Technological Growth, Asset Pricing, and Consumption Risk over Long Horizons

Stavros Panageas*
The Wharton School
University of Pennsylvania

Jianfeng Yu
The Wharton School
University of Pennsylvania

December, 2005

Abstract

In this paper we develop a theoretical model in order to understand comovements between asset returns and consumption over longer horizons. We develop an intertemporal general equilibrium model featuring two types of shocks: "small", frequent and disembodied shocks to productivity and "large" technological innovations, which are embodied into new vintages of the capital stock. The latter type of shocks affect the economy with significant lags, since firms need to make irreversible investments in the new types of capital and there is an option value to waiting. The model produces endogenous cycles, countercyclical variation in risk premia, and only a very modest degree of predictability in consumption and dividend growth as observed in the data. In the model, the conventional consumption CAPM holds conditionally. Yet, by conditioning down we show that its resulting unconditional version takes a form that resembles closely the version of the CAPM used in the literature on eventual or long run risk, and most closely Parker and Juliard (2005). We then use the model as a laboratory to show that in our simulated data the unconditional consumption CAPM performs badly, while its "long-horizon" version performs significantly better.

*Contact: Stavros Panageas, 2326 SH-DH, 3620 Locust Walk, Philadelphia PA 19106, USA. email: panageas@wharton.upenn.edu. We would like to thank Andy Abel, Ricardo Caballero, Adlai Fischer (the discussant at NBER), Motohiro Yogo, and participants of seminars at the Wharton Finance Faculty Lunch, the Swedish School of Economics (Hagen), the Helsinki School of Economics GSF, the University of Cyprus, the Athens University of Economics and Business, the University of Piraeus, and the NBER Summer Institute for very helpful discussions and comments.

Technological Growth, Asset Pricing, and Consumption Risk over Long Horizons

Abstract

In this paper we develop a theoretical model in order to understand comovements between asset returns and consumption over longer horizons. We develop an intertemporal general equilibrium model featuring two types of shocks: "small", frequent and disembodied shocks to productivity and "large" technological innovations, which are embodied into new vintages of the capital stock. The latter type of shocks affect the economy with significant lags, since firms need to make irreversible investments in the new types of capital and there is an option value to waiting. The model produces endogenous cycles, countercyclical variation in risk premia, and only a very modest degree of predictability in consumption and dividend growth as observed in the data. In the model, the conventional consumption CAPM holds conditionally. Yet, by conditioning down we show that its resulting unconditional version takes a form that resembles closely the version of the CAPM used in the literature on eventual or long run risk, and most closely Parker and Juliard (2005). We then use the model as a laboratory to show that in our simulated data the unconditional consumption CAPM performs badly, while its "long-horizon" version performs significantly better.

1 Introduction

The process of invention, development and diffusion of new technologies has been widely studied in the economic literature. Hardly anyone would dispute that technological progress is the most important factor in determining living standards over the long run. It appears equally plausible that the anticipation of the benefits of technological advancement is a key determinant of asset price movements during many periods of economic history.

Our goal in this paper is a) to illustrate how adoption of large technological innovations will lead to long lasting cycles in output and asset prices and b) to use our model in order to understand the empirical success of recent literature[1] that has re-ignited interest in consumption based asset pricing by emphasizing the correlation between returns and consumption growth over long horizons ("eventual consumption risk" or "long run risk").

In particular we attempt to simultaneously answer a set of related questions. First, why has asset pricing based on contemporaneous correlations between returns and consumption growth fared so poorly, while its longer horizon counterpart has performed better ? And second, how is that empirical success compatible with a consumption process that seems to have strong random walk components in the data ? Our analysis suggests a strong link between these phenomena and the delayed reaction of the economy to a technological shock, as has been recently documented in the macroeconomic literature[2].

The key idea behind our theoretical framework is that productivity growth comes in the form of two shocks. The first type are "small", frequent, disembodied shocks, that affect earnings in the entire economy. One should think of them as daily news that appear in the financial press (variations in the supply of raw materials, political decisions that affect production, bad weather etc.). However, these types of shocks do not fundamentally alter the technology used to produce output. The second type of shocks are Poisson arrivals of major technological innovations, like automobiles, the internet, just in time manufacturing etc.. These shocks will not affect the economy on impact, but only with a lag. The reason is that firms will need to make investments in order to take advantage of these innovations, since they are embodied in new types of capital. Given the

[1] See e.g. Bansal and Yaron [2004], Bansal, Dittmar, and Lundblad [2004] and for the purposes of this paper Parker and Julliard [2005] in particular.

[2] See Vigfusson [2004] and references therein.

irreversibility of the investment decisions, and the high relative cost of these new technologies on arrival, there is going to be an endogenous lag between the impact of the second type of shock and its positive effects on output. Importantly, we show that the process of adoption of new technologies leads to endogenous persistence and cycles, even though both all shocks in the model arrive in a *pure i.i.d.* fashion.

The link to asset pricing revolves around the "life cycle" of growth options over the cycle of technological adoption. On impact of a major technological shock, growth options emerge in the prices of all securities. We show that these growth options are riskier than assets in place. Hence, in the initial phases of the technological cycle expected returns in the stock market are higher, simply because most growth options have not been exercised. As time passes, firms start to convert growth options into assets in place, hence reducing the implicit riskiness of their stock. Eventually, the new technology enters the region of diminishing marginal returns at the aggregate, most growth options get exercised and expected returns become particularly low.

Hence, the model produces countercyclical variation in risk premia. On impact of a major technological shock, the economy is (by construction) below its stochastic trend and hence anticipations of strong growth over the long run are associated with high expected returns, as most growth options have not been exercised. Once a technology becomes widely adopted, both financial returns and anticipations of growth over the long run decline simultaneously, as growth options become converted into physical assets.

An implication of assuming two shocks of different nature is that we can use the "small" disembodied shocks to retain a strong random walk component in consumption over short horizons, while having expected asset returns be determined by the degree to which major technological shocks have been "absorbed" by the economy - a long lasting process -. This allows us to simultaneously explain the empirical failure of a short horizon CAPM and the empirical success of a consumption CAPM over longer horizons.

Given that the model allows for non-trivial cross sectional heterogeneity across firms, we are able to characterize the properties of returns in the cross section. We show that the returns to small/value firms will be particularly exposed to eventual consumption risk, as they can profit the most from the arrival from the new technologies. This helps us reproduce the findings of Parker and Julliard [2005] in a calibrated version of our model.

1.1 Relation to the literature

The literature closest to this paper is the growth-option based asset pricing literature (Cochrane [1996], Berk, Green, and Naik [1999],Berk, Green, and Naik [2004], Kogan [2001], Kogan [2004], Gomes, Kogan, and Zhang [2003], Carlson, Fisher, and Giammarino [2004], Zhang [2005], Cooper [2004]). To the best of our knowledge, the only other paper to address the cross section of expected returns in general equilibrium is Gomes, Kogan, and Zhang [2003]. The two most significant differences between their model and ours is a) the distinction between "embodied" and "disembodied" technological shocks, and b) the presence of a true timing decision as to the exercise of the growth options. Gomes, Kogan, and Zhang [2003] assume a single persistent and predictable technological shock. Options arrive in an i.i.d. fashion across firms, and the firms must decide "on the spot" if they want to proceed with the investment or not. By contrast in our model, all firms are presented with a (firm specific) opportunity to plant a tree at the same time. However, they have full discretion as to the timing. This is not a mere technicality. It is the very reason for a substantial simultaneity in the exercise of growth options that leads to our endogenous cycles. Alternatively put, in Gomes, Kogan, and Zhang [2003] cycles emerge out of the assumption of a trend stationary productivity process. In our model, total factor productivity is a random walk, as is the arrival of new technologies. Cycles emerge fully *endogenously* as the result of technological adoption. The most important practical payoff of our assumptions is that the consumption process in our model will have a strong random walk component, unlike Gomes, Kogan, and Zhang [2003], where the consumption process is completely predictable. In Gomes, Kogan, and Zhang [2003] a decomposition between trend and cycle in the Beveridge-Nelson tradition would attribute all variation in consumption to cyclical fluctuations and none to a random walk component. This is an unattractive feature if a model is to use consumption for pricing purposes as we do. It would also appear hard as a result to explain the failure of approaches based on short run vs. long run correlations. By contrast, we are able to decompose the consumption process into a strong non-degenerate stochastic trend and a cycle as is routinely done in the literature. This also allows us to drive a wedge between correlations of returns with short and long horizon consumption.

We also relate to the recent literature on long run risks (Bansal and Yaron [2004], Hansen, Heaton, and Li [2005], Bansal, Dittmar, and Lundblad [2004] among others), and in particular to Parker and Julliard [2005], which forms our main empirical benchmark. Papers in the long run risk

literature use an Epstein-Zin utility specification that introduces long run risk into the pricing kernel by construction. From that point on, the papers study the empirical correlations between long run consumption growth and cross sectional returns, in order to explain the cross section of returns. In this paper our purpose is different: We try to understand economically *why* the correlation between short run returns is stronger with long horizon consumption growth and *why* certain types of stocks correlate with eventual consumption risk stronger than others. All of our results emerge as a result of the modelling of *technology* and not preferences as in the existing literature. Importantly, we are able to link the presence of long run risks to recent findings in the macroeconomics literature on the delayed reaction of the economy to technological shocks (see Vigfusson [2004]). Hence our work should be viewed as an attempt to deepen our theoretical understanding of consumption risk over long horizons and it naturally complements the existing literature.

It is also notable, that we are able to obtain the correlation between short run returns and long horizon consumption risk to matter without having to assume recursive preferences, or deal with the difficult issues of cointegration between dividends and consumption (see Hansen, Heaton, and Li [2005]). In our framework the *conditional* consumption CAPM holds, and aggregate dividends are cointegrated with consumption[3]. Yet, when we "condition down" in order to obtain an unconditional CAPM in our model we obtain equations that are strikingly similar to Parker and Julliard [2005], Bansal and Yaron [2004], Hansen, Heaton, and Li [2005] and Bansal, Dittmar, and Lundblad [2004]. Hence, "eventual" or "long run" risk seems to be a concept that is more robust than the specific assumptions used to derive it. Moreover, showing that long run risk matters is perfectly consistent with a model that drives variations in expected returns from time varying betas.

Our paper also complements the work of Menzly, Santos, and Veronesi [2004]. In that paper the behavior of consumption and dividends are assumed exogenously. Consumption is a random walk and hence by construction there can be no difference between "short run" correlations of returns and consumption growth and their long horizon counterparts. Interestingly, our analysis will endogenously produce a process for the dividends of a firm that will resemble Menzly, Santos, and Veronesi [2004], in the sense that the total dividends of an individual firm will be cointegrated

[3]This statement requires qualification. In our model there are recurrent equity issuances. The total dividends paid to all investors is cointegrated with aggregate consumption.However, if an investor does not participate in equity issuances in order to keep her fraction of the company constant, then the dividend payments to that investor will become asymptotically negligible as a fraction of the total dividends of the company.

with aggregate consumption growth. However, given that our consumption process has some small predictable components, we can additionally characterize the differences between short horizon and long horizon correlations.

There is a vast literature in macroeconomics and growth that analyzes innovation, dissemination of new technologies or the impact of the arrival of new capital vintages. A partial listing would include Jovanovic and Rousseau [2004], Jovanovic and MacDonald [1994], Jovanovic and Rousseau [2003], Greenwood and Jovanovic [1999], Atkeson and Kehoe [1999], Atkeson and Kehoe [1993], Helpman [1998], Comin and Gertler [2003]. Our paper has a fundamentally different scope than this literature. In most of these models, uncertainty and the pricing of risk is not the focus of the analysis. By contrast these papers analyze innovation decisions in much greater depth than we do. The trade-off is that they cannot allow for sufficiently rich uncertainty, and an endogenous determination of the stochastic discount factor as is possible in the simpler setup of our paper. This is why most of this literature cannot be readily used for an in-depth asset pricing analysis, which is necessarily linked with the pricing of risk.

An important technical contribution of our work is that it provides a tractable solution to a general equilibrium model, where the micro-decsions are "lumpy" and exhibit optimal stopping features. The micro decision of the firm has a similar structure to the recent sequence of papers by Abel and Eberly [2003], Abel and Eberly [2002], Abel and Eberly [2004]. Just as firms in these papers adapt to the technological frontier at an optimally chosen time, firms in our framework decide on the optimal time to plant a new tree. Moreover, by having cross sectional heterogeneity only at the beginning of an epoch, we can aggregate over firms in a much simpler way than the existing literature (Caballero and Engel [1999], Caballero and Engel [1991], Caballero and Pindyck [1996], Novy-Marx [2003])

Several recent papers have also attempted to address issues specific to the recent upswing and crash in asset prices (Pastor and Veronesi [2004], Jermann and Quadrini [2002]). Our purpose in this paper is broader. We want to understand how technological growth interacts with asset prices, and how it leads to long cycles at a more general level than the specifics of a particular historical episode.

The structure of the paper is as follows: Section 2 presents the model and Section 3 the resulting equilibrium allocations. Section 4 presents the qualitative implications of the model, while section

5 presents quantitative implications. Section 6 concludes. All proofs are given in the appendix.

2 The model

2.1 Trees, Firms and Technological Epochs

2.1.1 Trees, Earnings, Epochs and the Firm's Optimization Problem

There exists a continuum of firms indexed by $j \in [0,1]$. Each firm owns a collection of trees that have been planted in different technological epochs[4], and its total earnings is just the sum of the earnings produced by the trees it owns. Each tree in turn produces earnings that are the product of three components: a) a vintage specific component that is common across all trees of the same technological epoch, b) a time invariant tree specific component and c) an aggregate productivity shock. To introduce notation, let $Y_{N,i,t}$ denote the earnings stream of tree i at time t, which was planted in the technological epoch $N \in (-\infty .. -1, 0, 1, .. + \infty)$. In particular, assume the following functional form for $Y_{N,i,t}$:

$$Y_{N,i,t} = \left(\overline{A}\right)^N \zeta(i)\theta_t \tag{1}$$

$\left(\overline{A}\right)^N$ captures the vintage effect. $\overline{A} > 1$ is a constant. $\zeta(\cdot)$ is a positive strictly decreasing function on $[0,1]$, so that $\zeta(i)$ captures a tree specific effect. θ_t is the common productivity shock and evolves as a geometric Brownian Motion:

$$\frac{d\theta_t}{\theta_t} = \mu dt + \sigma dB_t \tag{2}$$

where $\mu > 0, \sigma > 0$ are constants, and B_t is a standard Brownian Motion.

Technological epochs arrive at the Poisson rate $\lambda > 0$. Once a new epoch arrives, the index N becomes $N+1$, and every firm gains the option to plant a single tree of the new vintage at a time of its choosing. Since $\overline{A} > 1$, and N grows to $N+1$, equation (1) reveals that trees of a later epoch are on average "better" than previous trees.

Firm heterogeneity is introduced as follows: Once epoch N arrives, firm j draws a random number $i_{j,N}$ from a uniform distribution on $[0,1]$. This number informs the firm of the type of tree that it can plant in the new epoch. In particular a firm that drew the number $i_{j,N}$ can plant a tree

[4] We shall also use word "round" to refer to an epoch.

with tree specific productivity $\zeta(i_{j,N})$. These numbers are drawn in an i.i.d fashion across epochs: It is possible that firm j draws a low $i_{j,N}$ in epoch n, a high $i_{j,N+1}$ in epoch $N+1$ etc.

To simplify the setup, we shall assume that once an epoch changes, the firm loses the option to plant a tree that corresponds to any previous epoch. It can only plant a tree corresponding to the technology of the current epoch.

Let:

$$X_{j,t} = \sum_{n=-\infty..N} \overline{A}^n \zeta(i_{j,n}) 1_{\{\tilde{\chi}_{n,j}=1\}} \tag{3}$$

where N denotes the technological epoch at time t and $\tilde{\chi}_{n,j}$ is an indicator function that is 1 if firm j decided to plant a tree in technological epoch N and 0 otherwise. A firm's total earnings are then given by:

$$Y_{j,t} = X_{j,t}\theta_t$$

Any given firm determines the time at which it plants a tree in an optimal manner. Planting a tree at time t requires a fixed cost of q_t. This cost is the same for all trees of a given epoch and represents payments that need to be given to "gardeners" who will plant these trees. To keep with the usual assumptions of a Lucas tree economy, we shall assume that the company finances these fixed payments by issuing equity in the amount q_t.We provide more details on how these costs are determined in section 2.4.

Assuming complete markets, the firm's objective is to maximize its share price. Given that options to plant a tree arrive in an i.i.d fashion across epochs, there is no linkage between the decision to plant a tree in this epoch and any future epochs. Thus, the option to plant a tree can be studied in isolation in each round. Hence, the optimization problem of firm j in epoch N amounts to choosing the optimal stopping time τ:

$$P^o_{N,j,t} \equiv \sup_{\tau} E_t \left\{ 1_{\{\tau<\tau_{N+1}\}} \left[\left(\overline{A}^N \zeta(i_{j,N}) \int_{\tau}^{\infty} \frac{H_s}{H_t} \theta_s ds \right) - \frac{H_\tau}{H_t} q_\tau \right] \right\} \tag{4}$$

where Π_s is the (endogenously determined) stochastic discount factor, τ_{N+1} is the random time at which the next epoch arrives, while $P^o_{N,j,t}$ denotes the (real) option of planting a new tree in epoch N.

2.1.2 Firm Prices

Given the setup, a firm's price will consist of three components: a) the value of assets in place, b) the value of the growth option in the current technological epoch and c) the value of the growth options in all subsequent epochs. To see this, let:

$$P^A_{j,t} \equiv X_{j,t}\left(E_t \int_t^\infty \frac{H_s}{H_t}\theta_s ds\right) \tag{5}$$

denote the value of assets in place (with $X_{j,t}$ as defined in [3]). Then the price of firm j, assuming it has not planted a tree (yet) in technological round N is

$$P_{N,j,t} = P^A_{j,t} \mid P^o_{N,j,t} \mid P^f_{N,t} \tag{6}$$

where:

$$P^f_{N,t} = E_t\left(\sum_{n=N+1..\infty} \frac{H_{\tau_n}}{H_t} P^o_{n,j,\tau_n}\right)$$

and τ_n denotes the time at which technological round n arrives. The first term on the right hand side of (6) is the value of assets in place, while the second term is the value of the growth option in the current epoch. The third term is the value of all future growth options. Naturally, for a firm that has planted a tree in the current technological epoch there is no longer a "live" option and hence its value is given by[5]:

$$P_{N,j,t} = P^A_{j,t} + E_t\left(\sum_{n=N+1...\infty} \frac{H_{\tau_n}}{H_t} P^o_{n,j,\tau_n}\right)$$

2.2 Aggregation

The total output in the economy at time t is given by

$$Y_t = \int_0^1 Y_t(j)dj = \left(\int_0^1 X_{j,t}dj\right)\theta_t = X_t\theta_t \tag{7}$$

with $X_{j,t}$ defined in (3) and X_t defined as

$$X_t = \int_0^1 X_{j,t}dj \tag{8}$$

[5] Clearly, the $P^A_{j,t}$ in this formula will now reflect the fact that the assets of the company have been increased by the addition of an extra tree.

It will be particularly useful to introduce one extra piece of notation. Let $K_{N,t} \in [0,1]$ denote the mass of firms that have updated their technology in technological epoch N up to time t. We show formally later that $K_{N,t}$ will coincide with the index of the most profitable tree that has not been planted yet (in the current epoch)[6].

Since investment in new trees is irreversible, $K_{N,t}$ (when viewed as a function of time) will be an increasing process. Given the definition of $K_{N,t}$, the aggregate output is given as

$$Y_t = \left[\sum_{n=-\infty..N-1} \overline{A}^{(n-N)} \left(\int_0^{K_{n,\overline{\tau_n}}} \zeta(i)di \right) + \int_0^{K_{N,t}} \zeta(i)di \right] \overline{A}^N \theta_t$$

where $\overline{\tau_n} = \tau_{n+1}$ denotes the time at which epoch n ended (and epoch $n+1$ started). To analyze this decomposition it will be easiest to define

$$F(x) = \int_0^x \zeta(i)di$$

It can easily be verified that, $F_x \geq 0$ (since $\zeta(\cdot) > 0$) and $F_{xx} < 0$, (since $\zeta(\cdot)$ is declining). Hence $F(x)$ has the two key properties of a production function. Using the definition of $F(\cdot)$, Y_t can be rewritten as

$$Y_t = \left[\sum_{n=-\infty..N-1} \overline{A}^{(n-N)} F\left(K_{n,\overline{\tau_n}}\right) + F(K_{N,t}) \right] \overline{A}^N \theta_t \tag{9}$$

The aggregate output is thus the product of two components: A stationary component (inside the square brackets) and a trending component $\left(\overline{A}^N \theta_t\right)$ which captures the joint effects of aggregate technological progress and aggregate productivity growth. The term inside the square brackets is a weighted average of the contributions of the different vintages of trees towards the aggregate product. The weight on trees that were planted in previous epochs decays geometrically[7] at the rate $\overline{A}$. In this sense, $\overline{A}$ is simultaneously the rate of technological progress (in terms of new trees) and technological obsolescence (in terms of existing ones).

[6] To see why, consider two firms j and j'. Assume that in the current epoch firm j has drawn a lower index i_N^j than firm j', so that $i_{j,N} < i_{j',N}$. By assumption $\zeta(\cdot)$ is a decreasing function and hence $\zeta(i_{j,N}) > \zeta(i_{j',N})$. This in turn implies that firm j has the ability to plant a better tree than firm j'. Since the costs of planting a tree in the current epoch are the same for the two companies, company j will always choose to plant a tree no later than company j'. Simply put, firms that can profit more from the new technology (since they have drawn a low i_N) have a strictly higher opportunity cost of waiting.

