A Risk-centric Model of Demand Recessions and Macroprudential Policy

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This draft: October 24, 2017

Abstract

A productive capacity generates output and risks, both of which need to be absorbed by economic agents. If they are unable to do so, output and risk gaps emerge. Risk gaps close quickly: A decline in the interest rate increases the Sharpe ratio of the risky assets and equilibrates the risk markets. If the interest rate is constrained from below (or the policy response is slow), the risk markets are instead equilibrated via a decline in asset prices. However, the drop in asset prices also drags down aggregate demand and generates a recession, which further drags prices down, and so on. If investors are pessimistic about the recovery, the economy becomes highly susceptible to downward spirals due to the feedback between asset prices and aggregate demand. The fear of a recession with a downward asset price spiral also reduces the interest rate ("rstar") during the boom. If investors have belief disagreements, they take speculative positions that make the economy effectively extrapolative: raising optimism during the boom and pessimism during the recession. Speculation strengthens the downward spirals and exacerbates the recession. Speculation also motivates macroprudential policy: optimists take too much risk from a social point of view since they do not internalize their positive effect on asset prices and aggregate demand during recessions. Macroprudential policy can improve outcomes (in a Pareto and a belief-neutral sense), and is procyclical as the negative aggregate demand effect of prudential tightening is more easily offset by interest rate policy during booms than during recessions.

JEL Codes: **E00, E12, E21, E22, E30, E40, G00, G01, G11**

Keywords: Risk-gap, output-gap, aggregate demand, aggregate supply, liquidity trap, interest rates, "rstar," portfolio decisions, Sharpe ratio, monetary and macroprudential policy, heterogeneous beliefs, speculation, tail risk, endogenous volatility and jumps.

Summary at: http://voxeu.org/article/risk-intolerance-and-global-economy

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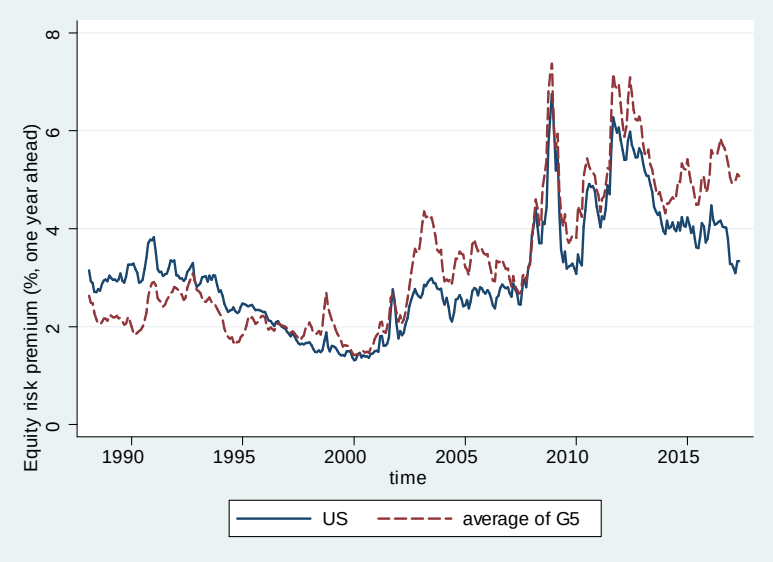


Figure 1: Solid line plots the (forward looking) equity risk premium for the US. Dashed line plots the unweighted average premium for the G5 countries (the US, Japan, the UK, Germany, France). Source: Constructed by Datastream as the median of nine different methods to calculate the ERP. Mean-based methods tend to give higher levels but similar shapes for the path of the ERP.

1. Introduction

Figure 1 shows an estimate of the path of the expected equity risk premium (ERP) for the U.S. and the average of the G5 countries. Several risk-intolerance patterns are apparent in this figure: (i) the ERP spiked during the subprime and European crises; (ii) the ERP remained elevated through much of the U.S. recovery; and (iii) at the global level there is little evidence that the ERP will go to pre-crisis levels any time soon. These risk market observations are not only important for asset pricing issues but also for macroeconomics. Central banks are acutely aware of the connection between risk markets and macroeconomic outcomes. For example, Cieslak and Vissing-Jorgensen (2017) conduct a textual analysis of 184 FOMC minutes during the 1994-2016 period and find extensive reference to stock market developments, which in turn had significant explanatory power for target rate changes. The rationale for these reactions highlighted the negative impact of severe stock markets declines on aggregate consumption and investment.

The implicit framework in these policy discussions is that a productive capacity generates output and risks, *both* of which need to be absorbed by economic agents. If they are unwilling or unable to do so, reinforcing output- and risk-gaps emerge that require appropriate policy responses to prevent severe downward spirals. In contrast with this dual-perspective, most New Keynesian macroeconomic modeling focuses on the output-gap component and relegates the risk-side to a secondary role or none at all. Our main goal in this paper is to provide a macroeconomic model that gives the risk-side a prominent role.

For this, we develop a continuous time macrofinance model with aggregate demand channels and speculative motives due to belief disagreements.[1] In this model, shocks interact with interest rate policy and its constraints in determining the output gap and the natural interest rate ("rstar"). Also, while the degree of optimism of economic agents is key in containing the fall during recessions, optimists' risk taking is potentially destabilizing, which generates a role for macroprudential policy.

The supply side of the (model-)economy is a stochastic AK model with capital-adjustment costs and sticky prices. The demand side has standard risk-averse consumer-investors that demand the goods and risky assets. In equilibrium, the volatility of their consumption is equal to "the Sharpe ratio" of capital (a measure of the risk-adjusted expected return in excess of the risk-free rate). Our analysis rests on the mechanism by which this *risk balance condition* is achieved. Economic agents only differ (when they do) in their beliefs with respect to the likelihood of a near-term recession or recovery. There are no financial frictions. Instead, we focus on "interest-rate frictions": factors that might constrain or delay the adjustment of the risk-free interest rate to shocks. For concreteness, we work with a zero lower bound on the interest rate.

The model has productivity shocks, which we use to generate asset price volatility. Our focus is on "volatility shocks," which we view as capturing a variety of factors that affect the risk premium. Specifically, the economy transitions between low and high volatility episodes. In the absence of interest-rate frictions, it is "rstar" that absorbs these shocks. The natural interest rate ensures that output is determined by the supply side of the economy. By Walras law, this also implies that the risk balance condition is satisfied. Put differently, "rstar" simultaneously closes the output gap and the risk gap. The output gap is closed by generating sufficiently high asset prices that convinces the agents to absorb the current productive capacity (via high consumption and investment), and the risk gap is closed by generating a sufficiently high Sharpe ratio that convinces them to hold the assets backed by (volatile) future productive capacity.

When the interest rate is constrained, it cannot accomplish both objectives. In our model, the risk markets are frictionless, whereas the goods markets are subject to nominal rigidities. This ensures that, when there is a conflict between the two objectives, the risk gaps close immediately whereas the output gaps remain and the economy experiences a demand-driven recession.

To fix ideas, consider a shock that increases volatility (or the risk premium). The immediate effect of this shock is to decrease the Sharpe ratio of capital. A risk gap develops, in the sense that the economy generates too much risk relative to what investors are willing to absorb. The natural response of the economy is a decrease in the interest rate, which increases the Sharpe

[1] By a macrofinance model we mean, following (and quoting) Brunnermeier and Sannikov (2016b): "Instead of focusing only on levels, the first moments, the second moments, and movements in risk variables are all an integral part of the analysis, as they drive agents' consumption, (precautionary) savings and investment decisions."

Also, while in our model heterogenous beliefs have a specific formulation, we intend to capture common features, especially on the positive analysis, of many mechanisms that generate effective heterogeneity in asset valuation.

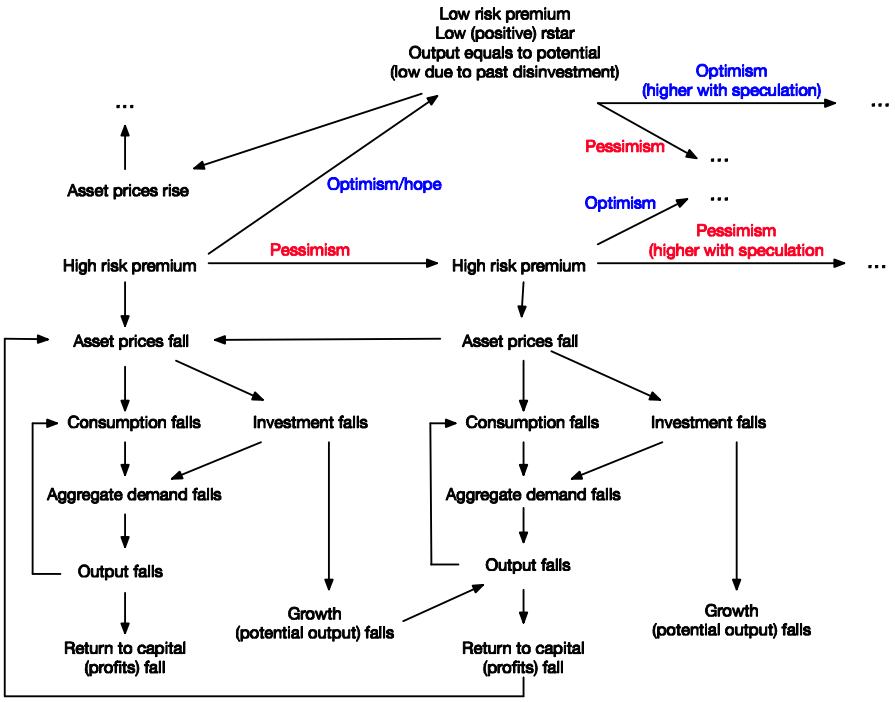


Figure 2: The dynamic equilibrium and the asset price-output feedbacks.

ratio and restores equilibrium in risk markets (as well as goods markets).

If there is a lower bound on the interest rate, the economy loses its natural line of defense. Instead, the risk markets are equilibrated via a decline in asset prices, which increases the Sharpe ratio via expected capital gains. However, the decline in asset prices lowers consumption through a wealth effect and investment through a standard valuation (marginal-Q) channel. This reduces aggregate demand and output, that is, the economy experiences a demand recession.

In a dynamic environment, the recession can be exacerbated by two feedback mechanisms. First, when the risk shock is somewhat persistent, the decline in output in future recessionary periods lowers expected profits, which exerts further downward pressure on asset prices. Second, the decline in current investment lowers the growth of potential output, which reduces expected profits and asset prices (even if there is no demand recession in future periods). In turn, the decline in asset prices feeds back into current consumption and investment, generating scope for severe spirals in asset prices and output. Figure 2 provides a graphical illustration of these dynamic feedback mechanisms.

We identify two conditions on investors' beliefs that amplify the dynamic feedbacks: *pessimism* and *belief disagreements*—that trigger *speculation*. Let's start with the former. If investors are pessimistic and interpret the decline in asset prices as a lasting one, then it will take a large drop in current asset prices to increase the Sharpe ratio and stabilize the risk markets. Consequently, the resulting decline in aggregate demand and output will be severe. If instead they are optimistic, and think of the decline in asset prices as largely temporary, then their per-

ceived Sharpe ratio will rise quickly with a decline in current asset prices, so only a limited asset price drop is required to restore equilibrium. Hence, the degree of pessimism is a critical state variable in our economy, not only because pessimism has a direct impact on asset valuations, but also because it mediates the strength of the dynamic feedbacks.

Consider now the role of belief disagreements and speculation. With disagreements, the economy's degree of pessimism depends on the share of wealth in the hands of pessimistic and optimistic investors. The value of rich optimists for the economy as a whole is high during recessions since they raise asset valuations, which in turn increases aggregate demand. However there is nothing in the economy that ensures this allocation of wealth. Disagreements also lead to speculation which may introduce undesirable correlations between the state of the economy and the relative wealth of optimists and pessimists. For example, if the main source of discrepancy during a boom is in the likelihood of a near-term recession, optimists will sell put options which will impoverish them precisely in the state of the economy that needs them the most. Or, if during a recession the discrepancy is about the speed of recovery, they will buy call options which will deplete their wealth if the recession lingers. That is, through relative wealth effects the economy becomes extrapolative: booms breed optimism and recessions breed pessimism.

Speculation amplifies the dynamic feedbacks during the recession, because the extrapolation that it induces has asymmetric effects on the economy. In case there is a recovery, the interest rate (optimally) rises to neutralize the effect of greater optimism on asset prices and output. If the recession persists, however, the interest rate is constrained and greater pessimism translates into lower prices and output. Moreover, as illustrated by Figure 2, the anticipation of this feature exacerbates the dynamic feedbacks and worsens the drop in current asset prices and output.

Speculation also motivates macroprudential policy, especially during the boom, because optimists' risk taking is associated with *aggregate demand externalities*. The depletion of optimists' wealth during a recession depresses asset prices and aggregate demand. Optimists do not internalize the effect of their portfolio risks on asset valuations (in subsequent periods), which leads to excessive risk taking from an aggregate point of view. We show that making optimistic agents behave as-if they were more pessimistic can lead to a *Pareto improvement* (that is, we evaluate investors' welfare according to their own beliefs). The policy is naturally procyclical as the tightening of prudential regulation always depresses aggregate demand (in the current period), but this effect can be easily offset with interest rate policy during booms but not during severe recessions. Moreover, with the passage of time, a credible macroprudential framework stabilizes the economy and eventually raises "rstar."

We also find that the drop in asset prices during the recession has implications for "rstar" during the boom. The fear of a switch into the high-volatility (recession) state raises the expected capital loss in the low-volatility (boom) state—and considerably so when pessimism or speculation is high and the feedback effects are strong. The interest rate then has to decline also in the low volatility state so as to increase the Sharpe ratio and equilibrate the risk markets. Hence, our model can generate low interest rates together with low volatility—similar to the

current macroeconomic environment—because agents fear downward price spirals triggered by a persistent increase in volatility. The strength of this mechanism is once again mediated by the current degree of pessimism, captured with the investors' belief that the economy will transition into the high-volatility state.

While we emphasize the effect of exogenous risk shocks, such as changes in volatility or pessimism, the model also generates endogenous price volatility. Without interest rate rigidities, the interest rate policy optimally mitigates the impact of risk shocks on asset prices. When the interest rate is constrained, these shocks translate into price volatility. With belief disagreements, speculation exacerbates endogenous price volatility further by creating fluctuations in investors' wealth shares. In recent work, Brunnermeier and Sannikov (2014) also obtain endogenous price volatility under a slightly different set of assumptions, but our model makes the additional prediction that volatility will be higher when the interest rate policy is constrained. This prediction lends support to the many unconventional tools aimed at reducing downward volatility, which the major central banks put in place once interest-rate policy was no longer available during the Great Recession.

Literature review. At a methodological level, our paper belongs in the new continuous time macrofinance literature started by the seminal work of Brunnermeier and Sannikov (2014, 2016a) and summarized in Brunnermeier and Sannikov (2016b) (see also Basak and Cuoco (1998); Adrian and Boyarchenko (2012); He and Krishnamurthy (2012, 2013); Di Tella (2012); Moreira and Savov (2017); Silva (2016)). This literature seeks to highlight the full macroeconomic dynamics induced by financial frictions, which force the reallocation of resources from high-productivity borrowers to low-productivity lenders after a sequence of negative shocks. While the structure of our economy shares many similarities with theirs, in our model there are no financial frictions, and the macroeconomic dynamics stem not from the supply side (relative productivity) but from the aggregate demand side.

Our paper is also related to an extensive New Keynesian literature that emphasizes the role of financial frictions and nominal rigidities in driving business cycle fluctuations (see, for instance, Bernanke et al. (1999); Curdia and Woodford (2010); Gertler and Karadi (2011); Gilchrist and Zakrajšek (2012); Christiano et al. (2014)). Like this literature, we focus on episodes with high risk premia, but we generate these episodes from changes in risk (or risk perceptions) as opposed to financial frictions. We also emphasize the role of beliefs (pessimism) as well as speculation in exacerbating risk-driven business cycles.

A strand of the literature emphasizes the role of "risk shocks" in exacerbating financial frictions (see, for instance, Christiano et al. (2014); Di Tella (2012)). We share with this literature the emphasis on uncertainty, but we focus on changes in aggregate risk—as opposed to idiosyncratic uncertainty—which increases risk premia even in absence of frictions. More broadly, there is an extensive recent empirical literature documenting the importance of uncertainty shocks in causing and worsening recessions (see, for instance, Bloom (2009)).

The interactions between risk shocks and interest rate lower bounds is also a central theme of the literature on safe asset shortages and safety traps (see, for instance, Caballero and Farhi (2017); Caballero et al. (2017b)). We extend this literature by analyzing recurrent business cycles with multiple shocks, speculation, as well as integrated interest-rate and macroprudential policies. In recent work, Del Negro et al. (2017) provide a comprehensive empirical evaluation of the different mechanisms that have put downward pressure on interest rate and argue convincingly that risk and liquidity considerations played a central role (see also Caballero et al. (2017a)). More broadly, the literature on liquidity traps is extensive and has been rekindled by the Great Recession (see, for instance, Tobin (1975); Krugman (1998); Eggertsson and Woodford (2006); Eggertsson and Krugman (2012); Guerrieri and Lorenzoni (2017); Werning (2012); Hall (2011); Christiano et al. (2015); Eggertsson et al. (2017); Rognlie et al. (2017); Midrigan et al. (2016); Bacchetta et al. (2016)). We extend this literature by focusing on the risk aspects (both shocks and mechanisms) behind the drop in the natural rate below its lower bound, as well as on the interaction between speculation and the severity of recessions.

Our results on macroprudential policy are related to a recent literature that analyzes the implications of aggregate demand externalities for the optimal regulation of financial markets. For instance, Korinek and Simsek (2016) show that, in the run-up to deleveraging episodes that coincide with a zero-lower-bound on the interest rate, welfare can be improved by policies targeted toward reducing household leverage. In Farhi and Werning (2017), the key constraint is instead a fixed exchange rate, and the aggregate demand externality calls for ex-ante regulation but also ex-post redistribution, in the form of a fiscal union. In these papers, heterogeneity in agents' marginal propensities to consume (MPC) is the key determinant of optimal macroprudential policy. The policy works by reallocating wealth across agents and states in a way that high-MPC agents hold relatively more wealth when the economy is more depressed due to deficient demand. The mechanism in our paper is different and works through *heterogeneous asset valuations.* In fact, we work with a log-utility setting in which all agents have the same marginal propensity to consume. The policy operates by transferring wealth to optimists during recessions, not because optimists spend more than other agents, but because they raise the asset valuations and induce *all investors to spend more* (while also increasing aggregate investment).[2]

Beyond aggregate demand externalities, the macroprudential literature is also extensive, and mostly motivated by the presence of pecuniary externalities that make the competitive equilibrium constrained inefficient (e.g., Caballero and Krishnamurthy (2003); Lorenzoni (2008); Bianchi and Mendoza (2013); Jeanne and Korinek (2010)). The friction in this case is not "nominal" and interest rate rigidities, but market incompleteness or collateral constraints that depend on asset prices (see Davila and Korinek (2016) for a detailed exposition). Macroprudential policy typically improves outcomes by mitigating fire sales that exacerbate financial frictions. The policy in our model also operates through asset prices but through a different channel. We

[2] Also, see Farhi and Werning (2016) for a synthesis of some of the key mechanisms that justify macroprudential policies in models that exhibit aggregate demand externalities.

show that a decline in asset prices is damaging not only because of the fire-sale reasons emphasized in this literature, but also because it lowers aggregate demand through standard wealth and investment channels. Moreover, our analysis does not feature the incomplete markets or collateral constraints that are central in this literature.

Our results are also related to a large literature that analyzes the effect of belief disagreements and speculation on financial markets (e.g., Lintner (1969); Miller (1977); Harrison and Kreps (1978); Varian (1989); Harris and Raviv (1993); Chen et al. (2002); Scheinkman and Xiong (2003); Fostel and Geanakoplos (2008); Geanakoplos (2010); Simsek (2013a,b); Iachan et al. (2015)). One strand of this literature emphasizes that disagreements can exacerbate asset price fluctuations by creating endogenous fluctuations in agents' wealth distribution (see, for instance, Basak (2000, 2005); Cao (2017); Xiong and Yan (2010); Kubler and Schmedders (2012); Korinek and Nowak (2016)). Our paper features similar forces but explores them in an environment in which output is not necessarily at its supply-determined level due to interest rate rigidities. In fact, our framework is similar to the models analyzed by Detemple and Murthy (1994); Zapatero (1998), who show that financial speculation between optimists and pessimists (with log utility) can increase the volatility of the interest rate. In our model, these results apply when the interest rate is unconstrained but they are modified if the interest rate is constrained in downward adjustments. In the latter case, speculation translates into (inefficient) fluctuations in asset prices as well as aggregate demand. Among other things, we show that (controlling for the average belief) speculation driven by belief disagreement depresses aggregate demand and lowers output during recessions. We also show that belief disagreements create scope for macroprudential policy.

The rest of the paper is organized as follows. In Section 2 we present an example that illustrates the main mechanism and motivates the rest of our analysis. Section 3 presents the general environment and defines the equilibrium. Section 4 characterizes the equilibrium in a benchmark setting with homogeneous beliefs. This section illustrates how risk premium shocks can lower asset prices and induce a demand recession, and how optimism helps to mitigate the recession. It also illustrates how the drop in asset prices during the recession lowers the interest rate during booms. Section 5 characterizes the equilibrium with belief disagreements, and illustrates how speculation exacerbates the recession. Section 6 establishes our normative results in two steps. Section 6.1 characterizes the value functions and illustrates the aggregate demand externalities. Section 6.2 analyzes the effect of introducing risk limits on optimists, and presents our results on (procyclical) macroprudential policy. Section 7 concludes and is followed by an appendix that contains the omitted derivations and proofs.

2. A Motivating Example

We first present a simple example that illustrates the workings of the aggregate demand mechanism and provides the motivation for our main model. Consider an economy with two dates, $t \in \{0, 1\}$, a single consumption good, and a single factor of production—capital. For simplicity, capital is fixed (i.e., there is no depreciation or investment) and it is normalized to one. *Potential* output is equal to capital's productivity, z_t, but the actual output can be below this level due to a shortage of aggregate demand, $y_t \leq z_t$. For simplicity, we assume output is equal to its potential at the last date, $y_1 = z_1$, and focus on the endogenous determination of output at the previous date, $y_0 \leq z_0$. We assume the productivity at date 1 is uncertain and log-normally distributed so that,

$$\log y_1 = \log z_1 \sim N\left(g - \frac{\sigma^2}{2}, \sigma^2\right). \tag{1}$$

We also normalize the initial productivity to one, $z_0 = 1$, so that g denotes the expected growth rate of productivity, and σ denotes its volatility.

The demand side is characterized by a representative household, who is endowed with the initial output as well as claims on future output. At date 0, she chooses how much to consume, c_0, and how to allocate her wealth across available assets. We assume there is a "market portfolio" that represents claims to the output at date 1 (the return to capital as well as profits), and a risk-free asset in zero net supply. We let Q and $r^k = \log \frac{z_1}{Q}$ denote, respectively, the price and the (log) return of the market portfolio, and r^f denote the log risk-free interest rate. The household invests a fraction of her wealth, ω^k, in the market portfolio, and the residual fraction, $1 - \omega^k$, in the risk-free asset. When asset markets are in equilibrium, the household will invest all of her wealth in the market portfolio, $\omega^k = 1$, and her portfolio demand will determine the price Q.

We assume the household has Epstein-Zin preferences with the discount factor given by $e^{-\rho}$, the elasticity of intertemporal substitution (EIS) equal to 1, and the relative risk aversion coefficient (RRA) given by γ.[3] The assumption on the EIS implies that the household consumes a fraction of her lifetime income,

$$c_0 = \frac{1}{1 + e^{-\rho}} (y_0 + Q). \tag{2}$$

In Appendix A.1, we also show that, up to a local approximation, the household's optimal optimal weight on the market portfolio is determined by,

$$\omega^k \sigma \simeq \frac{1}{\gamma} \frac{E\left[r^k\right] + \frac{\sigma^2}{2} - r^f}{\sigma}. \tag{3}$$

[3]For simplicity, in the main model we restrict attention to the special case with log utility, which implies EIS=1 and RRA=1. In the two-period model, we can be more general in terms of the RRA, which allows us to illustrate that our volatility shocks are meant to capture spikes in implied volatility, driven by risk aversion or realized volatility.

In words, household's optimal portfolio risk (left side) is proportional to "the Sharpe ratio" on the market portfolio (right side). The Sharpe ratio captures the reward per risk, where the reward is determined by the risk premium: the (log) expected return in excess of the (log) risk free rate. This is the standard risk-taking condition for mean-variance portfolio optimization, which applies exactly in continuous time. It applies approximately in the two-period model for arbitrary levels of the risk premium, and the approximation becomes exact for the level the risk premium that ensures equilibrium ($\omega^k = 1$).

In particular, substituting the equilibrium condition, $\omega^k = 1$, as well as the expected return on the market portfolio from Eq. (1), we obtain the (exact) risk balance condition,

$$\sigma = \frac{1}{\gamma} \frac{g - \log Q - r^f}{\sigma}. \tag{4}$$

In words, the equilibrium in asset markets requires the Sharpe ratio on the market portfolio (right side) to be sufficiently large to convince the investors to hold the risk generated by the productive capacity (left side).

The supply side of the economy is described by New-Keynesian firms that have preset fixed prices. These firms meet the available demand at these prices as long as it does not exceed their marginal costs (see Appendix A.2.2 for details). These features imply that output is determined by the aggregate demand for goods (consumption) up to the capacity constraint,

$$y_0 = c_0 \leq z_0. \tag{5}$$

Since prices are fully sticky, the real interest rate is equal to the nominal interest rate, which is controlled by the monetary authority. We assume that the interest rate policy attempts to replicate the supply-determined output level. However, there is a lower bound constraint on the interest rate, $r^f \geq 0$. Thus, the monetary policy is described by, $r^f = \max\left(r^{f*}, 0\right)$, where r^{f*} is the natural interest rate that ensures output is at its efficient level.

