A Structural Investigation of Monetary Policy Shifts*

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Abstract

This paper investigates the macroeconomic sources of the U.S. monetary policy shifts within a dynamic stochastic general equilibrium (DSGE) model. We introduce a threshold-type regime switching into the model that links the current state of monetary policy to the historical fundamental shocks by an autoregressive latent factor. This generates an important endogenous feedback mechanism between measured economic variables and the monetary policy stance. Estimation framework involves approximation of likelihood by our newly developed Kalman filter with Markov switching. Results show that Post-World War II U.S. monetary policy regime change is 76% driven by past monetary policy interventions. The extracted regime factor indicates that monetary policy is identified by repeated fluctuations between a more aggressive and a less aggressive regime, with the latter prevailing in the 1970s and during the recent crisis. To explore the policy implications of this endogenous feedback mechanism, we analyze the expectations formation effects for various scenarios of central bank's policy interventions and find that small but sustained interventions, comparing to large but short-lived ones, are more likely to shift agents' beliefs about policy regime and induce the changes in behavior that Lucas (1976) emphasizes.

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1 Introduction

Monetary policy behavior is purposeful and responds endogenously to the state of the economy. Empirical work by Clarida et al. (1998), Lubik and Schorfheide (2004) and Sims and Zha (2006) find that Taylor rule describing purposeful monetary policy behavior displays important time variation in the United States. Markov switching processes are embedded in DSGE models to explore these empirical findings. However, these work assume the monetary policy regime shifts as exogenous, and therefore in an important sense the work is inconsistent with a central tenet underlying the Taylor rule: monetary policy behavior is purposeful and reacts systematically to changes in the macroeconomic environment. For instance, most people who think that policy changed dramatically in late 1979 in the United States believe that it did so because inflation appeared to be running out of control, not because an exogenously evolving switching process happened to call for a change at that time. This belief calls for a model that makes the policy change as a purposeful response of central bank to the state of the economy. Davig and Leeper (2008) took a step toward resolving this inconsistency by building a new Keynesian model with threshold switching monetary policy. More specifically, policy regime switches when past inflation crosses a threshold value. As a practical matter, threshold model is more relevant to describing monetary policy rule than a time-varying transition probability model – an alternative method used by Diebold et al. (1994), Filardo (1994), and Filardo and Gordon (1998) to model endogenous regime switching. Our paper adopts their threshold switching setup of monetary policy rule, but introduces a different and novel approach of endogenous regime switching.

We use an approach first introduced by Chang et al. (2017) to model endogenous regime switching monetary policy, where an autoregressive latent factor determines regimes depending upon whether it takes a value above or below some threshold level. The endogeneity is captured by the feedback mechanism in model structure. First, federal funds rates systematically adjust to target variables consisting of inflations and output gaps, and policy intervention that reflects how federal funds rate reacts to non-target information. Second, the coefficient on inflation is an indicator function of the policy regime factor whose dynamic evolution depends on both past policy intervention and an exogenous idiosyncratic shock[1]. For example, if today the Fed sets the federal funds rate above the level implied by Taylor rule, this positive (contractionary) intervention predicts tomorrow's latent regime factor and hence policy regime through the feedback mechanism. Two economic effects arise from such a monetary policy intervention. First, a higher value of the federal funds rate is realized by the direct effects. Second, because the intervention carries information about future realizations of the systematic reactions of federal funds rate to policy targets, private agents who observe the policy intervention would adjust their expectations of future policy regime evolutions, which induces the expectations formation effects.

Figure 1 nicely motivates the aforementioned endogenous feedback mechanism. The upper subplot displays the actual federal funds rate and Taylor rule implied rate. The Taylor rule pro-

[1] See Section 7.1 for an important extension where we allow the policy regime factor to depend on both past policy intervention and all other non-policy structural shocks, and an exogenous idiosyncratic shock.

vides a fairly accurate summary[2] of post-World War II U.S. monetary policy. However, there exists several systematic, persistent discrepancies. Those are the policy interventions reflecting policy considerations of non-target information, calculated by subtracting Taylor rule implied rate from the actual federal funds rate. The shaded periods capture less aggressive regime from the estimation of our monetary DSGE model with endogenous feedback mechanism. Very intriguingly, we see that in the early 1980s, monetary policy was too tight comparing to Taylor rule, that is, the Fed was implementing contractionary policy intervention. Private agents observing the persistent contractionary intervention would infer that inflation control has become the Fed's top priority and gradually reinforce their belief of a switch from less aggressive regime to more aggressive regime. The adjustment of agents' belief affects their behavior and induce the expectations formation effects on the economy, which helps stabilize price levels in the 1980s.

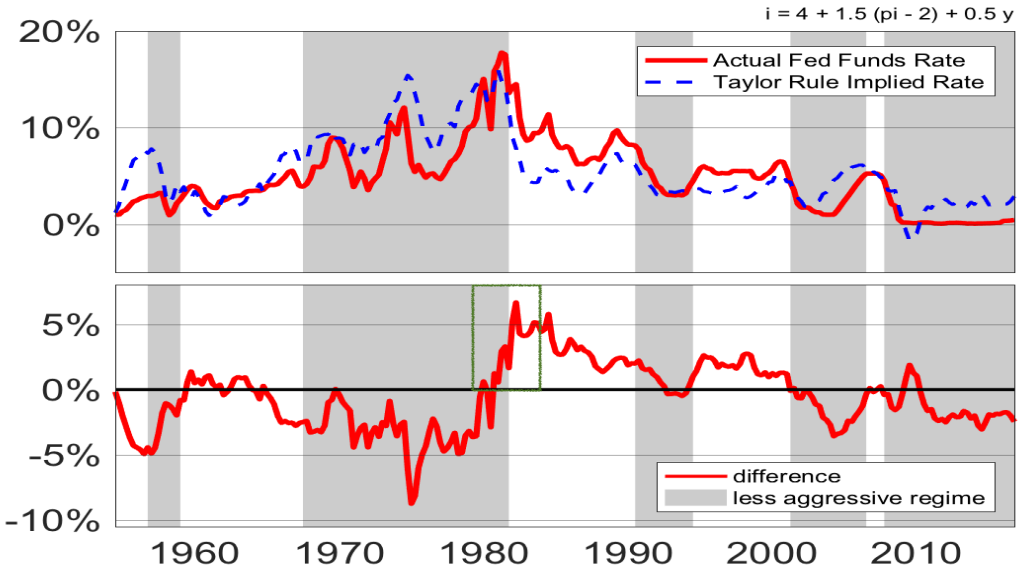


Figure 1: Relationship between monetary policy intervention and policy regime

In our model, the endogenous regime switching is captured by the correlation between current regime factor and past policy intervention. When the correlation is zero, endogenous feedback mechanism is turned off and our model reduces to the conventional exogenous regime switching monetary DSGE model. The correlation parameter explains that to what degree regime change is driven by past monetary policy intervention. Our estimation shows that monetary policy regime change is 76% triggered by past monetary policy interventions, while 24% by contemporaneous idiosyncratic shock. Our numerical experiments also show that if it is more triggered by past monetary policy interventions, the expectations formation effects become larger. This is because agents can utilize more information to form their expectations on future regime evolutions. These

[2] Data and parameters used to construct the Taylor rule are the same as in Taylor (1993), which are shown on the top-right of the plot: i denotes net federal funds rate, pi inflation and y output gap.

findings reinforce the claim by Leeper and Zha (2003) that small and sustained policy intervention may significantly shift agents' beliefs about policy regime and induce the changes in behavior that Lucas (1976) emphasizes.

Besides this substantive finding, we also make two methodological contributions. First, we bring the advanced reduced-form endogenous regime switching approach to the widely used structural models – rational expectations models – by macroeconomists. Prior to this paper, endogenous regime switching is studied separately by econometricians and macroeconomists. From an econometric standpoint, following the seminal paper by Hamilton (1989), Filardo (1994) and Filardo and Gordon (1998) have estimated Markov switching regressions with time-varying transition probabilities. Kim et al. (2008) consider a regression model where the state dependent coefficients are determined by an i.i.d. latent random variable that is contemporaneously correlated with innovation in regression error. More recently, Chang et al. (2017) introduces a less restrictive model with endogenous regime switching which is driven by an autoregressive latent factor correlated with the innovation to the observed time series. On the other hand, from an macroeconomic standpoint, solution methods for endogenous regime switching rational expectations models have only been available recently in Maih (2015), Foerster et al. (2016) and Barthélemy and Marx (2017). Especially, Barthélemy and Marx (2017) apply an implicit function theorem to numerically ensure both determinacy and accuracy of the approximate solution. However, those significant progresses on reduced-form endogenous regime switching literature haven't been introduced yet to construct and estimate rational expectations models with endogenous regime switching. Our paper contributes to advance in such a direction.

Second, we develop an approximate Kalman filter with Markov switching that, in conjunction with Barthélemy and Marx (2017)'s solution method, delivers an efficient way to estimate endogenous regime-switching DSGE models, either by classical or Bayesian approach. Our filter is not simulation-based, and hence more efficient than particle filter – an alternative way to estimate regime-switching DSGE models in literature. It is deterministic and allows stable numerical approximation of the likelihood function of state space representation with conditionally linear Gaussian structure. The latent regime factor can be readily extracted by using our filter. Furthermore, in order to compare model fit of endogenous regime switching DSGE model with exogenous regime switching and fixed-regime DSGE models, we develop a Bayesian Markov chain Monte Carlo scheme to simulate the latent regime factor as well as all DSGE parameters.

The remainder of the paper is organized as follows. Section 2 summarizes the relation to the literature. We discuss specifications, motivations, implications and limitations for different monetary policies – fixed regime, exogenous and endogenous switching rules. We claim that the endogenous regime switching in Chang et al. (2017) suits better than conventional time-varying transition probabilities model in the context of monetary policy switching. Analytical solution to a regime-switching Fisherian model with endogenous feedback mechanism is derived in Section 3. Section 4 embeds the endogenous feedback mechanism into a prototypical monetary DSGE model. In parallel with Section 2, solution methods for monetary DSGE models with three different setups of monetary policy rules are discussed in Section 5. Section 6 discusses estimation strategy and shows empirical results. Section 7 provides a roadmap for our future work including several potentially important extensions, and concludes.

2 Relation to the literature

The original Taylor rule is state-contingent in the sense that the policy instrument, *i.e.* federal funds rate, adjusts to the state of the economy, where a fixed set of parameters govern the degree of adjustment. In an environment with endogenous regime switching, the policy rule is state-contingent in this conventional sense, but also in a broader sense. Namely, the parameters governing the degree of adjustment of the federal funds rate to target variables are themselves a function of the economic state.

To understand monetary policy shifts with endogenous feedback mechanism, we first review fixed regime and exogenous regime switching monetary policy processes.

2.1 Original Taylor rule

The Great Moderation which started in the mid-1980s raised the notable "good luck" or "good policy" debate. The evidence on this question is mixed. But the profession resolved this largely by trusting its priors that Great Moderation was due to improved policy. Among many other papers on the issue of how the monetary policy is conducted, Taylor (1993) rule has become the most well-known and accepted functional form for monetary policy. It suggests that monetary policy behavior is purposeful and reacts systematically to changes in the macroeconomic environment. Rare is the paper now that posits an exogenous process for money growth and claims to offer practical policy advice. Although the Federal Reserve does not explicitly follow the Taylor rule, many analysts have argued that the rule provides a fairly accurate summary of US monetary policy under Paul Volcker and Alan Greenspan.[3] It has been well studied that Taylor rule in a new Keynesian model delivers a strong case for price stability, and hence output stability. It's also intended to increase the credibility of future actions by the central bank.

Following An and Schorfheide (2007), Davig and Doh (2014) and Foerster et al. (2016), we consider a modified Taylor rule with fixed regime specification:

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho_r} \left[\left(\frac{\Pi_t}{\Pi^*}\right)^{\alpha} \left(\frac{Y_t}{Y_t^*}\right)^{\gamma}\right]^{1-\rho_r} e_t \tag{1}$$

where R_t is the gross nominal interest rate, $\Pi_t = P_t/P_{t-1}$ is the gross inflation. The parameters α and γ measure the long-run reaction to inflation gap and output gap. The parameter ρ_r captures the smoothing motive of interest rates. Π^* is the target gross inflation rate, which in equilibrium coincides with the steady-state gross inflation rate. Steady-state nominal gross interest rate $R^* = r^*\Pi^*$ where r^* is the steady-state real gross interest rate. Y_t^* is the level of output that would prevail in

[3] See, e.g., Bernanke and Mihov (1995) and Clarida et al. (1998). Bernanke and Mihov (1995) shows that treating the federal funds rate as the instrument of monetary policy, as opposed to a money supply aggregate, is a reasonable approximation for most periods since 1965, except for the brief period of non-borrowed reserves targeting (1980-82) under Paul Volcker. Further, Goodfriend (1991) argues that even under the period of official reserves targeting, the Federal Reserve had in mind an implicit target for the Funds rate.

the absence of nominal rigidities. Finally, the monetary policy shock e_t stands for the unsystematic monetary policy component, following a lognormal distribution $\ln e_t \sim i.i.d.\mathbb{N}(0, \sigma_e^2)$. Taking logarithm of equation (1) yields

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1-\rho_r)[\alpha \hat{\Pi}_t + \gamma \hat{Y}_t] + \hat{e}_t \tag{2}$$

where $\hat{e}_t = \ln e_t$ has a normal distribution. The hat denotes the log-deviation around the steady state or, in the case of output, from a trend path. This usual Taylor rule is state-contingent that the nominal interest rate adjusts to the past interest rate, inflation gap and output gap. These policy targets are themselves functions of the underlying states of the economy. However, the parameters (α, γ, ρ_r) governing the systematic adjustment of policy instrument are constant. All deviations of $\hat{R}_t$ from $\rho_r \hat{R}_{t-1} + (1-\rho_r)[\alpha \hat{\Pi}_t + \gamma \hat{Y}_t]$ are folded into the exogenous shock $\hat{e}_t$.

2.2 Monetary policy with exogenous regime switching

Many economists agree that monetary policy in the United States has been relatively well managed from the time Paul Volcker took over the helm. It is also generally agreed that monetary policy was not so well managed in the fifteen years or so prior to Volcker. Empirical works by Clarida et al. (1998), Lubik and Schorfheide (2004) and Sims and Zha (2006) [4] supported the important time variation of Taylor rule in the United States, which calls for a model where policy parameters of Taylor rule could potentially change.

Theoretically, this kind of model is also motivated by Lucas (1976) critique. If policy has changed in the past, based on the rational expectations assumption, it will be expected to change again in the future. A Markov switching model allows agents to take account of this possibility. In an exogenous Markov switching framework, Leeper and Zha (2003) distinguish two effects from exogenous shocks. *Direct effects* capture the usual impacts of shocks in a fixed regime model where agents treat the policy regime as forever fixed in the future. *Expectations formation effects* arise whenever agents' rational expectations of future regime change induce them to alter their expectations functions. Expectations formation effects are the difference between the impact of a shock when regime can change and the impact when regime is forever fixed. The fact that expectations formation effects arise from the changes in agents' behaviors lie at the heart of Lucas's critique.

The policy regime change has important empirical implications. Clarida et al. (1998), Lubik and Schorfheide (2004) find that the monetary policy apparently followed in the 1970s was one that, when embedded in a DSGE model, would imply nonuniqueness of the equilibrium and hence vulnerability of the economy to "sunspot" fluctuations of arbitrarily large size. Their estimated policy rule for the later period, on the other hand, implied no such indeterminacy. These results

[4] Clarida et al. (1998) and Lubik and Schorfheide (2004) estimate fixed regime monetary DSGE over subsamples of pre-Volcker and Volcker-Greenspan periods. Sims and Zha (2006) estimate a reduced-form exogenous regime-switching model with potential switching in variances of monetary policy disturbance and structural shocks and also monetary policy coefficients.

apparently provide an explanation of the volatile and rising inflation of the 1970s and of its subsequent decline.

An exogenously switching rule extends equation (1) to:

$$\frac{R_t}{R^*_{s_t}} = \left(\frac{R_{t-1}}{R^*_{s_{t-1}}}\right)^{\rho_r} \left[\left(\frac{\Pi_t}{\Pi^*_{s_t}}\right)^{\alpha(s_t)} \left(\frac{Y_t}{Y^*_t}\right)^{\gamma}\right]^{1-\rho_r} e_t \tag{3}$$

Taking logarithm yields

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1-\rho_r)[\alpha(s_t)\hat{\Pi}_t + \gamma \hat{Y}_t] + \hat{e}_t \tag{4}$$

With regime switching, the steady states $R^*_{s_t}$ and $\Pi^*_{s_t}$ are potentially state-dependent, which will be explained in detail in Section 4. We let the weight on inflation switch between two values depending on the regime, s_t, but not the other parameters γ and ρ_r, to study the implications of regime switching on price level stabilization. The regime $\{s_t\}$ is a discrete-valued random variable sequence that follows a two-state Markov chain with transition probability matrix as

$$P_s = \begin{bmatrix} p_{00} & 1-p_{00} \\ 1-p_{11} & p_{11} \end{bmatrix},$$

where $p_{ii} = Pr(s_{t+1} = i | s_t = i), i = 0, 1$. The active, or more aggressive, regime corresponds to $s_t = 1$ and the passive, or less aggressive, regime corresponds to $s_t = 0$. This labeling implies $\alpha(0) < \alpha(1)$.

Now monetary policy consists of two different rules of the form in equation (4), with an exogenous Markov chain governing the dynamic switching between the rules. This makes the policy rule rather than just the policy instrument (the interest rate) state-contingent. In both equations (2) and (4), the parameter α is given exogenously. The key difference between the two specifications is that equation (4) introduces a new source of exogenous disturbance (the Markov chain governing s_t) to the economy with important implications for expectations formation effect.

But, conceptually, the exogenous regime switching is still inconsistent with the main idea underlying the Taylor rule: monetary policy behavior is purposeful and reacts endogenously to the states of the economy. The exogeneity would imply that the policy change, which most economists believe helped stabilize the high inflation during the 1980s, was due to an exogenously evolving switching process that happened to call for a change at that date, but not due to an endogenous reaction of the Federal Reserve to the state of economy. This will immediately lead to a logical contradiction when people use the exogenous regime switching monetary policy to justify that "good policy" is the cause for the Great Moderation, because what "good policy" do in this kind of model is nothing but waiting for the policy rule to be switched by an exogenous Markov process.

2.3 Monetary policy with endogenous regime switching

Most people who think that policy changed dramatically in late 1979 in the United States believe that it did so because inflation appeared to be running out of control, not because an exogenously

evolving switching process happened to call for a change at that date. This belief calls for a model that makes the policy change as a purposeful response of central bank to endogenous variables (e.g. inflation) in the economy.

Endogenous switching shares the feature of quantitatively important expectations formation effects with exogenous switching, but in a more realistic way. If monetary policy switches endogenously, as a response to the state of economy, between a more active and a less active reaction against inflation, agents' expectations and therefore the equilibrium outcomes always reflect the possibility that central bank can switch the policy regime in the future. The policy regime switches when the central bank's target variable cross specified threshold. The target variable is regarded as internal information to central bank, and unobservable to private agents. In this sense, a regime switching model with endogenous latent factor by Chang et al. (2017) suits perfectly to the application of monetary policy switching.

First, in an endogenous regime switching monetary DSGE model, transition probabilities are time varying which governs the evolution of regime strength. It is important to analyze how this additional information to agents would affect their expectations formation and hence macroeconomics dynamics. Second, a better understanding on why policy regime has shifted permits a better forecast of future regime evolutions. Third, it is also potentially important to study the selection effect on macroeconomic dynamics of central bank's selection on regimes.

An endogenous switching rule shares the same functional form with equations (3) and (4):

$$\frac{R_t}{R^*_{s_t}} = \left(\frac{R_{t-1}}{R^*_{s_{t-1}}}\right)^{\rho_r} \left[\left(\frac{\Pi_t}{\Pi^*_{s_t}}\right)^{\alpha(s_t)} \left(\frac{Y_t}{Y^*_t}\right)^{\gamma}\right]^{1-\rho_r} e_t$$

or taking logarithm,

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1-\rho_r)[\alpha(s_t)\hat{\Pi}_t + \gamma \hat{Y}_t] + \hat{e}_t$$

In the case of no persistency appears in the monetary policy shock e_t, *i.e.* $\hat{e}_t = \ln e_{t+1} = \sigma_e \varepsilon^e_{t+1}$, we can also write

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1-\rho_r)[\alpha(s_t)\hat{\Pi}_t + \gamma \hat{Y}_t] + \sigma_e \varepsilon^e_t \tag{5}$$

But now, the regime $\{s_t\}$ is not governed by an exogenous stochastic process. Instead, it is determined by a latent regime factor $\{w_t\}$ crossing an estimated threshold τ. Following Chang and Kwak (2016), we specify regime s_{t+1} as

$$s_{t+1} = \mathbb{1}\{w_{t+1} \geq \tau\} \tag{6}$$

The regime factor w_{t+1} follows an autoregressive process:

$$w_{t+1} = \phi w_t + v_{t+1} \tag{7}$$

where ϕ governs the degree of persistency in regime changes. Monetary policy intervention ε^e_t and next period shock v_{t+1} to regime factor are assumed to be jointly i.i.d. normal as

$$\begin{pmatrix} \varepsilon^e_t \\ v_{t+1} \end{pmatrix} =_d \mathbb{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \tag{8}$$

Referring to equation (5), we may view monetary policy intervention ε_t^e as a portion of monetary policy instrument, federal funds rate $\hat{R}_t$, that is not explained by all monetary policy target variables, $\hat{R}_{t-1}, \hat{\Pi}_t$ and $\hat{Y}_t$. Hence, ε_t^e represents all other factors that may affect monetary policy decision but not measured by those target variables. Central banks may take account of different economic conditions including commodity prices, sluggish labor market development, stock market and stance of fiscal policy for a monetary policy decision with the occasion. This monetary policy intervention then explicitly causes expectations formation effect through the feedback mechanism. A portion of the monetary policy intervention ε_t^e incurred today will affect the change in tomorrow's policy choice $\alpha(s_{t+1})$ through its endogeneity with the shock v_{t+1} to the regime factor w_{t+1} that determines monetary policy regime s_{t+1}. Further, the impact to the monetary policy regime will alter private agents' expectations of future macro dynamics.