[7] Note that the summation runs from $-\infty$ to $N-1$ and *not* from N to $+\infty$

2.3 Markets

As is typically assumed in "Lucas Tree" models, each firm is fully equity financed and the representative agent holds all its shares. Moreover, claims to the output stream of these firms are the only assets in positive supply, and hence the total value of positive supply assets in the economy is:

$$P_{N,t} = \int_0^1 P_{N,j,t} dj$$

Next to the stock market for shares of each company there exists a (zero net supply) bond market, where agents can trade 0-coupon bonds of arbitrary maturity. We shall assume that markets are complete[8] Since markets are complete, the search for equilibrium prices can be reduced to the search for a stochastic discount factor H_t, which will coincide with the marginal utility of consumption for the representative agent. (See Karatzas and Shreve [1998], Chapter 4)

2.4 Consumers, Gardeners, and Preferences

To keep with Lucas's analogy of "trees", we shall assume that trees can only be planted by "gardeners". The economy is populated by a continuum of identical consumers/gardeners that can be aggregated into a single representative agent. The representative agent owns all the firms in the economy, and is also the (competitive) provider of gardening services.

Gardeners have no disutility of labor per se, but they have disutility of *learning*. In order to plant a new tree they need to study a "gardening manual" on how to plant the new trees and this is associated with a fixed cost η_t in terms of the numeraire. Since they are contractors, they will not receive wages after they plant the tree. Moreover, the disutility of reading the manual is irreversible, and hence they require firms to pay up-front at least a lump sum amount of η_t in order to compensate them for reading the manual. The knowledge that they gain from planting a tree for company A is not reusable if they plant a tree for firm B, and hence they need to read a manual every time they plant a new tree. Finally, perfect competition for gardening services will ensure

[8] In particular there exist markets where agents can trade securities (in zero net supply) that promise to pay 1 unit of the numeraire when technological round N arrives. These markets will be redundant in general equilibrium, since agents will be able to create dynamic portfolios of stocks and bonds that produce the same payoff as these claims. However, it will be easiest to assume their existence throughout to guarantee ex-ante that markets are complete.

that firms can always get a worker to plant a tree[9] as long as they pay her η_t. Hence the prevailing fixed costs for planting a tree will be $q_t = \eta_t$ and will be identical for all firms.

We shall allow the agent's utility to exhibit (external) habit formation w.r.t. to the running maximum of consumption for both substantive and technical reasons that will become clear in the next subsection. The representative consumer's preference over consumption streams is characterized by a utility function of the form

$$U(C_t, M_t^C)$$

where:

$$M_t^C = \max_{s \leq t} \{C_s\} \tag{10}$$

denotes the running maximum of consumption up to time t, and $U\left(C_t, M_t^C\right)$ satisfies $U_C > 0$, $U_{CC} < 0$, $U_{M^C} < 0$, $U_{CM^C} > 0$.

The consumer maximizes expected discounted utility over consumption plans[10] in a complete market:

$$\max_{C_s} E_t \left[\int_t^\infty e^{-\rho(s-t)} U(C_s, M_s^C) ds \right] \tag{11}$$

$$s.t.$$

$$E_t \left(\int_t^\infty \frac{H_s}{H_t} C_s ds \right) \leq \int_0^1 P_{N,j,t} dj + E_t \left(\sum_{n=-\infty..N_t} \int_t^\infty \frac{H_s}{H_t} q_s dK_s^n \right) \tag{12}$$

Note that the representative consumer owns all the trees and receives gardening fees q_t every time a firm plants a tree.

[9] In technical terms, firms face an elastic supply curve for learning effort, much like the supply of capital in the standard real business cycle model is elastic.

[10] To be more precise, let $\int_t^{t+\Delta} dl_s$ denote the change in the trees that all gardeners have planted. Then, the representative agent's disutility is given by:

$$\max_{C_s, dl_s} E_t \left[\int_t^\infty e^{-\rho(s-t)} U(C_s, M_s^C) ds - \int_t^\infty e^{-\rho(s-t)} U_C(s) \eta(s) dl_s \right]$$

since we have to aggregate over the disutility caused by planting the trees. If we were to impose then the general equilibrium clearing condition $dl_s = dK_s$, then we can show that the $q_t = \eta_t$ in general equilibrium and all allocations and prices are unchanged. To save notation we have chosen to assume an elastic supply of gardening services at the outset.

2.5 Functional Forms and Discussion

Before proceeding, we need to make certain assumptions on functional forms, in order to solve the model explicitly. The assumptions that we make are intended either a) to allow for tractability or b) to ensure that the solution of the model satisfies certain desirable properties.

The first assumption on functional form concerns the utility $U\left(C_t, M_t^C\right)$. We shall assume that

$$U\left(C_t, M_t^C\right) = \left(M_t^C\right)^{\gamma} \frac{C_t^{1-\gamma}}{1-\gamma} = \frac{\left(\frac{C_t}{M_t^C}\right)^{-\gamma} C_t}{1-\gamma}, \quad \gamma > 1 \tag{13}$$

It can be easily verified that $U_C > 0$, $U_{CC} < 0$, $U_{CM^C} > 0$, $U_{M^C} < 0$. This utility is a special case of the utilities studied in Abel [1990] and exhibits both "envy" ($U_{M^C} < 0$) and catching up with the Joneses ($U_{CM^C} > 0$) in the terminology of Dupor and Liu [2003]. The main difference is that the habit index is in terms of the past consumption maximum, not some exponential average of past consumption as in Campbell and Cochrane [1999] or Chan and Kogan [2002]. Using the running maximum of consumption M_t^C as the habit index is particularly attractive for our purposes, because of the analytic tractability that it will allow. As most habit level specifications already proposed in the literature, it has the very attractive property that it is "cointegrated" with aggregate consumption in the sense that the difference between $\log(C_t)$ and $\log(M_t^C)$ will be stationary. Moreover, the ratio between C_t and M_t^C will be bounded between 0 and 1 (as is the surplus in Campbell and Cochrane [1999]).

At a substantive level, this utility specification will serve four purposes: First, it will allow us to match first and second moments of the equity premium. Second, it will imply that the growth cycles that will arise in the model will leave interest rates unaffected. To see this, note that

$$U_C = \left(\frac{C_t}{M_t^C}\right)^{\gamma}$$

In equilibrium, it will turn out that

$$\frac{C_t}{M_t^C} = \frac{\theta_t}{\max_{s<t} \theta_s} \tag{14}$$

The right hand side of (14) is unaffected by the investment decisions of firms and this will in turn be true for the mean of the stochastic discount factor and therefore the real interest rate. This is a key advantage of this specification. Without habit formation most of the effects of technological innovations will work through the real interest rate, which is unattractive[11].Third,

[11] See Campbell and Cochrane [1999].

keeping interest rates unaffected will overcome an additional severe problem of constant relative risk aversion (CRRA) utilities: The arrival of a technological innovation will boost growth expectations. Therefore, agents with CRRA utilities will try to smooth future consumption gains by dissaving. In general equilibrium, savings must remain at 0, and so real interest rates will have to rise in order to induce savings. With a coefficient of relative risk aversion above 1, the smoothing motive will be sufficiently strong as to push the interest rate so high, that the market price to earnings ratio will *decline*. This appears to be at odds with the stock market boom that was observed in periods of rapid technological innovation as the nineties and the twenties. Our specification will guarantee that increases in consumption growth are exactly counterbalanced by increases in the habit level, so that interest rates are completely unaffected by growth cycles. Finally, these preferences will imply constant relative risk aversion, so that we can isolate the effect of fluctuations in the relative weight of growth options on expected returns. Clearly, with time varying risk aversion, our results would look even stronger.

Our next choice of functional form concerns the specification of the gardening fees q_t. In general , we think of "gardening" services as compensation for the "know how" that is provided by experts who need to invent, create and install the new capital stock. Our choice for the functional form of these costs is motivated by four main considerations: First, we want the magnitude of this compensation to share the same trend as aggregate output. Second, we want to keep the amount of gardening services provided stationary. Third, we want to keep the gardening fees constant within each epoch, in order to keep the analysis simple, tractable, and provide a link to the partial equilibrium literature on growth options. Fourth, we want to avoid jumps in the marginal utility of consumption once a new epoch arrives.

To give a specification that satisfies all three objectives simultaneously, define

$$M_t = \max_{s \leq t} \theta_s \tag{15}$$

and let

$$q_t = q \overline{A}^N M_{\tau_N} \tag{16}$$

where $q > 0$ is a constant, $\overline{A}^N$ is the vintage specific productivity of trees in the current epoch and M_{τ_N} is the value of the historical maximum of θ_t at the start of the technological epoch. Note

that these costs will grow between epochs[12]. However they will stay constant within an epoch. Moreover, they will share the same trend growth as consumption[13].

A final assumption that is made purely for technical convenience is that

$$\zeta(i) = \zeta_0(1-i)^{\nu}, \quad i \in [0,1] \tag{17}$$

where $\zeta_0, \nu > 0$ are constants.

2.6 Equilibrium

The equilibrium definition is standard. It requires that all markets clear and that all actions are optimal.

Definition 1 *A competitive equilibrium is a set of stochastic processes* $\langle C_t, K_{n,t}, H_t \rangle$ *s.t.*

a) C_t *solves the optimization problem (11) subject to (12)*

b) Firms solve the optimization problem (4) and $K_{n,t}$ *is defined as:*

$$K_{n,t} = \int_0^1 \widetilde{\chi}_{n,j,t} dj \tag{18}$$

where $\widetilde{\chi}_{n,j,t}$ *is an indicator that takes the value* 1 *if firm* j *has updated its technology in epoch* n *by time* t *and* 0 *otherwise.*

c) The goods market clears:

$$C_t = Y_t \text{ for all } t \geq 0 \tag{19}$$

e) The markets for all assets clear

If one could determine the optimal processes $K_{n,t}$, then the optimal consumption process could be readily determined by (19) and this would in turn imply that the equilibrium stochastic discount factor is given by:

$$H_t = e^{-\rho t} U_C \tag{20}$$

This observation suggests that the most natural way to proceed in order to determine an equilibrium is to make a conjecture about the stochastic discount factor H_t, solve for the optimal

[12] Since N will grow when an epoch changes, and $M_{\tau_{N+1}}$ will be higher than M_{τ_N}

[13] At a fundamental level, it appears sensible to make the disutility associated with adjustment grow with the rate of technological advancement, since more complex units of the capital stock probably require more elaborate education of the experts, who install these units.

stopping times in equation (4), aggregate in order to obtain the processes $K_{n,t}$ for $n = N, ...\infty$, and verify that the resulting consumption process satisfies (20). This is done in section 3.

3 Equilibrium Allocations

We first start by making a guess about the stochastic discount factor in general equilibrium. In particular we assume that the equilibrium stochastic discount factor is:

$$H_t = e^{-\rho t}\left(\frac{\theta_t}{M_t}\right)^{-\gamma} \tag{21}$$

with M_t defined as in (15). In Proposition 1 in the appendix we present the solution to the firm's optimal stopping problem under this stochastic discount factor. We also show that the consumption and investment process that results at the aggregate will satisfy (14) and hence verify that it forms a competitive equilibrium.

The solution to the optimal stopping problem of the firm has an intuitive "threshold" form: A firm should update when the ratio of aggregate productivity θ_t to its running maximum at the beginning of the current epoch (M_{τ^N}) crosses the threshold $\overline{\theta}^{(j)}$ given by

$$\overline{\theta}^{(j)} = \frac{\Xi}{\zeta(i_{N,j})}$$

where $\Xi > 0$ is an appropriate constant given explicitly in the appendix. Formally:

$$\tau^*_{j,N} = \inf_{t>\tau_N}\left\{t : \frac{\theta_t}{M_{\tau^N}} \geq \frac{\Xi}{\zeta(i_{N,j})}\right\} \tag{22}$$

The optimal policies of the firms possess four desirable and intuitive properties: First, no firm will find it optimal to plant a tree immediately when the new epoch arrives, as long as[14]:

$$\frac{\Xi}{\zeta(0)} > 1 \tag{23}$$

[14]To see why this condition is sufficient to guarantee that no firm will immediately plant a new tree once a new epoch arrives, rewrite the optimal policy as:

$$\tau^* = \inf\left\{t : \frac{\theta_t}{\theta_{\tau_N}} = \frac{\overline{\theta}}{\theta_{\tau_N}}\right\}$$

where θ_{τ_N} is the value of θ_t evaluated at the beginning of the current epoch (τ_N). By (47)

$$\frac{\overline{\theta}}{\theta_{\tau_N}} = \frac{M_{\tau_N}}{\theta_{\tau_N}}\frac{\Xi}{\zeta(i_{N,j})} > 1$$

since $\frac{M_{\tau_N}}{\theta_{\tau_N}} \geq 1$ and $\min_{i_{N,j}} \frac{\Xi}{\zeta(i_{N,j})} = \frac{\Xi}{\zeta(0)} > 1$ by (23)

which we assume throughout. To see why this condition is sufficient to induce waiting, examine (22) and note that at the beginning of an epoch $\frac{\theta_{\tau_N}}{M_{\tau_N}} \leq 1$. Hence all firms (even the most productive one) will be "below" their investment thresholds.

Second, a key implication of (22) is that the firms that have the option to plant a more "productive" tree will always go first, since the threshold $\overline{\theta}^{(j)}$ will be lower for them. This is intuitive: A firm which can profit more from planting a tree has a higher opportunity cost of waiting and should always plant a tree first.

Third, and most importantly, there are going to be strong correlations between the optimal investment decisions of the firms. Conditional on $\frac{\theta_t}{M_{\tau_N}}$ reaching the relevant investment threshold $\frac{\Xi}{\zeta(0)}$ for the first firm, a number of other firms will also find it optimal to invest in very short time. This is simply because both $\zeta(i)$ is a continuous function of i and θ_t is a continuous function of time.

Finally, an attractive technical feature of the threshold nature of the individual firm policies is that they provide a simple way to aggregate over firms: The fraction of firms $K_{N,t}$ that have planted a tree in the current epoch is given by:

$$K_{N,t} = \max\left\{0, \zeta^{-1}\left(\frac{\Xi}{\left(\frac{M_t}{M_{\tau_N}}\right)}\right)\right\} \tag{24}$$

where $\zeta^{-1}(\cdot)$ denotes the inverse of the function ζ and $\frac{M_t}{M_{\tau_N}}$ is the ratio of the running maximum of θ_t to its level at the beginning of the epoch (M_{τ_N}). Hence a single state variable $\left(\frac{M_t}{M_{\tau_N}}\right)$ is a sufficient statistic for the fraction of firms that have invested in the current period.

In sum, the model implies two distinct regimes in terms of "planting" new trees at the aggregate: In the first, no firm finds it optimal to invest, while in the second regime a number of companies proceed with investment in new trees in close distance to each other.

Figure 1 gives a visual impression of these facts by plotting the impulse response function of a shock to N_t. I.e. it plots the response of

$$E\left(\log(C_{t+s+1}) - \log(C_{t+s})|N_t = n+1\right) - E\left(\log(C_{t+s+1}) - \log(C_{t+s})|N_t = n\right)$$

assuming that increments (shocks) to θ_t are set to 0 and N_t experiences no further shocks past time t.

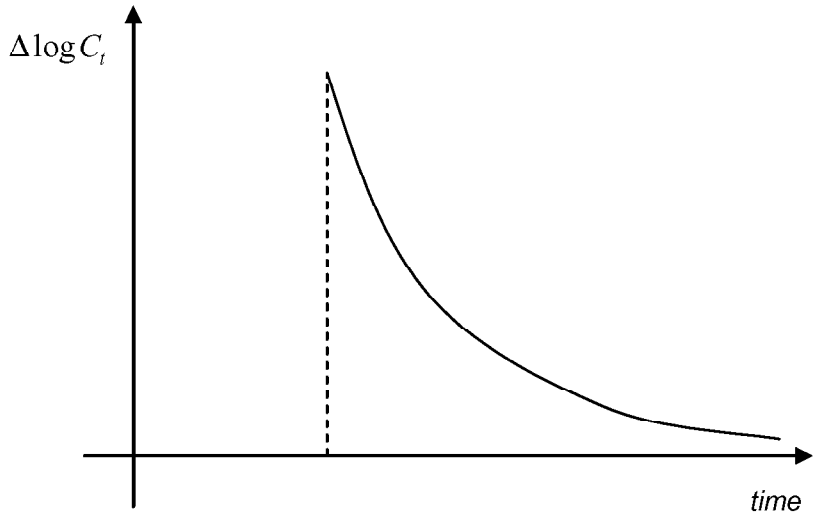


FIGURE 1: Impulse response function of a shock to the Poisson process N_t.

As can be seen, in the short run consumption is unaffected, as all firms are waiting to invest. Once however the threshold for the first firm is reached, then the growth rate of consumption peaks and starts to decline thereafter. The intuition for this decline is the following: the most profitable firms start investing first, and hence the most productive investment opportunities are depleted. This leaves less attractive investment opportunities unexploited and hence a moderation in the anticipated growth rate of the economy going forward. This delayed reaction of the economy to a major technological shock is consistent with recent findings in the macroeconomic literature (Vigfusson [2004]).

Another interesting implication of the behavior of aggregate consumption can be seen upon examining formula (9). Taking logs, this equation becomes:

$$\log(Y_t) = \log(C_t) = \log(\theta_t) + N \log(\overline{A}) + x_t \tag{25}$$

where we have defined x_t to equal:

$$x_t = \log\left(\frac{X_t}{\overline{A}^N}\right) = \log\left[\sum_{n=-\infty..N-1} \overline{A}^{(n-N)} F\left(K_{n,\overline{\tau_n}}\right) + F(K_{N,t})\right] \tag{26}$$

x_t is a geometrically declining average (at the rate $\frac{1}{\overline{A}}$) of the random terms $F(K_{n,\overline{\tau_n}})$. This means that x_t would behave exactly as an AR(1) process (across epochs) even if the terms $F(K_{n,\overline{\tau_n}})$ were

perfectly i.i.d. across epochs. As a matter of fact, one can show that there is small but positive persistence in these terms that further amplifies the persistence in x_t, and that x_t is a stationary process that has a stationary distribution.

Hence, the model is able to produce *endogenous* cycles, on top of the pure random walk stochastic trend $\log(\theta_t) + N\log(\overline{A})$ that we assumed at the outset. This will preserve a strong random walk component in consumption, which is desirable from an empirical point of view[15].

Finally, it will be useful for future reference to apply a Beveridge Nelson decomposition to $\log(C_t)$ in order to obtain:

$$\begin{aligned} \log(\theta_t) + N\log(\overline{A}) + E(x) &= \lim_{k\to\infty} \{E_t \log(C_{t+k}) - kE[\log(C_{t+1}) - \log(C_t)]\} = \\ &= \log(C_t) + \int_t^{\infty} [E_t\left(d\log(C_{t+j})\right) - E\left(d\log(C_{t+j})\right)] \end{aligned} \tag{27}$$

Combining (25), (26) and (27) gives:

$$x_t - E(x) = -\int_t^{\infty} [E_t\left(d\log(C_{t+j})\right) - E\left(d\log(C_{t+j})\right)]$$

This equation shows that $x_t - E(x)$ can be thought of as a measure of the distance between actual output and stochastic trend. Whenever that difference is negative, this means that the economy has not absorbed the full benefit of existing technology which is captured in the stochastic trend. Therefore future growth rates will be large. By contrast whenever $x_t - E(x)$ is positive, this means that the economy is above its trend line, and the future growth rates will be moderate.

Figure 2 illustrates the notions of trend and cycle, developed in this section.

4 Equilibrium Prices and Returns

4.1 Countercyclical Variation in Returns

[15] Existing general equilibrium models like Gomes, Kogan, and Zhang [2003] assume that total factor productivity follows a mean reverting process. Hence business cycles are effectively assumed at the outset. By contrast our model starts from a random walk assumption and derives the business cycle endogenously. Among many other advantages, this has the practical implication that consumption has a truly stochastic trend (random walk) that makes it unpredictable in contrast to Gomes, Kogan, and Zhang [2003].

