On the Correlation Structure of Microstructure Noise in Theory and Practice

Francis X. Diebold* Georg H. Strasser†

University of Pennsylvania University of Pennsylvania

October 4, 2007

Abstract

We use market microstructure theory to derive the cross-correlation function between latent returns and market microstructure noise. The cross-correlation at zero displacement is typically negative, and cross-correlations at nonzero displacements are positive and decay geometrically. When market makers are very risk averse, the cross-correlation pattern is inverted. The results may be useful for choosing among different market microstructure models and estimation of noise-robust measures of realized volatility.

1 Introduction

Recent years have seen substantial progress in asset return volatility measurement. In particular, so-called realized variances and covariances ("realized volatilities"), based on

*Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia, PA 19104-6297 (*fdiebold@sas.upenn.edu*)

†Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia, PA 19104-6297 (*georg@sas.upenn.edu*)

Keywords: market microstructure, high-frequency data, noise, cross-correlation structure

JEL Codes: G14, G20, D82, D83, C51

increasingly-available high-frequency data, have emerged as central for several reasons.[1] They are, for example, largely model-free (in contrast to traditional model-based approaches such as GARCH), they are computationally trivial, and they are in principle highly accurate.

A tension arises, however, linked to the last of the above desiderata. Econometric theory suggests the desirability of sampling as often as possible to obtain highly accurate estimates, but financial market reality suggests otherwise. In particular, the microstructure noise (MSN) such as bid/ask bounce associated with ultra-high-frequency sampling separates the observed and latent ("true") prices, contaminating the standard realized volatility estimator and rendering it potentially unreliable.

Early work (e.g., Andersen, Bollerslev, Diebold and Labys (2001), Andersen, Bollerslev, Diebold and Ebens (2001), Barndorff-Nielsen and Shephard (2002a), Barndorff-Nielsen and Shephard (2002b), Andersen, Bollerslev, Diebold and Labys (2003)) finessed the sampling issue by attempting to sample often, but not "too often," implicitly or explicitly using the volatility signature plot of Andersen, Bollerslev, Diebold and Labys (2000) to guide sampling frequency, typically resulting in use of five- to thirty-minute returns.[2]

Much higher-frequency data are usually available, however, so reducing the sampling frequency to insure against MSN potentially discards valuable information. To utilize all information, authors such as Bandi and Russell (2005) and Zhang, Mykland and Aït-Sahalia (2005) develop MSN-robust realized volatilities that use returns sampled at the highest available frequency, under an assumption of independence of the latent price and the MSN.

The independence assumption is potentially problematic, however, because MSN is the difference between observed and latent prices, so that a negative correlation between the latent price and the MSN would be intuitively expected, unless the observed price adjusts *instantaneously* to changes in latent price. Hence a new wave of research allows for correlation between observed and latent prices while continuing to use returns sampled at the highest available frequency, as for example in Aït-Sahalia, Mykland and Zhang (2005), Hansen and Lunde (2006), Barndorff-Nielsen, Hansen, Lunde and Shephard (2006a), Barndorff-Nielsen, Hansen, Lunde and Shephard (2006b) and Zhang (2006).

Motivated by this important recent financial econometric literature, in this paper we explore the financial *economics* of the relationship between latent price and MSN. In par-

[1] Several surveys are now available, ranging from the comparatively theoretical treatments of Andersen, Bollerslev and Diebold (2005) and Barndorff-Nielsen and Shephard (2007) to the applied perspective of Andersen, Bollerslev, Christoffersen and Diebold (2006).

[2] The volatility signature plot shows average daily realized volatility as a function of sampling frequency; when it begins to shoot upward, sampling is too frequent.

ticular, we characterize the cross-correlation structure between latent price and MSN, contemporaneously and dynamically, in several leading environments, including those of Roll (1984), Hasbrouck (2002), Glosten and Milgrom (1985), Kyle (1985) and Easley and O'Hara (1992).[3]

The payoff is several-fold. Among other things, our analysis enables us to assess the likely validity of the independence assumption, to offer explanations of observed cross-correlation patterns, to predict the existence of as-yet undiscovered patterns, and to suggest improved volatility estimation methods.

We proceed as follows. In section 2 we provide an overview of various market microstructure models and introduce our general framework, which nests a variety of such models, and we provide a generic (model-free) result on the nature of correlation between latent price and MSN. In sections 3 and 4 we provide a detailed analysis of models of private information. In section 3 we treat models with quickly-vanishing value of private information, and in section 4 we treat dynamic models of market maker learning, with slowly-vanishing value of private information. In section 5 we discuss the implications of our findings for the estimation of realized volatility, and in section 6 we conclude.

2 The Basic Framework

2.1 Price and Noise Processes

Let p_t^* denote the (natural logarithm of the) latent price of some asset in period t, and p_t the (natural logarithm of the) observed market price in period t.

Assumption 1 *The latent price*[4] *process follows a random walk with drift and ARCH(1) innovations.*[5]

$$p_t^* = p_{t-1}^* + \mu + \sqrt{h_t}\varepsilon_t \tag{1}$$

$$h_t = (1-\alpha)\sigma^2 + \alpha h_{t-1}\varepsilon_{t-1}^2 \tag{2}$$

$$\varepsilon_t \overset{iid}{\sim} (0,1). \tag{3}$$

[3]For insightful surveys of the key models, see O'Hara (1995) and Hasbrouck (2007).

[4]We interpret the latent price p_t^* as the price that an informed trader – more precisely the best-informed trader in the market – considers as the "fair" price.

[5]We could of course allow for more general forms of conditional heteroskedasticy than ARCH(1), which would be more tedious but otherwise straightforward.

The somewhat nonstandard notation in (1) – (3) is convenient in subsequent calculations, because it accounts for the conditional variance, h_t, and the innovation, ε_t, by two independent variables, and because it explicitly contains the unconditional variance, σ^2. Substituting $\nu_t = \sqrt{h_t}\varepsilon_t$ and $\omega = (1-\alpha)\sigma^2$ maps (1) – (3) into the standard ARCH(1) process

$$p_t^* = p_{t-1}^* + \mu + \nu_t \tag{4}$$

$$h_t = \omega + \alpha\nu_{t-1}^2 \tag{5}$$

$$\nu_t \sim WN(0, h_t). \tag{6}$$

Assumption 2 *The probability density function $\breve{f}(\varepsilon_t)$ is symmetric around its zero mean, monotonically increasing on $]-\infty; 0]$ and monotonically decreasing on $[0; \infty[$.*

Assumption 3 *The drift μ, the ARCH parameter $0 \leq \alpha < 1$ and $\breve{f}(\varepsilon_t)$ are public knowledge.*

We define *latent returns* as latent price changes net of drift, i.e.

$$\Delta p_t^* \equiv p_t^* - p_{t-1}^* - \mu = \sqrt{h_t}\varepsilon_t. \tag{7}$$

The unconditional variance of latent returns is therefore $Var(\Delta p_t^*) = E(h_t) = \sigma^2$. Analogously, the market returns, i.e. demeaned changes of the market price p_t, are

$$\Delta p_t \equiv p_t - p_{t-1} - \mu', \tag{8}$$

where μ' is the drift in market price. The effect of μ' has to equal the effect of μ in the long run, because otherwise the market price would diverge from the latent price over time. We rule out bubbles and set $\mu' = \mu$.

Although the latent price is exogenous, the market price is the outcome of the market participants' optimizing behavior. In general, the market price depends on the information available about the latent price. Let Ω_t denote all information available at time t. We assume perfect memory, so that $\Omega_i \subset \Omega_t$, $\forall i \leq t$. Further, we assume that given the information set Ω_t the market participants' optimizing behavior determines a unique, market price $p(\Omega_t)$, with corresponding market return $\Delta p(\Omega_t, \Omega_{t-1})$.

Microstructure *noise* (MSN) is the difference between the observed return and the latent return,[6] defined as

$$\Delta u_t \equiv \Delta p_t - \Delta p_t^*. \tag{9}$$

In the presence of MSN, the realized variance is generally a biased estimate of the variance of the latent return, because in the expression:

$$Var(\Delta p_t) = Var(\Delta p_t^* + \Delta u_t) = \sigma^2 + Var(\Delta u_t) + 2Cov(\Delta p_t^*, \Delta u_t), \tag{10}$$

the last two terms may be nonzero; see, for example, facts I and II in Hansen and Lunde (2006). The second term reflects, for example, the bid-ask-bounce. The independent noise literature accounts for this term but still ignores the covariance term. The third term captures asymmetric information, reflected in a simple (contemporaneous) dependence between noise and latent returns. Serial correlation of noise, or equivalently, a cross-correlation between noise and latent returns at nonzero displacement, would additionally require the use of robust estimators for both the variance and the covariance term. In this paper we determine what correlation and serial correlation market microstructure theory predicts, and how market microstructure theory can be useful for obtaining improved estimates of integrated variance.

2.2 Market Microstructure

In this section we provide a general setup, which contains most market microstructure models as special cases. As we will see, the key determinants of the shape of the cross-correlation function between latent returns and MSN are information and market structure.

Information matters in two ways: First, via its information content, and second, via the length of the time period while it is not publicly known and the length of the time period while it is valid. In this paper we discuss two polar cases: The case of short-lived private information, where all private information becomes public information after one period, and the case of persistent private information, where private information is revealed only after many periods.

Market structure determines the price updating rule, i.e. how, and how quickly, market prices respond to new information. Of particular importance is whether the market maker can quote prices dependent on the direction of trade, i.e. whether he can charge a spread,

[6] We assume throughout that market prices p_t adjust sufficiently fast such that the noise process Δu_t is covariance stationary.

because a spread lets prices react instantaneously.

We analyze limit-order markets, populated by informed and uninformed traders. There are many market makers[7] which are in perfect competition with each other, and which serve as counterparty to all trades. The timing of information and actions in any given period, t, which is infinitely often repeated, is as follows:

1. p_t^* changes randomly.

2. The market maker observes Ω_t which contains all public information at time t, at least previous transaction prices and trades, and the conditional variance up to the current period, i.e. $\{p_i, q_i, h_i, h_t\} \in \Omega_t \ \forall i < t$, where $q_i \in \{-1, 0, 1\}$ is an indicator of the direction of trade in period t.[8] Often Ω_t contains more information, in particular the previous latent price , i.e. $\{p_{t-1}^*\} \in \Omega_t$. Further, Ω_t may contain additional information about the *current* latent price, Δp_t^*, denoted by $\tilde{\Omega}_t$, for example, $\tilde{\Omega}_t = \{\text{sgn}(\varepsilon_t)\}$.

3. The market maker quotes a pricing scheme for period t, i.e. a mid price $p_t^{mid} > 0$ and a spread $2s_t \geq 0$, which may be zero. The market maker is bound to this quoted pricing scheme during period t up to an exogenously given commitment limit.[9]

4. Informed traders observe the latent price p_t^* and the market maker pricing scheme $\{p_t^{mid}, s_t\}$. If based on their private knowledge $p_t^* > p_t^{ask} \equiv p_t^{mid} + s_t$, then they demand to buy an infinite amount, whereas if $p_t^* < p_t^{bid} \equiv p_t^{mid} - s_t$, they try to (short-) sell an infinite amount. However, market makers fill the demand only up to their commitment limit. If a transaction takes place, the transaction price is $p_t = p_t^{ask}$ or $p_t = p_t^{bid}$, respectively. If neither buy nor sell is profitable, then informed traders don't trade.

5. Finally, uninformed traders trade randomly. For these traders buying at p_t^{ask} and selling at p_t^{bid} has equal probability, which allows market makers to earn the spread without risk. Denote this constant income per period by $\pi = \pi(s_t)$. Uninformed traders are the only source of revenue of the market maker. Depending on the model, the share of uninformed traders in periods of informed trading is either zero, exogenously given or random.

[7] At least there needs to be one market maker and many *potential* competitors.
[8] $q_t = +1$ in case of buy, $q_t = -1$ in case of a sell and $q_t = 0$ if no trade occurs.
[9] Throughout this paper we will assume that this limit is transacting one unit at the quoted price.

6. If private information is valid for only one period, then p_t^* becomes public information, i.e. $\{p_t^*\} \in \Omega_{t+1}$ and the market continues with step 1. If private information is persistent, no further information is revealed at this moment and the market continues with step 2.

Having detailed the market microstructure, we now describe the behavior of the market maker.

2.3 The Market Maker

Before trading occurs, the market maker has a belief about the distribution of p_t^*. Let the beliefs about p^* be summarized by the (prior) probability density function $f(p^*)$ and denote the corresponding cumulative distribution function with $F(\cdot)$. Let $\underline{p}$ and $\overline{p}$ denote the lower and upper ends of the support of p^* that the market maker has determined by previous experimentation.[10] $\breve{l}_n(\cdot)$ denotes the market maker's loss function

$$\breve{l}_n(x) = -\left|x\right|^n, \tag{11}$$

where $n \geq 1$ is the risk aversion factor. We define the market maker's per period loss as a function of disadvantageous differences between the latent price and the transaction price, which occur in the periods of informed trading. In periods without any informed trading the market maker's loss is zero.[11] When the distribution of p_t^* is expected to have a density $f(\cdot)$ and support $[\underline{p}, \overline{p}]$, the expected loss when the price is set at p_t is[12]

$$\begin{aligned} L_n(\underline{p}, \overline{p}, p_t) &= L_n\left(\underline{p}, \overline{p}, p_t, F(\cdot)\right) \\ &= E\left(l_n(p_t - p_t^*)\right) \\ &= -\int_{\underline{p}}^{\overline{p}} \left|p_t - p_t^*\right|^n f(p_t^*) dp_t^*. \end{aligned} \tag{12}$$

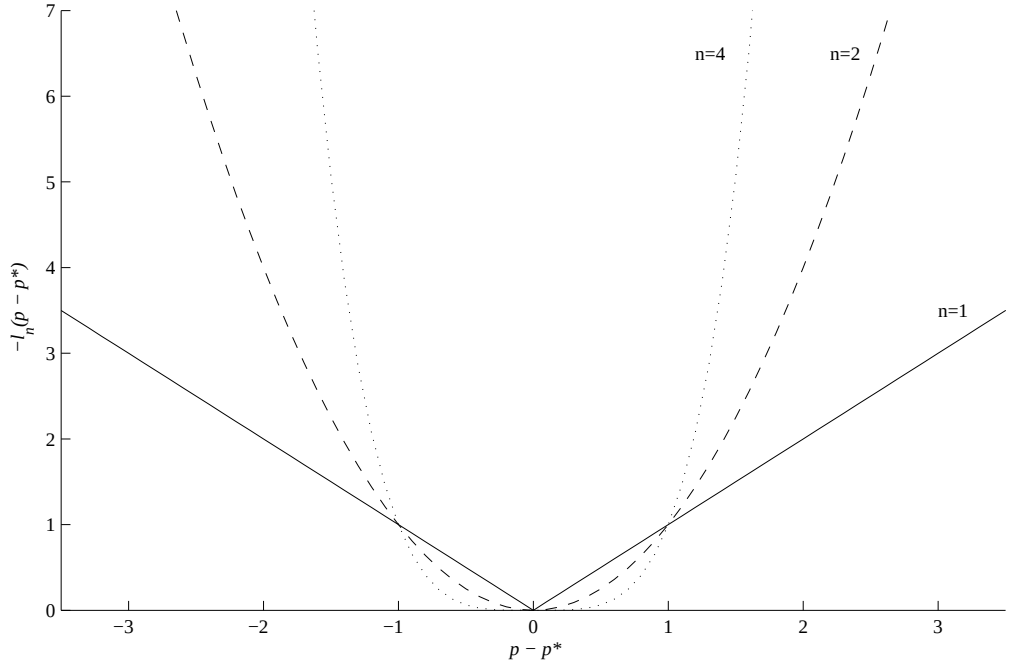
[10] In the first period, either $\underline{p}$ and $\overline{p}$ are known, or the market maker must set them to $\underline{p} = -\infty$ and $\overline{p} = \infty$.

[11] In Figure 1 we show the loss function (11) for various degrees of risk aversion. More comprehensive and realistic loss functions are possible, of course. For example, the loss function may be defined over *all* market maker income per period, not just over *deviations* from the income from uninformed trading. However, this would add extra notation without changing the effect of risk aversion on market maker behavior.

[12] $L_n(\underline{p}, \overline{p}, p_t)^{1/n}$ is related to the ℓ_n metric. However, it differs in that it is reweighted, and sums over infinitely many elements. In particular, for $n \to \infty$ we have

$$L_{n\to\infty}(\underline{p}, \overline{p}, p_t)^{1/n} = -sup\left\{\left|p_t - p_t^*\right|, p_t^* \in \left\{p | \underline{p} \leq p \leq \overline{p}, f(p) > 0\right\}\right\} = -sup\left\{\left|p_t - \overline{p}\right|, \left|p_t - \underline{p}\right|\right\}.$$

Figure 1: Market maker loss functions



The figure shows the loss function $-\breve{l}_n(x) = |p - p^*|^n$ for various degrees of risk aversion, n.

The expected loss, $E\left(l_n(p_t - p_t^*)\right)$, depends crucially on the support of p_t^*, i.e. on $\underline{p}$ and $\overline{p}$, and on the risk aversion parameter n. A well-known result is that the optimal choice for a risk neutral market maker ($n = 1$) is to set p_t equal to the median of $F(\cdot)$, and for a modestly risk averse market maker ($n = 2$) to the mean. Under extreme risk aversion ($n \to \infty$) the expected loss is minimized at the price in the middle of the support of $f(\cdot)$, i.e. $p_t = \frac{\underline{p}+\overline{p}}{2}$.[13] If $f(\cdot)$ is unbounded on one side, this p_t is infinite.

Between any two changes in the latent price the market maker chooses the pricing scheme p_t^{bid} and p_t^{ask} such that his discounted loss-adjusted profit

$$\Pi(\underline{p}, \overline{p}, t, F) = E\left[(1-\delta)\sum_{i=0}^{\infty}\delta^i \pi(s_{t+i})\right] + V_t(\underline{p}_t, \overline{p}_t, F_t) \tag{13}$$

is maximized. δ denotes the market maker's discount factor, and $V(\cdot)$ is the total expected loss from trades with informed traders.

Assumption 4 *Perfect competition among market makers implies that the market maker earns zero expected profit on each transaction, which pins down the spread* $2s_t \equiv p_t^{ask} - p_t^{bid} =$

[13] For $n \to \infty$ (12) degenerates to the sup norm. $L_{n\to\infty}(\underline{p}, \overline{p}, p_t)^{1/n}$ from footnote 12 is maximized by setting $p_t = \frac{\underline{p}+\overline{p}}{2}$. Thus, by monotonicity, also $L_{n\to\infty}(l, u, p_t)$ is maximized at $p_t = \frac{\underline{p}+\overline{p}}{2}$. See appendix C.

