Moral Hazard, Collateral and Liquidity[1]

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Abstract

We consider collateral as a means to relax rationing in a moral hazard set-up. Rationed firms must liquidate some of their assets. Assets are purchased by non-rationed firms but their borrowing capacity is also limited by moral hazard. The equilibrium price exhibits cash-in-the-market pricing and depends on the entire distribution of liquidity shocks in the economy. As moral hazard intensity varies, equilibrium price and level of collateral requirements are negatively related. Contrary to models where collateral requirements are exogenously specified, collateral has a stabilizing role on prices: For any given intensity of moral hazard problem, asset sales are smaller in quantity, and, in turn, equilibrium price is higher, in the presence of optimal collateral requirements. This price-stabilizing role implies that the ex-ante debt capacity of firms is higher with collateral, which in turn stabilizes prices further, resulting in an important feedback: Collateral reduces the proportion of ex-ante rationed firms and thus leads to greater market participation.

1 Introduction

Since the seminal contribution of Amihud and Mendelson (1986), the literature on asset pricing with trading frictions has burgeoned. Indeed, many would regard asset pricing with frictions as a new branch of financial economics. The general approach to research in this area is to posit or model the trading friction and analyze its impact on asset prices. On the one hand, the literature on market microstructure, starting with Glosten and Milgrom (1985) and Kyle (1985), has provided a foundation for trading frictions such as bid-ask spread and price impact by appealing to information asymmetry problem between traders and specialists or market-makers. On the other hand, a recent strand of literature (Gromb and Vayanos, 2002 and Brunnermeier and Pedersen, 2005) has recognized that the balance-sheet liquidity of traders is limited due to constraints such as collateral and margin requirements imposed by counterparties and financiers. This limited funding liquidity affects and is affected by the trading liquidity in markets.

A conclusion often reached, or at least implicitly conveyed, by this latter strand of literature is that such constraints are the source of drying up of liquidity in capital markets. In this paper, we depart from this strand of literature in that we recognize that constraints such as collateral and margin requirements are themselves an *endogenous* response to mitigate underlying agency problems between those who provide finance and those who receive finance, or more generally, between any two parties engaging in trade. Given this departure, we show that some important conclusions drawn by the current literature are overturned. In particular, collateral constraints and market illiquidity are both manifestations of underlying agency problems in our model, and market liquidity would be in fact far worse if collateral was not in in place to ameliorate agency problems.

Since the backdrop we have in mind is one of trading-based financial institutions which are typically highly levered, we focus on the agency problem of asset substitution or risk-shifting by borrowers (Jensen and Meckling, 1976).[1] The risk-shifting problem rations potential borrowers, that is, limits the maximum amount of financing they can raise from lenders. In this setting, we show that a collateral requirement – the pledging of cash that can be seized by financiers in case of borrower default – relaxes the extent of rationing. This simple agency-theoretic set-up forms the building block of our benchmark model.

To analyze asset-pricing implications, we cast this building block in a general equilibrium. Specifically, there is a continuum of firms which have undertaken some ex-ante financing (exogenous initially in the paper, endogenized later in the paper). The need to repay this ex-ante

[1] In this regard, we differ from Holmstrom and Tirole (1998) who consider the rent-seeking moral hazard problem to motivate limited funding liquidity. The rent-seeking problem is perhaps more relevant or appropriate for management of real assets. In case of financial assets, leverage and induced risk-shifting are more pertinent, as evidenced by the excessive borrowing and doubling-up strategies involved in a large number of events involving significant trading losses in derivatives and bond markets.

financing gives rise to liquidity shocks faced by firms in that asset liquidations may not be feasible on demand to meet these shocks. Thus, firms attempt to meet liquidity shocks by raising external financing, but its extent is limited due to the risk-shifting problem. Firms which are rationed by this agency problem attempt to relax the problem by pledging cash as collateral, which requires them to liquidate some or all of their assets. These liquidated assets are acquired by the set of remaining firms in the economy (as in the industry-equilibrium approach of Shleifer and Vishny, 1992). Though these remaining firms are able to meet their own liquidity shocks, they also potentially face the moral hazard problem, which limits their financing for asset purchase.[2]

. Thus, the liquidation price, determined by the market-clearing condition, reflects the so-called "cash-in-the-market" pricing (a term introduced by Allen and Gale, 1994): When a large number of firms are liquidating assets, market price is below the expected discounted cash flow and is affected by the distribution of liquidity in the economy.

Crucially, the entire general equilibrium is characterized by a single parameter of the economy which measures the (inverse) moral-hazard intensity, namely the extent of financing that can be raised by a firm per unit asset: (1) The moral-hazard intensity divides the set of firms into three categories – those that are rationed and fully liquidated, those that pledge collateral and are partially liquidated, and those that provide liquidity ("arbitrageurs") and purchase assets at fire-sale prices; (2) Through this division of firms, the moral-hazard intensity determines the equilibrium price at which assets are liquidated; and, finally (3) By determining the cost of liquidating an asset relative to the cost of funding it with external finance, the moral-hazard intensity determines the optimal level of collateral requirement for rationed firms.

The most important result that stems from this characterization is the following. As moral-hazard intensity *increases* (formally, the spread between the return on the good asset and the risk-shifting asset declines), firms' ability to raise financing is lowered and equilibrium levels of liquidity in the economy fall. In turn, market for assets clears at *lower prices*. This is simply the result that funding liquidity, measured by (inverse) moral-hazard intensity, affects market liquidity.[3] Simultaneously, to relax rationing, the optimal *collateral requirement increases*. To summarize, in the *cross-section of states, ranked by moral-hazard intensity*, the level of prices and the tightness of collateral requirement are *negatively* related. This, however, should not be construed as the causal effect of collateral requirement on prices. In fact, we show that if collateral requirements could not be imposed (for example, due to inability to pledge cash reserves or prevent their diversion), then equilibrium prices would be *lower* for any given level of

[2]We will allow the buying firms to pledge the assets that they buy, the financing they receive will be constrained by the risk shifting problem.

[3]Note however that unlike in the recent literature on funding and market liquidity, our measure of funding liquidity is based on the amount of financing that can be raised given an agency problem tied to external finance, and *not* by tightness of an exogenously specified constraint (such as collateral requirement) which is just a response to the agency problem.

moral-hazard intensity, compared to the case when collateral requirements are present.

Thus, when constraints such as collateral and margin requirements are designed endogenously to address underlying agency problems related to external finance, such constraints stabilize prices rather than cause drying up of liquidity. Illustrating this price-stabilizing role of collateral is a key contribution of our paper. It is also an important counterpoint to the literature on asset pricing with frictions, wherein as mentioned before, collateral constraints and margin requirements are generally perceived to exacerbate liquidity problems and price drops.

Furthermore, we show that this price-stabilizing role of collateral requirements has important welfare implications for ex-ante debt capacity of firms. In the preceding discussion, the ex-ante structure of liabilities undertaken by firms was treated as being given. We endogenize this structure by assuming that ex ante, firms are ranked by their initial wealth or capital levels and must raise incremental financing up to some fixed (identical) level in order to trade. The incremental financing is raised through short-term debt contracts that give lenders the ability to liquidate ex post in case promised payments are not met. While not critical to the overall thrust of our results, we show that this form of financing – which grants control to lenders in case of default (as in collateral and margin requirements) – is optimal from the standpoint of raising maximum ex-ante finance.

This augmentation of our benchmark model leads to an interesting, even if somewhat involved, fixed-point problem: On the one hand, the promised payment for a given amount of financing is decreasing in the level of liquidation prices in case of default;[4] on the other hand, the liquidation price is itself determined by the distribution of promised debt payments since these are the ex-post liquidity shocks faced by firms. We show that there is a unique solution to this fixed-point problem, characterized by the fraction of firms that are ex-ante rationed and by the mapping from moral-hazard intensity to price. In fact, the fixed-point is a contraction mapping and enables us to provide a recursive, constructive algorithm for the solution.

While the ex-ante rationing of firms renders analytical results on comparative statics difficult, numerical examples confirm our following conjectures. First, as the distribution of wealth or capital levels of firms declines in a first-order stochastic dominance (FOSD) sense, firms have to pledge more payment to raise given amount of financing, and, in turn, equilibrium price is lower. Second, as the distribution of quality of assets worsens in a FOSD sense, the distribution of moral-hazard intensity worsen too, firms face greater financing friction in future, the liquidation price is lower, and, in turn, equilibrium ex-ante financing requires higher debt payments. Note that the feedback between promised (ex-ante) debt payments and (ex-post) liquidation price produces an amplification effect of collateral in our model, but unlike the extant literature, this

[4]This argument is naturally reminiscent of the debt-capacity argument of Shleifer and Vishny (1992) since our model can be considered a general equilibrium variant of their analysis with endogenously modeled financing constraints.

amplification effect is positive for liquidity and efficiency: The presence of collateral requirements stabilizes liquidation prices as well as lowers promised debt payments, and these two effects feed on each other to produce an amplified reduction in ex-ante rationing. To conclude, in our augmented model too, endogenously designed collateral requirements enhance efficiency and trading by reducing the severity of agency problems tied to external finance. More generally, we conjecture that rather than being the source of illiquidity in markets, collateral is in fact the life-blood that sustains high levels of trading amongst financial institutions.

We would like to stress the important role played by financial leverage in our model. First, the ex-ante structure of leverage undertaken by firms determines the nature of ex-post liability-side shocks faced by these firms, which in our model constitute the liquidity shocks that necessitate asset sales and pledging of collateral. Second, leverage-based financing raises the possibility of risk-shifting agency problem, which limits funding liquidity, the most critical ingredient of our model. Finally, a combination of these two effects leads to the feature that illiquidity states in our model coincide with states in which quality of firms' assets has deteriorated and leverage on balance-sheets of institutions is high. These states are associated with significant agency problems between borrowers and lenders and lead to low market and funding liquidity.[5]

The paper is organized as follows. Section 2 discusses the related literature. Section 3 sets up the benchmark model and analyzes it to illustrate the price-stabilizing role of collateral. Section 4 augments the benchmark model to introduce the feedback between collateral and ex-ante debt capacity of firms. Section 5 discusses robustness issues. Section 6 concludes. All proofs not contained in the text are provided in Appendix 1. Appendix 2 presents the constructive algorithm to solve the fixed-point problem introduced in Section 4.

2 Related literature

The idea that asset prices may contain liquidity discounts when potential buyers are financially constrained dates back to Shleifer and Vishny (1992). That this effect would be more pronounced when assets are not easily re-deployable outside of the selling firm's industry traces its roots to the idea of asset-specificity of Williamson (1988). Since then, fire sales have been employed in finance models regularly, perhaps most notably by Allen and Gale (1994) to examine the link between limited market participation and asset price volatility, and Allen and Gale (1998) in providing a theoretical foundation for financial fragility observed in asset markets.

[5]Acharya and Pedersen (2005) document in Figure 1 of their paper and the related discussion that all significant (more than three standard deviation) illiquidity episodes in the US stock market during the period 1964-1999 have been preceded by significant asset-side shocks: 5/1970 (Penn Central commercial paper crisis), 11/1973 (oil crisis), 10/1987 (stock market crash), 8/1990 (Iraqi invasion of Kuwait), 4-12/1997 (Asian crisis) and 610/1998 (Russian default, LTCM crisis).

Empirically, the idea of fire sales has now found ample empirical evidence in a variety of different settings: in distressed sales of aircrafts in Pulvino (1998), in cash auctions in bankruptcies in Stromberg (2000), in creditor recoveries during industry-wide distress especially for industries with high asset-specificity in Acharya, Bharath and Srinivasan (2006), in equity markets when mutual funds engage in sales of similar stocks in Coval and Stafford (2006), and, finally, in an international setting where foreign direct investment increases during emerging market crises to acquire assets at steep discounts in the evidence by Krugman (1998), Aguiar and Gopinath (2005), and Acharya, Shin and Yorulmazer (2007).

At its roots, our model is closely linked to this literature on fire sales and industry equilibrium view of asset sales. The industry view makes clear that market prices depend on funding liquidity of potential buyers. More broadly, the overall approach and ambition of our paper in relating the distribution of liquidity shocks in an economy to equilibrium outcomes is closest to the seminal paper of Holmstrom and Tirole (1998). However, there are important differences between our papers, the two papers can be viewed as complementary.

First, Holmstrom and Tirole's model of limited funding liquidity is based on rent-seeking moral hazard. In contrast, we consider the risk-substitution moral hazard. It is our belief that rent-seeking is a more appropriate metaphor for agency problems affecting real or technological choices, whereas risk-substitution fits financial investment choices better. Second, Holmstrom and Tirole's model of liquidity shocks is motivated by investment shortfalls in operating assets, whereas our liquidity shocks are modeled as (endogenously derived) financial liabilities. These two important differences make Holmstrom and Tirole's model more suitable for understanding liquidity transfers and asset sales in the context of real assets, whereas our model is perhaps more apt for similar issues amongst financial intermediaries and in financial markets.

Somewhat more importantly, the primary goal of Holmstrom and Tirole's work is to provide a rationale for public provision of liquidity, and as a step towards that, characterize when private provision of liquidity may not be efficient. In contrast, our primary goal is to provide an agency-theoretic foundation for the recently much-examined requirement on the provision of collateral by financial intermediaries. After providing this foundation, we consider the implications of endogenously derived collateral requirements on market prices and liquidity. Hence, we view our most important conceptual contribution as being for the literature on asset prices with frictions, and our most significant modeling contribution as being its provision of a tractable model of endogenous collateral constraints. In this sense, our objectives are similar to those of Bernanke and Gertler (1989) who considered the role of real collateral, its role in ameliorating agency problems linked to real investments, and its implications for business cycle. In contrast to Bernanke and Gertler (1989), however, our focus is on financial collateral and its role in addressing agency problems in investment choices of financial intermediaries.

In terms of modeling details, we derive credit rationing based on a moral hazard problem and

collateral serves to relax such rationing for some firms in the economy. This finds exact parallel in an asymmetric information context in the rationing models of Stiglitz and Weiss (1981) and the corresponding signaling model with collateral of Bester (1985). Our choice of a moral hazard setting is driven purely by a desire for analytical simplicity, in perhaps much in the same vein as the choice made in Holmstrom and Tirole's work. Our assumptions that lead to optimality of a "hard" debt contract, with control rights given to lenders in case of default, mirror closely the work of Aghion and Bolton (1992), Hart and Moore (1994) and Hart (1995).

Our work is also related to the seminal work of Kiyotaki and Moore (1997) on credit cycles. In Kiyotaki and Moore (1997) and Krishnamurthy (2003), the underlying asset cannot be pledged because of inalienable human capital.[6] However, land can be pledged and has value both as a productive asset and as collateral. In our model, the underlying asset can be pledged but the amount pledged is endogenously determined by the moral hazard constraint. Further, the asset has a market price that determines its collateral value, this collateral value is determined by the equilibrium distribution of liquidity shocks. Hence we present an approach where the amount that can be pledged is endogenous due to moral hazard constraints.