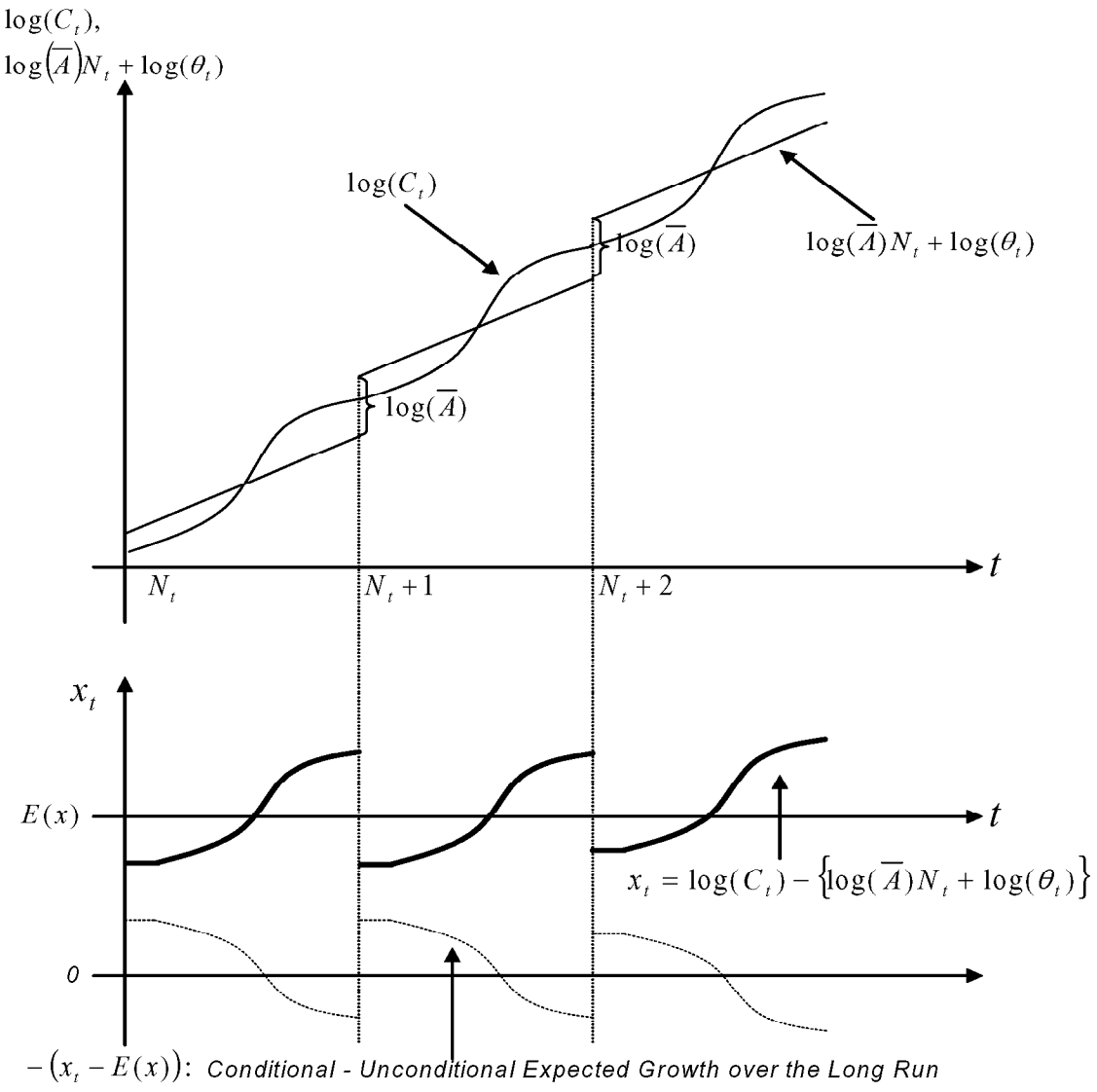


FIGURE 2: This figure depicts the trend $\log(\overline{A})N_t + \log(\theta_t)$ and the actual level of (log) consumption $\log(C_t)$, as well as the difference between the two. To illustrate the behavior of a "typical" path, we have set the Brownian increments (dB_t) to be equal to 0 so that $\log(\theta_t) = \left(\mu - \frac{\sigma^2}{2}\right) t$.

The price of a firm in general equilibrium is given by (6). Equation (6) decomposes the price of a firm in three components: 1) the value of assets in place, 2) The value of growth options in the current technological epoch and 3) The value of growth options in all subsequent technological epochs. In the appendix (Proposition 4) we give closed form expressions for both the value of a single firm and the value of the aggregate stock market.

In what follows we study the relative importance of these three terms at the aggregate. In particular, let the ratio of growth options in the aggregate stock market be defined as:

$$w_t^{o+f} = \frac{\left(\int_{K_{N,t}}^1 P_{N,j,t}^o dj\right) + P_{N,t}^f}{\left(\int_0^1 P_{N,j,t}^A dj\right) + \left(\int_{K_{N,t}}^1 P_{N,j,t}^o dj\right) + P_{N,t}^f}$$

We shall also denote w_t^o and w_t^f as the weight of current epoch growth options and future growth options respectively.

Lemma 1 *The fraction of total growth options w_t^{o+f} in the stock market is given by:*

$$w_t^{o+f} = \frac{e^{-x_t}}{f\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right) + e^{-x_t}} \tag{28}$$

where x_t was defined in equation (26) and $f\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right)$ is an appropriate positive function given explicitly in the appendix.

The expression (28) has a form similar to a standard logistic function. For our purposes, the most important advantage of this formula is that it gives us an explicit relation between x_t and w_t^{o+f}. To keep the formulas simple and build intuition it is simplest to expand in a Taylor fashion around $E(x)$ for fixed values of $\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}$. The Taylor expansion gives

$$\begin{aligned} \frac{w_t^{o+f}}{\overline{w}} &\simeq 1 - \xi\left[x_t - E(x)\right] = \\ &= 1 + \xi \int_t^\infty \left[E_t\left(d\log(C_{t+j})\right) - E\left(d\log(C_{t+j})\right)\right] \end{aligned} \tag{29}$$

for appropriate [16] $\overline{w}, \xi > 0$. The above equation shows that the weight of growth options is coun-

[16] Explicitly these constants are given by:

$$\begin{aligned} \overline{w} &= \frac{e^{-(E(x))}}{f\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right) + e^{-(E(x))}} < 1 \\ \xi &= \frac{f\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right)}{f\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right) + e^{-(E(x))}} < 1 \end{aligned}$$

tercyclical. This is intuitive: When the current level of consumption is below its stochastic trend, this is an indication that there is a large number of unexploited investment opportunities for firms. Accordingly, the relative weight of growth options at the level of the aggregate stock market will be substantial. By contrast, when consumption is above its trend level, this is the indication that the most profitable investment opportunities have been exploited, and the relative importance of growth options is moderate.

The importance of growth options in the aggregate stock market is of paramount importance for the asset pricing implications of the model in light of the following result:

Lemma 2 *The expected (excess) return of current epoch growth options* $(\mu^o - r)$ *is strictly larger than the expected (excess) return of future growth options* $\left(\mu^f - r\right)$*, which in turn is strictly larger than the expected (excess) return of assets in place* $\left(\mu^A - r\right)$*.*

Since the aggregate excess return is just a convex combination of the three returns, we obtain that the excess return one the aggregate stock market is:

$$\mu - r = \left(1 - w_t^{o+f}\right)\left(\mu^A - r\right) + w_t^{o+f}\left[\frac{w_t^o}{w_t^{o+f}}\left(\mu^o - r\right) + \frac{w_t^f}{w_t^{o+f}}\left(\mu^f - r\right)\right] \tag{30}$$

The closed form solutions in the appendix can be used to show that $\mu^A - r$ is a function of $\frac{\theta_t}{M_t}$ only, while the term inside the square brackets depends only on $\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}$. Combining (29) and (30) can be used to show that:

$$\begin{aligned} \mu_t - r &\simeq f_1\left(\frac{\theta_t}{M_t}\right) - f_2\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right)(x_t - E(x)) \\ &= f_1\left(\frac{\theta_t}{M_t}\right) + f_2\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right)\left(\int_t^\infty \left[E_t\left(d\log(C_{t+j})\right) - E\left(d\log(C_{t+j})\right)\right]\right) \end{aligned} \tag{31}$$

with $f_2\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right) > 0$. This formula shows how countercyclical variation in the relative importance of growth options translates into countercyclical variation in the expected return: When the economy is below its stochastic trend, there are numerous growth options, which are risky in light of Lemma 2. This pushes aggregate expected excess returns upwards. However, as growth opportunities get exploited, their relative importance and hence the expected excess returns in the stock market decline.

4.1.1 Predictability of returns and fundamentals in the time series

Equation (31) can help us understand the answer to two questions: First, which variables should predict excess returns ? And second, why don't valuation ratios like the wealth to consumption ratio and the P/D ratios predict consumption and dividend growth in the data ? The second question is of particular importance, since it appears to be the most important concern that comes to mind with models that feature predictable consumption / dividend growth.

In this subsection we provide some qualitative answers suggested by the model and leave the quantitative assessment for the next section.

The answer to the first question is a direct consequence of equation (31). This equation suggests that there are three (stationary) variables that will predict returns. The first is $x_t - E(x)$, the second is $\frac{M_t}{M_{\tau_N}}$ and the third is $\frac{\theta_t}{M_t}$. These three variables have substantially different degrees of persistence, are postitively correlated, and most importantly capture different economic forces of the model. The first variable is the most persistent one, and captures the distance between the current level of output and its stochastic trend. The second is somewhat less persistent and it maps in a one-to-one fashion to the number of companies that have planted a tree in the current epoch (see equation[24]) . The third is the least persistent of the three. It captures variations in habit and will affect both the P/E ratio, the excess returns and the spread between short-long bonds. This suggests that different state variables will be useful in predicting returns over different horizons.

Since these underlying state variables are not immediately observable in the data, the question then becomes whether there exist 3 "real-world" variables that can uncover the variation in these underlying state variables. The answer is yes. Given the structure of the model, interest rates are only affected by variations in $\frac{\theta_t}{M_t}$, while there is a one-to-one relation between the fraction of firms that have invested in the current epoch and $\frac{M_t}{M_{\tau_N}}$. Hence, the term premium will reveal variations in $\frac{\theta_t}{M_t}$, while variations in equity issuance activity (and investment) will allow us to uncover variations in $\frac{M_t}{M_{\tau_N}}$. Finally, in the appendix we show that the aggregate P/D ratio will reflect variation in all three components and in particular $x_t - E(x)$. Hence, according to the model, these three variables should all have independent explanatory power in uncovering variation in expected returns. There is a large empirical literature documenting the ability of each of these variables to predict returns[17].

One issue that arises with our analysis concerns the predictability of consumption and dividends.

[17] See e.g. page 393 in Cochrane [2005]

As we show in the quantitative section of the model, the variation in the P/D ratio is driven to a large extent by the discount factor and not the predictability of dividend payments. This may seem puzzling, since in our model, consumption and dividend growth have predictable components in contrast to Campbell and Cochrane [1999].

The resolution of the puzzle is that the variability of excess returns is positively correlated with anticipated consumption growth in light of (31). To build intuition and avoid issues related with the distinction between consumption and dividends, we consider a fictitious claim to aggregate consumption, whose price is given by P_t^C, and coincides with aggregate wealth. Using the same steps as in section 4.1, we can show that the returns to such a claim will be countercyclical. Now consider the well known log linearization:

$$\log\left(P_t^C\right) - \log(C_t) = \phi_1 + E_t \sum_{j=1}^{\infty} \varrho^{j-1} \left[\Delta c_{t+j} - E\left(\Delta c_{t+j}\right)\right] - E_t \sum_{j=1}^{\infty} \varrho^{j-1} r_{t+j} \tag{32}$$

where ϕ_1 is an appropriate constant and c_t is the log of aggregate consumption. Motivated by (31), suppose that we postulate its discrete time analog:

$$r_t = \beta_0 + \beta_1 E_t \sum_{j=1}^{\infty} \left[\Delta c_{t+j} - E\left(\Delta c_{t+j}\right)\right] + R\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right) \tag{33}$$

where β_0,β_1 are two positive constants and $R\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right)$ is an appropriate (remainder) function that depends on $\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}$ only. In our model the conditional expectation on the right hand side of (33) exhibits positive dependence across different points in time. We can approximate this property of the model by postulating that:

$$E_t\left(E_{t+T} \sum_{j=1}^{\infty} \left[\Delta c_{t+T+j} - E\left(\Delta c_{t+T+j}\right)\right]\right) = \upsilon^T \left(E_t \sum_{j=1}^{\infty} \left[\Delta c_{t+j} - E\left(\Delta c_{t+j}\right)\right]\right) \tag{34}$$

for some $\upsilon < 1$. In the model relation (34) will not hold exactly, only approximately. However, it will facilitate the calculations that follow and help convey the intuition. Combining (32), (33) and (34) we obtain:

$$\begin{aligned} \log\left(P_t^C\right) - \log(C_t) \quad = \quad & \phi_2 + E_t \sum_{j-1}^{\infty} \varrho^{j-1} \left[\Delta c_{t+j} - E\left(\Delta c_{t+j}\right)\right] \\ & -\frac{\beta_1}{1-\varrho\upsilon} \left(E_t \sum_{j=1}^{\infty} \left[\Delta c_{t+j} - E\left(\Delta c_{t+j}\right)\right]\right) - Z\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right) \end{aligned} \tag{35}$$

for an appropriate constant ϕ_2 and an appropriate function[18] $Z\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right)$. Given that both in the data and in the model $\varrho \backsim 1$ we shall approximate

$$E_t \sum_{j=1}^{\infty} \varrho^{j-1} \left[\Delta c_{t\,|\,j} - E\left(\Delta c_{t\,|\,j}\right)\right] \sim E_t \sum_{j=1}^{\infty} \left[\Delta c_{t\,|\,j} - E\left(\Delta c_{t\,|\,j}\right)\right] \tag{36}$$

Combining (35) and (36) gives:

$$\log\left(P_t^C\right) - \log(C_t) \sim \phi_2 + \left(1 - \frac{\beta_1}{1-\varrho\upsilon}\right) E_t \sum_{j=1}^{\infty} \left[\Delta c_{t+j} - E\left(\Delta c_{t+j}\right)\right] - Z\left(\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}\right)$$

The term $\left(1 - \frac{\beta_1}{1-\varrho\upsilon}\right) < 1$ captures the reason why the P^C/C will not be a particularly successful predictor of future consumption growth in this model. To make this case as extreme as possible suppose that $\beta_1 + \varrho\upsilon = 1$, so that $1 - \frac{\beta_1}{1-\varrho\upsilon} = 0$. Then, keeping $\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}$ fixed[19], one would obtain the result that the P^C/C ratio reveals no information on the economy's gap to its stochastic trend. Even if $\beta_1 + \varrho\upsilon$ is somewhat smaller than 1, the ability of any valuation ratio to predict consumption growth will be substantially attenuated. Finally, observe that even a small β_1 is enough to bring $\beta_1 + \varrho\upsilon$ close to 1, as long as $\varrho\upsilon$ is close to 1. Given that ϱ is already very close to 1, this implies that even moderate degrees of persistence in (34) will be enough to bring $\beta_1 + \varrho\upsilon$ close to 1. Intuitively, even a small sensitivity of the excess return to anticipations of long run growth (β_1) will be amplified by the infinite sum in (32).

To summarize, the countercyclical variation in returns will tend to substantially attenuate the ability of the wealth to consumption ratio to predict consumption growth. A similar argument applies to the P/D ratio and its ability to predict dividend growth. We quantify this statement in section 5. In the empirical literature Lettau and Ludvigson [2005] attribute the limited ability of the P/D ratio to predict dividend growth to the negative correlation between long run returns and dividends. Similarly, Larrain and Yogo [2005] find evidence in favor of countercyclical variation in returns.

We conclude this section with a discussion of the sign that we should expect in a regression of future expected returns on the current P/D ratio. One would anticipate that the P/D ratio will

[18] The function Z depends on $\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}$ because both of these processes are Markovian, and hence future expectations of them can at most depend on $\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}$.

[19] This is an important qualification, since there is always going to be some positive correlation between $\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}$ and $E_t \sum_{j=1}^{\infty} \left[\Delta d_{t+j} - E\left(\Delta d_{t+j}\right)\right]$, which will help predict dividend growth to some extent.

be high when growth options have not been exercised, and low if they have. Hence that should imply a positive relationship between expected excess returns and the P/D ratio, rather than the empirically observed negative relation. Nevertheless, as we show in section 5, when we perform the usual long horizon predictability regressions, we obtain a negative relationship.

The resolution of the puzzle lies in the difference between long horizon and instantaneous expected returns. The easiest way to see this is to consider an individual firm and study the evolution of its P/E ratio over a technological epoch: The top plot of figure 3 depicts the P/E ratio and the *instantaneous* expected return of the firm. Clearly, the two are positively correlated: As long as a firm has not planted a tree, the fraction of growth options in its price is large and so is its P/E ratio and its expected return in light of Lemma 2. Once the firm plants a tree, its P/E ratio experiences a discontinuous drop, and so does its instantaneous expected return. This reflects the transformation of growth options into assets in place.

To compare, the bottom plot depicts the P/E ratio against the *average instantaneous* expected return between t and $t+T$. The period T is the average time it takes to plant a tree, starting at the beginning of an epoch. Now, there is a negative relation between the P/E and the average expected return, at least before a firm decides to invest. The reason is that a high P/E is associated with both a high fraction of growth options but also with a short time to option exercise and accordingly a reversal in expected returns.

This qualitative pattern for the average expected return would hold as long as we averaged over any $T_1 > T$ periods. For intervals shorter than T we would obtain a hump shaped pattern for the average expected return and hence no clear positive or negative relationship. By aggregating over firms we can extend these results to the aggregate stock market, since the investment decisions of firms are strongly correlated. The main difference between the picture at the aggregate level and the individual firm level is that the decline in the P/E ratio does not occur in a discontinuous fashion, but is more gradual and smooth.

In conclusion, as long as we predict returns over long horizons, we should expect a negative relationship between the P/E return and expected returns. Over short horizons however, this relationship could be reversed and even be positive.

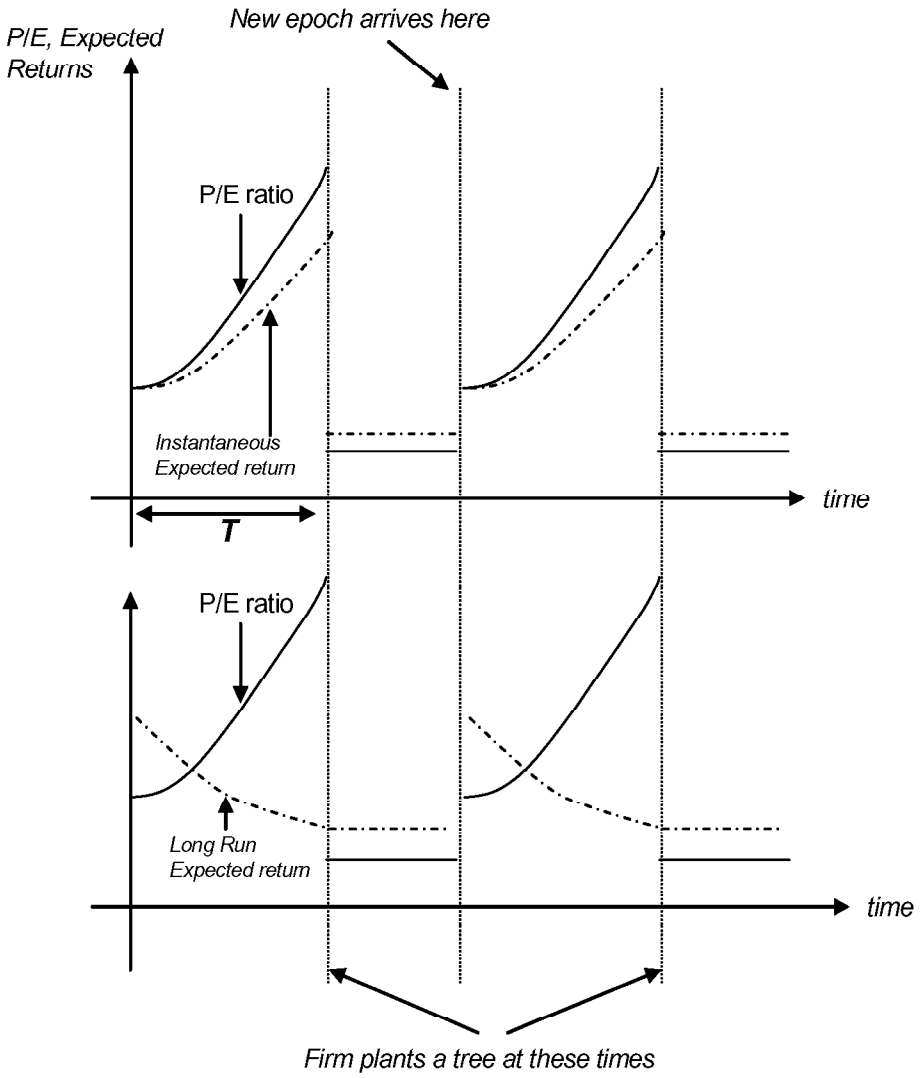


FIGURE 3: The top plot depicts the P/E ratio and the instantaneous expected return. The bottom plot depicts the P/E ratio against the average expected return over T periods, where T is the average time it takes to plant a tree. To pick a "typical" path we set the Brownian increments (dB_t) equal tp 0.

4.1.2 Cross Sectional Predictability and Eventual Consumption Risk

The last section developed the implications of the model for the aggregate stock market. We now turn to the cross sectional implications of the model. Our focus in this section will be to show jointly a) why the model is able to reproduce a size and a value premium and b) why these cross sectional phenomena can be explained by a consumption CAPM including "long run" consumption growth instead of quarterly consumption growth.