$2s(h_t)$, *and the market makers' revenue* $\pi(s_t)$ *from transactions with uninformed traders. The individual market maker takes this spread as given.*

Under assumption 4 the problem in (13) reduces to minimizing his expected loss from trades with informed traders, which can be written in recursive form as

$$\begin{aligned} V_t(\underline{p}_t, \overline{p}_t, F_t) &= \sum_{\tilde{\Omega}_{t+1}} P(\tilde{\Omega}_{t+1}) \max_{p^{bid}, p^{ask}} \Big[L_n(\underline{p}_{t+1}, p^{bid}, p^{bid}, \tilde{F}_{t+1}) \\ &+ \delta V_{t+1}(\underline{p}_{t+1}, p^{bid}, \tilde{F}_{t+1}|_{\underline{p}_{t+1}}^{p^{bid}}) \left(\tilde{F}_{t+1}(p^{bid}) - \tilde{F}_{t+1}(\underline{p}_{t+1}) \right) \\ &+ \delta V_{t+1}(p^{bid}, p^{ask}, \tilde{F}_{t+1}|_{p^{bid}}^{p^{ask}}) \left(\tilde{F}_{t+1}(p^{ask}) - \tilde{F}_{t+1}(p^{bid}) \right) \\ &+ \delta V_{t+1}(p^{ask}, \overline{p}_{t+1}, \tilde{F}_{t+1}|_{p^{ask}}^{\overline{p}_{t+1}}) \left(\tilde{F}_{t+1}(\overline{p}_{t+1}) - \tilde{F}_{t+1}(p^{ask}) \right) \\ &+ L_n(p^{ask}, \overline{p}_{t+1}, p^{ask}, \tilde{F}_{t+1}) \Big], \end{aligned} \tag{14}$$

where $\tilde{F}_{t+1}$ is the Bayesian update of F_t using information $\tilde{\Omega}_t$, $\tilde{F}(x)|_{x_1}^{x_2}$ is the distribution $\tilde{F}$ of x conditional on $x \in [x_1, x_2]$, $P(\tilde{\Omega})$ is the probability that the market maker observes the signal $\tilde{\Omega}$, and $p^{bid} \equiv p_{t+1}^{bid}(\underline{p}_{t+1}, \overline{p}_{t+1}, \tilde{F}_{t+1})$ and $p^{ask} \equiv p_{t+1}^{ask}(\underline{p}_{t+1}, \overline{p}_{t+1}, \tilde{F}_{t+1})$.

If there is no signal $\tilde{\Omega}$, and the spread is constant during the interval between two latent price changes, then the market maker's only choice variable is the location of the spread interval, p^{mid}, and the problem becomes independent of time. (14) simplifies to

$$\begin{aligned} V(\underline{p}, \overline{p}, F) = \max_{p^{mid}} & \Big[L_n\left(\underline{p}, p^{mid} - s, p^{mid} - s, F\right) \\ &+ \delta V\left(\underline{p}, p^{mid} - s, F|_{\underline{p}}^{p^{mid}-s}\right) \left(F\left(p^{mid} - s\right) - F\left(\underline{p}\right) \right) \\ &+ \delta V(p^{mid} - s, p^{mid} + s, F|_{p^{mid}-s}^{p^{mid}+s}) \left(F(p^{mid} + s) - F(p^{mid} - s) \right) \\ &+ \delta V\left(p^{mid} + s, \overline{p}, F|_{p^{mid}+s}^{\overline{p}}\right) \left(F\left(\overline{p}\right) - F\left(p^{mid} + s\right) \right) \\ &+ L_n\left(p^{mid} + s, \overline{p}, p^{mid} + s, F\right) \Big], \end{aligned} \tag{15}$$

where $p^{mid} \equiv p^{mid}(\underline{p}, \overline{p}, F)$.

The recursive problem (14) encompasses most cases that we discuss in this paper. Unfortunately, (14) and even (15) are hard to solve – in general the policy functions $p^{bid}(\cdot)$ and $p^{ask}(\cdot)$ are not available in closed form.[14]

In the following sections we look at specializations of the general setup (14) and examine

[14]For characterizations of the general solution see Aghion, Bolton, Harris and Jullien (1991) and Aghion, Espinosa and Jullien (1993).

the effect of various model setups on the cross-correlation function. We begin in section 3 with the one-period case $\delta = 0$, a case where private information becomes public, and worthless, after only one period. In contrast, in section 4 ($\delta \neq 0$) private information never expires, but also is not revealed until after many periods.

2.4 Cross-Correlations Between Latent Price and MSN

The focus of this paper is the *cross-correlation* between latent returns and noise contemporaneously and at all displacements.[15] Since latent returns, Δp_t^*, defined in (7) follow an exogenous process, this cross-correlation is entirely determined by the market return, $\Delta p_t(\Omega_t, \Omega_{t-1})$.

Proposition 1 ***(General cross-correlations)*** *Under assumption 1 the contemporaneous cross-correlation is positive only if the absolute market return in expectation exceeds the absolute expected latent return.*

$$Corr(\Delta p_t^*, \Delta u_t) > 0 \Leftrightarrow E(\Delta p_t(\Omega_t, \Omega_{t-1})\Delta p_t^*) > Var(\Delta p_t^*) \tag{16}$$

The cross-correlation at displacement $\tau \geq 1$ is positive if and only if the market price moves in expectation in the same direction as the latent price $\tau \geq 1$ periods before, i.e.

$$Corr(\Delta p_{t-\tau}^*, \Delta u_t) > 0 \Leftrightarrow E(\Delta p_t(\Omega_t, \Omega_{t-1})\Delta p_{t-\tau}^*) > 0. \tag{17}$$

If assumption 2 holds and $\Omega_t = \{\text{sgn}(\varepsilon_t), p_{t-1}^*, h_t\}$, then $\left|p_t - p_{t-1}^*\right| = R(h_t) > 0$ and

$$Corr(\Delta p_t^*, \Delta u_t) > 0 \Leftrightarrow E\left(R(h_t)\right) > \frac{Var(\Delta p_t^*)}{E\left(|\Delta p_t^*|\right)}. \tag{18}$$

Proof: See appendix B.

The importance of Proposition 1 stems from its generality. Without referring to any specific model of market participants' behavior, it nevertheless isolates the conditions on $\Delta p_t(\Omega_t, \Omega_{t-1})$ that determine the cross-correlation pattern.

[15] Throughout this paper, we refer to the "cross-correlation between latent returns and noise" simply as the "cross-correlation".

3 One-Period Private Information

Here we consider markets where p_t^* becomes public information at the end of each period, i.e. where p_{t-1}^* is known when the market maker decides on p_t. His recursive problem (14) therefore reduces to a sequence of single period ($\delta = 0$) problems. This restriction alone allows us to partially characterize the cross-correlation function at displacements $\tau \geq 1$. Because $E(\Delta p_{t-\tau}^* \Delta p_t^*) = 0 \ \forall \tau \geq 1$ we can write

$$\begin{aligned} Corr(\Delta p_{t-\tau}^*, \Delta u_t) &= -Corr(\Delta p_{t-\tau}^*, \Delta p_t^*) + Corr(\Delta p_{t-\tau}^*, \Delta p_t) \\ &= Corr(\Delta p_{t-\tau}^*, \Delta p_t), \end{aligned} \tag{19}$$

$\forall \tau \geq 1$. Because p_{t-1}^* is known at the beginning of time t, the market price in period t is p_{t-1}^* adjusted by the market maker's reaction $R(\cdot)$ to information $\tilde{\Omega}_t$ about p_t^*, i.e. $p_t = p_{t-1}^* + R(\tilde{\Omega}_t, h_t)$. Because $\tilde{\Omega}_t$ is unrelated to past latent price changes and $Corr(\Delta p_{t-1}^*, h_t) = 0$, (19) becomes for displacement $\tau = 1$

$$\begin{aligned} &Corr(\Delta p_{t-1}^*, \Delta p_t) \\ &= -Corr(\Delta p_{t-1}^*, \Delta u_{t-1}) + Corr(\Delta p_{t-1}^*, p_t - p_{t-2} - p_{t-1}^* + p_{t-2}^*) \\ &= -Corr(\Delta p_t^*, \Delta u_t) + Corr(\Delta p_{t-1}^*, p_{t-1}^* + R(\tilde{\Omega}_t, h_t) - p_{t-1}^*) \\ &= -Corr(\Delta p_t^*, \Delta u_t). \end{aligned} \tag{20}$$

Likewise, for displacements $\tau > 1$

$$\begin{aligned} &Corr(\Delta p_{t-\tau}^*, \Delta p_t) \\ &= -Corr(\Delta p_{t-\tau}^*, \Delta p_{t-1}^* + R(\tilde{\Omega}_t, h_t) - R(\tilde{\Omega}_{t-1}, h_{t-1})) = 0. \end{aligned} \tag{21}$$

From (19), (20) and (21) we conclude that in models of one-period private information the cross-correlation at displacement one has the opposite sign and same absolute value as the contemporaneous cross-correlation. Further, all cross-correlations at displacements larger than one are zero. In order to characterize the contemporaneous cross-correlation, we now turn to specific models.

3.1 Basic Bid-Ask Model Without Market Maker Information and Without Market Maker Optimization

We start with a simple bid-ask model (Roll 1984) with an exogenous latent price process (1). This latent price process determines the (endogenous) trade process. At the beginning of period t the market maker knows the previous period's latent price, p^*_{t-1}, but has no information about the latent price change, Δp^*_t, i.e. he has no extra information, $\tilde{\Omega}_t = \{\}$. By symmetry of $\breve{f}$ (assumption 2) his optimal choice of p^{mid}_t is therefore the known latent price of the previous period plus the drift

$$p^{mid}_t = p^*_{t-1} + \mu. \tag{22}$$

The bid and ask prices, p^{bid}_t and p^{ask}_t, follow

$$p^{tra}_t = p^{mid}_t + s_t q_t = p^*_{t-1} + \mu + s_t q_t, \tag{23}$$

where

$$p^{tra}_t = \begin{cases} p^{bid}_t & if \quad q_t = -1 \\ p^{ask}_t & if \quad q_t = +1 \end{cases},$$

and where q_t is a direction of trade indicator. We assume that ex ante buys ($q_t = +1$) and sells ($q_t = -1$) are equally likely, so that $E(q_t) = 0$ and $E(q_t|h_t) = 0$. Because p^*_t becomes public after trading in period t, all trades are serially uncorrelated, i.e. $E(q_t|q_{t-1}) = 0$. We therefore have

Proposition 2 ***(Cross-correlation with endogenous trading)***

$$Corr(\Delta p^*_t, \Delta u^{tra}_t) = \frac{1}{\sqrt{2}} \frac{E\left(s_t \sqrt{h_t} \varepsilon_t q_t\right) - \sigma^2}{\sigma \sqrt{E(s_t^2) + \sigma^2 - 2E(s_t \sqrt{h_t} \varepsilon_t q_t)}}, \tag{24}$$

$$Corr(\Delta p^*_{t-1}, \Delta u^{tra}_t) = -Corr(\Delta p^*_t, \Delta u^{tra}_t). \tag{25}$$

Proof: We have

$$\Delta p^*_t = \sqrt{h_t} \varepsilon_t \tag{26}$$

$$\Delta u^{tra}_t = s_t q_t - s_{t-1} q_{t-1} + \sqrt{h_{t-1}} \varepsilon_{t-1} - \sqrt{h_t} \varepsilon_t \tag{27}$$

$$\Delta u^{mid}_t = \sqrt{h_{t-1}} \varepsilon_{t-1} - \sqrt{h_t} \varepsilon_t \tag{28}$$

and

$$
\begin{aligned}
Var(\Delta p_t^*) &= \sigma^2 && (29)\\
Var(\Delta u_t^{tra}) &= 2E(s_t^2) + 2\sigma^2 - 4E(s_t\sqrt{h_t}\varepsilon_t q_t) && (30)\\
Var(\Delta u_t^{mid}) &= 2\sigma^2. && (31)
\end{aligned}
$$

This implies for the covariance of transaction prices

$$
\begin{aligned}
Cov(\Delta p_t^*, \Delta u_t^{tra}) &= E\left(\sqrt{h_t}\varepsilon_t\left(s_t q_t - s_{t-1}q_{t-1} + \sqrt{h_{t-1}}\varepsilon_{t-1} - \sqrt{h_t}\varepsilon_t\right)\right)\\
&= E\left(s_t\sqrt{h_t}\varepsilon_t q_t\right) - \sigma^2 && (32)
\end{aligned}
$$

and

$$
\begin{aligned}
Cov(\Delta p_{t-1}^*, \Delta u_t^{tra}) &= E\left(\sqrt{h_{t-1}}\varepsilon_{t-1}\left(s_t q_t - s_{t-1}q_{t-1} + \sqrt{h_{t-1}}\varepsilon_{t-1} - \sqrt{h_t}\varepsilon_t\right)\right)\\
&= -E\left(s_{t-1}\sqrt{h_{t-1}}\varepsilon_{t-1}q_{t-1}\right) + \sigma^2\\
&= -E\left(s_t\sqrt{h_t}\varepsilon_t q_t\right) + \sigma^2 = -Cov(\Delta p_t^*, \Delta u_t^{tra}), && (33)
\end{aligned}
$$

where we have used $E(q_t|s_t, \sqrt{h_{t-1}}, \varepsilon_{t-1}) = 0$, $E(\varepsilon_t|s_{t-1}, \sqrt{h_t}, q_{t-1}) = 0$, $E(\varepsilon_t|\varepsilon_{t-1}) = 0$ and the independence of h_t and ε_t. Using this we immediately obtain proposition 2. *Q.E.D.*

As the following proposition shows, the contemporaneous cross-correlation (24) can be bounded from above and below.

Proposition 3 ***(Bounds of contemporaneous cross-correlation under endogenous trading)***

$$
-\frac{1}{\sqrt{2}} \leq Corr(\Delta p_t^*, \Delta u_t^{tra}) \leq 0. \qquad (34)
$$

Proof: Negativity can be seen as follows. For uninformed traders, which trade randomly ($E(q_t|s_t\sqrt{(}h_t)\varepsilon_t) = 0$), we have $E(s_t\sqrt{h_t}\varepsilon_t q_t) = 0$. In contrast, informed traders buy ($q_t = +1$) only when $\sqrt{h_t}\varepsilon_t > s_t$ and sell ($q_t = -1$) only when $\sqrt{h_t}\varepsilon_t < -s_t$. Thus in a market of only informed trades $\sqrt{h_t}\varepsilon_t q_t > s_t \geq 0\ \forall t$. Therefore we can write

$$
\sigma^2 = E(h_t\varepsilon_t^2) = E(h_t\varepsilon_t^2 q_t^2) > E(s_t\sqrt{h_t}\varepsilon_t q_t) > E(s_t^2) > 0, \qquad (35)
$$

so in particular $\sigma^2 > E(s_t\sqrt{h_t}\varepsilon_t q_t)$. Combining informed and uninformed trades we have

$$\sigma^2 \geq E(s_t\sqrt{h_t}\varepsilon_t q_t), \tag{36}$$

which implies that the contemporaneous cross-correlation (24) is negative.

Further, (24) is bounded from below by $-1/\sqrt{2}$, which we prove by contradiction. Suppose this was not the case, then from (24) $\exists s(\cdot), \alpha, f(\cdot)$ such that

$$E\left(s_t\sqrt{h_t}q_t\varepsilon_t\right) - \sigma^2 < -\sigma\sqrt{E(s_t^2) + \sigma^2 - 2E(s_t\sqrt{h_t}q_t\varepsilon_t)}. \tag{37}$$

Squaring both sides and simplifying gives the condition

$$\left[E\left(s_t\sqrt{h_t}q_t\varepsilon_t\right)\right]^2 > \sigma^2 E(s_t^2). \tag{38}$$

Using (3) we obtain the condition

$$Cov\left(s_t^2, h_t\right) > Var\left(s_t\sqrt{h_t}q_t\varepsilon_t\right). \tag{39}$$

Let $X_t \equiv s_t\sqrt{h_t}$ be an arbitrary sequence. Then (39) can be written as $Cov\left(s_t^2, \frac{X_t^2}{s_t^2}\right) > Var\left(X_t q_t \varepsilon_t\right)$. Given X_t, $s_t = \sqrt{h_t}$ maximizes the lefthand side, without affecting the right-hand side. To see this, note that by the definition of the covariance

$$Cov\left(s_t^2, \frac{X_t^2}{s_t^2}\right) \leq \sqrt{Var\left(s_t^2\right) Var\left(\frac{X_t^2}{s_t^2}\right)}. \tag{40}$$

For our choice $s_t = \sqrt{h_t}$ (40) holds with equality, so we have indeed found the s_t which maximizes the covariance. Hence for (39) to be satisfied, it is necessary that

$$Var\left(h_t\right) > Var\left(h_t q_t \varepsilon_t\right). \tag{41}$$

Using (3) we obtain the equivalent condition

$$E\left(h_t q_t \varepsilon_t\right) > \sigma^2. \tag{42}$$

(42) cannot be satisfied, because for $s_t = \sqrt{h_t}$ the decision rule for the informed trade becomes $\varepsilon_t > 1$ for a buy and $\varepsilon_t < 1$, i.e. independent of h_t. Therefore $Cov(h_t, q_t\varepsilon_t) = 0$ and

thus

$$\sigma^2 = E(h_t) > E(h_t)E(|\varepsilon_t|) > E(h_t)E(q_t\varepsilon_t) = E(h_t q_t \varepsilon_t) - Cov(h_t, q_t\varepsilon_t) = E(h_t q_t \varepsilon_t), \quad (43)$$

which contradicts (42). *Q.E.D.*

Note that the lower bound holds with equality for mid prices ($s = 0$).[16] The contemporaneous cross-correlation is therefore less pronounced for transaction prices than for mid prices. The contemporaneous cross-correlation for mid prices is negative, because p_t^{mid} does not react at all to the change in the latent price in the same period.[17] It differs from negative one because market prices move in adjustment to the latent price change one period earlier.

Lagged cross-correlations are

$$Corr(\Delta p_{t-1}^*, \Delta u_t^{tra}) = \frac{1}{\sqrt{2}} \frac{-E\left(s_t\sqrt{h_t}\varepsilon_t q_t\right) + \sigma^2}{\sigma\sqrt{E(s_t^2) + \sigma^2 - 2E(s_t\sqrt{h_t}\varepsilon_t q_t)}} = -Corr(\Delta p_t^*, \Delta u_t^{tra}). \quad (44)$$

Since there is no further information to be revealed, all higher order cross-correlations are zero. We summarize these results in the upper two rows of table 1.