3 A regime-switching Fisherian model with endogenous feedback

In this section, we introduce the aforementioned endogenous feedback mechanism into a regime-switching Fisherian model. Analytical solution is derived and macroeconomic dynamics are studied.

3.1 The model

Following Davig and Leeper (2008), we consider a regime-switching Fisherian model. This simple model of inflation determination combines a Fisher equation with an interest rate rule for monetary policy. The Fisher equation can be derived from a perfectly competitive endowment economy with flexible prices and a one period nominal bond.[5] A linearized asset pricing equation for the nominal bond is given by:

$$i_t = \mathbb{E}_t \pi_{t+1} + \mathbb{E}_t r_{t+1} \tag{9}$$

where r_t denotes the real rate at t. The real rate evolves exogenously according to:

$$r_t = \rho_r r_{t-1} + \sigma_r \epsilon_t^r \tag{10}$$

where $0 \leq \rho_r < 1$, and ϵ_t^r is an i.i.d. standard normal distribution.

We assume a monetary policy process that permits the monetary authority to vary its response to inflation, depending on the state of economy. For example, a monetary authority may respond systematically more aggressively when their observed policy regime factor exceeds a particular threshold. The monetary policy is given as:

$$i_t = \alpha(s_t)\pi_t + \sigma_e \epsilon_t^e, \quad \alpha_1 > \alpha_0 > 1 \tag{11}$$

where the regime is determined by

$$s_t = 1\{w_t \geq \tau\} \tag{12}$$

and policy regime factor follows an AR(1) process

$$w_{t+1} = \phi w_t + v_{t+1} \tag{13}$$

with

$$\begin{pmatrix} \epsilon_t^e \\ v_{t+1} \end{pmatrix} =_d \mathbb{N}\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \tag{14}$$

[5] Since we derive the Fisher equation from an optimization problem for agents, the expected real interest rate appears in the Euler equation, hence the Fisher equation. The real interest rate equals the intertemporal marginal rate of substitution and measures how expensive today's consumption is relative to tomorrow. Because it is an intertemporal notion, the expected interest rate term arises. In a conventional Fisher equation, expected real interest rate is replaced to real interest rate at time t.

Using properties of bivariate normal random variables, from (13) and (14), we may rewrite the law of motion of regime factor as[6]:

$$w_{t+1} = \phi w_t + \rho \epsilon_t^e + \eta_{t+1}, \quad \eta_{t+1} =_d \mathbb{N}(0, 1-\rho^2) \tag{16}$$

This representation clearly displays the endogenous feedback mechanism through regime factor w_{t+1} from past monetary policy intervention ϵ_t^e to regime selection s_{t+1}. The endogeneity parameter ρ captures how strong the feedback channel is. If $\rho = 0$, the feedback channel is turned off and our model reduces to the conventional exogenous Markov switching case.[7] In this case, policy makers would select regime based only on realizations of exogenous idiosyncratic shocks. If $\rho = 1$, future regimes become deterministic because next period policy regime is determined only by the current policy interventions. Policy makers don't consider any idiosyncratic shocks beyond the information embedded in the past policy interventions. In general, we expect $0 < |\rho| < 1$.

Further, by iterating w_T backwards in (16), we can rewrite:

$$w_{t+1} = \sum_{u=1}^{\infty} \phi^u (\rho \epsilon_{T-u}^e + \eta_{T-u+1}) + \rho \epsilon_t^e + \eta_{t+1} \tag{17}$$

where $\eta_s =_d \mathbb{N}(0, 1-\rho^2)$ for all s. We interpret parameter ϕ as "policy persistence" since it captures how much all past policy interventions and idiosyncratic shocks would affect the decision on the policy regime for next period. The extreme case $\phi = 0$ implies that policy makers totally ignore the information of all past policy interventions and idiosyncratic shocks when they select policy regime. On the other hand, if $\phi = 1$, then central bankers would put same weights on information across all past periods. In this case, policy becomes most persistent.

3.2 Analytical solution

The equilibrium conditions (9)∼(14) form a system of expectational nonlinear difference equations. Solving this system is tantamount to solving a fixed point problem. Suppose that agents come across a decision rule to evaluate π_{t+1} given current information set. Mathematically, it means that π_{t+1} will be integrated out using its conditional distribution. Then, agents can solve π_{t+1} with the knowledge of all model equations. If the resulting solution shares the same form with the initial guess (*i.e.* initial decision rule), then it is regarded the true solution to the original system. Otherwise, agents will use the resulting solution as an updated initial guess and follow the

[6] We project v_{t+1} onto the direction of ϵ_t^e and obtain:

$$\begin{aligned} v_{t+1} &= \frac{cov(v_{t+1}, \epsilon_t^e)}{var(\epsilon_t^e)} \epsilon_t^e + \eta_{t+1} \\ &= \rho \epsilon_t^e + \eta_{t+1} \end{aligned} \tag{15}$$

where $\eta_{t+1} =_d \mathbb{N}(0, 1-\rho^2)$. Actually, $\rho \epsilon_t^e$ is the linear least squares estimator of v_{t+1} and η_{t+1} the estimation error.

[7] Chang et al. (2017) proves that when $\rho = 0$, (ρ, ϕ) has one to one correspondence to constant transition probabilities $(\bar{p}_{00}, \bar{p}_{11})$ in a conventional exogenous switching model.

same procedure until convergence to a fixed point. This solution approach is often called guess and verify (or undetermined coefficients).

We start with a guess[8] of true solution with undetermined coefficients:

$$\pi_{t+1} = a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}), p_{s_{t+1},1}(\epsilon^e_{t+1}))r_{t+1} + a_2(s_{t+1})\epsilon^e_{t+1} \tag{18}$$

where s_{t+1} is regime at period $t+1$, $p_{s_{t+1},0}(\epsilon^e_{t+1})$ the time-varying transition probability (henceforth TVTP) from regime s_{t+1} to regime 0 at period $t+2$, and $p_{s_{t+1},1}(\epsilon^e_{t+1})$ the TVTP from regime s_{t+1} to regime 1.[9] In this guess, monetary policy shock ϵ^e_{t+1} appears not only in the second term which has a direct effect on π_{t+1}, it also changes agents' beliefs, in the first term, on the distribution of next period regime through the endogenous mechanism.

We assume that agents and monetary authority can observe all history endogenous variables, exogenous shocks and states of regime, but not the history policy regime factor $\{w_k\}_{k=0}^t$ [10]. However, econometricians only observe realized endogenous variables; they would employ filtering technique to estimate other information. Under this information structure, agents solve the problem and obtain[11]

$$\pi_{t+1} = \underbrace{\frac{\rho_r}{\alpha(s_{t+1})}\frac{(\alpha_1-\alpha_0)p_{s_{t+1},0}(\epsilon^e_{t+1}) + \alpha_1\left(\frac{\alpha_0}{\rho_r} - \mathbb{E}p_{00}(\epsilon^e_{t+1})\right) + \alpha_0\mathbb{E}p_{10}(\epsilon^e_{t+1})}{(\alpha_1-\rho_r)\left(\frac{\alpha_0}{\rho_r} - \mathbb{E}p_{00}(\epsilon^e_{t+1})\right) + (\alpha_0-\rho_r)\mathbb{E}p_{10}(\epsilon^e_{t+1})}}_{a_1(s_{t+1},p_{s_{t+1},0}(\epsilon^e_{t+1}))} r_{t+1} \underbrace{-\frac{\sigma_e}{\alpha(s_{t+1})}}_{a_2(s_{t+1})}\epsilon^e_{t+1} \tag{19}$$

where random coefficient a_1 depends on s_{t+1} and ϵ^e_{t+1}. $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$ are two constants to be evaluated.

If we impose restriction of $\rho = 0$, (19) gives the solution for exogenous regime switching case:

$$\pi_{t+1} = \underbrace{\frac{\rho_r}{\alpha(s_{t+1})}\frac{(\alpha_1-\alpha_0)p_{s_{t+1},0} + \alpha_1\left(\frac{\alpha_0}{\rho_r} - p_{00}\right) + \alpha_0 p_{10}}{(\alpha_1-\rho_r)\left(\frac{\alpha_0}{\rho_r} - p_{00}\right) + (\alpha_0-\rho_r)p_{10}}}_{a_1(s_{t+1})} r_{t+1} \underbrace{-\frac{\sigma_e}{\alpha(s_{t+1})}}_{a_2(s_{t+1})}\epsilon^e_{t+1} \tag{20}$$

[8] This is a correct guess we obtained after a few failed trials. We believe the true solution should permit the additivity between r_{t+1} and ϵ^e_{t+1} because as a limited case, it must reduce to the known result (21) in literature. Also, it reflects that regime switching is the only nonlinear feature in the model. Conditional on regime, the model would be fully linear. It's worth noting that solution is unique under the structure of initial guess. Other forms of solutions might or might not exist.

[9] The TVTP will be computed in Appendix A.

[10] This assumption is critically important for the subsequent analysis. Two potentially important extensions arise by modifying the information structure. First, assume agents and monetary authority can both observe regime factor. In this case, deriving analytical solution is more challenging. But, it's interesting to see that by observing additional information of regime strength, how macroeconomic dynamics and expectations formation effect vary. Second, assume regime factor is observable to monetary authority but latent to agents. This is more realistic but raises the issue of how agents infer what regime factor is.

[11] Detailed derivation is provided in Appendix A.

Now, the random coefficient a_1 only depends on state s_{t+1} but not ϵ^e_{t+1}, which differs from the endogenous case.

Also, if $\rho = 0, p_{00}(\epsilon^e_{t+1}) = 1, p_{10}(\epsilon^e_{t+1}) = 0$, and $\alpha_0 = \alpha_1 = \alpha$, solution (19) reduces to the equilibrium inflation in fixed regime case:

$$\pi_{t+1} = \underbrace{\frac{\rho_r}{\alpha - \rho_r}}_{a_1} r_{t+1} \underbrace{- \frac{\sigma_e}{\alpha}}_{a_2} \epsilon^e_{t+1} \tag{21}$$

where both coefficients are deterministic, which differs from the above regime switching cases. This is the result of equation (9) in Davig and Leeper (2008). Therefore, our solution generalizes their result in a meaningful way.

3.3 Macroeconomic dynamics

By deriving the analytical solution, we aim to analyze how parameters

$$\Theta = (\phi, \rho, \tau; \alpha_0, \alpha_1, \sigma_e; \rho_r, \sigma_r)'$$

affect endogenous variable π_{t+1} and agents' expectations formation effect.

3.3.1 Price stabilization

Without loss of generality, suppose the economy is under regime 0 at period T and there will be an expansionary monetary policy shock at period $T+1$. Using solution (19), we compute next period inflation rate under different realizations of regime.

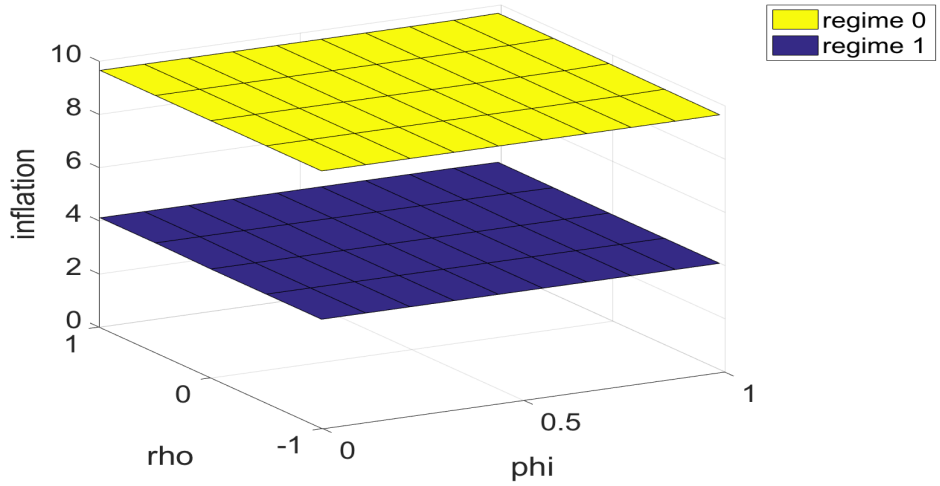


Figure 2: Realized inflation at each regime

From Figure 2, it is clear that when α_0 switches to α_1, the effect of monetary policy shock on inflation declines and monetary policy increasingly offsets the influence of the shock. Hence, a switch of monetary policy to the more aggressive regime helps stabilize price levels.

3.3.2 Expectations formation effect

We first define direct and expectations formation effects following Leeper and Zha (2003). Then, using our endogenous regime switching Fisherian model, we examine how does the expectations formation effect relate to policymakers' willingness to smooth policy change over time, as captured by ϕ (the larger the ϕ, the smoother the policy is), and also the endogeneity of regime switching, as captured by ρ.

The K-period forecast using solution (19), given information at T, is

$$\begin{aligned}\pi_{T+K} =& a_1(s_{T+K}, p_{s_{T+K},0}(\epsilon^e_{T+K}))\rho_r^K r_T + a_1(s_{T+K}, p_{s_{T+K},0}(\epsilon^e_{T+K}))\sum_{u=0}^{K-1}\rho_r^u \sigma_r \epsilon^r_{T+K-u} \\ &+ a_2(s_{T+K})\epsilon^e_{T+K} \end{aligned} \tag{22}$$

Direct effects and expectations formation effects from a policy intervention are computed from forecasts from fixed-regime and endogenous regime switching models conditional on a given intervention. Suppose the economy is under regime 0 at period T. Both of these forecasts are reported relative to a baseline forecast denoted by

$$\begin{aligned}\text{Baseline} &= \mathbb{E}(\pi_{T+K}|\mathcal{F}_T, s_t = 0, t = T+1, \ldots, T+K) \\ &= \mathbb{E}[a_1(s_{T+K} = 0, p_{00}(\epsilon^e_{T+K}))]\rho_r^K r_T \end{aligned} \tag{23}$$

Let I_T be a hypothetical intervention at time T, specified as a K-period sequence of exogenous policy actions, $I_T = \{\tilde{\epsilon}^e_{T+1}, \ldots, \tilde{\epsilon}^e_{T+K}\}$. Although the policy advisor chooses I_T, private agents treat it as random. Denote a forecast of $\{\pi_{T+K}\}$ conditional on I_T by $\mathbb{E}(\pi_{T+K}|I_T, \mathcal{F}_T, s_t = 0, t = T+1, \cdots, T+K)$. Direct effects of I_T relative to baseline are

$$\begin{aligned}\text{Direct effects} &= \mathbb{E}(\pi_{T+K}|I_T, \mathcal{F}_T, s_t = 0, t = T+1, \cdots, T+K) \\ &\quad - \mathbb{E}(\pi_{T+K}|\mathcal{F}_T, s_t = 0, t = T+1, \cdots, T+K) \\ &= a_1(s_{T+K} = 0, p_{00}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T + a_2(s_{T+K} = 0)\tilde{\epsilon}^e_{T+K} \\ &\quad - \mathbb{E}[a_1(s_{T+K} = 0, p_{00}(\epsilon^e_{T+K}))]\rho_r^K r_T \end{aligned} \tag{24}$$

In an exogenous regime switching model, when regime index is fixed, the model becomes linear. Hence, direct effect captures the linear effect from policy intervention. In our endogenous regime switching model, however, that is not true in general. Even if regime index is fixed for periods $T+1$ to $T+K$, monetary policy intervention can still affect the strength of that prevailing regime, through $p_{s_{T+K},0}(\tilde{\epsilon}^e_{T+K})$ in the first term of (24) and households' optimal behavior changes accordingly. This additional channel makes the direct effect include some nonlinear feature.

Under endogenous regime switching, the economy is highly nonlinear and the model has complicated solution (19). Now an intervention may trigger changes in agents' beliefs about policy regime which affects agents' expectations of future policy and their optimal choices. Total effects of policy combine direct effects with expectations formation effects, induced by changes in beliefs about regime. Total effects relative to the baseline projection are:

$$\begin{aligned}\text{Total effects} =&\mathbb{E}(\pi_{T+K}|I_T,\mathcal{F}_T)-\mathbb{E}(\pi_{T+K}|\mathcal{F}_T,s_t=0,t=T+1,\cdots,T+K)\\ =&[a_1(s_{T+K}=0,p_{00}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T+a_2(s_{T+K}=0)\tilde{\epsilon}^e_{T+K}]\cdot\mathbb{P}(s_{T+K}=0|s_T=0,I_T,\tilde{\epsilon}^e_T)\\ +&[a_1(s_{T+K}=1,p_{10}(\tilde{\epsilon}^e_{T+K}))\rho_r^K r_T+a_2(s_{T+K}=1)\tilde{\epsilon}^e_{T+K}]\cdot\mathbb{P}(s_{T+K}=1|s_T=0,I_T,\tilde{\epsilon}^e_T)\\ -&\mathbb{E}[a_1(s_{T+K}=0,p_{00}(\epsilon^e_{T+K}))]\rho_r^K r_T\end{aligned} \tag{25}$$

Direct effects equal to total effects when households believe regimes will remain the same in the future, *i.e.* imposing $\mathbb{P}(s_{T+K}=0|s_T=0,I_T,\tilde{\epsilon}^e_T)=1$. Finally, with direct effects and total effects computed relative to the same baseline, we isolate expectations formation effects, defined as the difference between (25) - (24):

$$\text{Expectations formation effect = total effects - direct effects} \tag{26}$$

It is understood that policy interventions may trigger changes in agents' beliefs about policy regime. In the model used by Leeper and Zha (2003), once agents *ad-hocly* realize that policy regime may change, they put constant weight on each regime conditional on past regime, which is a direct implication of using exogenous Markov chain for regime switching. In our model, we go one step further to endogenize the expectations formation effect. Upon observing the current regime index and policy intervention, agents will alter their beliefs about the strength of prevailing regime index and put time-varying weight on each regime. This smooth evolution, rather than discrete adjustment, of beliefs in future regime generates much richer implications on policy interventions.

Specification A. Suppose we are at period T under regime 0. The monetary authority decides to implement expansionary monetary policy intervention at next four periods, and they are interested to know what're the effects throughout the next sixteen periods[12]. That is,

$$I_T=\{\underbrace{-8,\ldots,-8}_{\text{4 periods}},\underbrace{0,\ldots,0}_{\text{12 periods}}\} \tag{27}$$

which is a large but short-lived policy intervention. To see how the effects depend on (ϕ,τ), we look at the simple exogenous regime switching case with $\rho=0$. Our model implies the following effects at period $T+1$. Figure 3 shows that:

(1) the direct effect, excluding the effect from regime switching, doesn't depend significantly on (ϕ,τ).

[12]Parameters are calibrated according to quarterly frequency. We choose a 4-year forecast horizon to coincide with a typical horizon the Federal Reserve considers in their routine policy making.

(2) The magnitude of expectations formation effect relies on the evolution of strength of prevailing regime, that is jointly determined by (ϕ, τ). Intuitively, if the strength of current prevailing regime gets larger, expectations formation effect is expected to be smaller. We look at two cases with $\tau \leq 0$ and $\tau > 0$.

(i) $\tau \leq 0$: The economy is currently at regime 0, that is $w_T < 0$. When $\tau \leq 0$, as policy persistence ϕ increases, a larger portion of negative w_T will be carried to next period, hence w_{T+1} would be more negative. That means, agents' belief of the economy staying at regime 0 is reinforced, or we say the "strength of regime 0" gets higher. As a result, the magnitude of expectations formation effect becomes smaller as shown in Figure 3.

(ii) $\tau > 0$: However, when $\tau > 0$, this mechanism becomes less transparent. For instance, in the situation of $0 \leq w_T < \tau$ (still at regime 0), more persistent policy would – counter-intuitively – push up w_{T+1} and lead to a greater probability of switching to regime 1, or we say the "strength of regime 0" gets lower, hence larger magnitude of expectations formation effect. In the other situation of $w_T < 0$, it works reversely and just as our intuition goes. The integrated consequence could be mixed. As Figure 3 shows, under current calibration, the first component outweighs and expectations formation effects increase in magnitude as ϕ increases.

The bottom line is: (ϕ, τ) jointly determine how the "strength of regime" evolves.[13]

[13]For other values of $\rho \neq 0$, graphs share similar patterns with Figure 3.