To show the first assertion, we start by defining the relative weight of growth options in a company's price:

$$w_t^{(j),o+f} = \frac{P^o_{N,j,t} + P^f_{N,t}}{P^A_{N,j,t} \mid P^o_{N,j,t} \mid P^f_{N,t}}$$

where $P^o_{N,j,t} = 0$ if company j has already planted a tree in round N. It will be easiest to assume that the firm under consideration has exercised its growth option in the current epoch, so that $P^o_{N,j,t} = 0$.Now let $X_{j,t}$ be defined as in (3) and let:

$$x_t^{(j)} = \log\left(\frac{X_{j,t}}{\overline{A}^N}\right)$$

The appendix shows that the relative importance of growth options in the price of company j (assuming it has exercised its current period growth option) is given by:

$$w_t^{(j),o+f} = \frac{e^{-x_t^{(j)}}}{g\left(\frac{\theta_t}{M_t}\right) + e^{-x_t^{(j)}}} = \frac{e^{-x_t}}{e^{\left(x_t^{(j)} - x_t\right)} g\left(\frac{\theta_t}{M_t}\right) + e^{-x_t}} \tag{37}$$

for an appropriate function g[20] and for x_t as defined in (26). Just as most empirical research we will be interested in deriving the properties of an asset (or portfolio) that is selected so that $x_t^{(j)} - x_t$ is constant at each point in time. We shall denote that distance as:

$$\kappa^{(j)} = x_t^{(j)} \quad x_t$$

$\kappa^{(j)}$. It is clear that sorting on size and value will produce sorts very similar to a sort based on $\kappa^{(j)}$. The reason is intuitive: Assume that all firms have exercised their current epoch growth options. Since all companies have the same future growth options, a higher index $\kappa^{(j)}$ indicates that a firm has planted more trees than the other firms in the economy and hence has a higher market value. Moreover, a higher ratio of market to book value of trees will also be associated with high values of

[20] If the company has not exercised its growth option in the current epoch, then g will also depend on $\zeta(i_{j,n})$.

$\kappa^{(j)}$, as it will indicate that a given firm has drawn more productive trees than the average firm in the past[21]. In sum a sort based on $\kappa^{(j)}$ is selecting companies with high Market/Book values and large size[22]. Additionally, taking account of current period growth options will allow for a separate short run momentum - long run contrarian effect, next to size effects[23].

Importantly, equation (37) shows that companies with higher indices $\kappa^{(j)}$ will have a relatively larger fraction of their market value associated with assets in place. This will drive their expected return down in light of Lemma 2. By contrast firms with low $\kappa^{(j)}$ indices will have higher expected returns, as most of their value will be in the form of growth options. These results resemble closely the results obtained in Gomes, Kogan, and Zhang [2003].

The new element in this paper is the relation between these cross sectional effects to eventual consumption risk. To achieve that, we first show that stocks with high $\kappa^{(j)}$ indices (i.e. large size/high M/B stocks) will tend to exhibit a more moderate covariance with eventual consumption risk.

A key result is the following:

[21] Theoretically, a sort based on market to book will produce a non monotone mapping to $\kappa^{(j)}$. Theoretically it is possible that a firm does not plant a tree for many epochs, so that its book value does not grow. However future growth options rise as each epoch changes making the market to book rise over time. Such a firm would have a low index $\kappa^{(j)}$, yet it would have a high value of market to book. However, in our calibrations we found the incidence of this effect to be relevant for an extremely small percentage of firms. In the stationary distribution size and market to book are strongly positively correlated.

[22] Even though the model is able to produce a size and a value effect, the two are virtually the same effect. Hence the model cannot produce a size and a value effect jointly. An extension of the model to include more idiosyncratic shocks would be able to produce the two effects separately, as in Berk, Green, and Naik [1999], Gomes, Kogan, and Zhang [2003] but we prefer to keep the model concise and focus on univariate sorts and their relation to eventual consumption risk as in Parker and Julliard [2005].

[23] If we perform a sort based on size, some companies will enter high size quantiles, because they have attractive current epoch growth options. If however, we sort again within each quantile based on returns in the recent past, then we will obtain a separate momentum effect: Controlling for size, a company that has experienced strong returns in the recent past is more likely to belong to a high size quantile because of a high current epoch growth option, rather than because of a rich set of assets in place. Interestingly, momentum will be short lived: over the long run the company is likely to invest, and hence its high expected returns will become low in the long run (contrarian effect), as figure 3 illustrates.

Lemma 3 *As long as*[24] $w_t^{(j),o+f} < \frac{1}{2}$ *the relative weight of growth options satisfies:*

$$\frac{\partial}{\partial \kappa^{(j)}} \left| \frac{\partial w_t^{(j),o+f}}{\partial x_t} \right| < 0$$

The simplest way to read this Lemma is as follows. Suppose that we select two stocks j and j', so that the first has either a lot of assets in place or has been fortunate to "draw" highly productive trees in the past (i.e. $\kappa(j)$ is large). However, the latter has a few assets in place or has drawn unproductive trees and hence has a low index $\kappa^{(j')}$. In light of the Lemma, the fraction of growth options in the price of the first asset will be less sensitive to variations in eventual consumption risk.

There is a straightforward intuition behind this result: Consider two firms and assume that the first has more assets in place than the "average" firm in the economy $\kappa^{(j)} > 0$ while the other has less $\kappa^{(j')} < 0$. We shall call the first the "large" firm, while we shall call the other the "small" firm. The variable x_t reveals information about the ratio of growth options to assets in place for the "average" firm. In times where the average firm in the economy can gain a lot by investing (i.e the economy is substantially below trend and x_t is low) this will be particularly true for the "small" firm rather than the "large" firm, and hence its value will be more sensitive to variations in x_t.

A first order Taylor expansion along the lines of the previous section, along with the observations in Lemmas 2 and 3, allows us to arrive at the following analog of (31)

$$\begin{aligned} \mu_t^{(j)} - r &\simeq g_1\left(\frac{\theta_t}{M_t}, \kappa^{(j)}\right) - g_2\left(\frac{\theta_t}{M_t}, \kappa^{(j)}\right)(x_t - F(x)) = \\ &= g_1\left(\frac{\theta_t}{M_t}, \kappa^{(j)}\right) + g_2\left(\frac{\theta_t}{M_t}, \kappa^{(j)}\right)\left(\int_t^\infty \left[E_t\left(d\log(C_{t+j})\right) - E\left(d\log(C_{t+j})\right)\right]\right) \end{aligned} \tag{38}$$

where $g_2 > 0$ and $\frac{\partial g_2}{\partial \kappa^{(j)}} < 0$, and $\frac{\partial g_1}{\partial \kappa^{(j)}} < 0$. Since $\frac{\theta_t}{M_t}$ reverts fast to its stationary mean $\overline{\frac{\theta_t}{M_t}}$, let us set $\frac{\theta_t}{M_t} = \overline{\frac{\theta_t}{M_t}}$ in the formula above, and condition on a value of $\kappa^{(j)}$. The discrete time analog of equation (38) is:

$$E_t\left(R_{t+\delta}^j - r\right) = g_1 + g_2 E_t\left[\sum_{j=t}^{\infty}\left[\Delta\log(C_j) - E(\Delta\log(C_j))\right]\right] \tag{39}$$

where $g_1 - g_1\left(\overline{\frac{\theta_t}{M_t}}, \kappa^{(j)}\right)$, $g_2 - g_2\left(\overline{\frac{\theta_t}{M_t}}, \kappa^{(j)}\right)$, $\Delta\log(C_j) - C_{j+\delta} - C_j$, and δ a short time interval (say a quarter). This equation is key since it produces two results: It asserts that expected

[24] In our simulations this will be the case for almost all companies, most all of the time.

"eventual" consumption risk will be a state variable, and that firms that posses a lot of preexisting assets will be more sensitive to variations in eventual consumption risk since $\frac{\partial g_2}{\partial \kappa^{(j)}} < 0$ and hence their returns will be more strongly correlated with variations in anticipated growth rates. Naturally, equation (39) is a *conditional* consumption CAPM. As Cochrane [1996] shows, the conditional CAPM implies an unconditional CAPM, as long as one uses appropriate conditioning variables as instruments. For short time intervals we can approximate the discount factor by[25] $\left(\frac{C_{t+\delta}}{C_t}\right)^{-\gamma} \simeq 1 - \gamma \Delta \log c_t$. Scaling the return of firm j by any affine function of eventual consumption risk gives the unconditional version of the CAPM:

$$\begin{aligned} E\left(R^j_{t+\delta} - r\right) &= \beta_1 cov(R^j_{t+\delta} - r, \Delta \log c_t) + \\ &\quad + \beta_2 cov\left[R^j_{t+\delta} - r, E_t\left(\sum_{j=t}^{\infty} [\Delta \log(C_j) - E(\Delta \log(C_j))]\right)\right] \\ &\quad + \beta_3 cov\left[R^j_{t+\delta} - r, (\Delta \log c_t)\, E_t\left(\sum_{j=t}^{\infty} [\Delta \log(C_j) - E(\Delta \log(C_j))]\right)\right] \end{aligned} \tag{40}$$

for appropriate fixed constants $\beta_1, \beta_2, \beta_3$. Since the conditional expectation of future consumption growth rates is persistent, it follows that:

$$E_t\left(\sum_{j=t}^{\infty} \Delta \log(C_j) - E(\Delta \log(C_j))\right) \simeq E_{t+\delta}\left(\sum_{j=t}^{\infty} \Delta \log(C_j) - E(\Delta \log(C_j))\right)$$

Moreover we shall truncate the infinite sum at a point T such that:

$$E_{t+\delta}\left(\sum_{j=T}^{\infty} \Delta \log(C_j) - E(\Delta \log(C_j))\right) < \varepsilon$$

for a sufficiently small ε. These calculations suggest that the second term in (40) is roughly equal to

$$cov\left[R^j_{t+\delta} - r, E_{t+\delta}\left(\sum_{j=t}^{T} \Delta \log(C_j) - E(\Delta \log(C_j))\right)\right] = cov\left[R^j_{t+\delta} - r, \sum_{j=t}^{T} \Delta \log(C_j)\right]$$

where we have used the law of iterated expectations and the fact that a covariance is invariant to the inclusion of the constant $TE(\Delta \log(C_j)$. In section 5 we show that this approximation to

[25] This approximation becomes exact in continuous time since the running maximum M^C has no quadratic variation and hence the covariance between increases in M^C and any return is exactly 0.

μ	0.010	γ	7	$\zeta(0)$	1
σ	0.033	ρ	0.05	s	2
λ	0.050	$\bar{A}$	1.55	e	32

TABLE 1: Parameters used for the calibration

the second term of (40) is very accurate. Moreover we show that the first and third term in (40) contribute very little to explaining the cross section and most of the weight is carried by the second term, which is the covariance that Parker and Julliard [2005] compute in their paper.

Summarizing, the reason why eventual consumption risk performs well in simulations is that small /value stocks are more strongly correlated with expected (and hence realized) future consumption growth, because of equation (39).

5 Calibration and quantitative implications

5.1 Matching unconditional moments

The model has 9 parameters. In order to calibrate it, we need to assign values to these parameters. Table 1 presents our choices. These parameters were chosen so as to match 22 unconditional moments. These unconditional moments include first and second moments of consumption growth, the one year real interest rate, the yearly equity premium, the log (P/E) ratio and the aggregate book to market ratio. These 10 time series moments were complemented by another 12 cross sectional moments, which correspond to the cross sectional distribution of size quantiles in the model. These are given in the bottom two rows of Table 5 along with their empirical counterparts.

As can be seen from the Tables 2 and 5 the model fit is satisfactory. Most time series moments are within 20-50% of their empirical counterparts. The only exception is the volatility of the one year interest rate, which is higher in the model than in the data. This is to be expected, since the utility specification contains (multiplicative) habit formation. The cross sectional distribution of size implied by the model is less disperse than in the data, especially so for the outlier portfolios.

	Data	Model
Mean of consumption growth	0.022	0.030
Volatility of consumption growth	0.033	0.033
Mean of 1-year zero coupon yield	0.018	0.029
Volatility of 1-year zero coupon yield	0.030	0.066
Mean of Equity Premium	0.042	0.036
Volatility of Equity Premium	0.177	0.208
Mean (log) Price to Earnings Ratio	3.091	3.496
Volatility of (log) Price to Earnings Ratio	0.280	0.335
Mean of Book to Market	0.668	0.707
Volatility of Book to Market	0.230	0.316

TABLE 2: Unconditional Moments of the model and the data. Data for consumption growth are from the Website of Robert Shiller. Consumption growth refers to (yearly) differences in log consumption. Both means and standard deviations are computed for the entire sample and for post 1918 data and then averaged to avoid issues with mismeasurement of consumption in pre World War I data. The rest of the data are from Chan and Kogan [2002] except for the mean and the volatility of the book to market, which is taken from Pontiff and Schall [1998]. The unconditional moments for the model are computed from a Monte Carlo Simulation involving 20000 years of data, dropping the initial 8000 to ensure that initial quantitites are drawn from their stationary distribution. We approximate the volatility of log consumption growth with its intantaneous quadratic variation σ.

5.2 Time series properties of aggregate consumption

We first turn to the time series properties of aggregate consumption. A useful visual depiction of the time series properties of (differences in log) consumption is afforded by the log-periodogram (see Hamilton [1994] for details). A flat log-periodogram is an indication of white noise, while a downward sloping log periodogram is an indication of time series dependence.

The top subplot of Figure 4 depicts the log periodogram for consumption growth in the data along with 5-95% confidence bands. The bottom subplot depicts the log periodogram obtained from simulations of the model and compares it to the data. As can be seen, the model performs remarkably well. The strong random walk component contained in the model implied consumption process allows us to match the subtle time series dependence of real-world consumption data.

From a skeptic's viewpoint it is also useful to have a sense of whether we can formally reject the hypothesis that consumption is a pure random walk. Figure 5 performs a Bartlett's periodogram based test. It depicts the cumulative periodogram along with 95% confidence bounds under the pure random walk hypothesis. This test reveals that one can formally reject the pure random walk hypothesis with yearly data. The figure also shows that it is frequencies between 0.10 and 0.30 where the cumulative periodogram exits its random walk confidence bounds indicating that the predictable components of consumption occur at cycles between 1/0.3=3.3 and 1/0.1=10 years.

5.3 The time series properties of the model

Table 3 demonstrates the results of predictability regressions of aggregate excess returns on the aggregate log P/D ratio, and compares it to the data. We simulate 100 years of data and obtain independent samples of such 100-year spans of artificial data. We run predictability regressions for each of these samples and report the average coefficient along with a 95% distribution band. We then compare these simulations to the equivalent point estimates in the data.

The coefficients in the simulations are about 1/3 of their empirical counterparts. Moreover, the empirical point estimates are within the 95% distribution band according to the model, with the exception of the longest horizons.

Another implication of the model that we discussed in section 4.1.1 is that different variables will be useful in predicting excess returns at different horizons. Table 4 quantifies this property of

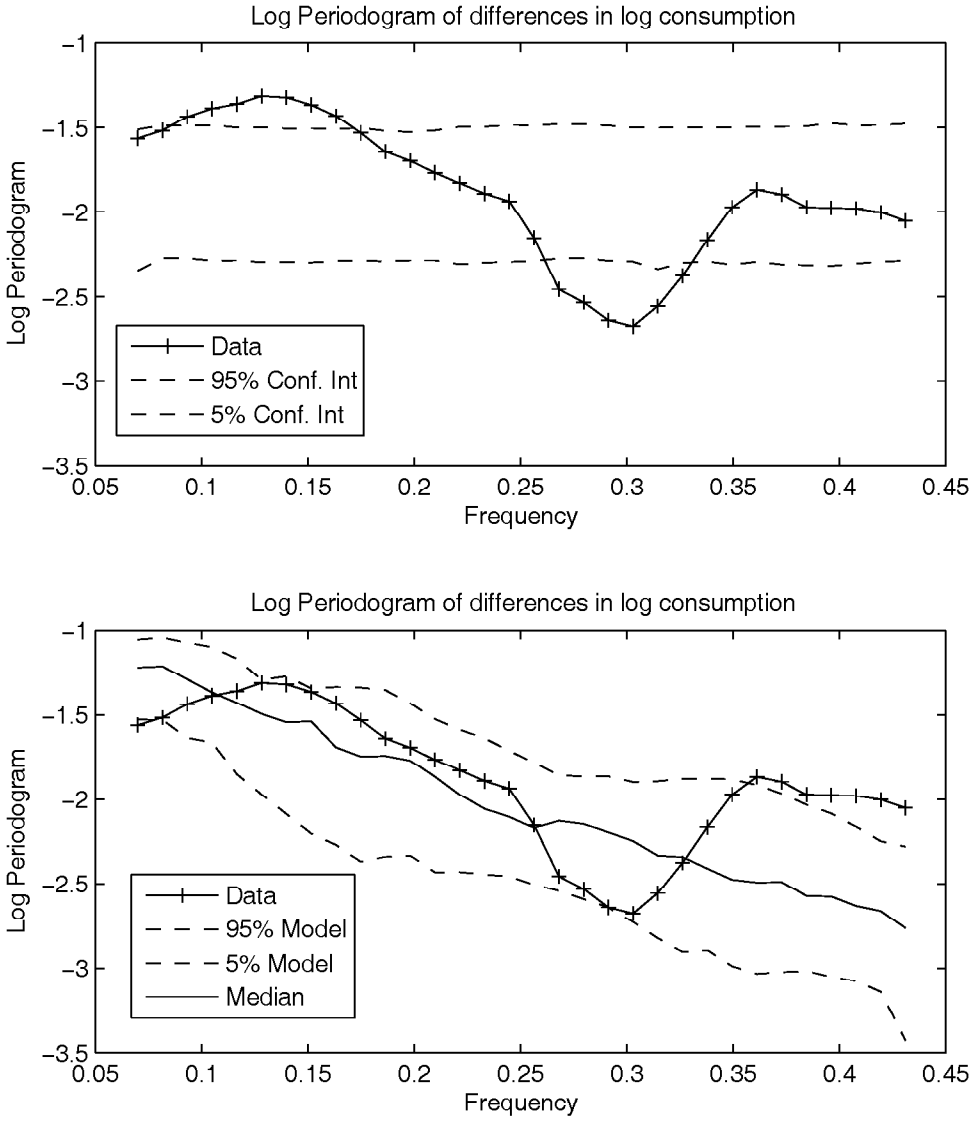


FIGURE 4: Log Periodogram of the consumption process for the data and the model. The top figure presents the log periodogram for post- world war I yearly differences in log consumption. Confidence bands are computed by bootstrapping the first differences in log consumption to produce 1000 artificial series with a length of 87 years. The bottom figure presents the same exercise for the model: 3000 years of simulated consumption growth were used to produce repeated series of a length of 87 years. The figure plots the median of these simulations, along with 5% and 95% range intervals. It also shows the log periodogram for the actual data. An equally weighted "nearest neighbor" kernel was used to perform the smoothing, equally weighting the 7 nearest frequencies.

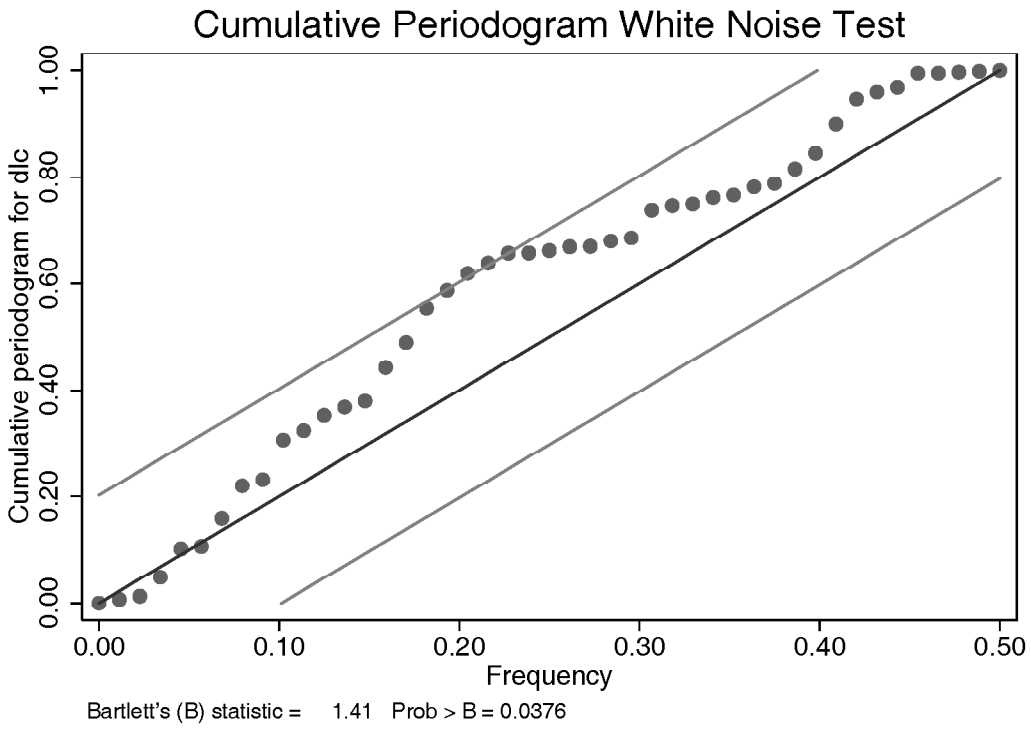


FIGURE 5: Bartlett's test for white noise based on the consumption cummulative periodogram. The test rejects the white noise hypothesis with a p-value of 0.0376

PE Predictive Ability

	Data		Model	
Horizon(years)	Coefficient	R-square	Coefficent	R-square
1	-0.120	0.040	-0.063	0.015
			(-0.282, 0.057)	(0.000, 0.071)
2	-0.300	0.100	-0.104	0.025
			(-0.413, 0.156)	(0.000, 0.099)
3	-0.350	0.110	-0.132	0.033
			(-0.553, 0.227)	(0.000, 0.141)
5	-0.640	0.230	-0.191	0.047
			(-0.650, 0.378)	(0.000, 0.181)
7	-0.730	0.250	-0.247	0.059
			(-0.919, 0.453)	(0.000, 0.225)

TABLE 3: Results of predictive Regressions. Excess returns in the aggregate stock market between t and $t+T$ for $T = 1, 2, 3, 5, 7$ are regressed on the P/E ratio at time t. A constant is included but not reported. The data column is from Chan and Kogan [2002]. The simulations were performed by drawing 100 time series of a length equal to the data and performing the same predictive regressions. For each draw out of the 100, we simulate 5000 years of data and only keep the last 100 years of data to run the regressions. We report the means of these simulations next to the data. The numbers in parentheses are the 95% confidence interval of the estimates obtained in the simulations.

the model. We run multiple regressions of excess returns on the lagged P/D ratio, the one year rate[26] and security issuance[27]. We report the partial correlation coefficients as implied by the model for thousands of independent 100-year samples.

