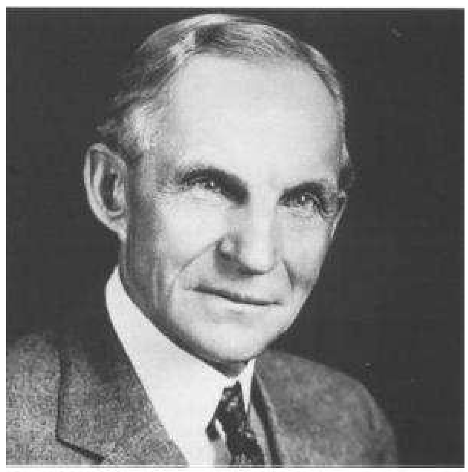
Suburbanization and the Automobile

Karen Kopecky and Ming Hon Suen

January 2004



Henry Ford

Version: May 2004

Suburbanization and the Automobile[a]

Karen Kopecky and Ming Hon Suen[b]

January 2004

Version: April 2004

Abstract

In 1910 the average American city was a small and densely populated place and less than one percent of Americans owned a car. By 1970, almost every family in the US owned at least one automobile. Not only did city size grow between 1910 and 1970, but city population became more evenly spread around the city center: suburbanization. A model of a linear city is developed in which agents choose both whether or not to own a car, and where to live. With declining automobile prices and rising incomes, the model is able to match the data on car ownership and decentralization for the period 1910 to 1970.

Keywords: automobiles, suburbanization, population density gradients

JEL Classification Nos: E10, 011, N12, R12

[a] *Updates:* http://troi.cc.rochester.edu/~kpky/SuburbandAuto.pdf

[b] *Contact Information:*

Karen A. Kopecky
Email: kpky@troi.cc.rochester.edu

Ming Hon Suen
Email: msen@troi.cc.rochester.edu

Harkness Hall
Department of Economics
University of Rochester
P.O. Box 270156
Rochester, NY 14627-0156, USA.
(585)275-5252 Fax: (585) 256-2309
P.O. Box 270156

1 Introduction

Suburbanization has been observed in cities throughout the US since the 1870's when street railways were first implemented. But technological progress in transportation made its biggest contributions during the twentieth century with the invention of the automobile and later the modern highway system. The adoption of the car as the dominant form of transportation in the US, combined with rising real income levels, encouraged movement to less dense areas where housing was more affordable. The goal here is to assess, quantitatively, the relationship between the invention and diffusion of the automobile and suburbanization.

1.1 Suburbanization

Suburbanization is defined as the increased dispersion of urban population over land area[1].

Measuring Suburbanization: To measure the extent of suburbanization, researchers have successfully adopted the following functional form to relate population density, d to distance from the city center, x:

$$d = ae^{-bx}. \tag{1}$$

The parameter, a is an estimate of the density (people/square mile) at the city center. The density gradient, b measures the rate of change of the density as the distance from the city center increases. The results from estimating equation (1) for two different cities, Chicago and Atlanta, are provided in Tables 1 and 2, respectively. Observe that Chicago is more densely populated than Atlanta, this can be seen by comparing the values of a across the two cities. The density gradient for Atlanta is much steeper than for Chicago in 1900. This can also be seen in Figure 1 which shows the population density functions for Atlanta in 1900, 1950, and 1970 and Chicago in 1900 and 1950. The amount of suburbanization that occurred over a period of time is measured by the percent decline in the population density gradient. In Chicago, the density gradient fell by 48% over the period 1900-50. In Atlanta, it decreased by 39% during the same period and 77% over the period 1900-70.

Facts: Suburbanization has been a widely observed phenomenon both in the US and in countries across the world. Decentralization of American cities was first observed in the second half of the nineteenth century.[2] During the period from 1910 to 1970, declining population density gradients have been observed for almost all US cities. It is during this period that suburbanization in the US occurred most intensively. This includes both cities with rising populations and those whose populations have been declining over the period. Estimates of the population density gradient b are shown in Table 3 for four US cities: Baltimore, Milwaukee, Philadelphia, and Rochester, and

[1] Hereafter, the terms 'suburbanization,' 'urban decentralization,' and 'urban sprawl' refer to the same phenomenon.

[2] See Mieszkowski and Mills (1993) for a survey of suburbanization.

in Table 4 for forty-one US cities, which were defined as metropolitan districts in 1900. The tables show the density gradients and the percentage change in the density gradients over the period 1900 to 1970. The density gradients have been declining for the sample as a whole from 1910 to 1970 with the largest decline occurring in the decade 1940-50. In the four city sub-sample the density gradients are declining throughout the entire period[3]. The density gradients declined by 77% over the period 1900 to 1970 for the forty-one US cities.