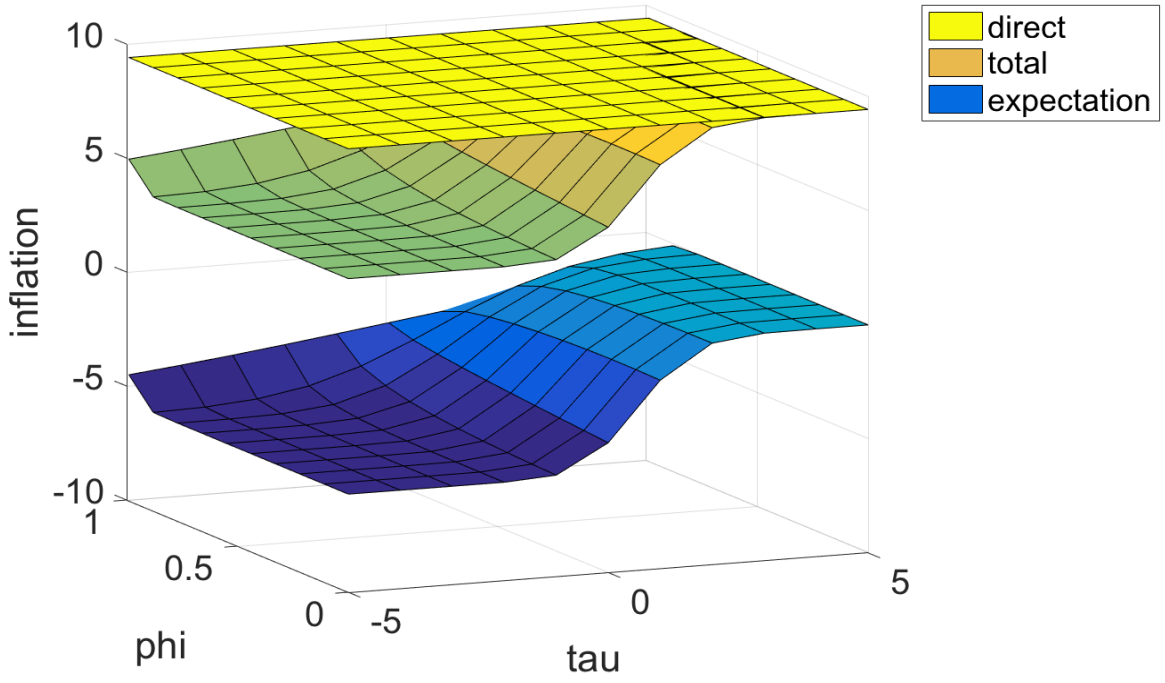


Figure 3: Impacts of expansionary policy intervention (currently at regime 0, with $\tau = 0$)

In order to analyze how the effects depend on ρ, we set $\tau = 0$ and show the following effects at period $T + 2$. Figure 4 indicates that: for fixed ϕ, as $\rho(> 0)$ increases, which means that policy intervention at period $T + 1$ conveys more information about policy regime at period $T + 2$, or more precisely, agents believe that negative realization of policy intervention at period $T + 1$ will significantly drive down (since $\rho > 0$) policy regime factor at period $T + 2$, policy regime will be less likely to switch. As a result, the expectations formation effect decreases in magnitude. When $\rho < 0$, the results will be reversed.

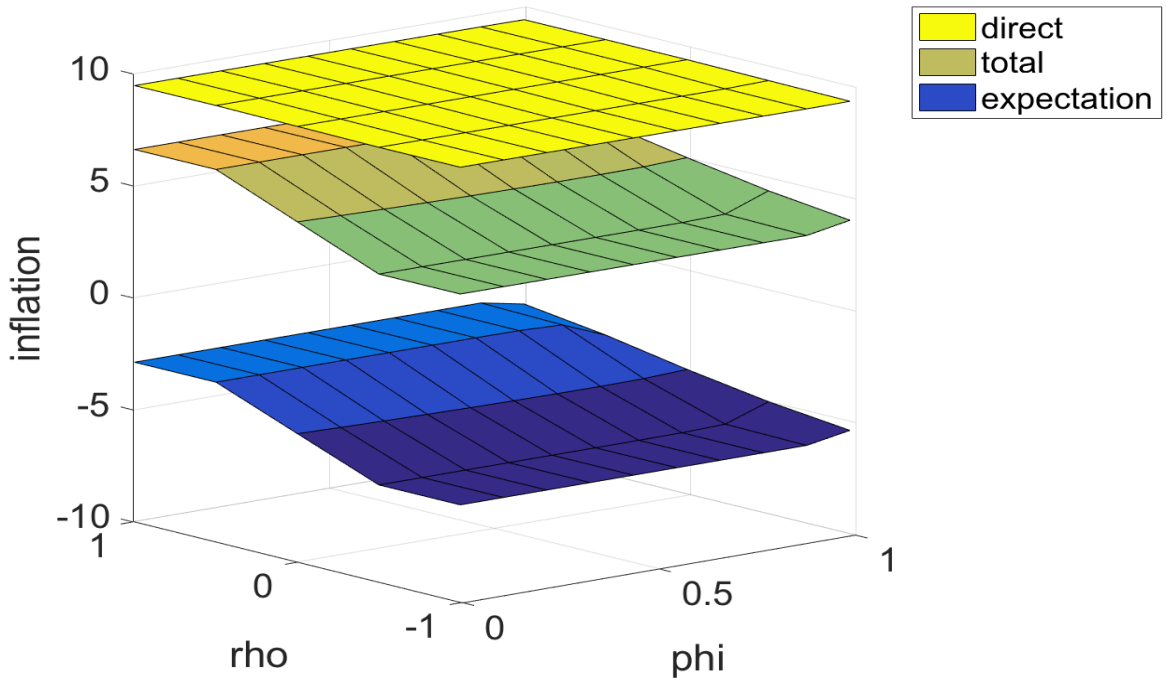


Figure 4: Impacts of expansionary policy intervention (currently at regime 0, with $\rho = 0$)

We can also show direct and total effects throughout the sixteen periods. We set $\phi = 0.88, \rho = 0.99, \tau = -2$. Figure 5 reveals that: First, there are jumps in inflation, because prices are flexible in Fisherian model and hence adjust immediately after policy shock realizes. Second, as households put certain weight on more aggressive regime which stabilizes price levels, the total effects would be smaller than direct effects. That is, expectations formation effects are negative. Suppose central bankers intend to raise inflation by 10% but neglect the existence of expectations formation effect when they assess and carry out the policy, then that policy may end with raising inflation only by 6% as implied in our model.

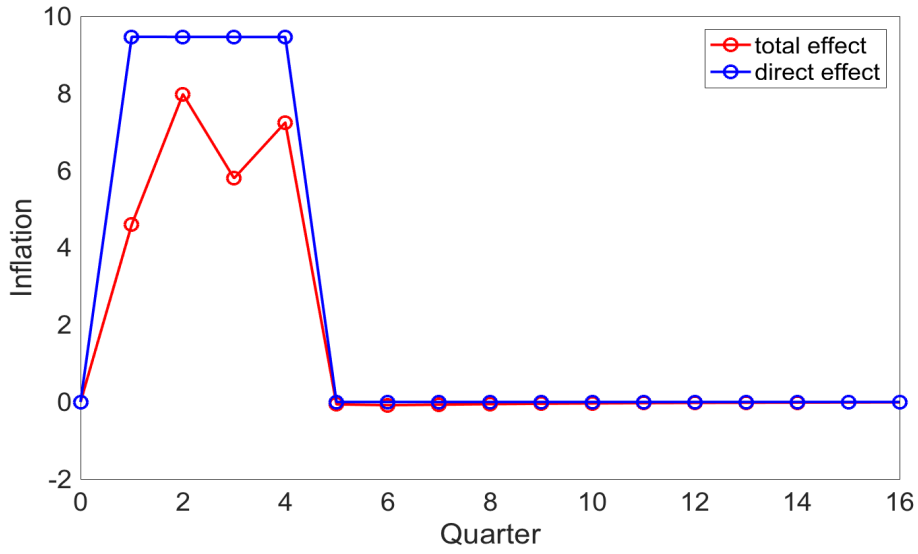


Figure 5: Impulse response of expansionary policy intervention

The cumulative impacts throughout the 16 quarters are shown in Figure 6. This large but short-lived policy intervention creates a relatively small size of expectations formation effect. At the end of 4 years, expectations formation effect is of size roughly 30% of direct effect.

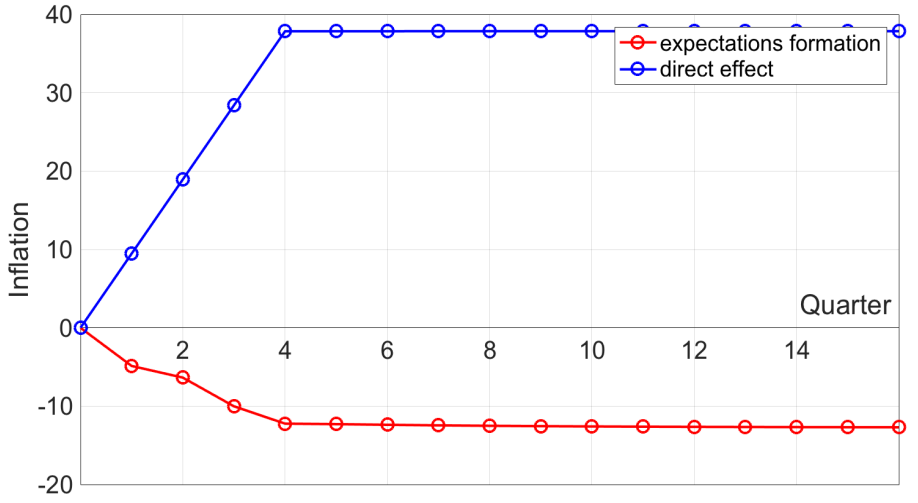


Figure 6: Cumulative impacts of expansionary policy intervention

Specification B. Now suppose the economy are at the same environment as in specification A, but central bankers would like to consider a sustained but relatively small policy intervention:

$$I_T = \underbrace{\{-2/3, -2/3, \ldots, -2/3\}}_{\text{16 periods}} \tag{28}$$

All effects share similar patterns as in figure 3 and 4 in terms of various values of (ρ, ϕ, τ). However, it generates different dynamics of effects as shown in Figure 7.[14] Direct effects are almost constant over periods. However, total effects are time-varying and gradually increasing because agents smoothly adjust their beliefs about the prevailing regime. Precisely, the sustained expansionary policy interventions make agents feel more confident about regime staying at 0. So, expectations formation effects are decreasing over time which offsets a portion of direct effects and leads to the increasing total effects.

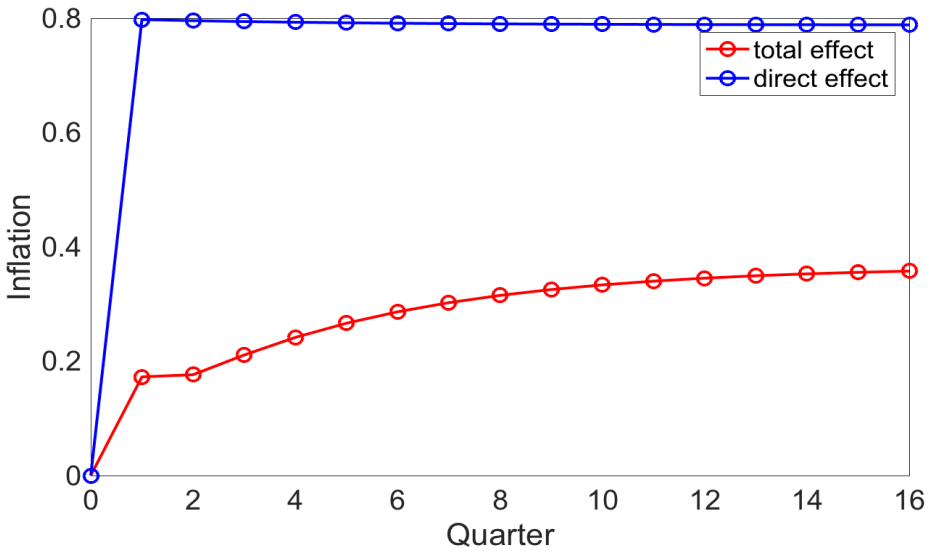


Figure 7: Impulse response of expansionary policy intervention

Although the expectations formation effects are decreasing over time, their cumulative magnitude relative to direct effects is more sizable than that in specification A. As Figure 8 shows, at the end of 4 years, the ratio of expectations formation effect to direct effect is roughly 60%, while that ratio is only around 30% in specification A. This significant difference illustrates that small but sustained interventions are likely to trigger important changes in agents' belief, while large but short-lived ones, contrary to the conventional view, tend not to trigger changes in beliefs that have quantitatively important implications.

[14] We set same parameter values as in specification A. That is, $\phi = 0.88$, $\rho = 0.99$ and $\tau = -2$.

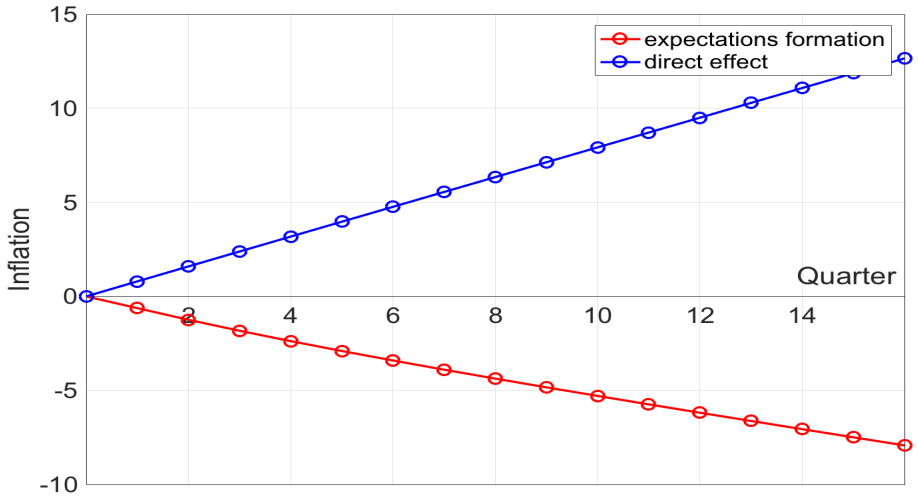


Figure 8: Cumulative impacts of expansionary policy intervention

4 A regime-switching monetary DSGE model with endogenous feedback

This section presents an endogenous regime-switching New Keynesian model with a relatively standard private sector specification, following closely the setups in Ireland (2004), An and Schorfheide (2007) and Davig and Doh (2014). This model has become a benchmark specification for the analysis of monetary policy and is analyzed in detail, for instance, in Woodford and Walsh (2005). The primary difference relative to these specifications is that the parameters in the monetary rule are subject to endogenous regime shifts. The shift happens when endogenous latent policy factor crosses threshold value.

The model economy includes a representative household, a representative firm that produces a final good, a continuum of monopolistically competitive firms that each produces an intermediate good indexed by $j \in [0, 1]$, and a monetary authority. For convenience, we abstract from investment and capital accumulation. This abstraction, however, does not affect any qualitative conclusions, as Clarida et al. (1999) discussed.

The Representative Household The representative household chooses consumption of a composite good C_t relative to a habit stock, hours worked L_t and debt holding B_t to maximize lifetime utility. We assume that the habit stock is given by the level of technology A_t. This assumption ensures that the economy evolves along a balanced growth path even if the utility function is additively separable in consumption and leisure.

$$\max \ \mathbb{E}_t \sum_{t=0}^{\infty} \xi_t \beta^t \left(\frac{(C_t/A_t)^{1-\epsilon}}{1-\epsilon} - L_t \right)$$

where $\beta \in (0, 1)$ is the discount factor, $\epsilon > 0$ is the coefficient of relative risk aversion, and ξ_t

corresponds to a preference shock affecting the discount factor. Also note that the scale factor for L_t is implicitly assumed as 1, following An and Schorfheide (2007). The intertemporal budget constraint is:

$$P_t C_t + Q_t B_t = B_{t-1} + W_t L_t + P_t D_t - P_t T_t$$

where P_t is the aggregate price level, Q_t the price of a zero-coupon bond at time t yielding 1 in period $t+1$, W_t the nominal wage per hour, D_t the real profits from ownership of firms, and T_t lump-sum taxes.

The Intermediate-Goods-Producing Firms Intermediate goods-producing firm j produces output Y_{jt} using labor L_{jt} as only input. The production technology is linear in labor for simplicity:

$$Y_{jt} = A_t L_{jt}$$

where A_t is common to all firms. The labor market is perfectly competitive, and firms are able to hire as much as demanded at the real wage.

To allow for a real effect of monetary policy, we introduce nominal rigidities *à la* Rotemberg (1982). The monopolistic intermediate goods-producing firms pay a real adjustment cost AC_{jt} when they adjust their price:

$$AC_{jt} = \frac{\varphi}{2}\left(\frac{P_{jt}}{\Pi^*_{s_t} P_{jt-1}} - 1\right)^2 Y_t$$

where $\varphi \geq 0$ governs the price stickiness in the economy, $\Pi^*_{s_t}$ denotes the regime-dependent steady-state gross inflation in regime s_t that coincides with the central bank's inflation target, and P_{jt} denotes the nominal price set by firm $j \in [0,1]$ at time t. The price adjustment cost is in terms of the final good Y_t. Each intermediate goods-producing firm maximizes the expected present value of real profits,

$$\max \sum_{s=0}^{\infty} \beta^s \lambda_{t+s} D_{jt+s}$$

where

$$\beta^s \lambda_{t+s} = \beta^s \frac{MU_{t+s}}{MU_t} = \beta^s \left(\frac{C_{t+s}}{C_t}\right)^{-\epsilon} \left(\frac{A_t}{A_{t+s}}\right)^{1-\epsilon}$$

is the stochastic discount factor for real profits. It measures the time t real value of a unit of the consumption good in period $t+s$ to the household, which is treated as exogenous by the firm. Real profits D_{jt} are

$$D_{jt} = \frac{P_{jt} Y_{jt}}{P_t} - \frac{W_t}{P_t} L_{jt} - \frac{\varphi}{2}\left(\frac{P_{jt}}{\Pi^*_{s_t} P_{jt-1}} - 1\right)^2 Y_t.$$

The Representative Final-Goods-Producing Firm The representative final-goods-producing firm purchases a continuum of intermediate goods indexed by $j \in [0,1]$ at prices P_{jt} and combines

them into a final good using the following constant-returns-to-scale technology in the familiar Dixit-Stiglitz form:

$$Y_t = \left[\int_0^1 Y_{jt}^{\frac{\theta_t - 1}{\theta_t}} dj \right]^{\frac{\theta_t}{\theta_t - 1}}$$

where $\theta_t > 1$ (hence $\frac{\theta_t - 1}{\theta_t} \in (0, 1)$) is the time-varying elasticity of substitution between goods. Notice that all intermediate goods enter the aggregation function symmetrically. This specific property implies a constant elasticity of substitution between intermediate goods, equal to $\theta_t, \forall t$. As a result of this, each intermediate-goods-producing firm possesses a certain degree of monopoly power in the good it produces. However, at the time dimension, θ_t is stochastic. The economic interpretation is that the variety of intermediate goods produced in the economy and their substitutability is constantly changing. As a result of this the monopoly power of each intermediate-goods-producing firm and therefore its desired markup, u_t, over marginal costs is also constantly changing. The steady-state markup can be solved as

$$u = \frac{\theta}{\theta - 1}$$

and θ is the steady-state elasticity of substitution. The profit maximization problem for the final-goods-producing firm yields a demand for each intermediate good given by:

$$Y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\theta_t} Y_t$$

Then, the zero-profit condition for the perfectly competitive final-goods-producing firm implies that the aggregate price level is:

$$P_t = \left[\int_0^1 P_{jt}^{1-\theta_t} dj \right]^{\frac{1}{1-\theta_t}}.$$

Monetary Policy As discussed in Section 2, the monetary authority sets the short-term nominal rate using the following modified Taylor rule:

$$\frac{R_t}{R^*_{s_t}} = \left(\frac{R_{t-1}}{R^*_{s_{t-1}}} \right)^{\rho_r} \left[\left(\frac{\Pi_t}{\Pi^*_{s_t}} \right)^{\alpha(s_t)} \left(\frac{Y_t}{Y_t^*} \right)^{\gamma} \right]^{1-\rho_r} e_t$$

or taking logarithm,

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1 - \rho_r)[\alpha(s_t)\hat{\Pi}_t + \gamma \hat{Y}_t] + \hat{e}_t$$

The inflation target $\Pi^*_{s_t}$ is regime-dependent. We suppose that the targeted nominal gross interest rate $R^*_{s_t}$ is chosen such that inflation equals its target in each regime in the absence of shocks. The regime $\{s_t\}$ is governed by an endogenous monetary policy latent factor $\{w_t\}$ specified in equations (6), (9), and (8).

Shocks The model economy is perturbed by four exogenous processes. Aggregate productivity evolves according to

$$\ln A_{t+1} = g_A + \ln A_t + \ln a_{t+1},$$

where

$$\ln a_{t+1} = \rho_a \ln a_t + \sigma_a \varepsilon^a_{t+1}.$$

with the innovation ε^a_{t+1} follows a standard normal distribution and $|\rho_a| < 1$. Thus on average technology grows at the rate g_A, and a_t captures exogenous fluctuations of the technology growth rate.

The preference shock, ξ_t follows a first order autoregressive process:

$$\ln \xi_{t+1} = \rho_\xi \ln \xi_t + \sigma_\xi \varepsilon^\xi_{t+1}$$

where $|\rho_\xi| < 1$ and the innovation ε^ξ_{t+1} follows a standard normal distribution. We denote by $\hat{\xi}_t$ the log deviation $\ln \xi_t$.

The time t markup follows:

$$\ln u_{t+1} = (1 - \rho_u) \ln u + \rho_u \ln u_t + \sigma_u \varepsilon^u_{t+1}$$

where $|\rho_u| < 1$, and the innovation ε^u_{t+1} follows a standard normal distribution. We denote by $\hat{u}_t$ the log deviation $\ln(u_t/u)$.

Finally, as defined above, the monetary policy shock

$$\hat{e}_{t+1} = \ln e_{t+1} = \sigma_e \varepsilon^e_{t+1}$$

where the innovation ε^e_{t+1} follows a standard normal distribution. The three innovations are independent of each other.