To characterize the equilibrium, first note that Eqs. (2) and (5) imply that there is a one-to-one relationship between output and the price of the market portfolio,

$$y_0 = e^{\rho} Q. \tag{6}$$

Intuitively, asset prices increase aggregate wealth and consumption, which in turn determines output. In particular, output is equal to its potential level, $y_0 = z_0 = 1$, if and only if the asset prices are at a particular level, $Q^* = e^{-\rho}$. Combining this expression with Eq. (4), we can also solve for the natural interest rate as,

$$r^{f*} = g + \rho - \gamma\sigma^2.$$

Suppose the initial parameters are such that $r^{f*} > 0$, so that the equilibrium features Q^*, r^{f*}

and full output, $y_0 = z_0 = 1$. Now suppose there is a "risk shock" that raises the volatility, σ, or risk aversion, γ. The immediate impact of this shock is to create an imbalance in the risk-market equilibrium condition (4). The economy produces too much risk (left side) relative to what households are willing to absorb (right side). In response, the monetary policy lowers the risk-free interest rate (as captured by the decline in r^{f*}), which tends to re-equilibrate the risk market condition (4). Intuitively, the monetary policy lowers the opportunity cost of risky investment and induces households to absorb risk.

Next suppose the shock is sufficiently large so that the natural interest rate becomes negative, $r^{f*} < 0$, and the actual interest rate becomes constrained, $r^f = 0$. In this case, the risk market condition is reestablished with a decline in the price of the market portfolio, Q, which increases the expected return on risky investment and induces investors to hold risk. However, the decline in Q reduces aggregate wealth and induces a demand-driven recession. Formally, we have,

$$\log y_0 = \rho + \log Q, \text{ where } \log Q = \begin{cases} \log Q^* = -\rho, & \text{if } \gamma\sigma^2 \leq g + \rho, \\ g - \gamma\sigma^2 < -\rho, & \text{otherwise.} \end{cases}$$

Note also that, in the constrained region, asset prices and output become sensitive to further changes in risk (σ, γ) and in beliefs about future prospects. For instance, an increase in the expected growth rate, g (optimism) increases asset prices and mitigates the recession.

While the two period model is useful to illustrate the basic mechanism by which risk shocks can induce a recession, it does not capture the richer mechanisms that arise from dynamic considerations. As Figure 2 in the introduction illustrates, (current and future) asset prices also affect expected profits. Put differently, g in the risk balance equation (4) is endogenous. Since g also affects current prices, there is scope for feedbacks between asset prices and output. Moreover, in this context heterogenous beliefs about future dynamics create speculative forces which have the potential to greatly exacerbate these feedbacks and justify macroprudential policy. We turn to the formal dynamic framework next.

3. General environment and equilibrium

In this section we introduce our general environment and define the equilibrium. In subsequent sections we will characterize this equilibrium in various special cases of interest. We start by describing the production and investment technology, as well as the risk-premium shocks that play the central role in our analysis. We then describe the firms' investment decisions, followed by the investors' consumption and portfolio choice decisions. Then, we introduce the nominal and the interest rate rigidities that ensure output is determined by aggregate demand. We finally introduce the goods and asset market clearing conditions and define the equilibrium.

Potential output and risk-premium shocks. The economy is set in infinite continuous time, $t \in [0, \infty)$, with a single consumption good and a single factor of production: capital. Let

$k_{t,s}$ denote the capital stock at time t and the aggregate state $s \in S$. Suppose that, when fully utilized, $k_{t,s}$ units of capital produces $Ak_{t,s}$ units of the consumption good. Hence, $Ak_{t,s}$ denotes the potential output in this economy. Capital follows the process,

$$\frac{dk_{t,s}}{k_{t,s}} = g_{t,s}dt + \sigma_s dZ_t \qquad \text{where } g_{t,s} \equiv \varphi\left(\iota_{t,s}\right) - \delta. \tag{7}$$

Here, $\iota_{t,s} = \frac{i_{t,s}}{k_{t,s}}$ denotes the investment rate, $\varphi\left(\iota_{t,s}\right)$ denotes the production function for capital (that will be specified below), and δ denotes the depreciation rate. The second equation defines the expected growth rate of capital (and potential output). The term, dZ_t, denotes the standard Brownian motion, which captures "aggregate productivity shocks."[4]

The states, $s \in S$, differ only in terms of the volatility of aggregate productivity, σ_s. For simplicity, suppose there are only two states, $s \in \{1, 2\}$, with $\sigma_1 < \sigma_2$ (see the extended working paper version Caballero and Simsek (2017b) for the general formulation with an arbitrary number of states). State $s = 1$ corresponds to a low-volatility state, whereas state $s = 2$ corresponds to a high-volatility state. At every instant, the economy in state s transitions into the other state $s' \neq s$ according to a Poisson process. We let λ_s^i denote the transition probability in state s (into the other state) according to investor $i \in I$. These transition probabilities will play a central role in the analysis, as they capture investors' optimism or pessimism. For instance, an investor with low λ_2^i is pessimistic in the sense that she expects the high risk conditions to persist. Likewise, an investor with high λ_1^i is pessimistic in the sense that she believes that, even though the economy currently features low risk, the high risk conditions are around the corner. We will set up the model for investors with potentially heterogeneous beliefs (and in fact, this will be the only possible source of heterogeneity). We will first analyze the special case with common beliefs (Section 4) and then investigate the effect of belief disagreements and speculation (Section 5).

Remark 1 (Interpreting the Volatility Shocks)**.** *We work with volatility shocks mainly because they lead to a tractable analysis. The key feature of these shocks is that they increase the risk premium on capital, and might push the economy into a liquidity trap in which the risk-free interest rate is at its lower bound. Many other shocks that increase the risk premium would lead to a similar analysis. In fact, we view the variance parameters, $\left\{\sigma_s^2\right\}_s$, as capturing in reduced form various unmodeled objective and subjective factors that might shift the risk premium (such as time-varying risk aversion, long-run risks, Knightian uncertainty, or financial panics).*

Investment and the growth-price relationship. There is a continuum of identical firms that manage capital. These firms rent capital to production firms (that will be described below) to earn the instantaneous rental rate, $R_{t,s}$. They also make investment decisions to maximize

[4] Note that fluctuations in $k_{t,s}$ generate fluctuations in potential output, $Ak_{t,s}$. We introduce Brownian shocks to capital, $k_{t,s}$, as opposed to the total factor productivity, A, since this leads to a slightly more tractable analysis. See Footnote 2 in Brunnermeier and Sannikov (2014) for an equivalent formulation in terms of shocks to A.

the value of capital. Letting $Q_{t,s}$ denote the price of capital, the firm's investment problem can be written as,

$$\max_{\iota_{t,s}} Q_{t,s}\varphi\left(\iota_{t,s}\right)k_{t,s} - \iota_{t,s}k_{t,s}.$$

Under standard regularity conditions for $\varphi\left(\iota\right)$, investment is determined by the optimality condition, $\varphi'\left(\iota_{t,s}\right) = 1/Q_{t,s}$ We will work with the special and convenient case proposed by Brunnermeier and Sannikov (2016b): $\varphi\left(\iota\right) = \psi \log\left(\frac{\iota}{\psi} + 1\right)$. In this case, we obtain the closed form solution,

$$\iota\left(Q_{t,s}\right) = \psi\left(Q_{t,s} - 1\right). \tag{8}$$

The parameter, ψ, captures the sensitivity of investment to asset prices.

Note also that the amount of capital produced is given by,

$$\varphi\left(\iota\left(Q_{t,s}\right)\right) = \psi q_{t,s}, \text{ where } q_{t,s} \equiv \log\left(Q_{t,s}\right). \tag{9}$$

The log price level, $q_{t,s}$, will simplify some of the expressions below. Combining Eq. (9) with Eq. (7), we also obtain an expression for growth,

$$g_{t,s} = \psi q_{t,s} - \delta. \tag{10}$$

In particular, unlike in the two period model, the expected growth rate of capital (and potential output) is now endogenous and depends on asset prices. Lower asset prices reduce investment, which in turn translates into lower growth and lower potential output in future periods. This mechanism will be a key source of amplification.

Capital price and return. As before, we assume there is a "market portfolio" that represents a claim on aggregate capital (more specifically, a claim on the firms that manage capital). The return on this portfolio depends on (among other things) the evolution of the value of aggregate capital, $Q_{t,s}k_{t,s}$. We next describe how $Q_{t,s}k_{t,s}$ evolves and how this translates into the return to capital.

Absent state transitions, the price of capital follows an endogenous but deterministic process,[5]

$$\frac{dQ_{t,s}}{Q_{t,s}} = \mu^Q_{t,s}dt \text{ for each } s \in \{1,2\}. \tag{11}$$

When investors have common beliefs (Section 4), the endogenous price drift will be zero, $\mu^Q_{t,s} = 0$: that is, the price of capital will be fixed with high and low volatility states, $\{Q_1, Q_2\}$. With belief disagreements (Section 5), there will be room for price dynamics due to changes in investors'

[5] In general, the price follows a diffusion process and this equation also features an endogenous volatility term, $\sigma^Q_{t,s}dZ_t$. In this paper, we assume that financial markets are complete, which (combined with our other assumptions) ensures that $\sigma^Q_{t,s} = 0$. See our companion paper for the analysis with incomplete markets, which features $\sigma^Q_{t,2} \neq 0$, that is, there is endogenous price volatility within the high-volatility state.

wealth shares. Combining Eqs. (7) and (11), the aggregate wealth (conditional on *no* transition) evolves according to

$$\frac{d\left(Q_{t,s}k_{t,s}\right)}{Q_{t,s}k_{t,s}} = \left(g_{t,s} + \mu^{Q}_{t,s}\right)dt + \sigma_s dZ_t. \tag{12}$$

It follows that, absent state transitions, the volatility of the market portfolio (claim on aggregate capital) is given by, σ_s. Likewise, the expected return on this portfolio conditional on no transition is given by,

$$r^{k}_{t,s} = \frac{R_{t,s} - \iota_{t,s}}{Q_{t,s}} + g_{t,s} + \mu^{Q}_{t,s}. \tag{13}$$

Here, the first term can be thought of as the "dividend yield," which captures the instantaneous rental rate of capital, $R_{t,s}$, as well as the investment costs. The second component is the capital gains conditional on no transition, which reflect the expected growth in aggregate wealth due to the growth of capital or price drift.

Eqs. $(11-13)$ describe the prices and returns conditional on there not being a state transition. If there is a transition at time t from state s into state $s' \neq s$, then the price of capital jumps from $Q_{t,s}$ to a potentially different level, $Q_{t,s'}$. Therefore, the aggregate wealth also jumps from $Q_{t,s}k_{t,s}$ to a potentially different level, $Q_{t,s'}k_{t,s}$, and the investors that hold the market portfolio experience instantaneous capital gains or losses that will be reflected in their portfolio problem.

Consumption and portfolio choice. There is a continuum of investors denoted by $i \in I$, who are identical in all respect except possibly their beliefs on state transitions, λ^{i}_{s}. Investors' beliefs are dogmatic: that is, they know each others' beliefs and they agree to disagree.

Investors continuously make consumption and portfolio allocation decisions. Each investor has access to three types of assets. First, the investor can invest in the market portfolio that we described above. Second, she can invest in a risk-free asset with return, $r^{f}_{t,s}$. The risk-free asset is in zero net supply. Third, the investor can also invest in a contingent Arrow-Debreu security that trades at the (endogenous) instantaneous price $p^{s'}_{t,s}$, and that pays 1 dollar if the economy transitions to the other state $s' \neq s$. These securities are also in zero net supply, and they ensure that the financial markets are complete.

Let $a^{i}_{t,s}$ denote the wealth level for investor i, at time t, in state s. For analytical tractability, we assume the investor has log utility. Let $c^{i}_{t,s}$ be the investor's consumption rate, $\omega^{k,i}_{t,s}$ denotes the fraction of her wealth she allocates to capital, and $\omega^{s',i}_{t,s}$ denotes the fraction of her wealth she allocates to the Arrow-Debreu security. The residual fraction of the investor's wealth, $1 - \omega^{k,i}_{t,s} - \omega^{s',i}_{t,s}$, is invested in the risk-free asset. The investor's optimization problem (at some time t and state s) can be written as,

$$V^i_{t,s}\left(a^i_{t,s}\right) = \max_{\left[\tilde{c}_{\tilde{t},\tilde{s}},\tilde{\omega}^k_{\tilde{t},\tilde{s}},\tilde{\omega}^{\tilde{s}'}_{\tilde{t},\tilde{s}}\right]_{\tilde{t}\geq t,\tilde{s}}} E^i_{t,s}\left[\int_t^\infty e^{-\rho\tilde{t}}\log\tilde{c}^i_{\tilde{t},\tilde{s}}d\tilde{t}\right] \tag{14}$$

$$\text{s.t.} \begin{cases} da^i_{t,s} = \left(a^i_{t,s}\left(r^f_{t,s} + \tilde{\omega}^k_{t,s}\left(r^k_{t,s} - r^f_{t,s}\right) - \tilde{\omega}^{s'}\right) - \tilde{c}_{t,s}\right)dt + \tilde{\omega}^k_{t,s}a^i_{t,s}\sigma_s dZ_t & \text{absent transition,} \\ a^i_{t,s'} = a^i_{t,s}\left(1 + \tilde{\omega}^k_{t,s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}} + \tilde{\omega}^{s'}_{t,s}\frac{1}{p^{s'}_{t,s}}\right) & \text{if there is a transition to state } s' \neq s. \end{cases} \tag{15}$$

Here, $E^i_{t,s}[\cdot]$ denotes the expectations operator that corresponds to the investor i's beliefs for state transition probabilities.

While problem (14) looks mouthful, its solution is relatively straightforward and follows from standard recursive techniques (see Appendix A.2.1). Specifically, the optimal consumption is given by,

$$c^i_{t,s} = \rho a^i_{t,s}. \tag{16}$$

In view of log utility, the investor consumes a constant fraction of her wealth. The optimal portfolio allocation to capital is determined by,

$$\omega^{k,i}_{t,s}\sigma_s = \frac{1}{\sigma_s}\left(r^k_{t,s} - r^f_{t,s} + \lambda^i_s\frac{1/a^i_{t,s'}}{1/a^i_{t,s}}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right). \tag{17}$$

Intuitively, she invests in capital up to the point at which the risk of her portfolio (left side) is equal to "the Sharpe ratio" of capital (right side). This is similar to the optimality condition in the two period model (cf. Eq. (3)) with the difference that the dynamic model also features state transitions. Our notion of the Sharpe ratio accounts for potential revaluation gains or losses from state transitions (the term, $\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}$) as well as the adjustment of marginal utility in case there is a transition (the term, $\frac{1/a^i_{t,s'}}{1/a^i_{t,s}}$).[6]

Finally, the optimal portfolio allocation to the contingent securities implies,

$$\frac{p^{s'}_{t,s}}{\lambda^i_s} = \frac{1/a^i_{t,s'}}{1/a^i_{t,s}}. \tag{18}$$

The portfolio weight, $\omega^{s',i}_{t,s}$, is implicitly determined as the level that ensures that this equality holds. The investor buys contingent securities up to the point at which the price-to-probability ratio of a state (or the state price) is equated to the investor's relative marginal utility in that state. Note that replacing (18) into (17) shows that investors allocate identical portfolio weights to capital, $\omega^k_{t,s}$ (which will be equal to one in equilibrium), and express their differences in beliefs through their holdings of contingent securities.

[6] The presence of state transitions makes the Sharpe ratio in our model slightly different than the common definition of the Sharpe ratio, which corresponds to the expected return in excess of the risk-free rate normalized by volatility.

Equilibrium in asset markets. Asset markets clearing requires that the total wealth held by investors is equal to the value of aggregate capital before and after the portfolio allocation decisions,

$$\int_I a^i_{t,s} di = Q_{t,s} k_{t,s} \text{ and } \int_I \omega^{k,i}_{t,s} a^i_{t,s} di = Q_{t,s} k_{t,s}. \tag{19}$$

Contingent securities are in zero net supply, which implies,

$$\int_I a^i_{t,s} \omega^{s',i}_{t,s} di = 0. \tag{20}$$

The market clearing condition for the risk-free asset (which is also in zero net supply) holds when conditions (19) and (20) are satisfied.

Nominal rigidities and aggregate demand. The supply side of our model features nominal rigidities similar to the standard New Keynesian model. We relegate the details to Appendix A.2.2 and describe the main implications relevant for our analysis. There is a continuum of monopolistically competitive production firms that rent capital from investment firms and produce intermediate goods (which are then converted into the final good). For simplicity, these production firms have preset prices that they never change. The firms meet the available demand (as long as they find it optimal to do so). In equilibrium, these features imply that output is determined by aggregate demand,

$$y_{t,s} = \eta_{t,s} A k_{t,s} = \int_I c^i_{t,s} di + k_{t,s} \iota_{t,s}, \text{ where } \eta_{t,s} \in [0,1]. \tag{21}$$

Here, $\eta_{t,s}$ denotes the instantaneous factor utilization rate for capital. We assume firms can increase factor utilization for free until $\eta_{t,s} = 1$ and they cannot increase it beyond this level (we relax the latter assumption in the extended working paper version). Aggregate demand corresponds to the sum of aggregate consumption and aggregate investment.

There are also lump sum taxes on the production firms' profits combined with linear subsidies to capital. In equilibrium, these features imply that the rental rate of capital is given by,

$$R_{t,s} = A \eta_{t,s}. \tag{22}$$

This also implies, $y_{t,s} = R_{t,s} k_{t,s}$, that is all output accrues to the agents in the form of return to capital, which simplifies our analysis.[7] Combining this expression with Eqs. (13), and using Eqs. (21) and (16), we also obtain the instantaneous return to capital conditional on no transition as,

$$r^k_{t,s} = \rho + g_{t,s} + \mu^Q_{t,s}, \text{ where } g_{t,s} = \psi q_{t,s} - \delta. \tag{23}$$

[7] Without this type of taxes and subsidies, firms would also make pure profits that are not necessarily linked to the capital they use in production. The analysis of the portfolio problem would then require introducing a second risky asset (claims on pure profits).

Hence, in equilibrium, the dividend yield from capital is the same as the consumption rate ρ.

Output-price relationship. Our analysis so far implies that there is a one-to-one relationship between output and the price of capital as in the two period model (cf. Eq. (6)). Specifically, combining Eqs. (16) and (19) implies that aggregate consumption is a constant fraction of aggregate wealth, $\int_I c^i_{t,s} di = \rho Q_{t,s} k_{t,s}$. Plugging this into Eq. (21), and using the investment equation (8), we obtain,

$$A\eta_{t,s} = \rho Q_{t,s} + \psi \left(Q_{t,s} - 1\right) = \left(\rho + \psi\right) Q_{t,s} - \psi.$$

Intuitively, output per capital (or factor utilization) depends on asset prices, because consumption depends on asset prices through a wealth effect and investment depends on asset prices through a standard marginal-Q channel. Rewriting this expression, we obtain,

$$q_{t,s} = q\left(\eta_{t,s}\right) = \log\left(\frac{A\eta_{t,s} + \psi}{\rho + \psi}\right), \tag{24}$$

which illustrates that full factor utilization, $\eta_{t,s} = 1$, obtains only if the price of capital is at a particular level $q^* \equiv q\left(1\right)$. This is the level of the price that ensures that the implied consumption and investment clears the goods market. Likewise, the economy features a demand recession, $\eta_{t,s} < 1$, if and only if the price of capital is strictly below q^*.[8]

Interest rate rigidity and monetary policy. Our assumption that production firms do not change their prices implies that the aggregate price level is fixed. The real risk-free interest rate is then equal to the nominal risk-free interest rate, which is determined by the interest rate policy of the monetary authority. We assume there is a lower bound on the nominal interest rate, which we take to be zero for convenience,[9]

$$r^f_{t,s} \geq 0. \tag{25}$$

In practice, this type of constraint emerges naturally from a variety of factors. The zero lower bound in particular can be motivated by the presence of cash in circulation (which we leave unmodeled for simplicity). Since cash offers zero interest rate, the monetary authority cannot lower the interest rate (much) below zero—a constraint that appeared to be binding for major central banks in the aftermath of the Great Recession.

We assume that the interest rate policy focuses on replicating the level of output that would obtain absent nominal rigidities subject to the constraint in (25). Appendix A.2.2 illustrates

[8] Recall that this is the price of effective capital; the value of actual equity adds the diffusion term to this price.

[9] In practice, the lower bound on the real interest rate seems to be slightly below zero due to steady-state inflation. We could also assume that firms set their prices at every period mechanically according to a predetermined inflation target. This formulation yields a very similar bound as in (25) and results in the same economic trade-offs. We normalize inflation to zero so as to economize on notation.

that, without nominal rigidities, capital is fully utilized, $\eta_{t,s} = 1$. Thus, we assume the interest rate policy follows the rule,

$$r^f_{t,s} = \max\left(0, r^{f,*}_{t,s}\right) \text{ for each } t \geq 0 \text{ and } s \in S. \tag{26}$$

Here, $r^{f,*}_{t,s}$ is recursively defined as the (instantaneous) natural interest rate that obtains when the (instantaneous) utilization is given by $\eta_{t,s} = 1$, and the monetary policy follows the rule in (26) at all future times and states.

Remark 2 (Interpretation of Price Stickiness)**.** *Our assumption that the aggregate price (or inflation) level is fixed is admittedly extreme. We should note, however, that making the prices more flexible does not necessarily circumvent the bound in* (25)*. In fact, if monetary policy follows an inflation targeting policy regime, then limited price flexibility leads to price deflation during a demand recession, which strengthens the bound in* (25) *and exacerbates the recession (see Werning (2012); Korinek and Simsek (2016); Caballero and Farhi (2017) for further discussion). We could capture this mechanism by allowing for some price flexibility, which would introduce a standard New-Keynesian Phillips curve into the model as in Werning (2012). We have chosen not to emphasize the deflationary spiral mechanism since the analysis is already involved with several other amplification mechanisms related to the endogeneity of (real) asset prices.*

Equilibrium in the goods market. Combining Eq. (26) with the output-price relationship (24), the goods market side of the economy can be summarized with,

$$q_{t,s} \leq q^*, r^f_{t,s} \geq 0, \text{ with at least one condition satisfied as equality.} \tag{27}$$

In particular, the equilibrium at any time and state takes one of two forms. If the natural interest rate is nonnegative, then the interest rate policy ensures that the price of capital is at the level consistent with full employment, $q_{t,s} = q^*$, capital is fully utilized, $\eta_{t,s} = 1$, and output is equal to its potential, $y_{t,s} = Ak_{t,s}$. Otherwise, the interest rate policy is constrained, $r^f_{t,s} = 0$, the price of capital is at a lower level, $q_{t,s} < q^*$, and output is determined by aggregate demand according to Eq. (24). We can now define the equilibrium as follows.

Definition 1. *The equilibrium is a collection of processes for allocations, prices, and returns such that capital and its price evolve according to Eqs.* (7) *and* (11)*, the investment firms maximize (cf. Eqs.* (16)*, the growth rate is given by Eq.* (10)*, the investors maximize (cf. Eqs.* $(16-18)$*), the asset markets clear (cf. Eqs.* (19) *and* (20)*), output is determined by aggregate demand (cf. Eqs.* (21) *and* (24)*), the return to capital (conditional on no transition) is given by Eq.* (23)*, the interest rate policy follows the rule in* (26)*, and the goods market clears (cf. Eq.* (27)*).*

For future reference, we also note that the first-best equilibrium obtains when price is at

its efficient level at all times and states, $q_{t,s} = q^*$. This also implies that the growth rate of output and the expected return to capital are constant and given by, respectively, $g = \psi q^* - \delta$ and $r^k = \rho + \psi q^* - \delta$ (see Eq. (23)). We next turn to the characterization of equilibrium with interest rate rigidities.

4. Common beliefs benchmark and amplification mechanisms

In this section, we analyze the equilibrium in a benchmark case in which all investors share the same belief, that is, $\lambda_s^i \equiv \lambda_s$ for each i. We also normalize the total mass of investors to one so that individual and aggregate allocations are the same. We use this benchmark to establish two amplification mechanisms that have no counterparts in the two period model. We also establish the comparative statics of the equilibrium with respect to investors' (common) belief, and illustrate that these amplification mechanisms are especially powerful when investors are pessimistic.

In view of the linear structure of the model, we conjecture that the price and the interest rate will remain constant within states, $Q_{t,s} = Q_s$ and $r_{t,s}^f = r_s^f$ (in particular, there is no price drift, $\mu_{t,s}^Q = 0$). Since the investors are identical, we also have $\omega_{t,s}^k = 1$ and $\omega_{t,s}^{s'} = 0$. In particular, the representative investor's wealth is equal to aggregate wealth, $a_{t,s} = Q_{t,s}k_{t,s}$. Combining this with Eq. (17) and substituting for $r_{t,s}^k$ from Eq. (23), we obtain the following risk balance conditions,

$$\sigma_s = \frac{\rho - \delta + \psi q_s + \lambda_s \left(1 - \frac{Q_s}{Q_{s'}}\right) - r_s^f}{\sigma_s} \text{ for each } s \in \{1, 2\}. \tag{28}$$

These equations are the dynamic counterpart to Eq. (4) in the two period model. They say that, in each risk state, the total risk in the economy (the left side) is equal to the Sharpe ratio perceived by the representative investor (the right side). Note that the Sharpe ratio accounts for the fact that the aggregate wealth (as well as the marginal utility) will change in case there is a state transition.[10]

The equilibrium is then characterized by finding four unknowns, $\left(Q_1, r_1^f, Q_2, r_2^f\right)$, that solve the two equations (28) together with the two goods market equilibrium conditions (27). We solve these equations under the following parametric restriction.