As can be seen, the total predictability of excess returns implied by the model is substantial once all three variables are included. Moreover, the relative importance of the various predictors declines as the horizon lengthens. Originally, the relative importance of the interest rate is (comparatively) large as revealed by the partial correlation coefficient associated with it. Over time, the importance of the real interest rate and security issuance declines compared to the importance of the log(P/D) ratio. This differential ability of various variables to predict excess returns at different horizons is exactly the conclusion reached by Fama and French [1989] and Cochrane [2005] (Chapter 20)

5.4 The cross-sectional implications

We turn now to one of the main implications of the model, namely the ability of eventual consumption risk to price the cross section. We focus on size sorted portfolios, since sorting on size or book to market produces very similar portfolios. Table 5 presents the average returns on size sorted portfolios and compares them to the data.

The difference in monthly returns between the highest and the lowest size portfolio in our model is about a third of the equivalent value in the data. Hence the model has a similar performance to Gomes, Kogan, and Zhang [2003] in terms of obtaining a sizable size premium.

The interesting aspect of the model proposed here is its ability to reproduce the results in Parker and Julliard [2005]. In particular, the top subplot of figure 6 demonstrates the inability of the regular consumption CAPM to account for the cross sectional patterns of average returns. It reports the predicted and the actual returns of 20 portfolios formed on size when one uses quarterly covariations between returns and consumption. The bottom subplot illustrates how the usage of 5-year consumption growth rates instead of quarterly consumption growth rates can account for the cross section in a significantly more satisfactory way.

This is hardly surprising in light of our derivations in section 4.1.2. To see the reason for

[26] Results are similar irrespective of whether we use a short term interest rate or the term spread.

[27] We define security issuance as aggregate equity issuance between t and $t+1$ divided by the post equity issuance aggregate market value.

Contribution of different variables to Predictability

Horizon (year)	Multiple R-squared	PE	Risk Free Rate	Equity Issuance
1	0.033	0.013	0.009	0.009
	(0.003, 0.104)	(0.000, 0.076)	(0.000, 0.037)	(0.000, 0.043)
2	0.046	0.021	0.013	0.008
	(0.002, 0.144)	(0.000, 0.117)	(0.000, 0.089)	(0.000, 0.033)
3	0.059	0.027	0.018	0.009
	(0.007, 0.160)	(0.000, 0.150)	(0.000, 0.113)	(0.000, 0.045)
5	0.082	0.038	0.024	0.011
	(0.010, 0.223)	(0.000, 0.184)	(0.000, 0.115)	(0.000, 0.055)
7	0.097	0.043	0.028	0.010
	(0.009, 0.289)	(0.000, 0.188)	(0.000, 0.155)	(0.000, 0.044)

TABLE 4: Results of predictive Regressions. Excess returns in the aggregate stock market between t and $t+T$ for $T = 1, 2, 3, 5, 7$ are regressed on the P/E ratio, one-year interest rate and the Equity Issurance at time t. A constant is included but not reported. The simulations were performed by drawing 100 time series of 100 years each and performing predictive regressions. For each draw out of the 100, we simulate 5000 years of data and only keep the last 100 years of data to run the regressions. We report the means of the R-square of the multiple regression and the partial R-square for each individual predictive variable. The numbers in parentheses represent the 95% confidence interval of the estimates obtained in the simulations.

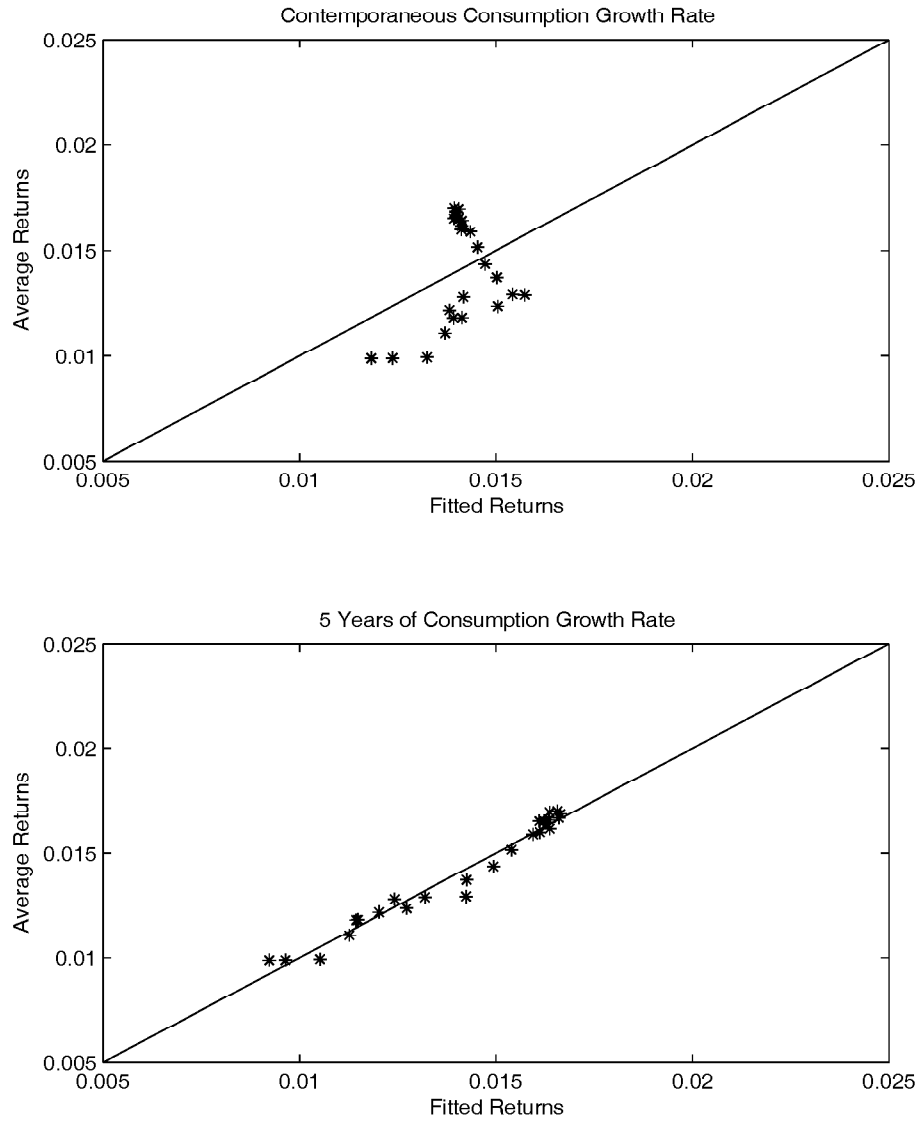


FIGURE 6: The consumption CAPM using 1-quarter consumption growth and 5-year consumption growth rates to evaluate the covariation between consumption growth and quarterly excess returns.

Portfolios formed on Size (Stationary Distribution)												
Deciles	1A	1B	2	3	4	5	6	7	8	9	10A	10B
Returns -Data	1.64	1.16	1.29	1.24	1.25	1.29	1.17	1.07	1.10	0.95	0.88	0.90
Returns -Simulated	0.70	0.69	0.68	0.67	0.66	0.64	0.63	0.62	0.61	0.60	0.60	0.59
Log Size - Data	1.98	3.18	3.63	4.10	4.50	4.89	5.30	5.73	6.24	6.82	7.39	8.44
Log Size - Simulated	3.34	3.80	4.07	4.31	4.56	4.83	5.10	5.39	5.69	6.01	6.29	6.60

TABLE 5: Portfolios sorted by size - model and data. The data are from Fama and French [1992], who report nominal montly returns, which are affected by the high inflation rates between 1963 and 1990.We report real montlhy returns for the simulations. For details on the number of simulations used see the caption to table 2. To compare, note that the average monthly inflation between 1963 and 1990 was about 0.8, and hence this number should be subtracted from the Fama-French returns in order to make them comparable to the simulated numbers.

the success of "eventual" consumption risk to account for cross sectional patterns, we evaluate the quality of the two approximations that we made in order to show the equivalence between the unconditional version of the conditional CAPM implied by our paper and the regressions that Parker and Julliard [2005] run. These were a) that the following equation holds approximately:

$$cov\left[R^j_{t+\delta}-r, E_t\left(\sum_{j=t}^{\infty}\left[\Delta\log(C_j)-E(\Delta\log(C_j))\right]\right)\right] \simeq cov\left[R^j_{t+\delta}-r, \sum_{j=t}^{T}\Delta\log(C_j)\right] \qquad (41)$$

and also that the first and third terms in equation (40) do not contribute much in terms of explaining the cross section. To establish the first assertion, we computed the correlation coefficient of the left vs. the right hand side of equation (41), which was 0.98 in the simulations when we set $T = 20$ quarters. Hence, the approximation error appears negligible. To establish the second assertion we provide figure 7, which supplies plots similar to figure 6. In the top subplot we use the first term of (40) to explain the cross section and in the middle subplot we use the first and the third term of (40). The bottom subplot uses only second term in (40), as approximated by the right hand side of (41). As can be seen clearly, the first and third terms of (40) contribute almost nothing to explaining the cross section, while the second term is particularly powerful.

This provides a potential explanation for the findings in Parker and Julliard [2005]: Over the short run, consumption growth is dominated by small disembodied shocks, which are white noise

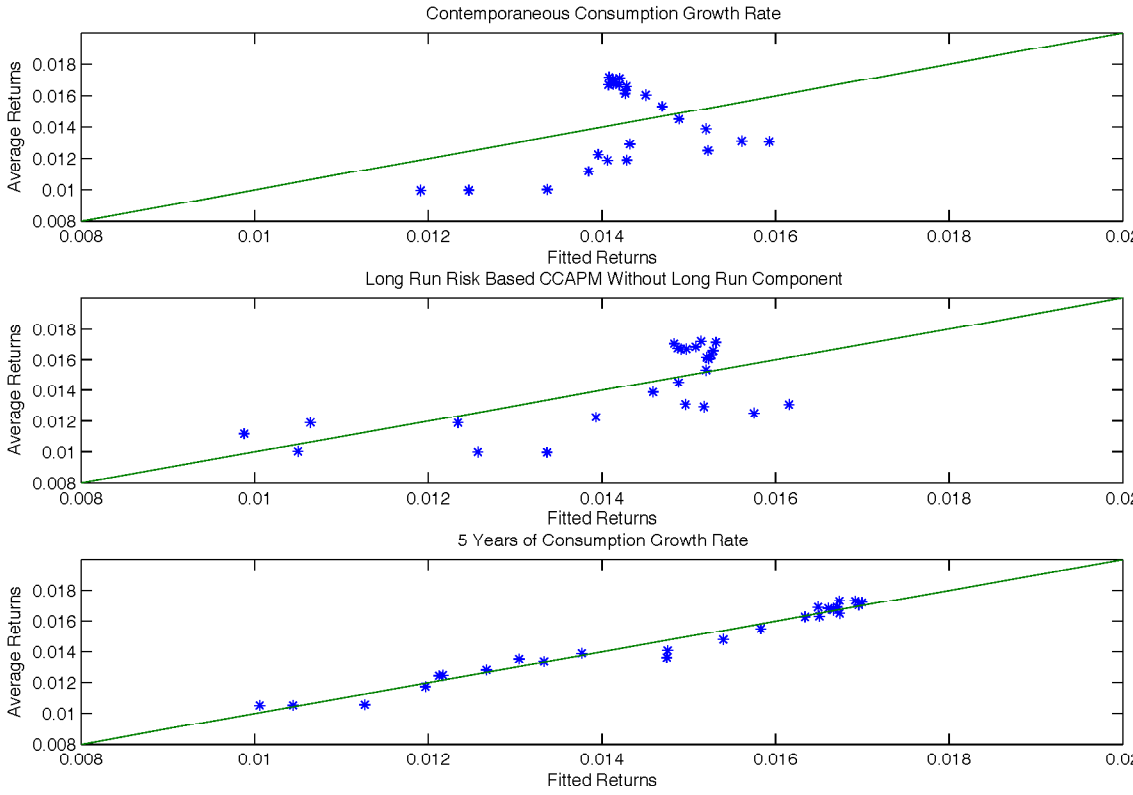


FIGURE 7: The first figure is the usual consumption based CAPM using the covariance between 1-quarter consumption growth rate the quaterly excess return. The third figure is the CCAPM using the covariance between 5-year consumption growth rates and quarterly excess return. The second figure evaluate the CCAPM with the first and the third therm of Equation (38), i. e. the covariance between quarterly excess return and contemporaneous consumption rate and the covariance between quarterly excess return and the product of contemporaneous consumption rate and lagged long run consumption. We first simulation the model for 20000 years, the keep the last 500 years for the cross-sectional regression. So the cross-sectional regression only uses 500 years of data. However, we tried several simulation, the cross-sectional results are similar.

shocks in nature. Only consumption growth over long horizons can reveal information about the degree to which the economy has absorbed a major technological shock. This is why the correlations with long horizon consumption reveal more information about the true risk of size sorted portfolios. Moreover, figures 7 and 6 illustrate that the success of eventual consumption risk in our model is that it allows us to obtain a very good approximation to the unconditional CAPM implied by "conditioning down" the conditional CAPM implied by equation (39).

5.5 The non-predictability of consumption / dividends by valuation ratios

A final issue to which we now turn concerns the ability of the P/D ratio to predict dividend growth. Given the closed form solutions of the model we can perform an "ideal" decomposition of the variability of the (log) P/D ratio into variation related to dividends (assuming constant returns) and variation that is related to return variation.

Specifically, using our closed form solutions, we computed the "fundamental" price P^f of a claim that would deliver the same dividends as the aggregate stock market, but imposing a *constant* rate of return for discounting dividends. That constant return was chosen so that the average (log) P/E ratio of the "fundamental" price exactly equal the (log) P/E ratio implied by our model.

Given the price of such a "fundamental" claim, we decomposed the variance of the log P/D ratio into its covariance with P^f and the difference $P - P^f$. I.e.:

$$var(\log(P/D)) = cov(\log(P/D), \log(P^f/D)) + cov(\log(P/D), \log(P/D) - \log(P^f/D)) \qquad (42)$$

This variance decomposition is the "ideal" version of the variance decompositions typically performed in the empirical literature, since the two components on the right hand side of equation (42) have to add up to 100% of the variance of the log P/D ratio. In a sequence of simulations of the model the average percentage of variation explained by the first term on the right hand side of (42) was about 33%.

This number should be viewed more as a theoretical number, since it can only be computed within the confines of the model, and would be incomputable with real data. It helps illustrate however, that most of the variation of the P/D ratio is due to variation of returns. It should also be viewed as an upper bound: If we were to perform a variance decomposition in the manner that is typically done in the empirical literature (i.e. truncating around 7-15 years, using aggregate

dividends instead of dividends per original share etc., then the fraction of the variation in the log P/D ratio, that is due to dividend growth, would appear to be less than 15%.

6 Conclusion

Why does consumption based asset pricing seem to work better over longer horizons, than over short horizons? In this paper we proposed a general equilibrium framework that served as a laboratory in order to investigate this question. The key ingredient of our model is the joint presence of "small" frequent disembodied productivity shocks and "large" infrequent embodied technology shocks. The first type of shocks affect the economy on impact and behave exactly as a random walk. The latter arrive also in a random walk fashion. However, there are delays between their impact and their effects on the economy.

This setup allowed us to obtain a consumption process that has strong random walk components, along with some small predictable parts. Importantly, the conditional expected returns are strongly affected by the endogenous economic cycles, and hence the predictable component of the consumption process. Intuitively, the extent to which the economy has absorbed a major technological shock, will be very informative for pricing purposes, as it will determine the relative weight of growth options in the prices of companies. In turn, the position of the economy with respect to this technological cycle will be revealed by subsequent consumption growth over long - not short horizons.

A key finding is that even though the regular consumption CAPM will hold *conditionally,* its *unconditional* version will amount to computing the same covariances between returns and long run growth as in Parker and Julliard [2005] (up to a close approximation). In turn the regressions run by Parker and Julliard [2005] are closely related to the literature on long run risk[28].

Hence, we are able to give a production based theory for the existence of predictable components in consumption, the nature of stochastic trends, and their implications for asset prices both in the time series and in the cross section. Importantly, our model appears able to match a number of key facts in the data, even with a constant relative risk aversion and non-Epstein Zin preferences.

We believe that incorporation of Epstein Zin preferences and/or time varying risk aversion in

[28] See the exposition in Malloy, Moskowitz, and Vissing-Jorgensen [2005] for details on this relation.

our framework would most likely strengthen our conclusions even further. We leave this to future research.

A Appendix

A.1 Propositions and Proofs

A.1.1 Proof of the Main Proposition

Proposition 1 *Define the constants* Z^*, γ_1, γ_1^* *and* Ξ *by*

$$Z^* = \frac{1}{\rho - \mu(1-\gamma) - \frac{\sigma^2}{2}\gamma(\gamma-1)} \tag{43}$$

$$\gamma_1 = \frac{\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(\rho+\lambda)} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} \tag{44}$$

$$\gamma_2 = \frac{-\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(\rho+\lambda)} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} \tag{45}$$

$$\gamma_1^* = \frac{\sqrt{\left(\mu - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2\rho} - \left(\mu - \frac{\sigma^2}{2}\right)}{\sigma^2} \tag{46}$$

$$\Xi = \frac{q}{Z^*}\frac{\gamma_1}{\gamma_1 - 1}\frac{\gamma_1^* - 1}{\gamma_1^* + \gamma - 1}$$

and assume that:

$$\begin{aligned} \gamma_1^* &> 1 \\ \gamma_2 &< 1 - \gamma \\ \frac{\Xi}{\zeta(0)} &> 1 \end{aligned}$$

Assume moreover that H_t *is given by (21), and* q_t *is given by (16). Then, firm* j *faced with the optimal stopping problem (4) will plant a tree the first time that* θ_t *crosses the threshold* $\overline{\theta}$

$$\overline{\theta} = M_{\tau_N}\frac{\Xi}{\zeta(i_{N,j})} \tag{47}$$

Formally, the optimal stopping time τ^* *is given by*

$$\tau^* = \inf\{t : \theta_t = \overline{\theta}\}$$

Finally, if firms follow the threshold policies of Proposition 1, then

$$\frac{C_t}{M_t^C} = \frac{\theta_t}{M_t} \tag{48}$$

with M_t, M_t^C *defined in (15) and (10). Therefore*

$$H_t = e^{-\rho t}U_C = e^{-\rho t}\left(\frac{C_t}{M_t^C}\right)^{-\gamma} = e^{-\rho t}\left(\frac{\theta_t}{M_t}\right)^{-\gamma}$$

as conjectured in (21).

This is the key Proposition of the paper. We start by assuming that the state price density is indeed given by:

$$H_t = e^{-\rho t}\left(\frac{\theta_t}{M_t}\right)^{-\gamma} \tag{49}$$

and prove that (47) provides the solution to the firm's optimal stopping problem. A useful intermediate first result is the following:

Lemma 4 *The conditional expectation:*

$$Z(\theta_t, M_t) = E_t\left[\int_t^{\infty} e^{-\rho(s-t)}\left(\frac{\theta_s}{M_s}\right)^{-\gamma}\theta_s ds\right] \tag{50}$$

can be computed explicitly as

$$Z\left(\theta_t, M_t\right) = Z^*\left(\frac{\theta_t}{M_t}\right)^{-\gamma}\theta_t\left[1+\frac{\gamma}{\gamma_1^*-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\right] \tag{51}$$

with Z^* and γ_1^* as defined in (43) and (46) respectively.

Proof. (Lemma 4) We give a sketch. One can verify directly that the function Z in equation (51) satisfies:

$$\left(\frac{\theta_t}{M_t}\right)^{-\gamma}\theta_t + \mu\theta Z_\theta + \frac{\sigma^2}{2}\theta^2 Z_{\theta\theta} - \rho Z = 0 \tag{52}$$

whenever $\theta_t < M_t$ and it also satisfies a reflection condition:

$$Z_M(\theta_t, M_t) = 0 \tag{53}$$

at $\theta_t = M_t$. By Ito's Lemma,

$$\begin{aligned} e^{-\rho T}Z\left(\theta_T, M_T\right) - e^{-\rho t}Z\left(\theta_t, M_t\right) &= \int_t^T e^{-\rho s}\left[\mu\theta Z_\theta + \frac{\sigma^2}{2}\theta^2 Z_{\theta\theta} - \rho Z\right]ds + \\ &+ \int_t^T e^{-\rho s}\sigma\theta_s Z_\theta dB_s + \int_t^T e^{-\rho s}Z_M dM_s \end{aligned} \tag{54}$$

Using (52) and (53), we can rewrite the above as:

$$\begin{aligned} e^{-\rho T}Z\left(\theta_T, M_T\right) - e^{-\rho t}Z\left(\theta_t, M_t\right) &= -\int_t^T e^{-\rho s}\left(\frac{\theta_s}{M_s}\right)^{-\gamma}\theta_s ds + \\ &+ \int_t^T e^{-\rho s}\sigma\theta_s Z_\theta dB_s \end{aligned}$$

Taking expectations, letting T go to infinity, and multiplying by $e^{\rho t}$ on both sides gives (50). ■

Corollary 2 *The value of assets in place for firm j is given by*

$$P_{j,t}^{\Lambda} = Z^* X_{j,t}\theta_t\left[1+\frac{\gamma}{\gamma_1^*-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\right]$$

Proof. (Corollary 2) Combine (51) and (5). ■

With this Lemma we are now in a position to discuss the solution to the firm's optimization problem.