Theories of Suburbanization: A variety of explanations of decentralization have been proposed. One popular theory is that the rich move to suburbs in order to avoid the disamenities of the inner cities, such as high crime rates and poor schools[4]. While another theory argues that government policies increase the attractiveness of suburban residential locations. For instance, Voith (1999) discusses how housing-related tax incentives, such as mortgage and property tax deduction, might induce people to demand more housing and hence move to suburban communities. These theories may be useful for explaining suburbanization observed in particular cities during particular time periods, but they cannot explain the more general trend of decentralization observed in the US since 1870 and in other countries.

A second theory of suburbanization focuses on the impact of technological progress in transportation which has reduced the costs of commuting over time. As the cost of commuting falls, higher income groups move further from the city center to enjoy more modern housing, more space, and/or more attractive communities. Beginning in the 1830's, several innovations in mass transportation, including omnibus, commuter railroads and streetcars were introduced into American cities. These improvements enhanced the mobility of city dwellers and helped to expand large cities like New York, Boston and Philadelphia[5]. Historians such as Warner (1962) and Ward (1971) take the view that the introduction of the streetcar in the 1850's and 1860's caused the first major movement, by wealthy individuals, to the suburbs. Yet, by far the biggest breakthrough in transportation came with the invention of the automobile. The extensive suburbanization observed during the twentieth century occurred simultaneously with the adoption of the automobile by Americans. Included in this adoption was the adaption of American cities to private vehicle commuting. This theory of suburbanization as occurring concurrently with technological progress in transportation, is developed and analyzed in the work of LeRoy and Sonstelie (1983). More recent discussions of the theory are included in Glaeser, Kahn and Rappaport (2000), and Glaeser and Kahn (2003).

[3]Balitmore, Milwaukee, Philadelphia, and Rochester are older than average US cities. Older cities tended to spread out earlier than younger cities. See Edmonston (1975) for a discussion of variations in density gradients across US cities.

[4]See for instance, Cullen and Levitt (1999), Mills and Lubuele (1999) and the references therein.

[5]See Taylor (1966) for a detailed historical account on these innovations.

1.2 Automobiles

The structure of automobile ownership in the United States changed immensely throughout the twentieth century. In 1906 approximately one tenth of Americans owned a car. In fact, automobiles were thought of more as a toy for the rich than as a realistic mode of transportation. By 1940, however, more than 44% of US families owned a car. By 1995, 92% of American families owned at least one car and 59% owned two or more.[6] The increase in car use and ownership can be seen in Figure 2, which shows the number of registered automobiles per capita in the United States from 1900 to 1993 and in Table 5 which presents the number of registered vehicles per person age twenty to sixty-four for the decennial years from 1900 to 1970. Figure 3 shows how car ownership evolved over the second half of the century.

What caused the rise in car ownership? Both ownership and number of cars owned are increasing with household income. As shown in Figure 4(a), car ownership ranges from less than 50% for the poorest income groups to over 90% for the richest, over the period 1952 - 65. Overall ownership increased from 65% to 74% during this period, largely due to the increase in the second and third quintiles.

There is a negative correlation between car ownership and prices. The quality-adjusted price of a new car decreased by 85% since 1906. Figure **??** shows the average price of a new automobile from 1906 to 2000. Figure 5(a) is a plot of the log average price. The price of a new automobile decreased at an average annual rate of approximately 2% throughout the period. The fastest rate of decrease occurred during the period 1906-40 with an average rate of 5.5%. The time cost of purchasing a new car fell by approximately 98% during the 1906-2000 period. In 1906 a worker earning the average wage would have to work approximately 453 weeks or more than 8 years to acquire enough earnings to afford to buy a new car. By 1920 the average wage earner could purchase a new car with approximately 1.6 years of earnings. In 2000 the worker could afford a new car with only 16 weeks of earnings. The ln time cost can be seen in Figure 6(a).[7] The figures depict the time cost for an individual earning the average wage.

As prices fall not only are families at lower and lower income levels able to afford automobiles but the purchasing of two and eventually more than two vehicles becomes a reasonable expenditure. Owning two cars then gives a two-headed household more freedom to move further away from central

[6]Sources: 1934-1936: Bureau of Labor Statistics, U.S. Monthly Labor Review, March 1940, 1995: Federal Highway Administration, National Personal Transportation Survey, Summary of Travel Trends.