In addition, our work is related to research on collateral due to Caballero and Krishnamurthy (2001). Caballero and Krishnamurthy essentially use a Holmstrom-Tirole approach to liquidity shocks (these are exogenous) and allow firms to post collateral in a manner similar to Kiyotaki and Moore (some assets can be collateralized but the underlying cash flows cannot be pledged). Again, our main innovation is to find a simple friction, namely risk-shifting, which we believe is important in financial markets, and to use this friction to motivate underlying features of our model. The main upshot is that this leads to a model with a single constraint that ties together all the relevant aspects of collateral and liquidity.

Our research is also related to the recent work on exogenous collateral requirements due to Gromb and Vayanos (2002), Brunnermeier and Pedersen (2005), Plantin and Shin (2006), and Anshuman and Viswanathan (2006). In Gromb and Vayanos (2002), agents can only borrow if each asset is separately and fully collateralized, i.e., borrowing is essentially riskless. In Brunnermeier and Pedersen (2005), the collateral requirement is similarly exogenous: a shock to prices leads to liquidity shocks, that, in turn, leads to liquidation by financial intermediaries who engage in risk management. It is unclear in these models why lenders engage in risk management and why collateral requirements are imposed when a liquidity shock is realized but no new fundamental information has arrived. Plantin and Shin (2006) consider a dynamic variant of this feedback effect focusing on application to the unwinding of carry trades and their precipitous effect on exchange rates.[7]

[6] Krishnamurthy (2003) differs from Kiyotaki and Moore (1997) in that all contingent claims on aggregate variables are allowed subject to collateral constraints.

[7] Morris and Shin (2004) present a model where traders are liquidated when an exogenous trigger price is reached and this trigger is different for each trader.

These papers explore how collateral requirements affect portfolio decisions and prices but do not model why these requirements are imposed in the first place. Anshuman and Viswanathan (2006) point out that the ability to renegotiate constraints can eliminate liquidity crises of the nature analyzed in these papers, unless some other frictions are present. Our paper presents one such friction in terms of the ability of financial intermediaries to substitute risks, which limits their borrowing capacity. Crucially, we show that while the exogenous view of collateral requirements often suggests them to be the culprit for fall in market liquidity, endogenous collateral requirements in fact have a stabilizing effect on prices (relative to when such requirements are absent).

Finally, this argument that financing constraints are endogenous reflection of agency problems rather than the source of capital flight and liquidity problems, echoes well with a similar hypothesis put forth by Diamond and Rajan (2001) in the context of financial crises. They argue that while the literature has found a positive relationship between the extent of short-term borrowing and incidence of crises, they may both be manifestations of the fact that the underlying economy has illiquid investments to start with. As they put it, "Banning short-term debt deals with symptoms rather than the underlying causes." In much the same vein, eliminating collateral constraints in our model actually results in lower market and funding liquidity.

3 Model

3.1 Benchmark model

Consider a continuum of firms (for example, financial intermediaries) that have all undertaken some borrowing at date 0. At date 1, these firms face financial liabilities such that firm i is required to pay back ρ_i to its existing creditors. The contract for borrowing is hard and if the payment is not made, then creditors take over the assets and liquidate them at market prices. The option to liquidate assets exists with firms also, however there is a timing mismatch: Assets can be liquidated only with a delay (at date $1\frac{1}{2}$). In this sense, firms' current liabilities generate future liquidity shocks. Depending on the specific nature of liabilities and assets, the delay could be intra-day (for example, in case of inter-bank borrowing and liquid assets) or several days (for example, in case of public debt and relatively illiquid assets). Firms have no internal liquidity and must raise new external finance at date 1 to pay off existing debt. We focus attention first on the date 1 aspects of the model, deriving the exact size and distribution of date 0 borrowing of firms in Section 4 (wherein we also provide a justification for the hard nature of borrowing we assume throughout). The time-line for the model, starting at date 1, is specified in Figure 1. All firm owners and creditors are risk-neutral and the risk-free rate of interest is zero.

After raising external finance at date 1, there is the possibility of moral hazard at the level of each firm. In particular, we consider asset-substitution moral hazard along the lines of Jensen and

Meckling (1976). Firm's existing investment is in an asset which is a positive net present value investment. However, after raising external financing at date 1 and also after asset sales at date $1\frac{1}{2}$, each firm can switch its investment to another asset. We denote the assets as j, $j \in \{1, 2\}$, yielding a cash flow at date 2 of $y_j > 0$ with probability $\theta_j \in (0, 1)$, and no cash flow otherwise. We assume that $\theta_1 < \theta_2$, $y_1 > y_2$, $\theta_1 y_1 \leq \theta_2 y_2$, and $\theta_1 y_1 \leq \rho_i$. In words, the first asset is riskier and has a higher payoff than the second asset, but the second asset has a greater expected value. Also, taking account of the financial liability at date 1, investing in the first asset is a negative net present value investment for all firms.[8]

We assume that the external finance at date 1 is raised in the form of debt with face value of f to be repaid at date 2. In our benchmark model, we suppress the market for asset sales at date $1\frac{1}{2}$. Then, the incentive compatibility condition to ensure that firm owners invest in asset $j = 2$ (that is, do not risk-shift to asset $j = 1$) requires that

$$\theta_2(y_2 - f) > \theta_1(y_1 - f). \tag{1}$$

This condition simplifies to an upper bound on the face value of new debt:

$$f < f^* \equiv \frac{(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)} \ . \tag{2}$$

The funding constraint for firm i requires that

$$\rho_i = \theta(f_i) f_i, \tag{3}$$

where f_i is the face value charged to firm i and $\theta(f_i)$ is the probability of high cash flow from the asset invested in when face value of debt issued is f_i. Note that this condition takes its specific form above because the lender cannot be paid any sum with probability $(1 - \theta_i)$.

Since (IC1) bounds the face value of debt that can provide incentives to invest in the better asset, we obtain credit rationing as formalized in the following Proposition. We stress that this result is by itself not new (see, for example, Stiglitz and Weiss, 1981). We elevate it to the level of a proposition as it forms the basis of our analysis that follows.

Proposition 1 *Firms with liquidity need ρ at date 0 that is greater than $\rho^* \equiv \theta_2 f^*$ cannot borrow, that is, they are credit-rationed in equilibrium.*

[8] It should be pointed out that we assume the shift between assets is at zero cost. The simplest interpretation could be a deterioration in the risk-management function of the financial intermediary. We discuss some other possibilities in Section 5.

To see this result, note first that $f^* < y_2$ so that borrowing up to face value f^* is indeed feasible in equilibrium provided it enables the borrowing firm to meet its funding needs. In other words, firms with $\rho \leq \rho^* \equiv \theta_2 f^*$ borrow, invest in the better asset, and simultaneously meet their funding constraint. Second, note that for $\rho > \rho^*$, investment is in the first, riskier asset. However, in this case funding constraint requires that the face value be $\hat{f} = \frac{\rho}{\theta_1}$ which is greater than y_1 for all $\rho > \rho^*$. That is, firms with liability ρ exceeding ρ^* cannot borrow and are rationed.

We assume in what follows that the continuum of firms is ranked by liquidity shocks ρ such that $\rho \sim g(\rho)$ over $[\rho_{\min}, \rho_{\max}]$, where $\rho^* \in [\rho_{\min}, \rho_{\max}]$. We also assume that $\theta_2 y_2 \leq \rho_{\max}$. Thus, Proposition 1 implies that firms in the range $(\rho^*, \rho_{\max}]$ are credit-rationed in our benchmark model.

3.2 Collateral

We extend this benchmark model to consider a role for collateral in the borrowing at date 1. Since firms have no internal liquidity, collateral can be provided only by offering to sell (at least) some assets at date $1\frac{1}{2}$. Hence, we now consider the market for asset sales which we had suppressed above. The analysis to follow is a natural counterpart to that of Bester (1985) who demonstrated the role of collateral in relaxing rationing (by screening) in the asymmetric information model of credit rationing by Stiglitz and Weiss (1981).

In particular, the borrowing contract now takes the form (f_i, k_i) for firm i where f_i is the face value of debt to be paid at date 2 in return for the funding provided ρ_i, and k_i is the amount of collateral that the firm must put up after meeting its liquidity shock ρ_i. The sequence of events is as follows. First, the firm attempts to raise financing against the contract (f_i, k_i) and if this can enable the firm to meet the liquidity shock ρ_i, it does so. After the liquidity shock has been met , collateral is provided at date $1\frac{1}{2}$ in the amount as per the borrowing terms agreed at date 1. In particular, the collateral is generated by liquidating a portion of the firm's investment. For now, we assume this liquidation occurs at an exogenously given price p per unit of the asset. Recall that the risk-shifting problem with respect to firm's assets (other than cash) arises after borrowing and after liquidations have taken place, so that p is the per unit liquidation price of asset $j = 2$ at date $1\frac{1}{2}$. We endogenize this liquidation price in the next subsection. The collateral is assumed to be invested in a storage technology between dates $1\frac{1}{2}$ and 2, whose rate of return is assumed to be zero (no liquidity or quality premium). Finally, assets pay off at date 2.

In essence, the above sequence of events captures the "liquidity" aspect of the liability side of firm's balance-sheet. The firm is unable to liquidate its assets to meet the liquidity shock at the very instant (intra-day, for example) the shock arises. The firm can however borrow and agree to provide collateral as part of the borrowing contract (by end of day, for example). Lenders rationally anticipate the price at which the firm can liquidate its assets. In other words, they set

the collateral requirement taking account of the liquidation value of firm's assets. Once collateral is provided, the firm continues its operations (beyond the day, for example) and now the likelihood of asset substitution arises.

Reverting to the model, to generate k_i units of collateral, the firm must sell a proportion α_i of its investment, given by $\alpha_i = k_i/p$. We focus on firms rationed in the benchmark model, that is, we consider here $\rho_i > \rho^*$. Dropping the subscripts i, in the presence of collateral, the firm's total cash position is $[k + (1-\alpha)y_j]$ at date 2 if asset j is chosen and this happens with probability θ_j, and simply k otherwise. Note that the payoff in the good state is declining in collateral k as long as the liquidation price p is lower than the cash flow y_j. We assume this now to be the case, so that putting up collateral is costly for the borrowing firm, and verify it later when we model the market for asset liquidations.

With collateral, the incentive compatibility constraint takes the form

$$\theta_2 \left[k + (1-\alpha)y_2 - f\right] > \theta_1 \left[k + (1-\alpha)y_1 - f\right]. \tag{4}$$

This simplifies to the condition

$$f < f^{**}(k) \equiv \left[k + \rho^* \left(1 - \frac{k}{p}\right)\right]. \tag{5}$$

An intuitively more appealing form of $f^{**}(k)$ is

$$f^{**}(k) = \left[\alpha p + (1-\alpha)\rho^*\right], \tag{6}$$

which can be interpreted as asset sales yielding p per unit, and borrowing yielding ρ^* per unit. As is clear from this expression, collateral will relax rationing only in the case where the unit price of asset sale p exceeds the borrowing capacity per unit of asset ρ^*.

Finally, the funding constraint for the firm is given by

$$\rho = \theta_2 f + (1-\theta_2)k, \tag{7}$$

and the conditions for the collateral requirement to be feasible are that (i) $k \geq 0$, and (ii) $\alpha = \frac{k}{p} \leq 1$.

Combining the incentive compatibility condition, funding constraint, and the two feasibility conditions yields the following proposition on optimal collateral requirement and the extent of its effect in relaxing credit rationing.

Proposition 2 *If the liquidation price p is lower than ρ^*, then no collateral requirement can relax credit rationing for firms with $\rho \in (\rho^*, \rho_{\max}]$.*

If the liquidation price p is greater than ρ^*, then collateral requirement relaxes credit rationing for firms with $\rho \in (\rho^*, p]$, and the collateral requirement takes the form

$$k(\rho) = \frac{(\rho - \rho^*)}{\left(1 - \frac{\rho^*}{p}\right)} \ . \tag{8}$$

Furthermore, the collateral requirement $k(\rho)$ is increasing in liquidity shock ρ and decreasing in liquidation price p, and the proportion of firms for which credit rationing is relaxed, $[p - \rho^*]$, is increasing in liquidation price p.

Essentially, the incentive compatibility and funding constraints yield the collateral requirement of the form stated in the proposition. Imposing the two feasibility constraints then yields that the liquidity shock ρ should be lower than liquidation price p for collateral to relax credit rationing.

The liquidation price p plays a crucial role in determining the size of collateral requirement. In particular, if liquidation price is low, then firms have to liquidate a large part of their existing investment. This lowers the cash flows of the firm and exacerbates the risk-substitution problem. To limit this, a lower face value of debt is required, and, then in turn, the funding constraint implies that collateral requirement must be raised. Finally, if liquidation price is higher then more firms that were otherwise rationed can be funded in equilibrium with collateral requirement.

Next, we introduce a market for liquidation of the asset at date $1\frac{1}{2}$ and study how it influences and is influenced by the equilibrium level of collateral requirement. Also, we assumed in the analysis above that $p \leq \rho_{\max}$. We verify below that this will indeed be the case under our maintained assumption $\theta_2 y_2 \leq \rho_{\max}$.

3.3 Market for asset sales

We assume that assets liquidated by firms that face rationing ($\rho > \rho^*$) are acquired by those that are not rationed ($\rho < \rho^*$). We consider standard market clearing for asset liquidation. An important consideration is that asset purchasers, by virtue of their lower liquidity shocks, may be able to raise liquidity not only against their existing assets but also against to-be-purchased assets.

Formally, suppose that a non-rationed firm with liquidity shock ρ acquires α additional units of assets. Then, the incentive-compatibility condition for the non-rationed firm (with rational expectation of its acquisition of assets) takes the form

$$\theta_2[(1+\alpha)y_2 - f] > \theta_1[(1+\alpha)y_1 - f], \tag{9}$$

which requires that the interest rate f satisfy the condition:

$$f < f^*(\alpha) = \frac{(1+\alpha)(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)} = \frac{(1+\alpha)\rho^*}{\theta_2}. \tag{10}$$

The total amount of liquidity available for asset purchase with such a non-rationed firm is thus given by[9]

$$l(\alpha, \rho) = [\theta_2 f^*(\alpha) - \rho] = [(1+\alpha)\rho^* - \rho]. \tag{11}$$

That is, the funding ability of a non-rationed firm consists of its spare debt capacity from existing assets, $(\rho^* - \rho)$, plus the liquidity that can be raised against assets to be acquired, $\alpha\rho^*$. The latter term arises because each unit of asset can command ρ^* of borrowing without incidence of moral hazard problem.