Full characterization The first-order conditions and market clearing conditions lead to the following nonlinear system of control variables and forward exogenous variables

$$\{Y_{t+1}, C_{t+1}, \Pi_{t+1}, R_{t+1}, s_{t+1}, w_{t+1}; A_{t+1}, a_{t+1}, \xi_{t+1}, u_{t+1}, e_{t+1}\},$$

endogenous and exogenous state variables

$$\{Y_t, C_t, \Pi_t, R_t, s_t, w_t, A_t, \xi_t, a_t, u_t, e_t\},$$

structural innovations and latent factor innovation

$$\{\varepsilon^a_{t+1}, \varepsilon^\xi_{t+1}, \varepsilon^u_{t+1}, \varepsilon^e_{t+1}, v_{t+1}\}.$$

That is,

$$\begin{aligned}\frac{\theta}{\theta-1} =&\theta u_t\left(\frac{C_t}{A_t}\right)^{\epsilon}-\varphi(\theta u_t-\theta+1)\frac{\Pi_t}{\Pi^*_{s_t}}\left[\frac{\Pi_t}{\Pi^*_{s_t}}-1\right]\\ &+\beta\varphi(\theta u_t-\theta+1)\mathbb{E}_t\left[\frac{Y_{t+1}/A_{t+1}}{Y_t/A_t}\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t}\right)^{-\epsilon}\frac{\Pi_{t+1}}{\Pi^*_{s_{t+1}}}\left(\frac{\Pi_{t+1}}{\Pi^*_{s_{t+1}}}-1\right)\right] \qquad (29)\end{aligned}$$

$$Y_t = C_t+\frac{\varphi}{2}\left[\frac{\Pi_t}{\Pi^*_{s_t}}-1\right]^2 Y_t \qquad (30)$$

$$\frac{R_t}{R^*_{s_t}} = \left(\frac{R_{t-1}}{R^*_{s_{t-1}}}\right)^{\rho_r}\left[\left(\frac{\Pi_t}{\Pi^*_{s_t}}\right)^{\alpha(s_t)}\left(\frac{Y_t}{Y^*_t}\right)^{\gamma}\right]^{1-\rho_r} e_t \qquad (31)$$

$$\mathbb{E}_t\left[\frac{\beta R_t}{\Pi_{t+1}}\left(\frac{C_t/A_t}{C_{t+1}/A_{t+1}}\right)^{\epsilon}\left(\frac{A_t}{A_{t+1}}\right)\left(\frac{\xi_{t+1}}{\xi_t}\right)\right]=1 \qquad (32)$$

$$s_{t+1} = \mathbb{1}\{w_{t+1}\geq\tau\} \qquad (33)$$

$$w_{t+1} = \phi w_t + v_{t+1} \qquad (34)$$

where the aggregate productivity A_{t+1}, technology fluctuation a_{t+1}, price markup shock u_{t+1}, monetary policy shock e_{t+1} and latent factor innovation v_{t+1} have the following law of motions:

$$\ln A_{t+1} = g_A+\ln A_t+\ln a_{t+1} \qquad (35)$$

$$\ln a_{t+1} = \rho_a\ln a_t+\sigma_a\varepsilon^a_{t+1} \qquad (36)$$

$$\ln \xi_{t+1} = \rho_\xi\ln \xi_t+\sigma_\xi\varepsilon^\xi_{t+1} \qquad (37)$$

$$\ln u_{t+1} = (1-\rho_u)\ln u+\rho_u\ln u_t+\sigma_u\varepsilon^u_{t+1} \qquad (38)$$

$$\ln e_{t+1} = \sigma_e\varepsilon^e_{t+1} \qquad (39)$$

$$\begin{pmatrix}\varepsilon^e_t\\ v_{t+1}\end{pmatrix} =_d \mathbb{N}\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1&\rho\\ \rho&1\end{pmatrix}\right) \qquad (40)$$

Equation (29) is a non-linearized new Keynesian Phillips curve reflecting the optimal price setting of intermediate firms, equation (30) reports market clearing conditions, equation (31) is the monetary policy rule and equation (32) is the Euler equation of households.

5 Solution methods to regime-switching monetary DSGE models

Corresponding to Section 2, we discuss solution methods for the above New Keynesian model with three different setup of monetary policies, namely, fixed-regime, exogenous regime-switching, and endogenous regime-switching policy rule. In particular, we'll use perturbation method since it is the common practice in recent literature on monetary DSGE models.

5.1 Monetary DSGE model with original Taylor rule

Full characterization In the absence of regime switching, the characterization of equations (29) to (40) reduces to the following:

$$\begin{aligned}\frac{\theta}{\theta-1} =&\theta u_t\left(\frac{C_t}{A_t}\right)^{\epsilon}-\varphi(\theta u_t-\theta+1)\frac{\Pi_t}{\Pi^*}\left[\frac{\Pi_t}{\Pi^*}-1\right]\\ &+\beta\varphi(\theta u_t-\theta+1)\mathbb{E}_t\left[\frac{Y_{t+1}/A_{t+1}}{Y_t/A_t}\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t}\right)^{-\epsilon}\frac{\Pi_{t+1}}{\Pi^*}\left(\frac{\Pi_{t+1}}{\Pi^*}-1\right)\right] \qquad (41)\end{aligned}$$

$$Y_t = C_t+\frac{\varphi}{2}\left[\frac{\Pi_t}{\Pi^*}-1\right]^2 Y_t \qquad (42)$$

$$\frac{R_t}{R^*}=\left(\frac{R_{t-1}}{R^*}\right)^{\rho_r}\left[\left(\frac{\Pi_t}{\Pi^*}\right)^{\alpha}\left(\frac{Y_t}{Y_t^*}\right)^{\gamma}\right]^{1-\rho_r}e_t \qquad (43)$$

$$\mathbb{E}_t\left[\frac{\beta R_t}{\Pi_{t+1}}\left(\frac{C_t/A_t}{C_{t+1}/A_{t+1}}\right)^{\epsilon}\left(\frac{A_t}{A_{t+1}}\right)\left(\frac{\xi_{t+1}}{\xi_t}\right)\right]=1 \qquad (44)$$

$$\ln A_{t+1} = g_A+\ln A_t+\ln a_{t+1} \qquad (45)$$

$$\ln a_{t+1} = \rho_a \ln a_t+\sigma_a\varepsilon^a_{t+1} \qquad (46)$$

$$\ln \xi_{t+1} = \rho_\xi \ln \xi_t+\sigma_\xi\varepsilon^\xi_{t+1} \qquad (47)$$

$$\ln u_{t+1} = (1-\rho_u)\ln u+\rho_u \ln u_t+\sigma_u\varepsilon^u_{t+1} \qquad (48)$$

$$\ln e_{t+1} = \sigma_e\varepsilon^e_{t+1} \qquad (49)$$

Steady states The system is in steady state if the shocks $\varepsilon^a_t, \varepsilon^\xi_t, \varepsilon^u_t, \varepsilon^e_t$ are zero. From market clearing condition (42) we obtain

$$C_t^* = Y_t^* \qquad (50)$$

Then, in the absence of nominal rigidities ($\varphi = 0$), aggregate output can be obtained from new Keynesian Phillips curve (41):

$$Y_t^* = C_t^* = (1/\theta)^{1/\epsilon}A_t \qquad (51)$$

which is the target level of output that appears in the monetary policy. Since the nonstationary technology process A_t induces a stochastic trend in output and consumption, it is convenient to express the model in terms of detrended variables $c_t = C_t/A_t$ and $y_t = Y_t/A_t$. The steady state can be represented in terms of the detrended variables c_t and y_t:

$$c^* = y^* = (1/\theta)^{1/\epsilon} \qquad (52)$$

As mentioned in Section 2.1, in equilibrium, the steady state inflation rate coincides with the target inflation rate chosen by the central bank. In practice, Π^* is often set to match the average inflation rate in the data. For instance, Ireland (2004) set Π^* equal to 1.0086, matching the average inflation

rate in the U.S. during 1948:1-2003:1, which equals 3.48 percent when annualized. Finally, Euler equation (44) implies the steady state relationship

$$R^* = r^*\Pi^* \quad \text{and} \quad \beta r^* = \exp(g_A) \tag{53}$$

Log-linearization To solve the nonlinear rational expectations system of (41)-(49), we can simply log-linearize it around the above defined deterministic steady states. Log-linearization approximation method is popular because it leads to a state-space representation of the DSGE model that can be empirically analyzed with the Kalman filter. The results are the canonical "trinity" system. The new Keynesian Phillips curve can be rewritten as:

$$\hat{\Pi}_t = \beta \mathbb{E}_t \hat{\Pi}_{t+1} + \kappa(\hat{y}_t + \frac{\hat{u}_t}{\epsilon}) \tag{54}$$

where $\kappa = \epsilon(\theta - 1)/\Pi^2\varphi$. The Euler equation is the standard IS curve:

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \epsilon^{-1}(\hat{R}_t - \mathbb{E}_t \hat{\Pi}_{t+1} - \mathbb{E}_t \hat{a}_{t+1} + \mathbb{E}_t \hat{\xi}_{t+1} - \hat{\xi}_t) \tag{55}$$

Finally, the monetary policy rule writes:

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1 - \rho_r)[\alpha \hat{\Pi}_t + \gamma \hat{y}_t] + \hat{e}_t \tag{56}$$

where

$$\hat{y}_t = \ln\left(\frac{y_t}{y^*}\right) = \ln\left(\frac{y_t A_t}{y^* A_t}\right) = \ln\left(\frac{Y_t}{Y_t^*}\right) = \hat{Y}_t.$$

Now, equations (54)-(56) combined with exogenous shocks (45)-(49) form an approximate linear rational expectations (henceforth LRE) system in control variables and forward exogenous variables

$$\{\hat{y}_{t+1}, \hat{\Pi}_{t+1}, \hat{R}_{t+1}; \hat{a}_{t+1}, \hat{\xi}_{t+1}, \hat{u}_{t+1}, \hat{e}_{t+1}\},$$

endogenous and exogenous state variables

$$\{\hat{y}_t, \hat{\Pi}_t, \hat{R}_t, \hat{a}_t, \hat{\xi}_t, \hat{u}_t, \hat{e}_t\},$$

that is driven by the vector of structural innovations

$$\varepsilon_t = (\varepsilon_t^a, \varepsilon_t^\xi, \varepsilon_t^u, \varepsilon_t^e)'.$$

We denote

$$\Theta = (\beta, \epsilon; \theta, \varphi; \rho_r, \alpha, \gamma; \rho_a, \sigma_a, \rho_\xi, \sigma_\xi, \rho_u, \sigma_u, \sigma_e)'$$

and assume it is known for the sake of analysis on solution methods.

Solution and Determinacy Following Sims (2002) which generalizes the methods by Blanchard and Kahn (1980), the LRE system can be rewritten in the canonical form

$$\mathcal{T}_0 X_{t+1} = \mathcal{T}_1 X_t + \mathcal{T}_2 \varepsilon_{t+1} + \mathcal{T}_3 \eta_{t+1} + C_0 \tag{57}$$

where

$$X_{t+1} = (\hat{y}_{t+1}, \hat{\Pi}_{t+1}, \hat{R}_{t+1}, \hat{a}_{t+1}, \hat{\xi}_{t+1}, \hat{u}_{t+1}, \hat{e}_{t+1})'$$

and

$$\eta_{t+1} = [\hat{y}_{t+1} - \mathbb{E}_t(\hat{y}_{t+1}), \hat{\Pi}_{t+1} - \mathbb{E}_t(\hat{\Pi}_{t+1})]'$$

are forecast errors. In the original version of Sims (2002) and the provided computer code Gensys, It provides the solution when under determinacy. In the case of indeterminacy, it fails to provide the complete set of all solutions, but only generates one solution associated with a warning message for indeterminacy. McCallum (1983) has argued that a model with an indeterminate set of equilibria is incompletely specified and he recommends a procedure, the minimal state variable (henceforth MSV) solution, for selecting one of the many possible equilibria in the indeterminate case. Although many literature derive conditions for equilibrium non-uniqueness and compute one particular solution, none, before Lubik and Schorfheide (2003), explicitly shows how to introduce sunspot shocks into a LRE model and compute the complete set of equilibria.

Following Lubik and Schorfheide (2004), we assume that in addition to the fundamental shocks ε_t (*i.e.* structural shocks), the agents observe an exogenous sunspot shock ζ_t.

It is well known that the above log-linear model can give rise to self-fulfilling expectations if the central bank does not raise the nominal interest rate aggressively enough in response to inflation. In this case not just the fundamental shock ε_t but also the sunspot shock ζ_t can influence the dynamics of output, inflation, and interest rates. More formally, since the model is linear and the only sources of uncertainty are the shocks ε_t and ζ_t, the forecast errors for output and inflation can be expressed as

$$\eta_t = \Psi_1 \varepsilon_t + \Psi_2 \zeta_t \tag{58}$$

where Ψ_1 is 2×4 and Ψ_2 is 2×1. All the solution algorithms for LRE systems then construct a mapping from the shocks to the forecast errors. Depending on the given parameter values Θ, three cases can be distinguished: (i) non-existence of a stable solution, (ii) existence of a unique stable solution (determinacy) in which Ψ_1 depends on the structural parameters Θ and $\Psi_2 = 0$, and (iii) existence of multiple stable solutions (indeterminacy) in which Ψ_1 is not uniquely determined by Θ and Ψ_2 can be non-zero. In general form, Lubik and Schorfheide (2003) solve equation (58) using a generalized complex Schur decomposition (QZ). The solution of the system will take the form

$$X_{t+1} = AX_t + B\varepsilon_{t+1} + C + D\zeta_{t+1} \tag{59}$$

Under determinacy, $D = 0$, hence sunspot shock drop out. For a detailed derivation of solution (59), please refer to Lubik and Schorfheide (2003) and (2004). Lubik and Schorfheide (2004) also extend the estimation of LRE models to the indeterminacy region of parameter space (see, in particular, section 3 in their paper).

5.2 Monetary DSGE model with exogenous switching monetary policy

Full characterization In the case of exogenous switching, the characterization of equations (29) to (40) reduces to the following:

$$\begin{aligned}\frac{\theta}{\theta-1} =&\theta u_t\left(\frac{C_t}{A_t}\right)^{\epsilon}-\varphi(\theta u_t-\theta+1)\frac{\Pi_t}{\Pi^*_{s_t}}\left[\frac{\Pi_t}{\Pi^*_{s_t}}-1\right]\\ &+\beta\varphi(\theta u_t-\theta+1)\mathbb{E}_t\left[\frac{Y_{t+1}/A_{t+1}}{Y_t/A_t}\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t}\right)^{-\epsilon}\frac{\Pi_{t+1}}{\Pi^*_{s_{t+1}}}\left(\frac{\Pi_{t+1}}{\Pi^*_{s_{t+1}}}-1\right)\right] \qquad (60)\end{aligned}$$

$$Y_t = C_t+\frac{\varphi}{2}\left[\frac{\Pi_t}{\Pi^*_{s_t}}-1\right]^2 Y_t \qquad (61)$$

$$\frac{R_t}{R^*_{s_t}} = \left(\frac{R_{t-1}}{R^*_{s_{t-1}}}\right)^{\rho_r}\left[\left(\frac{\Pi_t}{\Pi^*_{s_t}}\right)^{\alpha(s_t)}\left(\frac{Y_t}{Y^*_t}\right)^{\gamma}\right]^{1-\rho_r} e_t \qquad (62)$$

$$\mathbb{E}_t\left[\frac{\beta R_t}{\Pi_{t+1}}\left(\frac{C_t/A_t}{C_{t+1}/A_{t+1}}\right)^{\epsilon}\left(\frac{A_t}{A_{t+1}}\right)\left(\frac{\xi_{t+1}}{\xi_t}\right)\right] = 1 \qquad (63)$$

$$\ln A_{t+1} = g_A+\ln A_t+\ln a_{t+1} \qquad (64)$$

$$\ln a_{t+1} = \rho_a\ln a_t+\sigma_a\varepsilon^a_{t+1} \qquad (65)$$

$$\ln \xi_{t+1} = \rho_\xi\ln \xi_t+\sigma_\xi\varepsilon^\xi_{t+1} \qquad (66)$$

$$\ln u_{t+1} = (1-\rho_u)\ln u+\rho_u\ln u_t+\sigma_u\varepsilon^u_{t+1} \qquad (67)$$

$$\ln e_{t+1} = \sigma_e\varepsilon^e_{t+1} \qquad (68)$$

where regime s_t follows an exogenous Markov process with transition probability matrix:

$$P_s = \begin{bmatrix} p_{00} & 1-p_{00}\\ 1-p_{11} & p_{11}\end{bmatrix} \qquad (69)$$

Steady states Slightly different from the fixed regime case, this system is in steady state if the shocks $\varepsilon^a_t, \varepsilon^\xi_t, \varepsilon^u_t, \varepsilon^e_t$ are zero and also that there is no uncertainty about the inflation target, that is, $\Pi^*_0 = \Pi^*_1 = \Pi^*$. Analogous to the fixed regime case, we obtain steady states as below:

$$c^* = y^* = (1/\theta)^{1/\epsilon} \qquad (70)$$

and

$$R^* = r^*\Pi^* \quad \text{and} \quad \beta r^* = \exp(g_A) \qquad (71)$$

As long as we assume inflation target Π^* is constant over regimes, we'll have the whole steady-state equilibrium being independent of policy regime. This result is summarized formally as Proposition 5 in Liu et al. (2011). However, when we allow for

$$\Pi^*_0 \neq \Pi^*_1,$$

then equation (71) will imply

$$R_0^* = r^* \Pi_0^*;$$
$$R_1^* = r^* \Pi_1^*,$$

while equation $\beta r^* = \exp(g_A)$ and (70) remains to be true.

We'll focus on the case when we have deterministic steady states (*i.e.*, inflation target Π^* is constant over regimes) which are independent of policy regimes in the following analysis.

Solutions and determinacy Some early literature solve the nonlinear system using Markov-switching linear rational expectations (henceforth MSLRE) approach. They linearize the equilibrium conditions as if the parameters were constant over time. Discrete Markov chain processes are then annexed to certain parameters. Recent literature develop perturbation method for Markov-switching rational expectations model. We'll show the main ideas for both methods and discuss their difference.

5.2.1 Approach 1: Markov-switching linear rational expectations method

Farmer et al. (2009) show that the equilibria of MSRE models are of two types: minimal state variable (MSV) equilibria and non-fundamental equilibria. Non-fundamental equilibria may or may not exist. If a non-fundamental equilibrium exists, it is the sum of an MSV equilibrium and a secondary stochastic process (*i.e.*, the sunspot component). In another related paper (Farmer et al., 2011), they develop an efficient method for finding MSV equilibria in a general class of MSRE models, including those with lagged state variables (*e.g.* monetary policy with interest rate smoothing). Given the set of MSV equilibria, the earlier paper (Farmer et al., 2009) shows how to construct non-fundamental equilibria.

Previous authors, notably Leeper and Zha (2003), Svensson and Williams (2005), and Davig and Leeper (2007) had made some progress in developing methods to solve for the equilibria of MSRE models. But the techniques are not capable of finding all of the equilibria in a general class of MSRE models. We'll follow the idea in Farmer et al. (2009) to illustrate the MSLRE mothod.

Log-linearization The log-linearization around the above defined deterministic steady state leads to the canonical "trinity" system:

$$\hat{\Pi}_t = \beta \mathbb{E}_t \hat{\Pi}_{t+1} + \kappa(\hat{y}_t + \frac{\hat{u}_t}{\epsilon}) \tag{72}$$

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \epsilon^{-1}(\hat{R}_t - \mathbb{E}_t \hat{\Pi}_{t+1} - \mathbb{E}_t \hat{a}_{t+1} + \mathbb{E}_t \hat{\xi}_{t+1} - \hat{\xi}_t) \tag{73}$$

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1-\rho_r)[\alpha(s_t)\hat{\Pi}_t + \gamma \hat{y}_t] + \hat{e}_t \tag{74}$$

This system can be cast into the general form of MSRE models:

$$\mathcal{T}_0(s_{t+1}) X_{t+1} = \mathcal{T}_1(s_{t+1}) X_t + \mathcal{T}_2(s_{t+1}) \varepsilon_{t+1} + \mathcal{T}_3(s_{t+1}) \eta_{t+1} + C_0 \tag{75}$$

where regime s_t follows an exogenous Markov process with transition probability matrix:

$$P_s = \begin{bmatrix} p_{00} & 1-p_{00} \\ 1-p_{11} & p_{11} \end{bmatrix} \tag{76}$$

Minimal state variable solutions Farmer et al. (2011) develop an iterative techniques to construct MSV equilibria for MSRE system (75) and (76). Main results can be found in section 2 (in particular Theorem 1 and Algorithm 1) of their paper. MSV solutions will take forms of

$$\eta_t = \Psi(s_t)\varepsilon_t \tag{77}$$

Plugging (77) into (75) yields a solution:

$$X_{t+1} = A(s_{t+1})X_t + B(s_{t+1})\varepsilon_{t+1} + C + D(s_{t+1})\eta_{t+1} \tag{78}$$

It is analogous to the reduced-form vector-autoregression with regime switching, of the kind studied by Hamilton (1989) and Sims and Zha (2006). Of course, the difference is that coefficient matrices in (78) are functions of deep parameters.

Complete set of solutions (sunspot solutions) In LRE models, mean-square stability and bounded stability are the same.[15] However, in MSLRE models, Farmer et al. (2009) argued that mean-square stability (henceforth MSS) is the appropriate stability concept. In Proposition 1, they show how to obtain all MSS solutions other than the MSV solution. Their results permit an applied researcher to partition the parameter space of an economic model into determinate and indeterminate regions for a large class of Markov-switching rational expectations models and hence, to compute the likelihood for each regime. Details for derivation of MSS solutions are referred to Farmer et al. (2009).

To the best of my knowledge, recent literature on estimation of MSDSGE model still use MSV solutions with the algorithm provided in Farmer et al. (2011), *e.g.* Bianchi (2012), Bianchi and Ilut (2014) and Davig and Doh (2014).

Limitations As Foerster et al. (2016) pointed out, there are two main shortcomings with the MSLRE approach. First, this approach begins with a system of standard linear rational expectations equations that have been obtained from linearizing equilibrium conditions as though the parameters were constant over time. Discrete Markov chain processes are then annexed to certain parameters. As a consequence, the resultant MSLRE model may be incompatible with the optimizing behavior of agents in an original economic model with Markov-switching parameter. Second, because it builds on linear rational expectations model, the MSLRE approach dose not take into account higher-order coefficients in the approximation. Higher-order approximations improve the approximation accuracy and may be potentially important for certain economic questions.

[15] In LRE models, whether a first-order approximation is stable or not can be determined by verifying whether its largest absolute generalized eigenvalue is greater than or equal to one, a condition that holds for most concepts of stability.

