Risk Reallocation in OTC Derivatives Networks

Preliminary and incomplete, please do not circulate

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Abstract

Over-the-counter (OTC) derivatives markets are the key venue for quickly reallocating exposures to key risk factors such as interest rates, exchange rates, and credit amongst market participants. These markets are very large, and are characterized by a complex trading network with disperse prices. In this paper, we ask how the structure of the OTC derivatives trading network, the preferences and technologies of the participants, and the distribution of endowed exposures to the underlying risk factor, jointly determine the observed patterns of trade, post-trade exposures, and prices. We use detailed data from the DTCC to estimate the key parameters of our model. Finally, we use the model at estimated parameters to study comparative statics related to risk management and regulation.

Keywords: OTC markets, derivatives, networks, Credit Default Swaps

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1 Introduction

Over-the-counter (OTC) derivatives markets are the key venue for quickly reallocating exposures to key risk factors such as interest rates, exchange rates, and credit amongst market participants. These markets are very large, and are characterized by a complex trading network with disperse prices. In this paper, we ask how the structure of the OTC derivatives trading network, the preferences and technologies of the participants, and the distribution of endowed exposures to the underlying risk factor, jointly determine the observed patterns of trade, post-trade exposures, and prices. We use the CDS market as an example, and combine detailed data from the Deposit Trust and Clearing Corporation, covering almost all trades, with a model of the OTC trading network. In our model, we take the empirical network of bilateral connections as given. We then use the empirical observations on trades, resulting net positions, and prices to estimate the preference and technology parameters of our model. Finally, we use the model at estimated parameters to do comparative statics. These comparative statics illustrate the role of the network, preference and technology parameters, and endowments in determining equilibrium outcomes for trading and prices. They also allow us to do comparative statics that can help inform risk management and regulation.

We begin by describing the key stylized facts describing the OTC derivatives network. We group these observations into the following three categories: facts about the network, facts about positions and trades, and facts about prices. These categories correspond to specific objects in our model.

At each point in time t, the network is represented by a matrix of bilateral trading relationships, G_t. Element (i, j) of G is one if agents i and j have a trading relationship and zero otherwise. Using our data, we first establish that the trading network is fairly static through time, meaning that trading relationships are highly persistent. As such, we take the trading network as an exogenous primitive in our model. Another salient feature of G is that it is sparse. This indicates that a few traders (dealers) have many trading relationships, whereas most have only a handful. The dealer centric nature of the network is characteristic of OTC derivative markets and means that dealer preferences will play an important role in risk sharing and pricing inside of the model.

From 2010 onwards, we also observe all trades and accumulated positions in the U.S. CDS market. Our model keeps track of these gross trades and net positions separately, and in fact gross and net positions are subject to different risk limits in the model and in practice. There are two key features of the data that our model will be able to deliver as a natural equilibrium outcome. The first is that net positions across traders are heterogenous across traders in the market. The second and more subtle fact is that, for any given trader, there is

heterogeneity in the distribution of their *bilateral* net exposures — traders spread our their total net positions across their trading partners.

Finally, we observe all transaction prices in our network. As in the model, prices are disperse and vary across different counterparties. To fix ideas, consider an arbitrary date and all transactions on the on-the-run 5 year CDX investment grade index. Price dispersion in this example means that the price of CDS contracts for the 5 year on-the-run investment grade index varies across traders. We document the existence of price dispersion in the cross-section of contracts and also show that it is economically large. For instance, on average, the standard deviation of prices for a given contract is roughly 20% of the average price paid.

Motivated by these facts, we develop a parsimonious model of an OTC derivative trading network. We start with the empirical network of bilateral trading relationships. Our model is suitable for studying a general OTC derivatives market, but we focus on CDS to fix ideas and to bring our model to the DTCC data. Agents are endowed with heterogeneous amounts of a risky bond with exposure to default risk. To reallocate their exposure to default risk, and to earn trading profits, agents can enter into long or short positions in credit default swaps (CDS) with any of their connected counterparties. Agents are averse to two types of risk. First, they dislike exposure to the default risk in the underlying asset. Second, they also find it costly to engage in large trades with any single counterparty. We microfound the first cost as overall risk aversion and the second one as aversion to counterparty default risk.

In the model, a trading network defines which agent can trade with whom. When two agents can trade, then they agree on a CDS contract so that one agent provides insurance to the other. Agents take prices as given, and choose how much to insure. Hence, the amount insured and the volume of CDS contracts traded is endogenous, and prices are determined in equilibrium. Perfect risk-sharing is not achieved because of the cost of trading of large amounts with any counterparty, namely aversion to counterparty default risk. Furthermore, equilibrium prices reflect post-trade exposure in the underlying asset and therefore our model features price dispersion in equilibrium.

We allow for different values for these two costs of risk bearing across agents, and we will use empirical moments to estimate them. Heterogeneity in risk aversion drives aggregate default risk allocation between agents, while differences in the aversion to counterparty default risk determines the flow of default risk between different counterparties and how much agents exploit price dispersion to make trading profits. Therefore, risk aversion is an important determinant of net positions, and aversion to counterparty risk is important to determine gross positions. Therefore, we will use cross-sectional variation in quantities insured by ach agent along with each contracts' price to estimate aversion to counterparty risk. We will use time-series variation in net positions and overall exposures in the underlying

asset to estimate agents' risk aversion.

The paper proceeds as follows. In section 1.1 we briefly summarize the related literature. In section 2, we describe our data and the main stylized facts characterizing the OTC derivatives market. In section 3 we describe the setup of our model, define and characterize the trading equilibrium, and discuss the model's key properties and intuition. In section 4 we discuss how we can use the DTCC data to estimate the key preference and technology parameters of our model. Finally, we discuss future work and conclude in section 5.

1.1 Related literature

[TO BE COMPLETED]

There are several recent papers on networks and finance. The main contributions include Hou and Robinson (2006), Cohen, Frazzini, and Malloy (2008), Cohen and Frazzini (2008), Ahern and Harford (2014), Carvalho (2010), Ahern (2012), Kelly, Lustig, and Van Nieuwerburgh (2013), Oberfield (2013), Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013), Aobdia, Caskey, and Ozel (2014), Babus (2013), Babus and Kondor (2013), Biggio and La'O (2013), Carvalho and Gabaix (2013), Carvalho and Grassi (2014), Carvalho and Voigtlander (2014), Carvalho (2014), Farboodi (2014), Malamud and Rostek (2014), Denbee, Julliard, Li, and Yuan (2014), and Herskovic (2015). Finally, Allen and Babus (2009) present a detailed review of network models applied to finance.

In our empirical analysis, we document that the CDS derivatives network has a structure similar to a core-periphery network. There is a recent set of studies that endogenize network formation and find that a core-periphery network structure in equilibrium, when agents choose their connections unilaterally (Bala and Goyal 2000, Galeotti and Goyal 2010, Herskovic and Ramos 2015).

Recent papers on OTC markets: Duffie, Li, and Lubke (2011), Duffie, Scheicher, and Vuillemey (2014), Green, Hollifield, and Schurhoff (2007), Gofman (2015) and Atkeson, Eisfeldt, and Weill (2015).

Recent papers on CDS: Arora, Gandhi, and Longstaff (2012), Bolton and Oehmke (2013), Chen, Fleming, Jackson, Li, and Sarka (2011), Oehmke and Zawadowski (2013), Schachar (2012), Siriwardane (2015), and Atkeson, Eisfeldt, and Weill (2013).

2 Facts

In this section, we document the following facts about the CDS market: (i) the network of bilateral trading relationships is very static through time — agents typically trade with the same set of traders; (ii) the trading network is sparse in the sense that a few traders have many different trading connections, while most agents have only very few; (iii) the net amount of protection sold varies across agents; (iv) within an agent, net exposures across trading partners also varies, and (v) for the same contract, traders often face different prices. The first two of these facts will form the basis of our model in Section INSERT, while the last three will be equilibrium outcomes that our model aims to capture.

Description of Data

Our data comes from the DTCC, who provides trade processing services for every major dealer in credit default swaps. We have access to two complementary subsets of the DTCC's database: transactions and positions. Transactions represent flows in CDS, and positions represent stocks. For example, a new transaction between two counterparties also results in a new outstanding position between them. Future amendments or additional transactions on the same underlying reference entity simply add to their existing position. In practice, computing positions from transactions is quite complicated and is done using the DTCC's own internal algorithms.

For both transactions and positions, our data contains full information on the counterparties in the trade, the pricing terms, the swap type, the notional amount, the initiation date, and so forth. Within the DTCC's trade repository data, we are privy to any transaction or position that meets one of two conditions: (i) the underlying reference entity is a U.S. firm or (ii) at least one of the counterparties in the swap is registered in the U.S.. In addition, our data includes all North American index CDS transactions and positions (i.e. any where the reference entity is in the "CDX.NA." family). Taken together, our data effectively covers the entire CDS market for U.S. firms. The data begins in 2010 and is updated continuously on the Friday of each week.

2.1 Facts about the CDS Trading Network

To put some structure on our analysis, we begin by defining a simple undirected network, G_t, for each date t. There are $i = 1, \ldots, n$ different agents in the market, and G_t is an $n \times n$ matrix that summarizes the trading connections in the market at time t. In particular, element g_{ijt} of G_t is equal to one if agents i and j can trade with each other at time t and

g_{ijt} equals zero otherwise. In our empirical work, we say that agents i and j can trade with each other at time t (i.e. $g_{ijt} = g_{jit} = 1$) if they have an outstanding position as of time t.

The CDS Trading Network is Static Through Time

A salient feature of the CDS trading network is that it does not change very much through time. To be clear, this is not to say that the *net positions* of agents in the market are not dynamic; rather, it simply means that *who* agents can trade with does not really change. To quantify this fact, Panels A and B of Figure 1 plot the dynamics of the trading network. At each point in time, we summarize the network based on a few standard network statistics: (i) the degree of an agent, (ii) the eigenvector centrality (EVC) of a node, and (iii) the betweenness centrality of a node. The degree of an agent is one measure of connectedness, and it is simply the sum of that agent's connections, $\sum_j g_{ij}$. An agent has a large eigenvector centrality if it is connected to other agents who are also central to the network.[1] An agent i has a large betweenness centrality if a large proportion of the shortest paths between any two agents in the network pass through agent i.[2] Panel A of Figure 1 displays the degree distribution and the EVC distribution of all agents in the network through time, while Panel B focuses on the subset of dealers, as classified by the DTCC.

From Panel A of Figure 1, it is clear that agents do not really add or delete trading partners very often. Both the degree distribution and the distribution of eigenvector centrality measures are essentially flat through time. In Panel B of Figure 1, we plot the degree, eigenvector, and betweenness centrality for each of DTCC dealers. While there is slightly more time-series variation in the centrality of each dealer, the relative rankings of dealers is highly persistent, regardless of which centrality measure one chooses. Based on these two figures, it therefore seems reasonable to conclude that the network of CDS trading is fairly static.

