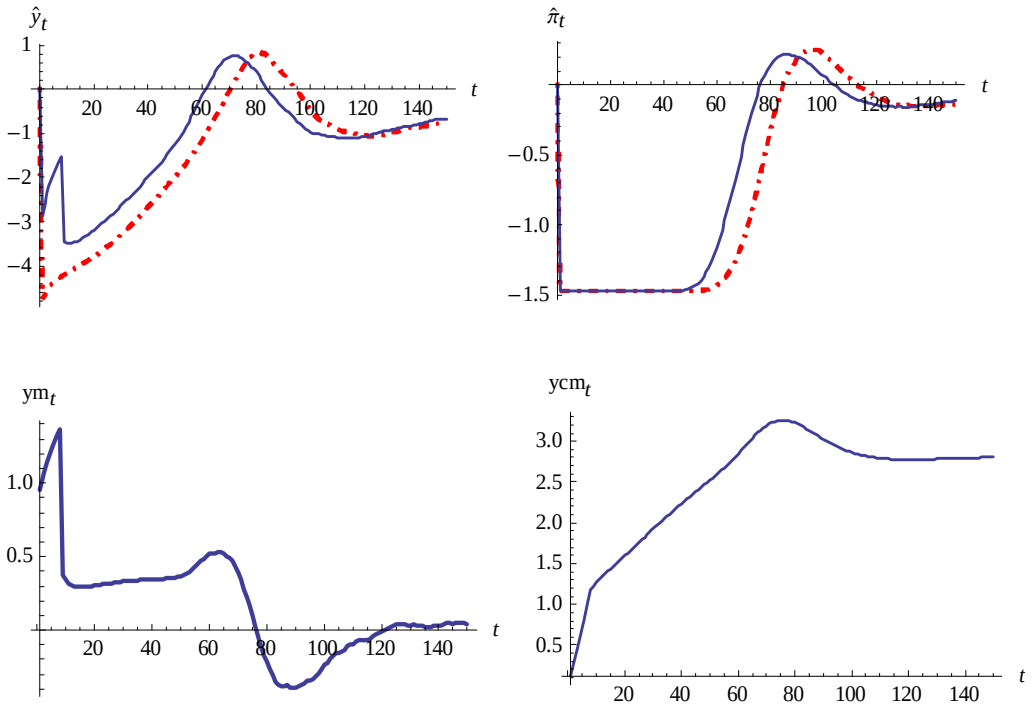


Figure 7: Small policy change when there is a unique steady state with learning gain parameter equal to 10%. Top panel: output and inflation paths under learning with policy change (solid line) and without policy change (dashed line) for temporary policy change. Bottom panel: distributed lag and cumulative output multipliers.

This last example shows that even in cases in which there is a unique steady state, fiscal policy can be important when there is a sufficiently large pessimistic expectations shock that drives the economy into recession and deflation and monetary policy to the ZLB. However the most dramatic results arise when the inflation lower bound is low enough to create the possibility of an additional stagnation steady state.

6 Further Results and Discussion

Our results raise two key questions. In the preceding Section we looked at the effectiveness of fiscal policy when expectations were subject to a pessimistic shock that would lead to convergence to the stagnation trap equilibrium in the absence of fiscal policy. Suppose, however, that fiscal policy is not contemplated until the economy has already converged to the trap. Can a fiscal stimulus still be effective is extracting the economy to the trap and returning it to the targeted steady state? The second question we consider is the size of the critical deflation rate below which a stagnation trap exists. Under our calibration this corresponds to an annual deflation rate of about 4% per year. Are there circumstances in which is milder deflation can result in a stagnation trap?

6.1 Escape from stagnation

Can a suitable fiscal stimulus return the economy to the targeted steady state if, as a result of a large pessimistic shock it has been allowed to converge to the stagnation steady state? This is clearly a worst-case setting given the parameters since we are also assuming that expectations have fully adapted to the stagnation steady state. To examine this question we use numerical simulations using the calibration of the previous Section, but now set the intercepts of the forecast rules so that mean forecasts correspond to mean inflation and output rates at the stagnation steady states. As in Table 1 we consider combinations of increased government spending levels G and policy length T

From Table 3 it can be seen that a fiscal stimulus can be successful in extracting the economy from the stagnation trap even if expectations have settled into levels consistent with the trap. However, the size of the stimulus is now very large – much larger than was required in Tables 1 and 2, when expectations were less pessimistic. The policy with the highest probability of success, around 71%, is a fiscal expansion that is both short and huge, e.g. a two-quarter stimulus at $G = 1.3$, a more than six-fold increase in G (and thus more than the steady state level of output!). A less huge, but still very large, stimulus of $G = 0.7$ for eight quarters, has about a 60% chance of success. Table 3 shows a general trade-off between magnitude and duration, but there are also some intriguing nonlinearities. For example, for the comparatively small stimulus of $G = 0.5$ the probability of success rises to about 50% for

very long durations of between 35 and 100 quarters, while for $G = 0.4$ the duration with best chance of success is near $T = 90$ quarters but is only about 17%. Similarly we find that huge one-quarter stimuli of more than 1.6 continue to have a greater than 50% chance of success, but the greatest chance of success is about 70% for around $G = 2.0$; larger one-quarter stimuli have somewhat smaller chances of success.[18].

$G\backslash T$	1	2	3	4	5	6	8	12
0.5	0	0	0	0	0	0	2	5
0.6	0	0	0	0	0	5	30	49
0.7	0	0	2	8	24	46	60	51
0.8	0	1	17	39	56	61	45	32
0.9	0	37	44	64	58	50	37	24
1.0	0	30	64	60	54	39	24	18
1.3	25	71	63	37	33	21	15	14
1.4	51	65	57	35	22	14	9	14
1.6	67	68	44	31	15	11	17	11

Table 3: Percentage of simulations in which fiscal policy successfully results in convergence to the targeted steady state starting from stagnation expectations. Based on 100 simulations.

The main conclusions of this exercise is that a sufficiently large, short stimulus can have a high probability of extracting the economy from a stagnation trap, even after expectations have adapted to the trap. However, a much smaller stimulus is needed if the policy is implemented earlier, when expectations are less pessimistic.

6.2 Deflation and calibration of the discount factor

The results of the previous Section emphasize the importance of the level of the inflation lower bound for the existence of a stagnation steady state. This brings us to a delicate issue, which is the appropriate calibration of the discount factor β. Our numerical results have used the standard quarterly calibration of $\beta = 0.99$. While this value is fairly standard, there are good

[18]There is no reason a priori to restrict the fiscal stimulus to be of fixed size over time until termination. We have not investigated other time profiles.

reason to consider alternative, higher, values. The historical average realized net real interest rate on US Treasuries bills is not more than 1% per annum. In an economy without growth this would correspond to a discount factor of about $\beta = 0.975$. Presumably one major reason that the lower discount factor $\beta = 0.99$ is often used in calibrated models is that the average net real rate of return on equities is much higher (e.g. 7% p.a.). With no consensus on how to resolve the equity premium puzzle, value of $\beta = 0.99$ might be viewed as a rough and ready compromise value.

However, in the context of liquidity traps, the ZLB and the existence of deflation traps with stagnation steady state, the calibration of β plays an important role and must be faced. Furthermore, for many households the risk-free interest rate may be the key rate of return in determining their consumption plans.

For $\beta = 0.99$ the critical π_L, when $\eta = 0$, corresponds to 1% deflation per quarter, i.e. to 4% per year. Because actual deflation rates in Japan and Europe (as well as the US even in 2009-2010) have been above these values, this suggests either that the inflation lower bound $\underline{\pi}$ is above π_L or that policy has prevented inflation from falling below the critical level. On the other hand if $\beta = 0.995$ or $\beta = 0.9975$ is the appropriate value to use in the consumption Euler equation, then the critical deflation rate is around 2% or 1% per year, in line with values that have occasionally been observed, e.g., in Japan in various periods since the 1990s.[19] The possibility of deflation falling below these levels would appear to be a serious concern, for example, in at least some countries in Europe in 2015 if the euro-area entered renewed recession.

[19] We note that Eggertsson (2010) uses a calibration of $\beta = 0.997$ in a model of the US economy during the Great Depression. During the Great Depression deflation reached 10% per year during the trough.

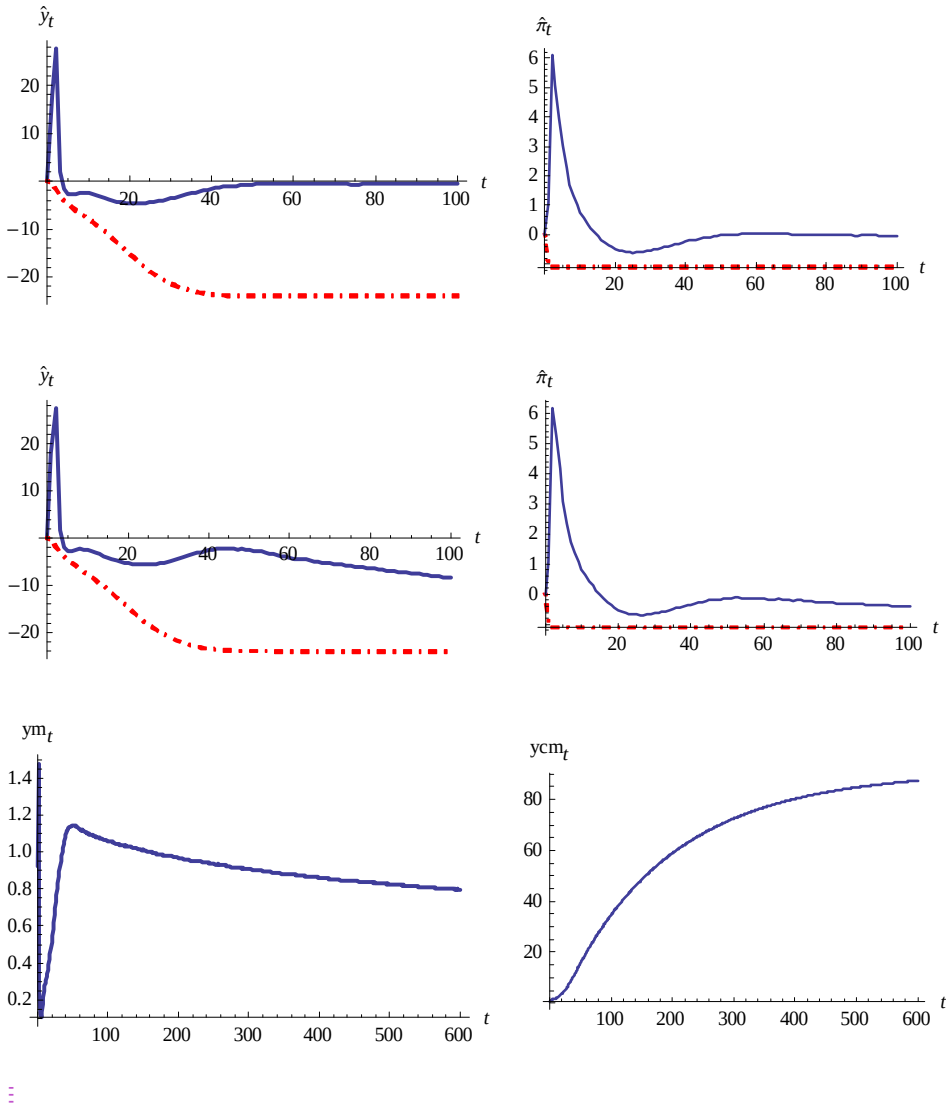


Figure 8: High discount factor, $\beta = 0.995$. The top two panels show paths under learning with policy change (solid line) and without policy change (dashed line). Top panel: means of paths with eventual convergence to targeted steady state under policy. Middle panel: mean paths with eventual convergence to deflation trap despite policy. Bottom panel: distributed lag and cumulative output multipliers across all paths.