5.2.2 Approach 2: Perturbation method

Foerster et al. (2016) use perturbation methods as described in Judd (1998) and Schmitt-Grohé and Uribe (2004) to solve Markov-switching DSGE models. To overcome the limitations of MSLRE method mentioned above, they work with the original MSDSGE model directly rather than taking a system of linear rational expectations equations with constant parameters as a short-cut.

Instead of linearization, the nonlinear rational expectations model (60)-(69) can be cast into the general form as below:

$$\mathbb{E}_t f(Z_{t+1}, Z_t, W_t, W_{t-1}, \chi\varepsilon_{t+1}, \varepsilon_t, \vartheta_{t+1}, \vartheta_t) = \mathbf{0}_{n_Z+n_W} \tag{79}$$

where $W_t \in \mathbb{R}^{n_W}$ is (endogenous or exogenous) predetermined variables, $Z_t \in \mathbb{R}^{n_Z}$ is non-predetermined (control) variables, $\varepsilon_t \in \mathbb{R}^{n_\varepsilon}$ is the vector of i.i.d. innovations to the exogenous predetermined variables, and $\chi \in \mathbb{R}$ is the perturbation parameter. ϑ_t is the vector of all Markov-switching parameters. Since the total number of equations in (79) is $n_Z + n_W$, the function f maps $\mathbb{R}^{2(n_Z+n_W+n_\varepsilon+n_\vartheta)}$ into $\mathbb{R}^{n_Z+n_W}$. In our case,

$$Z_t = (c_t, y_t, \Pi_t)', \quad W_t = (R_t, a_t, u_t, e_t)', \quad \vartheta_t = \alpha(s_t)$$

Consider model solutions of the form

$$Z_t = g(W_{t-1}, \varepsilon_t, \chi, s_t), \quad Z_{t+1} = g(W_t, \chi\varepsilon_{t+1}, \chi, s_{t+1}), \quad \text{and } W_t = h(W_{t-1}, \varepsilon_t, \chi, s_t) \tag{80}$$

where g maps $\mathbb{R}^{n_W+n_\varepsilon+1} \times \{0,1\}$ into $\mathbb{R}^{n_Z}$ and h maps $\mathbb{R}^{n_W+n_\varepsilon+1} \times \{0,1\}$ into $\mathbb{R}^{n_W}$.[16]

In general, we do not know the explicit functional forms of g and h; thus, we need to approximate them around the above defined steady state $(Z_{ss}, W_{ss}, \chi = 0)$. [17]

The first-order approximations to g and h around the steady state can be written in the following form

$$Z_t \approx g^{first}(W_{t-1}, \varepsilon_t, \chi, s_t) = Z_{ss} + \mathcal{D}_W g_{ss}(s_t)(W_{t-1} - W_{ss}) + \mathcal{D}_\varepsilon g_{ss}(s_t)\varepsilon_t + \mathcal{D}_\chi g_{ss}(s_t)\chi$$

and

$$W_t \approx h^{first}(W_{t-1}, \varepsilon_t, \chi, s_t) = W_{ss} + \mathcal{D}_W h_{ss}(s_t)(W_{t-1} - W_{ss}) + \mathcal{D}_\varepsilon h_{ss}(s_t)\varepsilon_t + \mathcal{D}_\chi h_{ss}(s_t)\chi$$

[16] Foerster et al. (2016) claims that in theory, one could allow g and h to depend on the entire history of the regimes. In practice, this is not tractable. The assumption that g and h depend only on the current regime corresponds to the notion of a MSV solution in Farmer et al. (2011). But, as Barthélemy and Marx (2017) writes, this can be extended to solutions depending on a finite number of past regimes by redefining regimes as the finite product of past regimes. What they actually rule out is the existence of history-dependent solutions.

[17] Foerster et al. (2016) prove that regime-specific steady state is not tractable naturally. If it's in that case, they partition the switching parameters into two subsets, for the subset where the switching parameters have an effect on the equilibrium conditions, this effect is considered as an additional exogenous shock and need to be perturbed. Correspondingly, a regime-independent steady state is defined based on the mean of the ergodic distribution across the switching parameter in this subset. Details are referred to section 2.2 and 2.3 in Foerster et al. (2016). Fortunately, however, the steady state is regime-independent in our model as pointed out in section 5.1: steady state.

By denoting $\Xi_t' = (W_{t-1}', \varepsilon_t', \chi)$ and $\Xi_{ss}' = (W_{ss}, \mathbf{0}_{n_\varepsilon}', 0)$, we can write the first-order approximation in a compact form

$$Z_t \approx g^{first}(\Xi_t, s_t) = Z_{ss} + \mathcal{D}g_{ss}(s_t)(\Xi_t - \Xi_{ss}) \tag{81}$$

and

$$W_t \approx h^{first}(\Xi_t, s_t) = W_{ss} + \mathcal{D}h_{ss}(s_t)(\Xi_t - \Xi_{ss}) \tag{82}$$

Analogously, the second-order approximation can be expressed as

$$Z_t \approx g^{second}(\Xi_t, s_t) = Z_{ss} + \mathcal{D}g_{ss}(s_t)(\Xi_t - \Xi_{ss}) + \frac{1}{2}\mathcal{H}g_{ss}(s_t)[(\Xi_t - \Xi_{ss}) \otimes (\Xi_t - \Xi_{ss})] \tag{83}$$

and

$$W_t \approx h^{second}(\Xi_t, s_t) = W_{ss} + \mathcal{D}h_{ss}(s_t)(\Xi_t - \Xi_{ss}) + \frac{1}{2}\mathcal{H}h_{ss}(s_t)[(\Xi_t - \Xi_{ss}) \otimes (\Xi_t - \Xi_{ss})] \tag{84}$$

The main computational objects, $\mathcal{D}g_{ss}(s_t)$, $\mathcal{D}h_{ss}(s_t)$, $\mathcal{H}g_{ss}(s_t)$ and $\mathcal{H}h_{ss}(s_t)$, can be found by solving systems of equations obtained through the chain rule using the first and second partial derivatives of equation (79). Foerster et al. (2016) shows that the bottleneck is to solve a quadratic system. Without regime switching, one can map the system of quadratic equations to the generalized Schur decomposition problem. However, with regime switching, that method is not feasible. Instead, Foerster et al. (2016) propose using Grobner bases to solve such a quadratic system and then they apply the MSS criterion to verify the existence and uniqueness of a stable approximation of the solution.

5.3 Monetary DSGE model with endogenous switching monetary policy

Full characterization In the case of endogenous switching, we'll directly consider the full characterization of equations (29) to (40):

$$\begin{aligned}\frac{\theta}{\theta-1} =& \theta u_t \left(\frac{C_t}{A_t}\right)^{\epsilon} - \varphi(\theta u_t - \theta + 1)\frac{\Pi_t}{\Pi^*_{s_t}}\left[\frac{\Pi_t}{\Pi^*_{s_t}} - 1\right] \\ &+\beta\varphi(\theta u_t - \theta + 1)\mathbb{E}_t\left[\frac{Y_{t+1}/A_{t+1}}{Y_t/A_t}\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t}\right)^{-\epsilon}\frac{\Pi_{t+1}}{\Pi^*_{s_{t+1}}}\left(\frac{\Pi_{t+1}}{\Pi^*_{s_{t+1}}} - 1\right)\right]\end{aligned} \tag{85}$$

$$Y_t = C_t + \frac{\varphi}{2}\left[\frac{\Pi_t}{\Pi^*_{s_t}} - 1\right]^2 Y_t \tag{86}$$

$$\frac{R_t}{R^*_{s_t}} = \left(\frac{R_{t-1}}{R^*_{s_{t-1}}}\right)^{\rho_r}\left[\left(\frac{\Pi_t}{\Pi^*_{s_t}}\right)^{\alpha(s_t)}\left(\frac{Y_t}{Y_t^*}\right)^{\gamma}\right]^{1-\rho_r} e_t \tag{87}$$

$$\mathbb{E}_t\left[\frac{\beta R_t}{\Pi_{t+1}}\left(\frac{C_t/A_t}{C_{t+1}/A_{t+1}}\right)^{\epsilon}\left(\frac{A_t}{A_{t+1}}\right)\left(\frac{\xi_{t+1}}{\xi_t}\right)\right] = 1 \tag{88}$$

$$s_{t+1} = \mathbb{1}\{w_{t+1} \geq \tau\} \tag{89}$$

$$w_{t+1} = \phi w_t + v_{t+1} \tag{90}$$

where the technology fluctuation a_t, price markup shock u_t, monetary policy shock e_t and latent factor innovation v_t have the following law of motions:

$$\ln A_{t+1} = g_A + \ln A_t + \ln a_{t+1} \tag{91}$$
$$\ln a_{t+1} = \rho_a \ln a_t + \sigma_a \varepsilon^a_{t+1} \tag{92}$$
$$\ln \xi_{t+1} = \rho_\xi \ln \xi_t + \sigma_\xi \varepsilon^\xi_{t+1} \tag{93}$$
$$\ln u_{t+1} = (1-\rho_u) \ln u + \rho_u \ln u_t + \sigma_u \varepsilon^u_{t+1} \tag{94}$$
$$\ln e_{t+1} = \sigma_e \varepsilon^e_{t+1} \tag{95}$$
$$\begin{pmatrix} \varepsilon^e_t \\ v_{t+1} \end{pmatrix} =_d \mathbb{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \tag{96}$$

Equation (85) is a non-linearized new Keynesian Phillips curve reflecting the optimal price setting of intermediate firms, equation (86) reports market clearing conditions, equation (87) is the monetary policy rule and equation (88) is the Euler equation of households.

Computing transition probabilities We are aware of the time-varying transition probability (henceforth TVTP) model which is an alternative to model endogenous regime switching. However, as pointed out in footnote 7, we believe that our methodology is more systematic and captures more information. For those who are more familiar with TVTP models, it is worth noting that our model also implies a TVTP. After obtaining the implied TVTP, the difference of the two methodologies is nothing but different functional forms of TVTP.

Taking logarithm of equation (87) and combining (95) yields

$$\hat{R}_t = \rho_r \hat{R}_{t-1} + (1-\rho_r)[\alpha(s_t)\hat{\Pi}_t + \gamma \hat{Y}_t] + \hat{e}_t$$

Also consider equations (89), (90) and (96),

$$s_{t+1} = \mathbb{1}\{w_{t+1} \geq \tau\} \tag{97}$$
$$w_{t+1} = \phi w_t + v_{t+1} \tag{98}$$
$$\begin{pmatrix} \varepsilon^e_t \\ v_{t+1} \end{pmatrix} =_d \mathbb{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \tag{99}$$

Suppose $|\rho| < 1$ and $|\phi| < 1$, we compute the transition probability for regime s_t following theorem 3.1 in Chang et al. (2017). The bivariate normality of ε^e_t and v_{t+1} implies

$$v_{t+1}|\varepsilon^e_t =_d \mathbb{N}(\rho\varepsilon^e_t, 1-\rho^2) \tag{100}$$

Define

$$z_{t+1} = \frac{v_{t+1} - \rho\varepsilon^e_t}{\sqrt{1-\rho^2}} = \frac{w_{t+1} - \phi w_t}{\sqrt{1-\rho^2}} - \rho\frac{\varepsilon^e_t}{\sqrt{1-\rho^2}},$$

where

$$\varepsilon^e_t = \frac{\hat{R}_t - \rho_r \hat{R}_{t-1} - (1-\rho_r)[\alpha(s_t)\hat{\Pi}_t + \gamma\hat{Y}_t]}{\sigma_e}$$

It follows that

$$p(z_{t+1}|w_t, \varepsilon_t^e) = p(z_{t+1}|w_t, \hat{R}_t, \hat{R}_{t-1}, \hat{\Pi}_t, \hat{Y}_t) = \mathbb{N}(0,1)$$

To simplify notation, we'll use ε_t^e instead of the set of endogenous variables in the following derivation. Then,

$$\begin{aligned}
&\mathbb{P}\{w_{t+1} < \tau | w_t, \varepsilon_t^e\} \\
&= \mathbb{P}\left\{ z_{t+1} < \frac{\tau - \phi w_t}{\sqrt{1-\rho^2}} - \rho \frac{\varepsilon_t^e}{\sqrt{1-\rho^2}} \middle| w_t, \varepsilon_t^e \right\} \\
&= \Phi_\rho \left(\tau - \phi w_t - \rho \varepsilon_t^e \right)
\end{aligned} \tag{101}$$

where we use $\Phi_\rho(x) = \Phi(x/\sqrt{1-\rho^2})$ to simplify notation. Now, note that the $AR(1)$ process w_t has the unconditional density:

$$w_t =_d \mathbb{N}\left(0, \frac{1}{1-\phi^2}\right) \tag{102}$$

or

$$w_t \sqrt{1-\phi^2} =_d \mathbb{N}(0,1) \tag{103}$$

Consequently, we have

$$\begin{aligned}
&\mathbb{P}\{s_{t+1} = 0 | s_t = 0, \varepsilon_t^e\} \\
=&\mathbb{P}\{w_{t+1} < \tau | w_t < \tau, \varepsilon_t^e\} \\
=&\mathbb{P}\{w_{t+1} < \tau | w_t \sqrt{1-\phi^2} < \tau \sqrt{1-\phi^2}, \varepsilon_t^e\} \\
=&\frac{\mathbb{P}\{w_{t+1} < \tau, w_t \sqrt{1-\phi^2} < \tau \sqrt{1-\phi^2} \mid \varepsilon_t^e\}}{\mathbb{P}(w_t \sqrt{1-\phi^2} < \tau \sqrt{1-\phi^2})} \\
=&\frac{\displaystyle\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \Phi_\rho \left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho \varepsilon_t^e \right) \varphi(x) dx}{\Phi(\tau \sqrt{1-\phi^2})}
\end{aligned}$$

where ε_t^e is the monetary policy innovation at current period, which can be simply computed after observing $(\hat{R}_t, \hat{R}_{t-1}, \hat{\Pi}_t, \hat{Y}_t)$. Therefore, we can rewrite the low-to-low transition probability as

$$\begin{aligned}
&\mathbb{P}\{s_{t+1} = 0 | s_t = 0, \varepsilon_t^e\} = \mathbb{P}\{s_{t+1} = 0 | s_t = 0, \hat{R}_t, \hat{R}_{t-1}, \hat{\Pi}_t, \hat{Y}_t\} \\
&= \frac{\displaystyle\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \Phi_\rho \left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho \frac{\hat{R}_t - \rho_r \hat{R}_{t-1} - (1-\rho_r)[\alpha(s_t)\hat{\Pi}_t + \gamma \hat{Y}_t]}{\sigma_e} \right) \varphi(x) dx}{\Phi(\tau \sqrt{1-\phi^2})}
\end{aligned} \tag{104}$$

Analogously, the high-to-low transition probability can be obtained as

$$\begin{aligned}
&\mathbb{P}\{s_{t+1}=0|s_t=1,\varepsilon_t^e\}=\mathbb{P}\{s_{t+1}=0|s_t=1,\hat{R}_t,\hat{R}_{t-1},\hat{\Pi}_t,\hat{Y}_t\}\\
&=\frac{\int_{\tau\sqrt{1-\phi^2}}^{\infty}\Phi_\rho\left(\tau-\frac{\phi x}{\sqrt{1-\phi^2}}-\rho\frac{\hat{R}_t-\rho_r\hat{R}_{t-1}-(1-\rho_r)[\alpha(s_t)\hat{\Pi}_t+\gamma\hat{Y}_t]}{\sigma_e}\right)\varphi(x)dx}{1-\Phi(\tau\sqrt{1-\phi^2})}
\end{aligned}\tag{105}$$

Finally, low-to-high and high-to-high transition probabilities are readily obtained as

$$\mathbb{P}\{s_{t+1}=1|s_t=0,\varepsilon_t^e\}=1-\mathbb{P}\{s_{t+1}=0|s_t=0,\varepsilon_t^e\}\tag{106}$$

$$\mathbb{P}\{s_{t+1}=1|s_t=1,\varepsilon_t^e\}=1-\mathbb{P}\{s_{t+1}=0|s_t=1,\varepsilon_t^e\}\tag{107}$$

Thus, we can also interpret endogenous regime switching in monetary policy as: the transition probabilities of monetary regimes to next period depend on current and past endogenous variables $\hat{R}_t,\hat{R}_{t-1},\hat{\Pi}_t,\hat{Y}_t$. This finding extends the conjecture by Davig and Leeper (2008), where they claim that the monetary regime change at next period is triggered only by current inflation $\hat{\Pi}_t$. We will illustrate, in the empirical section, that our interpretation of endogenous regime change is more consistent with real data.

With the TVTP computed, we can rewrite the full characterization as equations consists of four equilibrium conditions (85) - (88), technology process and three exogenous shocks (91)-(95), and the TVTPs (104)-(107). As we have already mentioned, this set of characterization is essentially equivalent to (85)-(96).

Perturbation method When we allow for endogenous transition probabilities, to the best of our knowledge, there are only two algorithms available for perturbation methods in recent literature. One is Barthélemy and Marx (2017), which follows Cho (2016). The essence of their algorithm is the same as that of Svensson and Williams (2005), which uses functional iteration technique to find the solution of a fix-point problem, through successive approximations of the solution. The other one is Maih (2015), which is closely related to Farmer et al. (2011). They use Newton algorithm and claim that it's more efficient than functional iteration algorithm. With this said, one advantage of Barthélemy and Marx (2017)'s algorithm is that they provide a way to check the determinacy condition, while Maih (2015) doesn't. Because for the endogenous regime switching monetary DSGE model we considered, we're interested in eventually evaluating how switching parameter would affect the determinacy condition, we choose to use Barthélemy and Marx (2017)'s algorithm (henceforth BM algorithm).

Unlike Foerster et al. (2016) (see equation (79)), BM algorithm doesn't require users to distinguish between predetermined and non-predetermined variables. Our model can be simply represented as

$$\mathbb{E}_t[f_{s_t}(Z_{t+1},Z_t,Z_{t-1},\chi\varepsilon_t)]=0\tag{108}$$

where [18]

$$Z_t=(\hat{y}_t,\hat{\Pi}_t,\hat{R}_t,\hat{a}_t,\hat{\xi}_t,\hat{u}_t,\hat{e}_t)'$$

[18] $\hat{c}_t$ is substituted using market clearing condition.

is a vector of endogenous and exogenous variables with dimension $n(=7)$, ε_t is an i.i.d. multi-dimensional innovations with dimension $n_\varepsilon(=4)$, and χ is a positive scalar. The transition probability of s_t from regime i to regime j follows (104)-(107). The set U^∞ represents the set of infinite sequences $(s^t, \varepsilon^t) = ((s_t, \varepsilon_t), (s_{t-1}, \varepsilon_{t-1}), \cdots)$. We describe a solution of the model as a continuous and bounded [19] function $g : U^\infty \to \mathcal{R}^n$ of all the past shocks and regimes[20], satisfying:

$$\mathcal{N}(g,\chi)(s^t,\varepsilon^t) = \mathbb{E}_t[f_{s_t}(Z_{t+1}, Z_t, Z_{t-1}, \chi\varepsilon_t)] \tag{109}$$

$$= \sum_j \int p_{s_t j}(\varepsilon_t^e) f_j(g(js^t, \varepsilon\varepsilon^t), g(s^t,\varepsilon^t), g(s^{t-1},\varepsilon^{t-1}), \chi\varepsilon_t) f(\varepsilon) d\varepsilon = 0 \tag{110}$$

We denote by $\mathcal{B}$ for the set of such continuous and bounded functions. In this functional framework, the problem can be reinterpreted as finding the zeros of the operator $\mathcal{N}$ acting on a bounded continuous function g in $\mathcal{B}$ and the scale parameter χ. This can be done by applying the implicit function theorem to the operator $\mathcal{N}(g,\chi)$.

Steady states We define a function $g_0 \in \mathcal{B}$ as $g_0(s^t,\varepsilon^t) = \overline{Z}_{s_t}$, where $\overline{Z}_{s_t}$ is a vector of potentially regime-dependent steady states for log-deviation variables Z_t. In the case of $\Pi_0^* = \Pi_1^* = \Pi^*$, we simply have $\overline{Z}_{s_t} = 0$. Note that to define Z_t we first need to define the steady states for detrended level variables, which we denote by Z_{ss}, [21] where

$$Z_{ss} = \left(\left(\frac{1}{\theta}\right)^{1/\epsilon}, \Pi^*, \frac{exp(g_A)\Pi^*}{\beta}, 1, 1, u, 1 \right)' \tag{111}$$

From the definition of steady states, we have $\mathcal{N}(g_0, 0) = 0$, *i.e.*, $(g_0, 0)$ is a reference point for $\mathcal{N}(g,\chi)$.

First-order By application of the implicit function theorem around the reference point $(g_0, 0)$, we know that the solution satisfies

$$g(\chi) = g_0 + \chi Z^1 + o(\chi) \tag{112}$$

where $Z^1 = \frac{\partial Z}{\partial \chi}|_{(\overline{Z}_{s_t}, 0)}$ is solution of

$$D_g\mathcal{N}(g_0,0)Z^1 = -D_\chi\mathcal{N}(g_0,0) \tag{113}$$

By equation (109), we compute:

$$D_\chi\mathcal{N}(g_0,0) = d(s_t)\varepsilon_t \tag{114}$$

[19] This relates to the stability concept for solutions and hence determinacy condition. Please refer to Definition 1 and Appendix B.2 in Barthélemy and Marx (2017) for details.

[20] Essentially, regime change should be regarded as an additional source of shock. But very different from the i.i.d. gaussian density for ε_t, s_t is a time-inhomogeneous Markov process.