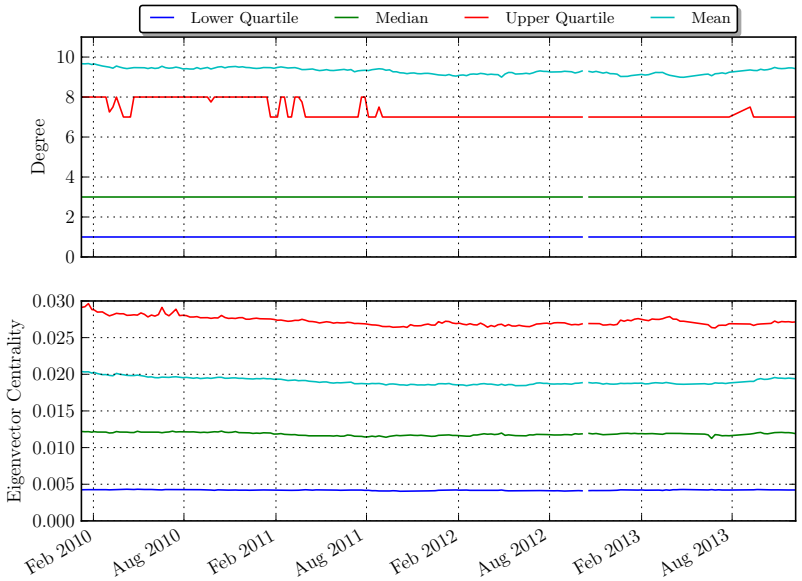
The persistent nature of the CDS network reflects the fact that setting up a trading relationship is costly. For instance, in order to trade with each other, two agents must have collateral agreements in place. Moreover, many dealers will set margins for clients based on their entire asset portfolio (e.g. interest rate swaps, corporate bonds, etc.), thus providing an additional incentive to trade all asset classes with a few counterparties. For these reasons, we will treat the CDS network as exogenous and static when solving our network model in Section INSERT.

[1] Formally, eigenvector centrality of node i is given by $\lambda C_i = \sum_j g_{ij} C_j$, where λ is a proportionality factor and C_i is the eigenvector centrality measure of node i. In practice, the vector of centrality measures $\mathbf{C}$ corresponds to the largest eigenvector of G and λ is the associated eigenvalue.

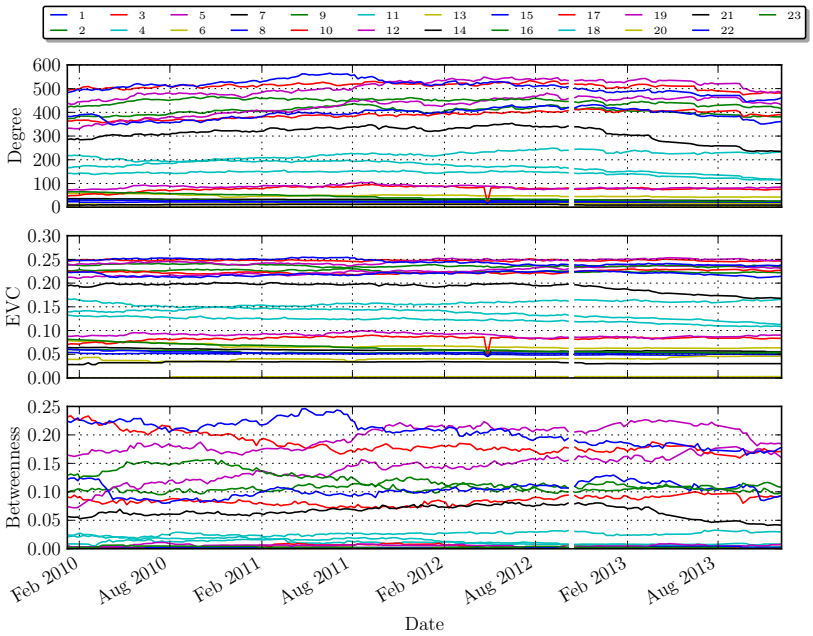
[2] For a formal definition, see Jackson (2008).

Figure 1: Dynamics of the CDS Trading Network

Panel A: Degree and Eigenvector Centrality of All Counterparties



Panel B: Degree, Eigenvector, and Betweenness Centrality of Dealers



Notes: This figure plots various centrality measures of the network of CDS trades, G_t, over time. Element g_{ijt} of G_t is equal to one if agents i and j can trade with each other at time t and g_{ijt} equals zero otherwise. We say that agents i and j can trade with each other at time t (i.e. $g_{ijt} = 1$) if they have an outstanding position as of time t. The top plot of Panel A plots the cross-sectional mean, median, lower quartile, and upper quartile of the degree distribution (number of connections for each agent) at each point in time. The bottom plot of Panel A is the same plot, but uses the cross-section of eigenvector centrality measures at each point in time. Panel B plots the degree, the eigenvector centrality, and betweenness centrality of each DTCC dealer through time (indexed from dealer 1 to 22). The time span is January 2010 to February 2014 and the frequency is weekly.

Table 1: Summary Statistics: Degree Distribution of CDS Trading Network

Degree	All Parties	Dealers	Non-Dealers
Min	1	1	1
Lower Quartile	1	27.6	1
Median	3	145.4	3
Mean	9.3	210.8	4.5
Upper Quartile	7.3	403.5	7
Max	492	492	15

Notes: This table provides summary statistics of the degree distribution in the network of CDS trades, G_t, over time. Element g_{ijt} of G_t is equal to one if agents i and j can trade with each other at time t and g_{ijt} equals zero otherwise. We say that agents i and j can trade with each other at time t (i.e. $g_{ijt} = 1$) if they have an outstanding position as of time t. The degree of an agent (node) i in the network is given by the number of its total trading connections, $\sum_{j=1}^{n} g_{ijt}$. The numbers reported in the table represent averages over the period January 2010 to June 2014. Dealers are classified according to the DTCC.

The CDS Trading Network is Sparse

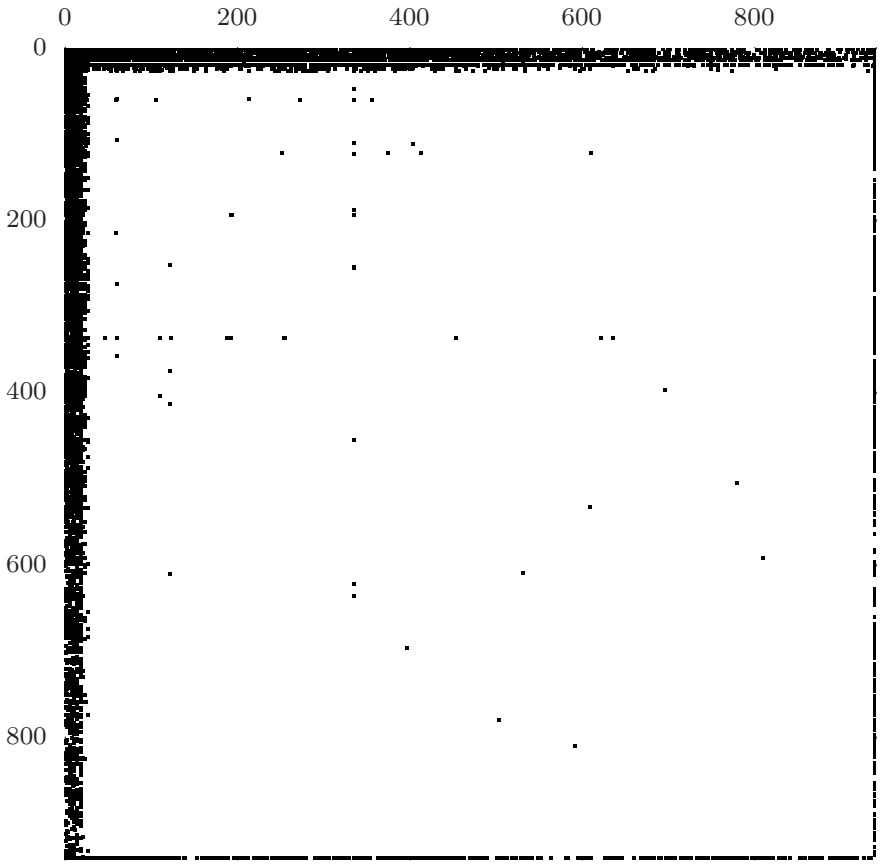
Figure 2 provides a visual snapshot of the CDS trading network G_t as of November 11, 2015. Each black square indicates that agent i and j are connected, whereas empty white space indicates there is no trading relationship. We index dealers from $i = 1, ...d$ where d is the total number of dealers in our data. Non-dealers are designated by the index $i = d + 1, ..., n$ where n is the total number of agents in the market. We use the last index $i = n$ is denote the central clearing party.

The first thing to notice from Figure 2 is that the matrix G_{ijt} is sparse. This is perhaps not so surprising given that OTC markets are dealer centric, with almost all trading volume passing through dealers. The willingness of dealers to face many different trading partners is therefore an important determinant of how effectively risk can flow through the CDS network.

Table 1 provides summary statistics of the degree distribution in the network G_{ijt} over time. Because of the dealer-centric nature of this network, we've also divided agents in the market into dealers and non-dealers. For non-dealers, trade is limited to a few partners, with the median non-dealer facing only three different agents. Non-dealer to non-dealer trades are extremely rare, so in essence, the median non-dealer faces three dealers. In stark contrast, the median dealer faces one hundred forty five different counterparties. Still, not all dealers are alike, as evidenced by the fact that the dealers in the 75-percentile dealer face nearly four hundred counterparties, whereas dealers in the 25-percentile face about 28 counterparties.

While end-users must rely heavily on dealers to transfer risk through the counterparty network, the size of the network itself is not all that large. One measure of network size is the so-called diameter, which measures the maximum number of steps between any two

Figure 2: Snapshot of CDS Trading Network



Notes: This figure plots the network of CDS trades, G_t, on $t = 11/5/2015$. Element g_{ijt} of G_t is equal to one if agents i and j can trade with each other at time t and g_{ijt} equals zero otherwise. We say that agents i and j can trade with each other at time t (i.e. $g_{ijt} = 1$) if they have an outstanding position as of time t. In the plot, we use a black square to designate all $g_{ijt} = 1$. Empty white space means that $g_{ijt} = 0$. The numbers along the left side and top of the plot index the different agents (i's) in the market. Dealers are indexed from $i = 1, ..., d$ where d is the total number of dealers. Non-dealers are then indexed from $i = d + 1, ..., n$ where n is the total number of agents in the market. The central clearing party corresponds to the last index in the set of agents, $i = n$.

counterparties. The diameter of the CDS trading network is four, owing to the fact that the *interdealer* network is itself highly connected. In this sense, because all of the dealers are connected, their respective subnetworks are not "far" apart.