In Figure 8 we redo the simulations using the higher discount factor of $\beta = 0.995$. For this policy experiment agents are assumed to use a gain parameter 10%. For these simulations we set $\underline{\hat{\pi}} = -0.011$, i.e. which is somewhat below the critical value, so that there are three steady states, including a low-level trap. For this value of $\underline{\hat{\pi}}$ in the stagnation steady state deflation is about 2.4% per annum. As in Figures 5 and 6 we set $\underline{\hat{c}} = -0.3$ which corresponds to a 30% reduction in consumption from the targeted steady state. Initial expectations following a presumed large pessimistic shock are set at $\hat{\pi}^e = -0.011$ and $\hat{Y}^e = -0.01$.

From Figure 8 one can see that in the absence of fiscal policy the economy would fall into a stagnation sate with deflation. For these parameters a short but very aggressive fiscal stimulus is needed to avoid stagnation. In these simulations we consider an increase in G from 0.20 to 0.40 for $T = 2$ periods.

As in the case of Figure 6, our fiscal policy does not guarantee that the stagnation state will be avoided. In fact, for these parameters, based on $10,000$ simulations, 61% of the with-policy simulations converged to the intended steady state, 32% eventually sank to the stagnation steady state, while 7% had not yet converged. In Figure 8 the top panel shows the mean paths of those simulations that under the policy converge to the intended steady state. The middle panel shows the mean paths of those paths that eventually under the policy converge to the stagnation steady state. For these cases we note that the process is typically very slow.

The bottom panel shows the multipliers averaged over all simulations. We remark that the cumulative multipliers are very high, even though the policy does not guarantee escape from eventual stagnation and deflation. An interesting avenue for future research would be the design of optimal fiscal and monetary policy under learning in our framework. Another less ambitious task would be finding a mix of policies that maximizes the chance of escaping the stagnation trap.

7 Conclusions

Calculation of the impact of fiscal policy in New Keynesian models usually assume rational expectations. Although this is standard, the rational expectation hypothesis assumes a great deal of knowledge on the part of agents and implicitly also assumes coordination of agents on these expectations. These criticisms of RE are particularly forceful when the government undertakes a

policy outside of the usual range of experience, such as a previously unanticipated increase in government spending for a specified duration in time. We have therefore studied fiscal policy when agents take account of the announced policy but when rational expectations is replaced by the assumption that agents forecast future values of endogenous market variables, i.e., future inflation and aggregate output using adaptive learning.[20]

Our results on fiscal policy depend sensitively on whether we are considering fiscal policy in normal times, when the ZLB on interest rates does not bind, or at times of low inflation and output when the ZLB binds. Our basic finding for normal times is that under adaptive learning the impact of the fiscal policy is front-loaded compared to RE, i.e. the strongest effects are towards the beginning of the policy. Furthermore, the positive effects during the policy implementation period can be partially offset by negative output effects when the policy comes to an end.

Outside normal times, when inflation and output may be very low, there are several issues that must be addressed. First, since we are assuming that private agents understand the monetary policy rule in which the interest rate depends on inflation and output, we must allow for the possibility that agents anticipate that future interest rates will be at the ZLB. In addition, we also impose lower bounds on inflation and consumption, which implies that if expectations of inflation and output enter the deflation trap region, then they will tend to converge to a stagnation steady state. In this framework there are two locally stable steady states with complementary domains of attraction: the steady state intended by policy and a stagnation steady state with mild deflation and a low level of output.

There is thus the possibility that a large pessimistic expectation shock (due, say, to a recent financial crisis) can push the economy to the ZLB and along a path to steady state stagnation and deflation. In this setting a fiscal stimulus can be particularly potent. We have seen that a sufficiently large temporary increase in government spending can increase output and inflation enough to prevent the economy being pulled into deflation and stagnation. Finally, we have seen that the needed government spending increase is much less if the policy is implemented early, before expectations have adapted to the stagnation trap.

[20] This approach was initiated by Evans, Honkapohja, and Mitra (2009).

Appendix 1: Model details

We develop here the model outlined in Section 2 following the analysis of Eusepi and Preston (2010).

Consumers

Next use the flow budget constraint and the NPG (no Ponzi game) condition to obtain an intertemporal budget constraint. Write

$$b_{t,i} = r_t b_{t-1,i} + \phi_{t,i},$$

where $r_t = R_{t-1}/\pi_t$. and

$$\phi_{t,i} = Y_{t,i} - C_{t,i} - G_{t,i}. \tag{28}$$

In (28) consumers are assumed to know that the government will run a balanced budget policy. Iterating (28) forward and imposing

$$\lim_{s\to\infty} \hat{E}_{t,i}(D_{t,t+s})^{-1} b_{t+s,i} = 0, \tag{29}$$

where

$$D_{t,t+s} = \prod_{i=1}^{s} r_{t+i},$$

with $r_{t+s} = R_{t+s-1}/\pi_{t+s}$. We obtain the life-time budget constraint of the household

$$0 = \Delta_{t-1,i} + \phi_{t,i} + \sum_{s=1}^{\infty} \hat{E}_t(D_{t,t+s})^{-1} \phi_{t+s,i},$$

where $\Delta_{t-1,i} = r_t b_{t-1,i}$ and

$$\phi_{t+s,i} = Y_{t+s,i} - C_{t+s,i} - G_{t+s,i}. \tag{30}$$

Because there is zero net government debt and we have representative agents, it follows that $\Delta_{t-1,i} = b_{t-1,i} = 0$ for all agents. Thus

$$0 = \phi_{t,i} + \sum_{s=1}^{\infty} \hat{E}_t(D_{t,t+s})^{-1} \phi_{t+s,i}, \tag{31}$$

Linearizing (31) we have

$$\begin{aligned} 0 &= \tilde{\phi}_{t,i} + \sum_{s=1}^{\infty} \beta^s \tilde{\phi}_{t+s,i} - \bar{\phi} \sum_{s=1}^{\infty} \beta^{s+1} \sum_{i=1}^{s} \tilde{r}_{t+i}, \\ \tilde{\phi}_{t+s,i} &= \tilde{Y}_{t+s,i} - \tilde{C}_{t+s,i} - \tilde{G}_{t+s,i}, \end{aligned}$$

where tilde denotes deviation from the non-stochastic steady state for each variable, for example, $\tilde{Y}_{t+s,i} = Y_{t+s,i} - \bar{Y}$. Note that here $\bar{\phi} = 0$ by market clearing, so that

$$\sum_{s=0}^{\infty} \beta^s \tilde{C}_{t+s,i} = \sum_{s=0}^{\infty} \beta^s (\tilde{Y}_{t+s,i} - \tilde{G}_{t+s,i}).$$

The next step is iterate the linearized Euler equation forward. We have

$$\tilde{C}_{t,i} = \hat{E}_{t,i} \tilde{C}_{t+1,i} - \beta \bar{C} \hat{E}_{t,i} \tilde{r}_{t+1} \text{ and}$$

$$\tilde{C}_{t,i} = \hat{E}_{t,i} \tilde{C}_{t+s,i} - \beta \bar{C} \hat{E}_{t,i} \sum_{i=1}^{s} \tilde{r}_{t+i}, \tag{32}$$

where $\tilde{C}_{t,i} = C_{t,i} - \bar{C}$ and $\tilde{r}_{t+i} = r_{t+i} - \bar{r}$. This can also be written as

$$\hat{C}_{t,i} = \hat{E}_{t,i} \hat{C}_{t+1,i} - \hat{E}_{t,i} \hat{r}_{t+1}, \text{ where}$$

$$\hat{C}_{t,i} = \frac{C_{t,i} - \bar{C}}{\bar{C}} \text{ and } \hat{r}_{t+1} = \frac{r_{t+1} - \bar{r}}{\bar{r}},$$

which yields

$$\hat{C}_{t,i} = \hat{E}_{t,i} \hat{C}_{t+s,i} - \hat{E}_{t,i} \sum_{i=1}^{s} \hat{r}_{t+i},$$

where we have used $\beta^{-1} = \bar{r}$.

Combining (32) with the linearized budget constraint in expectational form we get:

$$\begin{aligned} 0 &= \tilde{Y}_{t,i} - \tilde{C}_{t,i} - \tilde{G}_{t,i} + \sum_{s=1}^{\infty} \beta^s \hat{E}_{t,i} (\tilde{Y}_{t+s,i} - \tilde{G}_{t+s,i}) \\ &\quad - \sum_{s=1}^{\infty} \beta^s \tilde{C}_{t,i} - \beta \bar{C} \sum_{s=1}^{\infty} \beta^s \hat{E}_{t,i} \sum_{i=1}^{s} \tilde{r}_{t+i}. \end{aligned}$$

This yields the consumption function for consumer i

$$\begin{aligned}\tilde{C}_{t,i} &= (1-\beta)[\tilde{Y}_{t,i} - \tilde{G}_{t,i} + \sum_{s=1}^{\infty} \beta^s \hat{E}_{t,i}(\tilde{Y}_{t+s,i} - \tilde{G}_{t+s,i})] \\ &\quad -\beta \bar{C} \hat{E}_{t,i} \sum_{s=1}^{\infty} \beta^s \tilde{r}_{t+s}.\end{aligned}$$

This can also be written in proportional form yielding

$$\begin{aligned}\hat{C}_t &= (1-\beta)\left[\frac{\hat{Y}_t}{(\bar{C}/\bar{Y})} - \frac{\hat{G}_t}{(\bar{C}/\bar{G})} + \sum_{s=1}^{\infty} \beta^s \hat{E}_t \left(\frac{\hat{Y}_{t+s}}{(\bar{C}/\bar{Y})} - \frac{\hat{G}_{t+s}}{(\bar{C}/\bar{G})})\right)\right] \\ &\quad -\hat{E}_t \sum_{s=1}^{\infty} \beta^s \hat{r}_{t+s},\end{aligned}$$

which is (6) making use of the representative agent assumption.

The market clearing condition is

$$Y_t = C_t + G_t + \frac{\psi}{2}(\pi_t - \pi^*)^2$$

which at the targeted steady state $\pi_t = \pi^*$ is

$$\bar{Y} = \bar{C} + \bar{G}$$

It follows that the linear approximation around the targeted steady state is

$$\begin{aligned}\hat{Y}_t &= (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t, \\ \bar{g} &\equiv \frac{\bar{G}}{\bar{Y}} \text{ and } \frac{\bar{C}}{\bar{Y}} = 1 - \bar{g}.\end{aligned}$$

We thus have

$$\begin{aligned}\hat{C}_t &= (1-\beta)(1-\bar{g})^{-1}\hat{Y}_t + (1-\beta)(1-\bar{g})^{-1}\sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{Y}_{t+s} \\ &\quad -(1-\beta)\bar{g}(1-\bar{g})^{-1}\hat{G}_t - (1-\beta)\bar{g}(1-\bar{g})^{-1}\sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{G}_{t+s} \\ &\quad -\beta\hat{R}_t - \beta\hat{E}_t\sum_{s=1}^{\infty}\beta^s \hat{R}_{t+s} + \hat{E}_t\sum_{s=1}^{\infty}\beta^s \hat{\pi}_{t+s}. \qquad (33)\end{aligned}$$

For the contemporaneous interest rate rule $\hat{R}_t = \phi_\pi \hat{\pi}_t + \phi_y \hat{Y}_t$ we have

$$\begin{aligned}\hat{C}_t &= \left((1-\beta)(1-\bar{g})^{-1} - \beta\phi_y\right)\hat{Y}_t + \left((1-\beta)(1-\bar{g})^{-1} - \beta\phi_y\right)\sum_{s=1}^{\infty}\beta^s \hat{E}_t \hat{Y}_{t+s} \\ &\quad -\beta\phi_\pi\hat{\pi}_t + (1-\beta\phi_\pi)\sum_{s=1}^{\infty}\beta^s \hat{E}_t\hat{\pi}_{t+s}. \\ &\quad -(1-\beta)\bar{g}(1-\bar{g})^{-1}\hat{G}_t - (1-\beta)\bar{g}(1-\bar{g})^{-1}\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s} \qquad (34)\end{aligned}$$

Firms

As in the text

$$\begin{aligned}(1-\tau)(1-\theta)\bar{Y} + \bar{S}\bar{Y}\theta &= 0, \text{or} \\ (1-\tau)(1-\theta) + \bar{S}\theta &= 0\end{aligned}$$

We need to linearize around steady state $\bar{Y}, \bar{S}, \pi^*, \bar{Q}$ where $\bar{Q}$ is the steady state value of $Q_{t,t+1}$. Note that $\bar{Q} = \beta/\pi^*$.