[21] Note that in a model with trend, the steady state (gross) real interest rate, hence nominal interest rate, will be affected by the trend growth rate, in our case, g_A.

$$D_g\mathcal{N}(g_0,0)\frac{\partial Z}{\partial \chi} = \mathbb{E}_t^0\left[a(s_t,s_{t+1})\frac{\partial Z_{t+1}}{\partial \chi}\right] + b(s_t)\frac{\partial Z_t}{\partial \chi} + c(s_t)\frac{\partial Z_{t-1}}{\partial \chi} \tag{115}$$

where the subscript 0 in $\mathbb{E}_t^0$ denotes that the underlying transition probabilities of the expectations operator are constant:

$$\overline{p}_{ij} = p_{ij}(0)$$

and matrices a, b, c and d are regime-dependent matrices satisfying:

$$\underbrace{a(s_t,s_{t+1})}_{n\times n} = \partial_1 f_{s_t}(\overline{Z}_{s_{t+1}},\overline{Z}_{s_t},\overline{Z}_{s_{t-1}},0) \tag{116}$$

$$\underbrace{b(s_t)}_{n\times n} = \sum_j p_{s_tj}(0)\partial_2 f_{s_t}(\overline{Z}_j,\overline{Z}_{s_t},\overline{Z}_{s_{t-1}},0) \tag{117}$$

$$\underbrace{c(s_t)}_{n\times n} = \sum_j p_{s_tj}(0)\partial_3 f_{s_t}(\overline{Z}_j,\overline{Z}_{s_t},\overline{Z}_{s_{t-1}},0) \tag{118}$$

$$\underbrace{d(s_t)}_{n\times n_\varepsilon} = \sum_j [\partial p_{s_tj}(0) f_{s_t}(\overline{Z}_j,\overline{Z}_{s_t},\overline{Z}_{s_{t-1}},0) + p_{s_tj}(0)\partial_4 f_{s_t}(\overline{Z}_j,\overline{Z}_{s_t},\overline{Z}_{s_{t-1}},0)] \tag{119}$$

Thus, by applying the forward method in Cho (2016) to equation (113), we obtain

$$\begin{aligned} Z^1(s^t) &= \underbrace{R_\infty^{-1}(s_t)}_{n\times n}\underbrace{c(s_t)}_{n\times n} Z^1(s^{t-1}) - \underbrace{R_\infty^{-1}(s_t)}_{n\times n}\underbrace{d(s_t)}_{n\times n_\varepsilon}\varepsilon_t \\ &= \underbrace{(R_\infty^{-1}(s_t)c(s_t), R_\infty^{-1}(s_t)d(s_t))}_{n\times(n+n_\varepsilon)}\begin{pmatrix} Z^1(s^{t-1}) \\ \varepsilon_t \end{pmatrix} \\ &= \underbrace{A(s_t)}_{n\times(n+n_\varepsilon)}\begin{pmatrix} Z^1(s^{t-1}) \\ \varepsilon_t \end{pmatrix} \end{aligned} \tag{120}$$

where R_∞ is the limit of a recursively defined sequence of matrices. Please see details in equation (6) of Barthélemy and Marx (2017) for the definition of sequence $\{R_k\}$. Finally, the first-order perturbation solution (112) and (120) can be written in a compact form as follows

$$Z_t = \overline{Z}_{s_t} + \underbrace{A(s_t)}_{n\times(n+n_\varepsilon)}\begin{pmatrix} Z_{t-1} - \overline{Z}_{s_{t-1}} \\ \chi\varepsilon_t \end{pmatrix} \tag{121}$$

In particular, as mentioned in the above, in the case of $\Pi_0^* = \Pi_1^* = \Pi^*$, we simply have $\overline{Z}_{s_t} = 0$. Then, (121) is reduced to

$$Z_t = \underbrace{A(s_t)}_{n\times(n+n_\varepsilon)}\begin{pmatrix} Z_{t-1} \\ \chi\varepsilon_t \end{pmatrix} \tag{122}$$

which has a clean conditional linear gaussian error structure.

Second-order To obtain the second-order approximation, we take the second order of total differential to

$$\mathcal{N}(g(\chi),\chi)=0 \tag{123}$$

and apply again the forward method in Cho (2016). Detailed derivation can be found in Appendix E.2 in Barthélemy and Marx (2017). Solution is represented as follows

$$Z_t=\overline{Z}_{s_t}+\chi Z_t^1+\frac{\chi^2}{2}Z_t^2+o(\chi^2) \tag{124}$$

where

$$Z_t^1=\underbrace{R_\infty^{-1}(s_t)}_{n\times n}\underbrace{c(s_t)}_{n\times n}Z_{t-1}^1-\underbrace{R_\infty^{-1}(s_t)}_{n\times n}\underbrace{d(s_t)}_{n\times n_\varepsilon}\varepsilon_t=\underbrace{A_1(s_t)}_{n\times(n+n_\varepsilon)}\begin{pmatrix}Z_{t-1}^1\\ \varepsilon_t\end{pmatrix} \tag{125}$$

$$\begin{aligned}Z_t^2=&\underbrace{R_\infty^{-1}(s_t)}_{n\times n}\underbrace{c(s_t)}_{n\times n}Z_{t-1}^2+\underbrace{\alpha(s_t)}_{n\times n^2}Z_{t-1}^1\otimes Z_{t-1}^1+\underbrace{\beta(s_t)}_{n\times(n+n_\varepsilon)}Z_{t-1}^1\otimes\varepsilon_t+\underbrace{\gamma(s_t)}_{n\times n_\varepsilon^2}\varepsilon_t\otimes\varepsilon_t+\underbrace{\delta(s_t)}_{n\times n_\varepsilon^2}Vect(\Sigma)\\ =&\underbrace{R_\infty^{-1}(s_t)c(s_t)}_{n\times n}Z_{t-1}^2+\underbrace{A_2(s_t)}_{n\times(n+n_\varepsilon)^2}\begin{pmatrix}Z_{t-1}^1\\ \varepsilon_t\end{pmatrix}\otimes\begin{pmatrix}Z_{t-1}^1\\ \varepsilon_t\end{pmatrix}+\underbrace{\delta(s_t)}_{n\times n_\varepsilon^2}Vect(\Sigma)\end{aligned} \tag{126}$$

where $A_2(s_t)$ is constructed by using matrices α,β and γ. [22] Note that in this solution form, Z_t^1 and Z_t^2 are each defined iteratively, so we can't simply merge them into Z_t and rewrite the solution in a more compact way as we did in (121). The reason lies in the fact that we apply functional iteration technique twice to derive first-oder and second-order approximations separately. However, this not-so-clean solution form should not impose any trouble on the estimation practice. At each period, we simply compute Z_t^1 and Z_t^2 and substitute the results into equation (124). Essentially, it is still of a conditional quadratic gaussian error structure. [23]

[22] When coding, we don't need to construct $A_2(s_t)$. We could directly use α,β and γ to compute Z_t^2. I show this alternative representation to compare with the solution form of Maih (2015), which writes

$$Z_t=\overline{Z}_{s_t}+\underbrace{A_1(s_t)}_{n\times(n+n_\varepsilon+1)}\begin{pmatrix}Z_{t-1}-\overline{Z}_{s_{t-1}}\\ \varepsilon_t\\ \chi\end{pmatrix}+\frac{1}{2}\underbrace{A_2(s_t)}_{n\times(n+n_\varepsilon+1)^2}\begin{pmatrix}Z_{t-1}-\overline{Z}_{s_{t-1}}\\ \varepsilon_t\\ \chi\end{pmatrix}\otimes\begin{pmatrix}Z_{t-1}-\overline{Z}_{s_{t-1}}\\ \varepsilon_t\\ \chi\end{pmatrix} \tag{127}$$

[23] One way to work around the square terms of gaussian errors is as below:
(i) Introducing a dummy variable d_t and replacing ε_t in (125) and (126) with the dummy d_t;
(ii) Adding one more set of equations to the solution system. That is, $d_t=\varepsilon_t$.
In this way, the whole solution system is then driven by gaussian errors ε_t (now without square terms). The shocks are indirectly propagated into the system through d_t.

6 Estimation of monetary DSGE models

6.1 Contact with literature

The Great Moderation starting in the mid-1980s raised the famous "good luck" or "good policy" debate. To answer this question, literature have used the workhorse new Keynesian model with simple monetary policy rule – Taylor rule, which allows for a real effect of monetary policy. Lubik and Schorfheide (2004) use Bayesian approach (with Kalman filter) to estimate a fixed-regime monetary DSGE model over three subsample periods. They find that the monetary policy post 1982 is consistent with determinacy, whereas the pre-Volcker policy is not, which supports "good policy" argument. The fixed-regime monetary DSGE model is solved by the approach of Sims (2002), extended in Lubik and Schorfheide (2003).

In the spirit of Lucas (1976) critique, the empirical observations of time variation call for regime switching monetary policy: If policy has switched in the past, it might be expected to switch again in the future. Davig and Doh (2014) use Bayesian approach (with approximate Kalman filter of Kim et al. (1999)) to estimate a Markov-switching monetary DSGE model that allows shifts in the monetary policy reaction coefficients and structural shock volatilities. They find that a more aggressive monetary policy regime was in place after the Volcker disinflation and before 1970 than during the Great Inflation of the 1970s. Their estimates also indicate that a low-volatility regime has been in place during most of the sample period after 1984. They then connect the timing of the different regimes to a measure of inflation persistence. Bianchi (2012) uses Bayesian approach (with Gibbs sampling algorithm) to estimate a similar model as Davig and Doh (2014) but with additional interest rate smoothing parameter. His model also allows switching in both policy coefficients and shock variances. His estimates capture better features for recent crisis. To explore the role of agents' beliefs he applies counterfactual simulations and finds that: If, in the 1970s, agents had anticipated the appointment of an extremely conservative Chairman, inflation would have been lower. The large drop in inflation and output at the end of 2008 would have been mitigated if agents had expected the Federal Reserve to be exceptionally active in the near future. These two papers both use MSLRE method (Farmer et al., 2011) to solve the exogenous regime switching monetary DSGE models.

However, most people who think that policy changed dramatically in the early 1980s in the United States believe that it did so because inflation appeared to be running out of control, not because an exogenously evolving switching process happened to call for a change at that date. This belief calls for a model that makes the policy change as a purposeful response of central bank to endogenous variables (e.g. inflation) in the economy. Only recently, Maih (2015) and Barthélemy and Marx (2017) develop perturbation methods for solving endogenous regime switching DSGE models. Estimation of monetary DSGE model with endogenous regime switching hasn't been done yet.

6.2 Estimation strategy

6.2.1 State space representation

We use the first-order perturbation solution. This conditionally linear gaussian structure makes feasible the application of our newly developed Kalman filter with Markov switching. Algorithm for the filter is provided in Appendix B. Given the solution (122) for detrended variables, we connect to real data through a regime-dependent state space representation. First, we denote the observed real data as

$$\Upsilon_t = (YGR_t, INF_t, INT_t)'.$$

The sample period is from 1954:Q3 to 2016:Q4.

Quarterly per capita real GDP growth rate (in percentage) YGR_t can be linked to model variables as follows

$$\begin{aligned} YGR_t &= \frac{Y_t - Y_{t-1}}{Y_{t-1}} \times 100 \ \ \text{(definition)} \\ &= \left(\frac{Y_t/A_t}{Y_{t-1}/A_{t-1}} \frac{A_t}{A_{t-1}} - 1 \right) \times 100 \ \ \text{(detrending)} \\ &\approx \ln \left(\frac{y_t}{y_{t-1}} e^{g_A} a_t \right) \times 100 \\ &= (g_A + \hat{y}_t - \hat{y}_{t-1} + \hat{a}_t) \times 100 \\ &= g_A^{(Q)} + (\hat{y}_t - \hat{y}_{t-1} + \hat{a}_t) \times 100 \end{aligned} \tag{128}$$

Next, annualized net inflation rate (in percentage) INF_t can be represented as

$$\begin{aligned} INF_t &= 400(\Pi_t - 1) \ \ \text{(definition)} \\ &= 400(\Pi^* - 1) + 400[(\Pi_t - 1) - (\Pi^* - 1)] \\ &\approx \pi^{(A)} + 400\hat{\Pi}_t \end{aligned} \tag{129}$$

where $\pi^{(A)}$ is annualized net inflation target in percentage.

Finally, annualized net nominal interest rate (in percentage) INT_t can be represented as

$$\begin{aligned} INT_t &= 400(R_t - 1) \ \ \text{(definition)} \\ &= 400(R^* - 1) + 400[(R_t - 1) - (R^* - 1)] \\ &\approx 400 \left[g_A + \left(\frac{1}{\beta} - 1 + (\Pi^* - 1) \right) \right] + 400\hat{R}_t \\ &= 4g_A^{(Q)} + r^{(A)} + \pi^{(A)} + 400\hat{R}_t \end{aligned} \tag{130}$$

We collect the steady state vales as

$$\Upsilon_{ss}(s_t, \Theta)^{24} = (g_A^{(Q)}, \pi^{(A)}, 4g_A^{(Q)} + r^{(A)} + \pi^{(A)})'$$

[24]The vector becomes regime dependent if we allow regime specific inflation targets $\Pi_0^* \neq \Pi_1^*$.

For filtering convenience, we augment the state vector by including latent endogenous regime factor w_{t+1}. We also include dummy state variables $\hat{y}_{t-1}$. Therefore,

$$X_t = (\hat{y}_t, \hat{\Pi}_t, \hat{R}_t, \hat{a}_t, \hat{\xi}_t, \hat{u}_t, \hat{e}_t, w_{t+1}, \hat{y}_{t-1})'$$

Correspondingly, we also augment shock vector as

$$\varepsilon_t = (\varepsilon_t^a, \varepsilon_t^\xi, \varepsilon_t^u, \varepsilon_t^e, \eta_{t+1})'$$

Then, the measurement equation [25] is

$$\Upsilon_t = \Upsilon_{ss}(s_t, \Theta) + \underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{D} X_t + \nu_t, \quad \nu_t \sim \mathbb{N}(0, \Sigma_\nu) \tag{131}$$

where ν_t is measurement error term and $\Sigma_\nu = 0.2 diag(var(\Upsilon_t))$. Transition equations consist of solution for detrended variables (122) and law of motions for shocks. It can be simply represented as

$$X_t = \underbrace{G(s_t, \Theta)}_{9\times 9} X_{t-1} + \underbrace{M(s_t, \Theta)}_{9\times 5} \varepsilon_t, \quad \varepsilon_t \sim \mathbb{N}(0, \mathbf{1}) \tag{132}$$

where[26]

$$G(s_t, \Theta) = \begin{pmatrix} \underbrace{R_\infty^{-1}(s_t) c(s_t)}_{7\times 7} & \underbrace{0}_{7\times 2} \\ \underbrace{0}_{1\times 7} & \begin{matrix} \phi & 0 \end{matrix} \\ 1 \quad \underbrace{0}_{1\times 6} & \begin{matrix} 0 & 0 \end{matrix} \end{pmatrix} \tag{133}$$

and

$$M(s_t, \Theta) = \begin{pmatrix} \underbrace{R_\infty^{-1}(s_t) d(s_t)}_{7\times 4} & \underbrace{0}_{7\times 1} \\ \begin{matrix} 0 & 0 & 0 & \rho \end{matrix} & \sqrt{1-\rho^2} \\ \underbrace{0}_{1\times 5} & \end{pmatrix} \tag{134}$$

With the above state space representation, likelihood function can be evaluated by using our newly-developed Kalman filter with Markov switching, which combines Kalman filter with the modified Markov switching filter first introduced by Chang et al. (2017). Detailed algorithm is provided in Appendix B.

[25] Henceforth, we implicitly regard X_t as $100X_t$. Then, in (132), we need to scale ε_t by 100.

[26] Matrices $R_\infty^{-1}(s_t)c(s_t)$ and $R_\infty^{-1}(s_t)d(s_t)$ are from first-order perturbation solution. See equation (120) for details.

6.2.2 quasi-Bayesian MLE

As is well known, implementation of MLE can be particularly challenging in applications involving DSGE models. Several classes of problems can arise in minimizing the negative log-likelihood function: discontinuities, multiple local minima, and identification.

Quasi-Bayesian methods can be used in order to help induce curvature in the likelihood function. Assuming that we have obtained log-likelihood function $\log L(Y_{1:T}|\Theta)$ from the Kalman filter with Markov switching, quasi-Bayesian method involves augmenting the log-likelihood function with a prior distribution specified over Θ, denoted by $p(\Theta)$. Then, quasi-Bayesian ML estimator is

$$\hat{\Theta} = \underset{\Theta \in R(\Theta)}{\arg\min} - \log L(Y_{1:T}|\Theta) - \log p(\Theta) \tag{135}$$

Notice that the estimator corresponds to the mode of the log of the posterior distribution $p(\Theta|Y_{1:T})$. In the special case where the prior is diffuse or uninformative, the estimator $\hat{\Theta}$ converges to the classical ML estimator. When the prior is proper, the quasi-Bayesian estimate of Θ may be interpreted as the one obtained by maximization of a penalized log-likelihood function. The penalty $\log p(\Theta)$ depends on the strength of the researcher's prior about Θ and has the effect of "pulling" the estimator towards the mean of the prior density.

6.3 Empirical analysis

6.3.1 Prior distribution

The last three columns in Table 1 reports prior distributions for all parameters. The priors for all but those specific to our endogenous feedback mechanism are in line with previous results in the literature and are relatively loose. For example, the parameters α_0, α_1 of coefficients to inflation and annualized net inflation target $\pi^{(A)}$ are from Bianchi (2012), quarterly growth rate of technology $\gamma^{(Q)}$ is from Davig and Doh (2014). Other structural parameters are also close to those in literature. For instance, coefficient to output gap γ has the same prior distribution and mean as Bianchi (2012) but with a smaller standard deviation to reflect their estimation results and our stronger belief. Annualized net real interest rate $r^{(A)}$ has the same prior distribution and standard deviation but with a mean closer to the estimation result of Bianchi (2012). Risk aversion ϵ is set to be close to that of Davig and Doh (2014) but with a slightly larger mean and standard deviation. In a fully linearized model, steady-state elasticity of substitution between intermediate goods θ and level of price stickiness φ are not separately identified. Instead, an identified parameter κ captures their relationship is estimated. We use this information combined with our preliminary estimates from full MLE to set the prior for θ and φ. Autocorrelation coefficient and standard deviations for shocks are set to be relatively uninformative and loose as Beta(0.5,0.2) and Inverse Gamma(2,0.1). Finally, the most important new parameters in our endogenous feedback mechanism are set to be relatively uninformative.

6.3.2 Empirical results

Concerning the parameters of the Taylor rule, coefficients to inflation are both estimated to be larger than 1, hence both aggressive. This is in line with the monetary-fiscal interaction literature. In our model, we don't include fiscal policy, *i.e.* implicitly assume passive fiscal regime. Therefore, both monetary policy regimes are aggressive, with one more aggressive than the other. Other parameter estimates are in line with previous literature. For instance, annualized inflation target is estimated to be 1.84$\%$ and annualized real interest rate is 0.67$\%$. We take a more serious look at the estimates of parameters in endogenous feedback mechanism. Autocorrelation coefficient of latent regime factor ϕ is estimated to be 0.69, which is relatively persistent. Threshold level τ is estimated to be -0.27. The fact that it is below 0 implies that the unconditional probability of monetary policy being at more aggressive regime is larger than one half.

Most importantly, endogeneity parameter ρ is estimated to be 0.87. As we pointed out earlier, ρ^2 captures the contribution of past monetary policy intervention to the current monetary policy regime change. Therefore, we are readily to conclude that monetary policy shifts are 76$\%$ driven by past monetary policy interventions.

Table 1: Quasi-Bayesian Maximum Likelihood Estimate

	Posterior		Prior		
Parameter	Mode	(Std Err)	Density	Para(1)	Para(2)
α_0	1.1217	(0.0915)	G	1	(0.4)
α_1	1.7956	(0.0935)	G	1.8	(0.4)
γ	0.3165	(0.0008)	G	0.25	(0.05)
ρ_r	0.6517	(0.0152)	B	0.5	(0.2)
$\pi^{(A)}$	1.8483	(0.1514)	G	2	(0.2)
$\mathrm{r}^{(A)}$	0.6688	(0.0066)	G	0.8	(0.2)
ϵ	4.1287	(0.4792)	G	2	(0.5)
θ	6.4708	(2.5386)	G	8	(3)
φ	56.0391	(8.3187)	G	67	(10)
$\gamma^{(Q)}$	0.3987	(0.0006)	G	0.5	(0.1)
ρ_a	0.1771	(0.0069)	B	0.5	(0.2)
σ_a	0.0123	(0.00003)	I	2	(0.1)
ρ_ξ	0.9693	(0.0018)	B	0.5	(0.2)
σ_ξ	0.0500	(0.0002)	I	2	(0.1)
ρ_u	0.4799	(0.0005)	B	0.5	(0.2)
σ_u	0.0721	(0.00009)	I	2	(0.1)
σ_e	0.0103	(0.0001)	I	2	(0.1)
ϕ	0.6940	(0.0021)	B	0.5	(0.3)
τ	-0.2651	(0.3083)	N	0	(1)
ρ	0.8745	(0.0006)	N	0	(0.5)

Notes: Paras (1) and (2) list the means and standard deviations for Gamma, Beta, and Normal distributions; ν and s for the Inverse Gamma distribution, where $p(\sigma|\nu, s) \propto \sigma^{-\nu-1} e^{-\frac{\nu s^2}{2\sigma^2}}$

Figure 9 shows the extracted latent regime factor. Shaded periods represent the less aggressive regimes with regime factor being below the estimated threshold level. To facilitate the interpretation of the results, the two panels of Figure 10 display the actual annualized inflation and filtered monetary policy intervention. During a period characterized by high (low) inflation, a large negative (positive) monetary policy interventions reflecting the deviations from Taylor rule implies that the Fed is not responding strongly enough to inflation deviations. This empirical result is consistent with Bianchi (2012).