Assumption 1. $\sigma_2^2 > \rho + \psi q^* - \delta > \sigma_1^2$.

When this restriction holds (and additional assumptions are satisfied), there is an equilibrium in which the low-volatility state 1 features positive interest rates, efficient asset prices, and full factor utilization, $r_1^f > 0, q_1 = q^*$ and $\eta_1 = 1$, whereas the high-volatility state 2 features zero interest rates, lower asset prices, and imperfect factor utilization, $r_2^f = 0, q_2 < q^*$ and $\eta_2 < 1$.

[10] To see this, observe that the term, $\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s'}}$, in the equation is actually equal to, $\frac{Q_{t,s}}{Q_{t,s'}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}$. Here, $\frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}$ denotes the capital gains and $\frac{Q_{t,s}}{Q_{t,s'}}$ denotes the marginal utility adjustment when there is a representative investor (see (17)).

In particular, the analysis with common beliefs reduces to finding two unknowns, $\left(q_2, r_1^f\right)$, that solve the two risk balance equations (28) (after substituting $q_1 = q^*$ and $r_2^f = 0$).

Equilibrium in the high-volatility state. Using our conjecture, the risk balance equation (28) for the high-volatility state $s = 2$ can be written as,

$$\sigma_2 = \frac{\rho + \psi q_2 - \delta + \lambda_2 \left(1 - \frac{Q_2}{Q^*}\right)}{\sigma_2}. \tag{29}$$

In view of Assumption 1, if the price were at its efficient level, $Q_2 = Q^*$, the risk (the left side) would exceed the Sharpe ratio (the right side). As in the two period model, the economy generates too much risk relative to what the agents are willing to absorb at the constrained level of the interest rate. As before, the price of capital, Q_2, needs to decline to equilibrate the risk markets. Unlike in the two period model, however, the decline in the price of capital does not necessarily increase the Sharpe ratio, due to two destabilizing amplification mechanisms.

Amplification mechanisms. The first amplification mechanism comes from the output-price relation (cf. Eq. (24)). If the dividend yield from capital were kept constant, a decline in the current asset price would increase the dividend yield as well as the return—a stabilizing force. However, in our model the dividend yield is not constant and it is decreasing in the current price of capital. A lower price level reduces output, which reduces the rental rate of capital (see Eq. (22)), which in turn lowers dividends. In fact, the dividend yield term in Eq. (29) can be better understood by writing it as, $\frac{\rho Q_2}{Q_2} = \rho$ (see also Eq. (13)). It does not depend on the price because the cash flows in the numerator also decline proportionally with the price level. Hence, the output-price relation overturns an important stabilizing force from price declines, and opens the door for amplification of these declines.

The second amplification mechanism comes from the growth-price relation (cf. Eq. (10)). In particular, a decline in the current asset price also lowers investment, which reduces the growth of potential output and dividends, which in turn lowers the return to capital. The strength of this effect depends on the sensitivity of investment to asset prices, captured by the term ψq_2. Figure 2 in the introduction presents a graphical illustration of the two amplification mechanisms.

In view of these amplification mechanisms, one might wonder how the risk market ever reaches equilibrium once the price, Q_2, starts to fall below its efficient level, Q^*. The stabilizing force is captured by the last term in Eq. (29), $\lambda_2 \left(1 - \frac{Q_2}{Q^*}\right)$. A decline in the price of capital increases the expected capital gain from transition into the recovery state $s = 1$, which tends to increase the expected return to capital as well as the Sharpe ratio. Note that the stabilizing force is stronger when investors are more optimistic and perceive a higher transition probability into the recovery state, λ_2. In fact, to ensure that there exists an equilibrium with positive prices, we need a minimum degree of optimism, which is captured by the following assumption.

Assumption 2. $\lambda_2 \geq \lambda_2^{\min}$, where $\lambda_2^{\min}$ is the unique solution to the following equation over the range $\lambda_2 \geq \psi$:

$$\rho + \psi q^* - \delta + \lambda_2^{\min} - \psi + \psi \log\left(\psi/\lambda_2^{\min}\right) = \sigma_2^2.$$

Assumption 2 ensures that there is a unique positive solution to Eq. (29) (see Appendix A.3). When the assumption holds as strict inequality, the decline in prices increases the Sharpe ratio. In this case, the stabilizing capital gains force dominates the destabilizing endogenous output and growth mechanisms. When the condition is violated, a lower price level would lower the return further, which would trigger a downward spiral that would lead to an equilibrium with zero asset prices and output.[11] When the condition holds as equality, the stabilizing force barely balances the destabilizing mechanisms. As we will see below, the price and output in this case is very low and also very sensitive to further changes in beliefs.

Equilibrium in the low-volatility state. Using our conjecture, the risk balance equation (28) for the low-volatility state $s = 1$ can be written as,

$$\sigma_1 = \frac{\rho + \psi q^* - \delta + \lambda_1 \left(1 - \frac{Q^*}{Q_2}\right) - r_1^f}{\sigma_1}. \tag{30}$$

Given q_2, this equation determines the interest rate, r_1^f. Intuitively, given the expected return on capital (that depends on q_2, among other things), the interest rate adjusts to ensure that the risk-balance condition is satisfied with the efficient price level, $q_1 = q^*$. For our conjectured equilibrium, we also require that the implied interest rate to be nonnegative, $r_1^f \geq 0$. The following parametric condition ensures that this is the case.

Assumption 3. $\lambda_1 \leq \lambda_1^{\max}(q_2)$, where $\lambda_1^{\max}(q_2) \geq 0$ denotes the unique solution to the following equation with $q_2 < q^*$ that solves Eq. (29):

$$\rho + \psi q^* - \delta + \lambda_1 \left(1 - \frac{Q^*}{Q_2}\right) = \sigma_1^2.$$

That is, we need pessimism in the low-volatility state (captured by the transition probability) to be sufficiently low so that the fear of a transition into the high-volatility state does not push the economy into the interest rate lower bound. As expected, greater equilibrium price level in the high-volatility state, q_2, increases the upper bound for pessimism, $\lambda_1^{\max}(q_2)$.

Proposition 1. *Consider the model with two states, $s \in \{1, 2\}$, with common beliefs and Assumptions 1-3. The low-volatility state* 1 *features a nonnegative interest rate, efficient asset prices and full factor utilization, $r_1^f \geq 0, q_1 = q^*$ and $\eta_1 = 1$, whereas the high-volatility state* 2

[11] This is reminiscent of Werning (2012), who show that output approximates zero when the liquidity trap is expected to last forever—an extremely pessimistic scenario ($\lambda_2 = 0$). In our setting, even smaller doses of pessimism could push output to zero, since the destabilizing dynamics are stronger due to endogenous investment and growth (see Figure 2).

features zero interest rate, lower asset prices, and a demand-driven recession, $r_2^f = 0, q_2 < q^*$*, and* $\eta_2 < 1$*. The price level in state* 2 *is characterized as the unique solution to Eq.* (29)*, and the risk-free rate in state* 1 *is characterized by Eq.* (30)*.*

Comparative statics for the high-volatility state. We next establish comparative statics of the equilibrium, starting with the high-volatility state. First consider how a change in optimism, λ_2, affects the price of capital, q_2. Implicitly differentiating Eq. (29), we obtain,

$$\frac{dq_2}{d\lambda_2} = \frac{1 - Q_2/Q^*}{\lambda_2 Q_2/Q^* - \psi} > 0. \tag{31}$$

Here, the inequality follows since the denominator is nonnegative in view of Assumption 2 (see Appendix A.3). Hence, the effect of optimism on the price is determined by its direct effect on the expected return to capital captured in the numerator, which is positive. Intuitively, greater optimism increases the expected capital gains, which increases the asset price.

Next consider this expression for the special case in which optimism is at its lowest allowed level, $\lambda_2 = \lambda_2^{\min}$, so that Assumption 2 holds as equality. In this case, the denominator in Eq. (31) is zero, and we have $\frac{dq_2}{d\lambda_2} = \infty$. Hence, in the neighborhood of $\lambda_2 = \lambda_2^{\min}$, the recession is deep, and asset prices and output are extremely sensitive to further changes in beliefs due to the destabilizing endogenous output and growth mechanisms.

More generally, as Eq. (31) illustrates, the destabilizing mechanisms are more powerful when investors are more pessimistic. Hence, pessimism lowers asset prices not only because of its direct impact on asset valuations, but also because it strengthens the destabilizing feedback effects. Figure 3 illustrates these results for a particular parameterization.

Comparative statics for the low-volatility state. Note also that, as illustrated by Eq. (30), these changes that reduce the price in the high-volatility state, q_2, also reduce the interest rate in the low-volatility state, r_1^f. Lower prices in state 2 also lower asset prices and aggregate demand in state 1, which is countered by a lower interest rate. Moreover, the interest rate in the low-volatility state is also influenced by the beliefs in this state. Specifically, we have

$$\frac{dr_1^f}{d\lambda_1} = 1 - \frac{Q^*}{Q_2} < 0.$$

Figure 4 illustrates this result for a particular parameterization. For this exercise, we set $\lambda_2 = \lambda_2^{\min}$ so that the recession is severe and q_2 is low. We also set the exogenous shifter of the risk premium in the boom state to be much lower than in the recession state, $\sigma_1^2 = 0.01 < \sigma_2^2 = 0.1$ (so as to capture the current low volatility environment). This choice ensures that the first-best level of the interest rate in the boom state is quite high, $r_1^{f*} \simeq 7\%$. This is also the interest rate that obtains in equilibrium when pessimism is extremely low so there is no recession risk. The figure illustrates that, starting from this benchmark, small doses of pessimism can considerably

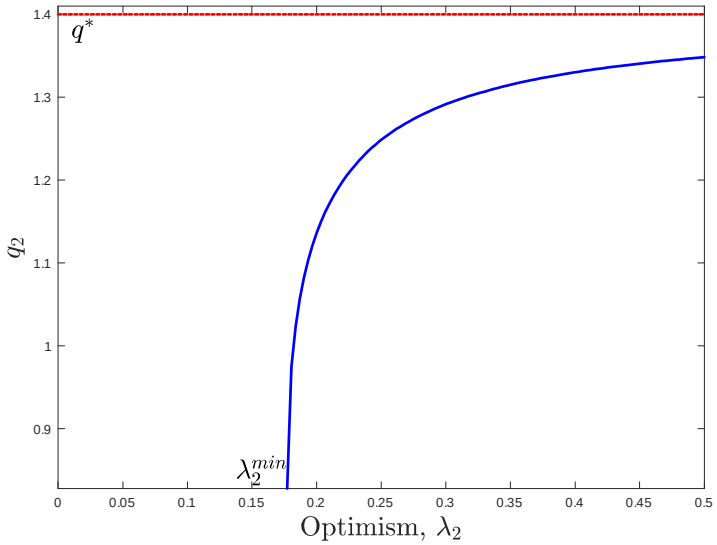


Figure 3: The effect of optimism on the asset price in state 2.

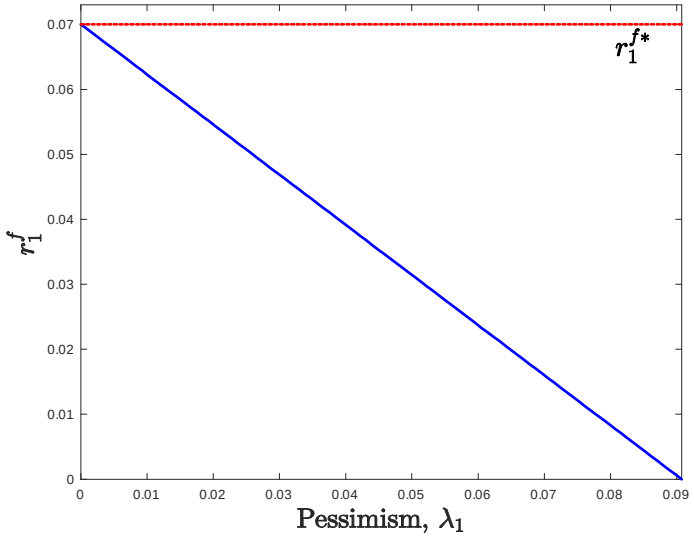


Figure 4: The effect of pessimism (in state 1) on the risk-free rate in the low-volatility state 1.

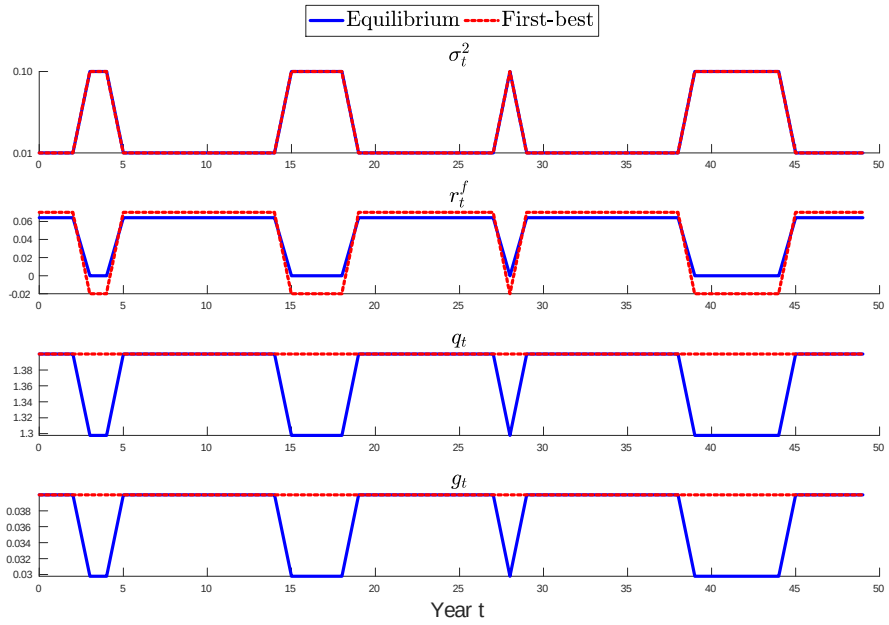


Figure 5: The evolution of the equilibrium variables with interest rate rigidity (solid line) and without interest rate rigidity (dashed line) over the medium run (50 years).

lower the risk-free interest rate, r_1^f. In particular, the equilibrium interest rate becomes zero for $\lambda_1^{\max} \simeq 0.09$, i.e., when the representative investor assigns about 9% probability to a risk-driven recession in a given year. How can a relatively small chance of a recession lower the interest rate by several percentage points? The intuition is that, as we discussed above, the price during the recession, q_2, is lowered considerably due to the destabilizing forces triggered by a combination of a risk shock and pessimism. The fear of a downward price spiral lowers the interest rate during the boom, even when the likelihood of the recession is relatively low.

Dynamics of equilibrium. We next fix the beliefs in both states and simulate the equilibrium for a particular realization of uncertainty. Figure 5 illustrates that, compared to the first-best benchmark, the interest rate is more compressed and the price of capital is more volatile. In the high-volatility state, the interest rate cannot decline sufficiently to close the risk gap, which leads to a drop in asset prices. This also lowers output as well as investment and expected growth. In the low-volatility state, the fear of transition into the recessionary high-volatility state keeps the interest rates lower than in the first-best benchmark. Note, however, that the effects are relatively mild. This is because the (common) belief we have chosen for the simulation comfortably satisfies Assumption 2 (as well as Assumption 3), which ensures that the amplification mechanisms are relatively weak. In the next section, we will see how introducing disagreements around this belief will generate further amplification and will lead to much more severe outcomes.

Endogenous Volatility As illustrated by Figure 5, the equilibrium features *endogenous volatility* in asset prices. To establish this as formally, we fix some $\Delta t > 0$ and consider the proportional change in the value of capital over this time interval, defined as,

$$\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}} \equiv \frac{\left(k_{t+\Delta t,s} Q_{t+\Delta t,s} - k_{t,s} Q_{t,s}\right)/\Delta t}{k_{t,s} Q_{t,s}}.$$

Corollary 1. *For any $s \in \{1,2\}$, the instantaneous (unconditional) variance of capital is,*

$$\lim_{\Delta t \to 0} Var_{t,s}\left(\frac{\Delta k_{t,s} Q_{t,s}/\Delta t}{k_{t,s} Q_{t,s}}\right) = \sigma_s^2 + \lambda_s \left(\frac{Q_{s'} - Q_s}{Q_s}\right)^2.$$

This is strictly greater than the instantaneous variance that would obtain in the first-best equilibrium without interest-rate frictions, σ_s^2.

Intuitively, when there is a shock to the risk premium, the interest rate policy changes the rate to mitigate the impact of the shock on asset prices. Interest rate rigidity reduces the ability of the policy to lean against risk premium shocks, which leads to endogenous volatility. As this intuition suggests, while we focus on risk premium (or volatility shocks), "beliefs shocks" that induce investors to revise their expectations would also create endogenous volatility in asset prices through the same mechanism. In the next section, we will see that speculation also exacerbates endogenous volatility, because it generates endogenous fluctuations in the effective belief that determines asset prices.

5. Belief disagreements and speculation

We next consider the equilibrium with belief disagreements. We show that *speculation* induced by belief disagreements creates further amplification and worsens the recession. While investors' beliefs are exogenously fixed, the extent of their speculation can be influenced by policy, which motivates our analysis of welfare and macroprudential policy in the next section.

We restrict attention to two types of investors, "optimists" and "pessimists", with beliefs denoted by, $\left\{\left(\lambda_1^i, \lambda_2^i\right)\right\}_{i \in \{o,p\}}$. We normalize the mass of each belief type to one so that $i = o$ and $i = p$ denotes, respectively, the representative optimist and pessimist (and the total set of investors I has mass equal to two). We assume the beliefs satisfy the following.

Assumption 4. $\lambda_2^o > \lambda_2^p$ and $\lambda_1^o \leq \lambda_1^p$.

This assumption ensures that optimists are more optimistic than pessimists in either state. Specifically, when the economy is in the high-volatility state, optimists find the transition into the low-volatility state relatively likely ($\lambda_2^o > \lambda_2^p$); when the economy is in the low-volatility state, optimists find the transition into the high-volatility state relatively unlikely ($\lambda_1^o \leq \lambda_1^p$).

To characterize the equilibrium, we define wealth-weighted average transition probability,

$$\overline{\lambda}_{t,s} \equiv \overline{\lambda}_s\left(\alpha_{t,s}\right) \equiv \alpha_{t,s}\lambda_s^o + \left(1-\alpha_{t,s}\right)\lambda_s^p, \text{ where } \alpha_{t,s}^o = \frac{a_{t,s}^o}{k_{t,s}Q_{t,s}}. \tag{32}$$

Here, $\alpha_{t,s}$ denotes optimists' wealth share, and it is the payoff-relevant state variable in this economy. The notation, $\overline{\lambda}_s\left(\alpha_{t,s}\right)$, describes the wealth-weighted average belief in state s as a function of optimists' wealth share, and $\overline{\lambda}_{t,s}$ denotes the belief at time t and state s. This belief is central to the analysis because the following analogue of the risk balance condition (28) holds in this setting (see Appendix A.4),

$$\sigma_s = \frac{1}{\sigma_s}\left(r_{t,s}^k - r_{t,s}^f + \overline{\lambda}_{t,s}\left(1 - \frac{Q_{t,s}}{Q_{t,s'}}\right)\right) \text{ for each } s \in \{1,2\}. \tag{33}$$

In particular, the equilibrium in risk markets are determined according to the wealth-weighted average belief. When $\alpha_{t,s}$ is greater, optimists exert a greater influence on asset prices.

It remains to characterize the evolution of optimists' wealth share, $\alpha_{t,s}$ (and thus, the evolution of $\overline{\lambda}_{t,s}$). In Appendix A.4, we further characterize investors' positions and find that $\omega_{t,s}^{k,o} = \omega_{t,s}^{k,p} = 1$. That is, investors continue to have the same exposure to the market portfolio, which is equal to one in equilibrium. Intuitively, since investors disagree about the jump probabilities, they settle these disagreements by adjusting their holdings of contingent securities as opposed to their exposure to the diffusion risk. In fact, we have the following closed form solution for optimists' equilibrium contingent positions [cf. Eq. $(A.21)$],

$$\omega_{t,s}^{s',o} = \lambda_s^o - \overline{\lambda}_{t,s} = \left(\lambda_s^o - \lambda_s^p\right)\left(1-\alpha_{t,s}\right). \tag{34}$$

Optimists take a positive position on a contingent security whenever her belief for the transition probability exceeds the weighted average belief. In view of Assumption 4, we further have, $\omega_{t,1}^{0,o} \leq 0$ and $\omega_{t,2}^{0,o} > 0$. In the boom state, optimists sell put options since they think transition into recession is unlikely. In the recession state, they buy call options since they believe the transition into recovery is likely.

Consistent with this interpretation, we also find that optimists' wealth share evolves according to [cf. Eqs. $(A.22)$ and $(A.23)$],

$$\begin{cases} \dot{\alpha}_{t,s} = -\left(\lambda_s^o - \lambda_s^p\right)\alpha_{t,s}\left(1-\alpha_{t,s}\right), & \text{if there is no state change,} \\ \alpha_{t,s'}/\alpha_{t,s} = \lambda_s^o/\overline{\lambda}_{t,s}, & \text{if there is a state change to } s'. \end{cases} \tag{35}$$

Here, $\dot{\alpha}_{t,s} = \frac{d\alpha_{t,s}}{dt}$ denotes the derivative with respect to time. In the boom state, optimists' wealth share drifts upwards due to the profits they make from selling put options, but it makes a downward jump if there is a transition into the recession state. In the recession state, optimists' wealth share drifts downwards due to the cost of the call options they purchase, but it makes an upward jump if there is a transition into the boom state.

These observations also imply that the weighted-average belief in (32) (that determines asset prices) is effectively *extrapolative*. As the boom (low-volatility) state persists, and optimists' wealth share increases, the aggregate belief becomes increasingly more optimistic. After a transition to a worse (high-volatility) state, the aggregate belief becomes more pessimistic. Conversely, the aggregate belief becomes more pessimistic as the recession state persists, and it becomes more optimistic after a transition into the boom state. As we will see, these endogenous extrapolation dynamics and their anticipation are behind the amplification mechanism in this setting.

The equilibrium is then characterized as follows. Regardless of the level of asset prices and output, Eq. (35) determines the evolution of investors' wealth shares. This in turn determines the weighted average belief, as well as its evolution [cf. Eq. (32)]. Given the characterization for the weighted-average belief, the equilibrium is determined by jointly solving the risk balance equation (33) and the goods market equilibrium condition (27). Solving these equations is slightly more involved than in the previous section since the weighted-average belief is generally not stationary, which implies the price of capital might also have a nonzero drift, $\mu^Q_{t,s}$ (although $\sigma^Q_{t,s}$ is zero as before).

To make progress, we suppose Assumptions 1-3 from the previous section hold according to both belief types. This ensures that, regardless of the wealth shares, state 2 features a zero interest rate, an inefficient asset price level, and imperfect factor utilization, $r^f_{t,2} = 0, q_{t,2} < q^*$, and $\eta_{t,1} < 1$, whereas state 1 features a positive interest rate, an efficient asset price level, and full factor utilization, $r^f_{t,1} > 0, q_{t,1} = q^*$, and $\eta_{t,1} = 1$. We next characterize this equilibrium starting with the high-volatility state.

Equilibrium in the high-volatility state. Consider the risk balance equation (33) for state $s = 2$. After substituting the return to capital from (23), and using $\mu^Q_{t,2} = \frac{dQ_{t,2}/dt}{Q_{t,2}} = \dot{q}_{t,2}$, we obtain,

$$\sigma_2 = \frac{1}{\sigma_2}\left(\rho + \psi q_2 - \delta + \dot{q}_{t,2} + \overline{\lambda}_{t,2}\left(1 - \frac{Q_2}{Q^*}\right)\right). \tag{36}$$

This expression is similar to its common-beliefs counterpart, Eq. (29), except for the term, $\dot{q}_{t,2}$, which captures the price drift conditional on no transition. This term enters the risk balance condition since it affects the expected return on capital. A negative price drift lowers the expected return and exerts a downward pressure on the equilibrium price. Conversely, a positive price drift increases the return and exerts an upward pressure.

To solve for the equilibrium, we combine Eqs. (35) and (36) to obtain a stationary differential equation,

$$\begin{aligned} \dot{q}_{t,2} &= -\left(\rho + \psi q_2 - \delta + \overline{\lambda}_2\left(\alpha_{t,2}\right)\left(1 - \frac{Q_2}{Q^*}\right) - \sigma_2^2\right), \\ \dot{\alpha}_{t,2} &= -\left(\lambda_2^o - \lambda_2^p\right)\alpha_{t,2}\left(1 - \alpha_{t,2}\right). \end{aligned} \tag{37}$$

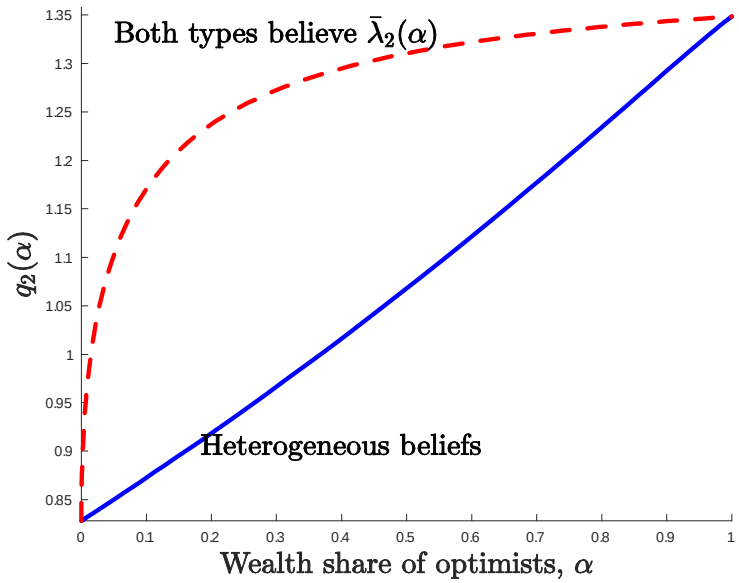


Figure 6: The solid line illustrates the equilibrium price function in state $s = 2$ under heterogeneous beliefs. The dashed line illustrates the price that would obtain if investors shared the wealth-weighted average belief in state $s = 2$.