Table 1: Cross-correlations $\rho_\tau = Corr(\Delta p_{t-\tau}^*, \Delta u_t)$ under one period private information, no market maker information ($\tilde{\Omega}_t = \{\}$)

exogenous process	price	ρ_0	ρ_1	ρ_τ $\tau > 1$
any	p^{mid}	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
price	p^{tra}	$-\frac{1}{\sqrt{2}} \leq \rho_0 < 0$	$-\rho_0$	0
trade	p^{tra}	$-\frac{1}{\sqrt{2}} \leq \rho_0 \leq \frac{1}{\sqrt{2}}$	$-\rho_0$	0

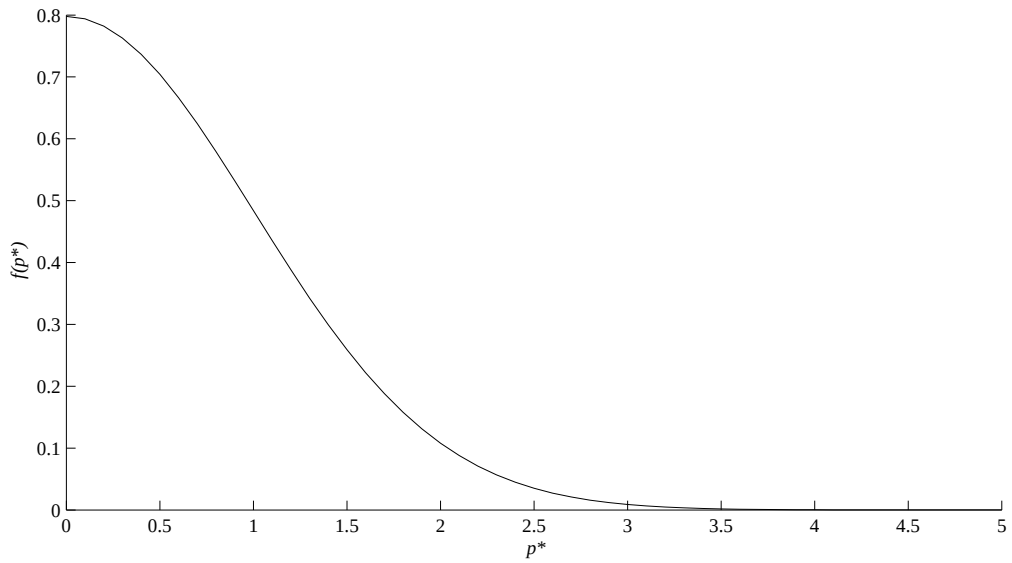
[16] $s = 0$ must also hold by assumption 4, if the market consisted only of uninformed traders.

[17] This is an instance of the price stickiness that Bandi and Russell (2006a) show to generate "mechanically" a negative contemporaneous cross-correlation.

3.2 Bid-Ask Model with Market Maker Optimization under Incomplete Information

In the previous subsection the market maker set prices without any information about the change in the latent price in period t. Under an exogenous price process we conclude that then the contemporaneous cross-correlation would have to be negative. Now suppose that the market maker observes a signal about the sign of Δp_t^*, namely $\tilde{\Omega}_t = \{\text{sgn}(\varepsilon_t)\}$, before setting his price p_t^{mid}. This enables him to change p_t^{mid} before any informed trader reacts to the latent price change. Suppose that the latent price process again follows (1). With the signal $\{\text{sgn}(\varepsilon_t)\}$ the market maker updates his prior belief $p_t^* \sim (p_{t-1}^* + \mu, h_t)$ summarized by the distribution $\breve{f}(\frac{p_t^* - p_{t-1} - \mu}{\sqrt{h_t}})$. The updated distribution $f(\cdot)$ differs from $\breve{f}(p_t^*)$ in that it is truncated from below or above at $p_t^* = p_{t-1}^* + \mu$ when $\text{sgn}(\varepsilon_t) > 0$ or $\text{sgn}(\varepsilon_t) < 0$, respectively.[18]

Figure 2: Expectation of latent price by market maker after a buy by an informed trader (normal distribution case)



The figure shows the density of the upper halfnormal distribution $f(p_t^*) = 2\varphi\left(\frac{p_t^* - p_{t-1}^* - \mu}{\sqrt{h_t}}\right)$, plotted with $p_{t-1}^* + \mu = 0$ and $h_t = 1$.

Example 1 *Suppose that without an observed signal latent price changes are normally distributed, i.e.* $\sqrt{h_t}\varepsilon_t | h_t, p_{t-1}^* \sim N(0, h_t)$. *Then, conditional on the signal* $\text{sgn}(\varepsilon_t) = +1$ *the*

[18] We assume that the market makers' beliefs make proper use of the available information, i.e. $f(\cdot)$ has a unique maximum at $p_{t-1}^* + \mu$, and is monotone on $]-\infty; p_{t-1}^* + \mu]$ and on $[p_{t-1}^* + \mu; \infty]$.

market maker's expected latent price density is the upper half-normal, i.e. $p_t^* \sim HN(p_{t-1}^* + \mu, h_t)$*, which is a normal distribution truncated from below at* $p_{t-1}^* + \mu$*. In Figure 2 we show the corresponding probability density function*

$$f(p_t^*) = 2\varphi\left(\frac{p_t^* - p_{t-1}^* - \mu}{\sqrt{h_t}}\right), \tag{45}$$

with support restricted to $[p_{t-1}^* + \mu, \infty[$*, where* φ *is the standard normal probability density function. Note that the knowledge of* $\text{sgn}(\varepsilon_t)$ *radically changes the market maker's belief. The distribution underlying his belief on the latent price is now highly skewed. It is this asymmetry that makes the price setting of the market maker dependent on his risk aversion.*

After observing this signal and the outcomes of period $t-1$, in particular p_{t-1}^*, the market maker quotes a bid and an ask price for the following period, i.e.

$$p_t^{tra} = p_{t-1}^* + \mu_t + s_t q_t + R(\text{sgn}(\varepsilon_t), h_t). \tag{46}$$

$R(\cdot)$ reflects by how much the market maker adjusts the mid price in response to the extra information $\text{sgn}(\varepsilon_t)$. $R(\cdot)$ depends on the market maker's risk aversion n.[19] The spread, s_t, is again taken as given by the market maker.

The problem of choosing $p_t^{mid} \equiv p(n)$ based on loss function (12) can be approximated by[20]

$$p(n) = \underset{x \in [\underline{p}, \overline{p}]}{\text{argmax}} - \int_{\underline{p}}^{x} (x - p^*)^n f(p^*) dp^* - \int_{x}^{\overline{p}} (p^* - x)^n f(p^*) dp^*. \tag{47}$$

[19] The extra information of the market maker disconnects the direction of the latent price change and the direction of trade. If the informed trade, then it must be that $p_t^* > p_t^{ask}$ or that $p_t^* < p_t^{bid}$. When $R(\cdot) = 0$, as in the previous subsection, the sign of the innovation, ε_t, pinned down the trading direction. For example, $\varepsilon_t > 0$ implied $p_t^* > p_t^{ask}$ in periods of informed trading. In this subsection, once the market maker observed $\text{sgn}(\varepsilon_t)$, his mid price quote, p_t^{mid}, already takes into account the expected change in p_t^*. Because his expectation of p_t^* could both be too high or too low, the sign of ε_t does not pin down the trading direction in periods of informed trading. Uninformed trades occur no matter what p_t^*, p_t^{ask} and p_t^{bid} are.

[20] This approximation is exact for $s = 0$ or, more generally, for

$$\int_{p(n)-s}^{p(n)} (p(n) - p^*)^n f(p^*) dp^* + \int_{p(n)}^{p(n)+s} (p^* - p(n))^n f(p^*) dp^* = 0.$$

For all densities $f(\cdot)$ which have all moments we can apply Leibnitz's rule and obtain the first order condition

$$\int_{\underline{p}}^{p(n)} (p(n) - p^*)^{n-1} f(p^*) dp^* - \int_{p(n)}^{\overline{p}} (p^* - p(n))^{n-1} f(p^*) dp^* = 0 \tag{48}$$

For some values of n, explicit solutions to (48) are available, which we list in proposition 4.[21]

Proposition 4 ***(Optimal Mid Price)***

$$\begin{aligned} p(1) &= \text{Median}(p_t^*) && (49) \\ p(2) &= \text{E}(p_t^*) && (50) \\ p(\infty) &= \text{Midsupport}(p_t^*). && (51) \end{aligned}$$

Proof: The first two equations are simply the well-known result that the median is the best predictor under linear (absolute) loss, whereas the mean is the best predictor under squared loss. (51) follows by rewriting (48) as a metric

$$\begin{aligned} &\lim_{n\to\infty} \left(\int_{\underline{p}}^{p(n)} (p(n) - p^*)^{n-1} f(p^*) dp^* \right)^{1/(n-1)} \\ &= \lim_{n\to\infty} \left(\int_{p(n)}^{\overline{p}} (p^* - p(n))^{n-1} f(p^*) dp^* \right)^{1/(n-1)}, \end{aligned} \tag{52}$$

which after taking the limit becomes

$$\sup_{p^* \in [\underline{p}, p(\infty)]} (p(\infty) - p^*) = \sup_{p^* \in [p(\infty), \overline{p}]} (p^* - p(\infty)) . \tag{53}$$

Hence

$$p(\infty) = \frac{\underline{p} + \overline{p}}{2}. \tag{54}$$

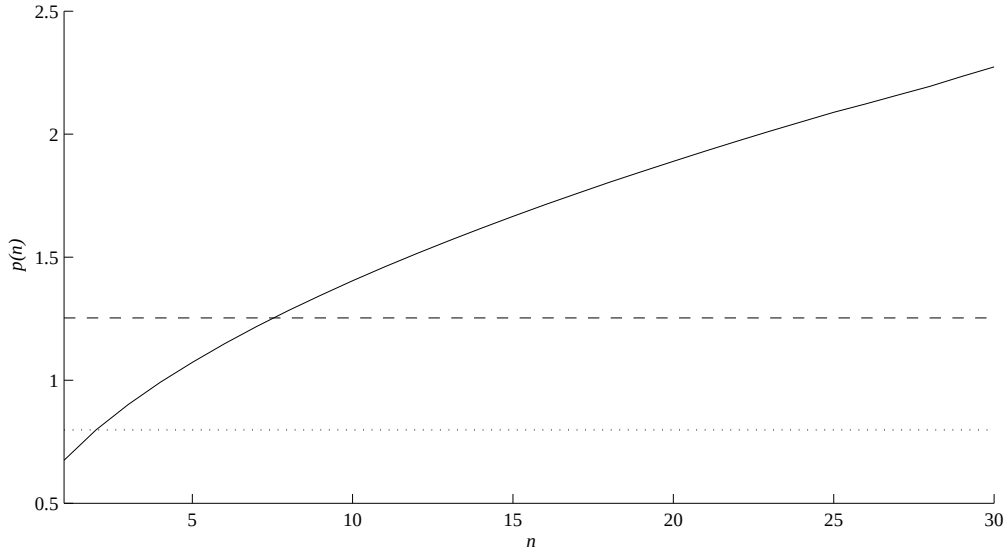
Q.E.D.

For distributions with finite support $p(\infty)$ is a finite number. Otherwise, it is positive or negative infinity for one-sided distributions or does not exist for distributions with positive

[21] We assume $n \geq 1$ throughout, because this implies realistic market maker preferences. However, (47) can be solved for any $n \geq 0$. In particular, $p(0)$ is the mode of $f(\cdot)$ (when $s = 0$) or the highest density (connected) region (when $s > 0$).

density over the entire real line. In example 1 we would get $p(\infty) = +\infty$.[22]

Figure 3: Optimal predictor $p(n)$ under a half normal distribution as a function of risk aversion n



The dotted line marks $E(\Delta p_t^*)$, the dashed line marks $\frac{Var(\Delta p_t^*)}{E(|\Delta p_t^*|)}$.

As risk aversion, n, grows, $p(n)$ moves monotonically from the median of $f(\cdot)$ to the midpoint of the support of $f(\cdot)$.[23] Figure 3 illustrates this for right-skewed distributions $f(\cdot)$ with infinite support such as the halfnormal distribution used in example 1. $p(n)$ increases in n, starting from the median for $n = 1$, monotonically without bound. If $f(\cdot)$ has finite support, $p(n)$ increases from the median monotonically toward an asymptote $p(\infty)$.

Analogously, for left-skewed distributions with infinite support, $p(n)$ decreases in n from the median monotonically without bound, and with finite support toward an asymptote $p(\infty)$. The asymptote is clearly visible in figure 5, in which we plot $p(n)$ for the triangular distribution defined on $[0, 1]$ shown in figure 4.

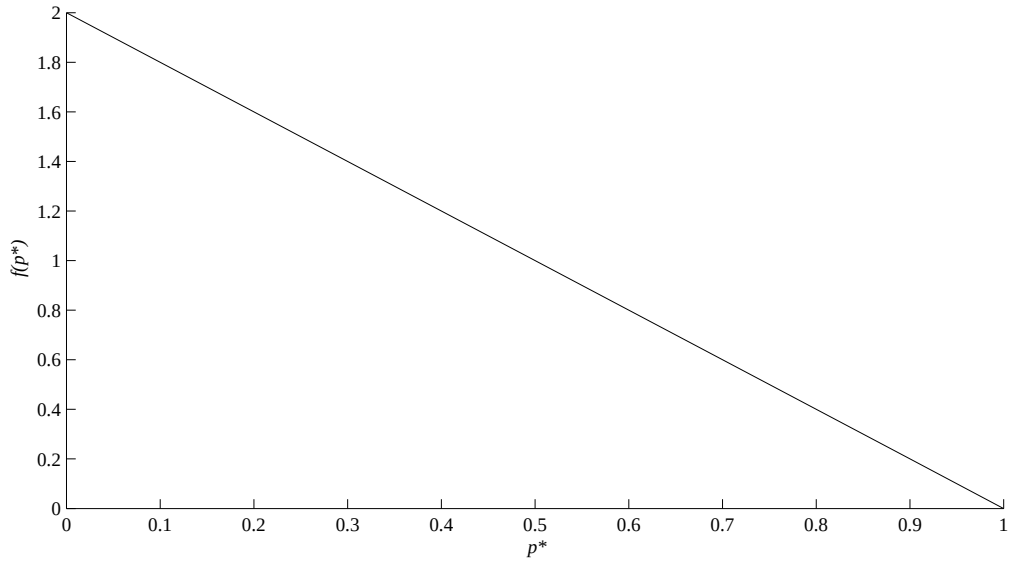
Therefore, for any $p(n) < \frac{\underline{p}+\overline{p}}{2}$ there is a risk aversion level n such that market makers will – after observing the signal $\text{sgn}(\varepsilon_t)$ – quote this price as p_t^{mid}. Hence for all distributions $f(\cdot)$ which satisfy

$$\frac{\underline{p}+\overline{p}}{2} > \frac{Var_{\check{f}}(\Delta p_t^*)}{E_{\check{f}}(|\Delta p_t^*|)} = \frac{\sigma^2}{E(\sqrt{h_t})E(|\varepsilon_t|)} \tag{55}$$

[22]For $n \notin \{1, 2, \infty\}$ no explicit solution exists, and for $n > 25$ even obtaining numerical solutions creates difficulty for non-trivial distribution functions $f(\cdot)$.

[23]See appendix C for a proof.

Figure 4: Expectation of latent price by market maker after a buy by an informed trader (tent distribution case)



The figure shows the density of the right-skewed triangular distribution $f(p_t^*) = \frac{2(\overline{p}-p_t^*)}{(\overline{p}-p_{t-1}^*-\mu)^2}$ with support $[p_{t-1}^* + \mu, \overline{p}]$ plotted with $p_{t-1}^* + \mu = 0$ and $\overline{p} = p_{t-1} + \mu + \sqrt{18h_t} = 1$.

a sufficiently large n gives

$$E\left(|R(\text{sgn}(\varepsilon_t), h_t)|\right) > \frac{Var_{\breve{f}}(\Delta p_t^*)}{E_{\breve{f}}\left(|\Delta p_t^*|\right)}, \tag{56}$$

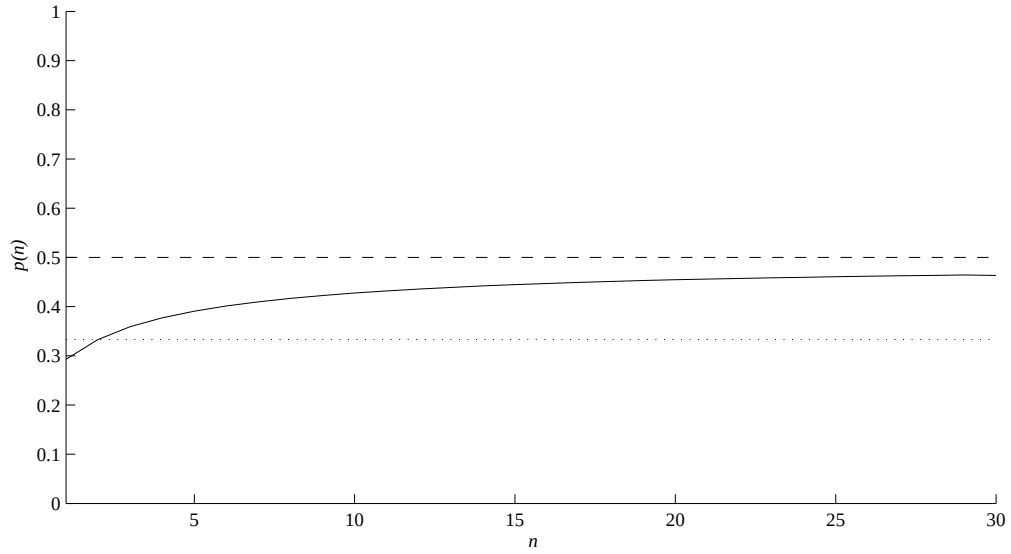
which by proposition 1 implies a positive contemporaneous cross-correlation.[24] For low risk aversion levels $\left|\Delta p_t^{mid}\right|$ is small and thus the contemporaneous cross-correlation is negative. We have therefore shown

Proposition 5 ***(Cross-correlation under endogenous trading and market maker information)*** *If $\Omega_t = \{\text{sgn}(\varepsilon_t), p_{t-1}^*, h_t\}$ and assumption 1, 2 and 3 hold, then the optimal $E\left(|R(\text{sgn}(\varepsilon_t), h_t)|\right)$ strictly increases in risk aversion, $n \geq 1$, without bound. If the distribution of innovations $\breve{f}$ induces beliefs with support $[\underline{p}, \overline{p}]$ satisfying condition (55), then $\exists n_0 > 1$ such that $\forall n > n_0 \quad Corr(\Delta p_t^*, \Delta u_t) > 0$.*

By the considerations at the beginning of section 3, the cross-correlation at displacement one has the opposite sign of the contemporaneous cross-correlation, and cross-correlations

[24]Note that condition (55) is not satisfied for $\breve{F}$ following a tent distribution, which leads to a belief with triangular distribution as in figure 4. The condition is satisfied for more dispersed $\breve{F}$, such as for the normal distribution.