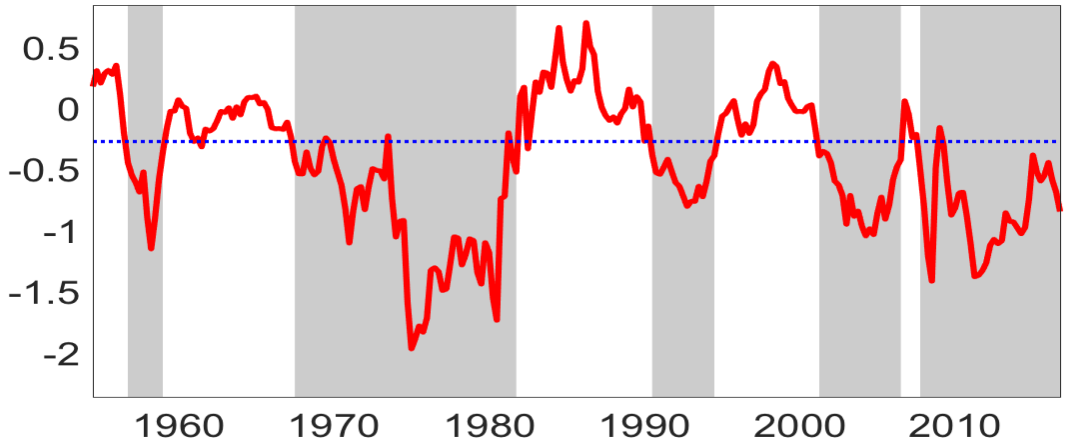


Figure 9: Extracted latent regime factor

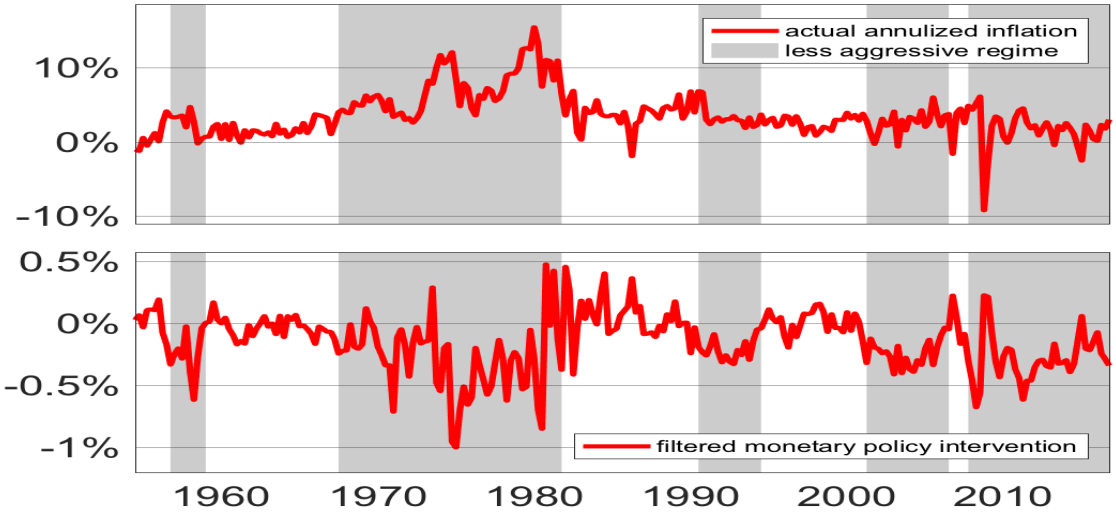


Figure 10: Actual inflation rate and filtered monetary policy intervention

The magnitude of filtered monetary policy intervention from our endogenous regime-switching

DSGE model is much smaller than that from the original Taylor rule, shown in Figure 1. This readily implies that endogenous regime switching is an important feature missing from the original Taylor rule. Monetary policy shocks can be essentially separated into two types: inter-regime monetary policy interventions and intra-regime switches. Again, we emphasize that our estimation of ρ implies that intra-regime monetary policy switches are largely driven by past inter-regime monetary policy interventions, which are small but sustained. This empirical result nicely support the claim by Leeper and Zha (2003) that small and sustained policy intervention may significantly shift agents' beliefs about policy regime and induce the changes in behavior that Lucas (1976) emphasizes. It is worth noting that the filtered monetary policy interventions are still very much persistent, magnitudes are smaller though comparing with Figure 1.

Monetary policy turns out to be more aggressive during the early years of the sample, from 1954 to 1967, except a short period on the second half of 1950s where inflation suddenly went up but soon controlled. During that time, Fed Chairman Martin had a clear goal in mind, that the Fed would raise interest rates in response to an overheated economy.

Towards the end of the 1960s, the latent regime factor started to fall below the threshold and monetary policy moved into the first significant and persistent less aggressive regime. As the inflation seemed to decrease in the early 1970s, latent regime factor climbed upwards to the threshold. However, as inflations raised again to a higher level in the mid-1970s, latent regime factor fell sharply which implies that the strength of less aggressive regime has been dramatically reinforced. The Fed Chairman Arthur F. Burns is often thought to be responsible for the high and variable inflation that prevailed during the 1970s. It is commonly accepted that on several occasions he had to succumb to the requests of the White House. In fact, the estimates show that for almost the entire duration of his mandate, the Fed followed a less aggressive regime. During these years, the more aggressive regime would have required a much higher federal funds rate.

This long period of less aggressive monetary policy ended around the second half of 1981, shortly after Paul Volcker took office in August 1979. Volcker was appointed with the precise goal of ending the high inflation. The jump in the latent regime factor to be significantly above the threshold confirms the widespread belief that he delivered on his commitment.

The estimation on the second half of our sample periods coincides very much with the results in Chang and Kwak (2016). Since 1981, monetary policy was more aggressive until the second half of 2007, right before the Great Recession, except for two relatively short periods following the recessions in 1991 and 2001. After the recent Great Recession, monetary policy has become less aggressive until the end of our sample periods.

To summarize, our results strongly support the idea that the appointment of Volcker marked a change in the Fed's inflation stance compared to the 1970s. Furthermore, the change in the regime is very much driven by the contractionary policy interventions around 1980s. At the same time, the evolution of regime factor captures repeated fluctuations between a more aggressive and a less aggressive regime for Post-World War II U.S. monetary policy.

7 Future work and conclusions

This work is still in progress. In this section, we point out our directions of future work and propose several interesting and potentially important extensions based on our current work.

7.1 Future work

Include all structural shocks In our current work, we assumed that latent regime factor shock is only correlated to past monetary policy intervention. This is motivated by the claim by Leeper and Zha (2003) that certain types of policy intervention may significantly shift agents' beliefs about policy regime and induce the changes in behavior and the observations from Figure 1. However, in the purpose of a fully structural investigation of monetary policy shifts, it's natural to extend to allow all past structural – both policy and non-policy – shocks to jointly determine current regime, that is

$$\begin{pmatrix} \varepsilon_{t-1} \\ v_t \end{pmatrix} \sim \mathbb{N}\left(\begin{pmatrix} 0_n \\ 0 \end{pmatrix}, \begin{pmatrix} I_n & \rho \\ \rho' & 1 \end{pmatrix} \right) \tag{136}$$

where $\varepsilon_t = [\varepsilon_{1,t}, \cdots, \varepsilon_{n,t}]'$ and $\rho = [\rho_1, \cdots, \rho_n]'$. By projecting v_t onto ε_t, we can rewrite v_t as

$$v_t = \sum_{k=1}^{n} \rho_k \epsilon_{k,t-1} + \sqrt{1 - \sum_{k=1}^{n} \rho_k^2}\eta_t, \quad \eta_t \sim \mathbb{N}(0,1) \tag{137}$$

where $\{\epsilon_{k,t-1}\}_{k=1}^n$ and η_t are all orthogonal and have unit variance. Note that

$$w_{t+h} = \phi^h w_t + \sum_{j=1}^{h} \phi^{h-j} v_{t+j}, \quad h \geq 1 \tag{138}$$

and hence forecast error of w_{t+h} can be computed as

$$Var_t(w_{t+h}) = Var(w_{t+h} - \mathbb{E}(w_{t+h}|\mathcal{F}_t)) \tag{139}$$

$$= \sum_{j=1}^{h} \phi^{2(h-j)} = \sum_{k=1}^{n} \underbrace{\rho_k^2 \sum_{j=1}^{h} \phi^{2(h-j)}}_{\text{k-th structural}} + \underbrace{\left(1 - \sum_{k=1}^{n} \rho_k^2\right) \sum_{j=1}^{h} \phi^{2(h-j)}}_{\text{non-structural}}, \quad h \geq 1 \tag{140}$$

It follows that the percent of the h-step ahead forecast error variance due to the k-th structural shock is given by ρ_k^2, which is independent of h. Hence, we interpret the magnitude of ρ_k^2 as "the contribution of k-th structural shock to the monetary policy shifts".

Study latent regime factor We can use the adaptive LASSO method to search among a bunch of macroeconomic variables for the principal factors that effectively explain the dynamics of policy

regime factor. Basically, it adds a penalty for model complexity to ordinary least squares regression, yielding solutions which are sparse in terms of the regression coefficients. Let w denotes monetary policy regime factor and X the set of potential candidate variables. Then the adaptive LASSO estimator is given as

$$\hat{\beta}_L = \arg\min_{\beta}(w - X\beta)'(w - X\beta) + \lambda \sum_{i=1}^{N} \frac{|\beta_i|}{\hat{\beta}_i} \tag{141}$$

where λ is a nonnegative regularization parameter, N the dimension of X and $|\hat{\beta}_i|$ the adaptive weight based on the OLS or ridge regression estimator $\hat{\beta}_i$, which is consistent for the true coefficient β_i.

Model fit When $\rho = 0$, our model reduces to an conventional exogenous regime switching monetary DSGE model. Further, if we impose $\alpha_0 = \alpha_1$, then it reduces to a fixed regime model. By employing Bayesian approach, we're able to compare model fit among these three nested setups.

7.2 Extensions

Different information structure We may consider different assumptions on the information structure. Currently, regime factor is assumed to be unobservable to both agents and monetary authority. First, if we assume agents and monetary authority can both observe regime factor. In this case, deriving analytical solution is more challenging. But, it's interesting to see that by observing this additional information of regime strength, how macroeconomic dynamics and expectations formation effect vary. Second, more realistically, if we assume regime factor is observable to monetary authority but latent to agents. This raises the issue of how agents infer and learn what regime factor is. Our model would provide a framework to study this asymmetric information structure.

Monetary-fiscal interaction In our current model, there're four structural shocks as candidates of driving forces to monetary policy regime change. Based on various theoretical and empirical findings in previous literature, it's natural and interesting to include fiscal policy into the DSGE model and consider it as another important candidate that drives choice of monetary policy regime.

7.3 Conclusions

We introduce a novel endogenous regime switching approach to a prototype monetary DSGE model. The endogeneity brings about a new feedback mechanism from past monetary policy interventions to central bank's current regime selection. To better illustrate how this mechanism works, we derive an analytical solution to a regime switching Fisherian model with the endogenous feedback mechanism. Numerical experiments show that this feedback mechanism would significantly affect the macroeconomic dynamics and expectations formation effect. In the empirical part, estimation shows that Post-World War II U.S. monetary policy regime change is 76% driven by past

monetary policy interventions. The extracted latent regime factor indicate that monetary policy is identified by repeated fluctuations between a more aggressive and a less aggressive regime, with the latter prevailing in the 1970s and during the recent crisis.

References

S. An and F. Schorfheide. Bayesian analysis of DSGE models. *Econometric reviews*, 26(2-4): 113–172, 2007.

J. Barthélemy and M. Marx. Solving endogenous regime switching models. *Journal of Economic Dynamics and Control*, 77:1–25, 2017.

B. S. Bernanke and I. Mihov. Measuring monetary policy. Technical report, National Bureau of Economic Research, 1995.

F. Bianchi. Regime switches, agents' beliefs, and post-world war ii us macroeconomic dynamics. *The Review of Economic Studies*, 2012.

F. Bianchi and C. Ilut. Monetary/fiscal policy mix and agents' beliefs. Technical report, National Bureau of Economic Research, 2014.

O. J. Blanchard and C. M. Kahn. The solution of linear difference models under rational expectations. *Econometrica: Journal of the Econometric Society*, pages 1305–1311, 1980.

Y. Chang and B. Kwak. Endogenous monetary-fiscal regime change in the united states. 2016.

Y. Chang, Y. Choi, and J. Y. Park. A new approach to model regime switching. *Journal of Econometrics*, 196(1):127–143, 2017.

S. Cho. Sufficient conditions for determinacy in a class of markov-switching rational expectations models. *Review of Economic Dynamics*, 21:182–200, 2016.

R. Clarida, J. Gali, and M. Gertler. Monetary policy rules and macroeconomic stability: evidence and some theory. Technical report, National bureau of economic research, 1998.

R. Clarida, J. Gali, and M. Gertler. The science of monetary policy: a new keynesian perspective. Technical report, National bureau of economic research, 1999.

T. Davig and T. Doh. Monetary policy regime shifts and inflation persistence. *Review of Economics and Statistics*, 96(5):862–875, 2014.

T. Davig and E. M. Leeper. Generalizing the taylor principle. *The American economic review*, 97 (3):607–635, 2007.

T. Davig and E. M. Leeper. Endogenous monetary policy regime change. In *NBER International Seminar on Macroeconomics 2006*, pages 345–391. University of Chicago Press, 2008.

F. X. Diebold, J.-H. Lee, and G. C. Weinbach. Regime switching with time-varying transition probabilities. *Business Cycles: Durations, Dynamics, and Forecasting*, pages 144–165, 1994.

R. E. Farmer, D. F. Waggoner, and T. Zha. Understanding markov-switching rational expectations models. *Journal of Economic theory*, 144(5):1849–1867, 2009.

R. E. Farmer, D. F. Waggoner, and T. Zha. Minimal state variable solutions to markov-switching rational expectations models. *Journal of Economic Dynamics and Control*, 35(12):2150–2166, 2011.

A. J. Filardo. Business-cycle phases and their transitional dynamics. *Journal of Business & Economic Statistics*, 12(3):299–308, 1994.

A. J. Filardo and S. F. Gordon. Business cycle durations. *Journal of Econometrics*, 85(1):99–123, 1998.

A. Foerster, J. F. Rubio-Ramz, D. F. Waggoner, and T. Zha. Perturbation methods for markov-switching dynamic stochastic general equilibrium models. *Quantitative Economics*, 7(2): 637–669, 2016.

M. Goodfriend. Interest rates and the conduct of monetary policy. In *Carnegie-Rochester conference series on public policy*, volume 34, pages 7–30. Elsevier, 1991.

J. D. Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica: Journal of the Econometric Society*, pages 357–384, 1989.

P. N. Ireland. Technology shocks in the new keynesian model. *Review of Economics and Statistics*, 86(4):923–936, 2004.

K. L. Judd. *Numerical methods in economics*. MIT press, 1998.

C.-J. Kim, C. R. Nelson, et al. *State-space models with regime switching: classical and Gibbs-sampling approaches with applications*, volume 2. MIT press Cambridge, MA, 1999.

C.-J. Kim, J. Piger, and R. Startz. Estimation of markov regime-switching regression models with endogenous switching. *Journal of Econometrics*, 143(2):263–273, 2008.

E. M. Leeper and T. Zha. Modest policy interventions. *Journal of Monetary Economics*, 50(8): 1673–1700, 2003.

Z. Liu, D. F. Waggoner, and T. Zha. Sources of macroeconomic fluctuations: A regime-switching dsge approach. *Quantitative Economics*, 2(2):251–301, 2011.

T. A. Lubik and F. Schorfheide. Computing sunspot equilibria in linear rational expectations models. *Journal of Economic dynamics and control*, 28(2):273–285, 2003.

T. A. Lubik and F. Schorfheide. Testing for indeterminacy: an application to US monetary policy. *The American Economic Review*, 94(1):190–217, 2004.

R. E. Lucas. Econometric policy evaluation: A critique. In *Carnegie-Rochester conference series on public policy*, volume 1, pages 19–46. North-Holland, 1976.

J. Maih. Efficient perturbation methods for solving regime-switching DSGE models. 2015.

B. T. McCallum. On non-uniqueness in rational expectations models: An attempt at perspective. *Journal of monetary Economics*, 11(2):139–168, 1983.

J. J. Rotemberg. Sticky prices in the united states. *Journal of Political Economy*, 90(6):1187–1211, 1982.

S. Schmitt-Grohé and M. Uribe. Solving dynamic general equilibrium models using a second-order approximation to the policy function. *Journal of economic dynamics and control*, 28 (4):755–775, 2004.

C. A. Sims. Solving linear rational expectations models. *Computational economics*, 20(1):1–20, 2002.

C. A. Sims and T. Zha. Were there regime switches in US monetary policy? *The American Economic Review*, 96(1):54–81, 2006.

L. Svensson and N. Williams. Monetary policy with model uncertainty: distribution forecast targeting. Technical report, National Bureau of Economic Research, 2005.

J. B. Taylor. Discretion versus policy rules in practice. In *Carnegie-Rochester conference series on public policy*, volume 39, pages 195–214. Elsevier, 1993.

M. Woodford and C. E. Walsh. Interest and prices: Foundations of a theory of monetary policy, 2005.

Appendices

A Derivation of analytical solution to a regime-switching Fisherian model

A.1 Analytical derivation

Under the assumed information structure, agents form their expectation of next period inflation by

$$\mathbb{E}_t\pi_{t+1} = \mathbb{E}(\pi_{t+1}|\mathcal{F}_t), \quad \mathcal{F}_t = \{i_k, \pi_k, r_k, \epsilon^r_k, \epsilon^e_k, s_k\}^t_{k=0} \tag{142}$$

Plugging (18) into (142), the second term of (18) vanishes because random variable s_{t+1} and ϵ^e_{t+1} are independent. Also notice that conditional on $\mathcal{F}_t$, r_{t+1} is independent of s_{t+1} and ϵ^e_{t+1}. Therefore, we obtain

$$\begin{aligned}
\mathbb{E}_t\pi_{t+1} &= \mathbb{E}(a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}), p_{s_{t+1},1}(\epsilon^e_{t+1}))r_{t+1}|\mathcal{F}_t) + 0 \\
&= \mathbb{E}(a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}), p_{s_{t+1},1}(\epsilon^e_{t+1}))|\mathcal{F}_t)\mathbb{E}(r_{t+1}|\mathcal{F}_t) \\
&= [\mathbb{E}(a_1(s_{t+1} = 0, p_{00}(\epsilon^e_{t+1}), p_{01}(\epsilon^e_{t+1}))|\mathcal{F}_t) \cdot p_{s_t,0}(\epsilon^e_t) \\
&+ \mathbb{E}(a_1(s_{t+1} = 1, p_{10}(\epsilon^e_{t+1}), p_{11}(\epsilon^e_{t+1}))|\mathcal{F}_t) \cdot p_{s_t,1}(\epsilon^e_t)] \cdot \rho_r r_t \\
&= [\mathbb{E}(a_1(s_{t+1} = 0, p_{00}(\epsilon^e_{t+1}), p_{01}(\epsilon^e_{t+1}))) \cdot p_{s_t,0}(\epsilon^e_t) \\
&+ \mathbb{E}(a_1(s_{t+1} = 1, p_{10}(\epsilon^e_{t+1}), p_{11}(\epsilon^e_{t+1}))) \cdot p_{s_t,1}(\epsilon^e_t)] \cdot \rho_r r_t
\end{aligned} \tag{143}$$

The last equality held by the independence of $\mathcal{F}_t$ and ϵ^e_{t+1}. Plugging (143) and (10) into (9), then combining (11) yields

$$\begin{aligned}
i_t =& [\mathbb{E}(a_1(s_{t+1} = 0, p_{00}(\epsilon^e_{t+1}), p_{01}(\epsilon^e_{t+1}))) \cdot p_{s_t,0}(\epsilon^e_t) \\
&+ \mathbb{E}(a_1(s_{t+1} = 1, p_{10}(\epsilon^e_{t+1}), p_{11}(\epsilon^e_{t+1}))) \cdot p_{s_t,1}(\epsilon^e_t) + 1] \cdot \rho_r r_t \\
=& \alpha(s_t)\pi_t + \sigma_e\epsilon^e_t
\end{aligned} \tag{144}$$

Taking one period forward, we can solve π_{t+1} as

$$\begin{aligned}
\pi_{t+1} =& \frac{\rho_r}{\alpha(s_{t+1})}[\mathbb{E}(a_1(s_{t+2} = 0, p_{00}(\epsilon^e_{t+2}), p_{01}(\epsilon^e_{t+2}))) \cdot p_{s_{t+1},0}(\epsilon^e_{t+1}) \\
&+ \mathbb{E}(a_1(s_{t+2} = 1, p_{10}(\epsilon^e_{t+2}), p_{11}(\epsilon^e_{t+2}))) \cdot p_{s_{t+1},1}(\epsilon^e_{t+1}) + 1]r_{t+1} \\
&- \frac{\sigma_e}{\alpha(s_{t+1})}\epsilon^e_{t+1}
\end{aligned} \tag{145}$$

Comparing (145) with (18), we match the unknown coefficients:

$$\begin{aligned}
&a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}), p_{s_{t+1},1}(\epsilon^e_{t+1})) \\
&= \frac{\rho_r}{\alpha(s_{t+1})}[\mathbb{E}(a_1(s_{t+2} = 0, p_{00}(\epsilon^e_{t+2}), p_{01}(\epsilon^e_{t+2}))) \cdot p_{s_{t+1},0}(\epsilon^e_{t+1}) \\
&+ \mathbb{E}(a_1(s_{t+2} = 1, p_{10}(\epsilon^e_{t+2}), p_{11}(\epsilon^e_{t+2}))) \cdot p_{s_{t+1},1}(\epsilon^e_{t+1}) + 1]
\end{aligned} \tag{146}$$

and

$$a_2(s_{t+1}) = -\frac{\sigma_e}{\alpha(s_{t+1})} \tag{147}$$

To determine a_1, we denote

$$C_0 = \mathbb{E}(a_1(s_{t+2} = 0, p_{00}(\epsilon^e_{t+2}), p_{01}(\epsilon^e_{t+2}))) \tag{148}$$

$$C_1 = \mathbb{E}(a_1(s_{t+2} = 1, p_{10}(\epsilon^e_{t+2}), p_{11}(\epsilon^e_{t+2}))) \tag{149}$$

which are two undetermined constants. Considering $s_{t+1} = 0$ and $s_{t+1} = 1$ of (146), taking expectation with respect to ϵ^e_{t+1}, then combining (148) and (149), we obtain:

$$C_0 = \mathbb{E}(a_1(s_{t+1} = 0, p_{00}(\epsilon^e_{t+1}), p_{01}(\epsilon^e_{t+1}))) = \frac{\rho_r}{\alpha_0}[C_0\mathbb{E}p_{00}(\epsilon^e_{t+1}) + C_1\mathbb{E}p_{01}(\epsilon^e_{t+1}) + 1] \tag{150}$$

$$C_1 = \mathbb{E}(a_1(s_{t+1} = 1, p_{10}(\epsilon^e_{t+1}), p_{11}(\epsilon^e_{t+1}))) = \frac{\rho_r}{\alpha_1}[C_0\mathbb{E}p_{10}(\epsilon^e_{t+1}) + C_1\mathbb{E}p_{11}(\epsilon^e_{t+1}) + 1] \tag{151}$$

Now, (151) implies that

$$\begin{aligned} C_1 &= \frac{\frac{\rho_r}{\alpha_1}\mathbb{E}p_{10}(\epsilon^e_{t+1})}{1 - \frac{\rho_r}{\alpha_1}\mathbb{E}p_{11}(\epsilon^e_{t+1})}C_0 + \frac{\frac{\rho_r}{\alpha_1}}{1 - \frac{\rho_r}{\alpha_1}\mathbb{E}p_{11}(\epsilon^e_{t+1})} \\ &= \frac{\rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}{\alpha_1 - \rho_r + \rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})}C_0 + \frac{\rho_r}{\alpha_1 - \rho_r + \rho_r\mathbb{E}p_{10}(\epsilon^e_{t+1})} \end{aligned} \tag{152}$$

Substituting (152) into (150), we solve

$$\begin{aligned} C_0 &= \frac{\alpha_1 + \rho_r(\mathbb{E}p_{10}(\epsilon^e_{t+1}) - \mathbb{E}p_{00}(\epsilon^e_{t+1}))}{\alpha_0\left(\frac{\alpha_1}{\rho_r} - 1 + \mathbb{E}p_{10}(\epsilon^e_{t+1})\right) - \alpha_1\mathbb{E}p_{00}(\epsilon^e_{t+1}) - \rho_r(\mathbb{E}p_{10}(\epsilon^e_{t+1}) - \mathbb{E}p_{00}(\epsilon^e_{t+1}))} \\ &= \frac{\alpha_1 + \rho_r(\mathbb{E}p_{10}(\epsilon^e_{t+1}) - \mathbb{E}p_{00}(\epsilon^e_{t+1}))}{(\alpha_1 - \rho_r)\left(\frac{\alpha_0}{\rho_r} - \mathbb{E}p_{00}(\epsilon^e_{t+1})\right) + (\alpha_0 - \rho_r)\mathbb{E}p_{10}(\epsilon^e_{t+1})} > 0 \end{aligned} \tag{153}$$

where $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$ can be obtained by taking expectation of TVTP with respect to ϵ^e_{t+1}. Details will be provided in section 2.2.