2.2 Facts about Net Positions

In this subsection we document a few facts about the net positions of traders in the CDS market. Throughout this section, we denote the net amount of protection sold by i to j as γ_{ij}. $z_i := \sum_j \gamma_{ij}$ is thus the net position of agent i. Unless otherwise stated, γ_{ij} and z_i are computed by accounting for notional exposure to all reference entities and maturities. While this certainly masks the actual risk exposures of agents, our purpose in this subsection is simply to get a high-level sense of how net positions vary both within the network and across agents.[3]

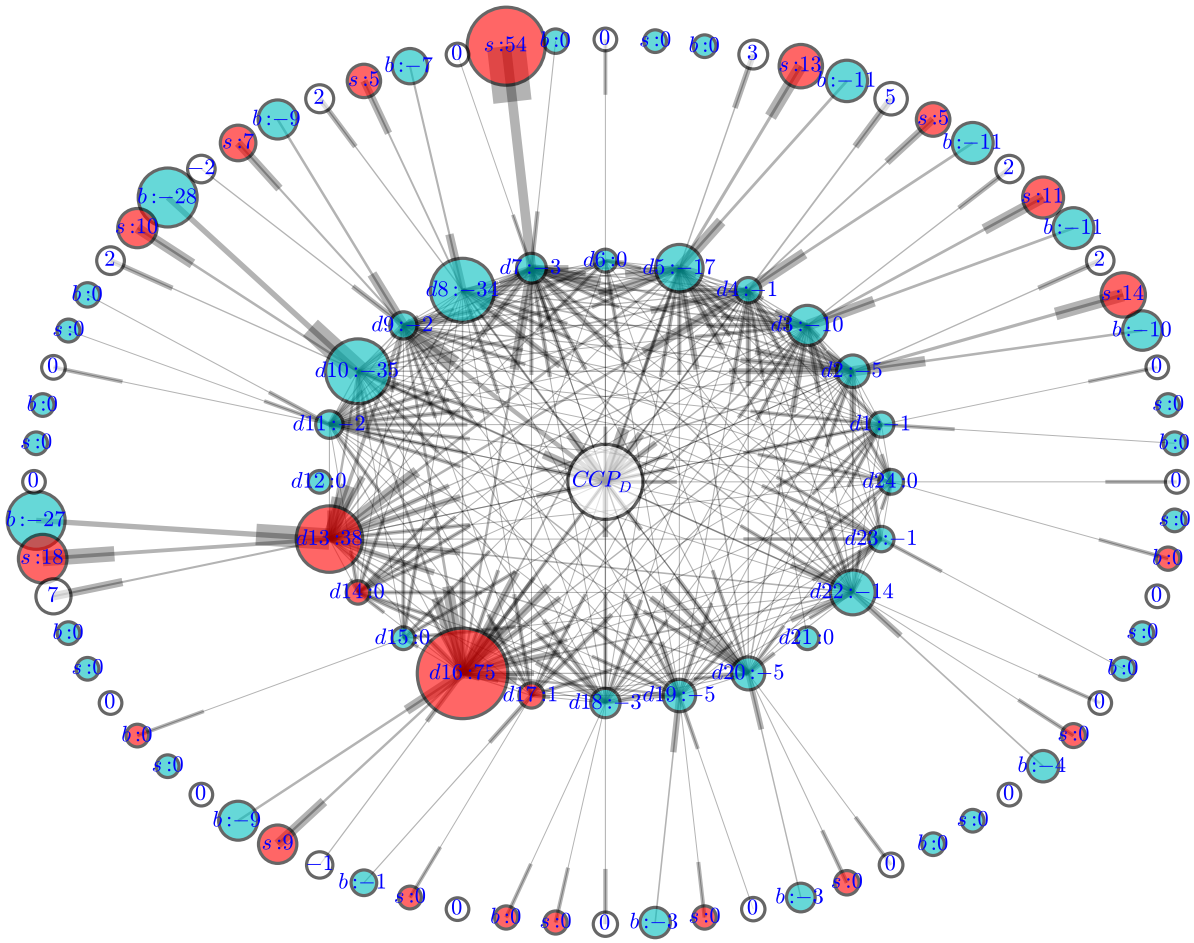
Net Positions (z_i) Vary Across Agents and Within Each Agent's Trading Network (γ_{ij})

Figure 3 provides a snapshot of how γ_{ij}'s and z_i's vary across agents as of February 28, 2014. Given the dealer centric nature of this market, we plot each of the dealers around the inner circle in Figure 3, with each dealer labeled by a "d" plus a random index (i.e. $d7$). The node at the center of the plot is the central clearing party, denoted by CCP_D. For each dealer, we divide their non-dealer trading partners ("end-users") into three types: (i) net buyers overall b, (ii) net sellers overall s, and (iii) the central clearing party. These three types are arranged around the outer edge of Figure 3. The lines between nodes in this plot contain information on the size of the notional exposure between agents (e.g. $|\gamma_{ij}|$) and the direction of notional exposure between agents (e.g. $sign(\gamma_{ij})$). A thick line denotes a large notional exposure. A thick stub at the end of a line between i and j indicates that j is net selling protection to i. Finally, the number next to each node's label in this plot corresponds to that agent's net overall exposure. Net sellers are colored red, net buyers are colored aqua, and the central clearing party is always white.

To fix ideas, consider dealer 16 ($d16$) in Figure 3. $d16$ is a net seller of protection overall, with a total net amount sold equal to $z_{d16} = \$75$ bn. Dealer 16 sells \$9 bn of protection to his end-users who are net buyers overall. This set of end-users is denoted by the node labeled "b" that is connected to $d16$. The thick stub at the end of the line connecting $d16$ and his

[3] In the model that we consider in Section INSERT, we consider exposure to only one reference entity. We interpret the reference entity in the model as being aggregate credit risk. One way to infer aggregate credit risk exposure from a large portfolio of net positions of reference entities is by computing standard statistics like market-β, etc.

Figure 3: Net Positions Across the CDS Network



Notes: This figure is one depiction of the network of bilateral exposures in the CDS market as of 2/28/2014. Dealers around arranged around the inner circle of the plot, with each dealer labeled by a "d" plus a random index (i.e. $d7$). The node at the center of the plot is the central clearing party, denoted by CCP_D. For each dealer, we divide their non-dealer trading partners ("end-users") into three types: (i) net buyers overall b, (ii) net sellers overall s, and (iii) the central clearing party. These three types are arranged around the outer edge of the plot. The lines between nodes in this plot contain information on the size of the notional exposure between agents (e.g. $|\gamma_{ij}|$) and the direction of notional exposure between agents (e.g. $sign(\gamma_{ij})$). A thick line denotes a large notional exposure. A thick stub at the end of a line between i and j indicates that j is net selling protection to i. Finally, the number next to each node's label in this plot corresponds to that agent's net overall exposure. Net sellers are colored red, net buyers are colored aqua, and the central clearing party is always white.

net buyers indicates that $d16$ is the net seller in the connection. Similarly, $d16$ buys \$9 bn of protection from his end-users who are net sellers overall, and buys \$1 bn of protection from the central clearing party.

The first thing to notice from Figure 3 is that net positions, z_i, vary *across* agents. On this date, the net position z_i of dealers ranges from -\$35 bn to \$75 bn. In addition, the net positions of each dealer's end-users is also very heterogenous. For example, dealer 7 faces a set of end-users who sell \$54 bn in total, whereas dealer 4 faces a set of end-users who sell only \$4 bn overall.

The second thing to notice from Figure 3 is that, for a given agent i, net exposures with i's trading partners γ_{ij} can vary quite a bit across j. The easiest way to see this is to focus only on the interdealer market in Figure 3. For example, dealer 16 has bilateral exposures with many different dealers, but these exposures vary in both size and sign. Based on the thickness and direction of the line connecting dealer 9 and dealer 16, we can infer that dealer 16 is a fairly large net seller of protection to dealer 9. Comparing this to the thickness of the line connecting dealer 16 and dealer 8, we can see that dealer 16 sells less protection to dealer 8 than dealer 9. On the other hand, dealer 16 is a net buyer of protection from dealer 7. The larger and simple takeaway is that, for each agent i, γ_{ij} is heterogeneous across i's trading partners.

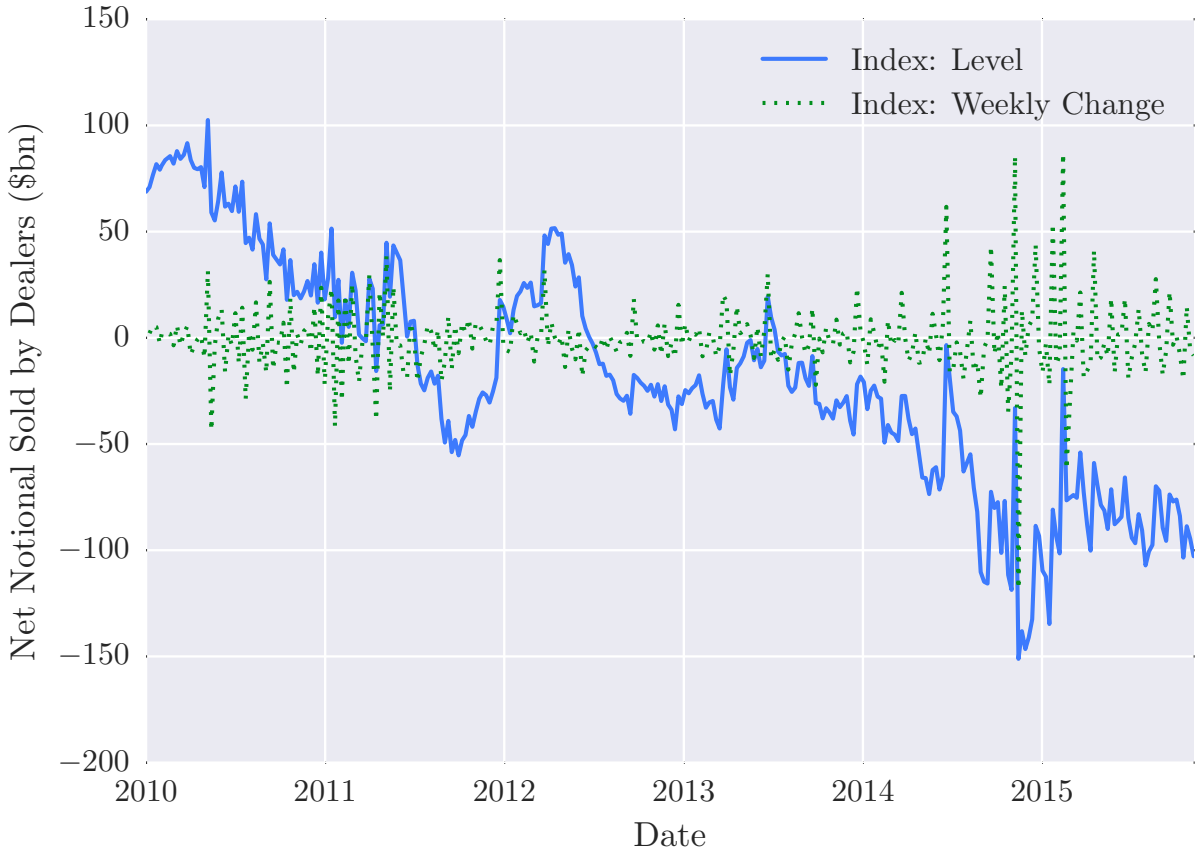
Net Positions Vary Through Time

Another important fact is that net positions of traders z_i fluctuate through time. As one way to make this point, Figure 4 displays the net notional amount of protection sold by DTCC dealers across all index swaps, both in levels and weekly changes. The first thing to notice is that the net position of dealers fluctuates quite a bit through time, as the average weekly change in dealers' net index positions is -\$500 million. There is also a great deal of volatility in weekly position changes. The standard deviation is \$17 billion, the minimum is -\$118 billion, and the maximum is \$87 billion. Because the average weekly position change for dealers is negative, the level of their net protection sold in indexes has been slowly declining since the beginning of the sample. This is not all that surprising given that dealers have been withdrawing as large net protection sellers in the aftermath of the 2008 Financial Crisis (Siriwardane (2015)).