It is useful to define the mark-up μ by equation (10). In the steady state $\mu = \theta(\theta-1)^{-1}$. Log-linearizing (10) gives

$$\hat{\mu}_t = -(\theta-1)^{-1}\hat{\theta}_t.$$

From above the mean real marginal cost is

$$\bar{S} = \frac{(\theta-1)(1-\tau)}{\theta} = (1-\tau)\mu^{-1}.$$

Next, we linearize (8) and obtain

$$\begin{aligned}0 &= (1-\tau)(1-\theta)\tilde{Y}_t - (1-\tau)(1-\theta)\theta\bar{Y}\left(\frac{P_{t,j}}{P_t} - 1\right) - (1-\tau)\bar{Y}\tilde{\theta}_t + \\ &\quad \theta\bar{Y}\tilde{S}_t + \bar{S}\theta\tilde{Y}_t - \bar{S}\bar{Y}\theta(1+\theta/\alpha)\left(\frac{P_{t,j}}{P_t} - 1\right) + \bar{S}\bar{Y}\tilde{\theta}_t - \\ &\quad \pi^*\psi\left(\frac{P_{t,j}}{P_{t-1,j}} - \pi^*\right) + (\pi^*)^2\hat{E}_{t,j}\left[\bar{Q}\psi\left(\frac{P_{t+1,j}}{P_{t,j}} - \pi^*\right)\right].\end{aligned}$$

where $\tilde{\theta}_t = \theta_t - \theta$, $\tilde{Y}_t = Y_t - \bar{Y}$, etc. Also, to the first order

$$\frac{P_{t+1,j}}{P_{t,j}} - 1 = \tilde{p}_{t+1,j} - \tilde{p}_{t,j},$$

where $\tilde{p}_t \equiv \ln P_t - (\pi^* - 1)t$ and $\tilde{p}_{t,j} \equiv \ln P_{t,j} - (\pi^* - 1)t$. It follows that

$$\begin{aligned} 0 \;=\; & (1-\tau)(1-\theta)\hat{Y}_t - (1-\tau)(1-\theta)\theta\left(\tilde{p}_{t,j} - \tilde{p}_t\right) - (1-\tau)\tilde{\theta}_t + \\ & \theta\tilde{S}_t + \bar{S}\theta\hat{Y}_t - \bar{S}\theta(1+\theta/\alpha)\left(\tilde{p}_{t,j} - \tilde{p}_t\right) + \bar{S}\tilde{\theta}_t - \\ & \pi^*\bar{Y}^{-1}\psi\left(\tilde{p}_{t,j} - \tilde{p}_{t-1,j}\right) + \hat{E}_{t,j}\left[\bar{Q}\bar{Y}^{-1}\left(\pi^*\right)^2\psi\left(\tilde{p}_{t+1,j} - \tilde{p}_{t,j}\right)\right]. \end{aligned}$$

In this expression the terms involving $\hat{Y}_t$ drop out because of the steady state relations. Then combine the terms involving $\tilde{p}_{t,j} - \tilde{p}_t$ and rearrange the coefficients using the steady state relations. Also combine the terms involving $\tilde{\theta}_t$ and use the log-linearization between $\hat{\mu}_t$ and $\hat{\theta}_t$. This yields the result

$$\tilde{p}_{t,j} - \tilde{p}_{t-1,j} = \beta\hat{E}_{t,j}(\tilde{p}_{t+1,j} - \tilde{p}_{t,j}) + \frac{\omega}{\bar{\psi}}(\tilde{p}_t - \tilde{p}_{t,j}) + \frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}(\hat{S}_t + \hat{\mu}_t), \qquad (35)$$

$$\text{where } \omega = \theta\bar{Y}\bar{S}\frac{\theta + \alpha(1-\theta)}{\alpha} \text{ and } \bar{\psi} = \psi\pi^*.$$

Here

$$\hat{S}_t = \frac{S_t - \bar{S}}{\bar{S}},\ \hat{\mu}_t = \frac{\mu_t - \mu}{\mu}.$$

Next, we use the back-shift operator technique on (35) (see pp. 393-5 of Sargent (1987)).[21] Taking expectations $\hat{E}_{t,j}$ of (35) and rearranging we get

$$\left[1 - \left(1 + \beta^{-1} + \frac{\omega}{\beta\bar{\psi}}\right)B + \beta^{-1}B^2\right]\hat{E}_{t,j}\tilde{p}_{t+1,j} = \hat{E}_{t,j}\left[-\frac{\omega}{\beta\bar{\psi}}\tilde{p}_t - \frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{S}_t + \hat{\mu}_t)\right].$$

The quadratic in B can be factored into the product $(1-\gamma_1 B)(1-\gamma_2 B)$ with roots $0 < \gamma_1 < 1 < \gamma_2$ satisfying

$$\gamma_1\gamma_2 = \beta^{-1} \text{ and } \gamma_1 + \gamma_2 = \beta^{-1}(1 + \beta + \omega\bar{\psi}^{-1}).$$

We write

$$(1-\gamma_1 B)(1-\gamma_2 B)E_{t,j}\tilde{p}_{t+1,j} = \hat{E}_{t,j}\left[-\frac{\omega}{\beta\bar{\psi}}\tilde{p}_t - \frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{S}_t + \hat{\mu}_t)\right]$$

[21] where $B^{-1}E_{t-1}x_{t+j} = E_{t-1}x_{t+j+1}$. As emphasized by Sargent, it is legitimate to operate on both sides of an equation by polynomials involving non-positive powers of B.

or

$$(B^{-1}-\gamma_1)(B^{-1}-\gamma_2)E_{t,j}\tilde{p}_{t-1,j}=\hat{E}_{t,j}\left[-\frac{\omega}{\beta\bar{\psi}}\tilde{p}_t-\frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{S}_t+\hat{\mu}_t)\right].$$

Operating on both sides by $(B^{-1}-\gamma_2)^{-1}$ we get

$$\begin{aligned}(B^{-1}-\gamma_1)E_{t,j}\tilde{p}_{t-1,j} &= \frac{1}{(B^{-1}-\gamma_2)}\hat{E}_{t,j}\left[-\frac{\omega}{\beta\bar{\psi}}\tilde{p}_t-\frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{S}_t+\hat{\mu}_t)\right]\\ &= \frac{\gamma_2^{-1}}{(1-\gamma_2^{-1}B^{-1})}\hat{E}_{t,j}\left[\frac{\omega}{\beta\bar{\psi}}\tilde{p}_t+\frac{\theta\bar{Y}\bar{S}}{\beta\bar{\psi}}(\hat{S}_t+\hat{\mu}_t)\right].\end{aligned}$$

Writing $(1-\gamma_2^{-1}B^{-1})^{-1}=1+\gamma_2^{-1}B^{-1}+\gamma_2^{-2}B^{-2}+\ldots$ and also using $\gamma_1\gamma_2=\beta^{-1}$ we obtain

$$\tilde{p}_{t,j}=\gamma_1\tilde{p}_{t-1,j}+\frac{\gamma_1}{\bar{\psi}}\left(\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\left[\omega\tilde{p}_{t+s}+\theta\bar{Y}\bar{S}(\hat{S}_{t+s}+\hat{\mu}_{t+s})\right]\right). \qquad (36)$$

as the evolution of the optimal price of firm j.

We now define $\tilde{\pi}_t=\tilde{p}_t-\tilde{p}_{t-1}$. Note that $\tilde{\pi}_t$ is the rate of inflation net of the target rate π^*. Using

$$\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{\pi}_{t+s}=\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s}-\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s-1}$$

and

$$\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s-1}=\tilde{p}_{t-1}+\gamma_2^{-1}\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s},$$

we obtain

$$\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{p}_{t+s}=(1-\gamma_2^{-1})^{-1}\sum_{s=0}^{\infty}\gamma_2^{-s}\hat{E}_{t,j}\tilde{\pi}_{t+s}+(1-\gamma_2^{-1})^{-1}\tilde{p}_{t-1}.$$

Plugging into (36) we obtain

$$\begin{aligned}\tilde{p}_{t,j} &= \gamma_1\tilde{p}_{t-1,j}+\frac{\gamma_1\omega}{\bar{\psi}(1-\beta\gamma_1)}\tilde{p}_{t-1}+\frac{\gamma_1\omega}{\bar{\psi}(1-\beta\gamma_1)}\sum_{s=0}^{\infty}(\beta\gamma_1)^s\hat{E}_{t,j}\tilde{\pi}_{t+s}+\\ &\quad\frac{\gamma_1\theta\bar{Y}\bar{S}}{\bar{\psi}}\sum_{s=0}^{\infty}(\beta\gamma_1)^s\hat{E}_{t,j}(\hat{S}_{t+s}+\hat{\mu}_{t+s}).\end{aligned}$$

Subtracting $\tilde{p}_{t-1,j}$ from both sides and collecting terms, and imposing the representative agent assumption, the coefficient of $\tilde{p}_{t-1,j}$ becomes

$$\begin{aligned}(\gamma_1 - 1) + \frac{\gamma_1 \omega}{\bar{\psi}(1-\beta\gamma_1)} &= \frac{(\gamma_1 - 1)(1-\beta\gamma_1) + (\omega/\bar{\psi})\gamma_1}{1-\beta\gamma_1} \\ &= \frac{\gamma_1(\omega/\bar{\psi} + \beta + 1 - \beta\gamma_1) - 1}{1-\beta\gamma_1} = \frac{\gamma_1\beta\gamma_2 - 1}{1-\beta\gamma_1} = 0.\end{aligned}$$

Note that

$$\gamma_1 = 1 - \frac{\gamma_1 \omega}{\bar{\psi}(1-\beta\gamma_1)}. \tag{37}$$

Using the representative agent assumption the resulting equation becomes

$$\tilde{\pi}_t = \frac{\gamma_1 \omega}{\bar{\psi}(1-\beta\gamma_1)} \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \tilde{\pi}_{t+s} + \frac{\gamma_1 \theta \bar{Y} \bar{S}}{\bar{\psi}} \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t (\hat{S}_{t+s} + \hat{\mu}_{t+s}). \tag{38}$$

Next, we work out the relationship between $\hat{S}_t$ and $\hat{Y}_t$, which will yield the Phillips curve. Using (3), (9) and the production function one obtains

$$S_t = \frac{\gamma}{\alpha}(1-g_t) Y_t^{\frac{1+\varepsilon}{\alpha}} A_t^{-\left(\frac{1+\varepsilon}{\alpha}\right)}.$$

so that

$$\hat{S}_t = \left(\frac{1+\varepsilon}{\alpha}\right)(\hat{Y}_t - \hat{A}_t) - \frac{\bar{g}}{1-\bar{g}}\hat{g}_t$$

Since

$$\hat{g}_t = \hat{G}_t - \hat{Y}_t,$$

it follows that

$$\hat{S}_t = \left(\frac{1+\varepsilon}{\alpha}\right)(\hat{Y}_t - \hat{A}_t) - \frac{\bar{g}}{1-\bar{g}}(\hat{G}_t - \hat{Y}_t). \tag{39}$$

Letting $\hat{\pi}_t \equiv \tilde{\pi}_t/\pi^*$ and substituting into (38) we finally obtain the Phillips curve

$$\begin{aligned}\hat{\pi}_t &= a_1 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{\pi}_{t+s} + a_2 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{Y}_{t+s} - a_3 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{A}_{t+s} \\ &\quad - a_4 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{G}_{t+s} + a_5 \sum_{s=0}^{\infty} (\beta\gamma_1)^s \hat{E}_t \hat{\mu}_{t+s}.\end{aligned} \tag{40}$$

where the coefficients a_i are defined as:

$$\begin{aligned} a_1 &= \frac{\omega\gamma_1}{\bar{\psi}(1-\beta\gamma_1)}; a_2 = \frac{\gamma_1\theta\bar{Y}\bar{S}}{\bar{\psi}\pi^*}\left(\frac{1+\varepsilon}{\alpha}+\frac{\bar{g}}{1-\bar{g}}\right); \\ a_3 &= \frac{(1+\varepsilon)\gamma_1\theta\bar{Y}\bar{S}}{\alpha\bar{\psi}\pi^*}; a_4 = \frac{\bar{g}\gamma_1\theta\bar{Y}\bar{S}}{(1-\bar{g})\bar{\psi}\pi^*}; a_5 = \frac{\gamma_1\theta\bar{Y}\bar{S}}{\bar{\psi}\pi^*}. \end{aligned}$$

Note that by (37) we have $a_1 = 1-\gamma_1$. Rearrange the above equation to get (14).