This system describes the joint evolution of the price and optimists' wealth share, $(q_{t,2}, \alpha_{t,2})$, conditional on there not being a transition. In Appendix A.4, we show that this system is saddle path stable. In particular, for any initial wealth share, $\alpha_{t,2} \in (0,1)$, there exists a unique equilibrium price level, $q_{t,2} \in [q^p, q^o)$, such that the solution satisfies $\lim_{t\to\infty} \alpha_{t,2} = 0$ and $\lim_{t\to\infty} q_{t,2} = q_2^p$. When $\alpha_{t,2} = 1$, the solution satisfies $q_{t,2} = q_t^o$.

Since the equilibrium system (37) is stationary, the equilibrium price can be written as a function of optimists' wealth share, that is, $q_{t,2} = q_2(\alpha)$ for some function $q_2 : [0,1] \to [q^p, q^o]$. In particular, we can eliminate time from the system in (37) (using the observation, $\dot{q}_{t,2} = q_2'(\alpha)\,\dot{\alpha}_{t,2}$), to obtain,

$$q_2'(\alpha)\left(\lambda_2^o - \lambda_2^p\right)\alpha(1-\alpha) = \rho + \psi q_2 - \delta + \overline{\lambda}_2(\alpha)\left(1 - \frac{Q_2}{Q^*}\right) - \sigma_2^2. \tag{38}$$

This provides an equivalent characterization of the price function as a solution to a differential equation in α-domain, together with the boundary conditions, $q_2(0) = q_2^p$ and $q_2(1) = q_2^o$. In Appendix A.4, we further show that the price function, $q_2(\alpha)$, is strictly increasing in α. As in the previous section, greater optimism increases the asset price.

Amplification from speculation. We are now in a position to present the main result in this section, which illustrates that speculation creates further amplification. To this end, we define $q_2^h(\alpha)$ as the solution to the risk balance equation in the common-beliefs benchmark [cf.

Eq. (29)] when all investors share the wealth-weighted average belief, $\overline{\lambda}_2(\alpha)$. Comparing the equilibrium price with this benchmark isolates the effect of speculation. In the appendix, we show that

$$q_2(\alpha) < q_2^h(\alpha) \text{ for each } \alpha \in (0,1). \tag{39}$$

That is, the equilibrium with speculation always features a lower equilibrium price (and a more severe recession).

Intuitively, speculation reshuffles optimists' wealth across states so that they become wealthier in case there is a transition into the boom state but they become poorer if the recession persists longer [cf. Eq. (35)]. The increase in optimists' wealth in the boom state does not increase asset prices since it is neutralized by monetary policy, which increases the interest rate and keeps the price of capital at its first-best level. The decline in optimists' wealth in the recession state, however, creates damage. Specifically, conditional on no transition, optimists' wealth share and the asset price drift downwards, $\dot{\alpha}_{t,2} < 0$ and $\dot{q}_{t,2} < 0$. Moreover, as illustrated by Eq. (36), the damage is anticipated by investors and lowers their expected return to capital. Thus, the current price needs to fall further to equilibrate the risk balance condition, which leads to a more severe recession.

Figure 6 illustrates the price function, $q_2(\alpha)$, for a particular parameterization. We chose the parameters so that pessimists' optimism in state 2 is at the lowest allowed level, $\lambda_2^p = \lambda_2^{\min}$ (see Assumption 2). This implies that, when optimists' wealth share is low, asset prices and output are very low due to the destabilizing feedbacks that we discussed in the previous section. The figure also illustrates that the price with belief disagreements differs sharply from the (appropriate) common belief benchmark. When investors share the same belief, there is no speculation and optimism improves the price considerably. With belief disagreements, optimism has a smaller impact since it comes bundled with speculation. This suggests that it is enough to have one group of highly pessimistic agents to unleash destabilizing dynamics.

Equilibrium in the low-volatility state. Following similar steps for the risk balance condition for the low-volatility state $s = 1$, we obtain,

$$r_1^f(\alpha) = \rho + \psi q^* - \delta + \lambda_1(\alpha)\left(1 - \frac{Q^*}{\exp(q_2(\alpha'))}\right) - \sigma_1^2 \text{ where } \alpha' = \frac{\alpha\lambda_1^o}{\overline{\lambda}_2(\alpha)}. \tag{40}$$

Here, $r_1^f(\alpha)$ denotes the interest rate when optimists' wealth share is equal to α. The interest rate depends (among other things) on the weighted average transition probability into the high-volatility state, $\lambda_1(\alpha)$, as well as the price level that would obtain after transition, $q_2(\alpha')$. The latter depends on the wealth-share of optimists after transition, α', which is smaller than α since optimists are selling put options. For our conjecture to be valid, we also require that $r_1^f(\alpha) \geq 0$ for each α. This condition holds because Assumptions 1-3 hold for pessimists (as well as optimists).

It is easy to check that the interest rate function, $r_1^f(\alpha)$, is increasing in optimists' wealth share, α, for two reasons. First, smaller α makes the wealth-weighted average belief assign a higher probability to a transition into the recession state $s = 2$, which decreases the interest rate (even if $q_{t,2}$ were kept constant). This effect is reminiscent of the analysis in Hall (2016), who argue that the decline in the wealth share of relatively optimistic (and risk tolerant) investors can explain some of the decline in the interest rate in recent years.[12] In our model, there is a second effect that operates in the same direction because the severity of the recession is endogenous. In particular, smaller α also reduces the price after a transition into the recession state, $q_2(\alpha')$, which further lowers the interest rate. The following result summarizes the characterization of equilibrium.[13]

Proposition 2. *Consider the model with two beliefs types. Suppose Assumptions 1-3 hold for each belief, and that beliefs are ranked according to Assumption 4. Then, optimists' wealth share evolves according to Eq.* (35)*. The equilibrium prices and interest rates can be written as a function of optimists' wealth shares,* $q_1(\alpha), r_1^f(\alpha), q_2(\alpha), r_2^f(\alpha)$*. At the high-volatility state,* $r_2^f(\alpha) = 0$ *and* $q_2(\alpha)$ *solves the differential equation* (36) *with* $q_2(0) = q_2^p$ *and* $q_2(1) = q_2^o$*. At the low-volatility state,* $q_1(\alpha) = q^*$ *and* $r_1^f(\alpha)$ *is given by Eq.* (40)*. The equilibrium price and interest-rate functions are increasing in optimists' wealth share. Moreover, speculation reduces the price and exacerbates the recession in state* $s = 2$*, that is, the price function satisfies the inequality in* (39)*.*

Dynamics of equilibrium. Figure 7 illustrates the dynamics of the equilibrium over a 50-year horizon for a particular realization of uncertainty. We choose the (objective) simulation belief to be in the "middle" of optimists' and pessimists' beliefs in terms of the relative entropy distance, which ensures that there is a non-degenerate long-run wealth distribution in which neither optimists nor pessimists permanently dominate.[14] This is the same belief that we use to calculate and simulate the equilibrium in the previous section, so comparing this figure with Figure 5 illustrates the very large effect of belief disagreements and speculation. Note that optimists'

[12] This mechanism is also present in Caballero and Farhi (2017), where the average pessimism during the low vol regime is so acute that the first-best level of r_1^f becomes negative (which they refer to as a "safety trap").

[13] It can also be checked that $r_1^f(\alpha) < r_1^{f,h}(\alpha)$ for each $\alpha \in (0,1)$, where $r_1^{f,h}(\alpha)$ denotes the interest rate that would obtain if investors shared the weighted average belief, $\overline{\lambda}_1(\alpha)$ (while keeping their beliefs in the other state unchanged). Hence, speculation in state 1 reduces the interest rate. Intuitively, the same amplification mechanism that lowers the price in state 2 is also operational in state 1. In this case, it translates into a low interest rate as opposed to a low price, since it is countered by monetary policy.

[14] Specifically, given two probability distributions $(p(\tilde{s}))_{\tilde{s}\in S}$ and $(q(\tilde{s}))_{\tilde{s}\in S}$, relative entropy of p with respect to q is defined as $\sum_{\tilde{s}} p(\tilde{s}) \log\left(\frac{p(\tilde{s})}{q(\tilde{s})}\right)$. Blume and Easley (2006) show that, in a setting with independent and identically distributed shocks (and identical discount factors), only investors whose beliefs have the maximal relative entropy distance to the true distribution survive. Since our setting features Markov shocks, we apply their result state-by-state to ensure that conditional probabilities satisfy the necessary survival condition. Specifically, for each state $s \in \{1,2\}$, we choose λ_s^{sim} so that the relative entropy of the conditional probability distribution for the next state with respect to optimists' beliefs (in the discrete-time approximation of the model) is the same as the relative entropy with respect to pessimists' beliefs.

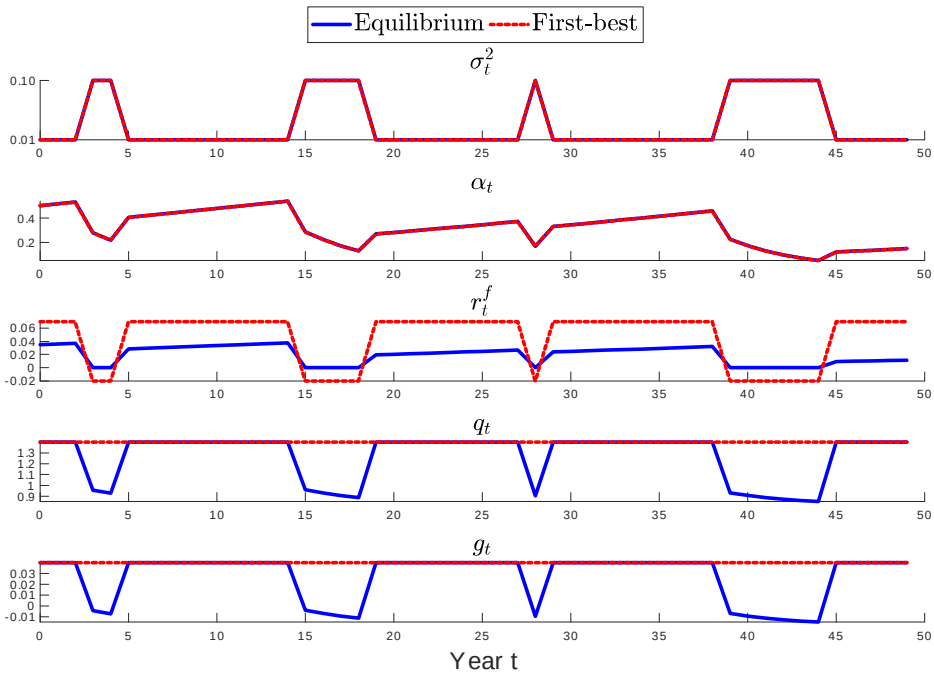


Figure 7: The evolution of the equilibrium variables with belief disagreements and interest rate rigidity (solid line) and without rigidity (dashed line) over the medium run (50 years).

wealth share grows in the low-volatility state but it declines when the economy switches to the high-volatility state. Note also that, compared to the common-belief benchmark, the price of capital declines by a much larger magnitude and the interest rate is much more compressed. This illustrates the additional amplification caused by speculation.

6. Welfare analysis and macroprudential policy

In this section we establish our normative results on macroprudential policy. To this end, we first characterize investors' value functions in equilibrium. This establishes the determinants of welfare in this setting and illustrates the aggregate demand externalities. We then show that, when investors have belief disagreements, the equilibrium can be Pareto improved by macroprudential policy that restricts optimists' risk taking. Throughout, we work with the model with two belief types, $\{o, p\}$, that we analyzed in the previous section.

6.1. Equilibrium value functions and aggregate demand externalities

In Appendix A.2.1, we show that the value function can be written as,

$$V_{t,s}^i\left(a_{t,s}^i\right) = \frac{\log\left(a_{t,s}^i/Q_{t,s}\right)}{\rho} + v_{t,s}^i. \tag{41}$$

Here, $v^i_{t,s}$ denotes the normalized value function per unit of capital stock. An investor that has twice the capital chooses the same portfolio weights and consumes twice the consumption state-by-state, which leads to the functional form in (41).

In Appendix A.5, we further characterize $v^i_{t,s}$ as the solution to the following differential equation system,

$$\rho v^i_{t,s} - \frac{\partial v^i_{t,s}}{\partial t} = \log\rho + q_{t,s} + \frac{1}{\rho}\begin{pmatrix} \psi q_{t,s} - \delta - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda^i_s - \overline{\lambda}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^i_s}{\overline{\lambda}_{t,s}}\right) \end{pmatrix} + \lambda^i_s\left(v^i_{t,s'} - v^i_{t,s}\right). \tag{42}$$

This expression illustrates the determinants of welfare. When there is a demand-driven recession (e.g., in the high-volatility state $s = 2$), a lower equilibrium price, $q_{t,s}$, reduces investors' welfare since it is associated with lower factor utilization, $\eta_{t,s}$. Note that welfare declines due to a decline in current consumption (captured by the term, $\log\rho + q_{t,s}$) as well as a decline in investment and consumption growth (captured by the term, $\psi q_{t,s} - \delta = g_{t,s}$). The risk premium, σ_s^2, also affects welfare through its influence on the risk-adjusted consumption growth. Finally, speculation among investors with belief disagreements also affects (perceived) welfare. This is captured by the term, $-\left(\lambda^i_s - \overline{\lambda}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^i_s}{\overline{\lambda}_{t,s}}\right)$, which is zero with common beliefs, and strictly positive with disagreements.

To facilitate our analysis of macroprudential policy, we also break down the value function into two components,

$$v_{t,s} = v^*_{t,s} + w_{t,s}. \tag{43}$$

Here, $v^*_{t,s}$ denotes *the first-best value function* that would obtain if there were no interest rate rigidity. It is characterized by solving Eq. (42) with $q_{t,s} = q^*$ for each t, s. The residual, $w_{t,s} = v_{t,s} - v^*_{t,s}$, denotes *the gap value function*, which captures the loss of value due to the interest rate rigidity and demand recessions. Using Eq. (42), the gap value function is characterized as the solution to the following differential equation,

$$\rho w^i_{t,s} - \frac{\partial w^i_{t,s}}{\partial t} = \left(1 + \frac{\psi}{\rho}\right)(q_{t,s} - q^*) + \lambda^i_s\left(w^i_{t,s'} - w^i_{t,s}\right). \tag{44}$$

This illustrates that the gap value captures the loss of welfare due to the price deviations from the first best. As we will see, the gap value functions are useful to understand the marginal effect of macroprudential policy on social welfare.

When investors share the same belief, the value function and its components are stationary, e.g., $v_{t,s} = v_s$. In Appendix A.5, we calculate these values in closed form (see Eq. $(A.27)$) and find that they depend on a weighted average of the price of capital in the two states, $(q_s)_{s\in\{1,2\}}$, as well as on the volatilities, $(\sigma_s)_{s\in\{1,2\}}$. The weights reflect time discounting and transition probabilities: they can be thought of as the "discounted expected time" the investor spends in one state relative to another. We show that the value in the recession state is lower than in the boom state, $v_2 < v_1$, precisely because the investor expects to spend more discounted time in

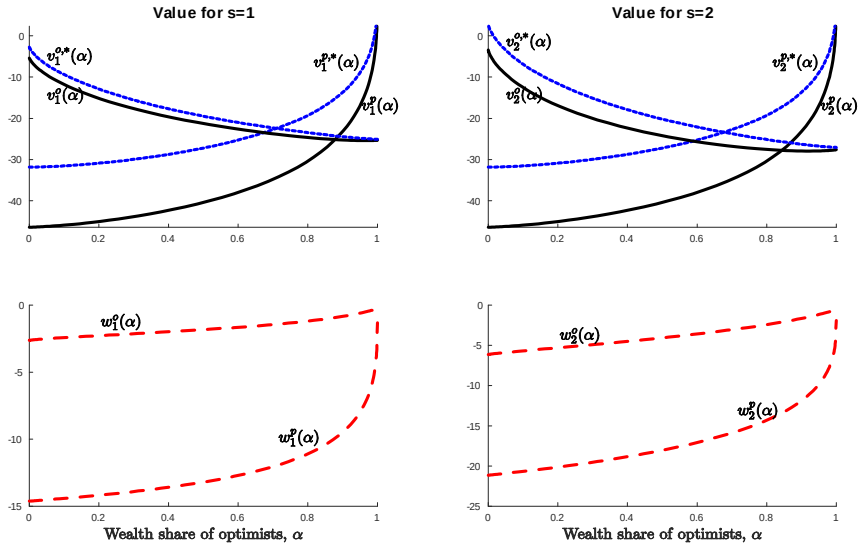


Figure 8: The plots illustrate the equilibrium value functions for each state and belief type. The solid lines are the actual value functions, $v_s^i(\alpha)$, the dotted lines are the first-best value functions, $v_s^{i,*}(\alpha)$, and the dashed lines (in the bottom panels) are the gap value functions, $w_s^i(\alpha)$.

state 2 that features both lower price of capital and higher volatility relative to the other state. For the same reason, we find that the gap value is negative in both states but more so in the recession state, $w_2 < w_1 < 0$.

With belief disagreements, the value function is not necessarily stationary since the price might have a drift. Recall that the equilibrium price in the high-volatility state is a function of optimists' wealth share, $q_2(\alpha)$. In Appendix A.5, we show that the equilibrium values and its components can also be written as a function of optimists' wealth share, $\left\{v_s^i(\alpha), v_s^{i,*}(\alpha), w_s(\alpha)\right\}_{s,i}$. We also characterize these value functions as solutions to differential equations in α-domain. Figure 8 illustrates the numerical solution for the equilibrium plotted in the earlier Figure 6.

The bottom panels of Figure 8 show that the gap value functions are increasing in the wealth share of optimists, α, which illustrates the *aggregate demand externalities*. Greater α increases the effective optimism, which in turn leads to a greater equilibrium asset price in the high-volatility state (see Figure 6). This improves the gap value function in this state by raising the aggregate demand and bringing the economy closer to the first-best benchmark (see Eq. (44)). It also improves the gap value function in the low-volatility state, because the economy can always transition into the high-volatility state, and these transitions are less costly when α is greater. Hence, increasing optimists' wealth share is always associated with positive aggregate demand externalities. Individual optimists that take risks (or pessimists that take the other side of these trades) do not internalize their effects on asset prices, which leads to inefficiencies and generates scope for macroprudential policy.

The top panels of Figure 8 illustrate that the first-best value functions are increasing in α for pessimists but they are decreasing in α for optimists. These effects can be understood via pecuniary externalities in contingent security markets. Increasing the mass of optimists increases the price of contingent securities that optimists purchase, while decreasing the price of contingent securities that pessimists purchase. This creates negative pecuniary externalities (or crowd-out effects) on optimists, and positive pecuniary externalities on pessimists.

Finally, note that the actual value function is the sum of the first-best and the gap value functions. For pessimists, the actual value is always increasing in α, since the two components move in the same direction. For optimists, this is not necessarily the case since the gap value is increasing in α whereas the first-best value is decreasing.

6.2. Macroprudential policy

We capture macroprudential policy as risk limits on optimists. Suppose, the planner can induce optimists to choose (instantaneous) allocations as if they have less optimistic beliefs. Specifically, optimists are constrained to choose allocations *as-if* they have the beliefs, $\lambda^{o,pl} \equiv \left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$, that satisfy, $\lambda_1^{o,pl} \geq \lambda_1^o$ and $\lambda_2^{o,pl} \leq \lambda_2^o$.[15] Pessimists continue to choose allocations according to their own beliefs. Throughout, we use $\lambda_s^{i,pl}$ to denote investors' as-if beliefs and λ_s^i to denote their actual beliefs (for pessimists, the two beliefs coincide). We also use the notations, $\overline{\lambda}_{t,s}^{pl} = \alpha_{t,s}\lambda_s^{o,pl} + (1-\alpha_{t,s})\,\lambda_s^{p,pl}$ and $\overline{\lambda}_s^{pl}(\alpha)$ to represent the weighted average as-if belief.

In Appendix A.6, we show that the planner can implement this policy by imposing (time-varying) inequality restrictions on optimists' portfolio weights, while allowing them to make unconstrained consumption-savings decisions. Specifically, the policy constrains optimists from taking too low a position on the contingent security that pays in the high-volatility state, $\omega_{t,1}^{2,o} \geq \underline{\omega}_{t,1}^{2,o}$ (restrictions on selling "put options"). It also constrains optimists from taking too high a position on the contingent security that pays in the low-volatility state, $\omega_{t,2}^{1,o} \leq \overline{\omega}_{t,2}^{1,o}$ (restrictions on buying "call options"). Finally, the policy also constrains optimists' position on capital not to exceed the market average, $\omega_{t,s}^{k,o} \leq 1$ (since otherwise optimists start to speculate by holding too much capital).

The characterization of equilibrium with policy is then the same as in Section 5. In particular, Eqs. (35) and (36) continue to hold with the only difference that investors' beliefs are replaced with their "as-if" beliefs, $\lambda_s^{i,pl}$.

To characterize the optimal policy, we assume the planner respects investors individual beliefs, that is, investors' expected values in equilibrium are calculated according to their own beliefs, λ_s^i. To trace the Pareto frontier, we also allow the planner to do a one-time wealth transfer among the agents at time zero. In Appendix A.6, we show that the planner's Pareto

[15] For simplicity, we restrict attention to time-invariant policies. The planner commits to a policy at time zero, $\left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$, and implements it throughout.

problem can then be reduced to,

$$\max_{\tilde{\lambda}^{o,pl}} \alpha_{0,s} v^o_{0,s} + (1-\alpha_{0,s})\, v^p_{0,s}. \tag{45}$$

Hence, the planner maximizes a wealth-weighted average of investors' normalized values. The relative wealth shares reflect the planner's relative Pareto weights

We further characterize $v^i_{0,s}$ as the solution to a differential equation [cf. Eq. $(A.34)$]. This is the analogue of Eq. (42) with the only difference that the portfolio weights on contingent securities (and the payoffs from these positions) are calculated according to investors' as-if beliefs, $\lambda^{i,pl}_s$, whereas the transition probabilities are calculated according to their actual beliefs, λ^i_s. As before, we also decompose the value function into first-best and gap value components, $v^i_{t,s} = v^{i,*}_{t,s} + w^i_{t,s}$.

We also show that the value function as well as its components can be written as a function of optimists' wealth shares. As in the case without policy, we denote the equilibrium price functions with $\{q_s(\alpha)\}_s$, individuals' value functions with $\left\{v^i_s(\alpha), v^{i,*}_s(\alpha), w^i_s(\alpha)\right\}_{s,i\in\{o,p\}}$.[16] The planner's value function is then a wealth-weighted average of individual value functions, $v^{pl}_s(\alpha) = \alpha v^o_s(\alpha) + (1-\alpha) v^p_s$. We also break this into first-best value and gap value, $v^{pl}_s(\alpha) = v^{pl,*}_s(\alpha) + w^{pl}_s(\alpha)$.

A key observation is that the marginal impact of the policy on the planner's first-best value function is zero,

$$\left.\frac{\partial v^{pl,*}_s(\alpha)}{\partial \lambda^{o,pl}}\right|_{\lambda^{o,pl}=\lambda^o} = 0. \tag{46}$$

This is because our model features complete markets and no frictions other than interest rate rigidities. Hence, the First Welfare Theorem applies to the first-best allocations that also correct for these rigidities (and features efficient output). This in turn implies that the marginal impact on the first-best value must be zero, since otherwise the first-best allocations could be Pareto improved by appropriately changing optimists' as-if beliefs. It follows that the marginal impact of the policy is determined by its marginal impact on the planner's gap value function, $w^{pl}_s(\alpha) = \alpha w^o_s(\alpha) + (1-\alpha) w^p_s(\alpha)$.

It remains to characterize how the policy affects investors' gap value functions. In Appendix A.6, we show that the gap value function solves the equation system,

$$\rho w^i_s(\alpha) = \left(1+\frac{\psi}{\rho}\right)(q_s(\alpha)-q^*) - \alpha(1-\alpha)\left(\lambda^{o,pl}_s - \lambda^p_s\right)\frac{\partial w^i_s(\alpha)}{\partial \alpha} + \lambda^i_s\left(w^i_{s'}\left(\alpha'\right) - w^i_s(\alpha)\right), \tag{47}$$

where $\alpha' = \alpha\frac{\lambda^{o,pl}_s}{\bar{\lambda}^{o,pl}_s(\alpha)}$. This follows from the earlier equation (44) after replacing optimists' wealth dynamics from Eq. (35) when they act according to their as-if beliefs, $\lambda^{o,pl}_s$. Note how the transition probability is calculated according to investors' actual beliefs, λ^i_s. The policy

[16] These functions also depend on the policy, $\lambda^{o,pl}$, which we suppress to simplify the notation.

influences the perceived gap values not because it changes investors' beliefs, but since it changes optimists' wealth dynamics, which in turn affects asset prices and the output gaps relative to the first-best. We next describe the effect of macroprudential policy in the boom state $s = 1$, assuming that there is no intervention in the other state. We then analyze the polar opposite case of macroprudential policy in the recession state $s = 2$, assuming no intervention in the boom state.

6.2.1. Macroprudential policy during the boom

Suppose the economy is currently in the boom state $s = 1$. The planner can use macroprudential policy in the current state, $\lambda_1^{o,pl} \geq \lambda_1^o$ (she can induce optimists to act as if transition into the recession is more likely), but not in the other state $\lambda_2^{o,pl} = \lambda_2^o$ (she cannot influence optimists' actions in the recession state). Finally, suppose we are in the special case in which the beliefs satisfy, $\lambda_1 \equiv \lambda_1^o = \lambda_1^p$ (so investors disagree only in the recession state). We obtain a sharp result for this case, and we show in numerical simulations that the result also applies when $\lambda_1^o < \lambda_1^p$.