2.3 Facts about Prices

Thus far our analysis has focused on the network structure of CDS markets and the dynamics of net notional positions. We now turn our attention to an important fact regarding pricing:

Figure 4: Net Position of DTCC Dealers



Notes: This figure plots the net notional amount of protection sold by dealers across all index swaps. The solid (blue) line is the level of net protection sold and the dashed (green) line is the weekly change in the amount of net protection sold. DTCC dealers are classified according to the DTCC. Data is weekly and ranges from January 2010 to November 2015.

for the *same* contract on the *same* day, market participants often trade at highly dispersed prices. To document this fact, we begin with a description of how pricing actually occurs in the CDS market. In theory, the fair value spread — the CDS spread — of a contract is set so that the buyer and the seller both have an expected net present value (NPV) of zero. In practice though, CDS contracts have fixed coupons standardized by contract and so there must be some money exchanged upfront to make the contract NPV zero. For instance, if the fair value spread is greater than the fixed coupon in the contract, then the buyer should be paying a higher premium for default insurance than is specified by the fixed coupon. In turn, the buyer will pay the seller some initial money upfront to make up for this difference. Thus, the fixed coupon and the upfront amount paid in the contract contains all of the relevant pricing information.

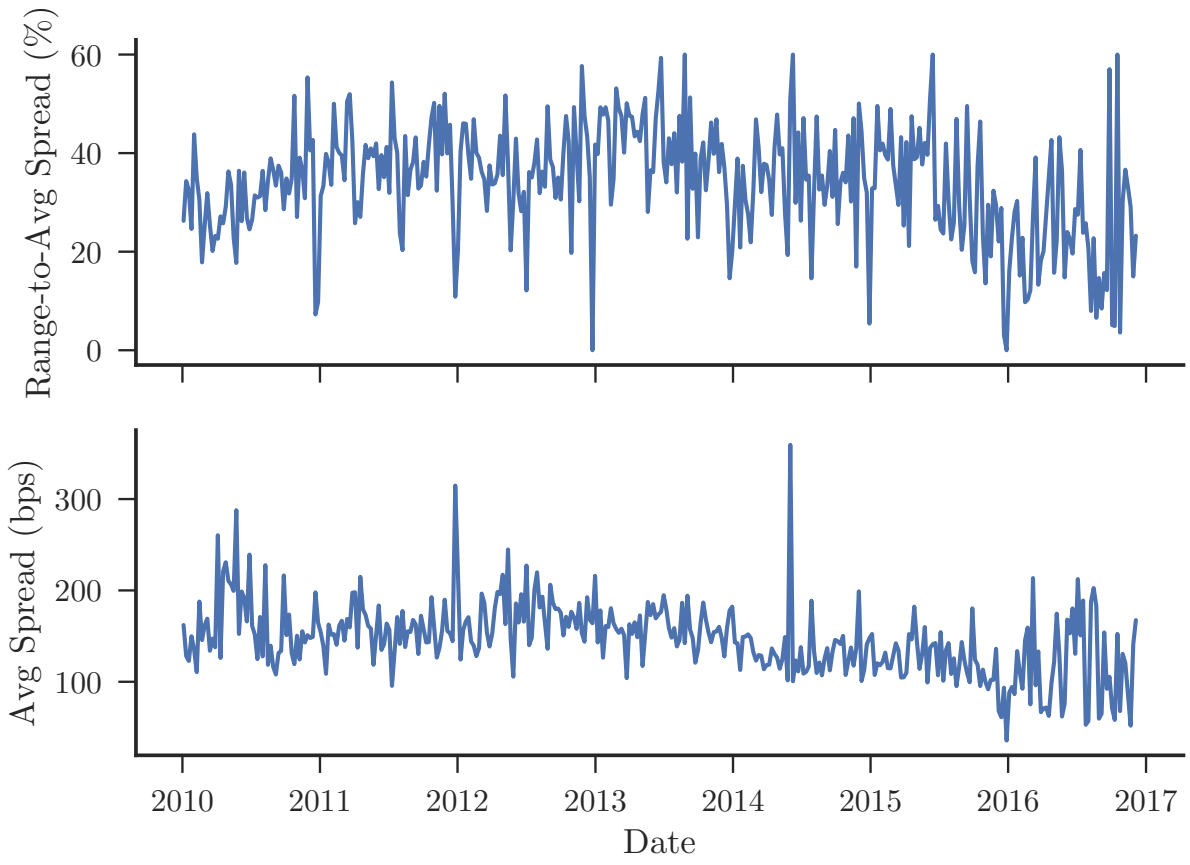
Denote the upfront paid (as a fraction of notional) by buyer j to seller i on transaction k by $R_k(i, j, n, c, m, t)$. Here, n is the underlying reference entity on the contract, c is the fixed coupon on the contract, m is the maturity, and t is the initiation date. To condense notation, define $\theta := (n, c, m)$. While this notation is somewhat cumbersome, we will abstract away from it when analyzing our model of the CDS market in Section INSERT. In our model, we consider only one contract type θ, which we interpret as exposure to aggregate risk. For now, the notation is necessary in order to make the simple point that traders often face different prices on the *same* underlying contract θ. On each date t, let $\Omega(\theta, t)$ be the set of transactions with the same (θ, t) and the size of this set is given by $\omega(\theta, t)$. We can then define the average upfront paid and the standard deviation of upfronts paid for the *same* contract as follows:

$$\begin{aligned} \overline{R}(\theta, t) &:= \frac{1}{\omega(\theta, t)} \times \sum_{k \in \Omega(\theta, t)} R_k(i, j, \theta, t) \\ \sigma(\theta, t) &:= \sqrt{\frac{1}{\omega(\theta, t)} \times \sum_{k \in \Omega(\theta, t)} \left[R_k(i, j, \theta, t) - \overline{R}(\theta, t) \right]^2} \end{aligned} \tag{1}$$

A natural measure of price dispersion for a given contract is $\sigma(\theta, t)$. If everyone in the market faces the same price for the same contract, then we would expect $\sigma(\theta, t) = 0$. An alternative definition of price dispersion that we consider is $\eta(\theta, t) := \sigma(\theta, t)/\overline{U}(\theta, t)$. Normalizing the within-contract volatility of prices in this fashion is one way to assess the economic significance of within-contract volatility. For instance, a volatility of 1 basis point will mean much more if the average upfront paid is 5 basis points versus 100 basis points.

At each point in time t, we obtain a cross-sectional distribution of $\sigma(\theta, t)$ and $\eta(\theta, t)$ across all contracts θ. This gives us a sense as to how dispersed prices are in the cross section of contracts. Figure 5 plots the average $\sigma(\theta, t)$ and $\eta(\theta, t)$ within each week. There

Figure 5: Price Dispersion of CDS Contracts



Notes: This figure summarizes two measures of price dispersion in CDS markets, $\sigma(\theta, t)$ and $\eta(\theta, t)$. $\sigma(\theta, t)$ is the standard deviation of prices for a contract θ on date t, and is defined formally in Equation (1). $\eta(\theta, t)$ is $\sigma(\theta, t)/\overline{U}(\theta, t)$, where $\overline{U}(\theta, t)$ is the average upfront amount paid for contract θ on date. Within each week, we plot the average price dispersion measure across all contract and time pairs. Data is weekly and spans January 2010 to November 2015. To be included in the analysis a contract must have at least five transactions on a given date.

are two important takeaways from this plot. The first is that there is clear dispersion in prices, as evidenced by the simple fact that $\sigma(\theta, t)$ is nonzero. The second is that this price dispersion is economically significant. On average, the within-contract volatility of prices is nearly 20% of the average price paid. This means, for instance, that for a contract with an average upfront of 100 basis points, the standard deviation of the price one might pay is 20 basis points.

3 Model

There is one type asset in the economy with a random payment given by $(1 - D)$, where D is a default component with mean μ and variance σ^2. There are n agents in the economy. An agent i is endowed with w_i units of the risky asset and can trade credit default swap (CDS) contracts before the aggregate default is realized. A CDS contract between agents i and j specifies that agent i pays D to agent j, and the cost of the contract is given by R_{ij}, which is determined in equilibrium. Let γ_{ij} be the number of contracts agent i sells to agent j, which can be either positive or negative depending on whether agent i is selling or buying insurance. Thus, a positive γ_{ij} implies that agent i is taking credit risk, increasing their exposure to the aggregate default risk in the underling asset.

The payment function for agent i is:

$$\pi_i = w_i(1 - D) + \sum_{j=1}^{n} \gamma_{ij}(R_{ij} - D). \tag{2}$$

There is network of trade connections that specifies which agents can trade with whom. The network is given by a n by n matrix G of zeros and ones, whose entry (i, j) is $g_{ij} \in \{0, 1\}$. If agent i and j can trade, then $g_{ij} = 1$, and $g_{ij} = 0$ otherwise. The network is symmetric, which means that if agent i can trade with agent j, then j can also trade with i. The trade structure being exogenous is consistent with our empirical finding that these connections do not vary much over time, as discussed in Section 2. Although the network of trade connections is exogenous, the quantities traded (γ's) are endogenously determined in equilibrium, which is also consistent with the observed change in the quantities traded over time.

We assume the following mean-variance preferences for agent i over a portfolio of CDS contracts:

$$\begin{aligned} U_i(\gamma_{i1}, \ldots, \gamma_{in}) &= \mathbb{E}[\pi_i] - \frac{\alpha_i}{2} V(\pi_i) - \frac{\phi_i}{2} \sum_{j=1}^{n} \gamma_{ij}^2 \\ &= w_i(1 - \mu) + \sum_{j=1}^{n} \gamma_{ij}(R_{ij} - \mu) - \frac{\alpha_i}{2} \left(w_i + \sum_{j=1}^{n} \gamma_{ij} \right)^2 \sigma^2 - \frac{\phi_i}{2} \sum_{j=1}^{n} \gamma_{ij}^2 \end{aligned} \tag{3}$$

where $\alpha_i > 0$ is a risk aversion parameter. As we will see, α_i will mainly affect net positions as it measure the of bearing default risk. There is also a cost of taking large gross positions with any one counterparty, as captured by $\phi_i > 0$. In the Appendix, we provide a microfoundation for ϕ_i. We introduce counterparty default risk and show that $\phi_i > 0$ captures agent i's

desire to diversify counterparty default risk or simply how agent i perceives the riskiness of counterparty default.

Agent i's optimization problem is given by:

$$\begin{aligned} \max_{z_i, \{\gamma_{ij}\}_{j=1}^n} \; & w_i(1-\mu) + \sum_{j=1}^{n} \gamma_{ij}(R_{ij}-\mu) - \frac{\alpha_i}{2}\left(w_i + z_i\right)^2 \sigma^2 - \frac{\phi_i}{2}\sum_{j=1}^{n}\gamma_{ij}^2 \\ s.t. \quad & \gamma_{ij} = 0 \text{ if } g_{ij} = 0 \\ & z_i - \sum_{j=1}^{n}\gamma_{ij} = 0, \end{aligned} \tag{4}$$

We model the CDS market payment structure including the insurance contract itself as in Atkeson, Eisfeldt, and Weill (2015). However, we have market clearing prices instead of nash bargaining between different counterparties, although the equilibrium outcome in our model looks similar to a nash bargain as we discuss in the next subsection. The main difference between both models is that we impose a network of trading connections and we assume that agents are averse to counterparty risk. Aversion to counterparty risk is equivalent to having convex costs of trading with any single counterparty or simply a smooth trade limit assumption (positive ϕ's). Furthermore, our framework is flexible enough to include heterogeneity in risk aversion (α's) and in aversion to counterparty risk (ϕ's).

Our modeling approach is also related to the work by Denbee, Julliard, Li, and Yuan (2014). We share similar type mean-variance preferences, however, in their model, banks choose liquidity levels taking as given other banks' liquidity. In our model, instead, agents choose how much to trade with each other taking prices as given. Although the network in both models is exogenous, the network weights in our model (i.e., the specific trade between counterparties—γ's) are endogenous while they are exogenous in their model. The main difference between both models is that they do not have equilibrium prices since they focus on the quantity of bank liquidity buffers. Price dispersion, however, is a key feature of CDS markets that we capture in our framework.