The pertinent question is: How many units of assets would this firm be prepared to buy as a function of the price p? Note that no firm would acquire assets at a price higher than their expected payoff under the better asset. Denoting this price as $\overline{p} = \theta_2 y_2$, we obtain the following demand function $\hat{\alpha}(p, \rho)$ for the firm. For $p > \overline{p}$, $\hat{\alpha} = 0$, and for $p \leq \overline{p}$, $\hat{\alpha}$ is set to its highest feasible value given the liquidity constraint:

$$p\ \hat{\alpha} = l(\hat{\alpha}, \rho), \tag{12}$$

which simplifies to

$$\hat{\alpha}(p, \rho) = \frac{(\rho^* - \rho)}{(p - \rho^*)}. \tag{13}$$

Thus, the total demand for assets for $p < \overline{p}$ is given by

$$D(p, \rho^*) = \int_{\rho_{\min}}^{\rho^*} \hat{\alpha}(p, \rho) g(\rho) d\rho = \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)}\ g(\rho) d\rho, \tag{14}$$

where we have stressed the dependence on (inverse) moral hazard intensity ρ^*.

Given this demand function for non-rationed firms, we can specify the market-clearing condition. Note that the total supply of assets up for liquidation is given by

$$S(p, \rho^*) = \int_{\rho^*}^{p} \frac{(\rho - \rho^*)}{(p - \rho^*)}\ g(\rho) d\rho + \int_{p}^{\rho_{\max}} g(\rho) d\rho. \tag{15}$$

[9]Note that if non-rationed firms want any additional liquidity beyond $f^*(\alpha)$, these firms would themselves have to pledge collateral and liquidate assets.

The two terms correspond respectively to (i) partial asset liquidations by firms with $\rho \in (\rho^*, p]$ to meet the collateral requirement, and (ii) complete liquidation of firms with $\rho \in (p, \rho_{\max}]$.

Then, the equilibrium price p^* equals the minimum of $\overline{p}$ and the solution to the market-clearing condition

$$D(p, \rho^*) = S(p, \rho^*). \tag{16}$$

In particular, if the solution to the market-clearing condition exceeds $\overline{p}$, then $p^* = \overline{p}$ and we have that $D(\overline{p}, \rho^*) > S(\overline{p}, \rho^*)$.

Before characterizing the behavior of the equilibrium price, it is useful to consider properties of the demand and supply functions. First, both demand and supply functions decline in price p. This is because as price increases, asset purchasers can only buy fewer assets given their limited liquidity. Simultaneously, as price increases, the collateral requirement also requires rationed firms to liquidate a smaller quantity of their assets. Hence, what is important is the behavior of excess demand function, $E(p, \rho^*) \equiv [D(p, \rho^*) - S(p, \rho^*)]$, as a function of price p.

The excess demand function can be rewritten as:

$$\begin{aligned} E(p, \rho^*) &= D(p, \rho^*) - S(p, \rho^*) & (17) \\ &= \int_{\rho_{\min}}^{p} \frac{(\rho^* - \rho)}{(p - \rho^*)} \, g(\rho) d\rho - \int_{p}^{\rho_{\max}} g(\rho) d\rho. & (18) \end{aligned}$$

Integrating this equation by parts yields that

$$E(p, \rho^*) = -1 + \frac{1}{(p - \rho^*)} \int_{\rho_{\min}}^{p} G(\rho) d\rho \tag{19}$$

where $G(\rho) = \int_{\rho_{\min}}^{p} g(\rho) d\rho$ and $G(\rho_{\min}) = 0$.

The condition that excess demand be zero (for $p < \overline{p}$), i.e., $E(p, \rho^*) = 0$, leads to the relationship

$$p = \rho^* + \int_{\rho_{\min}}^{p} G(\rho) d\rho. \tag{20}$$

If the solution to this equation exceeds $\overline{p}$, then we have $p^* = \overline{p}$.

From this representation of market-clearing condition, we observe that the price can never fall below the threshold level of ρ^*. This is because non-rationed firms can always raise ρ^* of liquidity against each additional unit of asset they purchase. Hence, at $p = \rho^*$, their demand for asset purchase is infinitely high.

Second, the price can never be higher than $\overline{p}$. Together, these facts guarantee an interior market-clearing price $p^* \in [\rho^*, \overline{p}]$.

Third, as intuition would suggest, the excess demand function is decreasing in price p.

And, finally, the key determinant of the market-clearing price is the extent of (inverse) moral hazard intensity ρ^*. This is the central parameter that drives all action in the model: It determines the partition of firms into rationed firms and non-rationed firms, the extent of buying power of non-rationed firms, and, also, the level of collateral requirement and thereby the size of asset liquidations.

The resulting equilibrium price satisfies the following proposition:

Proposition 3 *The market-clearing price for asset sales, p^*, depends on the (inverse) moral hazard intensity ρ^* in the following manner:*

(i) There exists a critical threshold $\hat{\rho}^* < \overline{p}$ such that $p^* = \overline{p}, \forall \rho^* \geq \hat{\rho}^*$; and,

(ii) For $\rho^* < \hat{\rho}^*$, $p^* \in [\rho^*, \overline{p})$, p^* is increasing in ρ^*, and $p^* = \rho^*$ only when $\rho^* = \rho_{\min}$. Therefore, in this region, there is an illiquidity discount, $[\overline{p} - p^*]$, whose size is declining in ρ^*.

When ρ^* is above a critical value $\hat{\rho}^* > \rho_{\min}$, assets are liquidated at their highest valuation: Few firms are rationed, buyers (non-rationed firms) have lot of liquidity and sellers (rationed firms) face the weakest possible collateral requirement. As moral hazard becomes worse, that is, ρ^* declines, there is not enough liquidity in the system to absorb the pool of assets being put up for liquidation at the highest price. Hence, the market-clearing price is lower than $\overline{p}$. Since assets are "cheap", non-rationed firms demand as much as possible of the liquidated assets with their entire available liquidity. On the supply side, as price falls, more firms are rationed, and rationed firms face tighter collateral requirements. As moral hazard keeps worsening (ρ^* becomes smaller), prices fall until they hit ρ^* eventually, and this happens when in fact ρ^* equals $\rho_{\min}$.

Note that the liquidation price exhibits "cash-in-the-market pricing" as in Allen and Gale (1994, 1998) since it depends on the overall amount of liquidity available in the system for asset purchase, which, in turn, is determined by the extent of moral hazard problem. The important message from this analysis is that whether a rationed firm can relax its own borrowing constraint or not by pledging collateral depends upon the liquidity of the potential purchasers of its assets (through the liquidation price) and on the liquidation of assets by other such rationed firms. The moral hazard parameter ρ^* partitions firms endogenously into liquidity providers and takers, based on the magnitude of their liquidity shocks, and one can think of the excess demand for the asset, $E(p, \rho^*) \equiv [D(p, \rho^*) - S(p, \rho^*)]$, given by equation (18), as an inverse measure of the *excess financial leverage in the system*.[10]

[10] These features of our model are essentially variants of the industry-equilibrium effects in Shleifer and Vishny

Another important observation is that part (ii) of Proposition 3 implies a natural link between funding liquidity of firms and liquidity of asset markets. Funding liquidity in our model is measured by ρ^*, the amount of financing that can be raised per unit of asset. Market illiquidity in our model can be measured as the fire-sale discount in prices, $[\overline{p} - p^*]$. The Proposition formally shows that funding liquidity and market illiquidity are negatively related. While the link here is only from funding liquidity to market liquidity, our augmented model of Section 4 will also formalize the reverse link from market liquidity to (ex-ante) funding liquidity. Unlike in the recent literature on funding and market liquidity (for example, Gromb and Vayanos, 2002, and Brunnermeir and Pedersen, 2005), our measure of funding liquidity is based on the amount of financing that can be raised given the risk-shifting problem tied to leverage (formally, by ρ^*), and *not* by tightness of an exogenously specified constraint (such as collateral requirement) which is just a response to this underlying problem. This distinction is quite important in the analysis that follows.

Reverting to our current model, we combine Proposition 3 with Proposition 2 to obtain the following natural result that collateral required of a rationed firm is higher when the perceived moral hazard is greater.

Proposition 4 *The collateral requirement $k(\rho)$ for a firm with liquidity shock ρ is decreasing in the (inverse) moral hazard intensity ρ^*.*

The importance of this result stems from the observation that it is not the higher levels of collateral requirement per se which cause the equilibrium price for asset sales to fall. Both are in fact manifestations of the same underlying problem, which is the limited funding liquidity of firms arising from a moral hazard problem. Indeed, if collateral was not posted by rationed firms, then their funding liquidity would be even lower and they would simply be forced to sell their entire pool of assets, and, in turn, the resulting market-clearing price would be even lower. We elaborate this point in detail in Section 3.4.

The following example which assumes a uniform distribution on liquidity shocks helps us illustrate these equilibrium relationships graphically.

Example: Suppose that $\rho \sim Unif[\rho_{\min}, \rho_{\max}]$ and $\overline{p} = \theta_2 y_2 = \rho_{\max}$. Then, solving the market-clearing condition $E(p, \rho^*) = 0$, yields the following equilibrium relationships:

1. If $\rho^* \geq \hat{\rho}^* \equiv \frac{1}{2}(\rho_{\min} + \rho_{\max})$, then the price for asset sales is $p^* = \rho_{\max}$;

(1992)'s model wherein borrowing involves collateral, and collateral induces asset liquidations. Crucially, however, the determinant of rationing and of the limited ability of buyers to purchase are both tied to the same underlying state variable, the extent of moral hazard problem.

2. Otherwise, that is, if $\rho^* < \frac{1}{2}(\rho_{\min} + \rho_{\max})$, then there is cash-in-the-market pricing and the price for asset sales is

$$p^* = \rho_{\max} - \sqrt{(\rho_{\max} - \rho_{\min})}\sqrt{(\rho_{\max} + \rho_{\min} - 2\rho^*)}.$$

3. In the cash-in-the-market pricing region, the equilibrium price p^* is increasing and convex in (inverse) moral hazard intensity ρ^*. In particular,

$$\frac{dp^*}{d\rho^*} = \frac{\sqrt{(\rho_{\max} - \rho_{\min})}}{\sqrt{(\rho_{\max} + \rho_{\min} - 2\rho^*)}} \; > \; 0,$$

and

$$\frac{d^2p^*}{d\rho^{*2}} = \sqrt{(\rho_{\max} - \rho_{\min})}(\rho_{\max} + \rho_{\min} - 2\rho^*)^{-\frac{3}{2}} \; > \; 0.$$

4. The collateral requirement $k(\rho)$ is given accordingly by Proposition 2 and the expressions for liquidation price p^* in the two regions (Points 1 and 2 above).

The price p^* and the collateral requirement $k(\rho)$ are illustrated in Figures 2 and 3. Figure 2 shows the cash-in-the-market pricing in asset market when funding liquidity is below $\hat{\rho}^{ast}$. Figure 3 in particular is striking.[11] As the moral hazard problem worsens (ρ^* falls), a smaller range of firms is able to relax rationing and at the same time these firms face increasingly steeper collateral requirement. Finally, Figure 4 plots market illiquidity, measured as the fire-sale discount in asset price, $[\overline{p} - p^*]$, as a function of the funding liquidity per unit of asset, ρ^*, for the example with uniform distribution of liquidity shocks. It illustrates that when funding liquidity is high, market liquidity is at its maximal level. As funding liquidity deteriorates and falls below $\hat{\rho}^*$, market becomes illiquid and increasingly so as funding liquidity deteriorates.

Interpretation of moral hazard intensity: What does it mean to vary the moral hazard parameter ρ^*? Recall that $\rho^* = \frac{\theta_2(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)}$, so that ρ^* is increasing in θ_2, the quality of the better asset. Thus, a decrease in ρ^* can be given the economically interesting interpretation of a deterioration in the quality of assets, for example, over the business cycle. Note that we are holding constant the quality of bad asset θ_1. So strictly speaking, if the better asset deteriorates in quality in a relative sense compared to the other asset during a business-cycle downturn, then the moral hazard problem gets aggravated. Thus, our model entertains a natural interpretation that during economic downturns and following negative shocks to the quality of assets, there is greater credit rationing and tighter collateral requirement in the economy. Accompanying these

[11] The parameters in Figure 3 are: $\theta_2 = 0.8$, $y_2 = 12.5$, giving $\rho_{\max} = 10$, and $\theta_1 = 0.2$, $y_1 = 20$, giving $\rho_{\min} = \theta_1 y_1 = 4$.

are lower prices for asset liquidations due to the deterioration in asset quality and the coincident deterioration in funding liquidity.

Interpretation of liquidity shocks: In our analysis, we assumed the liquidity shocks and their distribution were unrelated to the quality of assets. If a deterioration in the quality of assets is in fact associated with a worsening in the distribution of liquidity shocks, then the effects in our model are exacerbated.[12] Formally, this would mean a relationship between θ_2 and the distribution of liquidity shocks $g(\rho)$. We explore and build this link in Section 4 where we introduce and analyze the ex-ante (that is, date 0) structure of the model. Before we do so, however, we prove an important result which casts doubt over the recent claim in asset-pricing and liquidity literature that lack of liquidity and fall in prices in asset markets are attributable to constraints that financial intermediaries face, including collateral requirements. We show below that once collateral is recognized as an endogenous response to relax borrowing constraints, exactly the reverse is in fact true: All else equal, there would be more asset liquidations and lower prices if collateral requirements were not in place.

3.4 Price-stabilizing role of collateral

Suppose that there were no possibility of pledging collateral at all. In this case, firms with liquidity shocks below ρ^* would still be the candidate buyers of assets, but *all* firms with liquidity shocks above ρ^* would be forced to sell *all* of their assets. This is in contrast to the case with collateral where firms with liquidity shocks over the range $[\rho^*, p]$ engage only in partial asset sales, as they are able to meet their liquidity shock by borrowing against collateral. Thus, without collateral, the demand and supply functions for assets are given respectively as:

$$D(p, \rho^*) = \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p - \rho^*)}\, g(\rho) d\rho, \text{ and} \tag{21}$$

$$S_{nc}(p, \rho^*) = \int_{\rho^*}^{\rho_{\max}} g(\rho) d\rho, \tag{22}$$

where we have subscripted the supply to signify that it is different from its form under the case with collateral.

The equilibrium price p^*_{nc} then is the price p that clears the market, that is, sets the excess demand

$$E_{nc}(p, \rho^*) = [D(p, \rho^*) - S_{nc}(p, \rho^*)]$$

[12] On this point, the preceding of significant market illiquidity shocks by asset-side shocks, see the empirical evidence presented in footnote 5 from Acharya and Pedersen (2005).

to less than or equal to zero.