Then, C_1 is given by (152) and (153). Finally, substituting C_0 and C_1 into (146), we conclude

that

$$\begin{aligned}\pi_{t+1} =& \frac{\rho_r}{\alpha(s_{t+1})}\left[C_0 p_{s_{t+1},0}(\epsilon^e_{t+1}) + C_1 p_{s_{t+1},1}(\epsilon^e_{t+1}) + 1\right] r_{t+1} - \frac{\sigma_e}{\alpha(s_{t+1})}\epsilon^e_{t+1} \\ =& \frac{\rho_r}{\alpha(s_{t+1})}\left[\frac{(\alpha_1-\rho_r)p_{s_{t+1},0}(\epsilon^e_{t+1}) + \rho_r \mathbb{E}p_{10}(\epsilon^e_{t+1})}{\alpha_1 - \rho_r + \rho_r \mathbb{E}p_{10}(\epsilon^e_{t+1})} C_0 + \frac{\alpha_1 - \rho_r p_{s_{t+1},0}(\epsilon^e_{t+1}) + \rho_r \mathbb{E}p_{10}(\epsilon^e_{t+1})}{\alpha_1 - \rho_r + \rho_r \mathbb{E}p_{10}(\epsilon^e_{t+1})}\right] \\ & \cdot r_{t+1} - \frac{\sigma_e}{\alpha(s_{t+1})}\epsilon^e_{t+1} \\ =& \underbrace{\frac{\rho_r}{\alpha(s_{t+1})}\frac{(\alpha_1-\alpha_0)p_{s_{t+1},0}(\epsilon^e_{t+1}) + \alpha_1\left(\frac{\alpha_0}{\rho_r} - \mathbb{E}p_{00}(\epsilon^e_{t+1})\right) + \alpha_0 \mathbb{E}p_{10}(\epsilon^e_{t+1})}{(\alpha_1-\rho_r)\left(\frac{\alpha_0}{\rho_r} - \mathbb{E}p_{00}(\epsilon^e_{t+1})\right) + (\alpha_0-\rho_r)\mathbb{E}p_{10}(\epsilon^e_{t+1})}}_{a_1(s_{t+1}, p_{s_{t+1},0}(\epsilon^e_{t+1}))} r_{t+1} \underbrace{- \frac{\sigma_e}{\alpha(s_{t+1})}}_{a_2(s_{t+1})} \epsilon^e_{t+1}\end{aligned} \tag{154}$$

where random coefficient a_1 depends on s_{t+1} and ϵ^e_{t+1}. $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$ are two constants to be evaluated.

A.2 Numerical evaluation

To compute coefficient a_1 in (154), we need to evaluate $p_{s_{t+1},0}(\epsilon^e_{t+1}), \mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$. This can be done as follows.

We first compute the TVTP $p_{s_{t+1},0}(\epsilon^e_{t+1})$ using Theorem 3.1 in Chang et al. (2017). Main steps is shown in Section 5.3 in the context of endogenous regime switching monetary DSGE model. Let

$$\Phi_\rho(x) = \Phi(x/\sqrt{1-\rho^2})$$

we can obtain:

$$p_{00}(\epsilon^e_{t+1}) = \mathbb{P}(s_{t+2}=0|s_{t+1}=0,\epsilon^e_{t+1}) = \frac{\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)dx}{\Phi(\tau\sqrt{1-\phi^2})} \tag{155}$$

$$p_{10}(\epsilon^e_{t+1}) = \mathbb{P}(s_{t+2}=0|s_{t+1}=1,\epsilon^e_{t+1}) = \frac{\int_{\tau\sqrt{1-\phi^2}}^{\infty} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)dx}{1-\Phi(\tau\sqrt{1-\phi^2})} \tag{156}$$

Then,

$$p_{01}(\epsilon^e_{t+1}) = \mathbb{P}(s_{t+2}=1|s_{t+1}=0,\epsilon^e_{t+1}) = 1 - p_{00}(\epsilon^e_{t+1}) \tag{157}$$

$$p_{11}(\epsilon^e_{t+1}) = \mathbb{P}(s_{t+2}=1|s_{t+1}=1,\epsilon^e_{t+1}) = 1 - p_{10}(\epsilon^e_{t+1}) \tag{158}$$

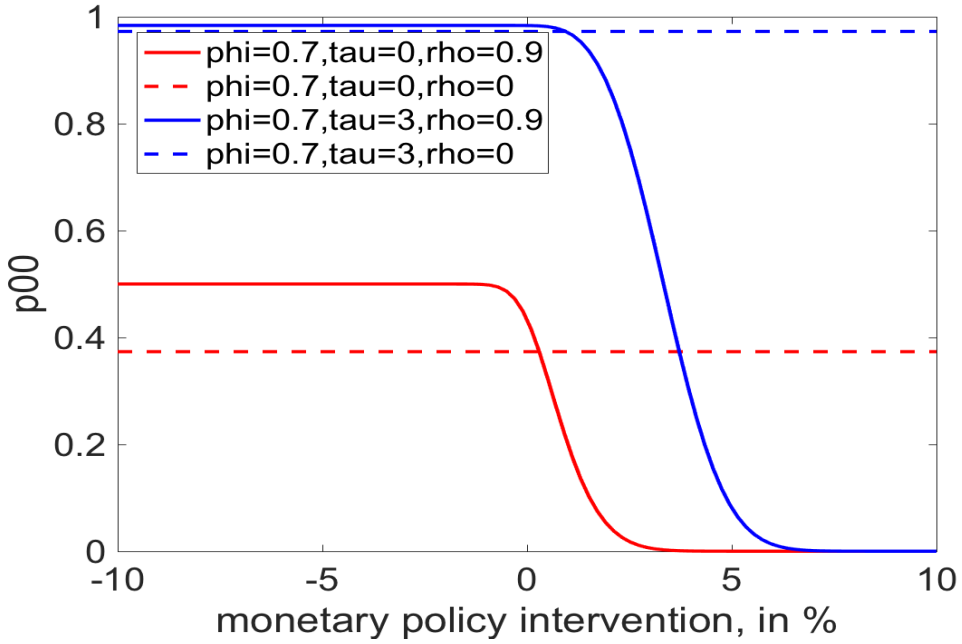


Figure 11: Time-varying transition probabilites

Figure 11 displays how TVTP is determined by parameters (ϕ, τ, ρ). If $\rho = 0$, (ϕ, τ) has one-to-one correspondence to the parameters in exogenous switching model, *i.e.* constant low-to-low and high-to-high probabilities $(\bar{p}_{00}, \bar{p}_{11})$. So, they determine the magnitude of transition probabilities. The endogeneity parameter ρ governs the fluctuation of transition probabilities.

Next, using the trick introduced in Chang et al. (2017), we can rewrite[27] (155) and (156) as bivariate normal distribution functions:

$$p_{00}(\epsilon^e_{t+1}) = \frac{\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)dx}{\Phi(\tau\sqrt{1-\phi^2})} = \frac{\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \int_{-\infty}^{\frac{\tau-\rho\epsilon^e_{t+1}}{\sqrt{1-\rho^2}}} f_1(x,y)dydx}{\Phi(\tau\sqrt{1-\phi^2})} \tag{159}$$

with

$$f_1(x,y) = \mathbb{N}\left(0, \begin{pmatrix} 1 & \frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} \\ \frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & 1 + \frac{\phi^2}{(1-\rho^2)(1-\phi^2)} \end{pmatrix}\right) \tag{160}$$

[27] Detailed derivations can be provided when needed.

and,

$$p_{10}(\epsilon^e_{t+1}) = \frac{\int_{\tau\sqrt{1-\phi^2}}^{\infty} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)dx}{1-\Phi(\tau\sqrt{1-\phi^2})} = \frac{\int_{-\infty}^{-\tau\sqrt{1-\phi^2}} \int_{-\infty}^{\frac{\tau-\rho\epsilon^e_{t+1}}{\sqrt{1-\rho^2}}} f_2(x,y)dydx}{1-\Phi(\tau\sqrt{1-\phi^2})} \tag{161}$$

with

$$f_2(x,y) = \mathbb{N}\left(0, \begin{pmatrix} 1 & -\frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} \\ -\frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & 1+\frac{\phi^2}{(1-\rho^2)(1-\phi^2)} \end{pmatrix}\right) \tag{162}$$

Using the same trick, we can rewrite $\mathbb{E}p_{00}(\epsilon^e_{t+1})$ and $\mathbb{E}p_{10}(\epsilon^e_{t+1})$ as trivariate normal distribution functions:

$$\begin{aligned}\mathbb{E}p_{00}(\epsilon^e_{t+1}) &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\tau\sqrt{1-\phi^2}} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)\varphi(\epsilon^e_{t+1})dxd\epsilon^e_{t+1}}{\Phi(\tau\sqrt{1-\phi^2})} \\ &= \frac{\int_{-\infty}^{\tau\sqrt{1-\phi^2}}\int_{-\infty}^{\tau/\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} f_3(x,y,\epsilon)d\epsilon dydx}{\Phi(\tau\sqrt{1-\phi^2})}\end{aligned} \tag{163}$$

with

$$f_3(x,y,\epsilon) = \mathbb{N}\left(0, \begin{pmatrix} 1 & \frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & 0 \\ \frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & \frac{1}{(1-\rho^2)(1-\phi^2)} & \frac{\rho}{\sqrt{1-\rho^2}} \\ 0 & \frac{\rho}{\sqrt{1-\rho^2}} & 1 \end{pmatrix}\right) \tag{164}$$

and,

$$\begin{aligned}\mathbb{E}p_{10}(\epsilon^e_{t+1}) &= \frac{\int_{-\infty}^{\infty}\int_{\tau\sqrt{1-\phi^2}}^{\infty} \Phi_\rho\left(\tau - \frac{\phi x}{\sqrt{1-\phi^2}} - \rho\epsilon^e_{t+1}\right)\varphi(x)\varphi(\epsilon^e_{t+1})dxd\epsilon^e_{t+1}}{1-\Phi(\tau\sqrt{1-\phi^2})} \\ &= \frac{\int_{-\infty}^{-\tau\sqrt{1-\phi^2}}\int_{-\infty}^{\tau/\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} f_4(x,y,\epsilon)d\epsilon dydx}{1-\Phi(\tau\sqrt{1-\phi^2})}\end{aligned} \tag{165}$$

with

$$f_4(x,y,\epsilon) = \mathbb{N}\left(0, \begin{pmatrix} 1 & -\frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & 0 \\ -\frac{\phi}{\sqrt{1-\rho^2}\sqrt{1-\phi^2}} & \frac{1}{(1-\rho^2)(1-\phi^2)} & \frac{\rho}{\sqrt{1-\rho^2}} \\ 0 & \frac{\rho}{\sqrt{1-\rho^2}} & 1 \end{pmatrix}\right) \tag{166}$$

Finally, using *mvncdf* function in Matlab, we are able to evaluate (154).

B Algorithm for Kalman filter with Markov switching

Consider the regime-dependent state space model (131) and (132) as follows

$$\Upsilon_t = \Upsilon_{ss}(s_t, \Theta) + D(s_t, \Theta)X_t + \nu_t, \quad \nu_t \sim \mathbb{N}(0, \Sigma_\nu) \tag{167}$$
$$X_t = G(s_t, \Theta)X_{t-1} + M(s_t, \Theta)\varepsilon_t, \quad \varepsilon_t \sim \mathbb{N}(0, \mathbf{1}) \tag{168}$$

where Θ stacks all structural parameters, $s_t = \mathbb{1}\{w_t \geq \tau\}$ is the regime index, and w_t is a latent factor that follows

$$w_t = \phi w_{t-1} + v_t \tag{169}$$

Endogeneity is introduced as in Chang et al. (2017) by assuming

$$\begin{pmatrix} \varepsilon^e_{t-1} \\ v_t \end{pmatrix} \sim \mathbb{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right) \tag{170}$$

where ε^e_t is the monetary policy shock. [28]

For filtering convenience, we have augmented the state vector X_t by including w_{t+1} as well as the shock vector ε_t by including η_{t+1}. Accordingly, we also augment the state transition system (168) by equation (169) rewritten as

$$w_{t+1} = \phi w_t + \rho\varepsilon^e_t + \sqrt{1-\rho^2}\eta_{t+1}, \quad \eta_{t+1} \sim \mathbb{N}(0, 1) \tag{171}$$

For notational ease, define the following probabilites

$$p^k_{t|t-1} = \mathbb{P}(s_t = k|Y_{1:t-1}), p^k_{t|t} = \mathbb{P}(s_t = k|Y_{1:t}), \quad k = 0, 1 \tag{172}$$

where $Y_{1:t}$ are the information of observables up to time t. The filtering algorithm can be summarized as follows:

1. **Initialization.** For $k = 0, 1$, initialize $(X^k_{0|0}, P^k_{0|0})$ using the invariant distribution and specify $p^k_{0|0}$.
2. **Recursion.** For $t = 1, \ldots, T$:
   (a) **Forecasting.** From initialization or previous updating step (182) and (183), we observe that $w_t|Y_{1:t-1}$ follows a mixture normal distribution [29]

$$\begin{aligned} p(w_t|Y_{1:t-1}) &= p^0_{t-1|t-1}p(w_t|s_{t-1} = 0, Y_{1:t-1}) + p^1_{t-1|t-1}p(w_t|s_{t-1} = 1, Y_{1:t-1}) \\ &= p^0_{t-1|t-1}\mathbb{N}(\mu^0_{t-1|t-1}, \sigma^0_{t-1|t-1}) + p^1_{t-1|t-1}\mathbb{N}(\mu^1_{t-1|t-1}, \sigma^1_{t-1|t-1}) \end{aligned} \tag{173}$$

[28] This implies that only past monetary policy shocks drive current regime change. A natural extension is to allow all past structural – both policy and non-policy – shocks to jointly determine current regime. We're currently working on this extension and more details will be provided in Section 7.1.

[29] σ_t denotes the variance.

where $(\mu^k_{t-1|t-1}, \sigma^k_{t-1|t-1})$ can be extracted from the last elements of $(X^k_{t-1|t-1}, P^k_{t-1|t-1})$. Hence, for $k = 0, 1$, we can compute predicted regime probabilities

$$\begin{aligned} p^1_{t|t-1} &= \mathbb{P}(w_t \geq \tau | Y_{1:t-1}) \\ &= p^0_{t-1|t-1}\mathbb{P}(w_t \geq \tau | s_{t-1} = 0, Y_{1:t-1}) + p^1_{t-1|t-1}\mathbb{P}(w_t \geq \tau | s_{t-1} = 1, Y_{1:t-1}) \\ &= p^0_{t-1|t-1}\left(1 - \Phi\left(\frac{\tau - \mu^0_{t-1|t-1}}{\sqrt{\sigma^0_{t-1|t-1}}}\right)\right) + p^1_{t-1|t-1}\left(1 - \Phi\left(\frac{\tau - \mu^1_{t-1|t-1}}{\sqrt{\sigma^1_{t-1|t-1}}}\right)\right) \end{aligned} \tag{174}$$

and $p^0_{t|t-1} = 1 - p^1_{t|t-1}$.
We then apply the Kalman filter forecasting step for unobservables to obtain

$$X^k_{t|t-1} = G(s_t = k)X^k_{t-1|t-1} \tag{175}$$
$$P^k_{t|t-1} = G(s_t = k)P^k_{t-1|t-1}G(s_t = k)' + M(s_t = k)\Sigma_\varepsilon M(s_t = k)' \tag{176}$$

(b) **Likelihood evaluation.** Apply the Kalman filter forecasting step for observables to obtain

$$\Upsilon^k_{t|t-1} = \Upsilon_{ss}(s_t = k) + D(s_t = k)X^k_{t|t-1} \tag{177}$$
$$F^k_{t|t-1} = D(s_t = k)P^k_{t|t-1}D(s_t = k)' + \Sigma_\nu \tag{178}$$

Then the period-t likelihood contribution can be computed as

$$\begin{aligned} p(\Upsilon_t | Y_{1:t-1}, \Theta) &= \sum_{s_t} p(\Upsilon_t | s_t, Y_{1:t-1})P(s_t | Y_{1:t-1}) \\ &= p_N(\Upsilon_t | \Upsilon^0_{t|t-1}, F^0_{t|t-1})p^0_{t|t-1} + p_N(\Upsilon_t | \Upsilon^1_{t|t-1}, F^1_{t|t-1})p^1_{t|t-1} \end{aligned} \tag{179}$$

where $p(\Upsilon_t | s_t = k, Y_{1:t-1}) = p_N(\Upsilon_t | \Upsilon^k_{t|t-1}, F^k_{t|t-1})$ is a multivariate normal distribution with mean $\Upsilon^k_{t|t-1}$ and covariance matrix $F^k_{t|t-1}$.

(c) **Updating.** Apply Bayes formula to update

$$P(s_t | Y_{1:t}) = P(s_t | \Upsilon_t, Y_{1:t-1}) = \frac{p(\Upsilon_t | s_t, Y_{1:t-1})P(s_t | Y_{1:t-1})}{p(\Upsilon_t | Y_{1:t-1})} \tag{180}$$

That is,

$$p^0_{t|t} = \frac{p_N(\Upsilon_t | \Upsilon^0_{t|t-1}, F^0_{t|t-1})p^0_{t|t-1}}{p(\Upsilon_t | Y_{1:t-1})} \tag{181}$$

and $p^1_{t|t} = 1 - p^0_{t|t}$.

Then apply the Kalman filter filtering step to obtain

$$X_{t|t}^k = X_{t|t-1}^k + P_{t|t-1}^k D(s_t = k)'(F_{t|t-1}^k)^{-1}(\Upsilon_t - \Upsilon_{t|t-1}^k) \tag{182}$$

$$P_{t|t}^k = P_{t|t-1}^k - P_{t|t-1}^k D(s_t = k)'(F_{t|t-1}^k)^{-1} D(s_t = k) P_{t|t-1}^k \tag{183}$$

from which we can easily compute the extracted filtered states of any order, including the autoregressive latent regime factor, by marginalization

$$\mathbb{E}(w_{t+1}|\mathcal{F}_t) = p_{t|t}^0 w_{t+1|t}^0 + p_{t|t}^1 w_{t+1|t}^1 \tag{184}$$

where $w_{t+1|t}^k$ is the last element in $X_{t|t}^k$.

3. **Quasi-Bayesian MLE.** The log-likelihood function can be written as

$$\log L(Y_{1:T}|\Theta) = \log p(\Upsilon_1) + \sum_{t=2}^{T} \log p(\Upsilon_t|Y_{1:t-1}) \tag{185}$$

Then, the quasi-Bayesian ML estimator $\hat{\Theta}$ of Θ is given by

$$\hat{\Theta} = \underset{\Theta \in R(\Theta)}{\arg\max} \log L(Y_{1:T}|\Theta) + \log p(\Theta) \tag{186}$$