Proposition 3. *Consider the model with two beliefs types that satisfy $\lambda_1^o = \lambda_1^p$. Consider the macroprudential policy in the boom state, $\lambda_1^{o,pl} \geq \lambda_1^o$ (and suppose $\lambda_2^{o,pl} = \lambda_2^o$). The policy increases the gap value according to each belief, that is,*

$$\left.\frac{\partial w_1^i(\alpha)}{\partial \lambda_1^{o,pl}}\right|_{\lambda_1^{o,pl}=\lambda_1^o} > 0 \text{ for each } i \in \{o,p\} \text{ and } \alpha \in (0,1).$$

The policy also increases the planner's value, $\left.\frac{\partial v_1^{pl}(\alpha)}{\partial \lambda_1^{o,pl}}\right|_{\lambda_1^o} = \left.\frac{\partial w_1^{pl}(\alpha)}{\partial \lambda_1^{o,pl}}\right|_{\lambda_1^o} > 0$. In particular, regardless of the planner's Pareto weight, there exists a Pareto improving macroprudential policy.

The result shows that macroprudential policy improves the gap value function according to optimists as well as pessimists. Therefore, it also increases the wealth-weighted average gap value. In view of Eq. (46), it also increases the social welfare and leads to a Pareto improvement.

To obtain a sketch proof for the result, consider the differential equation (47) for state $s = 1$ and an arbitrary belief type $i \in \{o,p\}$. Differentiating this expression with respect to policy, $\lambda_1^{o,pl}$, and evaluating at the no-policy equilibrium, $\lambda_1^{o,pl} = \lambda_1^o$, we obtain,

$$\begin{aligned}(\rho + \lambda_1)\frac{\partial w_1^i(\alpha)}{\partial \lambda_1^{o,pl}} &= \left[-\alpha(1-\alpha)\frac{\partial w_1^i(\alpha)}{\partial \alpha} + \lambda_1 \frac{\partial \alpha'}{\partial \lambda_1^{o,pl}}\frac{\partial w_2^i(\alpha')}{\partial \alpha'}\right] + \lambda_1 \frac{\partial w_2^i(\alpha')}{\partial \lambda_1^{o,pl}}, \\ &= \alpha(1-\alpha)\left[-\frac{\partial w_1^i(\alpha)}{\partial \alpha} + \frac{\partial w_2^i(\alpha)}{\partial \alpha}\right] + \lambda_1 \frac{\partial w_2^i(\alpha)}{\partial \lambda_1^{o,pl}}. \end{aligned} \tag{48}$$

Here, λ_1 denotes investors' common belief in state 1 (by assumption). The second line uses $\alpha' = \alpha \frac{\lambda_1^{o,pl}}{\lambda_1^{o,pl}(\alpha)}$. The two terms inside the brackets capture the direct effects of macroprudential policy on social welfare. Macroprudential policy effectively induces optimists to purchase more

insurance (or sell fewer puts). This reduces optimists' relative wealth share in state 1 but improves their relative wealth share in state 2. Moreover, using the equilibrium prices, one unit of decline in wealth share in state 1 is associated with one unit of increase in expected wealth share in state 2.

Next note that the gap value function in either state is increasing in optimists' wealth share $\frac{\partial w_1^i(\alpha)}{\partial \alpha}, \frac{\partial w_2^i(\alpha)}{\partial \alpha} > 0$ (see Figure 8). Hence, macroprudential policy always involves a trade-off. Intuitively, optimism is a scarce resource that could also be utilized immediately as well as in the future. Moving optimism across states via macroprudential policy is always associated with costs as well as benefits. However, the typical situation is such that optimism increases the social welfare more in the recession state $s = 2$, where it provides immediate benefits, as opposed to the boom state $s = 1$, where its benefits are realized in case there is a future transition into the recession state. For the special case with $\lambda_1^o = \lambda_1^p$, we in fact have $\frac{\partial w_1^i(\alpha)}{\partial \alpha} = \frac{\lambda_1}{\rho+\lambda_1} \frac{\partial w_2^i(\alpha)}{\partial \alpha} < \frac{\partial w_2^i(\alpha)}{\partial \alpha}$. Combining this with Eq. (48) provides a sketch-proof of Proposition 3. The actual proof in Appendix A.6 relies on the same idea but uses recursive techniques to establish the result formally.

The left panel of Figure 9 illustrates the result by plotting the change in the planners' value functions in the boom state resulting from a small macroprudential policy change (specifically, we start with the equilibrium with $\lambda_1^o = 0.03$ and set $\lambda_1^{o,pl} = 0.0305$). Note that the policy reduces the planner's first-best value function, since it distorts investors' allocations according to their own beliefs. However, the magnitude of this decline is small, illustrating the First Welfare Theorem (cf. Eq. (46)). Note also that the policy generates a relatively sizeable increase in the planner's gap value function. Moreover, this increase is sufficiently large that the policy also increases the actual value function and generates a Pareto improvement, illustrating Proposition 3.

Macroprudential policy improves welfare by internalizing the aggregate demand externalities. In the recession state $s = 2$, optimists improve asset prices, which in turn increases aggregate demand and brings output closer to the first-best level. Individual optimists do not internalize that they would improve asset prices and output by bringing more wealth into state 2. Macroprudential policy works by increasing optimists' insurance purchases, which increases their wealth share in the recession state. The result is reminiscent of the analysis in Korinek and Simsek (2016), in which macroprudential policy improves outcomes by inducing agents that have a high marginal propensity to consume (MPC) to bring more wealth into states in which there is a demand-driven recession. However, the mechanism here is different and operates via asset prices. In fact, in our setting, all investors have the same MPC equal to ρ. Optimists improve aggregate demand not because they spend more than pessimists, but because they increase asset prices and induce *all investors* to spend more, while also increasing aggregate investment and hence growth.

As this discussion suggests, the parametric restriction, $\lambda_1^o = \lambda_1^p$, is useful to obtain an analytical result but it does not play a central role. We suspect that Proposition 3 also holds

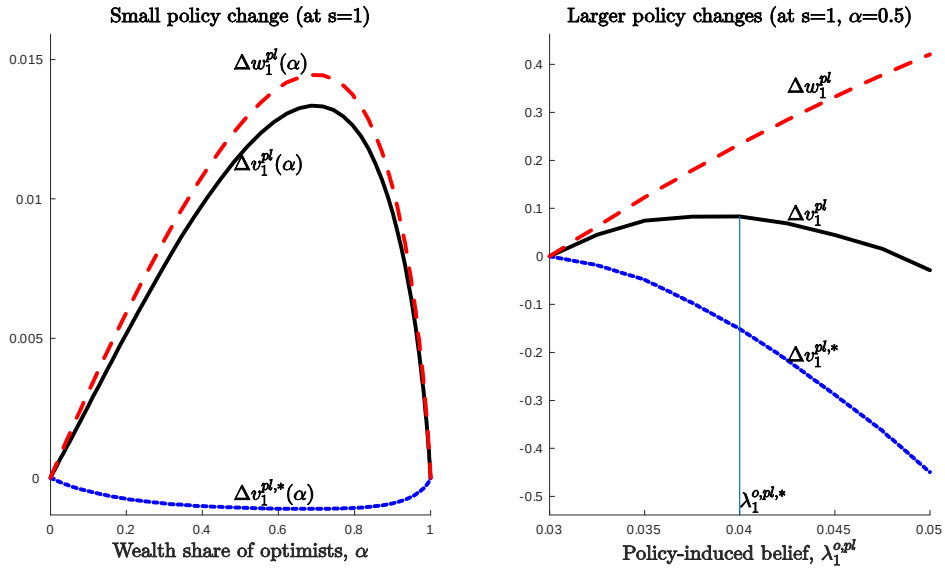


Figure 9: The left panel illustrates the effect of a small change in macroprudential policy in the boom state on the planner's value functions. The right panel illustrates the effect of larger policy changes.

absent this assumption, even though we are unable to provide a proof. In our numerical simulations, we have not yet encountered a counterexample. The results displayed in Figure 9 actually correspond to our earlier parameterization that features $\lambda_1^o < \lambda_1^p$.

Proposition 3 concerns a small policy change. The right panel of Figure 9 illustrates the effect of larger policies by plotting the changes in the planner's value as a function of the size of the policy (starting from no policy, $\lambda_1^{o,pl} = \lambda_1^o$). For this exercise, we fix the optimists' wealth share at a particular level, $\alpha = 1/2$. Note that, as the policy becomes larger, the gap value continues to increase whereas the first-best value decreases. Moreover, the decline in the first-best value is negligible for small policy changes but it becomes sizeable for large policy changes. The (constrained) optimal macroprudential policy obtains at an intermediate level, $\lambda_1^{o,pl,*} > \lambda_1^o$.

The figure also illustrates that the constrained optimal policy intervention is not too large (specifically, we have $\lambda_1^{o,pl,*} = 0.04$ where $\lambda_1^o = 0.03$). This is typically the case in our numerical simulations. The reason is that speculation generates high perceived utility for investors. Since macroprudential policy restricts speculation, the perceived costs quickly rise with the degree of the policy intervention, which implies that the optimal intervention is not too large.

Macroprudential policy according to a belief-neutral criterion. It is questionable whether the utility from speculation should be counted toward social welfare. A recent literature argues that the Pareto criterion is not the appropriate notion of welfare for environments with belief disagreements. If investors' beliefs are different due to mistakes (say, in Bayesian updating), then it is arguably more appropriate to evaluate investors' utility according to the

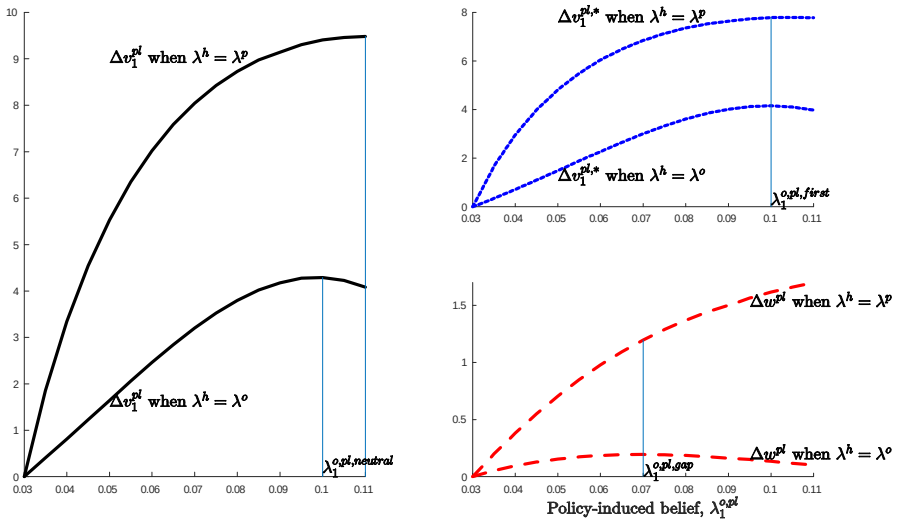


Figure 10: The left panel illustrates the effect of macroprudential policy on social welfare when all investors' value is calculated according to respectively optimists' or pessimists' belief. The panels on the right illustrate the effects on respectively the first-best and the gap value functions.

objective belief—which is common across the agents. Doing so would remove the speculative utility from welfare calculations, and it could lead to a constrained optimal policy that is much larger in magnitude. While reasonable, this approach faces a major challenge in implementation: whose belief should the policymaker use?

In recent work, Brunnermeier et al. (2014) offer a *belief-neutral welfare criterion* that circumvents this problem. The basic idea is to require the planner to evaluate social welfare according to a single belief, but also to make the welfare comparisons robust to the choice of the single belief. Specifically, their baseline criterion says that an allocation is belief-neutral superior to another allocation if it increases social welfare under every belief in the convex hull of investors' beliefs. Proposition 3 suggests their criterion can also be useful in this context since macroprudential policy increases the gap value according to each belief—that is, the gap-reducing welfare gains are belief neutral.

For a formal analysis, fix some $h \in [0,1]$ and let $v_s^i\left(\alpha; \lambda_1^{o,pl}, \lambda^h\right)$ denote the value function for an individual when the planner implements policy, $\lambda_1^{o,pl}$, and evaluates utility under the beliefs, $\lambda_s^h = \lambda_s^p + h\left(\lambda_s^o - \lambda_s^p\right)$.[17] As before, define the planner's value function, $v_s^{pl}\left(\alpha; \lambda_1^{o,pl}, \lambda^h\right)$, as the wealth-weighted average of individual's value functions. Then, given the wealth share α (that corresponds to a particular Pareto weight), the policy, $\lambda_1^{o,pl}$, is a belief-neutral improvement over some other policy, $\tilde{\lambda}_1^{o,pl}$, as long as it increases the planner's value according to each $h \in [0,1]$.

Figure 10 illustrates the belief-neutral optimal policy in the earlier example. The left panel

[17] This value function solves the differential equation system $(A.34)$ after replacing the actual beliefs, λ_s^i, with the planner's beliefs, λ_s^h.

plots the effect of the policy on the social welfare (given $\alpha = 1/2$) when the planner evaluates all investor's values under respectively pessimists' belief and optimists' belief. The social welfare evaluated under intermediate beliefs lie in between these two curves. As the figure suggests, tightening the policy towards $\lambda_1^{o,pl,neutral} = 0.1$ constitutes a belief-neutral improvement. In particular, the belief-neutral criterion supports a much larger policy intervention than the Pareto criterion (cf. Figure 9).

The right panel provides further intuition by breaking the social welfare into its two components, $v_1^{pl} = v_1^{pl,*} + w_1^{pl}$. The top right panel shows that tightening macroprudential policy towards the belief, $\lambda_1^{o,pl,first} = 0.1$, generates a belief-neutral improvement in the "first best" social welfare, $v_1^{pl,*}$. Speculation induces investors to deviate from the optimal risk sharing benchmark in pursuit of perceived speculative gains. However, these speculative gains are transfers from other investors, and they do not count towards social welfare when investors' values are evaluated under a common belief (regardless of whose belief is used). Hence, if there were no interest rate rigidities, a belief-neutral planner would eliminate almost all speculation.[18]

The bottom right panel shows the effects of policy on the gap value, w_1^{pl}, which captures the reduction in social welfare due to interest rate rigidities. Tightening the macroprudential policy towards the belief, $\lambda_1^{o,pl,gap} = 0.07$ increases the gap value according to both optimists and pessimists (illustrating Proposition 3). Beyond this level, tightening the policy improves the gap value according to pessimists but not according to optimists—who perceive smaller benefits from macroprudential policy since they find the transition into state 2 unlikely.

It follows that, up to the level, $\lambda_1^{o,pl,gap} = 0.07$—which constitutes a sizeable policy intervention—there is no conflict in belief-neutral policy objectives. Tightening the policy helps to rein in speculation while also improving the gap value, according to any belief. This might be a natural choice for a planner who focuses exclusively on closing the output gaps relative to the first best while remaining agnostic about whether speculation improves or reduces social welfare. Beyond this level, tightening the policy continues to generate belief-neutral welfare gains by reducing speculation and improving risk sharing, but it also reduces the gap value according to optimists.

Dynamics of equilibrium with policy. We next consider how macroprudential policy affects the dynamics of equilibrium variables. Figure 11 illustrates the evolution of equilibrium over a 50-year horizon when the planner implements the (belief–neutral) gap-value maximizing policy, $\lambda_1^{o,pl,gap} = 0.07$. For comparison, the figure also replicates the evolution of the equilibrium variables without policy from Figure 7. Note that macroprudential policy ensures optimists' wealth share drops relatively less when there is a transition into the high-volatility state. This in turn leads to greater asset prices and higher growth rate in the high-volatility

[18] An unconstrained planner that uses a common belief for welfare calculations would set, $\lambda_1^{o,pl} = \lambda_1^p \simeq 0.09$, so as to eliminate all speculation. Our constrained planner slightly overshoots this benchmark since she also corrects for the fact that she does not have access to macroprudential policy in state 2.

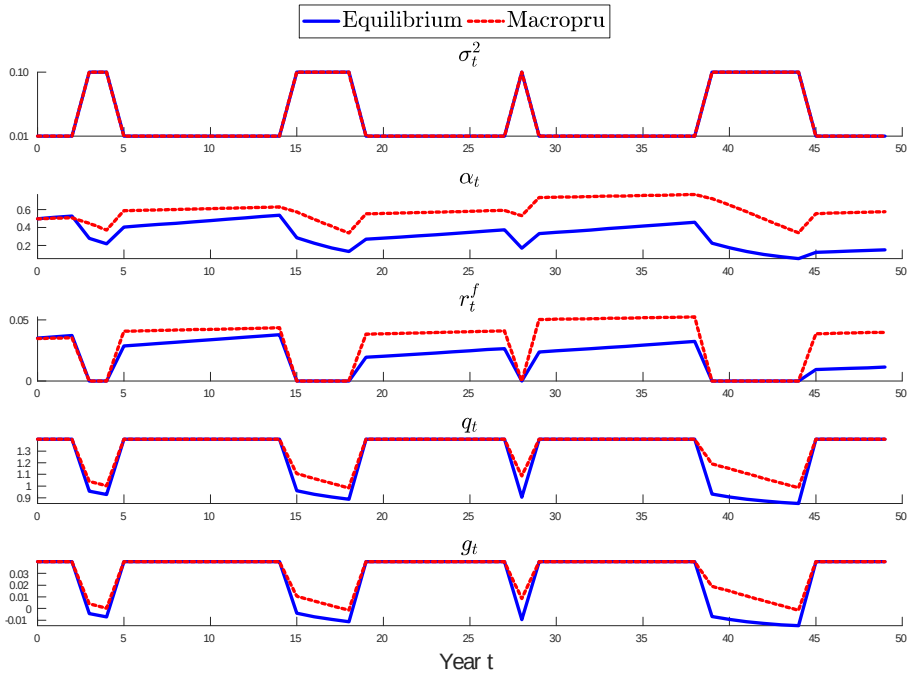


Figure 11: The evolution of the equilibrium variables without macroprudential policy (solid line) and with macroprudential policy in the boom state (dotted line) over the medium run (50 years).

state. However, macroprudential policy is not without its drawbacks. As the period between years 5-15 illustrates, the policy slows down the growth of optimists' wealth share when the economy remains in the low-volatility state.

The effect of macroprudential policy on the interest rate in the low-volatility state is rather subtle. On the one hand, for a fixed level of optimists' wealth share, the policy lowers the interest rate as it lowers aggregate demand. On the other hand, the policy also preserves optimists' wealth over time, which increases the interest rate. In our simulation illustrated in Figure 7, the latter effect dominates and macroprudential policy leads to a higher interest rate over time.

6.2.2. Macroprudential policy during the recession

The analysis so far concerns macroprudential policy in the boom state and maintains the assumption that $\lambda_2^{o,pl} = \lambda_2^o$. We next consider the polar opposite case in which the economy is currently in the recession state $s = 1$, and the planner can apply macroprudential policy in this state, $\lambda_2^{o,pl} \leq \lambda_2^o$ (she can induce optimists to act as if the recovery is less likely), but not in the other state, $\lambda_1^{o,pl} = \lambda_1^o$. We obtain a sharp result for the special case in which optimists' wealth share is sufficiently large.

Proposition 4. *Consider the model with two belief types. Consider the macroprudential policy in the recession state, $\lambda_2^{o,pl} \leq \lambda_2^o$ (and suppose $\lambda_1^{o,pl} = \lambda_1^o$). There exists a threshold, $\overline{\alpha} < 1$, such that if $\alpha \in (\overline{\alpha}, 1]$, then the policy reduces the gap value according to each belief, that is,*

$$\left.\frac{\partial w_2^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda_2^{o,pl}=\lambda_2^o} < 0 \text{ for each } i \in \{o,p\}.$$

Thus, for $\alpha \in (\overline{\alpha}, 1]$, the policy also reduces the planner's value, $\left.\frac{\partial v_s^{pl}(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda_2^o} = \left.\frac{\partial w_s^{pl}(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda_2^o} < 0.$

Thus, in contrast to Proposition 3, macroprudential policy in the bust phase can actually reduce the social welfare. The intuition can be understood by considering two counteracting forces. First, similar to the boom state, macroprudential policy in the recession state is potentially valuable by reallocating optimists' wealth from state $s = 1$ to state $s = 2$. Intuitively, optimists purchase too many call options that pay if there is a transition to the boom state but that impoverish them in case the recession persists. They do not internalize that, if they keep their wealth, they will improve asset prices if the recession lasts longer.

However, there is a second force that does not have a counterpart in the boom state: Macroprudential policy in the bust state also affects the current price level, with potential implications for social welfare. It can be seen that making optimists less optimistic in state 2 shifts the price function downward, $\frac{\partial q_2(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)} < 0$ (as in Figure 3 for common beliefs). Hence, the price impact of macroprudential policy is welfare reducing. Moreover, as optimists dominate the economy, $\alpha \to 1$, the price impact of the policy is still first order, whereas the beneficial effect from reshuffling optimists' wealth is second order. Thus, when optimists' wealth share is sufficiently large, the net effect of macroprudential policy is negative, illustrating Proposition 4.

This analysis also suggests that, even when the policy in the bust state exerts a net positive effect, it would typically increase the welfare by a smaller amount than a comparable policy in the boom state. Figure 12 illustrates this by plotting side-by-side the effects of a small policy change in either state. The left panel replicates the value functions from the earlier Figure 9, whereas the right panel illustrates the results from changing optimists' belief in the bust state by an amount that would generate a similar distortion in the first-best equilibrium as in our earlier analysis.[19] Note that a small macroprudential policy in state 2 has a smaller positive impact when optimists' wealth share is small, and it has a negative impact when optimists' wealth share is sufficiently large.

It is useful to emphasize that macroprudential policy does not have an adverse price impact in the boom state due to the interest rate response. Intuitively, as macroprudential policy reduces the demand for risky assets, the interest rate policy lowers the rate to dampen its effect on asset

[19] Specifically, we calibrate the belief change in state 2 so that the maximum decline in the planner's first-best value function is the same in both cases, $\max_\alpha \left|\Delta v_2^{pl,*}(\alpha)\right| = \max_\alpha \left|\Delta v_1^{pl,*}(\alpha)\right|$.

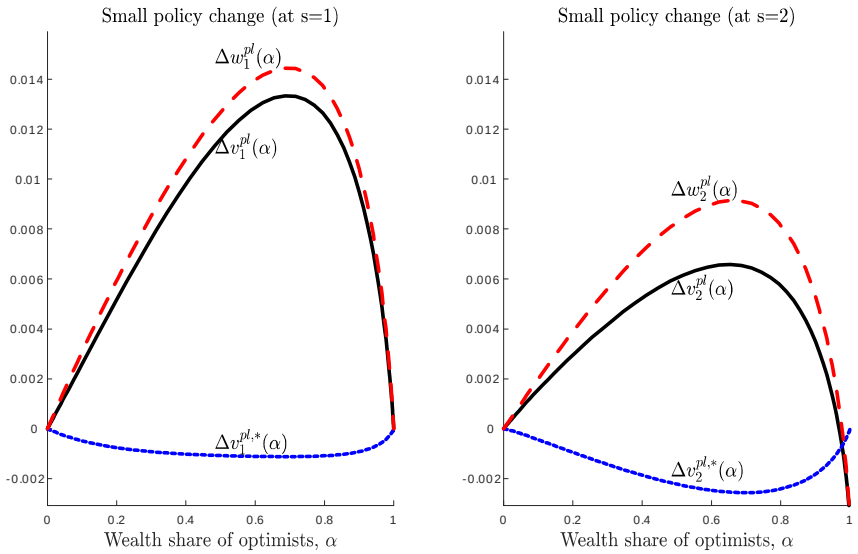


Figure 12: The left (resp. right) panel illustrates the effect of a small change in macroprudential policy in the boom (resp. recession) state.

prices and aggregate demand. In the recession state, the interest rate is already at zero, so the interest rate policy cannot neutralize the adverse effects of macroprudential policy.

Taken together, our analysis in this section provides support for *procyclical* macroprudential policy. In states in which output is not demand constrained (in our model, state 1), macroprudential policy that restricts high valuation investors' (in our model optimists') risk taking is desirable. This policy improves welfare by ensuring that high valuation investors bring more wealth to the demand-constrained states, which in turn increases asset prices and output. Its adverse price effects are countered by a reduction in the interest rate. In contrast, in states in which output is demand constrained (in our model, state 2), macroprudential policy has counteracting effects on social welfare. While the policy has the same beneficial effects as before, it also lowers asset prices and aggregate demand, which cannot be countered by the interest rate. The latter effect reduces the overall usefulness of macroprudential policy, and it could even reduce social welfare.

7. Final Remarks

We provide a macroeconomic framework where risk- and output–gaps are joint phenomena that feed into each other. The key tension in this framework is that asset prices have the dual role of equilibrating risk markets and supporting aggregate demand. When the dual role is inconsistent, the risk market equilibrium prevails. Interest rate policy works by taking over the role of equilibrating risk markets, which then leaves asset prices free to balance the goods markets. However, once interest rates reach a lower bound, the dual role problem reemerges

and asset prices are driven primarily by risk markets equilibrium considerations. This reduces aggregate demand and triggers a recession, which then feeds back negatively into asset prices. The drop in asset prices during recessions also reduces interest rates during booms. In this environment, the role of macroprudential regulation is to preserve the wealth of high-valuation investors during recessions, so as to reduce the gap between the asset prices that equilibrate the risk and goods markets when interest rate policy is no longer available.

Interest rate cuts work in our model by improving the market's Sharpe ratio. From this perspective, any policy that reduces market volatility should have similar effects, which renders support to the many such policies implemented during the aftermath of the subprime and European crises.