In this case, the excess demand function is given by

$$E_{nc}(p,\rho^*) \;=\; \int_{\rho_{\min}}^{\rho^*} \frac{(\rho^*-\rho)}{(p-\rho^*)}\, g(\rho)d\rho - \int_{\rho^*}^{\rho_{\max}} g(\rho)d\rho. \tag{23}$$

Integrating this equation by parts yields that

$$E_{nc}(p,\rho^*) \;=\; -[1-G(\rho^*)] + \frac{1}{(p-\rho^*)}\int_{\rho_{\min}}^{\rho^*} G(\rho)d\rho\ . \tag{24}$$

Thus, the excess demand being zero (for $p<\overline{p}$) leads to the variant of the condition (20) for the case with collateral:

$$p = \rho^* + \frac{1}{[1-G(\rho^*)]}\int_{\rho_{\min}}^{\rho^*} G(\rho)d\rho. \tag{25}$$

If the solution to this equation exceeds $\overline{p}$, then we have $p^*=\overline{p}$.

Then, we obtain the following characterization of the price in absence of collateral and its comparison to the price with collateral.

Proposition 5 *The market-clearing price for asset sales in absence of collateral, p^*_{nc}, depends on the (inverse) moral hazard intensity ρ^* in the following manner:*

(i) There exists a critical threshold $\hat{\rho}^*_{nc}<\overline{p}$ such that $p^*_{nc}=\overline{p}, \forall \rho^*\geq\hat{\rho}^*_{nc}$; and,

(ii) For $\rho^*<\hat{\rho}^*_{nc}$, $p^*_{nc}=\rho^*+\frac{1}{[1-G(\rho^*)]}\int_{\rho_{\min}}^{\rho^*}G(\rho)d\rho$, so that $p^*_{nc}\in[\rho^*,\overline{p})$, p^*_{nc} is increasing in ρ^*, and $p^*_{nc}=\rho^*$ only when $\rho^*=\rho_{\min}$.

In comparison to the price p^* for asset sales with collateral characterized in Proposition 3:

(iii) $p^*\geq p^*_{nc}$ for all values of ρ^*; and,

(iv) $\hat{\rho}^*<\hat{\rho}^*_{nc}$, so that the inequality in (iii) above is strict over at least some values of ρ^* below $\hat{\rho}^*_{nc}$.

The important message from this proposition is the following. Proposition 3 and Proposition 4 together imply that across states of the world, that is, as the moral hazard intensity ρ^* varies, equilibrium price and the level of collateral requirement are negatively correlated: Higher the collateral requirement, lower the prices. This result, however, should not be interpreted as being causal. Proposition 5 makes it clear that were collateral not pledgeable by some of the rationed firms, they would in fact be liquidated completely (as against partially) and equilibrium price would be even lower. The source of the negative correlation implied by Proposition 3 and Proposition

4 is thus not due to collateral requirements per se. Collateral, instead, is itself an endogenous response to the root cause of low prices and illiquidity in the market, namely the moral hazard problem (risk substitution by financial institutions, in our set-up) that limits the funding capacity of firms (borrowing capacity of financial institutions).

The implication of this endogeneity problem is particularly strong when one thinks about welfare costs that might arise from asset sales that occur at illiquidity discounts and what policy interventions, if any, might be desirable to minimize these costs. In our current set-up, asset sales merely result in transfers across firms and there are no other pecuniary externalities from fire-sale prices. We discuss in Section 5 the implications of introducing such allocation inefficiency from asset sales in our model.

Below, we revisit our earlier example which assumed a uniform distribution on liquidity shocks. We show explicitly that the price in absence of collateral is lower than the price in its presence.

Example (continued): Assume again that $\rho \sim Unif[\rho_{\min}, \rho_{\max}]$ and $\overline{p} = \theta_2 y_2 = \rho_{\max}$. Then, solving the market-clearing condition $E_{nc}(p, \rho^*) = 0$, yields the following equilibrium relationships:

1. If $\rho^* \geq \hat{\rho}^*_{nc} \equiv \frac{1}{(1+\sqrt{2})}(\rho_{\min} + \sqrt{2}\rho_{\max})$, then the price for asset sales in absence of collateral is $p^*_{nc} = \rho_{\max}$;
2. Otherwise, that is, if $\rho^* < \frac{1}{(1+\sqrt{2})}(\rho_{\min} + \sqrt{2}\rho_{\max})$, then there is cash-in-the-market pricing and the price for asset sales is
$$p^*_{nc} = \rho^* + \frac{(\rho^* - \rho_{\min})^2}{2(\rho_{\max} - \rho^*)}.$$
3. In the cash-in-the-market pricing region, the equilibrium price p^*_{nc} is increasing and convex in (inverse) moral hazard intensity ρ^*.
4. Importantly, the price with collateral p^* is (weakly) greater than the price in absence of collateral p^*_{nc}. In particular,
   - $p^*(\rho_{\min}) = p^*_{nc}(\rho_{\min}) = \rho_{\min}$;
   - $p^*(\rho) > p^*_{nc}(\rho)$ for all $\rho \in (\rho_{\min}, \hat{\rho}^*_{nc})$, and $\hat{\rho}^* < \hat{\rho}^*_{nc}$;[13]
   - Finally, $p^*(\rho) = p^*_{nc}(\rho) = \rho_{\max}$ for all $\rho \geq \hat{\rho}^*_{nc}$.

This relationship between the price with collateral, p^*, and the price without collateral, p^*_{nc}, is illustrated in Figure 5. Interestingly, p^* and p^*_{nc} both start at value of $\rho_{\min}$ when $\rho^* = \rho_{\min}$, but since p^* hits $\rho_{\max}$ before p^*_{nc} does, the price with collateral is more sensitive to ρ^* than the price without collateral, in the range where they are both below $\rho_{\max}$.

[13] This latter result is apparent from the expressions for $\hat{\rho}^*$ and $\hat{\rho}^*_{nc}$: $\hat{\rho}^*$ puts a weight of half each on $\rho_{\max}$ and $\rho_{\min}$, whereas $\hat{\rho}^*_{nc}$ puts a weight on $\rho_{\max}$ of $\frac{\sqrt{2}}{(1+\sqrt{2})}$, which is greater than half.

4 Ex-ante debt capacity and liquidity shocks

In this section, we provide an equilibrium setting that gives rise to the structure of liquidity shocks ρ_i assumed in our model so far. Before we move to modeling details, we provide a summary of what this section achieves.

We endogenize the structure of liquidity shocks in Section 4.1 by assuming that ex ante (at date 0), firms are ranked by their initial wealth or capital levels and must raise incremental financing up to some fixed (identical) level in order to trade. The incremental financing is raised through short-term debt contracts (payable at date 1) that give lenders the ability to liquidate ex post in case promised payments are not met. While not critical to the overall thrust of our results, we show in Section 4.4 that this form of financing – which grants control to lenders in case of default (as in collateral and margin requirements) – is optimal from the standpoint of raising maximum ex-ante finance.

This augmentation of the benchmark model leads to an interesting, even if somewhat involved, fixed-point problem: On the one hand, the promised payment for a given amount of financing is decreasing in the level of liquidation prices in case of default; on the other hand, the liquidation price is itself determined by the distribution of promised debt payments since these are the ex-post liquidity shocks faced by firms. We show in Section 4.2 that there is a unique solution to this fixed-point problem, characterized by the fraction of firms that are ex-ante rationed and the mapping from moral-hazard intensity to price. In fact, the fixed-point is a contraction mapping and enables us to provide a recursive, constructive algorithm for the solution (provided in Appendix 2). While the ex-ante rationing of firms renders analytical results on comparative statics difficult, numerical examples in Section 4.3 confirm some conjectures that follow naturally from our analysis.

4.1 The set-up

The augmented time-line is specified in Figure 6.

Suppose that at date 0, there is a continuum of firms with the starting wealth for firm i being w_i. The cumulative distribution function (cdf) of w_i is $Q(w_i)$ over $[0, w_{\max}]$ where $w_{\max}$ $= I - \theta_1 y_1$ (this assumption ensures that we borrow at least the value of the alternative asset so that there is a risk shifting problem). Each firm has access to an identical investment opportunity that pays off at date 2 and requires at date 0 an investment of I. We assume that this investment shortfall is externally financed via a debt contract with promised payment of ρ_i at date 1, against which creditors provide financing of $s_i = (I - w_i)$. Firms can attempt to meet the promised payment ρ_i through issuance of new debt and pledging of collateral (or what is equivalently in our model, through asset sales), as we describe below. If the payment ρ_i cannot be made at date 1 , then there is a transfer of control to creditors who liquidate the assets and collect the

proceeds. Since it is easier to work with the financing shortfall $s_i = I - w_i$, we note that the cdf of s_i is $R(s_i) = 1 - Q(I - s_i)$ over the support $[\theta_1 y_1, I]$.

The investment opportunity can yield in two periods (date 2) a cash flow y_2 with probability θ_2. However, at date 1, there is the possibility of moral hazard: Firm owners, if optimal to do so, may switch from the existing safer asset to the riskier asset, which yields a cash flow y_1 with probability θ_1, where we we assume as in our benchmark model that $\theta_1 < \theta_2$, $y_1 > y_2$, and $\theta_1 y_1 < \rho_i < \theta_2 y_2$. Viewed from date 0, θ_2 is uncertain: θ_2 has cdf $H(\theta_2)$ and probability density function (pdf) $h(\theta_2)$ over $[\theta_{\min}, \theta_{\max}]$, where we assume for simplicity that $\theta_{\min} y_2 \geq \theta_1 y_1$, that is, the worst-case expected outcome for the safer asset is no worse than that for the riskier asset. In fact we impose that

$$\theta_{\min} = \frac{\theta_1 y_1}{y_2}\left[1 + \sqrt{1 - \frac{y_2}{y_1}}\right], \tag{26}$$

this assumption ensures that maximum amount that can be borrowed is determined by ρ^* (which is always higher than $\theta_1 y_1$).

Conditional on the realization of θ_2 at date 1, firms may raise financing again in the form of debt in order to fund their "liquidity shock" - namely, the outstanding debt payment ρ_i. Thus, the date-0 structure of this augmented model maps one for one into the structure in our benchmark model where liquidity shocks were taken as given. In particular, the lower the realization of θ, the lower is $\rho^*(\theta_2)$ and hence the greater is the moral hazard problem; thus θ_2 indexes fundamental information that is related to the severity of the moral hazard problem. We show below that with this additional date 0 structure to the model, the initial distribution of wealth of firms w_i (or shortfall s_i) translates into an equilibrium distribution of their promised debt payments ρ_i.

Consider a particular realization of the quality of investment opportunity, say θ_2, at date 1. As shown in Proposition 1, firms with liquidity shocks up to $\rho^*(\theta_2) = \theta_2 f^*(\theta_2) = \frac{\theta_2(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)}$ are not rationed. That is, these firms can meet their outstanding debt payments at date 1, continue their investments, and possibly, also acquire more assets. Next, as shown in Proposition 2, firms with liquidity shocks over the range $[\rho^*(\theta_2), p^*(\theta_2)]$ are able to meet their liquidity shocks but only by pledging collateral from asset-sale proceeds. In other words, these firms can also meet their outstanding debt payments at date 1 and continue their investments, but do not have liquidity to acquire more assets. Finally, firms with liquidity shocks higher than $p^*(\theta_2)$ cannot meet their outstanding debt payments, and creditors liquidate these firms' assets.

Then, assuming that date 0 creditors are risk-neutral (like all other agents in our model), the amount of financing s_i that firm i can raise at date 0, satisfies creditors' individual rationality constraint:

$$s_i = (I - w_i) = \int_{\theta_{\min}}^{p^{*-1}(\rho_i)} p^*(\theta_2) h(\theta_2) d\theta_2 + \int_{p^{*-1}(\rho_i)}^{\theta_{\max}} \rho_i h(\theta_2) d\theta_2 \ , \tag{27}$$

which captures the fact that for low realizations of θ_2, the moral hazard is severe and firms end up being rationed, unable to meet their debt payments, and thus, liquidated, whereas for high realizations of θ_2, debt payments are met. The critical threshold determining whether θ_2 realization is "low" or "high" for firm i is given implicitly by the relation: $\rho_i = p^*(\theta_2)$. Also implicit in Equation (27) is the fact that some low wealth borrowers may be excluded as the amount owed $s_i = (I - w_i)$ may not be covered by the maximum amount available for payment the next period.

Note that given a price function $p^*(\theta_2)$ and financing s_i, equation (27) gives the face value ρ_i directly. However, we need to take account of Proposition 3 and recognize that the market-clearing price $p^*(\theta_2)$ itself depends upon the entire distribution of liquidity shocks ρ_i across firms. In case of their firm's default, creditors recover an amount that depends upon the asset liquidation price, and, thus on the liquidity shocks of *other* firms; in turn, each firm's debt capacity, and thereby their future liquidity shocks, depend on the (expectation over the) amount recovered. Thus the model can be viewed as a general equilibrium version of Shleifer and Vishny (1992) with ex-ante contracting and endogenous borrowing capacity determined by moral hazard problems.

With this background, we define the equilibrium of the ex-ante game. An important notational issue to bear in mind is that in the benchmark model, we assumed as exogenously given the distribution of liquidity shocks, $G(\rho)$, but in the augmented model, this distribution is induced by the distribution of financing needs, $R(s)$.

Definition: An equilibrium of the ex-ante borrowing game is (i) a pair of functions $\rho(s_i)$ and $p^*(\theta_2)$, which respectively give the promised face-value for raising financing s_i and equilibrium price given quality of assets θ_2, and (ii) a truncation point $\hat{s}$, which is the maximum amount of financing that a firm can raise in equilibrium, such that $\rho(s_i)$, $p^*(\theta_2)$ and $\hat{s}$ satisfy the following fixed-point problem.

1. For every θ_2, prices are determined by the industry equilibrium condition of Proposition 3:

$$p^*(\theta_2) \leq \rho^*(\theta_2) + \int_{\rho_{\min}}^{p^*(\theta_2)} \hat{G}(u)du \ , \tag{28}$$

where compared to equation (20), we have replaced distribution of liquidity shocks $G(\cdot)$ with the induced distribution $\hat{G}(\cdot)$ and also substituted the variable of integration ρ with u to avoid confusion with the function $\rho(s_i)$. In particular, $\hat{G}(u)$ is the truncated equilibrium distribution of liquidity shocks given by $\hat{G}(u) = \frac{R(\rho^{-1}(u))}{R(\hat{s})}$. Formally, the distribution of liquidity shocks, $\hat{G}(u)$, is induced by the distribution of financing amounts, $R(s)$, via the function Prob$[\rho(s_i) \leq u | s_i \leq \hat{s}]$. Note that as in case of equation (20), a strict ($<$) inequality leads to $p^*(\theta_2) = \overline{p}(\theta_2) = \theta_2 y_2$.