Temporary equilibrium

The IS curve under the contemporaneous interest-rate rule is obtained from combining the consumption function (34) with the market-clearing equation $\hat{Y}_t = (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t$. This yields

$$\begin{aligned} & \beta(1-\bar{g})\phi_\pi\hat{\pi}_t + \left(\beta+\beta(1-\bar{g})\phi_y\right)\hat{Y}_t \\ = \; & [(1-\bar{g})(1-\beta\phi_\pi)]\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{\pi}_{t+s} + [(1-\beta)-(1-\bar{g})\beta\phi_y]\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{Y}_{t+s} \\ & +\bar{g}\hat{G}_t - (1-\beta)\bar{g}\sum_{s=0}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s} \end{aligned} \tag{41}$$

Henceforth, we use the following short-hand notation

$$\begin{aligned} b_1 &\equiv (1-\bar{g})(1-\beta\phi_\pi), \\ b_2 &\equiv (1-\beta)-(1-\bar{g})\beta\phi_y. \end{aligned}$$

Let us now write the price setting equation (14) and the demand equation (41) (under subjective expectations) in matrix form. Let

$$M = \begin{pmatrix} 1-a_1 & -a_2 \\ \beta(1-\bar{g})\phi_\pi & \beta+\beta(1-\bar{g})\phi_Y \end{pmatrix} \tag{42}$$

Then

$$M\begin{pmatrix}\hat{\pi}_t\\ \hat{Y}_t\end{pmatrix} = \sum_{s=1}^{\infty}\begin{pmatrix} a_1(\beta\gamma_1)^s & a_2(\beta\gamma_1)^s \\ b_1\beta^s & b_2\beta^s\end{pmatrix}\begin{pmatrix}\hat{E}_t\hat{\pi}_{t+s}\\ \hat{E}_t\hat{Y}_{t+s}\end{pmatrix} + \begin{pmatrix}-a_4\\ \beta\bar{g}\end{pmatrix}\hat{G}_t +$$
$$\sum_{s=0}^{\infty}\begin{pmatrix} -a_3(\beta\gamma_1)^s & a_5(\beta\gamma_1)^s \\ 0 & 0\end{pmatrix}\begin{pmatrix}\hat{E}_t\hat{A}_{t+s}\\ \hat{E}_t\hat{\mu}_{t+s}\end{pmatrix} + \sum_{s=1}^{\infty}\begin{pmatrix}-a_4(\beta\gamma_1)^s\\ -(1-\beta)\bar{g}\beta^s\end{pmatrix}\hat{E}_t\hat{G}_{t+s}.$$

For the shock terms above we get

$$\begin{aligned}
&\sum_{s=0}^{\infty}\begin{pmatrix} -a_3(\beta\gamma_1)^s & a_5(\beta\gamma_1)^s \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \rho_A^s\hat{A}_t \\ \rho_\mu^s\hat{\mu}_t \end{pmatrix} \\
= &\begin{pmatrix} \sum_{s=0}^{\infty}\left(-a_3(\rho_A\beta\gamma_1)^s\hat{A}_t + a_5(\rho_\mu\beta\gamma_1)^s\hat{\mu}_t\right) \\ 0 \end{pmatrix} \\
= &\begin{pmatrix} \sum_{s=0}^{\infty}\left(-a_3(\rho_A\beta\gamma_1)^s\hat{A}_t + a_5(\rho_\mu\beta\gamma_1)^s\hat{\mu}_t\right) \\ 0 \end{pmatrix} \\
= &\begin{pmatrix} -a_3(1-\rho_A\beta\gamma_1)^{-1}\hat{A}_t + a_5(1-\rho_\mu\beta\gamma_1)^{-1}\hat{\mu}_t \\ 0 \end{pmatrix}.
\end{aligned}$$

Consider a change in government spending that is known to be temporary. We assume that initially, at $t = 0$, we are in the steady state corresponding to $G = \bar{G}$, and consider the following policy experiment, assumed fully credible and announced at the start of period 1:

$$G_t = \tau_t = \begin{cases} \bar{G}', t = 1, ..., T \\ \bar{G}, t \geq T + 1, \end{cases} \tag{43}$$

i.e., government spending and taxes are changed in period $t = 1$ and this change is reversed at a later period $T+1$. Thus, the experiment is one where the policy change is announced in period 1 to take place in the future for a fixed number T of periods. Denote the change in government spending by ΔG $(= \bar{G}' - \bar{G}\ )$ so that

$$\hat{G}_t = \begin{cases} \frac{\Delta G}{\bar{G}}, t = 1, ..., T \\ 0, t \geq T + 1. \end{cases}$$

We first consider the evolution of the learning economy during the period when the policy increase is in effect i.e. for periods $t = 1, ..., T$. Then we

have

$$\sum_{s=1}^{\infty}\left(\begin{array}{c}-a_4(\beta\gamma_1)^s\\-(1-\beta)\bar{g}\beta^s\end{array}\right)\hat{E}_t\hat{G}_{t+s} = \sum_{s=1}^{T-t}\left(\begin{array}{c}-a_4(\beta\gamma_1)^s\\-(1-\beta)\bar{g}\beta^s\end{array}\right)\frac{\Delta G}{\bar{G}} =$$
$$\left(\begin{array}{c}-a_4\sum_{s=1}^{T-t}(\beta\gamma_1)^s\\-(1-\beta)\bar{g}\sum_{s=1}^{T-t}\beta^s\end{array}\right)\frac{\Delta G}{\bar{G}} = \left(\begin{array}{c}-a_4\beta\gamma_1\frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1}\\-\beta(1-\beta)\bar{g}\frac{1-\beta^{T-t}}{1-\beta}\end{array}\right)\frac{\Delta G}{\bar{G}}.$$

Write the final form of the model when agents are learning in the following matrix form (which is true for $1 \leq t \leq T$)

$$M\left(\begin{array}{c}\hat{\pi}_t\\\hat{Y}_t\end{array}\right) = \sum_{s=1}^{\infty}\left(\begin{array}{cc}a_1(\beta\gamma_1)^s & a_2(\beta\gamma_1)^s\\b_1\beta^s & b_2\beta^s\end{array}\right)\left(\begin{array}{c}\hat{E}_t\hat{\pi}_{t+s}\\\hat{E}_t\hat{Y}_{t+s}\end{array}\right) +$$
$$\left(\begin{array}{c}-a_3(1-\rho_A\beta\gamma_1)^{-1}\hat{A}_t + a_5(1-\rho_\mu\beta\gamma_1)^{-1}\hat{\mu}_t\\0\end{array}\right) +$$
$$\left(\begin{array}{c}-a_4\beta\gamma_1\frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1}\\-\beta(1-\beta)\bar{g}\frac{1-\beta^{T-t}}{1-\beta}\end{array}\right)\frac{\Delta G}{\bar{G}} + \left(\begin{array}{c}-a_4\\\beta\bar{g}\end{array}\right)\frac{\Delta G}{\bar{G}}. \tag{44}$$

Note that when $t > T$, the model evolution under learning is governed by

$$M\left(\begin{array}{c}\hat{\pi}_t\\\hat{Y}_t\end{array}\right) = \sum_{s=1}^{\infty}\left(\begin{array}{cc}a_1(\beta\gamma_1)^s & a_2(\beta\gamma_1)^s\\b_1\beta^s & b_2\beta^s\end{array}\right)\left(\begin{array}{c}\hat{E}_t\hat{\pi}_{t+s}\\\hat{E}_t\hat{Y}_{t+s}\end{array}\right) +$$
$$\left(\begin{array}{c}-a_3(1-\rho_A\beta\gamma_1)^{-1}\hat{A}_t + a_5(1-\rho_\mu\beta\gamma_1)^{-1}\hat{\mu}_t\\0\end{array}\right). \tag{45}$$

since $\hat{G}_t = 0$ when $t > T$.

We consider PLMs of the same form as the MSV solution of the economy. One can solve the model under RE with fixed G_t to get a stochastic steady state of the form

$$\begin{aligned}\hat{\pi}_t &= f_\pi + d_{\pi A}\hat{A}_t + d_{\pi\mu}\hat{\mu}_t\\ \hat{Y}_t &= f_Y + d_{YA}\hat{A}_t + d_{Y\mu}\hat{\mu}_t\end{aligned}$$

where $\hat{A}_t, \hat{\mu}_t$ are observable processes (with known coefficients) given by (19) and (20). These can be used to construct forecasts $\hat{E}_t\pi_{t+s}$ and $\hat{E}_t\hat{Y}_{t+s}$ which

are then inserted into the model (44) to govern the evolution of the economy for the first T periods (and by (45) for periods after T).