In the model we take the interest rate friction to be a stark zero lower bound constraint, which can be motivated with standard cash-substitutability arguments. In practice, this constraint is neither as tight nor as narrowly motivated: Central banks do have some space to bring rates into negative territory, especially when macroeconomic uncertainty is rampant, but there are also many other frictions besides cash substitutability that can motivate downward rigidity in rates once these are already low (see, e.g., Brunnermeier and Koby (2016) for a discussion of the "reversal rate", understood as a level of rates below which the financial system becomes impaired). The broader points of the dual role of asset prices and their interactions with aggregate demand constraints during recessions would survive many generalizations of the interest rate friction. Similarly, one could also imagine situations that motivate ceilings on interest rates, in which case asset prices would overshoot and the productive capacity would become stretched.

In the main text, we also did not take a stand on whether optimists or pessimists are right about the transition probabilities. The reason is that core of our analysis does not depend on this. For example, we could think of optimists as rational and pessimists as Knightians (see, e.g., Caballero and Krishnamurthy (2008); Caballero and Simsek (2013)). Absent any direct mechanism to alleviate Knightian behavior during severe recessions, the key macroprudential point that optimists may need to be regulated during the boom survives this alternative motivation.

We deliberately kept our analysis of heterogeneity in asset valuation at an abstract level. With this ingredient our goal was to highlight the negative effect of (ex-ante) speculation on (ex-post) aggregate demand and growth. During booms, speculation boosts valuations and aggregate demand, but these effects can be offset by higher rates. In contrast, during severe recessions, speculation depresses valuations and aggregate demand, but these effects cannot be offset by interest rates cuts as these are constrained. In practice, this insight implies that (from a macroeconomic point of view) it doesn't matter whether banks are the optimists or pessimists: If the former, then their leverage during the boom (selling puts) is costly during recessions as they become impaired; if the latter, then their (fully collateralized) easy lending during booms (buying puts) implies that their optimistic borrowers will suffer losses during recessions.

As we noted earlier, our modeling approach belongs to the literature spurred by Brunnermeier and Sannikov (2014), although unlike that literature our analysis does not feature financial

frictions. However, if we were to introduce these realistic frictions in our setting, many of the themes in that literature would reemerge and become exacerbated by aggregate demand feedbacks. For instance, in an incomplete markets setting, optimists take leveraged positions on capital, and by doing so they induce endogenous volatility in asset prices and the possibility of tail events following a sequence of negative diffusion shocks that make the economy deeply pessimistic (we analyze the incomplete markets case in a companion paper, Caballero and Simsek (2017a)).

The model omits many realistic healing mechanisms that were arguably relevant for the Great Recession (as well as other deep recessions). For example, a financial crisis driven by a reduction in banks' net worth is typically mitigated over time as banks earn high returns and accumulate net worth (see Gertler et al. (2010); Brunnermeier and Sannikov (2014)). Likewise, household or firm deleveraging eventually loses its potency as debt is paid back (see Eggertsson and Krugman (2012); Guerrieri and Lorenzoni (2017)). Investment hangovers gradually dissipate as the excess capital is depleted (see Rognlie et al. (2017)). While these healing mechanisms are useful to understand the aftermath of the Great Recession, they raise the natural question of why the interest rates seem unusually low and the recovery (especially in investment) appears incomplete almost ten years after the start of the recession. Our paper illustrates how (objective and subjective) risk factors can drag the economy's recovery.

Conversely, the model also omits many sources of inertia that stem from financial markets. Throughout we have assumed that risk-markets clear instantly while goods markets are sluggish. In practice, risk-market have their own sources of inertia as financial institutions avoid or delay mark-to-market losses, liquidity evaporates, policy noise rises, and so on.

Finally, one feature of the aftermath of the subprime crisis is the present high valuation of risk-assets, that could appear to contradict the higher required equity risk premium observed in the data (see Figure 1). The model offers a natural interpretation for such a combination: While we focused exclusively on changes in the required risk-premium, there is also evidence that during this period both ρ and ψ have declined due to a variety of factors such as a worsening of the income distribution and an increase in monopoly rents (see, e.g., Gutiérrez and Philippon (2016)). Equation (24) shows that such declines require a higher valuation for any given level of output gap η, which is achieved via a drop in "rstar." In the interest rate constrained region the latter translates into a deeper recession, while once the economy has recovered the decline in"rstar" manifests in lower riskless interest rates and higher valuations. It may well be that this new high valuation full-employment equilibrium will bring about higher instability in the future by exacerbating speculation, which is a theme we intend to explore in future work.

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A. Appendix: Omitted Derivations and Proofs

This appendix presents the derivations and proofs omitted from the main text. The next appendix presents the proofs.

A.1. Omitted derivations in Section 2

Most of the analysis is provided in the main text. Here, we formally state the household's problem and derive the optimality conditions. Recall that the market portfolio is the claim to all output at date 1. Let $r^k(z_1) = \log\left(\frac{z_1}{Q}\right)$ denote the log return on this portfolio if the productivity is realized to be z_1. Since the payoff distribution is log normal, the return distribution is also log normal,

$$r^k(z_1) \sim N\left(g - \log Q - \frac{\sigma^2}{2}, \sigma^2\right). \tag{A.1}$$

The household takes the returns as given and solves the following problem,

$$\begin{aligned} & \max_{c_0, a_0, \omega^k} \log c_0 + e^{-\rho} \log U_1 \\ & \text{where } U_1 = \left(E\left[c_1(z_1)^{1-\gamma}\right]\right)^{1/(1-\gamma)} \\ \text{s.t.} \quad & c_0 + a_0 = y_0 + Q \\ \text{and} \quad & c_1(z_1) = a_0\left(\omega^k \exp\left(r^k(z_1)\right) + \left(1-\omega^k\right)\exp\left(r^f\right)\right). \end{aligned}$$

Here, $c_1(z_1)$ denotes total financial wealth, which equals consumption (since the economy ends at date 1). Note that the household has Epstein-Zin preferences with EIS coefficient equal to one and the RRA coefficient equal to $\gamma > 0$. The case with $\gamma = 1$ is equivalent to log utility as in the main text.

In view of the Epstein-Zin functional form, the household's problem naturally splits into two steps. Conditional on savings, a_0, the household solves a portfolio optimization problem, that is, $U_1 = R^{CE} a_0$, where

$$\begin{aligned} R^{CE} &= \max_{\omega^k}\left(E\left[\left(R^p(z_1)\right)^{1-\gamma}\right]\right)^{1/(1-\gamma)} \\ \text{and } R^p(z_1) &= \left(\omega^k \exp\left(r^k(z_1)\right) + \left(1-\omega^k\right)\exp\left(r^f\right)\right). \end{aligned} \tag{A.2}$$

Here, we used the observation that the portfolio problem is linearly homogeneous. The variable, $R^p(z_1)$, denotes the realized portfolio return per dollar, and R^{CE} denotes the optimal certainty-equivalent portfolio return. In turn, the household chooses asset holdings, a_0, that solve the intertemporal problem,

$$\max_{a_0} \log(y_0 + Q - a_0) + e^{-\rho}\log\left(R^{CE} a_0\right).$$

The first order condition for this problem implies Eq. (2) in the main text. That is, regardless of her certainty-equivalent portfolio return, the household consumes and saves a constant fraction of her lifetime wealth.

It remains to characterize the optimal portfolio weight, ω^k, as well as the resulting certainty-equivalent return, R^{CE}. Even though the return on the market portfolio is log-normally distributed (see Eq. $(A.1)$), the portfolio return, $R^p(z_1)$, is in general not log-normally distributed (since it is the sum of a log-normal variable and a constant). Following Campbell and Viceira (2002), we assume the household solves an

approximate version of the portfolio problem $(A.2)$ in which the log portfolio return is also normally distributed. Moreover, the mean and the variance of this distribution are such that the following identifies hold,

$$\begin{aligned} \pi^p &\equiv \omega^k \pi^k && \text{(A.3)} \\ \text{where } \pi^p &= \log E\left[R^p\right] - r^f \text{ and } \pi^k = \log\left(E\left[\exp\left(r^k\right)\right]\right) - r^f = E\left[r^k\right] - r^f + \frac{\sigma^2}{2}, \\ \text{and } \sigma^p &= \omega^k \sigma && \text{(A.4)} \\ \text{where } \left(\sigma^p\right)^2 &= var\left(\log R^p\right) \text{ and } \sigma^2 = var\left(r^k\right). \end{aligned}$$

Here, the first line says that the risk premium on the investor's portfolio (measured in log difference of expectations of gross returns) depends linearly on the investor's portfolio weight and the risk premium on the market portfolio. The third line says that the standard deviation of the (log) portfolio return depends linearly on the investor's portfolio weight and the standard deviation of the (log) return on the market portfolio.[20] These identities hold exactly in continuous time. In the two period model, they hold approximately when the period time-length is small. Moreover, they become exact for the level the risk premium that ensures equilibrium, $\omega^k = 1$, since in this case the portfolio return is actually log-normally distributed.

Taking the log of the objective function in problem $(A.2)$, and using the log-normality assumption, the problem can be equivalently rewritten as,

$$\log R^{CE} - r^f = \max_{\omega^k} \pi^p - \frac{1}{2}\gamma\left(\sigma^p\right)^2,$$

where π^p and σ^p are defined in Eqs. $(A.3)$ and $(A.4)$. It follows that, up to an approximation (that becomes exact in equilibrium), the investor's problem turns into standard mean-variance optimization. Taking the first order condition, we obtain Eq. (3) in the main text. Substituting $\omega^k = 1$ and $E\left[r^k\right] = g - \log Q - \frac{\sigma^2}{2}$ [cf. Eq. $(A.1)$] into this expression, we further obtain Eq. (4) in the main text.

A.2. Omitted derivations in Section 3

A.2.1. Recursive formulation of the portfolio problem

Note that the HJB equation corresponding to the portfolio problem (14) is given by,

$$\begin{aligned} \rho V_{t,s}^i\left(a_{t,s}^i\right) = \max_{\tilde{\omega}^k, \tilde{\omega}^{s'}, \tilde{c}} & \log \tilde{c} + \frac{\partial V_{t,s}^i}{\partial a}\left(a_{t,s}^i\left(r_{t,s}^f + \tilde{\omega}^k\left(r_{t,s}^k - r_{t,s}^f\right) - \tilde{\omega}^{s'}\right) - \tilde{c}\right) \\ & + \frac{1}{2}\frac{\partial^2 V_{t,s}^i}{\partial a^2}\left(\tilde{\omega}^k a_{t,s}^i \sigma_s\right)^2 + \frac{\partial V_{t,s}\left(a_{t,s}\right)}{\partial t} \\ & + \lambda_s^i\left(V_{t,s'}^i\left(a_{t,s}^i\left(1 + \tilde{\omega}^k \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}} + \frac{\tilde{\omega}^{s'}}{p_{t,s}^{s'}}\right)\right) - V_{t,s}^i\left(a_t\right)\right) \end{aligned} \quad \text{(A.5)}$$

[20] Note that there is a unique log-normal distribution for R^p that ensures these identities. Specifically, $\log R^p \sim N\left(r^f + \pi^p - \frac{\left(\sigma^p\right)^2}{2}, \left(\sigma^p\right)^2\right)$.

In view of the log utility, the solution has the functional form in (41), which we reproduce here,

$$V_{t,s}^{i}\left(a_{t,s}^{i}\right)=\frac{\log\left(a_{t,s}^{i}/Q_{t,s}\right)}{\rho}+v_{t,s}^{i}.$$

The first term in the value function captures the effect of holding a greater capital stock (or greater wealth), which scales the investors consumption proportionally at all times and states. The second term, $v_{t,s}^{i}$, is the normalized value function when the investor holds one unit of the capital stock (or wealth, $a_{t,s}^{i}=Q_{t,s}$). This functional form also implies,

$$\frac{\partial V_{t,s}^{i}}{\partial a}=\frac{1}{\rho a_{t,s}}\text{ and }\frac{\partial^{2}V_{t,s}^{i}}{\partial a^{2}}=\frac{-1}{\rho\left(a_{t,s}\right)^{2}}.$$

The first order condition for $\tilde{c}$ then implies Eq. (16). The first order condition for $\tilde{\omega}^{k}$ implies,

$$\frac{\partial V_{t,s}^{i}}{\partial a}a_{t,s}\left(r_{t,s}^{k}-r_{t,s}^{f}\right)+\lambda_{s}^{i}\frac{\partial V_{t,s'}^{i}\left(a_{t,s'}\right)}{\partial a}a_{t,s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}=-\frac{\partial^{2}V_{t,s}^{i}}{\partial a^{2}}\omega_{t,s}^{k}\left(a_{t,s}^{i}\sigma_{s}\right)^{2}.$$

After substituting for $\frac{\partial V_{t,s}^{i}}{\partial a}$, $\frac{\partial V_{t,s'}^{i}}{\partial a}\frac{\partial^{2}V_{t,s}^{i}}{\partial a^{2}}$ and rearranging terms, this also implies Eq. (17). Finally, the first order condition for $\tilde{\omega}^{s'}$ implies,

$$\frac{p_{t,s}^{s'}}{\lambda_{s}^{i}}=\frac{\frac{\partial V_{t,s'}^{i}\left(a_{t,s'}\right)}{\partial a}}{\frac{\partial V_{t,s}^{i}\left(a_{t,s}\right)}{\partial a}}=\frac{1/a_{t,s}^{i}}{1/a_{t,s}^{i}},$$

which is Eq. (18). This completes the characterization of the optimality conditions.

A.2.2. Description of the New Keynesian production firms

The supply side of our model features nominal rigidities similar to the standard New Keynesian setting. There is a continuum of measure one of production firms denoted by ν. These firms rent capital from the investment firms, $k_{t,s}\left(\nu\right)$, and produce differentiated goods, $y_{t,s}\left(\nu\right)$, subject to the technology,

$$y_{t,s}\left(\nu\right)=A\eta_{t,s}\left(\nu\right)k_{t,s}\left(\nu\right). \tag{A.6}$$

Here, $\eta_{t,s}\left(\nu\right)\in\left[0,1\right]$ denotes the firm's choice of capital utilization. We assume utilization is free up to $\eta_{t,s}\left(\nu\right)=1$ and infinitely costly afterwards (see our extended working paper version, in which we relax this assumption and allow for excess utilization at the cost of excess depreciation). The production firms sell their output to a competitive sector that produces the final output according to the CES technology, $y_{t,s}=\left(\int_{0}^{1}y_{t,s}\left(\nu\right)^{\frac{\varepsilon-1}{\varepsilon}}d\nu\right)^{\varepsilon/(\varepsilon-1)}$, for some $\varepsilon>1$. Thus, the demand for the firms' goods is given by,

$$y_{t,s}\left(\nu\right)=p_{t,s}\left(\nu\right)^{-\varepsilon}y_{t,s},\text{ where }p_{t,s}\left(\nu\right)=P_{t,s}\left(\nu\right)/P. \tag{A.7}$$

Here, $p_{t,s}\left(\nu\right)$ denotes the firm's relative price, which depends on its nominal price, $P_{t,s}\left(\nu\right)$, as well as the ideal nominal price index, $P_{t,s}=\left(\int P_{t,s}\left(\nu\right)^{1-\varepsilon}d\nu\right)^{1/(1-\varepsilon)}$.

We also assume there are subsidies designed the correct the inefficiencies that stem from the firm's monopoly power and markups. In particular, the government taxes the firm's profits lump sum, and redistributes these profits to the firms in the form of a linear subsidy to capital. Formally, we let $\Pi_{t,s}\left(\nu\right)$

denote the equilibrium pre-tax profits of firm ν (that will be characterized below). We assume each firm is subject to the lump-sum tax determined by the average profit of all firms,

$$T_{t,s} = \int_{\nu} \Pi_{t,s}(\nu)\, d\nu. \tag{A.8}$$

We also let $R_{t,s} - \tau_{t,s}$ denote the after-subsidy cost of renting capital, where $R_{t,s}$ denotes the equilibrium rental rate paid to investment firms, and $\tau_{t,s}$ denotes a linear subsidy paid by the government. We assume the magnitude of the subsidy is determined by the government's break-even condition,

$$\tau_{t,s} \int_{\nu} k_{t,s}(\nu)\, d\nu = T_{t,s}. \tag{A.9}$$

Without price rigidities, the firm chooses $p_{t,s}(\nu), k_{t,s}(\nu), \eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu)$, to maximize its (pre-tax) profits,

$$\Pi_{t,s}(\nu) \equiv p_{t,s}(\nu)\, y_{t,s}(\nu) - (R_{t,s} - \tau_{t,s})\, k_{t,s}(\nu), \tag{A.10}$$

subject to the supply constraint in $(A.6)$ and the demand constraint in $(A.7)$. The optimality conditions imply,

$$p_{t,s}(\nu) = \frac{\varepsilon}{\varepsilon - 1} \frac{R_{t,s} - \tau_{t,s}}{A} \text{ and } \eta_{t,s}(\nu) = 1.$$

That is, the firm charges a markup over its marginal costs, and utilizes its capital at full capacity. In a symmetric-price equilibrium, we further have, $p_{t,s}(\nu) = 1$. Using Eqs. $(A.6 - A.9)$, this further implies,

$$y_{t,s}(\nu) = y_{t,s} = A k_{t,s} \text{ and } R_{t,s} = \frac{\varepsilon - 1}{\varepsilon} A + \tau_{t,s} = A. \tag{A.11}$$

That is, output is equal to potential output, and capital earns its marginal contribution to potential output (in view of the linear subsidies).

We focus on the alternative setting in which the firms have a preset nominal price that is equal to one another, $P_{t,s}(\nu) = P$. In particular, the relative price of a firm is fixed and equal to one, $p_{t,s}(\nu) = 1$. The firm chooses the remaining variables, $k_{t,s}(\nu), \eta_{t,s}(\nu) \in [0,1], y_{t,s}(\nu)$, to maximize its (pre-tax) profits, $\Pi_{t,s}(\nu)$. We conjecture a symmetric equilibrium in which all firms choose the same allocation, $k_{t,s}, \eta_{t,s}, y_{t,s}$, output is determined by aggregate demand,

$$y_{t,s} = \eta_{t,s} A k_{t,s} = \int_{I} c^{i}_{t,s} di + k_{t,s} \iota_{t,s}, \text{ for } \eta_{t,s} \in [0,1], \tag{A.12}$$

and the rental rate of capital is given by,

$$R_{t,s} = A \eta_{t,s}. \tag{A.13}$$

To verify that the conjectured allocation is an equilibrium, first consider the case in which aggregate demand is below potential output, so that $y_{t,s} < A k_{t,s}$ and $\eta_{t,s} < 1$. In this case, the firms can reduce their capital input, $k_{t,s}(\nu)$, and increase their factor utilization, $\eta_{t,s}(\nu)$, to obtain the same level of production. Since factor utilization is free (up to $\eta_{t,s}(\nu) = 1$), the after tax cost of capital must be zero, $R_{t,s} - \tau_{t,s} = 0$. Since its marginal cost is zero, and its relative price is one, it is optimal for each firm to produce according to the aggregate demand, which verifies Eq. $(A.12)$. Using Eqs. $(A.8)$ and $(A.9)$, we further obtain, $\tau_{t,s} = A \eta_{t,s}$. Combining this with the requirement that $R_{t,s} - \tau_{t,s} = 0$ verifies Eq. $(A.13)$.

Next consider the case in which aggregate demand is equal to potential output, so that $y_{t,s} = Ak_{t,s}$ and $\eta_{t,s} = 1$. In this case, a similar analysis implies there is a range of equilibria with $\frac{R_{t,s}-\tau_{t,s}}{A} \leq 1$ and $R_{t,s} = A$. Here, the first equation ensures it is optimal for the firm to meet the aggregate demand. The second equation follows from the subsidy and the tax scheme. In particular, the frictionless benchmark allocation (*A*.11), that features $\frac{R_{t,s}-\tau_{t,s}}{A} = \frac{\varepsilon-1}{\varepsilon}$ and $R_{t,s} = A$, is also an equilibrium with nominal rigidities as long as the aggregate demand is equal to potential output.

A.3. Omitted derivations and proofs in Section 4

Proof of Proposition 1. Most of the proof is provided in the main text. It remains to show that Assumptions 1-3 ensure there exist a unique solution, $q_2 < q^*$ and $r_1^f \geq 0$, to Eqs. (29) and (30). To this end, we define,

$$f(q_2, \lambda_2) = \rho + \psi q_2 - \delta + \lambda_2 \left(1 - \frac{\exp(q_2)}{Q^*}\right) - \sigma_2^2.$$

The equilibrium price is the solution to, $f(q_2, \lambda_2) = 0$ (given λ_2). Note that $f(q_2, \lambda_2)$ is a concave function of q_2 with $\lim_{q_2 \to -\infty} f(q_2, \lambda_2) = \lim_{q_2 \to \infty} f(q_2, \lambda_2) = -\infty$. Its derivative is,

$$\frac{\partial f(q_2, \lambda_2)}{\partial q_2} = \psi - \lambda_2 \frac{\exp(q_2)}{Q^*}.$$

Thus, for fixed λ_2, it is maximized at,

$$q_2^{\max}(\lambda_2) = q^* + \log(\psi/\lambda_2).$$

Moreover, the maximum value is given by

$$\begin{aligned} f(q_2^{\max}(\lambda_2), \lambda_2) &= \rho - \delta + \psi(q^* + \log(\psi/\lambda_2)) + \lambda_2(1 - \exp(\log(\psi/\lambda_2))) - \sigma_2^2 \\ &= \rho - \delta + \psi q^* + \psi \log(\psi/\lambda_2) + \lambda_2 - \psi - \sigma_2^2. \end{aligned}$$

Next note that, by Assumption 1, the maximum value is strictly negative when $\lambda_2 = \psi$, that is, $f(q_2^{\max}(\psi), \psi) < 0$. Note also that $\frac{df(q_2^{\max}(\lambda_2), \lambda_2)}{d\lambda_2} = 1 - \frac{\psi}{\lambda_2}$, which implies that the maximum value is strictly increasing in the range $\lambda_2 \geq \psi$. Since $\lim_{\lambda_2 \to \infty} f(q_2^{\max}(\lambda_2), \lambda_2) = \infty$, there exists $\lambda_2^{\min}$ that ensures $f\left(q_2^{\max}\left(\lambda_2^{\min}\right), \lambda_2^{\min}\right) = 0$. By Assumption 1, the actual level of optimism satisfies $\lambda_2 \geq \lambda_2^{\min}$, which implies that $f(q_2^{\max}(\lambda_2), \lambda_2) \geq 0$. By Assumption 1, we also have that $f(q^*, \lambda_2) < 0$.

It follows that, for any $\lambda_2 \geq \lambda_2^{\min}$, there exists a unique $q_2 \in [q_2^{\max}, q^*)$ which ensures that solves the equation, $f(q_2, \lambda_2) = 0$. Our analysis also implies that the solution has the following feature, $\frac{\partial f(q_2, \lambda_2)}{\partial q_2} = \psi - \lambda_2 \frac{\exp(q_2)}{Q^*} \leq 0$, with equality only if $\lambda_2 = \lambda_2^{\min}$. This facilitates the comparative statics results in Section 4.

Next consider Eq. (30), which can be rewritten as

$$r_1^f = \rho + \psi q^* - \delta + \lambda_1 \left(1 - \frac{Q^*}{Q_2}\right) - \sigma_1^2.$$

Since $q_2 < q^*$, this expression is decreasing in λ_1. When $\lambda_1 = 0$, it is strictly positive by Assumption 1. As $\lambda_1 \to \infty$, it approaches $-\infty$. Thus, for any $q_2 < q^*$, there exists $\lambda_1^{\max}(q_2)$ such that $r_1^f \geq 0$ if and only if $\lambda_1 \in [0, \lambda_1^{\max}(q_2)]$. Note also that for any fixed $\lambda_1 > 0$, r_1^f is increasing in q_2. This implies that the upper bound for the transition probability, $\lambda_1^{\max}(q_2)$, is increasing in q_2, completing the proof. □

Proof of Corollary 1. Fix some $\Delta t > 0$ and let $s^{\Delta t}$ denote the random variable that is equal to s if there is no state transition over $[0, \Delta t]$, and s' if there is at least one state transition. The law of total variance implies,

$$Var_{t,s}\left(\frac{\Delta k_{t,s}Q_{t,s}/\Delta t}{k_{t,s}Q_{t,s}}\right) = E^{s^{\Delta t}}\left[Var_{t,s}\left(\frac{\Delta k_{t,s}Q_{t,s}/\Delta t}{k_{t,s}Q_{t,s}}|s^{\Delta t}\right)\right] + Var^{s^{\Delta t}}\left(E_{t,s}\left[\frac{\Delta k_{t,s}Q_{t,s}/\Delta t}{k_{t,s}Q_{t,s}}|s^{\Delta t}\right]\right). \tag{A.14}$$

Here, $E^{s^{\Delta t}}[\cdot]$ and $Var^{s^{\Delta t}}[\cdot]$ denote, respectively, the expectations and the variance operator over the random variable, $s^{\Delta t}$. We next calculate each component of variance.