The option to plant a tree in epoch N does not affect the option to plant a tree in any subsequent epoch. Therefore, the firm chooses its optimal strategy to plant a tree "epoch by epoch".

The individual firm takes the state price density (49) and the costs of planting a tree (16) as given. With these functional specifications, the optimization problem (4) becomes:

$$\begin{aligned} P^o_{N,j,t} &= \left(\frac{\theta_t}{M_t}\right)^{\gamma} \cdot \\ &\quad \cdot \max_{\tau} E_t \left\{ 1_{\{\tau \leq \tau_{N+1}\}} \left[\int_{\tau}^{\infty} e^{-\rho(s-t)} \zeta\left(i_{j,N}\right) \bar{A}^N \left(\frac{\theta_s}{M_s}\right)^{-\gamma} \theta_s ds - q \bar{A}^N M_{\tau_N} \left(\frac{\theta_\tau}{M_\tau}\right)^{-\gamma} e^{-\rho(\tau-t)} \right] \right\} \end{aligned}$$

Using Lemma 4 this optimization problem can be rewritten as

$$\begin{aligned} P^o_{N,j,t} &= \bar{A}^N \left(\frac{\theta_t}{M_t}\right)^{\gamma} M_{\tau_N} \cdot \qquad (55) \\ &\quad \cdot \max_{\tau \geq t} E_t \left[e^{-(\rho+\lambda)(\tau-t)} \left(\zeta\left(i_{j,N}\right) Z^* \left[1 + \frac{\gamma}{\gamma_1^* - 1} \left(\frac{\theta_\tau}{M_\tau}\right)^{\gamma+\gamma_1^*-1} \right] \frac{\theta_\tau}{M_{\tau_{N_t}}} \left(\frac{\theta_\tau}{M_\tau}\right)^{-\gamma} - q \left(\frac{\theta_\tau}{M_\tau}\right)^{-\gamma} \right) \right] \end{aligned}$$

To solve the optimization problem inside the square brackets we proceed as follows: We start by restricting our attention to trigger strategies, i.e. strategies where the firm invests the first time that the ratio $\frac{\theta_t}{M_{\tau_{N_t}}}$ crosses a threshold $\bar{\theta}$. Formally, consider strategies of the form:

$$\tau_{\bar{\theta}} = \inf\{s \geq t : \frac{\theta_s}{M_{\tau_{N_t}}} \geq \bar{\theta}\} \qquad (56)$$

The proof of the following result is standard and is omitted[29]:

$$E_t\left(e^{-(\rho+\lambda)(\tau_{\bar{\theta}}-t)}\right) = \left(\frac{\theta_t}{\bar{\theta} M_{\tau_{N_t}}}\right)^{\gamma_1} \qquad (57)$$

where γ_1 is defined in (44). Defining

$$v_{\tau_{\bar{\theta}}}(\theta_t, M_t, M_{\tau_{N_t}}; \bar{\theta}) = E_t \left[e^{-(\rho+\lambda)(\tau_{\bar{\theta}}-t)} \left(\zeta\left(i_{j,N}\right) Z^* \left[1 + \frac{\gamma}{\gamma_1^* - 1} \left(\frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{\bar{\theta}}}}\right)^{\gamma+\gamma_1^*-1} \right] \frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{N_t}}} \left(\frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{\bar{\theta}}}}\right)^{-\gamma} - q \left(\frac{\theta_{\tau_{\bar{\theta}}}}{M_{\tau_{\bar{\theta}}}}\right)^{-\gamma} \right) \right]$$

and using (57) we obtain:

$$\begin{aligned} v(\theta_t, M_t, M_{\tau_{N_t}}; \tau_{\bar{\theta}}) &= \zeta\left(i_{j,N}\right) Z^* \left[1 + \frac{\gamma}{\gamma_1^* - 1} \left(\frac{\bar{\theta}}{M_t / M_{\tau_{N_t}}} \wedge 1 \right)^{\gamma_1^*+\gamma-1} \right] \left(\frac{\bar{\theta}}{M_t / M_{\tau_{N_t}}} \wedge 1 \right)^{-\gamma} \left(\frac{\theta_t}{M_{\tau_{N_t}}}\right)^{\gamma_1} \bar{\theta}^{1-\gamma_1} \\ &\quad - q \left(\frac{\bar{\theta}}{M_t / M_{\tau_{N_t}}} \wedge 1 \right)^{-\gamma} \left(\frac{\theta_t}{M_{\tau_{N_t}}}\right)^{\gamma_1} \bar{\theta}^{-\gamma_1} \end{aligned}$$

[29] See e.g. Karatzas and Shreve [1991] page 197 for details.

We use the notation $x \wedge y = \min\{x, y\}$. The term

$$\frac{\bar{\theta}}{M_t/M_{\tau_{N_t}}} \wedge 1$$

captures the idea that the set of optimal stopping strategies that we consider satisfy either $\frac{\theta_\tau}{M_\tau} = 1$, or $\frac{\theta_\tau}{M_\tau} = \frac{\bar{\theta}}{M_t/M_{\tau_{N_t}}}$. Given the simple structure of the policies that we consider, finding the optimal stopping strategy in the constrained set (56) amounts to a simple one-dimensional optimization maximization over $\bar{\theta}$.

The next Lemma determines the optimal strategies in the constrained set (56). We omit the proof[30] both in order to save space, and because the steps of the proof are fairly straightforward differentiations. Most importantly, the next Lemma is only useful in that it will allow us to form a conjecture on the value function and the optimal stopping policies for arbitrary stopping policies. We then show via a formal verification theorem that the conjectured value function is indeed the appropriate value function.

It will be useful to refer to figure 8 that gives a graphical depiction of the regions described in the proof.

Lemma 5 *Let θ_1, $\underline{\theta}_2$ and $\bar{\theta}_2$ defined as*

$$\begin{aligned}
\underline{\theta}_2 &= \frac{q}{\zeta\left(i_{j,N}\right) Z^*} \frac{\gamma + \gamma_1}{\left(\gamma_1 + \gamma - 1\right) + \frac{\gamma\left(\gamma_1 - \gamma_1^*\right)}{\gamma_1^* - 1}} \\
\bar{\theta}_2 &\equiv \frac{q}{\zeta\left(i_{j,N}\right) Z^*} \frac{\gamma + \gamma_1}{\gamma_1 + \gamma - 1} \\
\theta_1 &= q \frac{1}{Z^*} \frac{\gamma_1}{\gamma_1 - 1} \frac{\gamma_1^* - 1}{\gamma_1^* + \gamma - 1} \frac{1}{\zeta\left(i_{j,N}\right)}
\end{aligned}$$

Then $\theta_1 < \underline{\theta}_2 < \bar{\theta}_2$ and there exists a unique positive solution to the nonlinear equation

$$h\left(\theta_2; M_t/M_{\tau_{N_t}}\right) \equiv \left(1 - \gamma_1 - \gamma\right)\theta_2 - \frac{\gamma\left(\gamma_1 - \gamma_1^*\right)}{\gamma_1^* - 1}\left(\frac{\theta_2}{M_t/M_{\tau_{N_t}}}\right)^{\gamma_1^* + \gamma - 1} \theta_2 + \left(\gamma + \gamma_1\right) q \frac{1}{\zeta\left(i_{j,N}\right) Z^*} = 0 \quad (58)$$

as long as $M_t/M_{\tau_{N_t}} > \underline{\theta}_2$. Let $\theta_2(M_t/M_{\tau_{N_t}})$ denote this unique positive solution. Then

$$\theta_2(M_t/M_{\tau_{N_t}}) < \bar{\theta}_2$$

and the solution to the problem

$$v(\theta_t, M_t, M_{\tau_{N_t}}) = \max_{\overline{\theta}} v_{\tau_{\bar{\theta}}}(\theta_t, M_t, M_{\tau_{N_t}}; \overline{\theta})$$

is given as follows:

In region I, i.e. when $\theta_t/M_{\tau_{N_t}} > \theta_2(M_t/M_{\tau_{N_t}})$, then:

$$v(\theta_t, M_t, M_{\tau_{N_t}}) = \zeta\left(i_{j,N}\right) Z^* \frac{\theta_t}{M_{\tau_{N_t}}} \left(\frac{\theta_t}{M_t}\right)^{-\gamma} + \frac{\gamma}{\gamma_1^* - 1} \zeta\left(i_{j,N}\right) Z^* \frac{\theta_t}{M_{\tau_{N_t}}} \left(\frac{\theta_t}{M_t}\right)^{\gamma_1^* - 1} - q\left(\frac{\theta_t}{M_t}\right)^{-\gamma}$$

[30] Available upon request

In region II, i.e. when $M_t/M_{\tau_{N_t}} \geq \underline{\theta}_2$ *and* $\theta_2(M_t/M_{\tau_{N_t}}) \geq \theta_t/M_{\tau_{N_t}}$, *then:*

$$\begin{aligned} v(\theta_t, M_t, M_{\tau_{N_t}}) &= \left[\zeta\left(i_{j,N}\right) Z^* \left(\frac{\theta_2}{M_t/M_{\tau_{N_t}}}\right)^{-\gamma} \theta_2^{1-\gamma_1} + \zeta\left(i_{j,N}\right) Z^* \frac{\gamma}{\gamma_1^* - 1} \left(\frac{\theta_2}{M_t/M_{\tau_{N_t}}}\right)^{\gamma_1^*-1} \theta_2^{1-\gamma_1}\right] \left(\frac{\theta_t}{M_{\tau_{N_t}}}\right)^{\gamma_1} \\ &\quad -q\left(\frac{\theta_2}{M_t/M_{\tau_{N_t}}}\right)^{-\gamma} \theta_2^{-\gamma_1} \left(\frac{\theta_t}{M_{\tau_{N_t}}}\right)^{\gamma_1} \end{aligned}$$

In region III, i.e. when $M_t/M_{\tau_{N_t}} \leq \theta_1$, *then*

$$v(\theta_t, M_t, M_{\tau_{N_t}}) = \frac{1}{\gamma_1 - 1} q \left(\frac{\theta_t}{M_{\tau_{N_t}}}\right)^{\gamma_1} \theta_1^{-\gamma_1}$$

In region IV, i.e. when $\theta_1 \leq M_t/M_{\tau_{N_t}} \leq \underline{\theta}_2$, *then*

$$v(\theta_t, M_t, M_{\tau_{N_t}}) = \zeta\left(i_{j,N}\right) Z^* \frac{\gamma + \gamma_1^* - 1}{\gamma_1^* - 1} \frac{M_t}{M_{\tau_{N_t}}} \left(\frac{M_t}{\theta_t}\right)^{-\gamma_1} - q\left(\frac{M_t}{\theta_t}\right)^{-\gamma_1}$$

The stopping region is $S = \{(\theta_t, M_t) : \theta_1 \leq \theta/M_{\tau_{N_t}} = M/M_{\tau_{N_t}} \leq \theta_2 \text{ or } \theta/M_{\tau_{N_t}} \geq \theta_2\}$. *The optimal stopping time is the first time* t *such that* (θ_t, M_t) *enters the stopping region.*

Figure 8 gives a graphical depiction of all the possible regions. Clearly $\theta_t \leq M_t$ and we need only concern ourselves with the region above the 45 degree line. The bold line depicts the boundary between the stopping and the continuation region. Given that this is a two dimensional optimal stopping problem the decision rule will be a mapping between (θ_t, M_t) and the discrete decision {stop - continue}. The depicted stopping region is the set of points where the firm would choose to invest. The complement of this region is the set of points where it would wait.

We are now ready to state and prove the key result:

Proposition 3 *The function* v *given in Lemma 5 is the value function of problem (55) and the optimal stopping policy given in Lemma 5 is optimal among all possible stopping policies*

Proof. (Proposition 3) As a first step we show that the function v satisfies the following properties:

$$v(\theta_s, M_s, M_{\tau_{N_t}}) \geq f(\theta_t, M_t, M_{\tau_{N_t}}) \tag{59}$$

where

$$\begin{aligned} f(\theta_t, M_t; M_{\tau_N}) &\equiv \zeta\left(i_{j,N}\right) Z^* \frac{\theta_t}{M_{\tau_N}} \left(\frac{\theta_t}{M_t}\right)^{-\gamma} \\ &\quad +\zeta\left(i_{j,N}\right) Z^* \frac{\gamma}{\gamma_1^* - 1} \left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*-1} \frac{\theta_t}{M_{\tau_N}} - q\left(\frac{\theta_t}{M_t}\right)^{-\gamma} \\ v_M(M_s, M_s, M_{\tau_{N_t}}) &\leq 0 \quad \text{for } \theta_t = M_t \end{aligned} \tag{60}$$

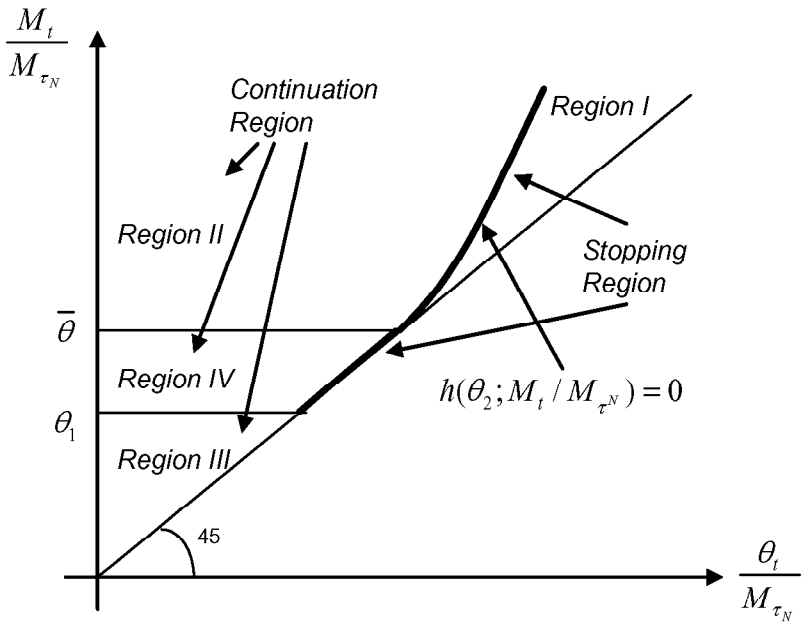


FIGURE 8: Depiction of the various regions. The continuation region is separated from the stopping region by the bold line.

$$\mathcal{A}v(\theta_s, M_s, M_{\tau_{N_t}}) \leq 0 \tag{61}$$

where $\mathcal{A}v = \frac{\sigma^2}{2}\theta^2 v_{\theta\theta} + \mu\theta v_\theta - (\rho + \lambda)\, v$. Moreover, we shall demonstrate that v is continuous and differentiable throughout its domain. Finally we will employ a verification Theorem for optimal stopping similar to (Øksendal [2003]) to conclude.

Equation (59) holds by construction of the value function $v(\theta_s, M_s, M_{\tau_{N_t}})$. To see this note that according to Lemma 5:

$$\begin{aligned} v(\theta_s, M_s, M_{\tau_{N_t}}) &= \max_{\overline{\theta}} v_{\tau_{\overline{\theta}}}(\theta_t, M_t, M_{\tau_{N_t}}; \overline{\theta}) \geq \\ &\geq v_{\tau_{\overline{\theta}}}(\theta_t, M_t, M_{\tau_{N_t}}; \frac{\theta_t}{M_{\tau_{N_t}}}) = \\ &= f(\theta_t, M_t, M_{\tau_{N_t}}) \end{aligned}$$

So we only need to check equation (60) and equation (61) in all the four regions and the smooth conditions.

In region I, $\theta_t / M_{\tau_{N_t}} > \theta_2\left(M_t / M_{\tau_{N_t}}\right)$, and hence

$$\begin{aligned} v_M(M_t, M_t, M_{\tau_{N_t}}) &= \zeta\left(i_{j,N}\right) Z^* \frac{\gamma}{M_{\tau_{N_t}}}\left(1 - \frac{\gamma_1^* - 1}{\gamma_1^* - 1}\right) - \gamma q \frac{1}{M_t} \\ &= -\gamma q \frac{1}{M_t} \\ &< 0 \end{aligned}$$

Moreover

$$\begin{aligned}\mathcal{A}v(\theta_t, M_t, M_{\tau_{N_t}}) &= -\zeta\left(i_{j,N}\right)\frac{Z^*}{Z}\frac{\theta_t}{M_{\tau_{N_t}}}\left(\frac{\theta_t}{M_t}\right)^{-\gamma} - \frac{\gamma\lambda}{\gamma_1^*-1}\zeta\left(i_{j,N}\right)Z^*\frac{\theta_t}{M_{\tau_{N_t}}}\left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*-1} - \frac{q}{Y}\left(\frac{\theta_t}{M_t}\right)^{-\gamma} \\ &= \left[-\zeta\left(i_{j,N}\right)\frac{Z^*}{Z}\frac{\theta_t}{M_{\tau_{N_t}}} - \frac{\gamma\lambda}{\gamma_1^*-1}\zeta\left(i_{j,N}\right)Z^*\frac{\theta_t}{M_{\tau_{N_t}}}\left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*+\gamma-1} - \frac{q}{Y}\right]\left(\frac{\theta_t}{M_l}\right)^{-\gamma} \quad (62)\end{aligned}$$

where

$$\begin{aligned}Z &= \frac{-1}{\frac{\sigma^2}{2}(\gamma-1)\gamma + \mu(1-\gamma) - \rho - \lambda} \\ Y &= \frac{1}{\frac{\sigma^2}{2}\gamma^2 - \gamma\left(\mu - \frac{\sigma^2}{2}\right) - \rho - \lambda}\end{aligned}$$

It is easy to check, by assumption $1-\gamma_2-\gamma>0$, that

$$\begin{aligned}Z &= \frac{1}{\frac{\sigma^2}{2}\left(\gamma_1+\gamma-1\right)\left(1-\gamma_2-\gamma\right)} > 0 \quad (63) \\ Y &= \frac{1}{\frac{\sigma^2}{2}\left(\gamma_1+\gamma\right)\left(\gamma_2+\gamma\right)}\end{aligned}$$

It is easy to check that $\gamma_1\gamma_2 = -\frac{2(\rho+\lambda)}{\sigma^2}$and $\gamma_1+\gamma_2 = 1-\frac{2\mu}{\sigma^2}$. Hence,

$$\frac{1}{Y} = \frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z} \quad (64)$$

From equation (58), we obtain:

$$\frac{\gamma\zeta\left(i_{j,N}\right)Z^*}{\gamma_1^*-1}\left(\frac{\theta_2}{M_t/M_{\tau_{N_t}}}\right)^{\gamma_1^*+\gamma-1}\theta_2 = \left(\gamma_1+\gamma-1\right)\theta_2\frac{\zeta\left(i_{j,N}\right)Z^*}{\gamma_1^*-\gamma_1} - q\frac{\gamma+\gamma_1}{\gamma_1^*-\gamma_1} \quad (65)$$

Hence, by $\theta_t/M_{\tau_{N_t}} > \theta_2\left(M_t/M_{\tau_{N_t}}\right)$, equation (64) and equation (65),

$$\begin{aligned}&-\zeta\left(i_{j,N}\right)\frac{Z^*}{Z}\frac{\theta_t}{M_{\tau_{N_t}}} - \frac{\gamma\lambda}{\gamma_1^*-1}\zeta\left(i_{j,N}\right)Z^*\frac{\theta_t}{M_{\tau_{N_t}}}\left(\frac{\theta_t}{M_t}\right)^{\gamma_1^*+\gamma-1} - \frac{1}{Y}q \\ \leq\ &-\zeta\left(i_{j,N}\right)\theta_2\frac{Z^*}{Z} - \lambda\left(\left(\gamma_1+\gamma-1\right)\theta_2\frac{\zeta\left(i_{j,N}\right)Z^*}{\gamma_1^*-\gamma_1} - q\frac{\gamma+\gamma_1}{\gamma_1^*-\gamma_1}\right) - \frac{1}{Y}q \\ =\ &\left[\frac{\lambda\left(\gamma_1+\gamma-1\right)}{\gamma_1\quad\gamma_1^*} - \frac{1}{Z}\right]Z^*\zeta\left(i_{j,N}\right)\theta_2 + q\left[-\frac{\left(\gamma+\gamma_1\right)\lambda}{\gamma_1\quad\gamma_1^*} - \frac{\gamma+\gamma_2}{1\quad\gamma\quad\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1\ \vert\ \gamma\quad 1}\frac{1}{Z}\right]\end{aligned}$$

To arrive from equation (62) to equation (61), we only need to show that

$$0 \geq \left[\frac{\lambda\left(\gamma_1+\gamma-1\right)}{\gamma_1-\gamma_1^*} - \frac{1}{Z}\right]Z^*\zeta\left(i_{j,N}\right)\theta_2 + q\left[-\frac{\left(\gamma+\gamma_1\right)\lambda}{\gamma_1-\gamma_1^*} - \frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z}\right] \quad (66)$$

We first want to show

$$\frac{\lambda\left(\gamma_1+\gamma-1\right)}{\gamma_1-\gamma_1^*} > \frac{1}{Z} \quad (67)$$

Actually, by equation (63), this is equivalent to showing that

$$\frac{\lambda(\gamma_1+\gamma-1)}{\gamma_1-\gamma_1^*} > (\gamma_1+\gamma-1)(1-\gamma_2-\gamma)\frac{\sigma^2}{2} \tag{68}$$

which amounts to showing that

$$(1-\gamma_2-\gamma)(\gamma_1-\gamma_1^*) \quad < \quad \frac{2\lambda}{\sigma^2} = (-\gamma_2+\gamma_1^*)(\gamma_1-\gamma_1^*) \tag{69}$$

Simplifying further, (69) becomes

$$\begin{aligned} -\gamma_2+\gamma_1^* &> 1-\gamma_2-\gamma \\ \gamma_1^* &> 1-\gamma \end{aligned}$$

which is clearly true. Hence, we have proven equation (67). Then, since $\theta_2 \leq \bar{\theta}_2 = q\frac{1}{Z^*}\frac{1}{\zeta(i_{j,N})}\frac{\gamma+\gamma_1}{(\gamma_1+\gamma-1)}$, $Z>0$, and the assumption $\gamma+\gamma_2-1<0$, we can show that

$$\begin{aligned} & \left[\frac{\lambda(\gamma_1+\gamma-1)}{\gamma_1-\gamma_1^*}-\frac{1}{Z}\right]Z^*\zeta(i_{j,N})\theta_2 + q\left[-\frac{(\gamma+\gamma_1)\lambda}{\gamma_1-\gamma_1^*}-\frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z}\right] \\ \leq & \left[\frac{\lambda(\gamma_1+\gamma-1)}{\gamma_1-\gamma_1^*}-\frac{1}{Z}\right]q\frac{\gamma_1+\gamma}{(\gamma_1+\gamma-1)} + q\left[-\frac{(\gamma+\gamma_1)\lambda}{\gamma_1-\gamma_1^*}-\frac{\gamma+\gamma_2}{1-\gamma-\gamma_2}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1}\frac{1}{Z}\right] \\ = & \left[\frac{\gamma+\gamma_2}{\gamma+\gamma_2-1}-1\right]q\frac{1}{Z}\frac{\gamma_1+\gamma}{\gamma_1+\gamma-1} \\ < & 0 \end{aligned}$$

Hence, we showed equation (66), and hence, $\mathcal{A}v(\theta_t, M_t, M_{\tau_{N_t}}) \leq 0$.