2. Given the price function $p^*(\theta_2)$, for every $s_i \in [0, \hat{s}]$, the face value ρ is determined by the

requirement that lenders receive in expectation the amount that is lent:

$$s_i = (I - w_i) = \int_{\theta_{\min}}^{p^{*-1}(\rho)} p^*(\theta_2) h(\theta_2) d\theta_2 + \int_{p^{*-1}(\rho)}^{\theta_{\max}} \rho h(\theta_2) d\theta_2. \tag{29}$$

3. The truncation point $\hat{s}$ for maximal financing is determined by the condition

$$\hat{s} \leq \theta_1 y_1 + \int_{\theta_{\min}}^{\theta_{\max}} p^*(\theta_2) h(\theta_2) d\theta_2 \ , \tag{30}$$

with a strict inequality implying that $\hat{s} = I - \theta_1 y_1$ (all borrowers are financed).

For future reference, we note that differentiating equality versions of Equations (28) and (29) yields alternative but equivalent conditions that

$$\frac{dp}{d\theta_2} = \frac{\frac{d\rho^*(\theta_2)}{d\theta_2}}{1 - \hat{G}(p)} \ \text{ if } p < \theta_2 y_2, \ \text{ else } \frac{dp}{d\theta_2} = y_2 \ , \tag{31}$$

and

$$\frac{d\rho}{ds_i} = \frac{1}{1 - H(p^{*-1}(\rho))} \ \text{ if } \rho \geq p^*(\theta_{\min}), \ \text{ else } \frac{d\rho}{ds_i} = 1 \ . \tag{32}$$

4.2 The solution

In this section, we show that there is a unique equilibrium to the ex-ante borrowing game given by the solution to the fixed-point problem stated above. We provide an explicit characterization of the solution.

Recall that $s_i = (I - w_i)$ is the financing amount and given a distribution of wealth w_i with cdf $Q(w_i)$; the distribution of s_i given by $R(s_i) = P[(I - w_i) \leq s_i]$ is thus known. As in the definition above, we will work with $R(s_i)$ directly, where $R(s_i)$ has support $[\theta_1 y_1, I]$. In what follows, we suppress the subscript i unless it is necessary. The maximum wealth financed is $\hat{s} = (I - \hat{w})$: in equilibrium, some borrowers with low wealth will be endogenously excluded.

It is easier to analyze the fixed-point problem by working with the inverse functions $s(\rho)$ and $\theta_2(p)$. $s(\rho)$ gives the financing raised ex ante for a given face-value ρ while $\theta_2(p)$ gives the realization of the state θ_2 for the given equilibrium price p. Since these are one-to-one functions, we can follow this approach. Notice that both ρ and p have the domain $[\theta_1 y_1, \theta_{\max} y_2]$ (one cannot have a face value higher than the highest possible price); it is possible that this highest price is not reached in equilibrium and we will account for this.

Our approach to solving the fixed point problem is as follows. Fix a maximal financing $\hat{s}$. First we invert Equation (28) and solve for $\theta_2(p)$: since this is an explicit quadratic equation we can solve for this variable, we also impose the constraint that price is always less that $\theta_2 y_2$. Given the solution to $\theta_2(p)$, we can substitute this into the differential equation for $s(\rho)$, Equation (32), this yields an integro-differential equation that has a unique solution. The maximum financing is then uniquely solved by the boundary condition in Equation (30).

Given the cdf of amount financed, $R(s)$, the cdf of face values conditional on financing being over the truncated support of amounts financed $[\theta_1 y_1, \hat{s}])$ is denoted as $\hat{G}(u)$, and is given by $\hat{G}(u) = \frac{R(s(u))}{R(\hat{s})}$, where $\hat{G}(u) = \text{Prob}[\rho \leq u | s \leq \hat{s}] = \text{Prob}[s(\rho) \leq s(u) | s \leq \hat{s}]$.

Define

$$L(p) = p - \int_{\theta_1 y_1}^{p} \hat{G}(\rho) d\rho, \tag{33}$$

where we have switched back to ρ as being the variable of integration.

Then, setting $L(p) = \rho^*(\theta_2)$ to satisfy equation (28) with equality, we obtain

$$\theta_2 \frac{(\theta_2 y_2 - \theta_1 y_1)}{(\theta_2 - \theta_1)} = L(p) \ , \tag{34}$$

which yields the following solution for θ_2 (we have a quadratic equation and pick the correct root)

$$\frac{(\theta_1 y_1 + L(p)) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4 y_2 L(p) \theta_1}}{2 y_2} \ . \tag{35}$$

Accounting for the fact that prices cannot be above $\theta_2 y_2$ (hence $\theta_2 \geq \frac{p}{y_2}$), we define $\theta_2(p)$ in terms of $s(\rho)$ as:

$$\theta_2(p) = \max \left\{ \frac{(\theta_1 y_1 + L(p)) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4 y_2 L(p) \theta_1}}{2 y_2}, \frac{p}{y_2} \right\} \tag{36}$$

on the domain $[\theta_1 y_1, \theta_{\max} y_2]$. Note that this equation defines $\theta_2(p)$ in terms of $s(\rho)$ since $L(p)$ depends on the function $\hat{G}(\rho) = \frac{R(s(\rho))}{R(\hat{s})}$.

Note also that if $p = \theta_1 y_1$, then Equation (36) is determined by Equation (35). In fact, it is the case from Equation (35) that $\theta_2(p) = \theta_{\min}$ as $L(\theta_1 y_1) = \theta_1 y_1$. At $p = \theta_{\max} y_2$, if there is no liquidity crisis, then we have $\theta_2(p) = \theta_{\max}$, the maximum possible price. If this is not true, then we have $\theta_2(\theta_{\max} y_2) > \theta_{\max}$ (there is a liquidity crisis in every possible state).

Next, we solve the differential equation implied by Equation (32) (which is itself equivalent to Equation 29):

$$\frac{ds}{d\rho} = 1 - H(\theta_2(\rho)), \tag{37}$$

where $H(\theta_2)$ is the cdf of θ_2. Since it is possible that $\theta_2(p) > \theta_{max}$ in Equation (37), we extend $H(\theta_2)$ by assuming that $H(\theta_2) = 1$ for $\theta_2 > \theta_{\max}$ (this is true and innocuous since $1 - H(\theta_2) = 0$ for such θ_2).

Substituting for $\theta_2(p)$ from equation (36), we obtain that

$$\frac{ds}{d\rho} = 1 - H\left(\max\left\{\frac{(\theta_1 y_1 + L(\rho)) + \sqrt{(\theta_1 y_1 + L(\rho))^2 - 4y_2 L(\rho)\theta_1}}{2y_2}, \frac{\rho}{y_2}\right\}\right) \tag{38}$$

with the end point constraint that $s(\theta_1 y_1) = \theta_1 y_1$.

This is a standard integro-differential equation of the form

$$\frac{ds}{d\rho} = f\left(\rho, \int_{\theta_1 y_1}^{\rho} \frac{R(s(u))}{R(\hat{s})} du\right) \tag{39}$$

with the endpoint constraint $s(\theta_1 y_1) = \theta_1 y_1$, and it has a unique solution if the function $f\left(\rho, \int_{\theta_1 y_1}^{\rho} \frac{R(s(u))}{R(\hat{s})} du\right)$ is Lipschitz in ρ.[14] This is indeed the case in our set-up, technical details of which are relegated to Appendix 1.

We now solve for the maximal financing $\hat{s}$, this is given by the condition

$$\hat{s} \leq \theta_1 y_1 + \int_{\theta_{\min}}^{\theta_{\max}} p(\theta_2) h(\theta_2) d\theta_2 \tag{40}$$

where $p(\theta_2)$ is the inverse function of $\theta_2(p)$ and $h(\theta_2)$ is the density of θ_2.

The left hand side of Equation (40) is $\theta_1 y_1$ at $\hat{s} = \theta_1 y_1$ and increasing in $\hat{s}$. The right hand side of Equation (40) is strictly greater than $\theta_1 y_1$ at $\hat{s} = \theta_1 y_1$ and decreasing in $\hat{s}$.[15] Either Equation (40) has a unique solution or no solution with strict inequality at $\hat{s}$, in that case there is no exclusion and $\hat{s} = I - \theta_1 y_1$.

This completes the proof that the solution exists and is unique.

We state all this in the following proposition.

Proposition 6 *There exists an unique equilibrium of the ex-ante borrowing game defined in Section 4.1. In particular, given a maximal borrowing amount $\hat{s}$, the borrowing function $s(\rho)$*

[14] Note that the generic function f for expressing the integro-differential equation is not to be confused with the face-value of debt in our benchmark model.

[15] To see this note that if we increase $\hat{s}$, we decrease $\hat{G}(\rho)$, which means we increase $L(p)$ and hence $\theta_2(p)$, therefore $p(\theta_2)$ decreases, therefore the right hand side of the Equation (40) decreases.

(financing as a function of face value borrowed) is the unique solution to the integro-differential equation

$$\frac{ds}{d\rho} = 1 - H\left(\max\left\{\frac{(\theta_1 y_1 + L(\rho)) + \sqrt{(\theta_1 y_1 + L(\rho))^2 - 4y_2 L(\rho)\theta_1}}{2y_2}, \frac{\rho}{y_2}\right\}\right) \tag{41}$$

with the end point constraint that $s(\theta_1 y_1) = \theta_1 y_1$. *Given* $s(\rho)$, *the inverse equilibrium price function* $\theta_2(p)$ *is uniquely given by*

$$\theta_2(p) = \max\left\{\frac{(\theta_1 y_1 + L(p) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4y_2 L(p)\theta_1}}{2y_2}, \frac{p}{y_2}\right\} \tag{42}$$

on the domain $[\theta_1 y_1, \theta_{\max} y_2]$.

The maximal borrowing amount is uniquely given by the boundary condition

$$\hat{s} \leq \theta_1 y_1 + \int_{\theta_{\min}}^{\theta_{\max}} p(\theta_2) h(\theta_2) d\theta_2 \tag{43}$$

where $p(\theta_2)$ *is implicitly a function of* $\hat{s}$.

It is important to note that the feedback between promised (ex-ante) debt payments and (ex-post) liquidation price produces an amplification effect of collateral in our model, but unlike the extant literature, this amplification effect is positive for liquidity and efficiency: The presence of collateral requirements stabilizes liquidation prices as well as lowers promised debt payments, and these two effects feed on each other to produce an amplified reduction in ex-ante rationing. To summarize, in our augmented model too, endogenously designed collateral requirements enhance efficiency and trading by reducing the severity of agency problems tied to external finance. More generally, we conjecture that rather than being the source of illiquidity in markets, collateral is in fact the life-blood that sustains high levels of trading amongst financial institutions.

4.3 Numerical examples

To understand the equilibrium and its comparative statics (which are difficult to derive analytically), we solve a numerical example. In the first comparative static, as the distribution of wealth or capital levels of firms declines in a first-order stochastic dominance (FOSD) sense, we show that firms pledge more payment to raise given amount of financing, and, in turn, equilibrium price is lower. In the second comparative static, as the distribution of quality of asset worsens in a FOSD sense, so does the moral-hazard intensity worsens, firms face greater financing friction

in future, the liquidation price is lower, and, in turn, equilibrium ex-ante financing requires higher debt payments.

Varying the distribution of ex-ante wealth or capital: This numerical example is constructed as follows:

1. Let $I = 1$, $w_{\min} = 0$, $y_1 = 4$, $y_2 = 1$, $\theta_1 = 0.05$, $w_{\max} = I - \theta_1 y_1 = 1 - 0.2 = 0.8$. Hence s has support $[0.2, 1]$.

2. Let $t = 0.8 = 1 - 0.2$ (which is also the value of $w_{\max}$) and suppose that

$$R(s) = 1 - (1 - \frac{s - 0.2}{t})^{1/\gamma} , \tag{44}$$

   where γ is a strictly positive number, $\gamma > 0$. A higher value of γ implies first-order stochastic dominance (FOSD); in fact for any truncation $\hat{s}$, a higher value of γ implies FOSD (Hopenhayn (1992) refers to this as monotone conditional dominance or MCD). The uniform distribution corresponds to the case $\gamma = 1$. Note that $E[s]$ is $s_{\min} + \frac{t\gamma}{1+\gamma}$ which is increasing in γ.

3. Suppose also that $H(\theta)$ is uniform on $[\theta_{\min}, \theta_{\max}]$ where $\theta_{\min} = 0.1(2 + \sqrt{3})$ and $\theta_{\max} = 0.9$.

Note that $\theta_{\min}$ is given by the formula $\frac{0.2}{1}[1 + \sqrt{1 - .0.25}]$, which yields the value of $\theta_{\min} = 0.1(2 + \sqrt{3})$.

We let γ take values $0.1, 0.5, 1.0, 5.0, 50, 100$ ($\gamma = 1$ being the uniform distribution). We show for these values the distributions of $\rho(s)$ in one plot (Figure 7) and $p(\theta)$ in another plot (Figure 8).

From Figure 7, note how $\hat{s}$ decreases as we increase the distribution of s in FOSD sense, $\hat{s}$ for $\gamma = 0.1$ being around 0.64 while that for $\gamma = 100$ being around 0.58. There is a greater likelihood of default with higher γ, hence it must be that prices are lower in the sense of FOSD, and therefore face values are higher. This results in higher liquidity shocks at date 1, hence lower prices which then feed backs to give rise to higher face values. Thus, with a higher value of γ, there is greater exclusion at the ex-ante borrowing stage.

The above intuition relating γ to default risk and price is confirmed in Figure 8, where we see that for $\gamma = 0.1$ prices are higher and the default region is smaller relative to the case where $\gamma = 100$. With a higher γ, liquidity shocks are more extreme and there are more fire sales in equilibrium, for the same level of fundamentals or the moral hazard problem (θ_2).

Thus, this numerical example reinforces the intuition provided by our theory and enables a comparative static exercise at the ex-ante borrowing stage.

Varying the distribution of moral-hazard intensity: This numerical example is to be completed.

4.4 Optimality of debt contracts with lender control

A key aspect of our model is the use of debt contracts where the lender has control. In particular, the lender calls the loan at time 0, inducing a liquidity shock. While this seems to correspond well to the nature of margins in financial contracts, we are unaware of any literature directly rationalizing such margin contracts. One may wonder why this makes sense and why renegotiation does not occur.[16]

We argue in this subsection that in a model of incomplete contracts that follows Aghion and Bolton (1992) (see also Hart and Moore (1994) and Hart (1995)), the contract with lender control maximizes the ex-ante financing available to investors. Our proof consists of two steps. First, we show that debt is the optimal contract. Second, we show that borrower control at date 1 is dominated by lender control at date 1.