For the first component, we have,

$$E^{s^{\Delta t}}\left[Var_{t,s}\left(\frac{\Delta k_{t,s}Q_{t,s}}{k_{t,s}Q_{t,s}}|s^{\Delta t}\right)\right] = e^{-\lambda_s \Delta t}\sigma_s^2 \Delta t + \left(1 - e^{-\lambda_s \Delta t}\right) O\left(\Delta t\right).$$

Here, the first term captures the variance conditional on there being no transition, $s^{\Delta t} = s$. The variance in this case comes from the Brownian motion for $k_{t,s}$. The second term captures the average variance conditional on there being a transition, $s^{\Delta t} = s'$. Here, the last term satisfies, $\lim_{\Delta t \to 0} O\left(\Delta t\right) = 0$. Dividing by Δt and evaluating the limit, we obtain,

$$\lim_{\Delta t \to 0} E^{s^{\Delta t}}\left[Var_{t,s}\left(\frac{\Delta k_{t,s}Q_{t,s}/\Delta t}{k_{t,s}Q_{t,s}}|s^{\Delta t}\right)\right] = \sigma_s^2. \tag{A.15}$$

For the second component, we have,

$$\begin{aligned} Var^{s^{\Delta t}}\left(E_{t,s}\left[\frac{\Delta k_{t,s}Q_{t,s}}{k_{t,s}Q_{t,s}}|s^{\Delta t}\right]\right) &= Var^{s^{\Delta t}}\left(E_{t,s}\left[\frac{\Delta Q_{t,s}}{Q_{t,s}}|s^{\Delta t}\right]\right) + O\left((\Delta t)^2\right), \\ &= (1 - \lambda_s \Delta t)\left(\frac{Q_s - \overline{Q}}{Q_s}\right)^2 + \lambda_s \Delta t \left(\frac{Q_{s'} - \overline{Q}}{Q_s}\right)^2 + O\left((\Delta t)^2\right), \\ &\quad \text{where } \overline{Q} = (1 - \lambda_s \Delta t) Q_s + \lambda_s \Delta t Q_{s'}. \end{aligned}$$

Here, $O\left((\Delta t)^2\right)$ denotes terms that satisfy, $\lim_{\Delta t \to 0} \frac{O\left((\Delta t)^2\right)}{\Delta t} = 0$. The first line uses the observation that, conditional on a path of state transitions, the expected level of investment is constant. Thus (for small Δt) the state transitions change the return only through their impact on the price level. The second line calculates the variance of price changes up to terms that are first order in Δt. Dividing the last line by Δt and evaluating the limit, we obtain,

$$\lim_{\Delta t \to 0} Var^{s^{\Delta t}}\left(E_{t,s}\left[\frac{\Delta k_{t,s}Q_{t,s}/\Delta t}{k_{t,s}Q_{t,s}}|s^{\Delta t}\right]\right) = \lambda_s \left(\frac{Q_{s'} - Q_s}{Q_s}\right)^2. \tag{A.16}$$

Combining Eqs. $(A.14)$, $(A.15)$ and $(A.16)$, the unconditional variance is given by, $\sigma_s^2 + \lambda_s \left(\frac{Q_{s'} - Q_s}{Q_s}\right)^2$, completing the proof. □

A.4. Omitted derivations and proofs in Section 5

We derive the equilibrium conditions that we state and use in Section 5. First note that, using Eq. (18), the optimality condition (17) can be written as,

$$\omega_{t,s}^{k,i}\sigma_s = \frac{1}{\sigma_s}\left(r_{t,s}^k - r_{t,s}^f + p_{t,s}^{s'}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right). \tag{A.17}$$

Combining this with the market clearing condition (19), we obtain,

$$\omega_{t,s}^{k,o} = \omega_{t,s}^{k,p} = 1. \tag{A.18}$$

Next note that by definition, we have

$$a_{t,s}^o = \alpha_{t,s}Q_{t,s}k_{t,s} \text{ and } a_{t,s}^p = (1-\alpha_{t,s})\,Q_{t,s}k_{t,s} \text{ for each } s \in \{1,2\}.$$

After plugging these into Eq. (18), using $k_{t,s} = k_{t,s'}$ (since capital does not jump), and aggregating over optimists and pessimists, we obtain,

$$p_{t,s}^{s'} = \overline{\lambda}_{t,s}\frac{Q_{t,s}}{Q_{t,s'}}, \tag{A.19}$$

where $\overline{\lambda}_{t,s}^o$ denotes the wealth-weighted average belief defined in (32). Combining Eqs. $(A.17), (A.18)$, and $(A.19)$, we obtain the risk balance condition (33) in the main text.

We next characterize investors' equilibrium positions. Combining Eq. (15) with Eqs. $(A.18)$ and $(A.19)$, investors' wealth after transition satisfies,

$$\frac{a_{t,s'}^i}{a_{t,s}^i} = \frac{Q_{t,s'}}{Q_{t,s}}\left(1 + \frac{\omega_{t,s}^{s',i}}{\overline{\lambda}_{t,s}}\right). \tag{A.20}$$

From Eq. (18), we have $\frac{p_{t,s}^{s'}}{\lambda_s^i} = \frac{1/a_{t,s'}^i}{1/a_{t,s}^i}$. Substituting this into the previous expression and using Eq. $(A.19)$ once more, we obtain,

$$\omega_{t,s}^{s',i} = \lambda_s^i - \overline{\lambda}_{t,s} \text{ for each } i \in \{o,p\}. \tag{A.21}$$

Combining this with Eq. (34), we obtain Eq. (34) in the main text.

Finally, we characterize the evolution of optimists' wealth share. After substituting $a_{t,s}^o = \alpha_{t,s}Q_{t,s}k_{t,s}$ and using Eq. $(A.21)$ (as well as $k_{t,s} = k_{t,s'}$), Eq. $(A.20)$ implies

$$\frac{\alpha_{t,s'}}{\alpha_{t,s}} = \frac{\lambda_s^o}{\overline{\lambda}_{t,s}}. \tag{A.22}$$

Thus, it remains to characterize the evolution of wealth conditional on no transition. This end, we combine Eq. (15) with Eqs. $(A.18), (23), (16)$ to obtain,

$$\frac{da_{t,s}^o}{a_{t,s}^o} = \left(g_{t,s} + \mu_{t,s}^Q - \omega_{t,s}^{s',i}\right)dt + \sigma_s dZ_t.$$

After substituting $a_{t,s}^o = \alpha_{t,s}Q_{t,s}k_{t,s}$, and using the observation that $\frac{dQ_{t,s}}{Q_{t,s}} = \mu_{t,s}^Q dt$ and $\frac{dk_{t,s}}{dk_{t,s}} = g_{t,s}dt + \sigma_s dZ_t$, we further obtain,

$$\frac{d\alpha_{t,s}}{\alpha_{t,s}} = -\omega_{t,s}^{s',o}dt = -\left(\lambda_s^o - \overline{\lambda}_{t,s}\right)dt. \tag{A.23}$$

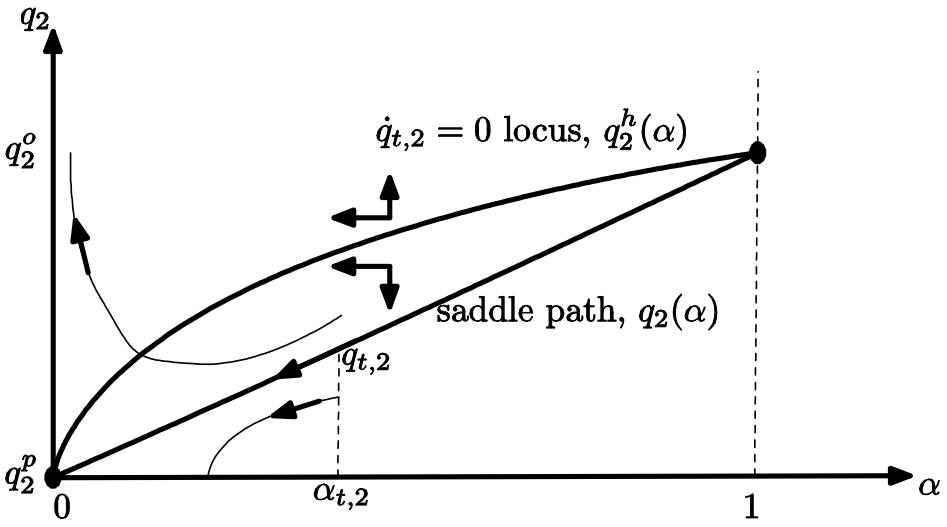


Figure 13: The phase diagram that describes the equilibrium with heterogeneous beliefs.

Combining Eqs. $(A.22)$ and $(A.23)$ implies Eq. (35) in the main text.

Proof of Proposition 2. We analyze the solution to the system in (37) using the phase diagram over the range $\alpha \in [0,1]$ and $q_2 \in [q_2^p, q_2^o]$. First note that the system has two steady states given by, $(\alpha_{t,2} = 0, q_{t,2} = q_2^p)$, and $(\alpha_{t,2} = 1, q_{t,2} = q_2^o)$. Next note that the system satisfies the Lipschitz condition over the relevant range. Thus, the vector flows that describe the law of motion do not cross. Next consider the locus, $\dot{q}_2 = 0$. By comparing Eqs. (36) and (33), we conclude that this locus is exactly the same as the price that would obtain if investors' shared the same wealth-weighted average belief, denoted by $q_2 = q_2^h(\alpha)$. Using our analysis in Section 4, we also find that $q_2^h(\alpha)$ is strictly increasing in α. Moreover, $q_2 < q_2^h(\alpha)$ implies $\dot{q}_{t,2} < 0$ whereas $q_2 > q_2^h(\alpha)$ implies $\dot{q}_{t,2} > 0$. Finally, note that $\dot{\alpha}_{t,2} < 0$ for each $\alpha \in (0,1)$.

Combining these observations, the phase diagram has the shape in Figure 13. This in turn implies that the system is saddle path stable. Given any $\alpha_{t,2} \in [0,1)$, there exists a unique solution, $q_{t,2}$, which ensures that $\lim_{t\to\infty} q_{t,2} = q_2^p$. We define the price function (the saddle path) as $q_2(\alpha)$. Note that the price function satisfies $q_2(\alpha) < q_2^h(\alpha)$ for each $\alpha \in (0,1)$, since the saddle path cannot cross the locus, $\dot{q}_{t,2} = 0$. Note also that $q_2(1) = q_2^o$, since the saddle path crosses the other steady-state, $(\alpha_{t,2} = 1, q_{t,2} = q_2^o)$. Finally, recall that $q_2 < q_2^h(\alpha)$ implies $\dot{q}_{t,2} < 0$. Combining this with $\dot{\alpha}_{t,2} < 0$, we further obtain $\frac{dq_2(\alpha)}{d\alpha} > 0$ for each $\alpha \in (0,1)$.

Next note that, after substituting $\dot{q}_{t,2} = q_2'(\alpha)\,\dot{\alpha}_{t,2}$, Eq. (37) implies the differential equation (38) in α-domain. Thus, the above analysis shows there exists a solution to the differential equation with $q_2(0) = q_2^p$ and $q_2(1) = q_2^o$. Moreover, the solution is strictly increasing in α, and it satisfies $q_2(\alpha) < q_2^h(\alpha)$ for each $\alpha \in (0,1)$. Note also that this solution is unique since the saddle path is unique.

Next consider Eq. (40) which characterizes the interest rate function, $r_1^f(\alpha)$. Note that $\frac{dr_1^f(\alpha)}{d\alpha} > 0$ since $\frac{dq_2(\alpha')}{d\alpha} > 0$ (recall that $\alpha' = \alpha\lambda_1^o/\overline{\lambda}_2(\alpha)$). Note also that $r_1^f(\alpha) > r_1^f(0) > 0$, where the latter inequality follows since Assumptions 1-3 holds for the pessimistic belief. Thus, the interest rate in state 1 is always positive, which verifies our conjecture and completes the proof. □

A.5. Omitted derivations in Section 6.1 on equilibrium values

This subsection derives the HJB equation that describes the normalized value function in equilibrium. It then characterizes this equation further for various cases analyzed in Section 6.1.

Characterizing the normalized value function in equilibrium. Consider the recursive version of the portfolio problem in $(A.5)$. Recall that the value function has the functional form in Eq. (41). Our goal is to characterize the value function per unit of capital, $v_{t,s}^i$ (corresponding to $a_{t,s}^i = Q_{t,s}$). To facilitate the analysis, we define,

$$\xi_{t,s}^i = v_{t,s}^i - \frac{\log Q_{t,s}}{\rho}. \tag{A.24}$$

Note that $\xi_{t,s}^i$ is the value function per unit wealth (corresponding to $a_{t,s}^i = 1$), and that the value function also satisfies

$$V_{t,s}^i\left(a_{t,s}^i\right) = \frac{\log\left(a_{t,s}^i\right)}{\rho} + \xi_{t,s}^i.$$

We first characterize $\xi_{t,s}^i$. We then combine this with Eq. $(A.24)$ to characterize our main object of interest, $v_{t,s}^i$.

After substituting the optimal consumption rule in (16) and Eq. (18) (as well as $a_{t,s}^i = 1$), we obtain the following version of the HJB equation in $(A.5)$,

$$\begin{aligned}\rho\xi_{t,s}^i = \log\rho + \frac{1}{\rho}\left(r_{t,s}^f + \omega_{t,s}^{k,i}\left(r_{t,s}^k - r_{t,s}^f\right) - \frac{1}{2}\left(\omega_{t,s}^{k,i}\right)^2\sigma_s^2 - \rho - \omega_{t,s}^{s',i}\right) \\ + \frac{\partial\xi_{t,s}^i}{\partial t} + \lambda_s^i\left(\frac{1}{\rho}\log\left(\frac{\lambda_s^i}{p_{t,s}^{s'}}\right) + \xi_{t,s'}^i - \xi_{t,s}^i\right).\end{aligned} \tag{A.25}$$

As we describe in Section 5, the market clearing conditions imply the optimal investment in capital and contingent securities satisfies, $\omega^k = 1$ and $\tilde{\omega}_{t,s}^{s',i} = \lambda_s^i - \overline{\lambda}_{t,s}$, and the price of the contingent security is given by, $p_{t,s}^{s'} = \overline{\lambda}_{t,s}\frac{1/Q_{t,s'}}{1/Q_{t,s}}$. Here, $\overline{\lambda}_{t,s}$ denotes the weighted average belief defined in (32). Using these conditions, the HJB equation becomes,

$$\begin{aligned}\rho\xi_{t,s}^i = \log\rho + \frac{1}{\rho}\left(\begin{array}{c} r_{t,s}^k - \rho - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda_s^i - \overline{\lambda}_{t,s}\right) + \lambda_s^i\log\left(\frac{\lambda_s^i}{\overline{\lambda}_{t,s}}\right)\end{array}\right) \\ + \frac{\partial\xi_{t,s}^i}{\partial t} + \lambda_s^i\left(\frac{1}{\rho}\log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right) + \xi_{t,s'}^i - \xi_{t,s}^i\right).\end{aligned} \tag{A.26}$$

After substituting the return to capital from (23), the HJB equation can be further simplified as,

$$\rho\xi_{t,s}^i = \left[\begin{array}{c} \log\rho + \frac{1}{\rho}\left(\begin{array}{c}\psi\log\left(Q_{t,s}\right) - \delta + \mu_{t,s}^Q - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda_s^i - \overline{\lambda}_{t,s}\right) + \lambda_s^i\log\left(\frac{\lambda_s^i}{\overline{\lambda}_{t,s}}\right)\end{array}\right) \\ +\frac{\partial\xi_{t,s}^i}{\partial t} + \lambda_s^i\left(\frac{1}{\rho}\log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right) + \xi_{t,s'}^i - \xi_{t,s}^i\right)\end{array}\right].$$

Here, the term inside the summation on the second line, $-\left(\lambda_s^i - \overline{\lambda}_{t,s}\right) + \lambda_s^i\log\left(\frac{\lambda_s^i}{\overline{\lambda}_{t,s}}\right)$, is zero when there are no disagreements, and it is strictly positive when there are disagreements. It captures the intuition that speculation increases the expected value for optimists as well as pessimists.

We finally substitute $v_{t,s}^i = \xi_{t,s}^i + \frac{\log Q_{t,s}}{\rho}$ (cf. $(A.24)$) into the HJB equation to obtain the differential

equation,

$$\rho v_{t,s}^i = \left[\begin{array}{c} \log\rho + \log\left(Q_{t,s}\right) + \frac{1}{\rho}\left(\begin{array}{c} \psi \log\left(Q_{t,s}\right) - \delta - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda_s^i - \overline{\lambda}_{t,s}\right) + \lambda_s^i \log\left(\frac{\lambda_s^i}{\overline{\lambda}_{t,s}}\right) \end{array} \right) \\ + \frac{\partial v_{t,s}^i}{\partial t} + \lambda_s^i \left(v_{t,s'} - v_{t,s}\right) \end{array} \right].$$

Here, we have canceled terms by using the observation that $\frac{\partial \xi_{t,s}^i}{\partial t} = \frac{\partial v_{t,s}^i}{\partial t} - \frac{1}{\rho}\frac{\partial \log Q_{t,s}}{\partial t} = \frac{\partial v_{t,s}^i}{\partial t} - \frac{1}{\rho}\mu_{t,s}^Q$. We have thus obtained Eq. (42) in the main text.

Solving for the value function in the common beliefs benchmark. Next consider the benchmark with common beliefs. In this case, the price level is stationary, $q_{t,s} = q_s$ for each s (see Section 4). Then, the HJB equation (42) implies the value functions are also stationary, $v_{t,s} = v_s$, with values that satisfy,

$$\rho v_s = \log\rho + q_s + \frac{1}{\rho}\left(\psi q_s - \delta - \frac{1}{2}\sigma_s^2\right) + \lambda_s\left(v_{s'} - v_s\right).$$

Consider the same equation for $s' \neq s$. Multiplying that equation with λ_s and the above equation with $(\rho + \lambda_{s'})$, and adding up, we obtain a closed form solution,

$$\begin{aligned} \rho v_s &= \log\rho + \overline{q}_s + \frac{1}{\rho}\left(\psi\overline{q}_s - \delta - \frac{1}{2}\overline{\sigma}_s^2\right), \qquad \text{(A.27)} \\ \text{where } \overline{q}_s &= \beta_s q_s + \left(1-\beta_s\right) q_{s'} \text{ and } \overline{\sigma}_s^2 = \beta_s\sigma_s^2 + \left(1-\beta_s\right)\sigma_{s'}^2, \\ \text{and } \beta_s &= \frac{\rho + \lambda_{s'}}{\rho + \lambda_{s'} + \lambda_s}. \end{aligned}$$

Here, the weights β_s and $1-\beta_s$ can be thought of as capturing the "discounted expected time" the economy spends in each state (note that the economy starts in state s and the investors discount the future at rate ρ). The value in a state is the sum of the utility from (the discounted average of) current consumption and the present value of the risk-adjusted growth rate. All else equal, the value is decreasing in the weighted average volatility, $\overline{\sigma}_s$, but it is increasing in the weighted-average price level, $\overline{q}_s$.

Note also that the weights (the discounted expected times) satisfy the following property,

$$\beta_s = \frac{\rho + \lambda_{s'}}{\rho + \lambda_{s'} + \lambda_s} > 1 - \beta_{s'} = \frac{\lambda_{s'}}{\rho + \lambda_s + \lambda_{s'}}.$$

Here, β_s (resp. $1-\beta_{s'}$) is the discounted time the investor spends in state s when she starts in state s (resp. in the other state s'). Thus, $\beta_s > 1-\beta_{s'}$ implies that the economy spends more discounted time in the state it starts with. Combining this observation with $q_2 < q_1 = q^*$ and $\sigma_2^2 > \sigma_1^2$, Eq. (A.27) implies $v_2 < v_1$. Intuitively, investors have a lower expected value when they are in the high-volatility state since they expect asset prices to be lower and the volatility to be higher.

Next note that $\{v_s^*\}_s$ is defined as the solution to the same equation system with $q_s = q^*$ for each s. The gap value, $w_s = v_s - v_s^*$, can be calculated by subtracting the corresponding equations for v_s an v_s^*. With some algebra, we obtain,

$$\rho w_s = \left(\overline{q}_s - q^*\right)\left(1 + \frac{\psi}{\rho}\right). \qquad \text{(A.28)}$$

That is, the gap value is proportional to the weighted-average price gap relative to the first best. Note also that we have $q_1 - q^* = 0$ and $q_2 - q^* < 0$. Since $\beta_s \in (0,1)$, this implies $w_s < 0$ for each $s \in \{1,2\}$. Since $\beta_2 > 1-\beta_1$, we further obtain $w_2 < w_1 < 0$.

Solving the value function with belief disagreements. With belief disagreements, the value function and its components, $\left\{v_{t,s}^{i}, v_{t,s}^{i,*}, w_{t,s}\right\}_{s,i}$, can be written as functions of optimists' wealth share, $\left\{v_{s}^{i}(\alpha), v_{s}^{i,*}(\alpha), w_{s}(\alpha)\right\}_{s,i}$, that solve appropriate ordinary differential equations.

Recall that the price level in each state can be written as a function of optimists' wealth shares, $q_{t,s} = q_s(\alpha)$ (where we also have, $q_1(\alpha) = q^*$). Plugging in these price functions, and using the evolution of $\alpha_{t,s}$ from Eq. (35), the HJB equation (42) can be written as,

$$\rho v_{s}^{i}(\alpha) = \begin{bmatrix} \log\rho + q_s(\alpha) + \frac{1}{\rho}\left(\begin{array}{c} \psi q_s(\alpha) - \delta - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda_s^i - \overline{\lambda}_s(\alpha)\right) + \lambda_s^i \log\left(\frac{\lambda_s^i}{\lambda_s^p + \alpha\Delta\lambda_s}\right) \end{array} \right) \\ -\frac{\partial v_s^i}{\partial\alpha}\left(\lambda_s^o - \lambda_s^p\right)\alpha(1-\alpha) + \lambda_s^i\left(v_{s'}^i\left(\alpha\frac{\lambda_s^o}{\overline{\lambda}_s(\alpha)}\right) - v_s^i(\alpha)\right) \end{bmatrix}.$$

For each $i \in \{o, p\}$, the value functions, $\left(v_s^i(\alpha)\right)_{s\in\{1,2\}}$, are found by solving this system of ODEs. For $i = 0$, the boundary conditions are that the values, $\left\{v_s^o(1)\right\}_s$, are the same as the values in the common belief benchmark characterized in Section 4 when all investors have the optimistic beliefs. For $i = p$, the boundary conditions are that the values, $\left\{v_s^p(0)\right\}_s$, are the same as the values in the common belief benchmark when all investors have the pessimistic beliefs.

Likewise, the first-best value functions, $\left(v_s^{i,*}(\alpha)\right)_{s\in\{1,2\}}$, are found by solving the analogous system after replacing $q_s(\alpha)$ with q^* (and changing the boundary conditions appropriately). Finally, after substituting the price functions into Eq. (44), the gap-value functions, $\left(w_s^i(\alpha)\right)_{s,i}$, are found by solving the following system (with appropriate boundary conditions),

$$\rho w_s^i(\alpha) = \left(1 + \frac{\psi}{\rho}\right)\left(q_s(\alpha) - q^*\right) - \frac{\partial w_s^i(\alpha)}{\partial\alpha}\left(\lambda_s^o - \lambda_s^p\right)\alpha(1-\alpha) + \lambda_s^i\left(w_{s'}\left(\alpha\frac{\lambda_s^o}{\overline{\lambda}_s(\alpha)}\right) - w_s(\alpha)\right).$$

Figure 8 in the main text plots the solution to these differential equations for a particular parameterization.

A.6. Omitted derivations in Section 6.2 on macroprudential policy

Recall that macroprudential policy induces optimists to choose allocations as if they have more pessimistic beliefs, $\lambda^{o,pl} \equiv \left(\lambda_1^{o,pl}, \lambda_2^{o,pl}\right)$, that satisfy, $\lambda_1^{o,pl} \geq \lambda_1^o$ and $\lambda_2^{o,pl} \leq \lambda_2^o$. We next show that this allocation can be implemented with portfolio restrictions on optimists. We then show that the planner's Pareto problem reduces to solving problem (45) in the main text. Finally, we derive the equilibrium value functions that result form macroprudential policy and present the proofs of Propositions 3 and 4.

Implementing the policy with risk limits. Consider the equilibrium that would obtain if optimists had the planner-induced beliefs, $\lambda_s^{o,pl}$. Using our analysis in Section 5, optimists' equilibrium portfolios are given by,

$$\omega_{t,s}^{k,o,pl} = 1 \text{ and } \omega_{t,s}^{s',o,pl} = \lambda_s^{o,pl} - \overline{\lambda}_{t,s}^{pl} \text{ for each } t, s. \tag{A.29}$$

We first show that the planner can implement the policy by requiring optimists to hold exactly these portfolio weights. We will then relax these portfolio constraints into inequality restrictions (see Eq. (*A*.31)).

Formally, an optimist solves the HJB problem (*A*.5) with the additional constraint (*A*.29). In view of log utility, we conjecture that the value function has the same functional form (41) with potentially

different normalized values, $\xi^o_{t,s}, v^o_{t,s}$, that reflect the constraints. Using this functional form, the optimality condition for consumption remains unchanged, $c_{t,s} = \rho a^o_{t,s}$ [cf. Eq. (16)]. Plugging this equation and the portfolio holdings in $(A.29)$ into the objective function in $(A.5)$ verifies that the value function has the conjectured functional form. For later reference, we also obtain that the optimists' unit-wealth value function satisfies [cf. Eq. $(A.24)$],

$$\begin{aligned}\xi^o_{t,s} &= \log\rho + \frac{1}{\rho}\left(r^f_{t,s} + \omega^{k,o,pl}_{t,s}\left(r^k_{t,s} - r^f_{t,s}\right) - \rho - \omega^{s',o,pl}_{t,s}\right) \\ &\quad -\frac{1}{2\rho}\left(\omega^{k,o,pl}_{t,s}\sigma_s\right)^2 + \frac{\partial \xi^o_{t,s}}{\partial t} + \lambda^o_s\left(\frac{1}{\rho}\log\left(\frac{a^o_{t,s'}}{a^o_{t,s}}\right) + \xi^o_{t,s'} - \xi^o_{t,s}\right).\end{aligned} \tag{A.30}$$

Here, $\frac{a^o_{t,s'}}{a^o_{t,s}} = 1 + \omega^{k,o,pl}_{t,s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}} + \frac{\omega^{s',o,pl}_{t,s}}{p^{s'}_{t,s}}$ in view of the budget constraints of problem $(A.5)$. Hence, the value function has a similar characterization as before [cf. Eq. $(A.25)$] with the difference that optimists' portfolio holdings reflect the constraints.