Solving agent i's optimization problem, we have:

$$\gamma_{ij} = \begin{cases} \frac{1}{\phi_i}\left[R_{ij} - \mu - \alpha_i(w_i + z_i)\sigma^2\right] & \text{if } g_{ij} = 1 \\ 0 & \text{if } g_{ij} = 0 \end{cases} \tag{5}$$

where

$$z_i = \sum_{j=1}^{n}\gamma_{ij} = \frac{\frac{1}{K_i}\sum_{j=1}^{n} g_{ij}(R_{ij}-\mu) - \alpha_i w_i \sigma^2}{\frac{\phi_i}{K_i} + \alpha_i \sigma^2} \tag{6}$$

and

$$K_i = \sum_{j=1}^{n} g_{ij}. \tag{7}$$

How much agent i is going to sell to agent j depends on the contract premium $(R_{ij} - \mu)$, on her risk aversion (α_i), on her post-trade exposure to aggregate default $(z_i + w_i)$, on the variance of the default (σ^2), and on agent i's aversion to counterparty default (ϕ_i). The net position of agent i in the CDS market is given by z_i, which depends on her risk aversion, her aversion to counterparty risk, and her pre-trade exposure. Agent i's net position also depend on the average contract price among agent i's neighbors.

To understand the difference between ϕ_i and α_i, we can show that the Lagrange multiplier (shadow price) on the last constraint of the optimization problem is equal to $\alpha_i(z_i + w_i)\sigma^2$. When agents i and j can trade, the number of contracts that i sells to j can be expressed as:

$$\gamma_{ij} = \frac{1}{\phi_i}\left(R_{ij} - \mu\right) - \frac{1}{\phi_i}\underbrace{\alpha_i(w_i + z_i)\sigma^2}_{\text{shadow price of } z_i}.$$

Therefore, risk aversion (α_i) determines how much agent i values net positions (shadow price of z_i), and aversion to counterparty risk (ϕ_i) determines how much the amount insured by the contract between agents i and j (γ_{ij}) depends on the contract price, holding the shadow price of net positions fixed. In other words, α_i drives agent i's net position, while ϕ_i defines how much agent i sells to and buys from each counterparty.

3.1 Equilibrium

We solve the model for a competitive equilibrium, in which agents maximize taking prices as given and all markets clear. In equilibrium, how much agent i wants to sell to agent j has to be the same amount that agent j wants to buy from agent i, and vice-versa. Hence, the market clearing conditions are:

$$\gamma_{ij} + \gamma_{ji} = 0 \qquad \forall i, j = 1, \ldots, n. \tag{8}$$

Definition. *A competitive equilibrium consists of spot market prices ($\{R_{ij}\}_{i,j=1,\ldots,n}$) and traded quantities ($\{\gamma_{ij}\}_{i,j=1,\ldots,n}$) such that (i) agents optimize, taking the network of trading connection and prices as given (equation 4), and (ii) markets clear (equation 8).*

Furthermore, we assume no transaction costs, and, therefore, the payment agent i receives

from selling a CDS contract to agent j is exactly the amount agent j pays for such contract:

$$R_{ij} = R_{ji}. \tag{9}$$

If agents i and j can trade (i.e., $g_{ij} = g_{ji} = 1$), we can use their optimality conditions from equation (5) and market clearing condition from equation (8) to solve for the equilibrium price of a contract between agents i and j:

$$R_{ij} - \mu = \underbrace{\left[\frac{\frac{1}{\phi_i}\alpha_i(w_i + z_i) + \frac{1}{\phi_j}\alpha_j(w_j + z_j)}{\frac{1}{\phi_i} + \frac{1}{\phi_j}}\right]\sigma^2}_{\text{contract premium}} \tag{10}$$

The contract premium, which is the contract price in excess to the expected default in the underlining asset, depends on an average of agents' post-trade exposure to default risk , as well as aggregate default variance and preferences parameters. As a result, whenever there are differences in post-trade exposures, there is price dispersion in the cross section of agents in equilibrium, even if agents have identical preferences. This result is consistent with Atkeson, Eisfeldt, and Weill (2015), however our model features additional sources of price dispersion absent in their framework. Our model generates price dispersion due to the structure of the trading network itself as it endogenously generates differences in post-trade exposures. Another source of price dispersion in our model is heterogeneity in risk aversion (α_i's) and heterogeneity in cost of trading with any counterparty (ϕ_i's). Therefore, our model generates cross-sectional dispersion of CDS prices, which is consistent with our empirical findings from Section 2.

The net positions, $\{z_i\}_i$, are determined in equilibrium. We can use equations (6) and (10) to solve for equilibrium net positions as a linear combination of initial exposures and net position of other agents:

$$z_i\left(1 + \frac{\phi_i}{\tilde{K}_i\alpha_i\sigma^2}\right) = -w_i + \sum_{j=1}^{n}\frac{\tilde{g}_{ij}\alpha_j}{\alpha_i}(z_j + w_j) \qquad \forall i = 1, \ldots, n \tag{11}$$

where $\tilde{g}_{ij} = g_{ij}\frac{\phi_i}{\phi_i+\phi_j}\frac{1}{\tilde{K}_i}$ and $\tilde{K}_i = \sum_{j=1}^{n} g_{ij}\frac{\phi_i}{\phi_i+\phi_j}$. The detailed derivation is in the Appendix.

According to equation (11), the net position of agent i depends not only on agent i's preferences (α_i and ϕ_i) and pre-trade exposure (w_i), but also on all other agents' preferences and pre-trade exposures. This is an equilibrium outcome of the model. When two agents trade, their contract premium as in equation (10) depends on the post-trade exposure and preferences of both trading parties. However, the contract premium itself affects post-trade

exposures of the agents trading, when they solve their respective optimization problems (equation 6). Therefore, the contract premium of a trade between i and j depends indirectly on the neighbors of both trading parties, because i and j trading relations with other counterparties affect the post-trade exposure of agents i and j and therefore the contract premium between i and j. Although, in equation (6), agent i's net position is determined by the average contract premium of her neighbors, the post-trade exposure of other agents in the economy affect agent i's post trade exposure through equilibrium prices. The ultimate effect on equilibrium post-trade exposures of all agents is captured by the net position in equation (11).

In equilibrium, net position of agent i (z_i) depends on her initial exposure (w_i) and on a linear combination of her neighbors' post-trade exposures, weighted by $\frac{\tilde{g}_{ij}\alpha_j}{\alpha_i}$. The weight $\tilde{g}_{ij}$ is a measure of relative aversion to counterparty default risk between agents i and j, and all these weights sum to one:

$$\sum_{j=1}^{n} \tilde{g}_{ij} = 1.$$

Whenever i and j are allowed to trade (i.e., $g_{ij} = 1$) and $\tilde{g}_{ij}$ is high, then it is less costly for agent j to bear counterparty risk relative to agent i. Hence, the term $\sum_{j=1}^{n} \frac{\tilde{g}_{ij}\alpha_j}{\alpha_i}(z_j + w_j)$ from equation (11) is an average of agent i's neighbors post-trade exposure, weighted by their relative aversion to counterparty default risk and by their relative overall risk aversion, with more weight being allocated to trade partners with either high overall risk aversion (high α_j) or low aversion to counterparty default risk (low ϕ_j). This means that, in equilibrium, agent i's post-trade exposure is closer to the post-trade exposures of her neighbors with high risk aversion or neighbors who can trade more easily and have low aversion to counterparty default risk.

How close agent i's post-trade exposure gets to the average of her neighbors depends on how averse agent i is to counterparty default risk. On the one hand, if agent i has low ϕ_i, then she can trade more easily and have her post-trade exposure closer to her neighbors in equilibrium. On the other hand, if ϕ_i is high, then it becomes costly for agent i to trade away from her initial exposure to a more diversified portfolio of CDS contracts, and her post-trade exposure remains similar to her pre-trade exposure. We can represent the relation between pre- and post-trade exposures by rewriting equation 11 as follows:

$$z_i + w_i = (1 - \lambda_i)w_i + \lambda_i \sum_{j=1}^{n} \frac{\tilde{g}_{ij}\alpha_j}{\alpha_i}(z_j + w_j)$$

where $\lambda_i = \frac{\tilde{K}_i\alpha_i\sigma^2}{\tilde{K}_i\alpha_i\sigma^2+\phi_i} \in (0, 1)$. Therefore, a positive aversion to counterparty default risk

(i.e., $\phi_i > 0$) allows agents to partially share the default risk with their trading partners (i.e., λ_i between 0 and 1). As a result, the pre-trade exposure of an agent in the network has less and less influence on the post-trade exposure of other agents as trading partners get further away from each other in the trading network. This decaying influence of pre-trade exposures on post-trade exposures becomes clear if we further develop the expression above and write post-trade exposures as a function of pre-trade exposures:

$$z_i + w_i = (1-\lambda_i)w_i + \lambda_i \left[\sum_{j=1}^{n} \tilde{g}_{ij} \frac{\alpha_j}{\alpha_i} (1-\lambda_j) w_j + \sum_{j=1}^{n} \sum_{s=1}^{n} \lambda_j \tilde{g}_{ij} \tilde{g}_{js} \frac{\alpha_s}{\alpha_i} (1-\lambda_s) w_s \right.$$
$$\left. + \sum_{j=1}^{n} \sum_{s=1}^{n} \sum_{k=1}^{n} \lambda_j \lambda_s \tilde{g}_{ij} \tilde{g}_{js} \tilde{g}_{sk} \frac{\alpha_k}{\alpha_i} (1-\lambda_k) w_k + \dots \right]$$

The equilibrium condition from equation (11) as well as the equation above shows that the equilibrium net position of agent i depends on the post-trade exposure of her neighbors, which in turn depend on the post-trade exposure of their own neighbors, and so on. We can write the system of equations described in equation (11) in matrix notation as:

$$\underbrace{\begin{bmatrix} 1 + \frac{\phi_1}{\tilde{K}_1 \alpha_1 \sigma^2} & 0 & \cdots & 0 \\ 0 & 1 + \frac{\phi_2}{\tilde{K}_2 \alpha_2 \sigma^2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & & 1 + \frac{\phi_n}{\tilde{K}_n \alpha_n \sigma^2} \end{bmatrix}}_{\equiv \Lambda^{-1}} z = -w + \underbrace{\begin{bmatrix} \frac{\tilde{g}_{11}\alpha_1}{\alpha_1} & \frac{\tilde{g}_{12}\alpha_2}{\alpha_1} & \dots & \frac{\tilde{g}_{1n}\alpha_n}{\alpha_1} \\ \frac{\tilde{g}_{21}\alpha_1}{\alpha_2} & \frac{\tilde{g}_{22}\alpha_2}{\alpha_2} & \dots & \frac{\tilde{g}_{2n}\alpha_n}{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\tilde{g}_{n1}\alpha_1}{\alpha_n} & \frac{\tilde{g}_{n2}\alpha_2}{\alpha_n} & \dots & \frac{\tilde{g}_{nn}\alpha_n}{\alpha_n} \end{bmatrix}}_{\equiv G^*} (z + w),$$

where $z = [z_1, \dots, z_n]'$ and $w = [w_1, \dots, w_n]'$ are column vectors of net positions and pre-trade exposures. We can solve the system of equations for the equilibrium net positions and post-trade exposures:

$$z = (\mathrm{I} - \Lambda G^*)^{-1} \Lambda (G^* - \mathrm{I}) w$$
$$z + w = (\mathrm{I} - \Lambda G^*)^{-1} (\mathrm{I} - \Lambda) w$$

3.2 Useful properties

Here we consider some special case in order to provide some intuition.