In region II, $M_t/M_{\tau_{N_t}} \geq \underline{\theta}_2$ and $\theta_2(M_t/M_{\tau_{N_t}}) \geq \theta_t/M_{\tau_{N_t}}$. A direct calculation yields

$$\mathcal{A}v(\theta_t, M_t, M_{\tau_{N_t}}) = 0$$

since $\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+\mu\gamma_1-(\rho+\lambda)=0$. It is equally straightforward to check that in region II, equation (60) is satisfied automatically, since $\theta_t < M_t$.

In region III, $M_t/M_{\tau_{N_t}} \leq \theta_1$,

$$v(\theta_t, M_t, M_{\tau_{N_t}}) = \frac{1}{\gamma_1-1}q\left(\frac{\theta_t}{M_{\tau_{N_t}}}\right)^{\gamma_1}\theta_1^{-\gamma_1}$$

Since $\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+\mu\gamma_1-(\rho+\lambda)=0$, we obtain

$$\mathcal{A}v(\theta_t, M_t; M_{\tau_{N_t}}) = 0$$

and

$$v_M(M_t, M_t; M_{\tau_{N_t}}) - 0$$

In region IV, $\theta_1 \leq M_t/M_{\tau_{N_t}} \leq \underline{\theta}_2$. Again, from $\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+\mu\gamma_1-(\rho+\lambda)=0$, we have

$$\mathcal{A}v(\theta_t, M_t, M_{\tau_{N_t}}) = 0.$$

To verify equation (60),

$$\begin{aligned} v_M(M_t, M_t, M_{\tau_{N_t}}) &= \zeta(i_{j,N}) Z^* \frac{\gamma+\gamma_1^*-1}{\gamma_1^*-1}(1-\gamma_1)\frac{1}{M_{\tau_{N_t}}} + \gamma_1 q \frac{1}{M_t} \\ &= \left[\zeta(i_{j,N}) Z^* \frac{\gamma+\gamma_1^*-1}{\gamma_1^*-1}(1-\gamma_1)\frac{M_t}{M_{\tau_{N_t}}} + \gamma_1 q\right]\frac{1}{M_t} \\ &\leq \left[\zeta(i_{j,N}) Z^* \frac{\gamma+\gamma_1^*-1}{\gamma_1^*-1}(1-\gamma_1)\theta_1 + \gamma_1 q\right]\frac{1}{M_t} \\ &= \left[(1-\gamma_1) q \frac{\gamma_1}{\gamma_1-1} + \gamma_1 q\right]\frac{1}{M_t} \\ &= 0 \end{aligned}$$

Having shown (59), (60), (61) in all regions of the state space, we are left with showing continuity and continuous differentiability of v with respect to θ_t. Since both of these conditions are satisfied in the interior of the four regions, it remains to check continuity and differentiability at the boundaries of regions I and II, II and IV and IV and III. This is a matter of straightforward computation of left and right limits and we leave the details of the computations out[31] in order to save space.

Given the properties of v it is now straightforward to take any stopping time τ and use Ito's formula to obtain:

$$\begin{aligned} e^{-(\rho+\lambda)\tau} v(\theta_\tau, M_\tau; M_{\tau_{N_t}}) &= e^{-(\rho+\lambda)t} v(\theta_t, M_t; M_{\tau_{N_t}}) + \int_t^\tau \mathcal{A}v(\theta_s, M_s; M_{\tau_{N_t}}) e^{-(\rho+\lambda)s} ds \\ &\quad + \int_t^\tau \sigma\theta_s v_\theta(\theta_s, M_s; M_{\tau_{N_t}}) e^{-(\rho+\lambda)s} dB_s + \int_t^\tau v_M(\theta_s, M_s; M_{\tau_{N_t}}) e^{-(\rho+\lambda)s} dM_s \end{aligned} \tag{70}$$

From equation (59), (60) and (61), we have

$$\begin{aligned} e^{-(\rho+\lambda)t} v(\theta_t, M_t; M_{\tau_{N_t}}) &\geq -E_t\left[\int_t^\tau \mathcal{A}v(\theta_s, M_s; M_{\tau_{N_t}}) e^{-(\rho+\lambda)s} ds\right] + E_t\left[e^{-(\rho+\lambda)\tau} v(\theta_\tau, M_\tau; M_{\tau_{N_t}})\right] \\ &\geq E_t\left[e^{-(\rho+\lambda)\tau} v(\theta_\tau, M_\tau; M_{\tau_{N_t}})\right] \geq E_t\left[e^{-(\rho+\lambda)\tau} f(\theta_\tau, M_\tau; M_{\tau_{N_t}})\right] \end{aligned}$$

That is,

$$v(\theta_t, M_t; M_{\tau_{N_t}}) \geq E_t\left[e^{-(\rho+\lambda)(\tau-t)} f(\theta_\tau, M_\tau; M_{\tau_{N_t}})\right]$$

Hence, the conjectured value function $\nu(\theta_t, M_t; M_{\tau_{N_t}})$ is an upper bound for all value functions, and it is obviously attainable by the proposed stopping rule. Hence it must be the value function of the optimal stopping problem (55). ■

[31] Available upon request.

This establishes that the stopping rule (47) is optimal. The next step is to establish that if firms follow the threshold policies just described, then equation (48) is satisfied.

To see this let $t^* \leq t$ be the first time such that $\theta_{t^*} = M_t$. Since in the proposed equilibrium firms plant new trees only at[32] $\theta_t = M_t$, then there is no new tree planted between time t^* and t. Hence,

$$\int_0^1 X_{j,t^*} dj = \int_0^1 X_{j,t} dj$$

Since

$$\theta_{t^*} \int_0^1 X_{j,t^*} dj \leq M_t^C \leq M_t \int_0^1 X_{j,t} dt = \theta_{t^*} \int_0^1 X_{j,t^*} dj$$

it follows that

$$M_t^C = M_t \int_0^1 X_{j,t} dt$$

Hence,

$$\frac{C_t}{M_t^C} = \frac{\theta_t \int_0^1 X_{j,t} dt}{M_t \int_0^1 X_{j,t} dt} = \frac{\theta_t}{M_t}$$

This concludes the proof of the main proposition.

Having constructed the equilibrium state price density, the rest of the verification that $\langle C_t, K_{n,t}, H_t \rangle$ constitute an equilibrium in the sense of definition 1 is standard. The reader is referred to Basak [1999] and the monograph of Karatzas and Shreve [1998] Chapter 4 for details.

A.1.2 Propositions and Proofs for section 4

Proposition 4 *Let* $\hat{\gamma}_1, \widehat{Z}$ *be given by*

$$\begin{aligned} \hat{\gamma}_1 &= \frac{\left(\frac{\sigma^2}{2} - \mu\right) + \sqrt{\left(\frac{\sigma^2}{2} - \mu\right)^2 + 2\sigma^2(\rho + \lambda\left(1 - \overline{A}\right))}}{\sigma^2} \\ \widehat{Z} &= -\frac{1}{\frac{\sigma^2}{2}\gamma_1(\gamma_1 - 1) + \mu\gamma_1 - \left[\rho + \lambda(1 - \bar{A})\right]} \end{aligned}$$

and assume that:

$$\hat{\gamma}_1 > 1, \ \widehat{Z} < 0$$

Then, the price of firm j *in technological epoch* N *is given by (6) where*

$$P_{j,t}^A = Z^* X_{j,t} \theta_t \left[1 + \frac{\gamma}{\gamma_1^* - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma + \gamma_1^* - 1}\right] \tag{71}$$

$$P_{N,j,t}^o = Z^* \bar{A}^N \theta_t \left[\frac{1}{\gamma_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma + \gamma_1 - 1}\left(\frac{M_t}{M_{\tau_N}}\right)^{\gamma_1 - 1}\left(\frac{q}{Z^*}\right)\psi(i_{j,N})^{-\gamma_1}\right]\left(1 - 1_{\{\widetilde{\chi}_{N,j}=1\}}\right) \tag{72}$$

$$P_{N,t}^f = Z^* \bar{A}^N \theta_t \left\{\frac{\lambda \bar{A} \hat{Z}}{\gamma_1 - 1}\left[\left(\frac{\theta_t}{M_t}\right)^{\gamma + \gamma_1 - 1} - \frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma + \hat{\gamma}_1 - 1}\right]\left(\frac{q}{Z^*}\right)\left(\int_0^1 \psi(i)^{-\gamma_1} di\right)\right\}$$

[32] Note that according to the proposed equilibrium all firms are always in Region III.

The constants $Z^*, \gamma_1^*, \gamma_1$ are given in Proposition 1. $X_{j,t}$ is given by (3), $1_{\{\tilde{\chi}_{N,j}=1\}}$ is the indicator function used in equation (3) that takes the value 1 if firm j has planted a tree in the current epoch and 0 otherwise and $\psi(i_{j,N})$ is given by

$$\psi(i_{j,N}) \equiv \frac{\Xi}{\zeta(i_{j,N})} = \frac{q}{Z^*}\frac{\gamma_1}{\gamma_1 - 1}\frac{\gamma_1^* - 1}{\gamma_1^* + \gamma - 1}\frac{1}{\zeta(i_{j,N})}$$

Proof. (Proposition 4) Let $P_{j,t}^A$ denote the value of asset in place, $P_{N,j,t}^o$ denote the value of current options and $P_{N,t}^f$ denote the value of all future update options for firm j. Then, by equation (51),

$$\begin{aligned} P_{N,j,t}^A &= \frac{X_{j,t} E_t\left[\int_t^\infty e^{-\rho(s-t)} M_s^\gamma \theta_s^{1-\gamma} ds\right]}{M_t^\gamma \theta_t^{-\gamma}} \\ &= X_{j,t} Z^* \left[\theta_t + \frac{\gamma}{\gamma_1^* - 1} M_t^{1-\gamma-\gamma_1^*} \theta_t^{\gamma+\gamma_1^*}\right] \end{aligned} \tag{73}$$

from which (71) follows. Let $\tau_{N,j}^*$ be optimal time for firm j to plant a new tree in the current epoch and τ_N be the time when epoch N arrived. Since $M_{\tau_{N,j}^*} - \theta_{\tau_{N,j}^*} - \psi(i_{j,N}) M_{\tau_N}$, then the current growth option has the following value,

$$\begin{aligned} P_{N,j,t}^o &= \frac{\bar{A}^N E_t\left(e^{-\rho\left(\tau_{N,j}^* - t\right)} I_{\tau_{N,j}^* \le \tau_{N_t+1}} \left[\zeta\left(i_{j,N}\right) \int_{\tau_{N,j}^*}^\infty e^{-\rho\left(s-\tau_{N,j}^*\right)} M_s^\gamma \theta_s^{1-\gamma} ds - q M_{\tau_N} M_{\tau_{N,j}^*}^\gamma \theta_{\tau_{N,j}^*}^{-\gamma}\right]\right)}{M_t^\gamma \theta_t^{-\gamma}} \left(1 - 1_{\{\tilde{\chi}_{N,j}=1\}}\right) \\ &= \bar{A}^N E_t\left\{e^{-(\rho+\lambda)\left(\tau_{N,j}^* - t\right)} \left[\zeta\left(i_{j,N}\right) Z^* \frac{\gamma + \gamma_1^* - 1}{\gamma_1^* - 1} \psi(i_{j,N}) - q\right] M_{\tau_N}\right\} M_t^{-\gamma} \theta_t^\gamma \left(1 - 1_{\{\tilde{\chi}_{N,j}=1\}}\right) \\ &= \bar{A}^N q \frac{1}{\gamma_1 - 1} M_{\tau_N} M_t^{-\gamma} \theta_t^\gamma E_t\left[e^{-(\rho+\lambda)\left(\tau_{N,j}^* - t\right)}\right] \left(1 - 1_{\{\tilde{\chi}_{N,j}=1\}}\right) \end{aligned}$$

where the second to last equality follows from $\psi(i_{j,N}) = \frac{q}{Z^*}\frac{\gamma_1}{\gamma_1 - 1}\frac{\gamma_1^* - 1}{\gamma_1^* + \gamma - 1}\frac{1}{\zeta(i_{j,N})}$. Since (see page 197 of Karatzas and Shreve [1991])

$$E_t\left[e^{-(\rho+\lambda)\left(\tau_{N,j}^* - t\right)}\right] = \left(\frac{\psi(i_{j,N})}{\theta_t / M_{\tau_N}}\right)^{-\gamma_1}$$

then, after some rearrangement, we obtain

$$\begin{aligned} P_{N,j,t}^o &= \bar{A}^N q \frac{1}{\gamma_1 - 1} M_{\tau_N} M_t^{-\gamma} \theta_t^\gamma \left(\frac{\psi(i_{j,N})}{\theta_t / M_{\tau_N}}\right)^{-\gamma_1} \left(1 - 1_{\{\tilde{\chi}_{N,j}=1\}}\right) \\ &= Z^* \bar{A}^N \theta_t \left[\frac{1}{\gamma_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1-1} \left(\frac{M_t}{M_{\tau_N}}\right)^{\gamma_1 - 1} \left(\frac{q}{Z^*}\right) \psi(i_{j,N})^{-\gamma_1}\right] \left(1 - 1_{\{\tilde{\chi}_{N,j}=1\}}\right) \end{aligned} \tag{74}$$

which is exactly equation (72).

Finally, since $M_{\tau_{N,j}^*} = \theta_{\tau_{N,j}^*} = \psi(i_{j,N}) M_{\tau_N}$, the value of all the future growth options is,

$$\begin{aligned} P_{N,j,t}^f &= \frac{E\left(E_t\left[\sum_{n=N+1}^\infty e^{-\rho\left(\tau_{n,j}^* - t\right)} I_{\tau_{n,j}^* \le \tau_{n+1}} \bar{A}^n \left(A(i_{j,n}) \int_{\tau_{n,j}^*}^\infty e^{-\rho\left(s-\tau_{n,j}^*\right)} M_s^\gamma \theta_s^{1-\gamma} ds - q M_{\tau_n} M_{\tau_{n,j}^*}^\gamma \theta_{\tau_{n,j}^*}^{-\gamma}\right)\right]\right)}{M_t^\gamma \theta_t^{-\gamma}} \\ &= \frac{\sum_{n=N+1}^\infty \bar{A}^n E\left(E_t\left[e^{-\rho\left(\tau_{n,j}^* - t\right)} I_{\tau_{n,j}^* \le \tau_{n+1}} \left(A(i_{j,n}) \int_{\tau_{n,j}^*}^\infty c^{-\rho\left(s-\tau_{n,j}^*\right)} M_s^\gamma \theta_s^{1-\gamma} ds - q M_{\tau_n}\right)\right]\right)}{M_t^\gamma \theta_t^{-\gamma}} \end{aligned}$$

where E denote the expectation with respect to the unknown draws $i_{j,n}$ in future periods. To calculate $P^f_{N,j,t}$, we first find the following conditional expectation for any $n > N$ and any given $\zeta(i_{j,n})$:

$$\begin{aligned}
& E_t\left[e^{-\rho\left(\tau^*_{n,j}-t\right)} I_{\tau^*_{n,j}\leq\tau_{n+1}}\left(\zeta(i_{j,n})\int_{\tau^*_{n,j}}^{\infty} e^{-\rho\left(s-\tau^*_{n,j}\right)} M_s^{\gamma}\theta_s^{1-\gamma} ds - qM_{\tau_n}\right)\right] \\
= \; & E_t\left[E_{\tau^*_{n,j}}\left[e^{-\rho\left(\tau^*_{n,j}-t\right)} I_{\tau^*_{n,j}\leq\tau_{n+1}}\left(\zeta(i_{j,n})\int_{\tau^*_{n,j}}^{\infty} e^{-\rho\left(s-\tau^*_{n,j}\right)} M_s^{\gamma}\theta_s^{1-\gamma} ds - qM_{\tau_n}\right)\right]\right] \\
= \; & E_t\left(e^{-(\rho+\lambda)\left(\tau^*_{n,j}-\tau_n\right)-\rho(\tau_n-t)} M_{\tau_n}\right)\left[\zeta(i_{j,n}) Z^* \frac{\gamma+\gamma_1^*-1}{\gamma_1^*-1}\psi(i_{j,n}) - q\right] \\
= \; & q\frac{1}{\gamma_1-1} E_t\left(e^{-(\rho+\lambda)\left(\tau^*_{n,j}-\tau_n\right)-\rho(\tau_n-t)} M_{\tau_n}\right) \\
= \; & q\frac{1}{\gamma_1-1} E_t\left(\left(\frac{\psi(i_{j,n})}{\theta_{\tau_n}/M_{\tau_n}}\right)^{-\gamma_1} e^{-\rho(\tau_n-t)} M_{\tau_n}\right) \\
= \; & q\frac{1}{\gamma_1-1}\psi(i_{j,n})^{-\gamma_1} E_t\left[e^{-\rho(\tau_n-t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right]
\end{aligned}$$

where the second to last equality follows from

$$E_{\tau_n}\left(e^{-(\rho+\lambda)\left(\tau^*_{n,j}-\tau_n\right)}\right) = \left(\frac{\psi(i_{j,n})}{\theta_{\tau_n}/M_{\tau_n}}\right)^{-\gamma_1}$$

$\tau_n \geq t$, and the property of iterated conditional expectation. Hence,

$$\begin{aligned}
P^f_{N,j,t} &= \frac{\sum_{n=N+1}^{\infty}\bar{A}^n E\left(q\frac{1}{\gamma_1-1}\psi(i_{j,n})^{\gamma_1} E_t\left[e^{\rho(\tau_n\ t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right]\right)}{M_t^{\gamma}\theta_t^{-\gamma}} \\
&= \frac{q}{\gamma_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma}\bar{A}^N\ E\left(\psi(i_{j,n})^{-\gamma_1}\right)\sum_{n=N+1}^{\infty}\bar{A}^{n-N} E_t\left[e^{-\rho(\tau_n-t)}\theta_{i_n}^{\gamma_1} M_{i_n}^{1-\gamma_1}\right] \\
&= \frac{q}{\gamma_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma}\bar{A}^N\ \left(\int_0^1\psi(i)^{-\gamma_1}di\right)\sum_{n=N+1}^{\infty}\bar{A}^{n-N} E_t\left[e^{-\rho(\tau_n-t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right] \\
&\equiv \frac{q}{\gamma_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma}\bar{A}^N\ \left(\int_0^1\psi(i)^{-\gamma_1}di\right)\cdot\hat{P}^f_{N,j,t}(\theta_t, M_t)
\end{aligned}$$

where

$$\begin{aligned}
\hat{P}^f_{N,j,t}(\theta_t, M_t) &\equiv \sum_{n=N+1}^{\infty}\bar{A}^{n-N} E_t\left[e^{-\rho(\tau_n-t)}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right] = & (75) \\
&= E_t\left(e^{-\rho(\tau_{N+1}-t)}\sum_{n=N+1}^{\infty}\bar{A}^{n-N}\left[e^{-\rho(\tau_n-\tau_{N+1})}\theta_{\tau_n}^{\gamma_1} M_{\tau_n}^{1-\gamma_1}\right]\right) & (76)
\end{aligned}$$