Using the MSV form of the PLM we get

$$\begin{aligned}\hat{E}_t\hat{\pi}_{t+s} &= f_\pi + d_{\pi A}\hat{E}_t\hat{A}_{t+s} + d_{\pi\mu}\hat{E}_t\hat{\mu}_{t+s} \\ &= f_\pi + d_{\pi A}\rho_A^s\hat{A}_t + d_{\pi\mu}\rho_\mu^s\hat{\mu}_t.\end{aligned}$$

Similarly,

$$\hat{E}_t\hat{Y}_{t+s} = f_Y + d_{YA}\rho_A^s\hat{A}_t + d_{Y\mu}\rho_\mu^s\hat{\mu}_t.$$

Consider the term below that needs to be evaluated in the first row of (44)

$$\begin{aligned}& a_1\sum_{s=1}^{\infty}(\beta\gamma_1)^s\hat{E}_t\hat{\pi}_{t+s} + a_2\sum_{s=1}^{\infty}(\beta\gamma_1)^s\hat{E}_t\hat{Y}_{t+s} \\ = \;& a_1\sum_{s=1}^{\infty}(\beta\gamma_1)^s(f_\pi + d_{\pi A}\hat{E}_t\hat{A}_{t+s} + d_{\pi\mu}\hat{E}_t\hat{\mu}_{t+s}) \\ & + a_2\sum_{s=1}^{\infty}(\beta\gamma_1)^s(f_Y + d_{YA}\rho_A^s\hat{A}_t + d_{Y\mu}\rho_\mu^s\hat{\mu}_t) \\ = \;& (a_1f_\pi + a_2f_Y)\frac{\beta\gamma_1}{1-\beta\gamma_1} + (a_1d_{\pi A} + a_2d_{YA})\frac{\rho_A\beta\gamma_1}{1-\rho_A\beta\gamma_1}\hat{A}_t \\ & + (a_1d_{\pi\mu} + a_2d_{Y\mu})\frac{\rho_\mu\beta\gamma_1}{1-\rho_\mu\beta\gamma_1}\hat{\mu}_t.\end{aligned}$$

Similarly consider the term below that is required to be evaluated in the second row of (44)

$$\begin{aligned}& b_1\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{\pi}_{t+s} + b_2\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{Y}_{t+s} \\ = \;& b_1\sum_{s=1}^{\infty}\beta^s(f_\pi + d_{\pi A}\hat{E}_t\hat{A}_{t+s} + d_{\pi\mu}\hat{E}_t\hat{\mu}_{t+s}) \\ & + a_2\sum_{s=1}^{\infty}(\beta\gamma_1)^s(f_Y + d_{YA}\rho_A^s\hat{A}_t + d_{Y\mu}\rho_\mu^s\hat{\mu}_t) \\ = \;& (b_1f_\pi + b_2f_Y)\frac{\beta}{1-\beta} + (b_1d_{\pi A} + b_2d_{YA})\frac{\rho_A\beta}{1-\rho_A\beta}\hat{A}_t \\ & + (b_1d_{\pi\mu} + b_2d_{Y\mu})\frac{\rho_\mu\beta}{1-\rho_\mu\beta}\hat{\mu}_t.\end{aligned}$$

We can obtain a mapping from the PLM to the ALM from (44) for the first T periods (and from (45) for periods after T).

We now combine terms of the right hand side of (44). The first row on the right hand side of (44) is given by

$$\begin{aligned}
&(a_1 f_\pi + a_2 f_Y)\frac{\beta\gamma_1}{1-\beta\gamma_1} + (a_1 d_{\pi A} + a_2 d_{YA})\frac{\rho_A\beta\gamma_1}{1-\rho_A\beta\gamma_1}\hat{A}_t \\
&+(a_1 d_{\pi\mu} + a_2 d_{Y\mu})\frac{\rho_\mu\beta\gamma_1}{1-\rho_\mu\beta\gamma_1}\hat{\mu}_t - a_3(1-\rho_A\beta\gamma_1)^{-1}\hat{A}_t \\
&+a_5(1-\rho_\mu\beta\gamma_1)^{-1}\hat{\mu}_t - a_4(\beta\gamma_1\frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1}+1)\frac{\Delta G}{\bar{G}}.
\end{aligned}$$

The second row on the right hand side of (44) is given by

$$\begin{aligned}
&(b_1 f_\pi + b_2 f_Y)\frac{\beta}{1-\beta} + (b_1 d_{\pi A} + b_2 d_{YA})\frac{\rho_A\beta}{1-\rho_A\beta}\hat{A}_t \\
&+(b_1 d_{\pi\mu} + b_2 d_{Y\mu})\frac{\rho_\mu\beta}{1-\rho_\mu\beta}\hat{\mu}_t - \beta\bar{g}[(1-\beta)\frac{1-\beta^{T-t}}{1-\beta}-1]\frac{\Delta G}{\bar{G}}.
\end{aligned}$$

This process gives us the mapping for the T-map as below.

We collect the terms for the intercept in preceding two equations. This gives two equations (46) and (47) to solve for the T-map for the intercept terms T_π, T_Y :

$$(1-a_1)T_\pi - a_2 T_Y = (a_1 f_\pi + a_2 f_Y)\frac{\beta\gamma_1}{1-\beta\gamma_1} - a_4(\beta\gamma_1\frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1}+1)\frac{\Delta G}{\bar{G}}, \tag{46}$$

$$\begin{aligned}
\beta(1-\bar{g})\phi_\pi T_\pi + (\beta + (1-\bar{g})\phi_y)T_Y \;&=\; (b_1 f_\pi + b_2 f_Y)\frac{\beta}{1-\beta} \\
&\quad -\beta\bar{g}[(1-\beta)\frac{1-\beta^{T-t}}{1-\beta}-1]\frac{\Delta G}{\bar{G}}.
\end{aligned} \tag{47}$$

Similarly, consider the terms involving $\hat{A}_t$

$$(1-a_1)T_{\pi A} - a_2 T_{YA} = (a_1 d_{\pi A} + a_2 d_{YA})\frac{\rho_A\beta\gamma_1}{1-\rho_A\beta\gamma_1} - a_3(1-\rho_A\beta\gamma_1)^{-1}, \tag{48}$$

$$\beta(1-\bar{g})\phi_\pi T_{\pi A} + (\beta + (1-\bar{g})\phi_y)T_{YA} = (b_1 d_{\pi A} + b_2 d_{YA})\frac{\rho_A\beta}{1-\rho_A\beta}. \tag{49}$$

Equations (48) and (49) are solved for the coefficients $T_{\pi A}$ and T_{YA}. Finally, consider terms involving $\hat{\mu}_t$

$$(1-a_1)T_{\pi\mu} - a_2 T_{Y\mu} = (a_1 d_{\pi\mu} + a_2 d_{Y\mu})\frac{\rho_\mu \beta\gamma_1}{1-\rho_\mu\beta\gamma_1} + a_5(1-\rho_\mu\beta\gamma_1)^{-1}, \quad (50)$$

$$\beta(1-\bar{g})\phi_\pi T_{\pi\mu} + (\beta + (1-\bar{g})\phi_y)T_{Y\mu} = (b_1 d_{\pi\mu} + b_2 d_{Y\mu})\frac{\rho_\mu\beta}{1-\rho_\mu\beta}. \quad (51)$$

and equations (50)-(51) are solved for the coefficients $T_{\pi\mu}$ and $T_{Y\mu}$.

These six equations (46), (47), (48), (49), (50), and (51) yield the mapping

$$(f_\pi, f_Y, d_{\pi A}, d_{YA}, d_{\pi\mu}, d_{Y\mu}) \to (T_\pi, T_Y, T_{\pi A}, T_{YA}, T_{\pi\mu}, T_{Y\mu})$$

from the PLM to the ALM in parameter space and the fixed points of the map correspond to the MSV REE solution. This is the T-mapping for periods $1, \ldots, T$. For periods $t > T$ the same equations, together with the requirement $\Delta G = 0$, give the T-map.

RE Solution with policy change

We need to compute the RE solution when the fiscal policy changes. To compute the effect of policy changes under RE, the Euler equation approach seems preferable owing to its simplicity.[22] We, therefore, compute the Euler equations for this framework. We first consider the IS curve equation. Imposing symmetry in equation (5), we obtain

$$\frac{C_t - \bar{C}}{\bar{C}} = E_t(\frac{C_{t+1} - \bar{C}}{\bar{C}}) - E_t(\frac{r_{t+1} - \bar{r}}{\beta^{-1}}),$$

or in proportional deviation form

$$\hat{C}_t = E_t\hat{C}_{t+1} - E_t\hat{r}_{t+1}$$

where

$$\hat{r}_{t+1} = \frac{r_{t+1} - \bar{r}}{\bar{r}}; \bar{r} = \beta^{-1}.$$

[22] Note that the effect of policy changes under RE in the New Keynesian model is easier to work out compared to the RBC model since the baseline New Keynesian framework does not have any lagged endogenous variables.

Then using (15), we obtain

$$\hat{Y}_t = E_t\hat{Y}_{t+1} - \frac{\bar{C}}{\bar{Y}}E_t\hat{r}_{t+1} + \frac{\bar{G}}{\bar{Y}}\left(\hat{G}_t - E_t\hat{G}_{t+1}\right) \tag{52}$$

as the IS curve in (proportional) deviation form. If we use the interest rate rule (18) in (52) above we obtain

$$(1 + \frac{\bar{C}}{\bar{Y}}\phi_y)\hat{Y}_t + \frac{\bar{C}}{\bar{Y}}\phi_\pi\hat{\pi}_t = E_t\hat{Y}_{t+1} + \frac{\bar{C}}{\bar{Y}}E_t\hat{\pi}_{t+1} + \frac{\bar{G}}{\bar{Y}}\left(\hat{G}_t - E_t\hat{G}_{t+1}\right) \tag{53}$$

Since

$$\frac{\bar{C}}{\bar{Y}} = 1 - \frac{\bar{G}}{\bar{Y}} \equiv 1 - \bar{g},$$

we can rewrite (53) as

$$\left(1 + (1-\bar{g})\phi_y\right)\hat{Y}_t + (1-\bar{g})\phi_\pi\hat{\pi}_t = E_t\hat{Y}_{t+1} + (1-\bar{g})E_t\hat{\pi}_{t+1} + \bar{g}\left(\hat{G}_t - E_t\hat{G}_{t+1}\right) \tag{54}$$

We now compute the one-step forward looking Phillips curve. From (35) we obtain (imposing symmetry)

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}(\hat{S}_t + \hat{\mu}_t).$$

Using (39) this may be rewritten as

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{(1+\varepsilon)\theta\bar{Y}\bar{S}}{\alpha\bar{\psi}}\hat{Y}_t + \frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}(\hat{\mu}_t - \frac{1+\varepsilon}{\alpha}\hat{A}_t) - \frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}\frac{\bar{g}}{1-\bar{g}}\hat{g}_t. \tag{55}$$

Using

$$\hat{g}_t = \hat{G}_t - \hat{Y}_t,$$

(55) reduces to

$$\hat{\pi}_t = \beta E_t\hat{\pi}_{t+1} + \frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}\left(\frac{1+\varepsilon}{\alpha} + \frac{\bar{g}}{1-\bar{g}}\right)\hat{Y}_t + \frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}(\hat{\mu}_t - \frac{1+\varepsilon}{\alpha}\hat{A}_t) - \frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}\frac{\bar{g}}{1-\bar{g}}\hat{G}_t. \tag{56}$$

Writing (54) and (56) in matrix form we get[23]

$$\begin{aligned}
&\begin{pmatrix} 1 & -\frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}\left(\frac{1+\varepsilon}{\alpha}+\frac{\bar{g}}{1-\bar{g}}\right) \\ (1-\bar{g})\phi_\pi & 1+(1-\bar{g})\phi_y \end{pmatrix}\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} \\
= &\begin{pmatrix} \beta & 0 \\ 1-\bar{g} & 1 \end{pmatrix}\begin{pmatrix} E_t\hat{\pi}_{t+1} \\ E_t\hat{Y}_{t+1} \end{pmatrix}+\begin{pmatrix} \frac{\theta\bar{Y}\bar{S}}{\bar{\psi}} & -\frac{(1+\varepsilon)\theta\bar{Y}\bar{S}}{\alpha\bar{\psi}} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{\mu}_t \\ \hat{A}_t \end{pmatrix} \\
&+\begin{pmatrix} -\frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}\frac{\bar{g}}{1-\bar{g}}\hat{G}_t \\ \bar{g}\left(\hat{G}_t-E_t\hat{G}_{t+1}\right) \end{pmatrix}.
\end{aligned}$$

Inverting the matrix on the left hand side of the above system we can obtain the system

$$\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix}=\Psi\begin{pmatrix} E_t\hat{\pi}_{t+1} \\ E_t\hat{Y}_{t+1} \end{pmatrix}+\digamma\begin{pmatrix} \hat{\mu}_t \\ \hat{A}_t \end{pmatrix}+\Gamma\begin{pmatrix} -\frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}\frac{\bar{g}}{1-\bar{g}}\hat{G}_t \\ \bar{g}\left(\hat{G}_t-E_t\hat{G}_{t+1}\right) \end{pmatrix}, \quad (57)$$

which can used to compute numerically the RE solution with the fiscal policy change. Thus, (57) gives the system under RE when the Taylor rule (18) is followed. We will be using this system to compute the RE solution when there is a change in government purchases (and a balanced budget).