Definition 1. *Two players i and j are path connected if there is a sequence of agents*

$\{s_1, s_2, \ldots, s_k\}$ *such that:*

$$g_{is_1} = g_{s_1 s_2} = \ldots = g_{s_{k-1} s_k} = g_{s_k j} = 1.$$

Proposition 1. *If $\phi_i = 0$ for every $i = 1, \ldots, n$, then any two path connected agents have the same post-trade exposure scaled by their risk aversion:*

$$\alpha_i(z_i + w_i) = \alpha_j(z_j + w_j)$$

for any i and j who are path connected.

Proof. Suppose players i and j are path connected, but

$$\alpha_i(z_i + w_i) \neq \alpha_j(z_j + w_j).$$

Then, there are two agents, say s and l, that are directly connected with each other (i.e., $g_{sl} = 1$) and have different post-trade exposure (i.e., $z_s + w_s \neq z_l + w_l$). If both agents are maximizing and their first-order conditions hold with equality, then we have that:

$$R_{sl} - \mu = \alpha_s(z_s + w_s)\sigma^2 = \alpha_l(z_l + w_l)\sigma^2 \implies \alpha_i(z_i + w_i) = \alpha_j(z_j + w_j)$$

□

Corollary 1. *If $\phi_i = 0$ and $\alpha_i = \alpha$ for every $i = 1, \ldots, n$, then any two path connected agents have the same post-trade exposure:*

$$z_i + w_i = z_j + w_j$$

for any i and j who are path connected.

Corollary 2. *If $\phi_i = 0$ and $\alpha_i = \alpha$ for every $i = 1, \ldots, n$, and all agents are path connected, then there is perfect risk-sharing in equilibrium and*

$$z_i + w_i = \frac{1}{n} \sum_j w_j.$$

Proof. We know that:

$$z_i + w_i = z_j + w_j = \overline{zw},$$

where $\overline{zw}$ is a constant. We also know that

$$\sum_j z_j = 0,$$

from the clearing conditions. □

Proposition 2. *If $\phi_i \to \infty$ for every $i = 1, \ldots, n$, then there is no trade in equilibrium, regardless of the network structure.*

Proof. From the first-order conditions, we get that $\gamma_{ij} = 0$ for any two agents i and j. □

3.3 Volatility

Here we derive some implications of our model for both time-series and cross-sectional dispersion of net positions, post-trade exposures and equilibrium prices. We also develop some implications for subnetwork cross-sectional price dispersion.

Time series variation Taking the network (G), risk aversion (α's) and trading costs (ϕ's) as given, assuming that the vector of pre-trade exposures (w) is a random variable with variance-covariance matrix given by Ω_w (i.e., $\text{Var}(w) = \Omega_w$), then the variance-covariance matrix of net positions and post-trade exposures are given by:

$$\begin{aligned}\text{Var}_w(z) &= (\text{I} - \Lambda G^*)^{-1}\Lambda(\text{I} - G^*)\Omega_w(\text{I} - G^{*\prime})\Lambda(\text{I} - \Lambda G^{*\prime})^{-1} \\ \text{Var}_w(z + w) &= (\text{I} - \Lambda G^*)^{-1}(\text{I} - \Lambda)\Omega_w(\text{I} - \Lambda)(\text{I} - \Lambda G^{*\prime})^{-1}\end{aligned}$$

We can write equilibrium prices from equation (10) as:

$$R_{ij} - \mu = \sigma^2 \frac{\phi_j}{\phi_i + \phi_j}\alpha_i(w_i + z_i) + \sigma^2 \frac{\phi_i}{\phi_i + \phi_j}\alpha_j(w_j + z_j) \tag{12}$$

which we can use to computed its variance and covariances using $\text{Var}_w(z + w)$.

Cross-sectional dispersion Conditional on observing pre-trade exposures and preference parameters, the cross-sectional dispersion of net positions is given by:

$$\hat{\text{Var}}_i(z) \equiv \frac{1}{n}\sum_{i=n}^{n} z_i^2 = \frac{1}{n} w'(\text{I} - G^{*\prime})\Lambda(\text{I} - \Lambda G^{*\prime})^{-1}(\text{I} - \Lambda G^*)^{-1}\Lambda(\text{I} - G^*)w$$

Similarly, the cross-sectional dispersion of post-trade exposures is given by:

$$\hat{\text{Var}}_i(z+w) \equiv \frac{1}{n}\sum_{i=n}^{n}(w_i+z_i)^2 = \frac{1}{n}w'(\text{I}-\Lambda)(\text{I}-\Lambda G^{*\prime})^{-1}(\text{I}-\Lambda G^{*})^{-1}(\text{I}-\Lambda)w$$

We can also derive the cross-sectional dispersion in equilibrium prices based on equation 12), however the expression gets longer.

Subnetwork cross-sectional price dispersion If we focus on one specific agent (say i), and assume that agent i's neighbors have all the same aversion to counterparty risk (say $\phi_j = \bar{\phi}$ for every j with $g_{ij} = 1$), then we can compute the price dispersion among her trading partners and get a very tractable and intuitive expression:

$$\hat{\text{Var}}_j(R_{ij}) \equiv \frac{1}{K_i}\sum_{j=1}^{n} g_{ij}\left(R_{ij} - \frac{1}{K_i}\sum_{s=1}^{n} g_{is}R_{is}\right)^2 = \left(\sigma^2\frac{\phi_i}{\phi_i+\bar{\phi}}\right)^2 \hat{\text{Var}}_j\left(\alpha_j(w_j+z_j)\right).$$

Hence, the cross-sectional dispersion that agent i faces with her trading connections (i's subnetwork) is proportional to the cross-sectional dispersion of of the post-trade exposures of her connections. Therefore, the specific position of agent in the network determines her insurance and trading opportunities in equilibrium.

3.4 Risk-sharing benchmark with complete network

Next, we present an example with identical agents e a complete trading network. Let assume that $\phi_i = \phi$ and $\alpha_i = \alpha$ for every agent $i = 1, \ldots, n$. Moreover, we assume that all agents can trade with each other and the network is complete (i.e., $g_{ij} = 1 \forall i, j$). In this case, we have:

$$\Lambda^{-1} = \left(1 + \frac{\phi}{n\alpha\sigma^2}\right)\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & 1 \end{bmatrix} \text{ and } G^* = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

The matrix G^* is idempotent (i.e., $G^{*2} = G^*$) and therefore the vector of net positions

becomes:

$$\begin{aligned} z &= (\mathrm{I} - \Lambda G^*)^{-1}\Lambda(G^* - \mathrm{I})w \\ &= \underbrace{\frac{n\alpha\sigma^2}{n\alpha\sigma^2+\phi}}_{\equiv\lambda}\left(\mathrm{I} - \frac{n\alpha\sigma^2}{n\alpha\sigma^2+\phi}G^*\right)^{-1}(G^* - \mathrm{I})w \\ &= \lambda\left(\mathrm{I} - \lambda G^*\right)^{-1}(G^* - \mathrm{I})w \\ &= \lambda\left(\mathrm{I} + \lambda G^* + \lambda^2 G^* + \lambda^3 G^* \ldots\right)(G^* - \mathrm{I})w \\ &= \lambda(G^* - \mathrm{I})w. \end{aligned}$$

Therefore the net position and total post-trade exposures are given by:

$$z_i = -\lambda\left(w_i - \frac{1}{n}\sum_j w_j\right),$$

and

$$z_i + w_i = (1-\lambda)w_i + \lambda\left(\frac{1}{n}\sum_j w_j\right),$$

where

$$\lambda = \frac{n\alpha\sigma^2}{n\alpha\sigma^2+\phi} = \frac{1}{1+\frac{\phi}{n\alpha\sigma^2}} \in (0,1).$$

The coefficient λ measures how much far the equilibrium allocation is from perfect risk sharing due to the cost of bearing counterparty risk.

3.5 Numerical examples

In this section, we discuss some numerical examples to highlight the differences between risk aversion and aversion to counterparty risk as well as other features of the model. In all examples, we assume that there are six agents with different pre-trade exposures. We are going to call two of them as sellers (s_1 and s_2) who have no pre-trade exposure ($w_{s_1} = w_{s_2} = 0$), two other player are named buyers (b_1 and b_2) with higher pre-trade exposure to the underlying asset ($w_{b_1} = w_{b_2} = 10$). The two remain agents are named dealers (d_1 and d_2) who have the economy-wide average pre-trade exposure ($w_{d_1} = w_{d_2} = 5$). These pre-trade exposures imply that buyers are more exposed to the default risk in the underlying asset and therefore value insurance more than other agents. So, one would expect buyers to be net buyers in the CDS makers, while sellers are expected to be net sellers as they are not as exposed to default risk and value insurance less. Dealers, however, are at the economy-wide

average of exposure to default in the underlying asset.

To illustrate how the model works, we first consider a benchmark case, in which all agents can trade with each other (complete network) and they have the same risk aversion of 10 and same aversion to counterparty risk of 1 (i.e., $\alpha_i = 10$ and $\phi_i = 1$ for every agent i). This is the risk-sharing benchmark studied in Subsection 3.4. In this case, dealers have zero net position ($z_{d_1} = z_{d_2} = 0$), buyers have negative net position ($z_{b_1} = z_{b_2} = -4.3$), and sellers have positive net position ($z_{s_1} = z_{s_2} = 4.3$). Although dealers have zero net position, they hold long positions with sellers and short positions with buyers, and endogenously become intermediaries. Buyers and sellers trade directly with each other as well, but they also use dealers as intermediaries to mitigate counterparty default risk. There is price dispersion in this risk-sharing benchmark, and the cross-sectional standard deviation of prices is 0.75.

Core-periphery dealer network

Next, we consider a different structure of bilateral trades. We assume that both sellers can trade with dealer 1, both buyers can trade with dealer 2, and dealers can trade with each other. This network structure roughly mimics the core-periphery CDS network described in Section 2. The matrix G in this example is given by:

$$G = \begin{array}{c} \overbrace{\begin{array}{cccccc} (d_1 & d_2 & s_1 & s_2 & b_1 & b_2) \end{array}} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Figure 6 represents this example graphically as well.