Figure 5: Optimal predictor $p(n)$ under triangular distribution on $[0,1]$ as a function of risk aversion n



The dotted line marks $E(\Delta p_t^*)$, the dashed line marks $\frac{Var(\Delta p_t^*)}{E(|\Delta p_t^*|)}$. The solid line is the solution to

$$p(n) = \underset{x \in [\underline{p}, \overline{p}]}{\operatorname{argmax}} - \int_{\underline{p}}^{x} (x - p^*)^n f(p^*) dp^* - \int_{x}^{\overline{p}} (p^* - x)^n f(p^*) dp^*.$$

Evaluated at $\underline{p} = 0$ and $\overline{p} = 1$ for the triangular distribution on $[0,1]$, the problem reduces to

$$p(n) = \underset{x \in [0,1]}{\operatorname{argmax}} - \int_{0}^{x} 2(1 - p^*)(x - p^*)^n dp^* - \int_{x}^{1} 2(1 - p^*)(p^* - x)^n dp^*,$$

which has the solutions $p(1) = \text{Median}(p_t^*) = 0.29$, $p(2) = \text{E}(p_t^*) = 0.33$, $\lim_{n \to \infty} \; p(n) = 0.5$.

at all higher displacements are zero.[25] We summarize these results in table 2.

Table 2: Cross-correlations $\rho_\tau = Corr(\Delta p^*_{t-\tau}, \Delta u^{mid}_t)$ under one period private information and incomplete market maker information ($\tilde{\Omega}_t = \{\text{sgn}(\varepsilon_t)\}$)

	ρ_0	ρ_1	ρ_τ, $\tau > 1$
high n, dispersed F	$\rho_0 > 0$	$\rho_1 < 0$	0
otherwise	$\rho_0 < 0$	$\rho_1 > 0$	0

Even though the contemporaneous cross-correlation can be positive for high risk aversion levels, the more realistic case is that it is negative. For the halfnormal distribution in figure 3, for example, we need $n \geq 8$; i.e. to generate a positive contemporaneous cross-correlation the risk aversion must be rather high. Clearly though, changes in risk aversion of the market maker have a fundamental impact on the cross-correlation. Hansen and Lunde (2006) note as their "Fact IV" that "the properties of the noise have changed over time." Since they base this observation on a comparison of year 2000 with year 2004 it is possible that the underlying cause is a change in risk aversion.

The link between properties of noise and risk aversion offers itself as a way to estimate the time path of risk aversion from the market price cross-correlation patterns. In stable periods with low risk aversion the contemporaneous cross-correlation is negative, but as uncertainty increases, contemporaneous cross-correlation increases as well. In periods of crisis this can lead to the extreme case of an inverted cross-correlation pattern that we have described in this section. The negative contemporaneous cross-correlation in Hansen and Lunde (2006), for example, indicates that during their sample period the risk aversion of market makers was rather low.

Note that if h_t is not observed by the market maker, then from his point of view all periods are identical. Every period the market maker gets the same type of new information ($\text{sgn}(\varepsilon_t) = 1$, or $\text{sgn}(\varepsilon_t) = -1$), thus s_t is the same in every period.

A small change in the model allows for time variation in spreads (Demsetz 1968). Based on our assumptions the information event $\text{sgn}(\varepsilon_t) = 0$ occurs with zero probability. If – contrary to the maintained assumptions – we assign nonzero probability mass to this event,

[25] Because the spread s is given to the market maker, p^{bid}_t and p^{ask}_t have the same cross-correlation pattern as p^{mid}_t.

then observing this information event ensures the market maker of no informed trading in this period. Therefore, the competitive spread in this period is zero. A subsequent $\text{sgn}(\varepsilon_t) = \pm 1$ then not only triggers a shift in p_t^{mid}, but also an increase in spread. Thus, allowing for $\text{sgn}(\varepsilon_t) = 0$ amplifies the market price reaction and increases the likelihood of a positive contemporaneous cross-correlation.

3.3 Bid-Ask Model with an Exogenous Trade Process

In the previous section informed traders acted rationally, trading only if it was profitable for them based on the observed latent price. Popular models turn this causality around: the latent price process is endogenous, determined by an exogenous trading process q_t, $q_t \in \{-1, +1\}$. With this modification the latent price equation (1) becomes

$$p_t^* = p_{t-1}^* + \mu_t + \lambda q_t + \sqrt{h_t}\varepsilon_t, \tag{57}$$

where the parameter $\lambda > 0$ captures the adverse selection effect, i.e. it measures how much private information is revealed to the market by a trade. This trading process itself may be driven by some private information about fundamentals, but also by any other rational or irrational cause. The total latent price change is thus $\Delta p_t^* = \lambda q_t + \sqrt{h_t}\varepsilon_t$.

This setup is closely related to the bid-ask model in Hasbrouck (2007). The drift, μ_t, is time-varying, which can be interpreted as publicly available information and q_t as the result of unobserved private information. ε_t is now unobserved by all market participants, $E(q_t|\varepsilon_t) = 0$ and $E(q_t|\mu_t) = 0$. Further we assume that trades are not serially correlated, i.e. $E(q_t|q_{t-1}) = 0$. At the time when s_t is determined, q_t is still unknown, thus $E(q_t s_t) = 0$. The transaction price process is given by (22) and (23). Using (57) equation (46) becomes

$$\begin{aligned} p_t^{tra} &= p_{t-1}^* + \mu_t + s_t q_t \\ &= p_t^* + (s_t - \lambda) q_t - \sqrt{h_t}\varepsilon_t, \\ p_t^{mid} &= p_t^* - \sqrt{h_t}\varepsilon_t. \end{aligned} \tag{58}$$

Typically, the adverse selection parameter, $\lambda > 0$, is assumed to be a publicly known constant. This type of model does not contain an utility maximizer, and all transaction price changes are hard-wired with the latent price and the direction of trade.

The cross-correlation pattern for transaction prices is given in the following proposition.

Proposition 6 ***(Cross-correlation in the generalized Roll model)*** *In the generalized Roll (1984) model the contemporaneous cross-correlation is*

$$Corr(\Delta p_t^*, \Delta u_t^{tra}) = \frac{\lambda(E(s_t) - \lambda) - \sigma^2}{\sqrt{2}\sqrt{\lambda^2 + \sigma^2}\sqrt{E(s_t^2) - 2\lambda E(s_t) + \lambda^2 + \sigma^2}}, \tag{59}$$

where

$$\left|Corr(\Delta p_t^*, \Delta u_t^{tra})\right| \leq \frac{1}{\sqrt{2}}. \tag{60}$$

The cross-correlation at displacement one equals

$$Corr(\Delta p_{t-1}^*, \Delta u_t^{tra}) = -Corr(\Delta p_t^*, \Delta u_t^{tra}). \tag{61}$$

All cross-correlations at higher displacements are zero.

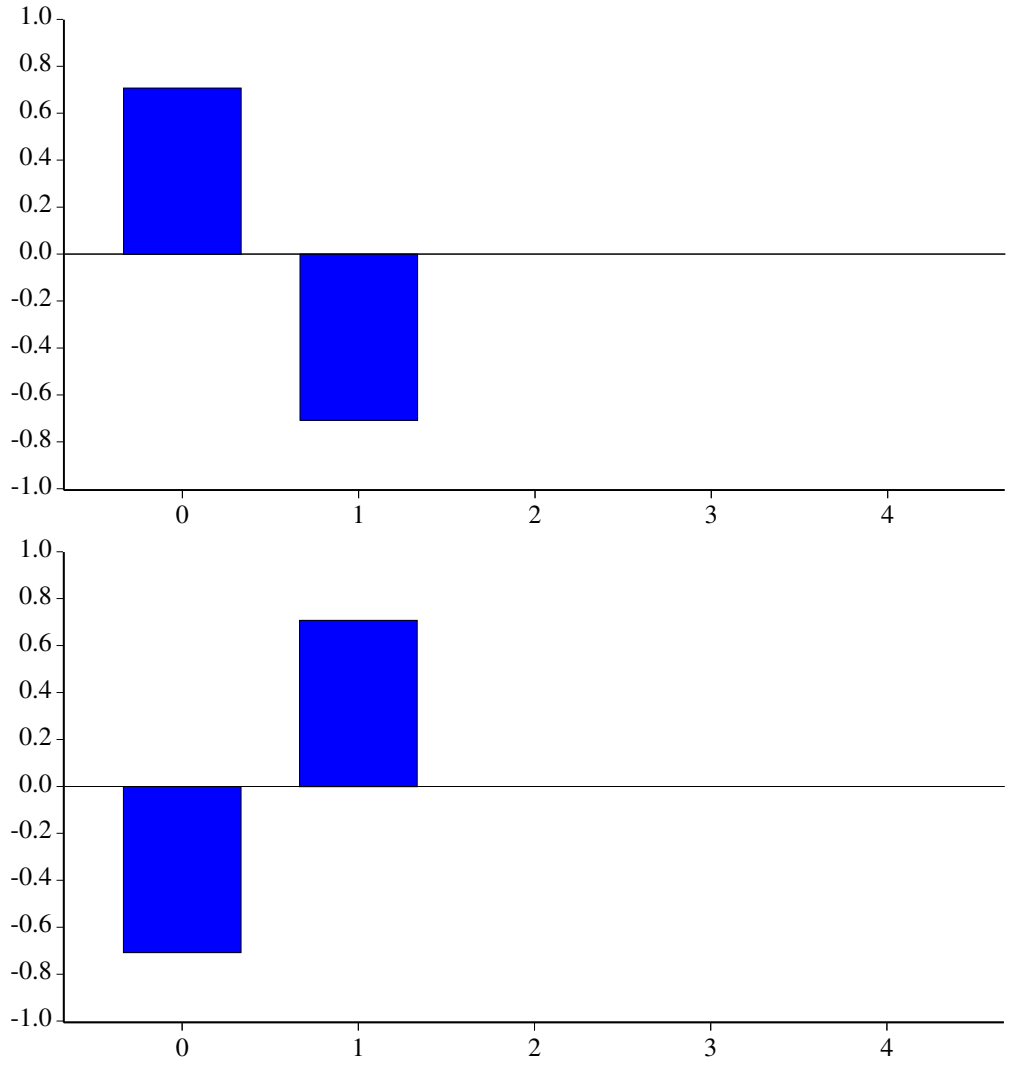
Proof: See appendix D.

The cross-correlation with transaction prices is positive if $E(s_t)$ is sufficiently larger than the adverse selection parameter λ, i.e. if the transaction price reacts more strongly to a trade than the latent price. In particular, the spread effect has to exceed the negative covariance component caused by the random price change ε_t. Note, however, that with $E(s_t) > \lambda$ the average trader in expectation incurs a loss with every transaction. As Hasbrouck (2007) suggests, this may be justified by trader's liquidity needs.

Setting $s_t = 0\ \forall t$ in proposition 6 we obtain a negative cross-correlation for the mid price, $Corr(\Delta p_t^*, \Delta u_t^{mid}) = -\frac{1}{\sqrt{2}}$. Further, without random price shocks ($\sigma = 0$) the competitive spread is constant s. Then $Corr(\Delta p_t^*, \Delta u_t^{tra}) = -\frac{\lambda(s-\lambda)}{|\lambda|\sqrt{2}|s-\lambda|} = \frac{1}{\sqrt{2}}\text{sgn}|s - \lambda|$ and all that matters about the spread is whether it is bigger or smaller than λ. If $s = \lambda$ or if $\lambda = 0$ then all cross-correlations are zero. As Diebold (2006) notes, for s sufficiently large the model predicts a positive contemporaneous cross-correlation. We summarize the results in the first and third row of table 1. The upper panel of figure 6 illustrates the cross-correlation function for a sufficiently wide spread ($s > \lambda > 0$), and the lower panel does so for a small spread ($0 \leq s < \lambda$).

The exogenous trade model takes the spread s_t, the sequence of trades q_t and the adverse selection parameter λ as exogenously given. We need therefore some additional theory to pin down the sign of (59). If λ is observable, a rational market marker could block all trade involving a loss for him by setting $s_t = \lambda\ \forall t$. This would guarantee him a profit of $\pi(\lambda) > 0$ per period, which would violate the zero profit condition, i.e. assumption (4). Therefore $s_t < \lambda$ for at least some t, which by (59) implies a negative contemporaneous

Figure 6: Cross-correlation function $\rho(\tau)$ of the transactions price in a generalized Roll-type model



The upper panel shows the typical cross-correlation pattern under a wide spread ($s > \lambda > 0$), the lower panel for a small spread ($0 \leq s < \lambda$). In both panels $\sigma = 0$.

cross-correlation.

The situation changes if λ is stochastic. Consider a market as in the previous section, but with $\lambda_t > 0$ being stochastic and, for simplicity, without any other latent price innovations, i.e. $\varepsilon_t = 0\ \forall t$, and with constant variance σ_2^2 and constant spread s. That is, (57) is replaced by

$$\begin{aligned} p_t^* &= p_{t-1}^* + \mu_t + \lambda_t q_t, \\ \lambda_t q_t &\sim (0, \sigma_2^2), \end{aligned} \tag{62}$$

where $\lambda_t q_t$ follows a symmetric probability distribution which is monotone on $]-\infty, 0]$ and $[0, \infty[$. Conditional on the trade direction q_t, the distribution of $\lambda_t q_t$ is truncated at zero, thus λ_t alone follows a highly skewed distribution. (62) resembles (1) with ε_t replaced by $\lambda_t q_t$. The latent price process (57) implies that exogenous reasons unrelated to p_t^* drive the traders' trading plans, for example liquidity needs, and that because of these reasons they are willing the trade at *any* price. At the same time, their joint actions (q_t) change the latent price by λ. Then, by the reasoning of section 3.2, there exists a minimal risk aversion level n_0 such that all $n > n_0$ generate a spread $E(s) > \frac{E(\lambda^2)}{E(|\lambda|)} = \frac{\sigma_2^2}{E(|\lambda|)}$. Thus there exists n_0 such that all $n > n_0$ generate a positive contemporaneous cross-correlation and a negative cross-correlation at displacement one. Note that unlike in section 3.2 positive contemporaneous cross-correlation obtains even though the market maker does not observe a signal.

In summary, positive contemporaneous and negative lagged cross-correlations occur for sufficiently high risk aversion (1) under an exogenous latent price process if a signal is observed, and (2) under an exogenous trade process. Thus, whereas a negative cross-correlation is not "inevitable" as Bandi and Russell (2006a) rightly wonder, a positive contemporaneous cross-correlation is indeed not likely. Various market arrangements and sampling speeds can dampen the negative cross-correlation to zero, but the sign maintains except in the two aforementioned cases.

4 Persistent Private Information

The previous section considered a sequence of one period problems. In this section we focus on the effect of multiple periods, when the latent price changes infrequently. Now private information about the latent price is no longer automatically revealed after one period. The market maker can learn this private information by setting prices and observing the

resulting trades. As the market maker "learns by experimentation"[26] over time, the value of private information of the informed trader slowly vanishes. Although there are many possible interactions of strategic actions by market participants, we will see that they all share the same cross-correlation sign pattern and differ only in the absolute value of the cross-correlation.

We assume in this section that market makers are risk neutral ($n = 1$) and limit our discussion to the mid price in order to study the learning effect in isolation. In all models in this section there is only one single latent price change at a known time from an initial publicly known level, e.g. at the beginning of the day,[27] which allows studying the effect of one latent price change in isolation. Otherwise the timing is as described in section 2.2.

In general we work with the latent price process as in (1). With $\delta \neq 0$ the recursive problem (14) is hard to solve, and in particular there are in general no closed form policy functions p_t^{bid} and p_t^{ask}. Therefore we follow the market microstructure literature by discussing interesting polar cases, which can be solved because $f(p_t^*)$ is degenerate.

4.1 Signal Certainty

We start with a dynamic model where as in section 3 the signal is always correct and strategic considerations are ruled out. As before, the market maker does not observe p_t^*, but only signals which allow him to narrow down the range of the current p_t^* level. Unlike before private information is persistent, so p_t^* is not revealed at the end of period t. The market maker observes the response of traders to his previous price quote and uses this signal to revise his quote. Because the latent price does not move after the initial jump, the market maker can use the entire sequence of signals to learn the latent price p_t^* over time. His optimization task is to quote prices that minimize his losses by learning about p_t^* as quickly as possible.

The market maker's learning speed depends on the reliability of the signal. Let us start with a situation where the signal is known to be free of noise and strategic manipulation by market participants. To learn as much as possible the market maker minimizes the length of the interval in which p_t^* may be located. In the special case of a constant spread during

[26] Aghion et al. (1991), Aghion et al. (1993)

[27] With "day" we mean the average time between two changes in the latent price, which could be a few hours, or several days. Engle and Patton (2004) and Owens and Steigerwald (2005), for example, find evidence of multiple information arrivals during a calendar day.

the interval between two latent price changes he solves (15) with $n = 1$:

$$
\begin{aligned}
V(\underline{p}, \overline{p}, F) = \max_{p^{mid}} \quad & \left[- \int_{\underline{p}}^{p^{mid}-s} (p^{mid} - s - p^*) f(p^*) dp^* \right. \\
+ \quad & \delta V \left(\underline{p}, p^{mid} - s, F|_{\underline{p}}^{p^{mid}-s} \right) \left(F\left(p^{mid} - s\right) - F\left(\underline{p}\right) \right) \\
+ \quad & \delta V(p^{mid} - s, p^{mid} + s, F|_{p^{mid}-s}^{p^{mid}+s}) \left(F(p^{mid} + s) - F(p^{mid} - s) \right) \\
+ \quad & \delta V \left(p^{mid} + s, \overline{p}, F|_{p^{mid}+s}^{\overline{p}} \right) \left(F\left(\overline{p}\right) - F\left(p^{mid} + s\right) \right) \\
- \quad & \left. \int_{p^{mid}+s}^{\overline{p}} (p^* - p^{mid} - s) f(p^*) dp^* \right]. \qquad (63)
\end{aligned}
$$

The key difference relative to the market maker problem in section 3 is that (63) is forward looking.[28]

Our simplifying assumptions force the contemporaneous cross-correlation to be negative. For $s = 0$ it is negative one by construction, because p_t cannot react instantaneously to p_t^*. Note that if we allowed for high risk aversion and a signal $\{sgn(\varepsilon_t)\}$, or for high risk aversion and a truly exogenous trading process, then just as in section 3 a positive contemporaneous cross-correlation would obtain.

Because p_t^* by assumption does not change during learning and because learning takes several periods, the contemporaneous cross-correlation is larger in absolute value than lagged cross-correlations. Likewise, the sign of cross-correlation at lag one and higher is positive, because the more the market maker learns, the closer p_t gets to p_t^*, and the more noise shrinks to zero. Compared to section 3 the absolute value of the cross-correlation at lag one is dampened toward zero, because the information is not fully revealed. This is picked up by cross-correlation at higher order lags, which are now strictly positive and which decrease toward zero as the displacement increases. We summarize these qualitative results in the first row of table 3. The key difference of persistent private information relative to quickly-vanishing private information is the slowly decaying cross-correlation function.