We get

$$\begin{aligned}
\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} &= \sum_{i=0}^{\infty}\Psi^i\left(\digamma E_t\begin{pmatrix} \hat{\mu}_{t+i} \\ \hat{A}_{t+i} \end{pmatrix}+\Gamma H_{t+i}\right), \text{ where} \\
H_t &= \begin{pmatrix} -\frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}\frac{\bar{g}}{1-\bar{g}}\hat{G}_t \\ \bar{g}\left(\hat{G}_t-E_t\hat{G}_{t+1}\right) \end{pmatrix},
\end{aligned}$$

which can be written as

$$\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix}=\sum_{i=0}^{\infty}\Psi^i\digamma\begin{pmatrix} \rho_\mu & 0 \\ 0 & \rho_A \end{pmatrix}^i\begin{pmatrix} \hat{\mu}_t \\ \hat{A}_t \end{pmatrix}+\sum_{i=0}^{\infty}\Psi^i\Gamma H_{t+i}. \quad (58)$$

The first term on the right-hand side of (58) is the MSV solution when government spending is constant. It takes the form

$$\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix}_{MSV}=\begin{pmatrix} \bar{d}_{\pi A} \\ \bar{d}_{YA} \end{pmatrix}\hat{A}_t+\begin{pmatrix} \bar{d}_{\pi\mu} \\ \bar{d}_{Y\mu} \end{pmatrix}\hat{\mu}_t,$$

[23]Note that $\frac{\theta\bar{Y}\bar{S}}{\bar{\psi}}=\frac{(\theta-1)\bar{Y}(1-\tau)}{\bar{\psi}}$ which is denoted by ξ (with $\tau=0$) in Eusepi-Preston (2010) and is set equal to 0.06 in footnote 11, p. 243, of their paper.

where

$$\begin{pmatrix} \bar{d}_{\pi A} \\ \bar{d}_{YA} \end{pmatrix} = (M - N_A)^{-1} cons_A,$$
$$\begin{pmatrix} \bar{d}_{\pi \mu} \\ \bar{d}_{Y\mu} \end{pmatrix} = (M - N_\mu)^{-1} cons_\mu,$$

and

$$N_A = \begin{pmatrix} a_1 \frac{\rho_A \beta \gamma_1}{1-\rho_A \beta \gamma_1} & a_2 \frac{\rho_A \beta \gamma_1}{1-\rho_A \beta \gamma_1} \\ b_1 \frac{\rho_A \beta}{1-\rho_A \beta} & b_2 \frac{\rho_A \beta}{1-\rho_A \beta} \end{pmatrix}, cons_A = \begin{pmatrix} -a_3(1-\rho_A \beta \gamma_1)^{-1} \\ 0 \end{pmatrix},$$
$$N_\mu = \begin{pmatrix} a_1 \frac{\rho_\mu \beta \gamma_1}{1-\rho_\mu \beta \gamma_1} & a_2 \frac{\rho_\mu \beta \gamma_1}{1-\rho_\mu \beta \gamma_1} \\ b_1 \frac{\rho_\mu \beta}{1-\rho_\mu \beta} & b_2 \frac{\rho_\mu \beta}{1-\rho_\mu \beta} \end{pmatrix}, cons_\mu = \begin{pmatrix} a_5(1-\rho_\mu \beta \gamma_1)^{-1} \\ 0 \end{pmatrix}.$$

The second term gives the modification due to changes in government spending and it is calculated as follows. For $t = 1$ we have

$$\begin{aligned} HRE_1 &\equiv \sum_{s=0}^{\infty} \Psi^i \Gamma H_{1+i} = \sum_{s=0}^{T-2} \Psi^i \Gamma \bar{H}_1 + \Psi^{T-1} \Gamma \bar{H}_2 \\ &= (1 - \Psi^{T-1})(1-\Psi)^{-1} \bar{H}_1 + \Psi^{T-1} \Gamma \bar{H}_2, \text{ where} \\ \bar{H}_1 &= \begin{pmatrix} -\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \frac{\Delta G}{1-\bar{g}} \\ 0 \end{pmatrix} \text{ and } \bar{H}_2 = \begin{pmatrix} -\frac{\theta \bar{Y} \bar{S}}{\bar{\psi}} \frac{\Delta G}{1-\bar{g}} \\ -\Delta G \end{pmatrix}. \end{aligned}$$

In general,

$$HRE_t = (1 - \Psi^{T-t})(1-\Psi)^{-1} \bar{H}_1 + \Psi^{T-t} \Gamma \bar{H}_2$$

for $t = 1, \ldots, T-1$. For $t = T$ we have

$$HRE_T = \Gamma \bar{H}_2$$

and

$$HRE_t = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $t \geq T + 1$.

In total, the RE solution is the sum of the MSV solution with constant government spending plus the term HRE_t.

Appendix 2: Details of model with lower bounds

Proofs of Propositions

The proof of the Proposition 1 uses the following result:

Lemma 3 *Let*

$$\begin{aligned}\phi(C;\pi,\psi,G) &= C\gamma A^{-\frac{1+\varepsilon}{\alpha}}\left(C+G+\frac{\psi}{2}(\pi-\pi^*)^2\right)^{\frac{1+\varepsilon}{\alpha}} \\ &\quad -\alpha(1-\tau)(1-\theta^{-1})\left(C+G+\frac{\psi}{2}(\pi-\pi^*)^2\right).\end{aligned}$$

Let $\underline{G}=\left((A\alpha/\gamma)(1-\tau)(1-\theta^{-1})\right)^{\frac{\alpha}{1+\varepsilon}}$. *Then provided* $G>\underline{G}$, *there exists* $\check{\psi}>0$ *such that for all* $0<\psi<\check{\psi}$ *the function* $\phi(C;\pi,\psi,G)$ *is strictly monotonically increasing in* C *and for given* $\pi,\psi<\check{\psi}$ *and* $G>\underline{G}$ *we have* $\lim_{C\to\infty}\phi(C;\pi,\psi,G)=+\infty$.

Proof of Lemma 3: Computing the derivative we have

$$\begin{aligned}\left.\frac{d\phi}{dC}\right|_{\psi=0} &= \gamma C\frac{1+\varepsilon}{\alpha A}\left(A^{-1}(C+G)\right)^{\frac{1+\varepsilon}{\alpha}-1}+\gamma A^{-1}(C+G)^{\frac{1+\varepsilon}{\alpha}} \\ &\quad -\alpha(1-\tau)(1-\theta^{-1}).\end{aligned}$$

If $G>\underline{G}$ then $\left.\frac{d\phi}{dC}\right|_{\psi=0}>0$ for all $C\geq 0$. Since $d\phi/dC$ is continuous in ψ, then result follows. ■

Proof of Proposition 1: First suppose the inflation lower bound satisfies $\underline{\pi}<\pi_L$. (i) $\pi=\pi^*$, with $R=\beta^{-1}\pi^*$ satisfy (25) and (27). From (23) and (24) we have $0=\phi(C;\pi^*,\psi,G)$. We have $\phi(0;\pi^*,\psi,G)=-\alpha(1-\tau)(1-\theta^{-1})G<0$. Since we are assuming $G>\underline{G}$ it follows from the Lemma that there is a unique $C=\bar{C}$ that solves (23) and (24).

(ii) $\pi=\pi_L$, with $R=1+\eta$ satisfy (25) and (27). From (23) and (24) we have $(\pi_L-1)\pi_L(1-\beta)\alpha\psi\theta^{-1}=\phi(C;\pi_L,\psi,G)$. For ψ sufficiently small the term $(\pi_L-1)\pi_L(1-\beta)\alpha\psi\theta^{-1}$ can be made arbitrarily close to zero. Since $\phi(0;\pi_L,\psi,G)=-\alpha(1-\tau)(1-\theta^{-1})\left(G+\frac{\psi}{2}(\pi_L-\pi^*)^2\right)<0$, the Lemma again applies and there is a unique $C=C_L$ that solves (23) and (24).

(iii) $\pi=\underline{\pi}$ with $R=1+\eta$ satisfy (25) and (27) provided $C=\underline{C}$. We thus need to establish that (26) holds with strict inequality, i.e. that

$$(\underline{\pi}-1)\underline{\pi}(1-\beta)\alpha\psi\theta^{-1}>\phi(\underline{C};\underline{\pi},\psi,G)$$

As in part (ii) the Lemma implies that, given $G > \underline{G}$ and ψ sufficiently small, there exists $\tilde{C} > 0$ such that

$$(\underline{\pi} - 1)\underline{\pi}(1-\beta)\alpha\psi\theta^{-1} = \phi(\tilde{C}; \underline{\pi}, \psi, G)$$

Thus for consumption lower bound $\underline{C} < \tilde{C}$ (26) holds with strict inequality.

It is straightforward to see that there is no other steady state. Suppose first that there is a steady state at π with $\underline{\pi} < \pi < \pi_L$ or $\pi > \pi^*$. Then $R > \beta^{-1}\pi$. By (25) this implies $C = \underline{C}$. But there exists C_π such that (23) is satisfied and since we can assume $\underline{C} < C_\pi$ it follows from the Lemma that (26) holds with strict inequality. However this implies $\pi = \underline{\pi}$, which contradicts our assumption. If instead $\pi_L < \pi < \pi^*$. Then $R < \beta^{-1}\pi$. But this contradicts (25).

Next, suppose If $\pi_L < \underline{\pi} < \pi^*$. By (26) there cannot be a steady state at $\pi < \underline{\pi}$. Clearly there is again a steady state at $\pi = \pi^* \geq 1$. This is the unique steady state since steady states π with $\underline{\pi} < \pi < \pi^*$ or $\pi > \pi^*$ can be ruled out using the above arguments. ■

Proof of Proposition 2: For the consumption function we employ equation (33), which with G_{t+s}, held constant can be written in the form

$$\hat{C}_t = \left(\frac{1-\beta}{1-\bar{g}}\right)\left[\hat{Y}_t + \sum_{s=1}^{\infty}\beta^s \hat{E}_t\hat{Y}_{t+s}\right] - \left[\beta\hat{R}_t + \beta\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{R}_{t+s}\right] + \left[\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{\pi}_{t+s}\right].$$

Assuming steady-state learning, we have $\hat{E}_t\hat{Y}_{t+s} = f_Y$ and $\hat{E}_t\hat{\pi}_{t+s} = f_\pi$. With the forward looking interest rate rule $\hat{R}_t = \phi_\pi\hat{E}_t\hat{\pi}_{t+1}$ we also have $\hat{R}_t = \hat{E}_t\hat{R}_{t+s} = \phi_\pi f_\pi$ for all s locally near the targeted steady state. Thus near the targeted steady state

$$\hat{C}_t = \left(\frac{1-\beta}{1-\bar{g}}\right)\left(\hat{Y}_t + \frac{\beta}{1-\beta}f_Y\right) - \beta\phi_\pi f_\pi - \frac{\beta^2}{1-\beta}\phi_\pi f_\pi + \frac{\beta}{1-\beta}f_\pi. \quad (59)$$

Assuming that government spending and the shocks are constant in equation (14), the Phillips curve is

$$\begin{aligned}(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t &= a_1\sum_{s=1}^{\infty}(\beta\gamma_1)^s\hat{E}_t\hat{\pi}_{t+s} + a_2\sum_{s=1}^{\infty}(\beta\gamma_1)^s\hat{E}_t\hat{Y}_{t+s} \\ &= a_1 f_\pi\frac{\beta\gamma_1}{1-\beta\gamma_1} + a_2 f_Y\frac{\beta\gamma_1}{1-\beta\gamma_1}.\end{aligned}$$

The third equation is the linearized market-clearing condition (15). It can be noted that at the steady state $a_1 = 1 - \gamma_1$. For steady state learning, the system giving the temporary equilibrium map $(\hat{\pi}, \hat{Y}) = T(f_\pi, f_Y)$ takes the linearized form with coefficient matrix DT at the targeted steady state given by[24]

$$\begin{pmatrix} \hat{\pi}_t \\ \hat{Y}_t \end{pmatrix} = DT \begin{pmatrix} f_\pi \\ f_Y \end{pmatrix}, \text{ where}$$

$$DT = \begin{pmatrix} 2 - \frac{1-\beta}{1-\beta\gamma_1} + \bar{g}(\phi_\pi - 1) - \phi_\pi & 1 + \beta(-1 + \frac{a_2}{1-\beta\gamma_1}) \\ \frac{(\phi_\pi - 1)(\bar{g}-1)}{1-\beta} & 1 \end{pmatrix}.$$

E-stability is determined by the eigenvalues of $DT - I$ and holds if these have negative real parts. It is easily verified that $Tr(DT - I) < 0$ and $\det(DT - I) > 0$ when ϕ_π is sufficiently large, which implies E-stability of the targeted steady state.