Next, we evaluate the equilibrium under four different preference parameterizations:

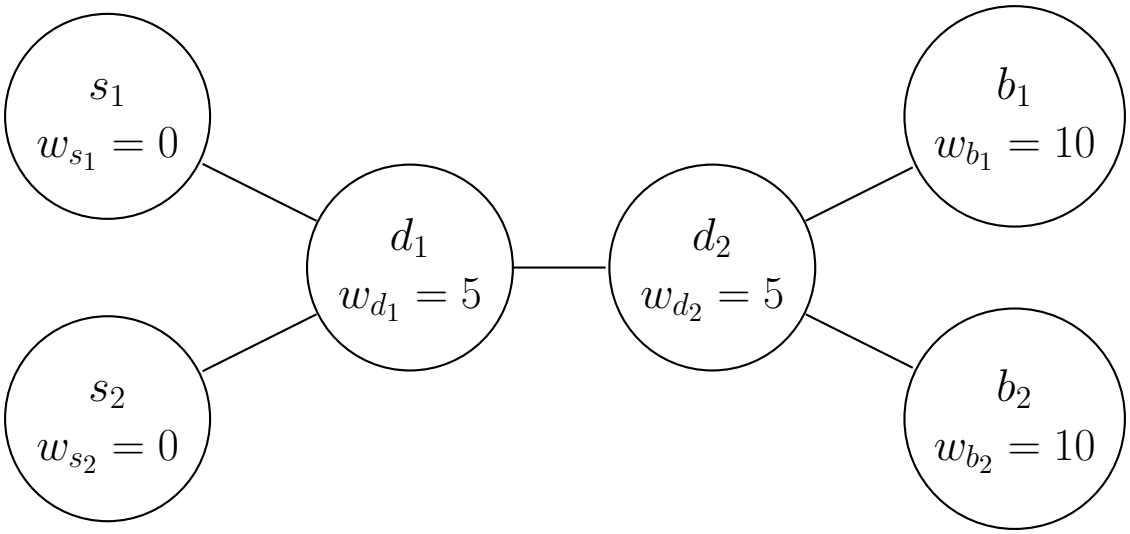
(A) Same risk aversion and aversion to counterparty default, $\alpha_i = 10$ and $\phi_i = 1$ for every $i = d_1, d_2, s_1, s_2, b_1, b_2$.

(B) Dealers have lower aversion to counterparty risk: $\alpha_i = 10$ and $\phi_i = 1$ for $i = s_1, s_2, b_1, b_2$, and $\alpha_i = 10$ and $\phi_i = 0.25$ for $i = d_1, d_2$.

(C) Dealers have lower risk aversion: $\alpha_i = 10$ and $\phi_i = 1$ for $i = s_1, s_2, b_1, b_2$, and $\alpha_i = 2$ and $\phi_i = 1$ for $i = d_1, d_2$.

Figure 6: Example of core-periphery dealer network

This picture represent a network with six players. The two players on the left are seller 1 (s_1) and seller 2 (s_2) and their pre-trade exposure is $w_{s_1} = w_{s_2} = 0$. The two players in the middle are dealer 1 (d_1) and dealer 2 (d_2) and their pre-trade exposure is $w_{d_1} = w_{d_2} = 5$. The two players on the right are buyer 1 (s_1) and buyer 2 (s_2) and their pre-trade exposure is $w_{s_1} = w_{s_2} = 10$.



(D) Dealers have lower risk aversion and lower aversion to counterparty risk: $\alpha_i = 10$ and $\phi_i = 1$ for $i = s_1, s_2, b_1, b_2$, and $\alpha_i = 2$ and $\phi_i = 0.25$ for $i = d_1, d_2$.

The equilibrium outcome under these four parameterizations are reported in Table 2 in Panels A, B, C, and D, respectively.

Under parameterization A, the cross-sectional standard deviation of equilibrium prices is 4.12. Clearly, there is much more price dispersion if compared to the risk-sharing benchmark, in which all agents are connected with each other as buyers and seller no longer trade directly with each other. Equilibrium price dispersion translates into trading profits for dealers who endogenously become intermediaries. Dealers' trading profits are compensation for bearing counterparty default risk.

If dealers have lower aversion to counterparty default risk (parameterization B), then they become more efficient intermediaries. Post-trade exposures in the underlying asset are closer to perfect risk-sharing, and price dispersion is significantly lower (1.77). In this case, dealers have low net positions, but large gross positions. Interestingly, when dealers have lower aversion to counterparty default risk, they end up with lower utility (parameterization A vs. B). Holding prices constant, a decrease in ϕ_i should increase agent i's utility. Lower aversion to counterparty risk makes it easier for agent i to trade with her counterparties, allowing her to improve risk allocation and to make more trading profits from price dispersion. However,

Table 2: Example: equilibrium evaluation

	risk aversion (α_i)	counterparty risk (ϕ_i)	pre-trade exposure (w_i)	post-trade exposure ($w_i + z_i$)	net position (z_i)	gross position $\left(\sum_{j=1}^{n} \lvert\gamma_{ij}\rvert\right)$
Panel A: Same risk aversion and aversion to counterparty default						
Dealer 1	10.00	1.00	5.00	3.75	−1.25	6.25
Dealer 2	10.00	1.00	5.00	6.25	1.25	6.25
Seller 1	10.00	1.00	0.00	1.88	1.88	1.88
Seller 2	10.00	1.00	0.00	1.88	1.88	1.88
Buyer 1	10.00	1.00	10.00	8.12	−1.88	1.88
Buyer 2	10.00	1.00	10.00	8.12	−1.88	1.88
$\sigma(R_{ij})$	4.12					
Panel B: Dealers with lower aversion to counterparty risk						
Dealer 1	10.00	0.25	5.00	4.40	−0.60	10.23
Dealer 2	10.00	0.25	5.00	5.60	0.60	10.23
Seller 1	10.00	1.00	0.00	2.71	2.71	2.71
Seller 2	10.00	1.00	0.00	2.71	2.71	2.71
Buyer 1	10.00	1.00	10.00	7.29	−2.71	2.71
Buyer 2	10.00	1.00	10.00	7.29	−2.71	2.71
$\sigma(R_{ij})$	1.77					
Panel C: Dealers with lower risk aversion						
Dealer 1	2.00	1.00	5.00	5.21	0.21	2.29
Dealer 2	2.00	1.00	5.00	11.46	6.46	8.96
Seller 1	10.00	1.00	0.00	0.52	0.52	0.52
Seller 2	10.00	1.00	0.00	0.52	0.52	0.52
Buyer 1	10.00	1.00	10.00	6.15	−3.85	3.85
Buyer 2	10.00	1.00	10.00	6.15	−3.85	3.85
$\sigma(R_{ij})$	3.32					
Panel D: Dealers with lower aversion to counterparty risk and lower risk aversion						
Dealer 1	2.00	0.25	5.00	6.79	1.79	5.13
Dealer 2	2.00	0.25	5.00	11.11	6.11	13.03
Seller 1	10.00	1.00	0.00	0.84	0.84	0.84
Seller 2	10.00	1.00	0.00	0.84	0.84	0.84
Buyer 1	10.00	1.00	10.00	5.21	−4.79	4.79
Buyer 2	10.00	1.00	10.00	5.21	−4.79	4.79
$\sigma(R_{ij})$	1.49					

we observe the opposite in equilibrium. Under parameterization B, the post-trade exposures of dealers' neighbors are closer perfect risk-sharing because dealers pass more default risk from buyers to sellers. This results in better risk-sharing, lower dispersion in post-trade exposures and lower price dispersion in equilibrium. As a result, dealers do not benefit much from having a low ϕ as the economy-wide risk-sharing benefits decrease their profit margins.

If dealers have lower risk aversion (parameterization C), then they are willing to bear more default risk than other agents. Buyers are initially more exposed to default risk and want to buy insurance, while sellers want to sell insurance. However, dealers 1 and 2 are willing to bear more risk and want to become net sellers in this economy due to their low risk aversion. This creates an asymmetry in dealers' equilibrium net positions because of their position in the trading network. Dealer 1 is not directly connected to buyers, and has to use dealer 2 as an intermediary to reach them. Dealer 1 also faces a demand for insurance from the two sellers she is connected to. In equilibrium, dealer 1 is a net seller ($z_{d_1} > 0$) and does not pass much default risk to sellers 1 and 2. Dealer 2 acts as an intermediary between buyers and dealer 1, but her unique position in the network allows her to absorb most of the default risk in the economy. In equilibrium, dealer 2 post-trade exposure is the highest of all agents ($w_{d_2} + z_{d_2} = 11.46$).

Under parameterization D, not only dealers have lower risk aversion but they also have lower aversion to counterparty default risk. Similar to parameterization B, dealers are more efficient intermediaries, and, as a result, post-trade exposures are closes to perfect risk-sharing and price dispersion is low (1.49). Parameterization D also shares some features of parameterization C as dealers are net sellers of insurance and dealer 2 bear most of the default risk in the underlying asset.

These four parameterizations highlight the differential roles of risk aversion and counterparty risk aversion inside of the model. As in a Walrasian market, risk aversion is a key determinant of who becomes more or less exposed to aggregate default risk. Aversion to counterparty risk dictates the flow of default risk through the system and thus is important in determining how prices are dispersed in the economy. The numerical examples also show that the impact of risk aversion and counterparty risk aversion on equilibrium is influenced by the underlying structure of bilateral connections in the market.

4 Estimation

In this section, we discuss the identification and estimation of the model parameters α_i's and ϕ_i's.

The parameter ϕ_i is important in determining specific portfolio allocation of bilateral

positions of agent i, but is less important for the total net position. A low aversion to counterparty risk (i.e., low ϕ_i) means that agent i can exploit price dispersion by going long and short with her counterparties without affecting her total post-trade exposure, and as a result a low ϕ_i agent takes greater advantage of any profitable trades. Hence, ϕ_i can be estimated based on how much differences in CDS prices drive her portfolio allocation.

A cross-sectional regression of agent i's holdings on CDS price in excess to the average price, controlling for her net position, can be used to estimate the ϕ_i. Using Equation (5), and taking the difference between how much agent i buys from agent j and s, we have that:

$$\gamma_{ij} - \gamma_{is} = \frac{1}{\phi_i} \left(R_{ij} - R_{is} \right).$$

Taking the average over s and using the fact that $z_i = \sum_{s=1}^{n} \gamma_{is}$, we have that:

$$\gamma_{ij} = \frac{z_i}{K_i} + \frac{1}{\phi_i} \left(R_{ij} - \frac{1}{K_i} \sum_{s=1}^{n} R_{is} \right). \tag{13}$$

The equation above can be used to estimate ϕ_i from the data.

Once we have estimates for ϕ's, we can use agents' first-order conditions to compute post-trade exposures scaled by risk aversion and default variance, namely the shadow price of z_i defines as $\hat{z}_i = \alpha_i(w_i + z_i)\sigma^2$. This is an interesting object because it is the marginal value of one additional unit of net position, i.e. insurance against the underlying asset. To calculate $\hat{z}_i$, we start from equations (5) and (6), and rewrite them as:

$$\hat{z}_i = \alpha_i(w_i + z_i)\sigma^2 = \frac{\sum_{j=1}^{n} \frac{g_{ij}}{\phi_i}(R_{ij} - \mu) - z_i}{\sum_{j=1}^{n} \frac{g_{ij}}{\phi_i}},$$

which we can compute from the data given the estimates for ϕ's and that we observe prices and net positions.

The parameter α_i affects more the total post-trade exposure than the specific portfolio allocation of bilateral trades of agent i. High risk aversion (i.e., high α_i) means that agent i places high value on insurance and use the CDS contracts to achieve a lower exposure to default risk in the underlying asset. Thus, high α_i implies that changes in pre-trade exposure (w_i) should result in more changes in the net positions in order to optimize the post-trade exposure ($z_i + w_i$). This motivates us to study the time series of one agents' trades, along with proxies for the default risk exposures, to get an estimate of α_i.

Using Equation (5), and summing over j we have that:

$$z_i = \frac{1}{\phi_i} \sum_{j=1}^{n} g_{ij} \left(R_{ij} - \mu \right) - \frac{\alpha_i}{\phi_i} K_i (w_i + z_i) \sigma^2 \tag{14}$$

If we observe $\sum_{j=1}^{n} g_{ij} \left(R_{ij} - \mu \right)$, z_i, and $(z_i + w_i)$ over time, then, for each agent i, we can run a time-series regression of an agent's net position on the average CDS prices and total post-trade exposure. The coefficient of this regression would be $\frac{\alpha_i \sigma^2 K_i}{\phi_i}$, which allows us to get an estimate for $\alpha_i \sigma^2$, given K_i and ϕ_i.