Since pessimists are unconstrained, their optimality conditions are unchanged. It follows that the equilibrium takes the form in Section 5 with the difference that investors' beliefs are replaced by their as-if beliefs, $\lambda^{i,pl}_s$. This verifies that the planner can implement the policy using the portfolio restrictions in $(A.29)$.

We next show that the portfolio constraints in $(A.29)$ can be relaxed to the following inequality restrictions,

$$\begin{aligned}\omega^{k,o,pl}_{t,s} &\leq 1 \text{ for each } s, \\ \omega^{2,o,pl}_{t,1} &\geq \underline{\omega}^{2,o}_{t,1} \equiv \lambda^{o,pl}_1 - \overline{\lambda}^{pl}_{t,1} \text{ and } \omega^{1,o,pl}_{t,2} \leq \overline{\omega}^{1,o}_{t,2} \equiv \lambda^{o,pl}_2 - \overline{\lambda}^{pl}_{t,2}.\end{aligned} \tag{A.31}$$

In particular, we will establish that all inequality constraints bind, which implies that optimists optimally choose the portfolio weights in Eq. $(A.29)$. Thus, our earlier analysis continues to apply when optimists are subject to the more relaxed constraints in $(A.31)$.

The result follows from the assumption that the planner-induced beliefs are more pessimistic than optimists' actual beliefs, $\lambda^{o,pl}_1 \geq \lambda^o_1$ and $\lambda^{o,pl}_2 \leq \lambda^o_2$. To see this formally, note that the optimality condition for capital is given by the following generalization of Eq. (17),

$$\omega^{k,o,pl}_{t,s}\sigma_s \leq \frac{1}{\sigma_s}\left(r^k_{t,s} - r^f_{t,s} + \lambda^o_s\frac{a^o_{t,s}}{a^o_{t,s'}}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}}\right) \text{ and } \omega^{k,o,pl}_{t,s} \leq 1, \tag{A.32}$$

with complementary slackness. Note also that,

$$\lambda^o_s\frac{a^o_{t,s}}{a^o_{t,s'}}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s}} = \lambda^o_s\frac{\overline{\lambda}^{pl}_{t,s}}{\lambda^{o,pl}_s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s'}} \geq \overline{\lambda}^{pl}_{t,s}\frac{Q_{t,s'}-Q_{t,s}}{Q_{t,s'}} \text{ for each } s.$$

Here, the equality follows from Eq. $(A.35)$ and the inequality follows by considering separately the two cases, $s \in \{1,2\}$. For $s = 2$, the inequality holds since $Q_{t,s'} - Q_{t,s} > 0$ and the beliefs satisfy, $\lambda^o_s \geq \lambda^{o,pl}_s$. For $s = 1$, the inequality holds since $Q_{t,s'} - Q_{t,s} < 0$ and the beliefs satisfy, $\lambda^{o,pl}_s \geq \lambda^o_s$. Note also that in equilibrium the return to capital satisfies the risk balance condition [cf. Eq. (33)],

$$\sigma_s = \frac{1}{\sigma_s}\left(r^k_{t,s} - r^f_{t,s} + \overline{\lambda}^{pl}_{t,s}\left(1 - \frac{Q_{t,s}}{Q_{t,s'}}\right)\right).$$

Combining these expressions implies, $\sigma_s \leq \frac{1}{\sigma_s}\left(r^k_{t,s} - r^f_{t,s} + \lambda^o_s \frac{a^o_{t,s}}{a^o_{t,s'}} \frac{Q_{t,s'} - Q_{t,s}}{Q_{t,s}}\right)$, which in turn implies the optimality condition $(A.32)$ is satisfied with $\omega^{k,o,pl}_{t,s} = 1$. A similar analysis shows that optimists also choose the corner allocations in contingent securities, $\omega^{2,o,pl}_{t,1} = \underline{\omega}^{2,o}_{t,1}$ and $\omega^{1,o,pl}_{t,2} = \overline{\omega}^{1,o}_{t,2}$, verifying that the portfolio constraints $(A.29)$ can be relaxed to the inequality constraints in $(A.31)$.

Simplifying the planner's problem. Recall that, to trace the Pareto frontier, we allow the planner to do a one-time wealth transfer among the agents at time 0. Let $V^i_{t,s}\left(a^i_{t,s} | \left\{\lambda^{o,pl}_t\right\}\right)$ denote type i investors' expected value in equilibrium when she starts with wealth $a^i_{t,s}$ and the planner commits to implement the policy, $\left\{\lambda^{o,pl}_t\right\}$. Then, the planner's Pareto problem can be written as,

$$\max_{\tilde{\lambda}^{o,pl}, \tilde{\alpha}_{0,s}} \gamma^o V^o_{0,s}\left(\tilde{\alpha}_{0,s} Q_{0,s} k_{0,s} | \tilde{\lambda}^{o,pl}\right) + \gamma^p V^p_{0,s}\left(\left(1 - \tilde{\alpha}_{0,s}\right) Q_{0,s} k_{0,s} | \tilde{\lambda}^{o,pl}\right). \tag{A.33}$$

Here, $\gamma^o, \gamma^p \geq 0$ (with at least one strict inequality) denote the Pareto weights, and $Q_{0,s}$ denotes the endogenous equilibrium price that obtains under the planner's policy.

Next recall that the investors' value function with macroprudential policy has the same functional form in (41) (with potentially different $\xi^o_{t,s}, v^o_{t,s}$ for optimists that reflect the constraints. After substituting $a^i_{t,s} = \alpha^i_{t,s} k_{t,s} Q_{t,s}$, the functional form implies,

$$V^i_{t,s} = v^i_{t,s} + \frac{\log\left(\alpha^i_{t,s}\right) + \log\left(k_{t,s}\right)}{\rho}.$$

Using this expression, the planner's problem $(A.33)$ can be rewritten as,

$$\max_{\tilde{\lambda}^{o,pl}, \tilde{\alpha}_{0,s}} \left(\gamma^o v^o_{0,s} + \gamma^p v^p_{0,s}\right) + \frac{\gamma^o \log\left(\tilde{\alpha}^o_{0,s}\right) + \gamma^p \log\left(1 - \tilde{\alpha}^o_{0,s}\right)}{\rho} + \frac{\left(\gamma^o + \gamma^p\right)\log\left(k_{0,s}\right)}{\rho}.$$

Here, the last term (that features capital) is a constant that doesn't affect optimization. The second term links the planner's choice of wealth redistribution, $\alpha^o_{0,s}, \alpha^p_{0,s}$, to her Pareto weights, γ^o, γ^p. Specifically, the first order condition with respect to optimists' wealth share implies $\frac{\gamma^o}{\gamma^p} = \frac{\alpha_{0,s}}{1-\alpha_{0,s}}$. Thus, the planner effectively maximizes the first term after substituting γ^o and γ^p respectively with the optimal choice of $\alpha_{0,s}$ and $1 - \alpha_{0,s}$. This leads to the simplified problem (45) in the main text.

Characterizing the value functions with macroprudential policy. We first show that the normalized value functions, $v^i_{t,s}$, are characterized as the solution to the following differential equation system,

$$\rho v^i_{t,s} - \frac{\partial v^i_{t,s}}{\partial t} = \log\rho + q_{t,s} + \frac{1}{\rho}\left(\begin{array}{c} \psi q_{t,s} - \delta - \frac{1}{2}\sigma^2_s \\ -\left(\lambda^{i,pl}_s - \overline{\lambda}^{pl}_{t,s}\right) + \lambda^i_s \log\left(\frac{\lambda^{i,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\right) \end{array}\right) + \lambda^i_s\left(v^i_{t,s'} - v^i_{t,s}\right). \tag{A.34}$$

This is a generalization of Eq. (42) in which investors' positions are calculated according to their as-if beliefs, $\lambda^{i,pl}_s$, but the transition probabilities are calculated according to their actual beliefs, λ^i_s.

First consider the pessimists. Since they are unconstrained, their value function is characterized by solving the earlier equation system $(A.30)$. In this case, equation $(A.34)$ also holds since it is the same as the earlier equation.

Next consider the optimists. In this case, the analysis in Section 5 and Appendix A.4 applies with as-if beliefs. In particular, we have,

$$\frac{a^o_{t,s'}}{a^o_{t,s}} = \frac{\alpha_{t,s'}}{\alpha_{t,s}}\frac{Q_{t,s'}}{Q_{t,s}} = \frac{\lambda^{o,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\frac{Q_{t,s'}}{Q_{t,s}}. \tag{A.35}$$

Plugging this expression as well as Eq. $(A.29)$ into Eq. $(A.30)$, optimists' unit-wealth value function satisfies,

$$\begin{aligned}\xi^o_{t,s} &= \log\rho + \frac{1}{\rho}\left(\begin{array}{c} r^k_{t,s} - \rho - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda^{o,pl}_s - \overline{\lambda}^{pl}_{t,s}\right) + \lambda^o_s \log\left(\frac{\lambda^{o,pl}_{t,s}}{\overline{\lambda}^{pl}_{t,s}}\right)\end{array}\right) \\ &\quad + \frac{\partial \xi^o_{t,s}}{\partial t} + \lambda^o_s\left(\frac{1}{\rho}\log\left(\frac{Q_{t,s'}}{Q_{t,s}}\right) + \xi^o_{t,s'} - \xi^o_{t,s}\right),\end{aligned}$$

This is the same as Eq. $(A.30)$ with the difference that the as-if beliefs, $\lambda^{o,pl}_s$, are used to calculate their positions on (and the payoffs from) the contingent securities, whereas the actual beliefs, λ^o_s, are used to calculate the transition probabilities. Using the same steps after Eq. $(A.30)$, we also obtain $(A.34)$ with $i = o$.

We next characterize the first-best and the gap value functions, $v^{i,*}_{t,s}$ and $w^i_{t,s}$, that we use in the main text. By definition, the first-best value function solves the same differential equation $(A.34)$ after substituting $q_{t,s} = q^*$. It follows that the gap value function $w^i_{t,s} = v^i_{t,s} - v^{i,*}_{t,s}$, solves,

$$\rho w^i_{t,s} - \frac{\partial w^i_{t,s}}{\partial t} = \left(1 + \frac{\psi}{\rho}\right)(q_{t,s} - q^*) + \lambda^i_s\left(w^i_{t,s'} - w^i_{t,s}\right),$$

which is the same as the differential equation (44) without macroprudential policy. The latter affects the path of prices, $q_{t,s}$, but it does not affect how these prices translate into gap values.

Note also that, as before, the value functions can be written as functions of optimists' wealth share, $\left\{v^i_s(\alpha), v^{i,*}_s(\alpha), w_s(\alpha)\right\}_{s,i}$. For completeness, we also characterize the differential equations that these functions satisfy in equilibrium with macroprudential policy. Combining Eq. $(A.34)$ with the evolution of optimists' wealth share conditional on no transition, $\dot{\alpha}_{t,s} = -\left(\lambda^{o,pl}_s - \lambda^p_s\right)\alpha_{t,s}(1-\alpha_{t,s})$, the value functions, $\left(v^i_s(\alpha)\right)_{s,i}$, are found by solving,

$$\rho v^i_s(\alpha) = \left[\begin{array}{c} \log\rho + q_s(\alpha) + \frac{1}{\rho}\left(\begin{array}{c}\psi q_s(\alpha) - \delta - \frac{1}{2}\sigma_s^2 \\ -\left(\lambda^{i,pl}_s - \overline{\lambda}^{pl}_{t,s}\right) + \lambda^i_s\log\left(\frac{\lambda^{i,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\right)\end{array}\right) \\ -\frac{\partial v^i_s}{\partial\alpha}\left(\lambda^{o,pl}_s - \lambda^p_s\right)\alpha(1-\alpha) + \lambda^i_s\left(v^i_{s'}\left(\alpha\frac{\lambda^{o,pl}_s}{\overline{\lambda}^{pl}_{t,s}}\right) - v^i_s(\alpha)\right)\end{array}\right],$$

with appropriate boundary conditions. Likewise, the first-best value functions, $\left(v^{i,*}_s(\alpha)\right)_{s\in\{1,2\}}$, are found by solving the analogous system after replacing $q_s(\alpha)$ with q^*. Finally, combining Eq. (44) with the evolution of optimists' wealth share, the gap-value functions, $\left(w^i_s(\alpha)\right)_{s,i}$, are found by solving Eq. (47) in the main text.

Proof of Proposition 3. For this and the next proof, we find it useful to work with the transformed

state variable,

$$b_{t,s} = \log\left(\frac{\alpha_{t,s}}{1-\alpha_{t,s}}\right), \text{ which implies } \alpha_{t,s} = \frac{1}{1+\exp\left(-b_{t,s}\right)}. \tag{A.36}$$

The variable, $b_{t,s}$, varies between $(-\infty, \infty)$ and provides a different measure of optimism, which we refer to as "bullishness." Note that there is a one-to-one relation between optimists' wealth share, $\alpha_{t,s} \in (0,1)$, and the bullishness, $b_{t,s} \in \mathbb{R} = (-\infty, +\infty)$. Optimists' wealth dynamics in (35) become particularly simple when expressed in terms of the bullishness,

$$\begin{cases} \dot{b}_{t,s} = -\left(\lambda_s^{o,pl} - \lambda_s^p\right), & \text{if there is no state change,} \\ b_{t,s'} = b_{t,s} + \log\lambda_s^{o,pl} - \log\lambda_s^p, & \text{if there is a state change.} \end{cases} \tag{A.37}$$

With a slight abuse of notation, we also let $q_s(b)$ and $w_s^i(b)$ denote respectively the price function and the gap value function in terms of the bullishness.

Note also that, since $\frac{db}{d\alpha} = \frac{1}{\alpha(1-\alpha)}$, we have the identities,

$$\frac{\partial q_2(b)}{\partial b} = \alpha(1-\alpha)\frac{\partial q_2(\alpha)}{\partial b} \text{ and } \frac{\partial w_s^i(b)}{\partial b} = \alpha(1-\alpha)\frac{\partial w_s^i(\alpha)}{\partial \alpha}. \tag{A.38}$$

Using this observation, the differential equation for the price function, Eq. (38), can be written in terms of bullishness as,

$$\frac{\partial q_2(b)}{\partial b}\left(\lambda_s^{o,pl} - \lambda_s^p\right) = \rho + \psi q_2 - \delta + \overline{\lambda}_2(\alpha)\left(1 - \frac{Q_2}{Q^*}\right) - \sigma_2^2. \tag{A.39}$$

Likewise, the differential equation for the gap value function, Eq. (47) can be written as bullishness,

$$\rho w_s^i(b) = \left(1+\frac{\psi}{\rho}\right)(q_s(b) - q^*) - \left(\lambda_s^{o,pl} - \lambda_s^p\right)\frac{\partial w_s^i(b)}{\partial b} + \lambda_s^i\left(w_{s'}^i(b') - w_s^i(b)\right). \tag{A.40}$$

We next turn to the proof. To establish the comparative statics of the gap value function, we first describe it as a fixed point of a contraction mapping. Recall that, in the time domain, the gap value function solves the HJB equation (44). Integrating this equation forward, we obtain,

$$w_s^i(b_{0,s}) = \int_0^\infty e^{-\left(\rho+\lambda_s^i\right)t}\left(\left(1+\frac{\psi}{\rho}\right)(q_s(b_{t,s}) - q^*) + \lambda_s^i w_{s'}^i(b_{t,s'})\right)dt, \tag{A.41}$$

for each $s \in \{1,2\}$ and $b_{0,s} \in \mathbb{R}$. Here, $b_{t,s}$ denotes the bullishness conditional on there not being a transition before time t, whereas $b_{t,s'}$ denotes the bullishness if there is a transition at time t. Solving Eq. $(A.37)$ (given as-if beliefs, $\lambda^{i,pl}$) we further obtain,

$$\begin{aligned} b_{t,s} &= b_{0,s} - t\left(\lambda_s^{o,pl} - \lambda_s^p\right), \\ b_{t,s'} &= b_{0,s} - t\left(\lambda_s^{o,pl} - \lambda_s^p\right) + \log\lambda_s^{o,pl} - \log\lambda_s^p. \end{aligned} \tag{A.42}$$

Hence, Eq. $(A.41)$ describes the value function as a solution to an integral equation given the closed form solution for bullishness in $(A.42)$.

Let $B\left(\mathbb{R}^2\right)$ denote the set of bounded value functions over $\mathbb{R}^2$. Given some continuation value

function, $\left(\tilde{w}_s^i(b)\right)_s \in B\left(\mathbb{R}^2\right)$, we define the function, $\left(T\tilde{w}_s^i(b)\right)_s \in B\left(\mathbb{R}^2\right)$, so that

$$T\tilde{w}_s^i(b_{0,s}) = \int_0^\infty e^{-(\rho+\lambda_s^i)t}\left(\left(1+\frac{\psi}{\rho}\right)(q_s(b_{t,s})-q^*)+\lambda_s^i\tilde{w}_{s'}^i(b_{t,s'})\right)dt, \tag{A.43}$$

for each s and $b_{0,s} \in \mathbb{R}$. Note that the resulting value function is bounded since the price function, $q_s(b_{t,s})$, is bounded (in particular, it lies between q_p and q^*). It can be checked that operator T is a contraction mapping with respect to the sup norm. In particular, it has a fixed point, which corresponds to the gap value function, $\left(w_s^i(b)\right)_s$.

We next show that the value function has strictly positive derivative with respect to bullishness or optimism. To this end, we first note that the value function is differentiable since it solves the differential equation (47). Next, we implicitly differentiate the integral equation $(A.41)$ with respect to $b_{0,s}$, and use Eq. $(A.42)$, to obtain,

$$\frac{\partial w_s^i(b_{0,s})}{\partial b} = \int_0^\infty e^{-(\rho+\lambda_s^i)t}\left(\left(1+\frac{\psi}{\rho}\right)\frac{\partial q_s(b_{t,s})}{\partial b}+\lambda_s^i\frac{\partial w_{s'}^i(b_{t,s'})}{\partial b}\right)dt. \tag{A.44}$$

Note from Eq. $(A.39)$ that the derivative of the price function, $\frac{\partial q_s(b)}{\partial b}$, is bounded. Thus, Eq. $(A.44)$ describes the derivative of the value function, $\frac{\partial w_s^i(b_{0,s})}{\partial b}$, as a fixed point of a corresponding operator T^{∂_b} over bounded functions (which is related to but different than the earlier operator, T). This operator is also a contraction mapping with respect to the sup norm. Since $\frac{\partial q_s(b_{t,s})}{\partial b} > 0$ for each b, and $\lambda_s^i > 0$ for each s, it can further be seen that the fixed point satisfies, $\frac{\partial w_s^i(b_{0,s})}{\partial b} > 0$ for each b and $s \in \{1,2\}$. Using Eq. $(A.38)$, we also obtain $\frac{\partial w_s^i(\alpha)}{\partial \alpha} > 0$ for each $\alpha \in (0,1)$ and $s \in \{1,2\}$.

Next consider the comparative statics of the fixed point with respect to macroprudential policy. We implicitly differentiate the integral equation $(A.41)$ with respect to $\lambda_1^{o,pl}$, and use Eq. $(A.42)$, to obtain,

$$\begin{aligned}
\frac{\partial w_1^i(b_{0,1})}{\partial \lambda_1^{o,pl}} &= \int_0^\infty e^{-(\rho+\lambda_1^i)t}\lambda_1^i\left(\frac{\partial w_2^i(b_{t,2})}{\partial \lambda_1^{o,pl}}+\frac{\partial w_2^i(b_{t,2})}{\partial b}\frac{db_{t,2}}{d\lambda_1^{o,pl}}\right)dt,\\
\frac{\partial w_2^i(b_{0,2})}{\partial \lambda_1^{o,pl}} &= \int_0^\infty e^{-(\rho+\lambda_1^i)t}\lambda_2^i\frac{\partial w_1^i(b_{t,1})}{\partial \lambda_1^{o,pl}}dt.
\end{aligned}$$

Note also that, using Eq. $(A.42)$ implies, $\frac{db_{t,2}}{d\lambda_1^{o,pl}} = -t + \frac{1}{\lambda_1^{o,pl}}$. Plugging this into the previous system, and evaluating the partial derivatives at $\lambda_1^{o,pl} = \lambda_1$, we obtain,

$$\begin{aligned}
\frac{\partial w_1^i(b_{0,1})}{\partial \lambda_1^{o,pl}} &= h(b_{0,1}) + \int_0^\infty e^{-(\rho+\lambda_1^i)t}\lambda_1^i\frac{\partial w_2^i(b_{t,2})}{\partial \lambda_1^{o,pl}}dt, \qquad (A.45)\\
\frac{\partial w_2^i(b_{0,2})}{\partial \lambda_1^{o,pl}} &= \int_0^\infty e^{-(\rho+\lambda_1^i)t}\lambda_2^i\frac{\partial w_1^i(b_{t,1})}{\partial \lambda_1^{o,pl}}dt,\\
\text{where } h(b_{0,1}) &= \int_0^\infty e^{-(\rho+\lambda_1^i)t}\lambda_1^i\frac{\partial w_2^i(b_{t,2})}{\partial b}\left(-t+\frac{1}{\lambda_1^o}\right)dt.
\end{aligned}$$

Note that the function, $h(b)$, is bounded since the derivative function, $\frac{\partial w_2^i(b)}{\partial b}$, is bounded (see $(A.44)$). Hence, Eq. $(A.45)$ describes the partial derivative functions, $\left(\frac{\partial w_s^i(b)}{\partial \lambda_1^{o,pl}}|_{\lambda_1^{o,pl}=\lambda_1^o}\right)_s$, as a fixed point of a corresponding operator T^{∂_λ} over bounded functions (which is related to but different than the earlier

operator, T). Since $h(b)$ is bounded, it can be checked that the operator T^{∂_λ} is also a contraction mapping with respect to the sup norm. In particular, it has a fixed point, which corresponds to the partial derivative functions.

The analysis so far applies generally. We next consider the special case, $\lambda_1^o = \lambda_1^p$, and show that it implies the partial derivatives are strictly positive. In this case, $\lambda_1^i = \lambda_1$ for each $i \in \{o, p\}$. In addition, Eq. $(A.42)$ implies $b_{t,2} = b_{0,2}$. Using these observations, for each $b_{0,1}$, we have,

$$\begin{aligned} h(b_{0,1}) &= \frac{\partial w_2^i(b_{0,2})}{\partial b} \int_0^\infty e^{-(\rho+\lambda_1)t} \lambda_1 \left(-t + \frac{1}{\lambda_1}\right) dt \\ &= \frac{\partial w_2^i(b_{0,2})}{\partial b} \left(-\frac{\lambda_1}{\rho+\lambda_1}\frac{1}{\rho+\lambda_1} + \frac{1}{\rho+\lambda_1}\right) > 0. \end{aligned}$$

Here, the inequality follows from our earlier result that $\frac{\partial w_2^i(b_{0,2})}{\partial b} > 0$. Since $h(b) > 0$ for each b, and $\lambda_s^i > 0$, it can further be seen that the fixed point that solves $(A.45)$ satisfies $\frac{\partial w_s^i(b)}{\partial \lambda_s^{o,pl}} > 0$ for each b and $s \in \{1, 2\}$. Using Eq. $(A.38)$, we also obtain $\frac{\partial w_s^i(\alpha)}{\partial \lambda_1^{o,pl}} > 0$ for each $\alpha \in (0, 1)$ and $s \in \{1, 2\}$. □

Proof of Proposition 4. A similar analysis as in the proof of Proposition 3 implies that the partial derivative function, $\frac{\partial w_s^i(b)}{\partial\left(-\lambda_2^{o,pl}\right)}$, is characterized as the fixed point of a contraction mapping over bounded functions (the analogue of Eq. $(A.45)$ for state 2). In particular, the partial derivative exists and it is bounded. Moreover, since the corresponding contraction mapping takes continuous functions into continuous functions, the partial derivative is also continuous over $b \in \mathbb{R}$. Using Eq. $(A.38)$, we further obtain that the partial derivative, $\frac{\partial w_s^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}$, is continuous over $\alpha \in (0, 1)$.

Next note that $w_s^i(1) \equiv \lim_{\alpha\to 1} w_s^i(\alpha)$ exists and is equal to the value function according to type i beliefs when all investors are optimistic. In particular, the asset prices are given by $q_1 = q^*$ and $q_2 = q^o$, and the transition probabilities are evaluated according to type i beliefs. Then, following the same steps as in our analysis of value functions in Appendix A.5, we obtain,

$$\begin{aligned} w_s^i(1) &= \left(1 + \frac{\psi}{\rho}\right)\left(\beta_s^i q_s^o + \left(1 - \beta_s^i\right) q_{s'}^o - q^*\right), \\ \text{where } \beta_s^i &= \frac{\rho + \lambda_{s'}^i}{\rho + \lambda_{s'}^i + \lambda_s^i}. \end{aligned}$$

Here, β_s^i denotes the expected discount time the investor spends in state s according to type i beliefs. We consider this equation for $s = 2$ and take the derivative with respect to $\left(-\lambda_2^{o,pl}\right)$ to obtain,

$$\frac{\partial w_s^i(1)}{\partial\left(-\lambda_2^{o,pl}\right)} = \left(1 + \frac{\psi}{\rho}\right)\beta_s^2 \frac{dq_s^o}{d\left(-\lambda_2^{o,pl}\right)} < 0.$$

Here, the inequality follows since reducing optimists' optimism reduces the price level in the common belief benchmark (see Section 4).

Note that the inequality, $\frac{\partial w_s^i(1)}{\partial\left(-\lambda_2^{o,pl}\right)} < 0$, holds for each state s and each belief type i. Using the continuity of the partial derivative function, $\frac{\partial w_s^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}$, we conclude that there exists $\overline{\alpha}$ such that $\left.\frac{\partial w_s^i(\alpha)}{\partial\left(-\lambda_2^{o,pl}\right)}\right|_{\lambda^{o,pl}=\lambda^o} < 0$ for each i, s and $\alpha \in (\overline{\alpha}, 1)$, completing the proof. □