Next, consider the steady state π_L. The lower bound on the interest rate is binding locally near π_L, so we impose the constraint $\hat{R} = 0$ and evaluate the other variables at their low steady state values and impose $\psi = 0$. Local stability of the low steady state is determined by the eigenvalues of DT at the steady state. It can be computed that

$$DT = \begin{pmatrix} \frac{a_2(1-\bar{g})}{(1-\beta)(1-a_1)} + \frac{\beta\gamma_1 a_1}{(1-\beta\gamma_1)(1-a_1)} & \frac{a_2}{(1-\beta\gamma_1)(1-a_1)} \\ \frac{1-\bar{g}}{1-\beta} & 1 \end{pmatrix}$$

and that $\det(DT - I) < 0$ since $a_2 > 0$ and $0 < \beta, a_1, \gamma_1, \bar{g} < 1$. By continuity of eigenvalues, it follows that the low steady state π_L is unstable also for sufficiently small $\psi > 0$.

Next, consider the trap steady state, where the bounds $\underline{\pi}$ and $\underline{C}$ are strictly binding as described in part (iii) of Proposition 1. Clearly, from the interest rate rule (see Figure 3) $\exists \breve{\pi}$ such that $\hat{\pi}_t = \underline{\pi}$ for all $\underline{\pi} \leq \pi \leq \breve{\pi}$. Now also $\exists \breve{C}$ such that $\hat{C}_t = \underline{C}$ for $\underline{C} \leq C \leq \breve{C}$ and $\underline{\pi} \leq \pi \leq \breve{\pi}$. By market clearing, $\hat{Y}_t = (1 - \bar{g})\underline{C} + (\psi/2)\underline{\pi}^2$ in this region. It follows that $\partial\hat{\pi}_t/\partial f_\pi = \partial\hat{\pi}_t/\partial f_Y = 0$ and $\partial\hat{Y}_t/\partial f_\pi = \partial\hat{Y}_t/\partial f_Y = 0$ implying E-stability of the trap steady state. ■

We remark that $\det(DT - I) < 0$ implies that the π_L steady state has local dynamics under learning that take the form of a saddle.[25]

[24] Mathematica routine for the details is available on request.

[25] Stability of the targeted steady state and instability of the π_L steady state have also

Construction of Phase Diagram of Global E-stability Dynamics

We here give the additional details for constructing and interpreting Figure 4. In constructing this Figure we ignore the impact of exogenous shocks, so that we set $\hat{A}_t = \hat{\mu}_t = 0$. Consequently, the forecast rule coefficients ϕ_π and ϕ_Y consist only of the two intercepts f_π and f_Y, which allows us to illustrate global learning dynamics using a 2-dimensional figure. Under real-time learning the least-square updating equations at the end of Section 2.4 simplify and are replaced by

$$\begin{aligned} f_{\pi,t} &= f_{\pi,t-1} + \kappa\left(\pi_t - f_{\pi,t-1}\right) \\ f_{Y,t} &= f_{Y,t-1} + \kappa\left(Y_t - f_{Y,t-1}\right). \end{aligned}$$

It is know that these real-time learning dynamics are, for small gains $\kappa > 0$, approximated by the E-stability equations given below.

The nonlinear market-clearing equation, where variables are expressed in terms of proportional deviations from the targeted steady state, is given by

$$\hat{Y}_t = (1-\bar{g})\hat{C}_t + \frac{\psi}{2\bar{Y}}\hat{\pi}_t^2. \tag{60}$$

We use this rather than the linearized market-clearing equation because we are looking at global dynamics that include regions around all three of the steady states. In Figure 4 we set $\pi^* = 1$ and thus $\hat{\pi}_t = \pi_t - 1$. In the absence of lower bound constraints the temporary equilibrium equation for the Phillips curve is given by

$$(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t = (a_1 f_\pi + a_2 f_Y)\frac{\beta\gamma_1}{1-\beta\gamma_1}, \tag{61}$$

where $\hat{\pi}^e = f_\pi$ and $\hat{Y}^e = f_y$. Similarly, in the absence of lower bound constraints, the temporary equilibrium equation for consumption is given by (59), with the one change that $\hat{Y}_t$ is replaced by output net of price adjustment costs $\hat{Y}_t - \frac{\psi}{2\bar{Y}}\hat{\pi}_t^2$. These adjustment costs are zero, to first order, near the targeted steady state.

been observed for the version of the model in which price adjustment costs are formulated in terms of utility losses. See, for example, Benhabib, Evans, and Honkapohja (2014).

To allow for the lower bound constraints, consider first the incorporation of the ZLB into the consumption function. This gives

$$\begin{aligned}\hat{C}_t &= \left(\frac{1-\beta}{1-\bar{g}}\right)\left(\hat{Y}_t - \frac{\psi}{2\bar{Y}}\hat{\pi}_t^2\right) + \frac{\beta}{1-\bar{g}} f_Y - \beta \max[\phi_\pi f_\pi, \beta - 1] \quad (62)\\ &\quad -\frac{\beta^2}{1-\beta}\max[\phi_\pi f_\pi, \beta - 1] + \frac{\beta}{1-\beta} f_\pi.\end{aligned}$$

The temporary equilibrium for $(\hat{Y}_t, \hat{C}_t, \hat{\pi}_t)$ is given by equation (60) together with (61) and (62), where the latter two equations are interpreted as inequalities subject to lower bound constraints and holding with complementary slackness. That is, (61) holds unless $\hat{\pi}_t < \underline{\pi}$, in which case $\hat{\pi}_t = \underline{\pi}$, and (62) holds unless $\hat{C}_t < \underline{C}$, in which case $\hat{C}_t = \underline{C}$.

In defining the temporary equilibrium map we incorporate one additional constraint: an upper bound on π_t, which is needed to ensure existence of a unique temporary equilibrium. This is needed because of the quadratic market-clearing equation. For given $\hat{\pi}^e$ there are two solutions, provided $\hat{Y}^e$ is not too large, and we choose the one with the smaller inflation rate, which is the economically relevant solution: this is the solution in which higher $\hat{Y}^e$ gives higher $\hat{Y}_t$ and $\hat{\pi}_t$. If instead $\hat{Y}^e$ is sufficiently large no temporary equilibrium solution exists to our equations. To cover this case we then replace (61) with $\hat{\pi}_t = \pi^{\sup}$. In Figure 4 we choose $\pi^{\sup} = 0.05$.

This procedure defines the temporary equilibrium map

$$\left(\hat{\pi}, \hat{Y}\right) = T\left(\hat{\pi}^e, \hat{Y}^e\right)$$

giving the realized values of $\hat{\pi}_t$ and $\hat{Y}_t$ given expectations $\hat{\pi}^e$ and $\hat{Y}^e$. The three steady states correspond to the fixed points of this map. E-stability dynamics are given by

$$\frac{d}{d\tau} T\left(\hat{\pi}^e, \hat{Y}^e\right) = T\left(\hat{\pi}^e, \hat{Y}^e\right) - \left(\hat{\pi}^e, \hat{Y}^e\right),$$

where τ represents "notional" time, which can, however, be linked to real time t according to the equation $\tau \approx \kappa t$.Figure 4 plots the vector field generated by $T\left(\hat{\pi}^e, \hat{Y}^e\right) - \left(\hat{\pi}^e, \hat{Y}^e\right)$. This vector field shows the paths of expectations $\left(\hat{\pi}^e, \hat{Y}^e\right) = (f_\pi, f_Y)$ under the simple learning rule given above.

Appendix 3: Temporary equilibria with lower bounds

We her develop the model details when the economy with exogenous shocks is subject to interest rate, inflation and output lower bounds, and fiscal policy is included. We start by focusing on the interest rate lower bound (ZLB). The Phillips curve is unaffected by the ZLB. Using the calculations after (45) in Appendix 1 we have, using the first row of (44), the Phillips Curve

$$\begin{aligned}(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t &= (a_1 f_\pi + a_2 f_Y)\frac{\beta\gamma_1}{1-\beta\gamma_1} \\ &+ \left[(a_1 d_{\pi A} + a_2 d_{YA})\frac{\rho_A\beta\gamma_1}{1-\rho_A\beta\gamma_1} - \frac{a_3}{1-\rho_A\beta\gamma_1}\right]\hat{A}_t \\ &+ \left[(a_1 d_{\pi\mu} + a_2 d_{Y\mu})\frac{\rho_\mu\beta\gamma_1}{1-\rho_\mu\beta\gamma_1} + \frac{a_5}{1-\rho_\mu\beta\gamma_1}\right]\hat{\mu}_t \\ &- a_4(\beta\gamma_1\frac{1-(\beta\gamma_1)^{T-t}}{1-\beta\gamma_1}+1)\frac{\Delta G}{\bar{G}},\end{aligned} \tag{63}$$

where the term in $\frac{\Delta G}{\bar{G}}$ is set to zero for $t > T$. It is convenient to write this as

$$(1-a_1)\hat{\pi}_t - a_2\hat{Y}_t = TPCfn(d, f, \hat{A}_t, \hat{\mu}_t, t)$$

where $d' = (d_{\pi A}, d_{YA}, d_{\pi\mu}, d_{Y\mu})$ and $f' = (f_\pi, f_Y)$.

For the IS curve we start by combining (33) with $\hat{Y}_t = (1-\bar{g})\hat{C}_t + \bar{g}\hat{G}_t$, which yields

$$\begin{aligned}\beta\hat{Y}_t &= \bar{g}\hat{G}_t + (1-\beta)\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{Y}_{t+s} - (1-\beta)\bar{g}\sum_{s=0}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s} \\ &-(1-\bar{g})\sum_{s=1}^{\infty}\beta^s\hat{E}_t(\hat{R}_{t+s-1} - \hat{\pi}_{t+s}).\end{aligned}$$

We write this as

$$\beta\hat{Y}_t + (1-\bar{g})\beta\hat{R}_t = ISG_t + ISPY_t + ISR_t \tag{64}$$

where $ISPY_t = (1-\beta)\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{Y}_{t+s} + (1-\bar{g})\sum_{s=1}^{\infty}\beta^s\hat{E}_t\hat{\pi}_{t+s}$, $ISG_t = \bar{g}\hat{G}_t - (1-\beta)\bar{g}\sum_{s=0}^{\infty}\beta^s\hat{E}_t\hat{G}_{t+s}$ and $ISR_t = -(1-\bar{g})\sum_{s=2}^{\infty}\beta^s\hat{E}_t\hat{R}_{t+s-1}$. $ISPY_t$ and ISG_t

are given by

$$\begin{aligned} ISPY_t &= \frac{\beta(1-\bar{g})}{1-\beta} f_\pi + \beta f_Y + ((1-\bar{g})d_{\pi A} + (1-\beta)d_{YA}) \frac{\rho_A \beta}{1-\rho_A \beta} \hat{A}_t \\ &\quad + ((1-\bar{g})d_{\pi\mu} + (1-\beta)d_{Y\mu}) \frac{\rho_\mu \beta}{1-\rho_\mu \beta} \hat{\mu}_t \end{aligned}$$

and

$$ISG_t = \begin{cases} \beta \Delta G - \beta(1-\beta^{T-t})\Delta G \text{ if } t \leq T \\ 0 \text{ if } t \geq T+1. \end{cases}$$

For ISR_t we need to determine when agents expect the ZLB to apply in the future. For simplicity we assume $\rho_A = \rho_\mu = \rho$ and $0 \leq \rho < 1$. It then can be shown that there are four cases, depending on $\hat{A}_t$ and $\hat{\mu}_t$. Let

$$\begin{aligned} K &= \hat{A}_t(\phi_\pi d_{\pi A} + \phi_Y d_{YA}) + \hat{\mu}_t(\phi_\pi d_{\pi\mu} + \phi_Y d_{Y\mu}) \\ L &= \frac{\beta(1+\eta)}{\pi^*} - 1 - \phi_\pi f_\pi - \phi_Y f_Y. \end{aligned}$$

Note that K depends on $\hat{A}_t$ and $\hat{\mu}_t$ and that under learning both K and L depend on the PLM parameter estimates, which are evolving over time.