A special case obtains if the market maker needs more than one period ($d > 1$) to process the signal he observes. For example, suppose that the market maker needs $d = 2$ periods to react to a signal. Further suppose that the latent price changes every T-th period, and label the period of the latent price change with $t = 0$. Then the market market will change his quotes for the first time in $t = 2$, and then, assuming that traders react without any delay, again in $t = 4$, $t = 6$, and so on. Because his quote in the interim period is unchanged,

[28] For a solution of (63) see appendix F and Aghion et al. (1991), p. 637.

Table 3: Cross-correlations $\rho_\tau = Corr(\Delta p^*_{t-\tau}, \Delta u^{mid}_t)$ with persistent private information, risk neutrality

signal	traders strategic	ρ_0	ρ_1	ρ_τ $\tau > 1$
certain	no	$\rho_0 < 0$	$\lvert\rho_0\rvert > \rho_1 > 0$	> 0
noisy	no	$\frac{-1}{\sqrt{X(r,T)}}$	$\frac{\left(-e^{-r}+1\right)^2}{\sqrt{X(r,T)}}$	$\frac{\left(-e^{-r\tau}+e^{-r(\tau-1)}\right)^2}{\sqrt{X(r,T)}}$
noisy	yes	$-\sqrt{1-\frac{1}{T}}$	$\frac{1}{\sqrt{T(T-1)}}$	$\frac{1}{\sqrt{T(T-1)}}$

the trading activity in these periods does not provide any additional information. The noise pattern is therefore $-\Delta p^*_t, 0, \Delta p_2, 0, \Delta p_4, \ldots$. Whereas the variance of Δp^*_t is unchanged, the (unconditional) variance of noise is smaller, but more than half of the variance of the standard $d = 1$ case. The cross-correlation function $(\rho'_0, \rho'_1, \rho'_2, \ldots)$ is then $(a\rho_0, 0, a\rho_1, 0, a\rho_2 \ldots)$, where $a > 1$ and $\rho_0, \rho_1, \ldots$ refer to the first row of table 3.

4.2 Noisy Signal and Non-strategic Traders

None of the models so far accounts for signal uncertainty and strategic behavior. Here we do so. We discuss three standard models and characterize their implied cross-correlation patterns.

Consider first a market in which the market maker observes only a noisy signal of whether p^*_t has changed, but in which traders do not yet behave strategically. The market maker then has to learn both about the quality of the signal and about the latent price. Glosten and Milgrom (1985) describe a market maker who does not know whether he is trading with an informed or an uninformed trader and thus cannot tell whether his (only) signal of the latent price change, the direction of trade q_t, contains any valuable information. For example, the market maker cannot tell whether a "buy" originates from an informed trader, in which case it would indicate an increase in the latent price, or whether it is just a random trade of an uninformed trader. Thus, this noisy "buy" increases the likelihood of an increase in the latent price less than a "buy" in the environment of section 4.1. Because of the lower information content in the signal, the initial reaction of the market maker to the signal is smaller.

Glosten and Milgrom (1985) show that if learning is costless, the expectations of market makers and traders converge as the number of trades increases.[29] Because the uncertainty of whether a trade reflects the information of the informed traders or not, the market maker adjust only *partially* to the price indicated by this signal. Therefore, whereas the cross-correlations have the same signs as under signal certainty of section 4.1, all absolute values are dampened toward zero. The correlation is less than one even if $s = 0$.

Easley and O'Hara (1992) additionally consider the information conveyed by periods of no trading. Their model is more abstract, but gives the same cross-correlation pattern as Glosten and Milgrom (1985), and has the advantage that this pattern can be derived explicitly. Suppose at the beginning of the trading day randomly a new latent price p_t^* is drawn. Let this price be either high ($\overline{p}$) or low ($\underline{p}$), each with probability $\frac{1}{2}$.[30] The unconditionally expected latent price is therefore $E(p^*) = \frac{\underline{p}+\overline{p}}{2}$. The two possible latent price levels, $\underline{p}$ and $\overline{p}$, and their probability 1/2 are publicly known, but the actual realization is not. A publicly known percentage of traders is watchful and occasionally observes – jointly as a group – the latent price p_t^*, thereby becoming informed traders.

Informed traders always trade, because $p^{ask} \leq \overline{p}$ and $p^{bid} \geq \underline{p}$. In periods when watchful traders do not observe any information, they do not trade, thus their trade share is zero. Uninformed traders trade randomly; in any given period they randomly either buy, sell or do not trade at all. The direction-of-trade signal, q_t, in this model is thereby uncertain in two ways. Not only does the market maker not know if a specific trade originates from informed traders, thereby being informative; the market maker does not even know if there *are* any informed traders. Since the market maker loses in every trade with an informed trader, he has an incentive to find out the latent price. To do so, he updates in a Bayesian manner his belief about the probability that nobody observed a signal, and that some traders observed a high latent price or a low latent price, using his information set of all previous quotes and trades, $\Omega_t = \{q_t, p_t, q_{t-1}, p_{t-1}, \ldots, q_0, p_0\}$. Even non-trading intervals contain information about p^*, because they lower the probability that watchful traders have observed the latent price at the beginning of the trading day and therefore the probability of informed trading.[31] Easley and O'Hara (1992) show that transaction prices converge to the latent price in clock time at exponential rates. Denote $\beta_{t,\{\overline{p}\}}$ the belief at time t that a high latent price has been observed,

[29] But see Aghion et al. (1991) for situations in which learning stops before reaching p^*. In that case the cross-correlations cut off at some $\tau > 1$ despite persistent private information.

[30] The case discussed here corresponds to $\delta = 0.5$ and $V^* = \frac{\underline{p}+\overline{p}}{2}$ in Easley and O'Hara (1992).

[31] A variation of this is the model of Diamond and Verrecchia (1987), where short selling constraints cause periods of no trading to be a noisy signal of a low latent price.

$\beta_{t,\{\underline{p}\}}$ the belief that a low latent price has been observed and $\beta_{t,\{\}}$ the belief that nobody has observed any signal, all conditional on $\Omega_{t-1} \cup \{q_t\}$. For the case of informed traders having observed the low latent price $\underline{p}$, the market maker sets under perfect competition

$$\begin{aligned} p_t^{bid} - \underline{p} &= \beta_{t,\{\underline{p}\}}(1-\beta_{t,\{\}})\underline{p} + \beta_{t,\{\overline{p}\}}(1-\beta_{t,\{\}})\overline{p} + \beta_{t,\{\}}\frac{\underline{p}+\overline{p}}{2} - \underline{p} \\ &= \left(\beta_{t,\{\overline{p}\}} + \frac{\beta_{t,\{\}}}{2}\right)(\overline{p}-\underline{p}), \end{aligned} \tag{64}$$

with $\beta_{t,\{\overline{p}\}} = exp(-r_1 t)$, $\beta_{t,\{\}} = exp(-r_2 t)$ and $r_1, r_2 > 0$. Hence for large t the bid price p_t^{bid} converges to $(\underline{p})$ almost surely at the exponential rate $r = min(r_1, r_2)$ in clock time.

$$p_t^{bid} \overset{a.s.}{\to} \underline{p}. \tag{65}$$

An analogous result applies to the convergence of the ask price to $\underline{p}$.

Note that there are fewer trades during trading days in which no trader has observed the latent price than during trading days in which some have, because the watchful traders do not trade without information. Hence the results strictly apply only to calendar time sampling (CTS). With tick time sampling (TTS) we miss the no-trade periods, which reveal information to the market maker, too. Since TTS effectively compresses time by dropping periods without a trade, the convergence rate will be higher, but exponential convergence to the true value still obtains.

Proposition 7 ***(Cross-correlations in Easley-O'Hara model)***

The contemporaneous cross-correlation in the Easley and O'Hara (1992) model is

$$Corr\left(\Delta p_t^*, \Delta u_t^{mid}\right) = \frac{-1}{\sqrt{X(r,T)}} > -\frac{1}{\sqrt{2}}, \tag{66}$$

where $X(r,T)$ is decreasing in both r and T and has a minimum at 2.[32]

Likewise, the lagged cross-correlation ($1 \le \tau < T$) is positive, even for $\tau > 1$.

$$Corr\left(\Delta p_{t-\tau}^*, \Delta u_t^{mid}\right)$$

[32]
$$X(r,T) = 1 + e^{-2r}(e^r-1)^2\frac{(e^{-2r})^{T-1}-1}{e^{-2r}-1} + e^{-2r(T-1)} \ge 2.$$
The variance of the noise Δu_t^{mid} is $\frac{\sigma^2}{T}X(r,T)$.

$$
\begin{aligned}
&= \frac{\left(-e^{-r\tau}+e^{-r(\tau-1)}\right)^2}{\sqrt{X(r,T)}} \\
&< \frac{\left(-e^{-r\tau}+e^{-r(\tau-1)}\right)^2}{\sqrt{2}}.
\end{aligned} \tag{67}
$$

Proof: See appendix G.

Unsurprisingly because of the assumption of risk neutrality, the contemporaneous correlation is negative, and approaches its minimum for large r and T. For $T = 1$ the contemporaneous cross-correlation is equal to the cross-correlation with one period private information and zero spread. Furthermore, for $\tau \geq 1$,

$$
\begin{aligned}
&Corr\left(\Delta p^*_{t-\tau}, \Delta u^{mid}_t\right) \\
&= \left(\frac{1}{e^{2r}}\right)^{\tau-1} Corr\left(\Delta p^*_{t-1}, \Delta u^{mid}_t\right).
\end{aligned} \tag{68}
$$

That is, the cross-correlation decays geometrically to zero. After T the information about Δp^*_0 becomes public, thus for $\tau > T$ we get $Corr\left(\Delta p^*_{t-\tau}, \Delta u^{mid}_t\right) = 0$.

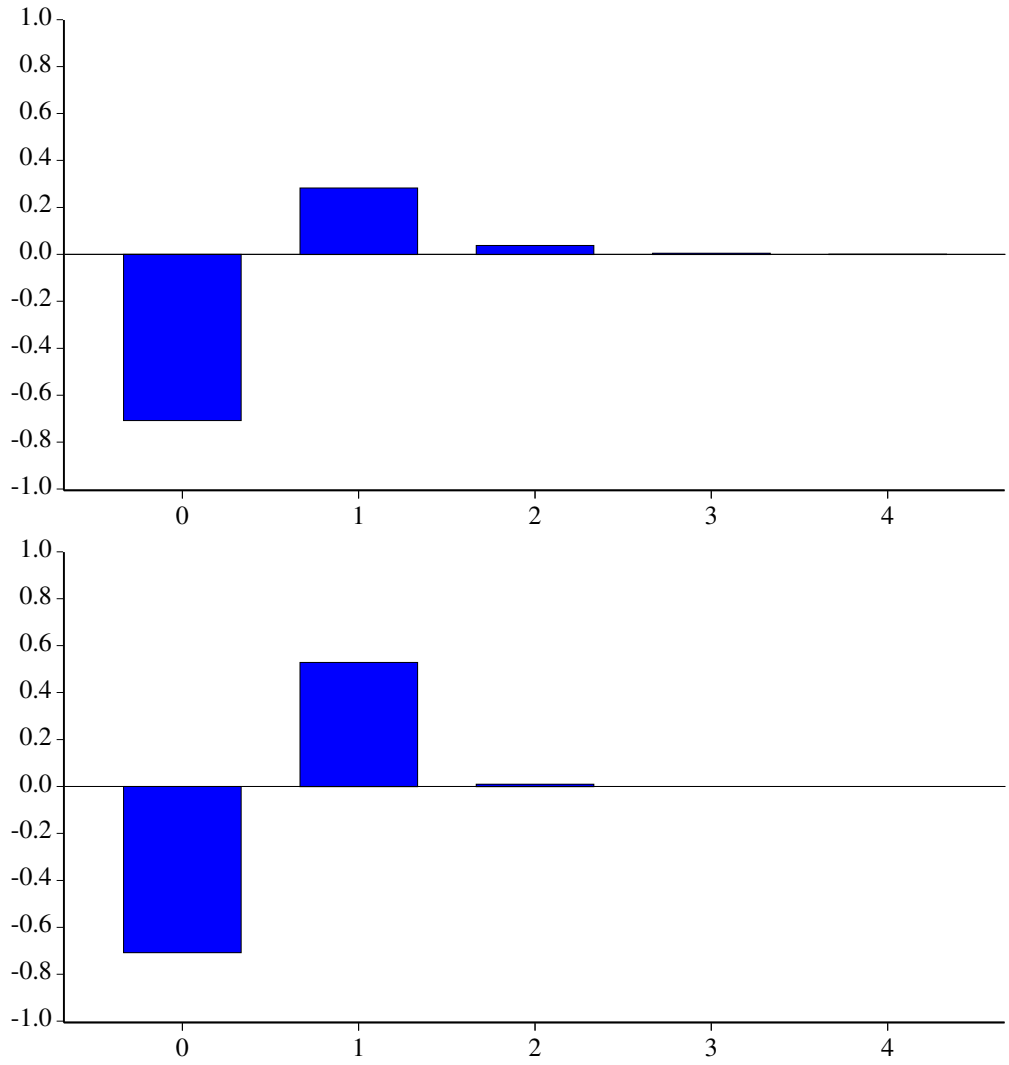
In Figure 7 we graph this cross-correlation function. We show the cross-correlation pattern for a convergence rate of $r = 1$ in the upper panel, and for a convergence rate of $r = 2$ in the lower panel.

4.3 Noisy Signal and Strategic Traders

Because the informed trades are not directly visible, and private information does not automatically become public after one period, informed traders can act strategically. The aim of strategic behavior of informed traders is to make the signals about p^*_t conveyed by their orders as noisy as possible, while still executing the desired trades. By mimicing uninformed traders they keep the market maker unaware of new information, i.e. unaware of the change in p^*_t. Since the market maker observes order flow imbalances – and possibly other variables such as order size –, and uses this to detect informed trading, the informed traders stretch their orders over a long time period such that detecting any significantly abnormal trading pattern becomes difficult. The market maker will, of course, notice the imbalance in (large) trades over time by sequentially updating his belief about p^* based on the history of trades. In this way the market maker still learns about p^*_t, but, because of the strategic behavior of traders, at the slowest possible rate.

Markets of this type have been described in Kyle (1985) and Easley and O'Hara (1987).

Figure 7: Cross-correlation function $\rho(\tau)$ of the mid price in a Easley and O'Hara (1992)-type model



The upper panel shows the typical cross-correlation pattern under adjustment speed $r = 1$, the lower panel for adjustment speed $r = 2$. In both panels $X = 1$.

In the following we discuss the cross-correlation function implied by the Kyle (1985) model under zero spread ($s = 0$). Suppose the latent price jumps in every T-th period, and remains constant for the next $T-1$ periods. Likewise Δp_0^* remains private information for T periods, after which it becomes public knowledge. The jumps follow the distribution

$$\Delta p_0^* \sim (0, \sigma^2). \tag{69}$$

p^* can be interpreted as the liquidation value of the asset at the end of the trading period T, with $p^* = p(0) + \Delta p_0^*$. Informed traders observe p^* perfectly, but the market maker and unformed traders do not. Knowing the end-of-day latent price, the informed traders trade whenever it is profitable during the day; that is, they buy whenever the market price is above the end of day value, and they sell otherwise. The prices can move up and down during the day because uninformed (noise) traders buy and sell randomly at any price for exogenous reasons. Thus the informed traders accumulate profit during the trading time, at the cost of the market maker and uninformed traders.

Here, the market maker does not maximize a particular objective function, he merely ensures market efficiency, i.e. sets the market price $p(t)$ such that it equals the expected latent price p^*, given the observed trading volume from informed and uninformed traders. The only optimizer in this model is the (risk neutral) informed trader who optimally spreads his orders over the day to minimize the (unfavorable) price reaction of the market maker. Thereby he maximizes his expected total daily profit using his private information and taking the rule $\Delta p_t(\Omega_t)$ market makers use to set prices as given. Effectively, the informed trader trades most when the sensitivity of prices to trading quantity is small. The key feature of the Kyle (1985) model for our purposes is that market prices pick up information at a constant rate, and that all information is reflected in the price by the end of the trading day.

Under the assumption of linear reaction functions for demanded quantity and prices, Kyle (1985) shows that the market price approaches the latent price linearly, compared to exponential decay in Easley and O'Hara (1992). The reason for this difference is that the market maker in Easley and O'Hara (1992) updates his believes in a Bayesian manner, whereas in Kyle (1985) the market maker's actions are constrained to market clearing.

Proposition 8 ***(Cross-correlations in Kyle model)***

The contemporaneous cross-correlation in Kyle (1985) is

$$Corr\left(\Delta p_t^*, \Delta u_t^{mid}\right) = -\sqrt{1 - \frac{1}{T}}, \tag{70}$$

and the cross-correlations at displacements $\tau \geq 1$ are

$$Corr\left(\Delta p^*_{t-\tau}, \Delta u^{mid}_t\right) = \frac{1}{\sqrt{T(T-1)}}. \tag{71}$$

Proof: See appendix H.

The constant cross-correlations at displacements $\tau \geq 1$ stem from the strategic behavior of traders, which spreads their information equally over time. This maximizes the time it takes the market maker to include the entire latent price change in his quotes. The more periods, the more pronounced is the negative contemporaneous correlation, and the smaller are the lagged correlations. After period T the information about Δp^*_0 becomes public; thus for $\tau > T$ we get $Corr\left(\Delta p^*_{t-\tau}, \Delta u^{mid}_t\right) = 0$. In Figure 8 we plot these cross-correlation functions. We show in the upper and lower panels the cross-correlation function under frequent changes in the latent price ($T = 2$), and for less frequent changes ($T = 5$), respectively.

In Table 3 we compare the market microstructure models with persistent private information. Markets with nonstrategic traders display decaying lagged cross-correlations (row 2). Conversely, markets with strategic traders display constant lagged cross-correlations (row 3).