Alternatively, we can jointly estimate ϕ_i and α_i using a GMM procedure. For example, equation (5), assuming time-variation, becomes:

$$\gamma_{ij}^t = \frac{1}{\phi_i} \left(R_{ij}^t - \mu^t \right) - \frac{\alpha_i \sigma^2}{\phi_i} (w_i^t + z_i^t) \tag{15}$$

for each j that i is connected to. We might use equation (15) to run a panel regression for each agent i to jointly estimate $\frac{1}{\phi_i}$ and $\frac{\alpha_i \sigma^2}{\phi_i}$. We can also try to match the cross-sectional and time-series volatility measures discussed in Subsection 3.3.

5 Conclusion

[TO BE COMPLETED]

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Appendix

A Model with counterparty default risk

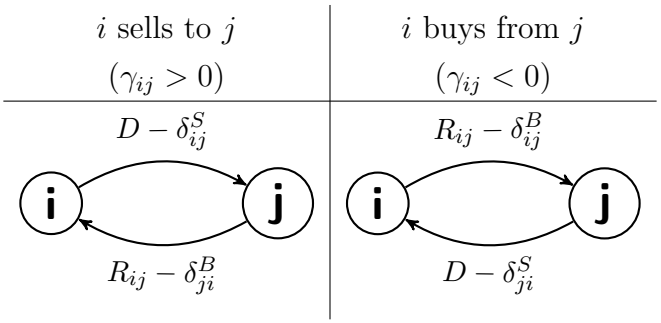
There is one asset in the economy with a random payment given by $(1 - D)$, where D is a default component with mean μ and variance σ^2. There are n agents in the economy. An agent i is endowed with w_i units of the risky asset and can trade CDS contracts before the payment is realized.

Let γ_{ij} be the number of contracts agent i sells to agent j at a price R_{ij}. Then, the payment function for agent i is:

$$\pi_i = w_i(1 - D) + \sum_{j=1}^{n} \gamma_{ij}(R_{ij} - D). \tag{16}$$

The above specification assumes that the counterparty does not default on the CDS contract. A CDS contract between i and j specifies that i has to pay agent j an amount D and such contract costs R_{ij} to agent j.

In this contract, however, there are two types of default risk: The seller can default on the insured amount (D), and the buyer can default on the paying the full price of the contract (R_{ij}). If agent i is a seller (i.e., $\gamma_{ij} > 0$ for some j), then agent i receives $R_{ij} - \delta^B_{ji}$ from agent j, where δ^B_{ji} is how much agent j (the CDS buyer) defaults on agent i. Agent i pays a total amount of $D - \delta^S_{ij}$ to agent j, where δ^B_{ji} is how much agent i (the CDS seller) defaults on agent j. The case in which agent i is a CDS buyer is analogous. When we take into account the counterparty default risk, then the total amount transferred are represented by:



Given the default structure, the payment function becomes:

$$\pi_i = w_i(1 - D) + \sum_{j=1}^{n} \gamma_{ij}(R_{ij} - D) + \underbrace{\sum_{b \in \mathcal{S}_i} \gamma_{ib}(\delta^S_{ib} - \delta^B_{bi})}_{\text{counterparty risk from selling}} + \underbrace{\sum_{s \notin \mathcal{S}_i} \gamma_{is}(\delta^S_{si} - \delta^B_{is})}_{\text{counterparty risk from buying}}, \tag{17}$$

where $\mathcal{S}_i = \{j \in N : \gamma_{ij} > 0\}$ is the set of players that agent i sells to (i's buyers).

We define Δ_b and Δ_s as the normalized differences in default by i's buyers and sellers:

$$\delta^S_{ib} - \delta^B_{bi} = \sqrt{\varphi_i}\Delta_b$$
$$\delta^S_{si} - \delta^B_{is} = \sqrt{\varphi_i}\Delta_s,$$

where $\{\Delta_j\}_j$ are independent from each other with mean zero and variance normalized to one. Under this specification, a mean-variance preference yields the benchmark objective function in equation 3:

$$\begin{aligned} U_i(\gamma_{i1}, \ldots, \gamma_{in}) &= \mathbb{E}[\pi_i] - \frac{\alpha_i}{2} V(\pi_i) \\ &= w_i(1-\mu) + \sum_{j=1}^{n} \gamma_{ij}(R_{ij} - \mu) - \frac{\alpha_i}{2}\left(w_i + \sum_{j=1}^{n}\gamma_{ij}\right)^2 \sigma^2 - \underbrace{\frac{\alpha_i \varphi_i}{2}}_{\equiv \frac{\phi_i}{2}} \sum_{j=1}^{n} \gamma_{ij}^2 \end{aligned}$$

We interpret φ_i as how much agent i is able to effectively enforce the contract between agent i and any other counterparty. When $\varphi_i = 0$, agent i faces no counterparty default risk, however for high values of φ_i the countryparty default risk becomes higher for agent i. For example, if we interpret agent i as a bank or a hedge fund manager, then φ_i measures the ability of her team of lawyers to make sure all the counterparties fulfill the CDS contract conditions. Hence, φ_i measures agent i's ability to mitigate counterparty default risk for agent i, measured in units of risk aversion.

B Equilibrium derivation

The net positions, $\{z_i\}_i$, are determined in equilibrium. We can use equations (6) and (10), and solve for equilibrium net positions:

$$R_{ij} - \mu = \sigma^2 \left[\frac{\phi_j \alpha_i}{\phi_i + \phi_j}(w_i + z_i) + \frac{\phi_i \alpha_j}{\phi_i + \phi_j}(w_j + z_j)\right]$$
$$\sum_{j=1}^{n} g_{ij}(R_{ij} - \mu) = \sigma^2 \sum_{j=1}^{n} g_{ij}\left[\frac{\phi_j \alpha_i}{\phi_i + \phi_j}(w_i + z_i) + \frac{\phi_i \alpha_j}{\phi_i + \phi_j}(w_j + z_j)\right]$$

Let $\tilde{g}_{ij} = g_{ij}\frac{\phi_i}{\phi_i+\phi_j}\frac{1}{\tilde{K}_i}$ and $\tilde{K}_i = \sum_{j=1}^n g_{ij}\frac{\phi_i}{\phi_i+\phi_j}$, then:

$$\sum_{j=1}^n g_{ij}(R_{ij}-\mu) = \sigma^2(z_i+w_i)\sum_{j=1}^n \tilde{K}_j\alpha_i\tilde{g}_{ji} + \sigma^2\sum_{j=1}^n \tilde{K}_i\alpha_j\tilde{g}_{ij}(z_j+w_j)$$

Using:

$$\sum_{j=1}^n g_{ij}(R_{ij}-\mu) = z_i(\phi_i + K_i\alpha_i\sigma^2) + K_i\alpha_i\sigma^2 w_i$$

from equation (6) and using the fact that $g_{ij} = g_{ji}$, we have:

$$z_i(\phi_i + K_i\alpha_i\sigma^2) + K_i\alpha_i\sigma^2 w_i = \sigma^2(z_i+w_i)\sum_{j=1}^n \tilde{K}_j\alpha_i\tilde{g}_{ji} + \sigma^2\sum_{j=1}^n \tilde{K}_i\alpha_j\tilde{g}_{ij}(z_j+w_j)$$

$$z_i(K_i + \frac{\phi_i}{\alpha_i\sigma^2}) + K_i w_i = (z_i+w_i)\sum_{j=1}^n \tilde{K}_j\tilde{g}_{ji} + \sum_{j=1}^n \tilde{K}_i\tilde{g}_{ij}(z_j+w_j)\frac{\alpha_j}{\alpha_i}$$

$$z_i(K_i - \sum_{j=1}^n \tilde{K}_j\tilde{g}_{ji} + \frac{\phi_i}{\alpha\sigma^2}) = -w_i\left(K_i - \sum_{j=1}^n \tilde{K}_j\tilde{g}_{ji}\right) + \sum_{j=1}^n \tilde{K}_i\tilde{g}_{ij}(z_j+w_j)\frac{\alpha_j}{\alpha_i}$$

Notice that

$$K_i - \sum_{j=1}^n \tilde{K}_j\tilde{g}_{ji} = \sum_{j=1}^n\left(g_{ij} - g_{ji}\frac{\phi_j}{\phi_i+\phi_j}\right) = \sum_{j=1}^n g_{ij}\frac{\phi_i}{\phi_i+\phi_j} = \sum_{j=1}^n \tilde{g}_{ij} = \tilde{K}_i$$

The system becomes:

$$z_i\left(\tilde{K}_i + \frac{\phi_i}{\alpha_i\sigma^2}\right) = -w_i\tilde{K}_i + \sum_{j=1}^n \tilde{K}_i\frac{\tilde{g}_{ij}\alpha_j}{\alpha_i}(z_j+w_j)$$

$$z_i\left(1 + \frac{\phi_i}{\tilde{K}_i\alpha_i\sigma^2}\right) = \sum_{j=1}^n \frac{\tilde{g}_{ij}\alpha_j}{\alpha_i}z_j - \underbrace{\left(w_i - \sum_{j=1}^n \frac{\tilde{g}_{ij}\alpha_j}{\alpha_i}w_j\right)}_{\equiv w_i^{ex}} \qquad \forall i = 1,\ldots,n,$$

which is equation (11). In matrix notation this system of equations becomes:

$$\underbrace{\begin{bmatrix} 1+\frac{\phi_1}{\tilde{K}_1\alpha_1\sigma^2} & 0 & \cdots & 0 \\ 0 & 1+\frac{\phi_2}{\tilde{K}_2\alpha_2\sigma^2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & & 1+\frac{\phi_n}{\tilde{K}_n\alpha_n\sigma^2} \end{bmatrix}}_{\equiv\Lambda^{-1}}\underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}}_{\equiv z} = -\underbrace{\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}}_{\equiv w} + \underbrace{\begin{bmatrix} \frac{\tilde{g}_{11}\alpha_1}{\alpha_1} & \frac{\tilde{g}_{12}\alpha_2}{\alpha_1} & \ldots & \frac{\tilde{g}_{1n}\alpha_n}{\alpha_1} \\ \frac{\tilde{g}_{21}\alpha_1}{\alpha_2} & \frac{\tilde{g}_{22}\alpha_2}{\alpha_2} & \ldots & \frac{\tilde{g}_{2n}\alpha_n}{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\tilde{g}_{n1}\alpha_1}{\alpha_n} & \frac{\tilde{g}_{n2}\alpha_2}{\alpha_n} & \ldots & \frac{\tilde{g}_{nn}\alpha_n}{\alpha_n} \end{bmatrix}}_{\equiv G^*}\begin{bmatrix} z_1+w_1 \\ z_2+w_2 \\ \vdots \\ z_n+w_n \end{bmatrix},$$

and we can solve for equilibrium net positions as follows:

$$\begin{aligned}
\Lambda^{-1}z &= -w + G^*(z+w) \\
z &= \Lambda G^* z + (\Lambda G^* w - \Lambda w) \\
(\mathrm{I} - \Lambda G^*)z &= (\Lambda G^* w - \Lambda w) \\
z &= (\mathrm{I} - \Lambda G^*)^{-1}(\Lambda G^* w - \Lambda w) \\
z &= (\mathrm{I} - \Lambda G^*)^{-1}\Lambda(G^* - \mathrm{I})w
\end{aligned}$$