The ZLB $R \geq 1+\eta$ binds if and only if the forecasted interest rate based on the Taylor rule satisfies

$$\hat{E}_t \hat{R}^*_{t+s} = \phi_\pi \hat{E}_t \hat{\pi}_{t+s} + \phi_Y \hat{E}_t \hat{Y}_{t+s} \leq \frac{\beta(1+\eta)}{\pi^*} - 1.$$

It is then easy to see that this holds when $K\rho^s \leq L$. There are four cases:

1. The ZLB never binds (in anticipation) if $K\rho^s > L$. This happens when $K \geq 0, L < 0$ or if $K, L < 0$ and $\rho K > L$ or if $K = 0, L \neq 0$ or if $K > 0$ and $L = 0$.

2. The ZLB holds for all $s \geq 1$ if $K \leq 0, L \geq 0$ or $K, L > 0, \rho K \leq L$.

3. The ZLB holds for all $s \geq \hat{s}$, where $\hat{s}$ is the smallest integer below $\hat{s}^* = \ln(L/K)/\ln(\rho)$, if $K, L > 0, \rho K > L$.

4. The ZLB holds for all $1 \leq s \leq \hat{s}$ if $K, L < 0, \rho K \leq L$.

The value of ISR_t depends on the case. In case 1 we have

$$\begin{aligned} ISR1_t &= -\frac{\beta^2\rho(1-\bar{g})}{1-\beta\rho}\left[\hat{A}_t(\phi_\pi d_{\pi A}+\phi_Y d_{YA})+\hat{\mu}_t(\phi_\pi d_{\pi\mu}+\phi_Y d_{Y\mu})\right] \\ &\quad -(\phi_\pi f_\pi+\phi_Y f_Y)\frac{\beta^2(1-\bar{g})}{1-\beta}. \end{aligned}$$

In case 2 we have

$$ISR2_t = \frac{\beta^2(1-\bar{g})}{1-\beta}\left(1-\beta(1+\eta)\right).$$

In case 3 we have

$$\begin{aligned} ISR3_t &= -\frac{\beta^2\rho(1-\bar{g})\left(1-(\beta\rho)^{\hat{s}-1}\right)}{1-\beta\rho}\left[\hat{A}_t(\phi_\pi d_{\pi A}+\phi_Y d_{YA})+\hat{\mu}_t(\phi_\pi d_{\pi\mu}+\phi_Y d_{Y\mu})\right] \\ &\quad -(\phi_\pi f_\pi+\phi_Y f_Y)\frac{\beta^2(1-\bar{g})(1-\beta^{\hat{s}-1})}{1-\beta}+(1-\bar{g})\frac{1-\beta(1+\eta)}{1-\beta}\beta^{\hat{s}+1}. \end{aligned}$$

In case 4 we have

$$\begin{aligned} ISR4_t &= -\frac{\beta(1-\bar{g})(\beta\rho)^{\hat{s}}}{1-\beta\rho}\left[\hat{A}_t(\phi_\pi d_{\pi A}+\phi_Y d_{YA})+\hat{\mu}_t(\phi_\pi d_{\pi\mu}+\phi_Y d_{Y\mu})\right] \\ &\quad -(\phi_\pi f_\pi+\phi_Y f_Y)\frac{(1-\bar{g})\beta^{\hat{s}+1}}{1-\beta}+(1-\bar{g})\frac{\beta^2\left(1-\beta(1+\eta)\right)\left(1-\beta^{\hat{s}-1}\right)}{1-\beta}. \end{aligned}$$

We can now solve for the tentative temporary equilibrium values $\hat{\pi}_t^{tent}, \hat{Y}_t^{tent}$ for $\hat{\pi}_t, \hat{Y}_t$ that would obtain if none of the lower bounds at time t apply. These are given by

$$\begin{aligned} (1-a_1)\hat{\pi}_t^{tent}-a_2\hat{Y}_t^{tent} &= TPC(d,f,\hat{A}_t,\hat{\mu}_t,t) \\ (1-\bar{g})\beta\phi_\pi\hat{\pi}_t^{tent}+\beta(1+\phi_Y(1-\bar{g}))\hat{Y}_t^{tent} &= TIS(d,f,\hat{A}_t,\hat{\mu}_t,t) \end{aligned}$$

where $TIS(d,f,\hat{A}_t,\hat{\mu}_t,t) = ISG_t + ISPY_t + ISR_t$ and $ISR_t = ISRn_t$ for case $n = 1,2,3,4$. Under real-time learning we use the time t estimates of d, f.

We next need to incorporate the lower bounds on inflation, consumption and interest rate in the time t temporary equilibrium. The consumption lower bound gives an output lower bound. In proportional terms the lower

bound $\underline{C}$ is $\hat{C}_{\inf} = \frac{\underline{C}-\bar{C}}{\bar{C}}$. From $\hat{Y} = (1-\bar{g})\hat{C} + \bar{g}\hat{G}$ this gives the lower bound on $\hat{Y}$ of

$$\begin{aligned} \underline{\hat{Y}}_t &= (1-\bar{g})\hat{C}_{\inf} + \bar{g}\hat{G}_t, \text{ or} \\ \underline{\hat{Y}}_t &= (1-\bar{g})\hat{C}_{\inf} + \frac{G_t - \bar{G}}{\bar{Y}}. \end{aligned}$$

Note that $\underline{\hat{Y}}_t = (1-\bar{g})\hat{C}_{\inf}$ after the fiscal policy stimulus has ended.

First we check for the ZLB at time t. Assuming $\phi_\pi \hat{\pi}_t^{tent} + \phi_Y \hat{Y}_t^{tent} \geq \beta(1+\eta) - 1$ so that the ZLB at t does not bind then set $\pi\pi_t = \hat{\pi}_t^{tent}$, $YY_t = \hat{Y}_t^{tent}$ and $RR_t = \phi_\pi \hat{\pi}_t^{tent} + \phi_Y \hat{Y}_t^{tent}$. If instead $\phi_\pi \hat{\pi}_t^{tent} + \phi_Y \hat{Y}_t^{tent} < \beta(1+\eta) - 1$ then we set $RR_t = \beta(1+\eta) - 1$ and set YY_t and $\pi\pi_t$ to solve

$$(1-a_1)\pi\pi_t - a_2 YY_t = TPC(d, f, \hat{A}_t, \hat{\mu}_t, t) \quad (65)$$

$$(1-\bar{g})\beta\left(\beta(1+\eta) - 1\right) + \beta YY_t = TIS(d, f, \hat{A}_t, \hat{\mu}_t, t). \quad (66)$$

Next, if $\pi\pi_t < \underline{\pi}$ ("situation 1") then we calculate $YYnew_t$ and $RRnew_t$ by simultaneously solving (64) with $\hat{Y}_t = YYnew_t$ and $RRnew_t = \phi_\pi \underline{\pi} + \phi_Y YYnew_t$. If $RRnew_t > \beta(1+\eta)-1$ then the situation 1 step has ended and we set $\pi\pi_t = \underline{\pi}$, $YY_t = YYnew_t$ and $RR_t = RRnew_t$. If instead $RRnew_t \leq \beta(1+\eta)-1$ then set $\pi\pi_t = \underline{\pi}$, YY_t is set to solve (66), and $RR_t = \beta(1+\eta)-1$. If now $YY_t < \underline{\hat{Y}}_t$ then $\pi\pi_t = \underline{\pi}$, $YY_t = \underline{\hat{Y}}_t$ and $RR_t = \max(\phi_\pi \underline{\pi} + \phi_Y \underline{\hat{Y}}_t, \beta(1+\eta) - 1)$.

It remains to consider situation 2 in which $\pi\pi_t \geq \underline{\pi}$ and $YY_t < \underline{\hat{Y}}_t$.[26] We set $YY_t = \underline{\hat{Y}}_t$, $\pi\pi_t$ to solve (65) with $YY_t = \underline{\hat{Y}}_t$, and $RR_t = \max(\phi_\pi \pi\pi_t + \phi_Y \underline{\hat{Y}}_t, \beta(1+\eta) - 1)$.

The resulting values for YY_t, $\pi\pi_t$ and RR_t are the temporary equilibrium values for $\hat{Y}_t, \hat{\pi}_t$ and $\hat{R}_t$. We remark that we have not assumed that firms and households restrict forecasts to obey the consumption and inflation lower bounds. This seems natural since households may not be aware of these aggregate lower bound constraints. Under adaptive learning expectations of future inflation and output will have to eventually obey these lower bounds.

Consumption and output at the inflation lower-bound

If inflation and inflation expectations are at the inflation lower bound, i.e. $\hat{\pi}_t = \hat{E}_t \hat{\pi}_{t+s} = \underline{\hat{\pi}}$, where $\underline{\pi} < \pi_L$, then interest rates and expected interest

[26] It is assumed below that $a_2 > 0$ and $0 < a_1 < 1$. This is satisfied in the calibrated cases and ensures that $\pi\pi_t > \underline{\pi}$.

rates are also at their lower bound, i.e. $\hat{R}_t = \hat{E}_t \hat{R}_{t+s} = \frac{\beta(1+\eta)}{\pi^*} - 1 \equiv \hat{R}_{ZLB}$. Inserting these into the consumption function (33) we obtain

$$(1-g)\hat{C}_t = (1-\beta)\hat{Y}_t + \beta f_{Y,t} - \frac{\beta(1-\bar{g})}{1-\beta}(\hat{R}_{ZLB} - \hat{\underline{\pi}}).$$

Here we have simplified by ignoring the impact on expected output of the exogenous shocks $\hat{A}_t, \hat{\mu}_t$, i.e. we are setting $\hat{E}_t Y_{t+s} = f_{Y,t}$. This approximation is reasonable since in the temporary equilibrium at the inflation lower bound the shocks do not affect output. Combining this equation with the linearized market-clearing equation $\hat{Y}_t = (1-\bar{g})\hat{C}_t$ gives

$$\hat{Y}_t = f_{Y,t} - \frac{1-\bar{g}}{1-\beta}(\hat{R}_{ZLB} - \hat{\underline{\pi}}) \text{ where}$$

$$\hat{R}_{ZLB} - \hat{\underline{\pi}} = \frac{\beta(1+\eta) - \underline{\pi}}{\pi^*} = \frac{\pi_L - \underline{\pi}}{\pi^*} > 0$$

since we assume $\underline{\pi} < \pi_L$.

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