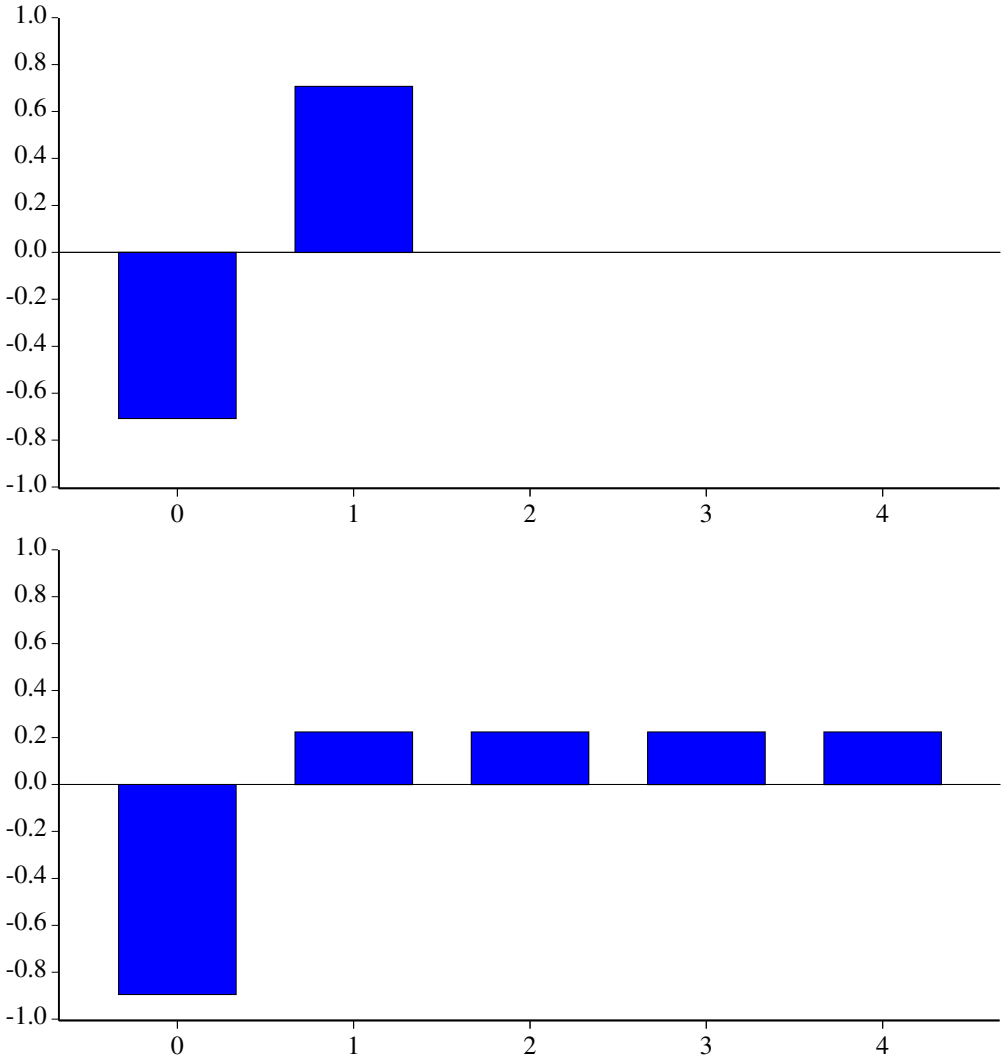
4.4 Frequent Price Changes and Slowly Vanishing Private Information

So far in this section we have discussed stylized models, where the old latent price becomes public information at the beginning of the trading day before any new permanent latent price shift. In general, however, the latent price may change again permanently before the old latent price becomes fully publicly known. In this case, if the latent price process follows (1), the old p^*_{t-1} still contains information about the new p^*_t. As p^*_{t-1} itself is not precisely known as well, the entire history of prices contains information about p^*_t.

The contemporaneous cross-correlation is unaffected by frequent latent price changes, in particular it is not affected by a future latent price change, so the results for contemporaneous cross-correlation remain valid.

Cross-correlations at nonzero displacements, however, are affected. If as before the market maker's signal is $\text{sgn}(p_t - p^*_t)$, and $Corr(p^*_t, \Delta p^*_{t-\tau}) > 0$, then Ω_t contains new information about $p^*_{t-\tau}$ beyond Ω_{t-1}. By (17) the signs of the cross-correlations at nonzero displacements remain unchanged even if p^*_t changes frequently. But the more often p^*_t changes

Figure 8: Cross-correlation function $\rho(\tau)$ of the mid price in a Kyle (1985)-type model



The upper panel shows the typical cross-correlation pattern under frequent changes in the latent price ($T = 2$), the lower panel for less frequent changes ($T = 5$).

during $[t, t-\tau]$, the closer to zero is the correlation $Corr(p_t^*, \Delta p_{t-\tau}^*)$, and likewise the cross-correlation between latent returns and noise. Overall, slowly decaying private information keeps the cross-correlation sign pattern unchanged, but dampens the absolute values toward zero.

5 Econometric Implications

We have already drawn econometric implications insofar as we have shown that market microstructure models predict rich cross-correlation patterns between latent prices and microstructure noise (MSN), which have yet to be investigated empirically. Quite apart from suggesting avenues for new empirical exploration, however, our results have important implications for "traditional" volatility estimation, on which we focus in this section.

5.1 Effect of Sampling Frequency

We have thus far focused primarily on sampling at exactly the rate taken by the market maker to react to new information in the previously sampled price. In practice, the sampling speed of the market maker is unknown. In general, tick time seems to be the best choice for estimation that accounts for correlation between noise and latent returns (Griffin and Oomen 2006).[33]

The differences in performance of sampling methods are largely caused by market microstructure effects as well, and it is therefore not surprising that the choice of the sampling method affects the cross-correlations. If we sample m times during an interval of no changes in both latent and market prices, then we record each latent return - noise pair m times. The cross-correlation function is then a step function, but it still has the shape obtained at slower sampling speeds. For example, the cross-correlations up to displacement $m-1$ all share the sign of the contemporaneous cross-correlation. Since all models predict a sign change in cross-correlations between displacement zero and displacement one, observing a cross-correlation at displacement one with the same sign as the contemporaneous cross-correlation is evidence of sampling faster than prices evolve, so the sampling rate should be reduced.

A longer sampling interval dampens cross-correlations toward zero. As an example suppose that in the model with one period private information of section 3.3 we sample only

[33] Under an independent noise assumption, transaction time sampling is more reliable (Oomen 2006b).

every n-th tick, denoted by $p_{t,n}$ and $p^*_{t,n}$. Let $p_{t,i}$, $i = 1, \ldots, n-1$ be the unsampled market prices of the omitted ticks, and $p^*_{t,i}$ the corresponding latent prices. For ease of exposition set $\alpha = 0$ and thus $s_t = s$. Then (57) is replaced by

$$\Delta\hat{p}^*_t = \sum_{i=1}^{n} \Delta p^*_{t,i} = \sum_{i=1}^{n} \lambda q_{t,i} + \varepsilon_{t,i} + \mu_{t,i}, \tag{72}$$

and from (97) the noise is

$$\begin{aligned} \Delta\hat{u}^{tra}_t &= \sum_{i=1}^{n} \Delta u^{tra}_{t,i} \\ &= (s-\lambda)(q_{t,n} - q_{t-1,n}) + \varepsilon_{t-1,n} - \varepsilon_{t,n}. \end{aligned} \tag{73}$$

The covariance $Cov(\Delta\hat{p}^*_t, \Delta\hat{u}^{tra}_t) = \lambda(s-\lambda) - \sigma^2$ and noise variance $Var(\Delta\hat{u}_t) = 2(s-\lambda)^2 + 2\sigma^2$ are therefore unchanged. However, the variance of latent returns increases to $Var(\Delta\hat{p}^*_t) = n(\lambda^2 + \sigma^2)$. Thus

$$\begin{aligned} Corr(\Delta\hat{p}^*_t, \Delta\hat{u}^{tra}_t) &= \frac{\lambda(s-\lambda) - \sigma^2}{\sqrt{2n}\sqrt{\lambda^2+\sigma^2}\sqrt{(s-\lambda)^2+\sigma^2}} \\ &= \frac{1}{\sqrt{n}} Corr(\Delta p^*_t, \Delta u^{tra}_t). \end{aligned} \tag{74}$$

That is, reducing the sampling rate shrinks the contemporaneous cross-correlation toward zero. The same result applies to lagged cross-correlations.

As discussed at the beginning of section 3, the cross-correlations at displacements zero and one have opposite signs. As equation (74) shows, combining several ticks into one sample blurs this sign change. Hansen and Lunde (2006) find a negative contemporaneous cross-correlation between returns and noise in their data. This effect fades with the number of ticks that are combined into one market price sample, which suggests that lagged correlations indeed work in the opposite direction, i.e. being zero or positive. Increasing the sampling interval averages the initial market price underreaction with later price readjustments, thereby dampening the cross-correlation to zero.

Overall, then, the sampling frequency does not change the sign pattern of cross-correlations, but can severely impact its absolute values. At very slow sampling rates the cross-correlations become empirically indistinguishable from zero, which can be useful when analysis is being done on the assumption of independent noise. At higher sampling frequencies the cross-correlation of noise needs to be taken into account. In the next subsection

we show that this can be done parsimoniously using the structure imposed by market microstructure theory.

5.2 Implications for Volatility Estimation

In the introduction we highlighted the key issue of estimation of integrated volatility using high-frequency data, the potential problems of the first-generation estimator (simple realized volatility) in the presence of MSN, and subsequent attempts to "correct" simple *realized volatilities* (RV).

In a leading development, for example, Hansen and Lunde (2006) suggest making realized volatility robust to serial correlation via HAC methods. If q terms are included in the calculation of this estimator, then it is exact for any MA(q) process. Whereas this approach is very flexible, it might be inefficient in the context of market MSN. Because of the often unsatisfactory performance of these nonparametric estimators, especially in finite samples, Bandi and Russell (2006b) suggest sophisticated ways to improve their finite sample properties.

Here we suggest a different approach, utilizing market microstructure theory. As we have seen, dynamic market microstructure models imply that noise decays geometrically over time after displacement one, with two polar cases of immediate decay (as in section 3) and no decay (as in section 4.3). We suggest using this knowledge to construct a superior estimator of integrated volatility, using the parameter restrictions on the moving average coefficients obtained from the market microstructure models.

Suppose Δp_t follows an MA(∞) process in the innovations for the latent price,

$$\Delta p_t = \alpha_0 \varepsilon_t + \alpha_1 \sum_{\tau=1}^{\infty} \alpha_2^{\tau} \varepsilon_{t-\tau}. \tag{75}$$

This form of Δp_t accommodates very persistent cross-correlations, similar to the idea behind the sequence of examples in Oomen (2006a). We can decompose latent returns into market returns and noise

$$\Delta p_t^* = \Delta p_t + (1-\alpha_0)\,\varepsilon_t - \alpha_1 \sum_{\tau=1}^{\infty} \alpha_2^{\tau} \varepsilon_{t-\tau} = \varepsilon_t. \tag{76}$$

Using this, we have a simple formula for the integrated volatility (IV) of latent returns,

$$IV = E\left[(\Delta p_t^*)^2\right] = E\left[(\Delta p_t - \Delta u_t)^2\right] = E(\Delta p_t^2) + (1+\alpha_0)(1-\alpha_0)IV - \alpha_1^2 \sum_{\tau=1}^{\infty} \alpha_2^{2\tau} IV. \tag{77}$$

Solving for IV, using $\sum_{\tau=1}^{\infty} \alpha_2^{2\tau} = \frac{\alpha_2^2}{1-\alpha_2^2}$, and simplifying yields

$$IV = \frac{1-\alpha_2^2}{\alpha_0^2(1-\alpha_2^2)+\alpha_1^2\alpha_2^2} \cdot E(\Delta p_t^2). \tag{78}$$

Standard RV is consistent for $E(\Delta p_t^2)$; hence a consistent estimator for IV is

$$\hat{IV} = \frac{1-\hat{\alpha}_2^2}{\hat{\alpha}_0^2(1-\hat{\alpha}_2^2)+\hat{\alpha}_1^2\hat{\alpha}_2^2} \cdot RV, \tag{79}$$

where $\hat{\alpha}_0$, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are consistent estimators. Such estimators are easily obtained, for example, in a GMM framework using three moments.

The upshot thus far is that exploitation of market microstructure theory delivers a simple consistent estimator of integrated volatility. One might of course counter that so too does the purely statistical approach of Hansen and Lunde (2006). However, although a thorough exploration is beyond the scope of this paper, we conjecture that our economic estimator is more efficient, because its correction to RV involves estimation of only three parameters.

The result is even simpler in the framework of Easley and O'Hara (1992),

$$\Delta p_t \approx 0 \cdot \varepsilon_t + \sum_{\tau=1}^{\infty} \left(-e^{-r\tau} + e^{-r(\tau-1)}\right) \varepsilon_{t-\tau} = (e^r - 1) \sum_{\tau=1}^{\infty} e^{-r\tau} \varepsilon_{t-\tau}. \tag{80}$$

This is an approximation because we have assumed throughout that the latent price is revealed after T periods, suggesting an MA(T) process, whereas in (80) we use MA(∞). It follows that the integrated variance can be consistently estimated by

$$\hat{IV} = \frac{1-e^{-2\hat{r}}}{1-2e^{-\hat{r}}+e^{-2\hat{r}}} \cdot RV, \tag{81}$$

which requires a consistent estimator of only one parameter, the rate of learning r.

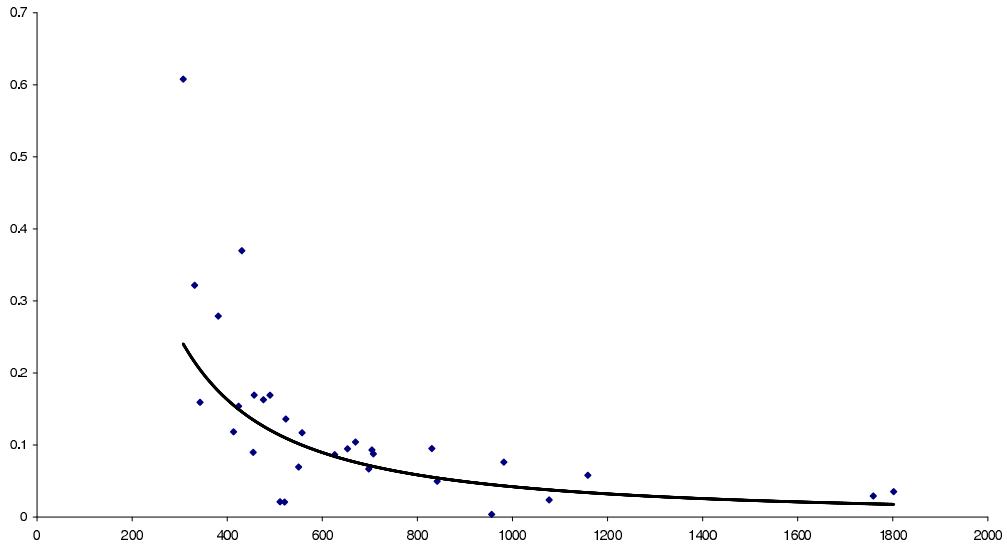
The expression for the integrated variance (81) offers a structural interpretation to estimates of noise and integrated variance, such as the results reported in table 3 of Hansen and Lunde (2006). The Easley and O'Hara (1992) model predicts that the noise decreases as the learning rate of the market maker increases. Slow learning implies a very persistent cross-correlation between noise and latent returns, and hence persistent autocorrelation of noise, so that fluctuations in noise tend to dominate the integrated variance.

Figure 9 provides some indication for this relationship. It is based on the noise-to-integrated-variance ratios reported by Hansen and Lunde (2006), which are unfortunately

derived under the assumption of independent noise. The ratio of noise to integrated variance shrinks with the number of price-changing quotes per day. If the number of times that the market maker changes his price quote during a trading day is indicative of his speed of learning, then MSN indeed decreases as the learning rate of the market maker increases. Thus, even though these ratios may not be directly applicable, they seem to support the Easley and O'Hara (1992) model.

Further, since the bias of realized variance due to noise seems to have declined over recent years (Hansen and Lunde's fact III) the learning rate r must have increased. Meddahi (2002)'s finding that the standard deviation of this bias is large relative to the integrated variance suggests that the learning rate itself fluctuated considerably around its increasing trend.

Figure 9: Relationship between noise-to-integrated-variance ratio and number of quotes per day



The vertical axis measures the noise to signal ratio as 100 times noise divided by integrated variance under the assumption of independent noise. The horizontal axis is the number of quotes per day with a price change. Data is for the year 2000 for 30 stocks listed on NYSE and NASDAQ. Data are from Hansen and Lunde (2006), Tables 1 and 3. The solid line is a fitted power trend line.

6 Concluding Remarks

The recent realized volatility literature provides statistical insights into microstructure noise (MSN) and its effects. In this paper we have provided complementary *economic* insights,

treating MSN not simply as a nuisance, but rather as the result of financial economic decisions, which we seek to understand. In that regard, we derived the predictions of economic theory regarding correlation between latent price and MSN; we characterized and contrasted the entire cross-correlation functions corresponding to a variety of market environments, with a variety of results.

Some results are generic. In particular, cross-correlations between latent price and MSN at displacements greater than zero have sign opposite to that of the contemporaneous correlation.

Some results are not generic but nevertheless quite robust to model choice. In particular, almost all models predict negative contemporaneous correlation between latent price and MSN, so long as the risk aversion of market makers is not too high.

Finally, some results are highly model-specific. In particular, the cross-correlation patterns and absolute magnitudes depend critically on the frequency of latent price changes, the presence of bid/ask bounce, the timing of information and actions, and the market maker's degree of risk aversion.

We believe firmly in the interaction of theory and data, with each beneficially disciplining the other in the advancement of science. We hope that the results of this paper will prove useful in both directions. In particular, we have argued that data-based cross-correlation patterns between latent price and MSN should help determine (if not definitively resolve) the comparative merits of various economic microstructure models, and conversely, that our theoretical cross-correlation results may lead to improved volatility estimators.

Looking to the future, we can envision some novel uses and extensions of our results. For example, the rate of decay of cross-correlations might be used to assess the extent to which strategic traders are active in the market, and the sign and size of the contemporaneous correlation might be used to allow one to assess the degree of market maker risk aversion. Indeed market maker risk aversion might be time-varying, with associated time-varying cross-correlation structure between latent price and MSN. During crises, for example, market makers may be more risk averse, as borrowing and hedging possibilities are reduced. If so, the "normal pattern" of negative contemporaneous cross-correlation and positive higher-order cross-correlations might switch to a "crisis pattern" of positive contemporaneous cross-correlation and negative higher-order cross-correlations. Such possibilities await future empirical exploration.

References

Aghion, P., Bolton, P., Harris, C. and Jullien, B., 1991, "Optimal learning by experimentation," *Review of Economic Studies*, 58, 621–654.

Aghion, P., Espinosa, M. P. and Jullien, B., 1993, "Dynamic duopoly with learning through market experimentation," *Economic Theory*, 3, 517–539.

Aït-Sahalia, Y., Mykland, P. and Zhang, L., 2005, "Ultra high frequency volatility estimation with dependent microstructure noise," *NBER Working Paper*, 11380.

Andersen, T. G., Bollerslev, T., Christoffersen, P. F. and Diebold, F. X., 2006, Volatility and correlation forecasting, *in* G. Elliot, C. W. J. Granger and A. Timmermann (eds.), *Handbook of Economic Forecasting*, Vol. 1 of *Handbooks in Economics 24*, North-Holland, Amsterdam, chapter 15, pp. 777–878.

Andersen, T. G., Bollerslev, T. and Diebold, F. X., 2005, Parametric and nonparametric volatility measurement, *in* L. P. Hansen and Y. Aït-Sahalia (eds.), *Handbook of Financial Econometrics*, Handbooks in Finance, North-Holland, Amsterdam.