Intuitively, in absence of lender control, the borrower can always invoke the moral hazard problem and strategically renegotiate the lender down to $\rho^*(\theta_2)$. This would lower the payoffs available to lenders. In contrast, with lender control, the maximum amount available to lenders via collaterization and asset sales is $p^*(\theta_2) \geq \rho^*(\theta_2)$. Hence, lender control yields much higher payoffs to the lender ex post. Ex-ante, it is thus in the borrower's interest to give control rights to the lender and raise as much ex-ante debt financing as possible.[17] In effect, this rationalizes the use of margin contracts which give the lender the power to call the loan (as is common in financial markets). We formalize this intuition next.

To prove our results, we make two assumptions in the spirit of Aghion and Bolton (1992) and Hart and Moore (1994).

Assumption C1: Courts can verify whether the state 0 occurs or whether $\{y_1, y_2\}$ occurs, however they cannot distinguish between states $\{y_1, y_2\}$.

[16] Diamond (2004) in his Presidential address also discusses why short term debt may resolve incentive problems. He focuses on an environment where the collective action problem makes it hard to renegotiate short term debt and leads to a run on the firm, ex ante this may be better for the borrower. Diamond and Rajan (2001) present a similar argument to Diamond (2004).

[17] Note that our model differs from the standard Aghion and Bolton (1992) model in that borrower's ability to invoke the moral hazard problem gives the borrower too much power ex post. The only way to limit this is to give the lender the ex-post control rights.

This assumption essentially states that there is some coarseness in the enforcement ability of courts. While contracts can distinguish between low and high states, they cannot discriminate between different high states.

Assumption C2: While the interim state θ_2 is observable, it is not contractible.

This assumption is similar to Aghion and Bolton (1992) and forces the contract designer to give control conditional on the state θ_2 to either the lender or the borrower. We believe that this assumption is justifiable because hedge funds have complex portfolio strategies with many liquid and illiquid positions – while the lender and hedge fund may agree on a valuation, courts may find it difficult to verify this.

Assumption C3: Payments at date 1 (ex-post states) cannot be bigger than the maximum payoff in that state or smaller than 0.

This is a standard assumption that limits liability and does not allow payments in excess of what is available.

These three assumptions essentially deliver the result that we want. From Assumption C1, the optimal contract must be a pair $\{0, \rho_i\}$ that pays off the same amount whether states y_1 or y_2 occur (we do not formally prove this).

Assumption C2 implies that we have to compare borrower control or lender control in every state. With borrower control, if $\rho_i \leq f^*(\theta_2)$, the borrower will honor the contract and the lender will get $\theta_2\rho_i$. However, if $\rho_i > f^*(\theta_2)$, then the borrower will credibly threaten to switch to the bad project. Hence, the lender will renegotiate the claim to $\rho_i = f^*(\theta_2)$ and obtain only $\rho^*(\theta_2)$ $= \theta_2 f^*(\theta_2)$. Hence with borrower control, the lender gets $\max[\theta_2\rho_i, \rho^*(\theta_2)]$.

In contrast, with lender control, the lender can threaten the borrower with liquidation at market prices. Hence, in this case, the lender gets $\max[\rho_i, p^*(\theta_2)]$, where $p^*(\theta_2) \geq \rho^*(\theta_2)$ with $>$ inequality in states with sufficiently high θ_2 and $\rho_i > \theta_2\rho_i$.

Thus, borrowing with control rights allocated to the lender always generates higher ex-post payoff to the lender and thus greater ex-ante borrowing capacity for the borrower. We state this as a Proposition.

Proposition 7 *Under assumptions (C1)–(C3), the optimal contract is debt and lender control always yields a greater region of financed projects than borrower control.*

Proposition 7 provides a rationalization for the structure of financing contracts for trading intermediaries where the moral hazard of risk-shifting is most pertinent. Lenders lend to borrowers and call the loan on interim information unless collateral is posted. This contract gives strong

ex-post control to the lender but reduces the borrower's ability to choose among risky projects and renegotiate.

One may also ask why the borrower cannot buy back debt at market prices (though no renegotiation is allowed). Effectively this is worse than borrower control as the borrower pays $\theta_2\rho_i$ for a face value of ρ_i if $\rho_i \leq f^*(\theta_2)$ (which is identical to that with borrower control). If $\rho_i > f^*(\theta_2)$, the borrower will shift to the more risky project since the debt cannot be renegotiated, and the lender will be bought out for $\theta_1\rho_i$ which is the value of claim. Notice that this regime does not give the lender enough ex post power and is in fact worse than the borrower control as the risk shifting occurs in equilibrium. Hence it is dominated ex ante by lender control. In practice, borrowers cannot buyout lenders at market prices when there is a margin call, they have to post the margin or be liquidated. This commitment feature that occurs in our model through the assignment of bargaining rights to lenders is a critical aspect of our model.

More importantly, in the context of this paper, the Proposition rationalizes the contract structure that we have employed in our analysis, this matches the margin financing closely.

5 Robustness issues

In this section, we discuss some of the important assumptions that have gone into our analysis and attempt to understand how robust the model is to these assumptions.

In our model, liquidity shocks at date 1 arise due to liabilities undertaken by firms at date 0. In principle, such liabilities can be foreseen and hence potentially hedged by firms through management of asset duration, holding of liquid assets, and pre-arrangement of lines of credit (as in Holmstrom and Tirole, 1998). Changing asset duration and holding liquidity can be economically expensive. The lines of credit generally contain a Material Adverse Change (MAC) clause, which allows the provider of the line to revoke access in case the borrower's condition has deteriorated sufficiently (see Sufi, 2006 for empirical evidence that this clause is invoked in practice). Indeed, one reason why such clauses feature in optimal contracting of the line of credit is precisely to avoid agency problems tied to borrower-lender relationships. In our view, modeling liquidity shocks as liability shocks is thus a metaphor for the residual asset-liability mis-match on the balance-sheet of firms.

Especially in our context of financial intermediaries, the liquidity shocks can be more broadly interpreted as arising due to change in the mark-to-market valuations of financial securities such as swaps where ex-ante contract values are zero, but ex post, depending upon the realization of underlying price shocks, the valuation may result in liabilities or cash outflows. Such shocks cannot be hedged perfectly as that would be tantamount to completely undoing the position undertaken through the security in the first place.

A related question to that about lines of credit is why the non-rationed firms do not lend to the rationed firms. One rationale to believe such lending would occur is that players within the financial sector understand financial assets better and may have superior monitoring technology to lend to other financial firms compared to dispersed lenders (such as purchasers of commercial paper, for example). Such superior peer-monitoring skills have been employed in the literature to provide a micro-foundation for the existence of inter-bank lending (Rochet and Tirole, 1996). Since improved monitoring mitigates the opportunity to engage in asset substitution, such lending would in general improve efficiency. However, equilibrium between the market for lending and the market for acquiring assets ensures that funding illiquidity persists at least when the moral-hazard intensity is sufficiently severe. The reason for this is that if there is limited funding in the system as a whole, then asset markets will clear only at fire-sale prices, and if this is the case, potential lenders – who are also potential asset acquirers – would be willing to provide financing only at rates that ensure them the same return as the purchase of cheap assets. Since the face-value of loans would be constrained by the risk-shifting problem, only limited financing would be possible in equilibrium.[18]

The converse of this question is why the financiers in our model (assumed to be the dispersed type) do not participate in the market for assets. Again, they might lack the expertise or sophistication to operate complex financial assets. On the one hand, liquidation to such inefficient users would result in allocation inefficiencies in the model. On the other hand, such users would not find prices attractive (relative to non-rationed industry insiders) unless fire-sale discount becomes relatively steep. To summarize, as in the original models of Williamson (1988) and Shleifer and Vishny (1992), the idea of asset-specificity is key to ensuring that there is limited participation by financiers in the market for assets.

Importantly, this specificity also justifies why collateral required to be posted by financiers is generally in the form of cash or treasury securities, rather than in the form of risky assets underlying the business activities of borrowers. Financiers may not operate or risk-manage complex assets well if collateral of such form is held on their own books, and if held on the books of borrowers, financiers may lack the expertise to monitor well and constrain the risk-shifting problem. Thus, hair-cuts or discounts to market valuations charged when posting collateral, are increasing in the complexity and riskiness of assets being posted and become steeper during times of tight liquidity, that is, when borrower-lender agency problems are exacerbated.

Finally, we acknowledge that our choice of risk-shifting technology from asset 2 to asset 1,

[18] See Acharya and Yorulmazer (2007) for modeling of such linkages between markets for financial and real assets. Further, Acharya, Gromb and Yorulmazer (2007) argue that in cases where a large number of players are liquidity takers and only a handful remain as potential liquidity providers, these providers may act strategically and charge higher than competitive lending rates in order to force greater asset sales and extract further price discounts. This argument has also been made in the context of predatory trading in capital markets by Brunnermeier and Pedersen (2005) and Carlin, Lobo and Viswanathan (2007).

as merely switching from a stream of risky to even riskier cash flows without incurring any costs or without engaging in any trades, has the flavor of risk-shifting in the context of real assets. Put another way, in the case of financial assets, one would ideally want the shift of assets to arise because of the sale of risky asset and the purchase of even riskier asset, and potentially clear the markets at date 2 from such shifts. Our choice has been based primarily on simplicity and parsimony. Nevertheless, we believe there are at least a few justifications and interpretations that accredit the choice. First, the shift in assets could represent simply a deterioration in the risk-management function of the financial intermediary, for example, not constraining traders from following doubling-up strategies and allowing (or even encouraging) them to put additional capital at risk so as to "gamble for resurrection." Second, the riskier technology could in fact be outside of the traditional assets invested by the financial sector. Given the risk-shifting incentive, institutions may be willing to pay positive price for the option-value of an asset that otherwise represents a negative net-present value investment. The sellers of such assets from outside of the traditional financial sector may only be too willing to be the recipient of this benefit. An example here would be the "reaching for yield" behavior attributed in recent times to hedge funds and other players as their alphas from previously successful strategies have been eroded by competition. The growth in markets for alternative risks and the "excess" in funding of sub-prime mortgages are again cases in point.

6 Conclusion

In this paper, we have made a first attempt at providing an agency-theoretic micro-foundation to collateral constraints in a set-up where asset-pricing implications of such constraints can be studied. We would like to reiterate that our general equilibrium variant of the Shleifer and Vishny (1992) industry-equilibrium model is characterized entirely by a single parameter, the moral-hazard intensity, which drives the extent of financing friction, the optimal level of collateral requirement designed to ameliorate the moral hazard, and finally, the level of equilibrium prices. This characterization is crucial to understanding that tight collateral constraints are not the source of market illiquidity, but that both are in fact manifestations of underlying agency problems tied to external finance. Indeed, we argued that markets may be far more illiquid in the absence of such constraints.

In ongoing work, we are examining the case where the quality of assets or fundamentals (formally, θ_2) is only imperfectly observed by different firms, and the price serves to aggregate this information. This feature has the potential to produce a further amplification effect in the model and sharper falls in market and funding liquidity when fundamentals deteriorate (say in a first-order stochastic dominance sense), compared to our benchmark model, and perhaps more consistently with the sharp nature of observed liquidity crises. Separately, we are also

examining the possibility of contagion across asset markets, when there is uncertainty about portfolio composition of financial institutions and there is a shock in fundamentals to some of the assets. Such uncertainty, resulting from the opaqueness of increasingly complex balance-sheets of trading institutions (and to an extent, necessary for them to prevent erosion of their "alphas"), has been argued to be a significant contributor to market and funding liquidity problems. The spillover of the recent sub-prime mortgage collapse onto broader credit markets present a case in point. Finally, we are addressing the welfare question as to whether collateral requirements set bilaterally between counterparties in over-the-counter products are efficient from a system-wide perspective. In particular, collateral has a price-stabilizing effect as shown in this paper, and if this pecuniary externality is not fully internalized in the bilateral setting of collateral requirements, then there may be a role for centralized counterparties and exchanges in ensuring that these requirements are not set inefficiently high.

We believe that such pursuits represent only the tip of the iceberg and much work remains in integrating agency-theoretic corporate-finance issues into main-stream asset-pricing literature, especially in the context of understanding liquidity issues. Financial markets feature a wide range of institutional arrangements such as credit enhancements, trading through AAA-rated counterparties and special-purpose vehicles, bilateral and multilateral netting arrangements in derivative contracts, and many others. The simple building blocks of this paper can serve as a useful starting point for modeling and understanding the effect of these institutional arrangements on trading liquidity and asset prices. This appears to us to be a fertile territory for future research.

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Appendix 1

Proof of Proposition 3: Note that

$$E(p,\rho^*) = D(p,\rho^*) - S(p,\rho^*) \quad (45)$$

$$= \int_{\rho_{\min}}^{p} \frac{(\rho^* - \rho)}{(p-\rho^*)}\, g(\rho)d\rho - \int_{p}^{\rho_{\max}} g(\rho)d\rho, \quad (46)$$

which can also be expressed after integration by parts as:

$$E(p,\rho^*) = -1 + \frac{1}{(p-\rho^*)} \int_{\rho_{\min}}^{p} G(\rho)d\rho. \quad (47)$$

Now, consider $E(\overline{p},\rho^*)$. As $\rho^* \to \rho_{\min}$, $E(\overline{p},\rho^*) < 0$; as $\rho^* \to \overline{p}$, $E(\overline{p},\overline{p}) > 0$; and, finally,

$$\frac{\partial E(\overline{p},\rho^*)}{\partial \rho^*} = \frac{1}{(\overline{p}-\rho^*)^2} \int_{\rho_{\min}}^{\overline{p}} G(\rho)d\rho \; > \; 0. \quad (48)$$

It follows that there exists a unique critical value $\hat{\rho}^* \in (\rho_{\min}, \overline{p})$ such that $E(\overline{p},\hat{\rho}^*) = 0$ and $E(\overline{p},\rho^*) < 0$ if and only if $\rho^* < \hat{\rho}^*$.

In turn, we obtain that the market-clearing price $p^* = \overline{p}, \forall \rho^* \geq \hat{\rho}^*$ and $p^* < \overline{p}$ otherwise, in which case p^* satisfies equation (20).

Note also that if $p = \rho^*$, then $E(p,\rho^*) > 0$ for all $\rho^* > \rho_{\min}$. Hence, we must have $p^* \geq \rho^*$ with equality arising only when $\rho^* = \rho_{\min}$ (see equation 20).