Now, we only need to find $\hat{P}^f_{N,j,t}$. It will be easiest to re-express $\hat{P}^f_{N,j,t}$ in recursive form:

$$\hat{P}^f_{N,j,t}(\theta_t, M_t) = E_t\left(e^{-\rho(\tau_{N+1}-t)}\left[\bar{A}\theta_{\tau_{N+1}}^{\gamma_1} M_{\tau_{N+1}}^{1-\gamma_1} + \bar{A}\hat{P}^f_{N,j,t}(\theta_t, M_t)\right]\right) \quad (77)$$

We shall drop the subscripts in $\hat{P}^f_{N,j,t}$ to avoid excessive notation. In light of (77), the Bellman equation for

$\hat{P}^f$ is

$$\begin{aligned} 0 &= \frac{\sigma^2}{2}\theta_t^2\hat{P}^f_{\theta\theta} + \mu\theta_t\hat{P}^f_{\theta} - (\rho+\lambda)\hat{P}^f + \lambda\left[\bar{A}\theta_t^{\gamma_1}M_t^{1\ \gamma_1} + \bar{A}\hat{P}^f\right] \\ &= \frac{\sigma^2}{2}\theta_t^2\hat{P}^f_{\theta\theta} + \mu\theta_t\hat{P}^f_{\theta} - \left[\rho+\lambda-\lambda\bar{A}\right]\hat{P}^f + \lambda\bar{A}\left[\theta_t^{\gamma_1}M_t^{1-\gamma_1}\right] \end{aligned}$$

The solution to this second order ODE[33] is

$$\hat{P}^f_{N,j,t}(\theta_t, M_t) = \lambda\bar{A}\hat{Z}\theta_t^{\gamma_1}M_t^{1-\gamma_1} + \theta_t^{\hat{\gamma}_1}F(M_t)$$

where $\hat{\gamma}_1$ is the positive root of the following equation (note: $\gamma_1 > \gamma_1^* > \hat{\gamma}_1$)

$$\frac{\sigma^2}{2}\gamma^2 + \left(\mu - \frac{\sigma^2}{2}\right)\gamma - \left[\rho+\lambda-\lambda\bar{A}\right] = 0$$

and

$$\hat{Z} = \frac{-1}{\frac{\sigma^2}{2}\gamma_1(\gamma_1-1) + \mu\gamma_1 - \left[\rho+\lambda-\lambda\bar{A}\right]}$$

To determine $F(M_t)$ we shall use a standard reflection condition at $\theta_t = M_t$, namely:

$$\hat{P}^f_M(\theta_t, M_t) = 0 \text{ at } \theta_t = M_t$$

Employing this boundary condition yields

$$F_M(M_t) = (\gamma_1 - 1)\lambda\bar{A}\hat{Z}M_t^{-\hat{\gamma}_1}$$

Integrating:

$$F(M_t) = (\gamma_1 - 1)\lambda\bar{A}\hat{Z}\int_{M_t}^{\infty} M_t^{-\hat{\gamma}_1}$$

together with the assumption $\hat{\gamma}_1 > 1$ gives:

$$F(M_t) = -\lambda\bar{A}\hat{Z}\frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1}M_t^{1-\hat{\gamma}_1}$$

That is,

$$\begin{aligned} \hat{P}^f_{N,j,t}(\theta_t, M_t) &- \lambda\bar{A}\hat{Z}\theta_t^{\gamma_1}M_t^{1-\gamma_1} - \lambda\bar{A}\hat{Z}\frac{\gamma_1-1}{\hat{\gamma}_1-1}\theta_t^{\hat{\gamma}_1}M_t^{1-\hat{\gamma}_1} \\ &= \lambda\bar{A}\hat{Z}\theta_t^{\gamma_1}M_t^{1-\gamma_1}\left[1 - \frac{\gamma_1-1}{\hat{\gamma}_1-1}\theta_t^{\hat{\gamma}_1-\gamma_1}M_t^{\gamma_1-\hat{\gamma}_1}\right] \end{aligned} \tag{78}$$

Hence, the value of all the future growth options is

$$\begin{aligned} P^f_{N,j,t} &= \frac{q}{\gamma_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma}\bar{A}^N\left(\int_0^1 \psi(i)^{-\gamma_1}di\right)\cdot\hat{P}^f_{N,j,t}(\theta_t, M_t) = \\ &= \frac{q}{\gamma_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma}\bar{A}^N\left(\int_0^1 \psi(i)^{-\gamma_1}di\right)\lambda\bar{A}\hat{Z}\theta_t^{\gamma_1}M_t^{1-\gamma_1}\left[1 - \frac{\gamma_1-1}{\hat{\gamma}_1-1}\theta_t^{\hat{\gamma}_1-\gamma_1}M_t^{\gamma_1-\hat{\gamma}_1}\right] \qquad (79) \\ &= Z^*\bar{A}^N\theta_t\left\{\frac{\lambda\bar{A}\hat{Z}}{\gamma_1-1}\left[\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1-1} - \frac{\gamma_1-1}{\hat{\gamma}_1-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\hat{\gamma}_1-1}\right]\left(\frac{q}{Z^*}\right)\left(\int_0^1 \psi(i)^{-\gamma_1}di\right)\right\} \qquad (80) \end{aligned}$$

[33] We have excluded solutions that explode near $\theta_t = 0$.

This completes the proof of the proposition. ■

Proof. (Lemma 1) The value of the entire stock market is:

$$P_{N,t} = \int_0^1 P_{N,j,t}dj = \int_0^1 P_{N,j,t}dj = \int_0^1 P_{j,t}^A dj + \int_0^1 P_{N,j,t}^o dj + P_{N,t}^f$$

Using Proposition 4, the first term is equal to:

$$\int_0^1 P_{j,t}^A dj = Z^* X_t \theta_t \left[1 + \frac{\gamma}{\gamma_1^* - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\right]$$

where X_t is given in (8). Similarly:

$$\int_0^1 P_{N,j,t}^o dj = Z^* \bar{A}^N \theta_t \left[\frac{1}{\gamma_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1-1}\left(\frac{M_t}{M_{\tau_N}}\right)^{\gamma_1-1}\left(\frac{q}{Z^*}\right)\left(\int_{K_{N,t}}^1 \psi(i)^{-\gamma_1}di\right)\right]$$

Substituting these formulas into the definition of w_t^{o+f}, using the definition of x_t in equation (26), and simplifying gives the result with f given by:

$$f = \frac{\left[1 + \frac{\gamma}{\gamma_1^*-1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1^*-1}\right]}{\Gamma}$$

and Γ is given by:

$$\begin{aligned}\Gamma &= \left[\frac{1}{\gamma_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1-1}\left(\frac{M_t}{M_{\tau_N}}\right)^{\gamma_1-1}\left(\frac{q}{Z^*}\right)\left(\int_{K_{N,t}}^1 \psi(i)^{-\gamma_1}di\right)\right] \\ &+ \frac{\lambda\bar{A}\hat{Z}}{\gamma_1 - 1}\left[\left(\frac{\theta_t}{M_t}\right)^{\gamma+\gamma_1-1} - \frac{\gamma_1 - 1}{\hat{\gamma}_1 - 1}\left(\frac{\theta_t}{M_t}\right)^{\gamma+\hat{\gamma}_1-1}\right]\left(\frac{q}{Z^*}\right)\left(\int_0^1 \psi(i)^{-\gamma_1}di\right)\end{aligned}$$

Note that $K_{N,t}$ is a function of $\frac{M_t}{M_{\tau_N}}$ only, and hence Γ itself is a function of $\frac{\theta_t}{M_t}, \frac{M_t}{M_{\tau_N}}$. ■

Proof. (Lemma 2) A feature of the model is that risk aversion is constant[34]. Hence the Sharpe ratio is a constant equal to $\gamma\sigma$, and the instantaneous excess return on any asset is given by:

$$\mu^{(j)} - r = \gamma\sigma\sigma_{j,t}$$

where:

$$\sigma_{j,t} = \left(\frac{\sigma\theta_t}{P_{N,j,t}}\right)\frac{\partial P_{N,j,t}}{\partial\theta_t}$$

Hence in order to establish the result is suffices to compare the instantaneous volatility of assets in place, current epoch growth options and future growth options. Using Proposition 4 and starting with the volatility

[34] The running maximum of a diffusion has no quadratic variation, and hence all of the quadratic variation of the pricing kernel is driven by variations in θ_t and not M_t.

of asset in place, we obtain:

$$\begin{aligned}\sigma_{j,t}^{A} &= \left(\frac{\sigma\theta_t}{P_{N,j,t}^{A}}\right)\frac{\partial P_{N,j,t}^{A}}{\partial\theta_t} \\ &= \sigma\left[w_1+(1-w_1)\left(\gamma+\gamma_1^*\right)\right] \\ &= \sigma\left[\left(\gamma+\gamma_1^*\right)+w_1\left(1-\gamma-\gamma_1^*\right)\right] \end{aligned} \tag{81}$$

where

$$\begin{aligned} w_1 &= \frac{\theta_t}{\theta_t+\frac{\gamma}{\gamma_1^*-1}M_t^{1-\gamma-\gamma_1^*}\theta_t^{\gamma+\gamma_1^*}} \\ &= \frac{\gamma_1^*-1}{\gamma_1^*-1+\gamma\left(\frac{M_t}{\theta_t}\right)^{1-\gamma-\gamma_1^*}} \end{aligned}$$

It is clear that $1\geq w_1\geq 0$. For the volatility of the current epoch growth option, we obtain similarly

$$\begin{aligned}\sigma_{j,t}^{o} &= \left(\frac{\sigma\theta_t}{P_{N,j,t}^{o}}\right)\frac{\partial P_{N,j,t}^{o}}{\partial\theta_t} \\ &= \sigma\left(\gamma+\gamma_1\right)\end{aligned} \tag{82}$$

For the volatility of future options,

$$\begin{aligned}\sigma_{j,t}^{f} &= \left(\frac{\sigma\theta_t}{P_{N,j,t}^{f}}\right)\frac{\partial P_{N,j,t}^{f}}{\partial\theta_t} \\ &= \sigma\left[w_3\left(\gamma+\gamma_1\right)+\left(1-w_3\right)\left(\gamma+\hat{\gamma}_1\right)\right] \\ &= \sigma\left[\left(\gamma+\hat{\gamma}_1\right)+w_3\left(\gamma_1-\hat{\gamma}_1\right)\right] \end{aligned} \tag{83}$$

where

$$\begin{aligned} w_3 &= \frac{\theta_t^{\gamma+\gamma_1}M_t^{1-\gamma-\gamma_1}}{\theta_t^{\gamma+\gamma_1}M_t^{1-\gamma-\gamma_1}-\frac{\gamma_1\ \ 1}{\hat{\gamma}_1-1}\theta_t^{\gamma+\hat{\gamma}_1}M_t^{1-\gamma-\hat{\gamma}_1}} \\ &= \frac{1}{1-\frac{\gamma_1-1}{\hat{\gamma}_1-1}\left(\frac{M_t}{\theta_t}\right)^{\gamma_1-\hat{\gamma}_1}} \end{aligned}$$

Since $\gamma_1\geq\hat{\gamma}_1$ and $w_3\leq 0$, equation (82) and (83) implies

$$\sigma_{j,t}^{f}\leq\sigma\left(\gamma+\hat{\gamma}_1\right)<\sigma\left(\gamma+\gamma_1\right)=\sigma_{j,t}^{o}$$

To develop the relation between the volatility of assets in place and future growth options, we define the function $d\left(x\right)$ as follows,

$$\begin{aligned} d\left(\frac{M_t}{\theta_t}\right) &\equiv \frac{\sigma_{j,t}^{A}-\sigma_{j,t}^{f}}{o} \\ &= \left[\left(\gamma+\gamma_1^*\right)+w_1\left(1-\gamma-\gamma_1^*\right)\right]-\left[\left(\gamma+\hat{\gamma}_1\right)+w_3\left(\gamma_1-\hat{\gamma}_1\right)\right] \\ &= \left(\gamma_1^*-\hat{\gamma}_1\right)+\frac{\gamma_1^*-1}{\gamma_1^*-1+\gamma\left(\frac{M_t}{\theta_t}\right)^{1-\gamma-\gamma_1^*}}\left(1-\gamma-\gamma_1^*\right)-\frac{\hat{\gamma}_1-1}{\hat{\gamma}_1-1-\left(\gamma_1-1\right)\left(\frac{M_t}{\theta_t}\right)^{\gamma_1-\hat{\gamma}_1}}\left(\gamma_1-\hat{\gamma}_1\right) \end{aligned}$$

It is clear that

$$\begin{aligned} d(1) &= (\gamma_1^* - \hat{\gamma}_1) + \frac{\gamma_1^* - 1}{\gamma_1^* - 1 + \gamma}(1 - \gamma - \gamma_1^*) - \frac{\hat{\gamma}_1 - 1}{\hat{\gamma}_1 - 1 - (\gamma_1 - 1)}(\gamma_1 - \hat{\gamma}_1) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} d'(x) &= -\frac{(\gamma_1^* - 1)\gamma}{(\gamma_1^* - 1 + \gamma x^{1-\gamma-\gamma_1^*})^2}(1 - \gamma - \gamma_1^*)^2 - \frac{(\hat{\gamma}_1 - 1)(\gamma_1 - 1)}{[\hat{\gamma}_1 - 1 - (\gamma_1 - 1)x^{\gamma_1 - \hat{\gamma}_1}]^2}(\gamma_1 - \hat{\gamma}_1)^2 \\ &< 0 \end{aligned}$$

It follows that $d(x) < 0$ for all $x > 1$. That is, $\sigma_{j,t}^A \leq \sigma_{j,t}^f$ and the equality holds only if $\theta_t = M_t$. Therefore, we have

$$\sigma_{j,t}^A \leq \sigma_{j,t}^f < \sigma_{j,t}^o$$

and the equality holds only if $\theta_t = M_t$. ■

Proof. (Lemma 3) Note that in light of (37):

$$w_t^{(j),o+f} = \frac{e^{-x_t}}{e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right) + e^{-x_t}} = \frac{1}{1 + \frac{e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right)}{e^{-x_t}}}$$

Hence $w_t^{(j),o+f} < \frac{1}{2}$ if and only if:

$$\frac{e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right)}{c^{-x_t}} > 1 \tag{84}$$

Keeping $\kappa^{(j)}$ fixed, differentiate (37) w.r.t. x_t to obtain:

$$\frac{\partial w_t^{(j),o+f}}{\partial x_t} = -\frac{e^{-x_t} e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right)}{\left(e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right) + e^{-x_t}\right)^2}$$

so that:

$$\begin{aligned} \frac{\partial}{\partial \kappa^{(j)}}\left|\frac{\partial w_t^{(j),o+f}}{\partial x_t}\right| &= e^{-x_t} \frac{e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right)\left(e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right) + e^{-x_t}\right) - 2\left[e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right)\right]^2}{\left(e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right) + e^{-x_t}\right)^3} = \\ &= e^{-x_t} \frac{e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right)\left[e^{-x_t} - e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right)\right]}{\left(e^{\kappa^{(j)}} g\left(\frac{\theta_t}{M_t}\right) + e^{-x_t}\right)^3} < 0 \end{aligned}$$

The last inequality follows from (84). ■

References

Abel, A. B. (1990): "Asset Prices under Habit Formation and Catching Up with the Joneses," *American Economic Review*, 80(2), 38–42.

Abel, A. B., and J. C. Eberly (2002): "Q for the Long Run," mimeo. The Wharton School and Kellog.

——— (2003): "Investment, Valuation, and Growth Options," mimeo. The Wharton School and Kellog.

——— (2004): "Q Theory Without Adjustment Costs Cash Flow Effects Without Financing Constraints," mimeo. The Wharton School and Kellog.

Atkeson, A., and P. J. Kehoe (1993): "Industry evolution and transition: the role of information capital," Federal Reserve Bank of Minneapolis, Staff Report: 162, 1993.

——— (1999): "Models of Energy Use: Putty-Putty versus Putty-Clay," *American Economic Review*, 89(4), 1028–1043.

Bansal, R., R. Dittmar, and C. Lundblad (2004): "Consumption, Dividends and the Cross-Section of Equity Returns," forthcoming, Journal of Finance.

Bansal, R., and A. Yaron (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59(4), 1481–1509.

Basak, S. (1999): "On the Fluctuations in Consumption and Market Returns in the Presence of Labor and Human Capital: An Equilibrium Analysis," *Journal of Economic Dynamics and Control*, 23(7), 1029–1064.

Berk, J. B., R. C. Green, and V. Naik (1999): "Optimal Investment, Growth Options, and Security Returns," *Journal of Finance*, 54(5), 1553–1607.

——— (2004): "Valuation and Return Dynamics of New Ventures," *Review of Financial Studies*, 17(1), 1–35.

Caballero, R. J., and E. M. R. A. Engel (1991): "Dynamic (S, s) Economies," *Econometrica*, 59(6), 1659–1686.

——— (1999): "Explaining Investment Dynamics in U.S. Manufacturing: A Generalized (S, s) Approach," *Econometrica*, 67(4), 783–826.

Caballero, R. J., and R. S. Pindyck (1996): "Uncertainty, Investment, and Industry Evolution," *International Economic Review*, 37(3), 641–662.

Campbell, J. Y., and J. H. Cochrane (1999): "By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior," *Journal of Political Economy*, 107(2), 205–251.

Carlson, M., A. Fisher, and R. Giammarino (2004): "Corporate Investment and Asset Price Dynamics: Implications for the Cross Section of Returns," *Journal of Finance*, 59(6), 2577–2603.

Chan, Y. L., and L. Kogan (2002): "Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices," *Journal of Political Economy*, 110(6), 1255–1285.

Cochrane, J. H. (1996): "A Cross-Sectional Test of an Investment-Based Asset Pricing Model," *Journal of Political Economy*, 104(3), 572 621.

——— (2005): *Asset pricing*. Princeton University Press, Princeton and Oxford.

Comin, D., and M. Gertler (2003): "Medium Term Business Cycles," C.V. Starr Center for Applied Economics, New York University, Working Papers.

Cooper, I. (2004): "Asset Pricing Implications of Non-convex Adjustment Costs and Irreversibility of Investment," Journal of Finance, forthcoming.

Dupor, B., and W.-F. Liu (2003): "Jealousy and Equilibrium Overconsumption," *American Economic Review*, 93(1), 423–428.

Fama, E. F., and K. R. French (1989): "Business Conditions and Expected Returns on Stocks and Bonds," *Journal of Financial Economics*, 25(1), 23–49.

——— (1992): "The Cross-Section of Expected Stock Returns," *Journal of Finance*, 47(2), 427–465.

Gomes, J., L. Kogan, and L. Zhang (2003): "Equilibrium Cross Section of Returns," *Journal of Political Economy*, 111(4), 693–732.

Greenwood, J., and B. Jovanovic (1999): "The Information-Technology Revolution and the Stock Market," *American Economic Review*, 89(2), 116–122.

Hamilton, J. D. (1994): *Time series analysis*. Princeton University Press, Princeton.

Hansen, L. P., J. C. Heaton, and N. Li (2005): "Consumption Strikes Back?: Measuring Long Run Risk," Working Paper, University of Chicago.

Helpman, E. (1998): *General purpose technologies and economic growth*. MIT Press, Cambridge and London.

Jermann, U., and V. Quadrini (2002): "Stock Market Boom and the Productivity Gains of the 1990s," NBER Working Papers: 9034.

Jovanovic, B., and G. M. MacDonald (1994): "The Life Cycle of a Competitive Industry," *Journal of Political Economy*, 102(2), 322–347.

Jovanovic, B., and P. L. Rousseau (2003): "Two Technological Revolutions," *Journal of the European Economic Association*, 1(2-3), 419–428.

——— (2004): "Interest Rates and Initial Public Offerings," NBER Working Papers: 10298.

Karatzas, I., and S. E. Shreve (1991): *Brownian motion and stochastic calculus*, vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edn.

——— (1998): *Methods of mathematical finance*, vol. 39. Springer-Verlag.

Kogan, L. (2001): "An Equilibrium Model of Irreversible Investment," *Journal of Financial Economics*, 62(2), 201–245.

——— (2004): "Asset Prices and Real Investment," *Journal of Financial Economics*, 73(3), 411–431.

Larrain, B., and M. Yogo (2005): "Does Firm Value Move Too Much to be Justified by Subsequent Changes in Cash Flow?," Working Paper, The Wharton School.

Lettau, M., and S. C. Ludvigson (2005): "Expected Returns and Expected Dividend Growth," *Journal of Financial Economics*, 76(3), 583–626.

Malloy, C., T. Moskowitz, and A. Vissing-Jorgensen (2005): "Long-Run Stockholder Consumption Risk and Asset Returns," Working Paper, Chicago GSB.

Menzly, L., T. Santos, and P. Veronesi (2004): "Understanding Predictability," *Journal of Political Economy*, 112(1), 1–47.

Novy-Marx, R. (2003): "An equilibrium model of investment under uncertainty," mimeo. Chicago GSB.

Øksendal, B. (2003): *Stochastic differential equations*. Springer-Verlag.

Parker, J. A., and C. Julliard (2005): "Consumption Risk and the Cross Section of Expected Returns," *Journal of Political Economy*, 113(1), 185–222.

Pastor, L., and P. Veronesi (2004): "Was There a Nasdaq Bubble in the Late 1990s?," NBER Working Papers: 10581.

Pontiff, J., and L. D. Schall (1998): "Book-to-Market Ratios as Predictors of Market Returns," *Journal of Financial Economics*, 49(2), 141–160.

Vigfusson, R. J. (2004): "The Delayed Response To A Technology Shock. A Flexible Price Explanation," Working paper, The Federal Reserve Board.

Zhang, L. (2005): "The Value Premium," *Journal of Finance*, 60(1), 67–103.