Andersen, T. G., Bollerslev, T., Diebold, F. X. and Ebens, H., 2001, "The distribution of realized stock return volatility," *Journal of Financial Economics*, 61(1), 43–76.

Andersen, T. G., Bollerslev, T., Diebold, F. X. and Labys, P., 2000, "Great realizations," *Risk*, 13(March), 105–108. Reprinted in J. Danielsson (ed.), The Value-at-Risk Reference, London: Risk Publications, 2008.

Andersen, T. G., Bollerslev, T., Diebold, F. X. and Labys, P., 2001, "The distribution of realized exchange rate volatility," *Journal of the American Statistical Association*, 96, 42–55. Reprinted in N. Shephard (ed.), Stochastic Volatility: Selected Readings, Oxford: Oxford University Press, 2005.

Andersen, T. G., Bollerslev, T., Diebold, F. X. and Labys, P., 2003, "Modeling and forecasting realized volatility," *Econometrica*, 71, 579–625.

Bandi, F. M. and Russell, J. R., 2005, "Microstructure noise, realized volatility, and optimal sampling," Working paper, Graduate School of Business, University of Chicago.

Bandi, F. M. and Russell, J. R., 2006a, "Comment," *Journal of Business and Economic Statistics*, 24, 167–173.

Bandi, F. M. and Russell, J. R., 2006b, "Market microstructure noise, integrated variance estimators, and the accuracy of asymptotic approximations," Working paper, Graduate School of Business, University of Chicago.

Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A. and Shephard, N., 2006a, "Designing realised kernels to measure the ex-post variation of equity prices in the presence of

noise," mimeo, University of Oxford, Nuffield College.

Barndorff-Nielsen, O. E., Hansen, P. R., Lunde, A. and Shephard, N., 2006b, "Subsampling realised kernels," mimeo, University of Oxford, Nuffield College.

Barndorff-Nielsen, O. E. and Shephard, N., 2002a, "Econometric analysis of realized volatility and its use in estimating stochastic volatility models," *Journal of the Royal Statistical Society B*, 64(2), 253–280.

Barndorff-Nielsen, O. E. and Shephard, N., 2002b, "Estimating quadratic variation using realized variance," *Journal of Applied Econometrics*, 17, 457–477.

Barndorff-Nielsen, O. E. and Shephard, N., 2007, Variation, jumps, market frictions and high frequency data in financial econometrics, *in* R. Blundell, P. Torsten and W. K. Newey (eds.), *Advances in Economics and Econometrics, Theory and Applications, Ninth World Congress*, Vol. 43/3 of *Econometric Society Monographs*, Cambridge University Press, Cambridge, chapter 10.

Demsetz, H., 1968, "The cost of transacting," *Quarterly Journal of Economics*, 82(1), 33–53.

Diamond, D. W. and Verrecchia, R. E., 1987, "Constraints on short-selling and asset price adjustment to private information," *Journal of Financial Economics*, 18, 277–311.

Diebold, F. X., 2006, "Comment," *Journal of Business and Economic Statistics*, 24, 181–183.

Easley, D. and O'Hara, M., 1987, "Price, trade size, and information in securities markets," *Journal of Financial Economics*, 19, 69–90.

Easley, D. and O'Hara, M., 1992, "Time and the process of security price adjustment," *Journal of Finance*, 47, 576–605.

Engle, R. F. and Patton, A. J., 2004, "Impacts of trades in an error-correction model of quote prices," *Journal of Financial Markets*, 7, 1–25.

Glosten, L. R. and Milgrom, P. R., 1985, "Bid, ask, and transaction prices in a specialist market with heterogeneously informed traders," *Journal of Financial Economics*, 14, 71–100.

Griffin, J. E. and Oomen, R. C., 2006, "Sampling returns for realized variance calculations: tick time or transaction time?," Working Paper, forthcoming Econometric Reviews.

Hansen, P. R. and Lunde, A., 2006, "Realized variance and market microstructure noise," *Journal of Business and Economic Statistics*, 24(2), 127–161.

Hasbrouck, J., 2002, "Stalking the efficient price in empirical microstructure specifications," *Journal of Financial Markets*, 5, 329–339.

Hasbrouck, J., 2007, *Empirical Market Microstructure*, Oxford University Press, New York.

Kyle, A. S., 1985, "Continuous auctions and insider trading," *Econometrica*, 53, 1315–1336.

Meddahi, N., 2002, "A theoretical comparison between integrated and realized volatility," *Journal of Applied Econometrics*, 17(5), 479–508.

O'Hara, M., 1995, *Market Microstructure Theory*, Cambridge, Cambridge.

Oomen, R. C. A., 2006a, "Comment," *Journal of Business and Economic Statistics*, 24, 195–202.

Oomen, R. C. A., 2006b, "Properties of realized variance under alternative sampling schemes," *Journal of Business and Economic Statistics*, 24, 219–237.

Owens, J. and Steigerwald, D. G., 2005, "Inferring information frequency and quality," *Journal of Financial Econometrics*, 3(4), 500–524.

Roll, R., 1984, "A simple implicit measure of the effective bid-ask spread in an efficient market," *Journal of Finance*, 39, 1127–1139.

Zhang, L., 2006, "Efficient estimation of stochastic volatility using noisy observations: A multi-scale approach," *Bernoulli*, 12(6), 1019–1043.

Zhang, L., Mykland, P. A. and Aït-Sahalia, Y., 2005, "A tale of two time scales: Determining integrated volatility with noisy high frequency data," *Journal of the American Statistical Association*, 100, 1394–1411.

A Proof: Risk Aversion Increases in n

Because $\breve{l}_n(x)$ defined in (11) is concave $\forall n \geq 1$, the loss function satisfies the Jensen inequality and hence represents risk averse preferences.

Further, risk aversion increases in n, because there exists a concave function ψ such that $l_{n+1}(x) = \psi[l_n(x)]$, namely

$$\psi(y) = -(-y)^{\frac{n+1}{n}}, \tag{82}$$

restricted to $y \leq 0$. It follows that $l_{n+1}(x)$ is more risk averse than $l_n(x)$.

B Proof of Proposition 1: Model-free Cross-Correlations

For determining the sign of the cross-correlation it is sufficient to look at the cross-covariances, which by definition have the same sign. The *contemporaneous cross-covariance* between latent returns and noise is

$$\begin{aligned} Cov(\Delta p_t^*, \Delta u_t) &= Cov(p_t^* - p_{t-1}^* - \mu_t, p_t - p_t^* - p_{t-1} + p_{t-1}^*) \\ &= E(\Delta p_t^*(\Delta p_t(\Omega_t, \Omega_{t-1}) - \Delta p_t^*)) \\ &= E(\Delta p_t(\Omega_t, \Omega_{t-1})\Delta p_t^*) - E(\Delta p_t^{*2}) \\ &= E(\Delta p_t(\Omega_t, \Omega_{t-1})\Delta p_t^*) - Var(\Delta p_t^*) - (E(\Delta p_t^*))^2 \\ &= E(\Delta p_t(\Omega_t, \Omega_{t-1})\Delta p_t^*) - Var(\Delta p_t^*), \end{aligned} \tag{83}$$

which immediately implies the first part of proposition 1.

A more specific expression can be derived for the case of symmetric $f(\cdot)$ (assumption 2) and $\Omega_t = \{\text{sgn}(\varepsilon_t), p_{t-1}^*, h_t\}$. This implies that $\left|p_t - p_{t-1}^*\right| = R(h_t) > 0 \ \forall t$, where $R(h_t)$ is a function of only h_t. After $\Delta p_t^* > 0$ we have $\Delta p_t = -p_{t-1} + p_{t-1}^* + R(h_t)$, and after $\Delta p_t^* < 0$ we have $\Delta p_t = -p_{t-1} + p_{t-1}^* - R(h_t)$. Therefore

$$\begin{aligned} E\left(\Delta p_t \Delta p_t^*\right) &= \frac{1}{2} E\left(\left(-p_{t-1} + p_{t-1}^* + R(h_t)\right)\Delta p_t^* \,|\Delta p_t^* > 0\right) \\ &+ \frac{1}{2} E\left(\left(-p_{t-1} + p_{t-1}^* - R(h_t)\right)\Delta p_t^* \,|\Delta p_t^* < 0\right) \\ &= \frac{1}{2} E\left(R(h_t)\Delta p_t^* \,|\Delta p_t^* > 0\right) + \frac{1}{2} E\left(-R(h_t)\Delta p_t^* \,|\Delta p_t^* < 0\right) \\ &= E\left(R(h_t)\right) E\left(|\Delta p_t^*|\right) \end{aligned} \tag{84}$$

Plugging (84) into (83) gives

$$Cov(\Delta p_t^*, \Delta u_t) = E\left(R(h_t)\right) E\left(|\Delta p_t^*|\right) - Var(\Delta p_t^*). \tag{85}$$

Thus in this special case the contemporaneous cross-covariance is positive if

$$E\left(R(h_t)\right) > \frac{Var(\Delta p_t^*)}{E\left(|\Delta p_t^*|\right)}. \tag{86}$$

Similarly, the *lagged cross-covariance* between latent returns $\tau \geq 1$ one period ago, $\Delta p_{t-\tau}^* = p_{t-\tau}^* - p_{t-\tau-1}^* - \mu_{t-\tau-1}$, and noise, $\Delta u_t = p_t - p_t^* - p_{t-1} + p_{t-1}^*$, is

$$\begin{aligned} & Cov(\Delta p_{t-\tau}^*, \Delta u_t) \\ &= Cov(p_{t-\tau}^* - p_{t-\tau-1}^* - \mu_{t-\tau}, p_t - p_t^* - p_{t-1} + p_{t-1}^*) \\ &= E(\Delta p_{t-\tau}^*(\Delta p_t - \Delta p_t^*)) \\ &= E(\Delta p_{t-\tau}^* \Delta p_t(\Omega_t)) - E(\Delta p_{t-\tau}^* \Delta p_t^*) \\ &= E(\Delta p_{t-\tau}^* \Delta p_t(\Omega_t)), \end{aligned} \tag{87}$$

which implies the second part of proposition 1.

C Proof: Increased Risk Aversion Moves Optimal Price Toward the Midpoint of the Support of *f()*

In this appendix we show that if $f(\cdot)$ is without loss of generality right-skewed then $p(n)$ is increasing in n, $\forall n \geq 1$. First, note that $p(n)$, $p(n) \in [l, u]$ is continuous, because if $\underline{(p)}$ or $\overline{(p)}$ are infinite, we can always replace these bounds with a function of n, thereby making the domain of p compact. As $f(\cdot)$ and all components of the integral are continuous functions, the theorem of the maximum gives continuity of $p(n)$.

Next, to evaluate how the optimal price $p(n)$ responds to changes in risk aversion n, take the total differential of (48) and rearrange to obtain

$$\frac{dp(n)}{dn} = \frac{1}{n-1} \times \left\{ - \int_{l}^{p(n)} \left(p(n) - p^*\right)^{n-1} \ln\left(p(n) - p^*\right) f(p^*) dp^* \right.$$

$$+ \int_{p(n)}^{u} (p^* - p(n))^{n-1} \ln(p^* - p(n)) f(p^*) dp^* \Bigg\} \Bigg/$$
$$\left\{ \int_{l}^{p(n)} (p(n) - p^*)^{n-2} f(p^*) dp^* + \int_{p(n)}^{u} (p^* - p(n))^{n-2} f(p^*) dp^* \right\}. \tag{88}$$

In the following argument we will use that $f(\cdot)$ is monotone. As in example 1 we will assume without loss of generality that $f(\cdot)$ is monotonically decreasing. This means $f(\cdot)$ is right-skewed on $[l, u]$, which occurs if the market maker has some information that the latent price has increased. Under this assumption (88) is positive. To see this, note first that both terms in denominator are positive. To economize notation we replace $p \equiv p(n)$, $d \equiv p(n) - l$ and $x \equiv p^*$. The numerator can be broken up into three parts:

$$\begin{aligned}
&- \int_{l}^{p} (p-x)^{n-1} \ln(p-x) f(x) dx + \int_{p}^{u} (x-p)^{n-1} \ln(x-p) f(x) dx \\
&= - \int_{p-d}^{p-1} (p-x)^{n-1} \ln(p-x) f(x) dx + \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) f(x) dx \\
&- \int_{p-1}^{p} (p-x)^{n-1} \ln(p-x) f(x) dx + \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) f(x) dx \\
&+ \int_{p+d}^{u} (x-p)^{n-1} \ln(x-p) f(x) dx.
\end{aligned} \tag{89}$$

The first term, which exists only for $d > 1$, gives

$$\begin{aligned}
&- \int_{p-d}^{p-1} (p-x)^{n-1} \ln(p-x) f(x) dx + \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) f(x) dx \\
&= - \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) f(2p-x) dx \\
&\quad + \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) f(x) dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{p+1}^{p+d} (x-p)^{n-1} \ln(x-p) \left[-f(2p-x)+f(x)\right] dx \\
&\geq \int_{p+1}^{p+d} (x-p)^{n-1} \ln(d) \left[-f(2p-x)+f(x)\right] dx \\
&= -\int_{p-d}^{p-1} (p-x)^{n-1} \ln(d) f(x) dx + \int_{p+1}^{p+d} (x-p)^{n-1} \ln(d) f(x) dx. \qquad (90)
\end{aligned}
$$

The second term is for $d \geq 1$

$$
\begin{aligned}
&- \int_{p-1}^{p} (p-x)^{n-1} \ln(p-x) f(x) dx + \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) f(x) dx \\
&= -\int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) f(2p-x) dx \\
&\quad + \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) f(x) dx \\
&= \int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) \left[f(x)-f(2p-x)\right] dx \geq 0. \qquad (91)
\end{aligned}
$$

For $d < 1$ the last inequality of the calculations for the second term be replaced by

$$
\begin{aligned}
&\int_{p}^{p+1} (x-p)^{n-1} \ln(x-p) \left[f(x)-f(2p-x)\right] dx \\
&\geq \int_{p}^{p+d} (x-p)^{n-1} \left[f(x)-f(2p-x)\right] dx \ln(d) \geq 0. \qquad (92)
\end{aligned}
$$

And for the last term we can write

$$
-\int_{p+d}^{u} (x-p)^{n-1} \ln(x-p) f(x) dx > -\int_{p+d}^{u} (x-p)^{n-1} \ln(d) f(x) dx. \qquad (93)
$$

Using (90), (91) and (93), (89) becomes

$$\begin{aligned}
(89) &> \left[-\int_{p-d}^{p-1}(p-x)^{n-1}f(x)dx+\int_{p+1}^{u}(x-p)^{n-1}f(x)dx\right]\ln(d)\\
&> \left[-\int_{p-d}^{p-1}(p-x)^{n-1}f(x)dx-\int_{p-1}^{p}(p-x)^{n-1}f(x)dx\right.\\
&+ \left.\int_{p}^{p+1}(x-p)^{n-1}f(x)dx+\int_{p+1}^{u}(x-p)^{n-1}f(x)dx\right]\ln(d)\\
&= \left[-\int_{p-d}^{p}(p-x)^{n-1}f(x)dx+\int_{p}^{u}(x-p)^{n-1}f(x)dx\right]\ln(d)\\
&= 0, \qquad (94)
\end{aligned}$$

where the inequality follows from the monotonicity of $f(\cdot)$, and the last equality follows from the first order condition (48).

Likewise, for $d < 1$ we have

$$\begin{aligned}
(89) &> \left[-\int_{p-d}^{p}(p-x)^{n-1}f(x)dx+\int_{p}^{u}(x-p)^{n-1}f(x)dx\right]\ln(d)\\
&= 0. \qquad (95)
\end{aligned}$$

Therefore the numerator is positive and

$$\frac{dp(n)}{dn} > 0 \qquad (96)$$

for right-skewed distributions. Combining this with the fact that $p(1) = \text{Median}(p^*)$ and $p(\infty) = \text{Midsupport}(p^*)$ we conclude that $p(n)$ monotonically increases from the median to the midpoint of the support of latent price distribution $f(\cdot)$, if $f(\cdot)$ is right-skewed. Analogously, for left-skewed $f(\cdot)$, $p(n)$ monotonically *de*creases from the median to the midpoint of the support.

D Proof of Proposition 4: Correlation in the Generalized Roll (1984) Model

¿From the setup of the generalized Roll (1984) model (Hasbrouck 2007), given by (57) and (58) the noise (9) is

$$\begin{aligned} \Delta u_t^{tra} &= s_t q_t - s_{t-1} q_{t-1} - \lambda(q_t - q_{t-1}) - \sqrt{h_t}\varepsilon_t + \sqrt{h_{t-1}}\varepsilon_{t-1}, \\ \Delta u_t^{mid} &= -\sqrt{h_t}\varepsilon_t + \sqrt{h_{t-1}}\varepsilon_{t-1}. \end{aligned} \tag{97}$$

The cross-covariance for the transaction price is therefore

$$\begin{aligned} Cov(\Delta p_t^*, \Delta u_t^{tra}) &= E\left((\lambda q_t + \sqrt{h_t}\varepsilon_t)(s_t q_t - s_{t-1} q_{t-1} - \lambda(q_t - q_{t-1}) - \sqrt{h_t}\varepsilon_t + \sqrt{h_{t-1}}\varepsilon_{t-1})\right) \\ &= \lambda(E(s_t) - \lambda) - \sigma^2. \end{aligned} \tag{98}$$

Further, the variances are

$$Var(\Delta p_t^*) = Var(\lambda q_t + \sqrt{h_t}\varepsilon_t) = E(\lambda^2 q_t^2) + E(h_t \varepsilon_t^2) - (E(\sqrt{h_t}\varepsilon_t))^2 = \lambda^2 + \sigma^2 \tag{99}$$

$$Var(\Delta u_t^{tra}) = 2E(s_t^2) - 4\lambda E(s_t) + 2\lambda^2 + 2\sigma^2. \tag{100}$$

Thus

$$Corr(\Delta p_t^*, \Delta u_t^{tra}) = \frac{\lambda(E(s_t) - \lambda) - \sigma^2}{\sqrt{2}\sqrt{\lambda^2 + \sigma^2}\sqrt{E(s_t^2) - 2\lambda E(s_t) + \lambda^2 + \sigma^2}} \leq \frac{1}{\sqrt{2}}, \tag{101}$$

where we have used that trades are not serially correlated.

$|Corr(\Delta p_t^*, \Delta u_t^{tra})| < \frac{1}{\sqrt{2}}$ can be shown by squaring both sides. After some algebra we obtain $-\lambda Var(s_t) < \sigma^2 E(s_t^2)$, which holds for any path of s_t. In particular, for $s_t = 0\ \forall t$ we have $Corr(\Delta p_t^*, \Delta u_t^{tra}) = -\frac{1}{\sqrt{2}}$.