When the price p^* is lower than $\overline{p}$, it is given implicitly by the market-clearing condition $E(p^*(\rho^*),\rho^*) = 0$, so that by applying the implicit-function theorem to equation (46), we obtain

$$\frac{dp^*}{d\rho^*} = -\frac{\left(\frac{\partial E}{\partial \rho^*}\right)}{\left(\frac{\partial E}{\partial p}\right)}\,, \quad (49)$$

where all derivatives are evaluated at (p^*,ρ^*). First, we obtain that

$$\frac{\partial E(p,\rho^*)}{\partial p} = -\int_{\rho_{\min}}^{p} \frac{(\rho^* - \rho)}{(p-\rho^*)^2}\, g(\rho)d\rho - g(p) + g(p) \quad (50)$$

$$= -\frac{1}{(p-\rho*)} \int_{p}^{\rho_{\max}} g(\rho)d\rho \; < \; 0\,, \quad (51)$$

where we have substituted the market-clearing condition $E(p^*(\rho^*), \rho^*) = 0$ in the second equation. Next, observe that

$$\frac{\partial E}{\partial \rho^*} = \int_{\rho_{\min}}^{p} \frac{(p-\rho)}{(p-\rho^*)^2}\, g(\rho)d\rho \;>\; 0. \tag{52}$$

Thus, we obtain that p^* is increasing in ρ^* for $\rho^* < \hat{\rho}^*$. $\diamondsuit$

Proof of Proposition 5: The proof of parts (i) and (ii) of the Proposition follow quite closely the steps in the proof of Proposition 3. Hence, we prove only parts (iii) and (iv) below.

Using equations (18) and (23), we obtain that for $\rho^* = \rho_{\min}$, $p^* = p^*_{nc} = \rho_{\min}$. We also obtain from these equations that

$$E(p, \rho^*) - E_{nc}(p, \rho^*) = \int_{\rho^*}^{p} \frac{(\rho^* - \rho)}{(p-\rho^*)}\, g(\rho)d\rho + \int_{\rho^*}^{p} g(\rho)d\rho, \tag{53}$$

which simplifies to yield

$$E(p, \rho^*) - E_{nc}(p, \rho^*) = \int_{\rho^*}^{p} \frac{(p-\rho)}{(p-\rho^*)}\, g(\rho)d\rho \;>\; 0, \text{ for } \; p > \rho^*. \tag{54}$$

Now, by definition, for $p^*_{nc} < \overline{p}$, $E_{nc}(p^*_{nc}, \rho^*) = 0$. It follows then that $E(p^*_{nc}, \rho^*) > 0$. Since $E(p, \rho^*)$ is decreasing in p (see the proof of Proposition 3), it follows that if $E(p^*, \rho^*) = 0$, then $p^* > p^*_{nc}$, else if $E(p^*, \rho^*) > 0$, then $p^* = \overline{p} > p^*_{nc}$. A similar argument implies that if $p^*_{nc} = \overline{p}$, then $p^* = \overline{p}$ as well, and part (iii) of the proposition is proved.

Finally, part (iv) of the proposition relies on the proof of part (iii) and the additional observation that just as $\frac{\partial E(p,\rho^*)}{\partial p} < 0$, we also have that

$$\frac{\partial E_{nc}(p, \rho^*)}{\partial p} = -\int_{\rho_{\min}}^{\rho^*} \frac{(\rho^* - \rho)}{(p-\rho^*)^2}\, g(\rho)d\rho \;<\; 0. \;\diamondsuit \tag{55}$$

Completion of Proof of Proposition 6:

We now fill the Lipschitz continuity for the function $f\left(\rho, \int_{\theta_1 y_1}^{\rho} \hat{G}(u)du\right)$. To prove this we need to show that $|f(\rho') - f(\rho)| < K|\rho' - \rho|$. We know that if f is differentiable with bounded derivative $f'(\rho) \leq L$, the f is Lipschitz with constant $K < L$. We assume the cdfs H and G are continuously differentiable and have bounded derivatives. We know that the function $\theta(p)$ is continuously differentiable except at the first θ where no liquidity crisis occurs. But this poses no problem for Lipschitz continuity as if $\theta(\hat{\rho})$ is this point and $\hat{\rho} \in (\rho, \rho')$,

$$|f(\rho') - f(\rho)| \leq |f(\rho') - f(\hat{\rho})| - |f(\hat{\rho}) - f(\rho| \tag{56}$$

and we can use the Lipschitz continuity in each region. We also have to worry about our extensions of H and G. For H, our definition that $H(\theta(\rho) = 1$, if $\theta > \theta_{\max}$ maintains differentiability of H.

The extension that $\hat{G}(\rho) = 0$ if $s(\rho) > \hat{s}$ does causes a discontinuity at $\hat{s}$ but again if $\rho(\hat{s}) \in (\rho(s), \rho(s'))$, we can use the triangle inequality and in each region, there is Lipschitz continuity. Hence we have proved that $f\left(\rho, \int_{\theta_1 y_1}^{\rho} \hat{G}(\rho)\right)$ is Lipschitz and therefore from Theorem 2.2 (page 57) in Granas and Dugunji (2000), there is a unique solution that is a contraction. $\diamondsuit$

Appendix 2

Recursive Algorithm to solve the integro-differential equation

Find the initial value of $\hat{s}$ on $[\theta_1 y_1, I]$ as follows,

$$\hat{s} = \min\left\{I, \int_{\theta_{\min}}^{\theta_{\max}} \theta_2 y_2 h(\theta_2) d\theta_2\right\} \tag{57}$$

where we have used the fact that $\theta_2 y_2$ is the highest possible price in each state.

The recursive algorithm is very simple. Start with $s(\rho) = \rho$ on $[\theta_1 y_1, \theta_{\max} y_2]$.

Again define on the domain $[\theta_1 y_1, \theta_{\max}]$, $\hat{G}(u) = \frac{R(s(u))}{R(\hat{s})}$, where for $s(u) > \hat{s}$, $\hat{G}(u) = 0$.

Define

$$L(p) = p - \int_{\theta_1 y_1}^{p} \hat{G}(\rho) d\rho \tag{58}$$

and solve for $\theta_2(p)$ as

$$\theta_2(p) = \max\left\{\frac{(\theta_1 y_1 + L(p) + \sqrt{(\theta_1 y_1 + L(p))^2 - 4 y_2 L(p)\theta_1}}{2 y_2}, \frac{p}{y_2}\right\} \tag{59}$$

on the correct domain $[\theta_1 y_1, \theta_{\max} y_2]$.

Use this solution $\theta_2(p)$ to integrate the function

$$s(\rho) = \theta_1 y_1 + \int_{\theta_1 y_1}^{\rho} [1 - H(\theta_2(t))] dt \tag{60}$$

on the domain $[\theta_1 y_1, \theta_{\max} y_2]$ to find a new $s(\rho)$ (here we note that $\theta_2(p)$ may go higher than $\theta_{\max}$ if the highest price does not occur, hence we extend $H(\theta_2)$ by assuming $H(\theta_2) = 1$ if $\theta_2 > \theta_{\max}$, this is innocuous as $1 - H(s(\rho))$ is 0 is such a region).

Use this new $s(\rho)$ in Equation (58) this will converge.

The algorithm for finding $\hat{s}$ is as follows. Start with $\hat{s} = I$ and then solve for $\hat{s}$ in $[\theta_1 y_1, I]$ in

$$\hat{s} = \theta_1 y_1 + \int_{\theta_{\min}}^{\theta_{\max}} p(\theta_2) h(\theta_2) d\theta_2 \tag{61}$$

or there is inequality at $\hat{s} = I$.

If $\hat{s}_1$ is interior, use this value and redo algorithm, then at $\hat{s}_1$, Equation (61) will have an $<$ inequality, therefore the new solution will be $\hat{s}_2 > \hat{s}_1$. This procedure gives a sequence going back and forth which converges to $\hat{s}$.

t = 1		t = 1 1/2	t = 2
LIQUIDITY SHOCKS	**DEBT FINANCING**	**MARKET FOR ASSETS AT PRICE *p***	**MORAL HAZARD PROBLEM**
	• Firms with low ρ_i : - Borrow, face value f_i	• Buy assets	• Choose between safe and risky asset • Assets pay off, debt is due
• Firm ***i*** receive liquidity shock ρ_i	• Firms with moderate ρ_i : - Borrow, face value f_i - Pledge collateral k_i	• Liquidate some assets to meet collateral requirement	• Choose between safe and risky asset • Assets pay off, debt is due
	• Firms with high ρ_i : - Credit-rationed	• Are entirely liquidated	

Figure 1: Timeline of the benchmark model.

Figure 2: Equilibrium price p^* as a function of (inverse) moral hazard intensity

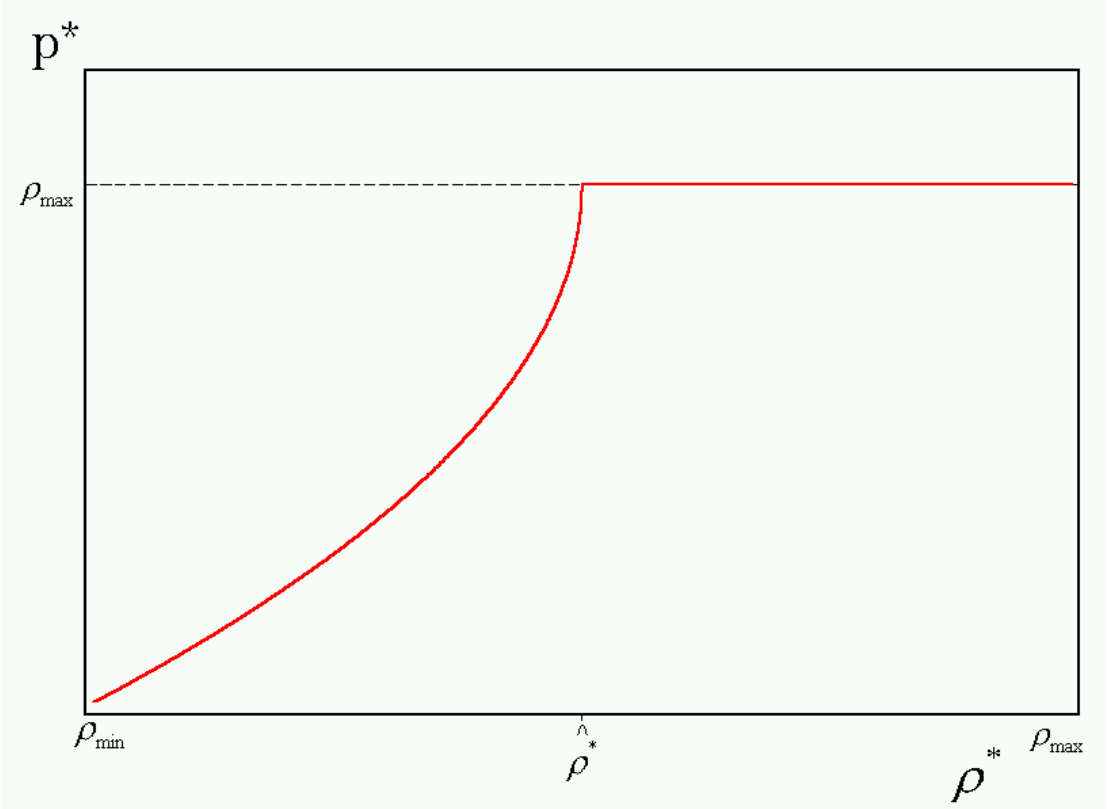


Figure 3: Equilibrium collateral constraint k(ρ) as a function of (inverse) moral hazard intensity ρ