The contemporaneous cross-correlation is positive if and only if $\lambda \in \left[\frac{E(s_t) - \sqrt{(E(s_t))^2 - 4\sigma^2}}{2}, \frac{E(s_t) + \sqrt{(E(s_t))^2 - 4\sigma^2}}{2}\right]$. A necessary condition for this is $E(s_t) > 2\sigma$. Further, if there is no random noise ($\sigma = 0$), then the contemporaneous cross-correlation is positive if $0 < \lambda < E(s_t)$.

Cross-covariances at nonzero displacements are as before

$$\begin{aligned} &Cov(\Delta p_{t-1}^*, \Delta u_t^{tra}) \\ &= E((\lambda q_{t-1} + \sqrt{h_{t-1}}\varepsilon_{t-1})(s_t q_t - s_{t-1} q_{t-1} - \lambda(q_t - q_{t-1}) - \sqrt{h_t}\varepsilon_t + \sqrt{h_{t-1}}\varepsilon_{t-1})) \end{aligned}$$

$$= \quad -\lambda(E(s_t) - \lambda) + \sigma^2 = -Cov(\Delta p_t^*, \Delta u_t^{tra}), \tag{102}$$

and for displacements $\tau > 1$ we have

$$\begin{aligned} &Cov(\Delta p_{t-\tau}^*, \Delta u_t^{tra}) \\ &= \quad E((\lambda q_{t-\tau} + \sqrt{h_{t-\tau}}\varepsilon_{t-\tau})(s_t q_t - s_{t-1} q_{t-1} - \lambda(q_t - q_{t-1}) - \sqrt{h_t}\varepsilon_t + \sqrt{h_{t-1}}\varepsilon_{t-1})) = 0 \end{aligned} \tag{103}$$

E Zero Profit Condition under Perfect Competition

If the market maker can quote bid and ask independently, he can effectively condition on q_t and therefore $\text{sgn}(\Delta p_t^*)$. Choosing p^{mid} and s independently is equivalent to choosing p^{bid} and p^{ask} independently. For the ask price the problem is

$$p(n)^{ask} = \underset{x \in [l,u]}{\text{argmax}} - \int_x^u (p^* - x)^n f(p^*) dp^*, \tag{104}$$

which is maximized at $p^{ask}(n) = \infty$, because absent any competition the market maker has no incentive to do any loss-bringing trades with informed traders. But competition with other market makers drives profit down to zero. Assuming that the zero profit condition must hold in expectation for each buy and sell independently, the spread $2s_t$ from (12), (13) and (14) is determined by[34]

$$-\int_{p^{ask}}^u \left(p^* - p^{ask}\right)^n f(p^*) dp^* + \pi(s) = 0. \tag{105}$$

Clearly, as in section 3.2 as $n \to \infty$ the ask price p^{ask} grows as well, without bound if $f(\cdot)$ has unbounded support. The expression for the bid price is analogous; the bid price falls with n. Because bid and ask price just offset one another, $\Delta p^{mid} = 0$, and thus $Corr(\Delta p_t^*, \Delta u_t^{mid}) = -1$. The optimal $|\Delta p_t^{tra}|$ is strictly increasing in the risk aversion n without bound. For uninformed trades, buys and sells are equally likely, thus for these trades $Corr(\Delta p_t^*, \Delta u_t^{tra}) < 0$. Rational informed traders trade only when $p^* > p^{ask}$ and $p^* < p^{bid}$; thus for them $Corr(\Delta p_t^*, \Delta u_t^{tra}) < 0$ as well. Overall, $Corr(\Delta p_t^*, \Delta u_t^{tra}) < 0 \ \forall n$.

In the following we show that s is uniquely determined as a competitive outcome. For simplicity, we assume risk neutrality $n = 1$. Under perfect competition the zero profit

[34] We assume here $\pi(s) = \pi$. Later in the appendix we consider the more general $\pi(s)$ case and show that p^{ask} is unique.

condition says that the market maker's losses by trades with informed traders are exactly offset by the spread earned from trades with uninformed traders. An increase in the spread benefits the market maker in two ways: it increases his spread income from uninformed traders and reduces his losses to the informed traders. This can be written as

$$-\int_{-\infty}^{p^{bid}} \left(p^* - p^{bid}\right) f(p^*)dp^* + \int_{p^{ask}}^{\infty} \left(p^{ask} - p^*\right) f(p^*)dp^*$$

$$= \left(\int_{p^{bid}}^{p^{ask}} f(p^*)dp^*\right) \frac{p^{ask} - p^{bid}}{2} \equiv U. \tag{106}$$

Using the symmetry of the expected density $f(p^*)$ around p^{mid}, (106) becomes a problem of setting p^{mid} and s.

$$-\int_{-\infty}^{p^{mid}-s} \left(p^* - p^{mid} + s\right) f(p^*)dp^* = \left(\int_{p^{mid}-s}^{p^{mid}} f(p^*)dp^*\right) s. \tag{107}$$

Then, with F denoting the cumulative density function of f (whose expected value is assumed to exist)

$$-\left(E(p^*)\left|_{-\infty}^{p^{mid}-s} \cdot F(p^{mid} - s) - (p^{mid} - s)F(p^{mid} - s)\right.\right)$$

$$= \left(F(p^{mid}) - F(p^{mid} - s)\right) s \tag{108}$$

where $E(p^*)\,|_{-\infty}^{p}$ denotes the expectation of p^* over the distribution $f(p^*)$ restricted to the interval $[-\infty, p]$ (but $f(\cdot)$ is in no way required to be positive on the entire interval). p^{mid} is given by the optimal learning rule. Then (108) is one equation in the one unknown s.

$$p^{mid} - E(p^*)\left|_{-\infty}^{p^{mid}-s} = \frac{sF(p^{mid})}{F(p^{mid} - s)}\right. \tag{109}$$

The left-hand side (LHS) is monotonically increasing in s from some positive number to positive infinity. The right-hand side (RHS) is monotonically increasing in s, from 0 to positive infinity. It can be shown that the RHS increases faster than the LHS and that this difference in slope does not go to zero as s becomes larger. Differentiating (107) using Leibnitz's rule, we get:

$$-\left(p^{mid} - s - p^{mid} + s\right) f(p^{mid} - s)\left(\frac{-1}{2}\right) - \int_{-\infty}^{p^{mid}-s} \frac{1}{2} f(p^*)dp^*$$

$$< -f(p^{mid} - s)\left(\frac{-1}{2}\right)s + \frac{1}{2}\left(\int_{p^{mid}-s}^{p^{mid}} f(p^*)dp^*\right) \tag{110}$$

$$\begin{aligned} -F(p^{mid} - s) &< \frac{s}{2}f(p^{mid} - s) + F(p^{mid}) - F(p^{mid} - s) \\ 0 &< sf(p^{mid} - s) + F(p^{mid}) \end{aligned} \tag{111}$$

(111) is a true statement. So (111) shows two things: Firstly, the RHS in (107) is increasing faster than the LHS. And secondly, the difference in slope is always at least $F(p^{mid}) > 0$. Hence we have proven that there is a single crossing and s is determined uniquely.

If the LHS in (109) is very small because the support of the distribution became very small by learning, then s must be small as well. Hence, as market makers learn, the spread s in the market shrinks. Clearly, if one market maker learned too slowly relative to his peers, he would make losses at least until the next change in p^*.

F Example of Optimal Learning

Assuming no discounting ($\delta = 1$), risk neutrality ($n = 1$), zero spread ($s = 0$) the general recursive problem (15) simplifies to,

$$\begin{aligned} V(l,u) = \max_{p^{mid}} \Bigg[& -\int_l^{p^{mid}} (p^{mid} - p^*)f(p^*)dp^* \\ & + V(l, p^{mid})F(p^{mid}) + V(p^{mid}, u)\left(1 - F(p^{mid}\right) \\ & - \int_{p^{mid}}^{u} (p^* - p^{mid})f(p^*)dp^* \Bigg]. \end{aligned} \tag{112}$$

To simplify the problem further, we assume as in the example in Aghion et al. (1991) that $f(\cdot)$ is uniform. Then, the position of the interval $[l, u]$ does not matter, only the interval length, $m = u - l$, is relevant. (112) becomes

$$\begin{aligned} V(m) &= \max_{\alpha} \quad -\frac{1}{m}\frac{1}{2}(\alpha m)^2 + V(m)\alpha \\ &\quad - \frac{1}{m}\frac{1}{2}\left((1-\alpha)m\right)^2 + V(m)(1-\alpha) \\ &= \max_{\alpha} \quad V(m) - \frac{m}{2}\left(\alpha^2 + 1 - 2\alpha + \alpha^2\right) \end{aligned}$$

$$= \max_{\alpha} \quad V(m) - m\left(\alpha^2 - \alpha + \frac{1}{2}\right). \tag{113}$$

From the first order condition we find the maximum

$$\alpha = 1/2, \tag{114}$$

thus optimal learning is achieved by repeated bisections.

This result just a special case of $f(\cdot)$ in (112) being symmetric. By symmetry, the term in brackets in (112) is symmetric around the symmetric point of $f(\cdot)$. Further, by monotonicity of $f(\cdot)$ on each side of the symmetry point we have that the optimal p^{mid} equals the median, and midpoint of support of $f(\cdot)$. For asymmetric $f(\cdot)$ the solutions are specific to the shape of $f(\cdot)$ and have to be determined numerically.

G Cross-Correlations in a Easley and O'Hara (1992) Model

The latent price process switches between two states of equal probability

$$p_t^* = \begin{cases} u & \text{with} \quad P(p_t^* = u) = 1/2 \\ l & \text{with} \quad P(p_t^* = l) = 1/2 \end{cases}. \tag{115}$$

Therefore,

$$\Delta p_0^* = \begin{cases} u - l & \text{with} \quad P(\Delta p_0^* = u - l) = 1/4 \\ 0 & \text{with} \quad P(\Delta p_0^* = 0) = 1/2 \\ l - u & \text{with} \quad P(\Delta p_0^* = l - u) = 1/4 \end{cases} \tag{116}$$

Denoting the number of periods through which new information remains private by T and assuming zero spread ($s = 0$) we can write

$$E(\Delta p_t^*) = 0 \tag{117}$$

$$E(\Delta p_t^{mid} - \Delta p_t^*) = 0 \tag{118}$$

$$E((\Delta p_t^*)^2) = \left(\frac{(u-l)^2}{4} + \frac{(l-u)^2}{4}\right)\frac{1}{T} = \frac{(u-l)^2}{2T} \equiv \frac{\sigma^2}{T} \tag{119}$$

$$E(\Delta p_t^* \Delta p_t^{mid}) = 0 \tag{120}$$

The contemporaneous cross-covariance ($\tau = 0$) is

$$\begin{aligned} Cov\left(\Delta p_t^*, \Delta u_t^{mid}\right) &= E\left(\Delta p_t^* (\Delta p_t^{mid} - \Delta p_t^*)\right) \\ &= E\left(\Delta p_t^* \Delta p_t^{mid}\right) - E\left(\Delta p_t^* \Delta p_t^*\right) \\ &= -\frac{\sigma^2}{T}. \end{aligned} \tag{121}$$

Using the result from Easley and O'Hara (1992) that transaction prices converge to the latent price at an exponential rate we get for $1 \leq \tau < T$

$$\begin{aligned} Cov\left(\Delta p_{t-\tau}^*, \Delta u_t^{mid}\right) &= E\left(\Delta p_{t-\tau}^* (\Delta p_t^{mid} - \Delta p_t^*)\right) \\ &= E\left(\Delta p_{t-\tau}^* \Delta p_t^{mid}\right) - E\left(\Delta p_{t-\tau}^* \Delta p_t^*\right) \\ &= \frac{\sigma^2}{T}\left(-e^{-r\tau} + e^{-r(\tau-1)}\right)^2. \end{aligned} \tag{122}$$

The variance of the noise Δu_t^{mid} is

$$\begin{aligned} & E\left((\Delta p_t^{mid} - \Delta p_t^*)^2\right) \\ &= E\left((\Delta p_t^{mid})^2\right) - 2E\left(\Delta p_t^{mid} \Delta p_t^*\right) + E\left((\Delta p_t^*)^2\right) \\ &= \frac{\sigma^2}{T} + \frac{1}{T}\left[\sum_{\tau=1}^{T-1} \sigma^2 \left(-e^{-r\tau} + e^{-r(\tau-1)}\right)^2 + \sigma^2 e^{-2r(T-1)}\right] \\ &= \frac{\sigma^2}{T}\left[1 + e^{-2r}(e^r - 1)^2 \sum_{\tau=0}^{T-2} \left(e^{-2r}\right)^\tau + e^{-2r(T-1)}\right] \\ &= \frac{\sigma^2}{T}\left[1 + e^{-2r}(e^r - 1)^2 \frac{(e^{-2r})^{T-1} - 1}{e^{-2r} - 1} + e^{-2r(T-1)}\right]. \end{aligned} \tag{123}$$

Denoting the term in brackets by $X(r, T)$ we get for the contemporaneous cross-correlation

$$\begin{aligned} Corr\left(\Delta p_t^*, \Delta u_t^{mid}\right) &= \frac{-\frac{\sigma^2}{T}}{\sqrt{\frac{\sigma^2}{T}}\sqrt{\frac{\sigma^2}{T} X(r,T)}} \\ &= \frac{-1}{\sqrt{X(r,T)}} > -\frac{1}{\sqrt{2}}, \end{aligned} \tag{124}$$

and for the lagged cross-correlation

$$\begin{aligned} Corr\left(\Delta p^*_{t-\tau}, \Delta u^{mid}_t\right) &= \frac{\frac{\sigma^2}{T}\left(-e^{-r\tau}+e^{-r(\tau-1)}\right)^2}{\sqrt{\frac{\sigma^2}{T}}\sqrt{\frac{\sigma^2}{T}X(r,T)}} \\ &= \frac{\left(-e^{-r\tau}+e^{-r(\tau-1)}\right)^2}{\sqrt{X(r,T)}} \\ &> \frac{\left(-e^{-r\tau}+e^{-r(\tau-1)}\right)^2}{\sqrt{2}}. \end{aligned} \tag{125}$$

H Cross-Correlations in a Kyle (1985) Model

In this section we derive cross-correlations for the model of Kyle (1985). In order to present a closed form solution we use continuous time, $t \in [0,T]$, but note that Kyle (1985) discussed the discrete time case as well.

Suppose every T periods the latent price jumps, and remains constant for the next T periods. Δp^*_0 remains private information for $T \geq 1$ periods, after which it becomes public knowledge. The jumps follow the distribution

$$\Delta p^*_0 \sim (0, \sigma^2). \tag{126}$$

p^* can be interpreted as the liquidation value of the asset at the end of the trading period T, with $p^* = p(0) + \Delta p^*_0$. Assuming a zero spread ($s = 0$) from the continuous auction equilibrium in Kyle (1985) the market price change at time t in absence of noise trading is[35]

$$dp(t) = \frac{\sigma^2}{\sigma(t)}(p^* - p(t))dt = \frac{1}{T-t}(p^* - p(t))dt, t \in [0,T] \tag{127}$$

where $\sigma(t) = (T-t)\sigma^2$ measures the informativeness of prices. Hence,

$$\frac{dp(t)}{dt} = \frac{1}{T-t}(p^* - p(t)) = \frac{1}{T-t}p^* - \frac{1}{T-t}p(t). \tag{128}$$

This simple inhomogeneous linear differential equation of degree one has the solution

$$p(t) = p(0) + \frac{t}{T}(p^* - p(0)), \tag{129}$$

[35]This uses the assumption of Kyle (1985) that the reaction functions for quantity demanded and prices are linear. There may exist nonlinear solutions as well.

and so

$$\Delta p_t^{mid} = \frac{1}{T}(p^* - p(0)) = \frac{1}{T}\Delta p_0^*. \tag{130}$$

Further,

$$\begin{aligned} E(\Delta p_t^*) &= 0 && (131)\\ E(\Delta p_t^{mid} - \Delta p_t^*) &= 0 && (132)\\ E((\Delta p_t^*)^2) &= \frac{\sigma^2}{T} && (133)\\ E(\Delta p_t^* \Delta p_t^{mid}) &= \frac{\sigma^2}{T^2} && (134) \end{aligned}$$

The contemporaneous cross-covariance ($\tau = 0$) is therefore

$$\begin{aligned} Cov\left(\Delta p_t^*, \Delta u_t^{mid}\right) &= E\left(\Delta p_t^* (\Delta p_t^{mid} - \Delta p_t^*)\right)\\ &= E\left(\Delta p_t^* \Delta p_t^{mid}\right) - E\left(\Delta p_t^* \Delta p_t^*\right)\\ &= \frac{\sigma^2}{T^2} - \frac{\sigma^2}{T}\\ &= \frac{\sigma^2}{T}\left(\frac{1}{T} - 1\right), \end{aligned} \tag{135}$$

and the lagged cross-covariance ($1 \leq \tau < T$)

$$\begin{aligned} Cov\left(\Delta p_{t-\tau}^*, \Delta u_t^{mid}\right) &= (E\left(\Delta p_{t-\tau}^* (\Delta p_t^{mid} - \Delta p_t^*)\right)\\ &= E\left(\Delta p_{t-\tau}^* \Delta p_t^{mid}\right) - E\left(\Delta p_{t-\tau}^* \Delta p_t^*\right)\\ &= E\left(\Delta p_{t-\tau}^* \frac{1}{T} \Delta p_0^*\right) \end{aligned} \tag{136}$$

(136) equals

$$\frac{\sigma^2}{T^2} \tag{137}$$

for $T > \tau \geq 1$, and zero otherwise.

The variance of the noise Δu_t^{mid} is

$$\begin{aligned} &E\left((\Delta p_t^{mid} - \Delta p_t^*)^2\right)\\ &= E\left(\left(\frac{1}{T}\Delta p_0^* - \Delta p_t^*\right)^2\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma^2}{T^2} + \frac{\sigma^2}{T} - \frac{2\sigma^2}{T^2} \\
&= \frac{\sigma^2}{T}\left(1 - \frac{1}{T}\right) \qquad (138)
\end{aligned}
$$

Therefore we get for the contemporaneous cross-correlation

$$
\begin{aligned}
Corr\left(\Delta p_t^*, \Delta u_t^{mid}\right) &= \frac{-\frac{\sigma^2}{T}\left(\frac{1}{T} - 1\right)}{\sqrt{\frac{\sigma^2}{T}}\sqrt{\frac{\sigma^2}{T}\left(1 - \frac{1}{T}\right)}} \\
&= -\sqrt{1 - \frac{1}{T}}, \qquad (139)
\end{aligned}
$$

and for the lagged cross-correlation for $T > \tau \geq 1$

$$
\begin{aligned}
Corr\left(\Delta p_{t-\tau}^*, \Delta u_t^{mid}\right) &= \frac{\frac{\sigma^2}{T^2}}{\sqrt{\frac{\sigma^2}{T}}\sqrt{\frac{\sigma^2}{T}\left(1 - \frac{1}{T}\right)}} \\
&= \frac{1}{\sqrt{T(T-1)}} < \frac{1}{\sqrt{2}}. \qquad (140)
\end{aligned}
$$