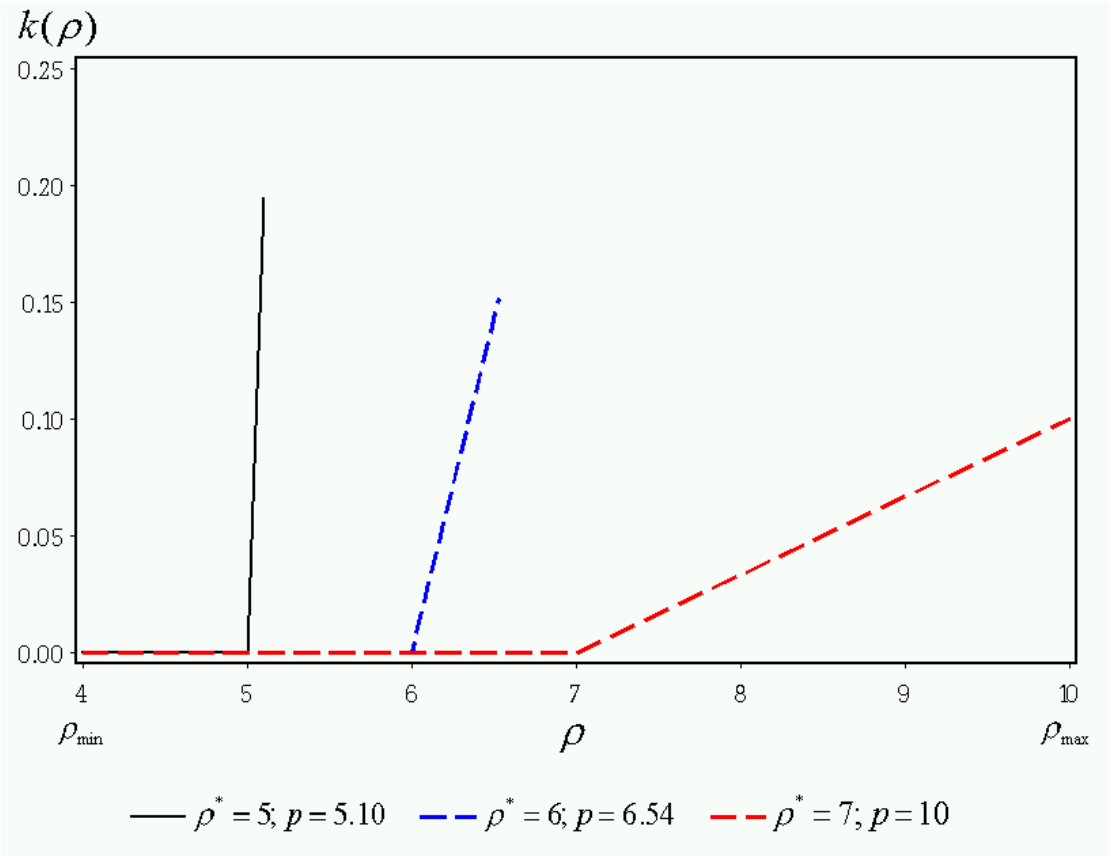


Figure 4: The relationship between market (il)liquidity and funding liquidity

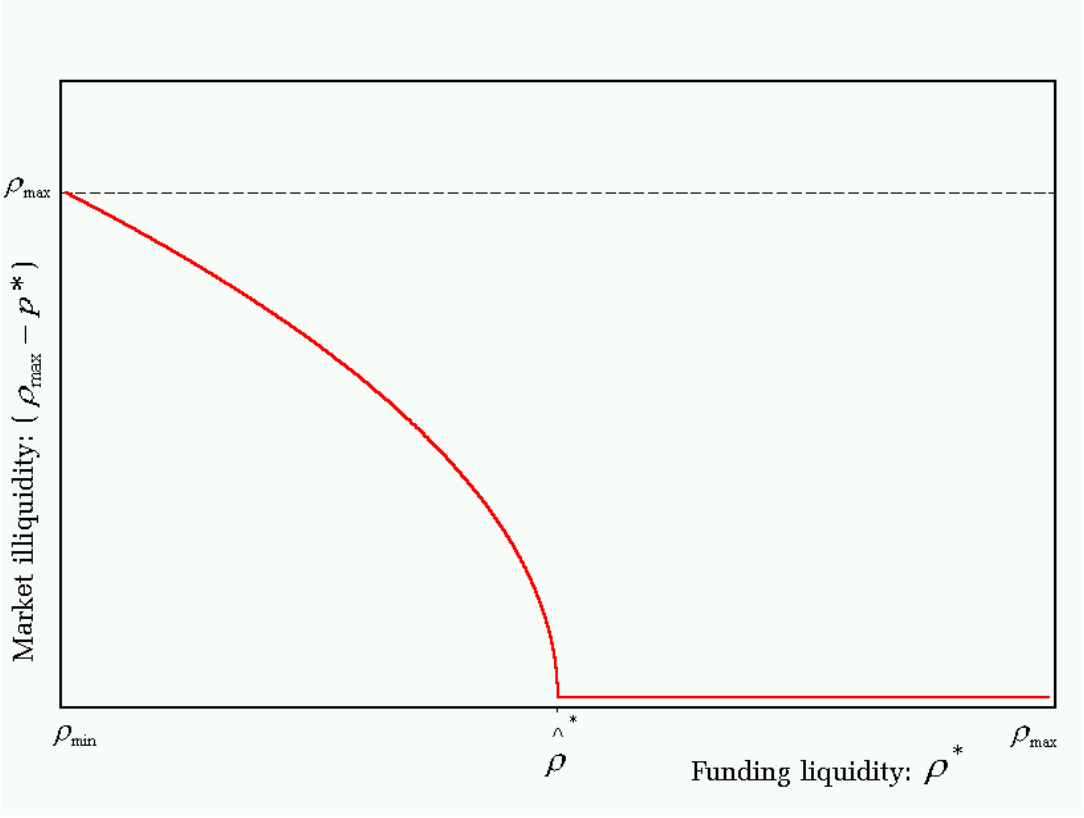
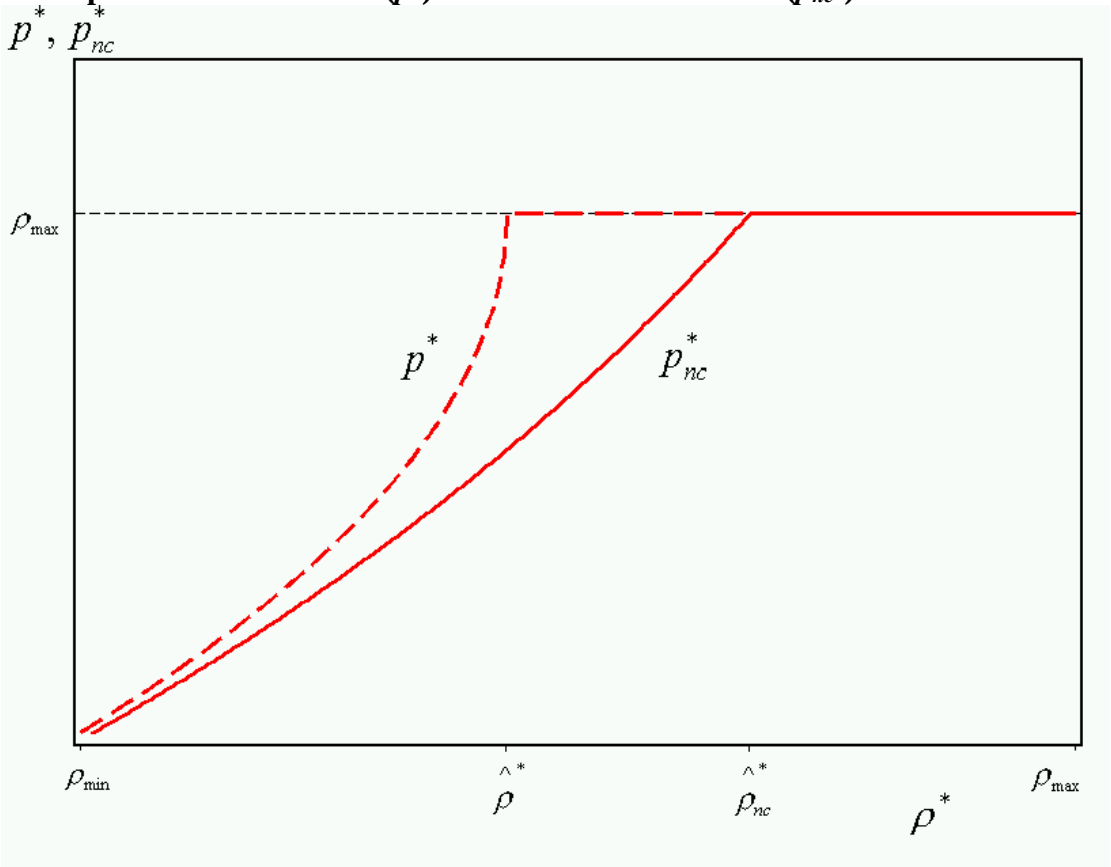
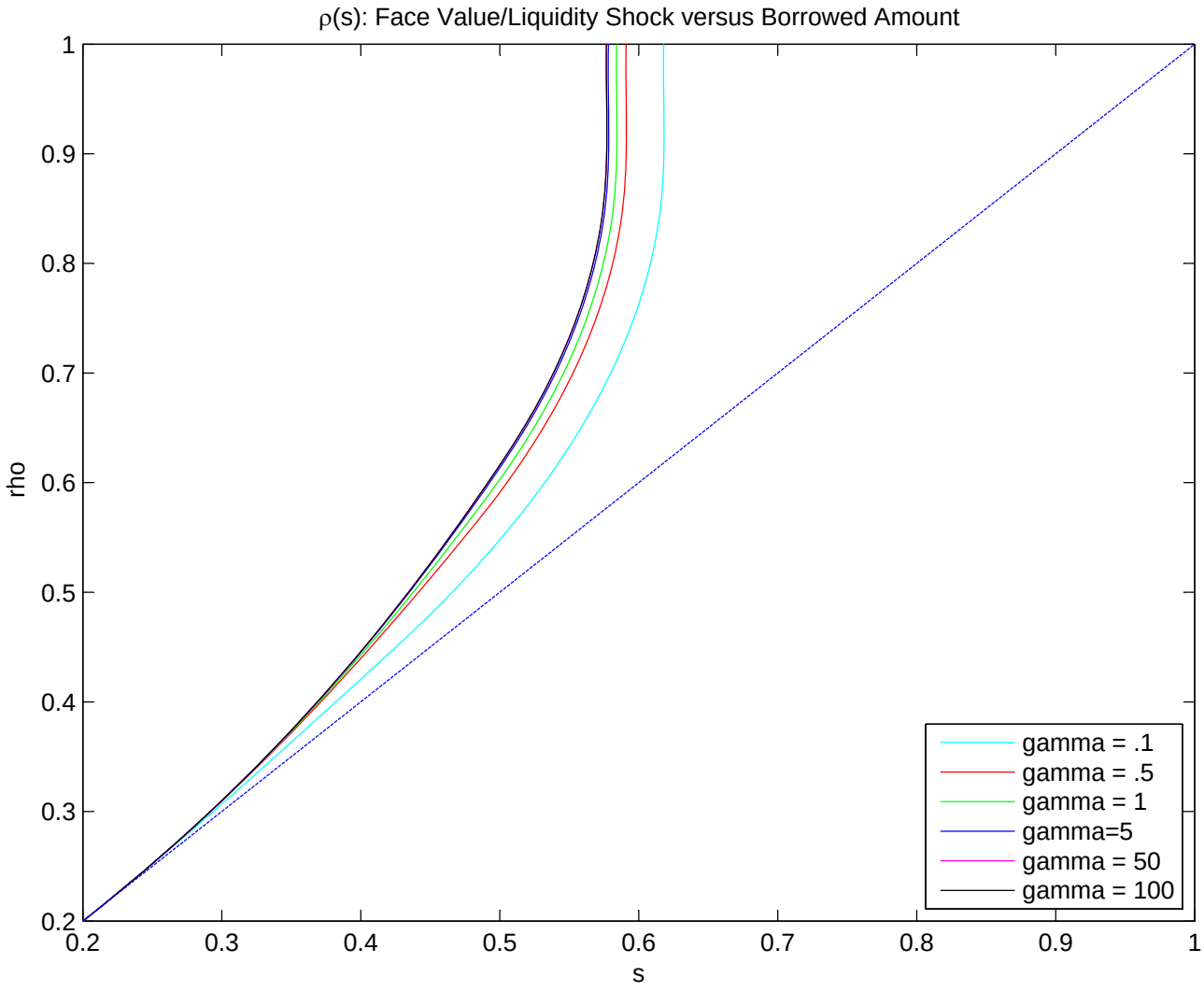


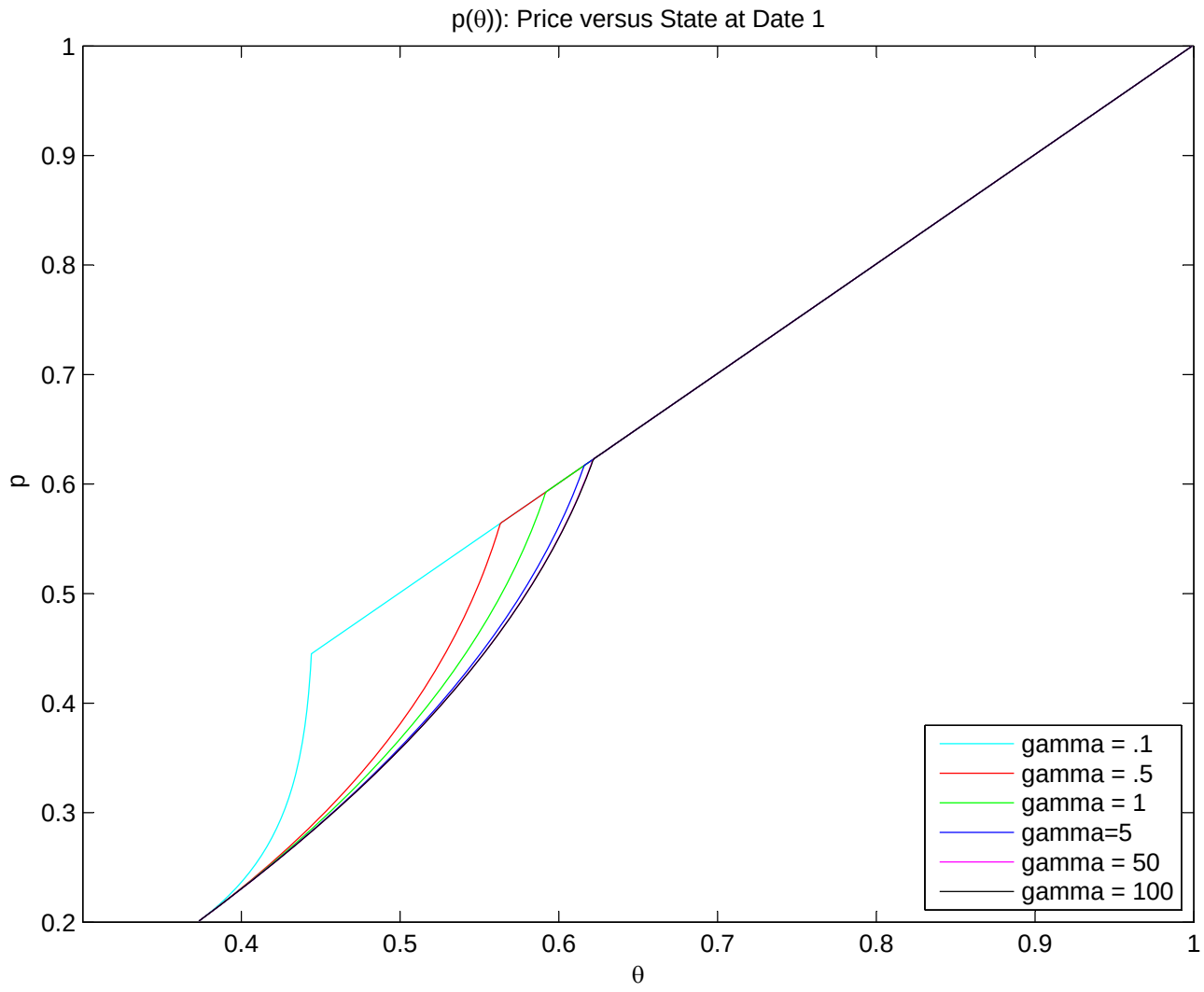
Figure 5: Equilibrium price as a function of (inverse) moral hazard intensity ρ^* in the presence of collateral (p^*) and absence of collateral (p_{nc}^*)



t = 0	t = 1	t = 1 1/2	t = 2
FIRST ROUND OF BORROWING	**DEBT DUE** **ADDITIONAL DEBT FINANCING** **REALIZATION OF ASSET QUALITY θ_2**	**MARKET FOR ASSETS** **AT PRICE $p(\theta_2)$**	**MORAL HAZARD PROBLEM**
	• Firms with low ρ_i : - Borrow again	• Buy assets	• Choose between safe and risky asset • Assets pay off, debt is due
• Firm ***i*** with wealth w_i raises $(I - w_i)$ with debt of face value ρ_i	• Firms with moderate ρ_i : - Borrow again - Pledge collateral	• Liquidate some assets to meet collateral requirement	• Choose between safe and risky asset • Assets pay off, debt is due
	• Firms with high ρ_i : - Credit-rationed	• Are entirely liquidated	

Figure 6: **Timeline of the augmented model.**


ρ(s): Face Value/Liquidity Shock versus Borrowed Amount
rho
s
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
gamma = .1
gamma = .5
gamma = 1
gamma=5
gamma = 50
gamma = 100


p(θ)): Price versus State at Date 1
p
θ
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.4
0.5
0.6
0.7
0.8
0.9
1
gamma = .1
gamma = .5
gamma = 1
gamma=5
gamma = 50
gamma = 100