[7] The time cost of purchasing a car is obtained by dividing the price of automobiles by labor income. The value for 1906 is then normalized to one. Data on automobile prices are collected from various sources: For the period 1906 - 1940, data reported by Raff and Trajtenberg (1995) are used; for 1947 - 1983, data are obtained from Gordon (1990); and for 1967 - 2000, the data is taken from Ward (2002). The Raff and Trajtenberg (1995) and Gordon (1990) price series are of quality-adjusted prices. The series from Ward (2002) is the price for a comparable car. Labor income is the average hourly wages of workers. This is computed with the help of the US real wage indices reported in Williamson (1995). Here it is assumed that agents work 40 hours a week.

locations. This argument is supported by Figure 7 which depicts the positive relationship between car ownership and distance from the city center. The same figure also shows that families living further away from the city center are more likely to own more than one car, than families living closer to the center. Another piece of evidence is Figure 8, which shows car ownership by city versus suburban location. When compared to those living in the central cities, residents in suburban areas have a higher rate of ownership and own more cars. This phenomenon becomes more predominant over time. By 1961, only 6% of those living in the central cities of the 12 largest SMSA's have two or more cars, as compared to 17% in the suburbs. Ten years later, the two figures rise to 15% and 41%, respectively.

1.3 The Goal

The goal is to develop a model that can be used to explain the relationship between automobile use versus other modes of transportation, and the changes in the density gradients of cities over time. By explicitly modelling individuals' car choice, the impact of decreasing automobile prices on suburbanization and car-ownership can be assessed quantitatively.

2 The Model

Begin with a simple model of car ownership and location choice. In the model an agent can choose his mode of transportation and residential site. An agent can either take a bus, which is publicly owned and operated, or purchase and use a car. The bus in the model serves as a proxy for all forms of public transportation that are relatively cheaper and slower than the automobile. When agents make their location choices, they take into consideration two factors: the cost of commuting and the cost of housing. The desire to save on commuting costs pulls them closer to the employment center. This in turns generates a large demand for housing around the center and bids up rents. Optimal location choice thus involves a balance between the two. By owning a car, one can spend less time commuting from a given location. This induces agents to spread to neighboring suburbs and enjoy larger living spaces. But not every one will *choose* to own a car. The price of a car serves as a fixed cost that screens out those with lower incomes. As income rises, car prices decline, and the cost of commuting by car relative to taking the bus rises, automobiles become more affordable and attractive, and this promotes suburbanization.

2.1 The Environment

Consider a linear city located on the positive half of the real line. The city is of unit width so that the density of land at each location is equal to one. The upper boundary of the city is determined endogenously. Land beyond this boundary is used for agriculture.[8] All production activities take place at the origin, or the city center.[9] The city is inhabited by a continuum of agents. The size of the total population is normalized to unity. Each agent is characterized by an ability, λ, drawn from a distribution $F(\lambda)$ defined on a finite support $[\lambda_{\min}, \lambda_{\max}]$. The mean of the ability distribution is normalized to unity.

There are three types of goods in the economy: final goods, automobiles, and land.[10] Land in the city is owned by a landowner who collects all the rent and spends it on consumption. During each period, the agents must commute from their residential location to their workplace at the origin. There are two modes of transportation in the city: car and bus.

2.2 Bus-user's Problem

If an agent with ability λ chooses to commute by bus, then he chooses consumption in goods, c, consumption in land services, l, and location, x, to solve the static problem (P1), taking the rent function $q(\cdot)$ as given.

$$V^b(\lambda) = \max_{c,l,x} \{\alpha \ln c + (1-\alpha) \ln l\} \tag{P1}$$

subject to

$$c + q(x)\, l = w\lambda - t(w\lambda, x),$$

$$x \geq 0,$$

where $q(x)$ is the rent at location x, w is the market wage for an efficiency unit of labor, and $t(w\lambda, x)$ is the cost for an agent with ability λ to travel distance x by bus. The last inequality restricts the location choice of bus-users to the positive half of the real line. The transportation cost function, $t(\cdot)$, is assumed to be (i) twice continuously differentiable, (ii) increasing in both arguments, and (iii) satisfying $t(w\lambda, 0) > 0$ for all $w\lambda \geq 0$. The second assumption implies that the time cost of commuting is increasing in income. Since agents spend all their non-commuting time at work, those with higher wages have a higher opportunity cost of commuting. The third assumption implies that the cost function includes a fixed cost that is independent of distance.

Conditional on any given location $x \geq 0$, expenditures on goods and land by a bus-user with ability λ are given by

$$c_b(\lambda, x) = \alpha [w\lambda - t(w\lambda, x)] \tag{2}$$

and

$$q(x)\, l_b(\lambda, x) = (1-\alpha)[w\lambda - t(w\lambda, x)]. \tag{3}$$

[8] Note that the agricultural sector is not modeled here nor do agents have any demand for agricultural goods. The agricultural sector exists only to serve as an alternative-user of land. As discussed below, this helps to pin down the boundary of the city.

[9] Firms do not use land for production.

[10] 'Land' and 'housing' are not differentiated in this paper.

Let $x_b(\lambda)$ denote the agent's optimal location choice. This location choice is characterized by the first-order condition

$$-t_2(w\lambda, x) = q'(x)\, l_b(\lambda, x).^{11} \tag{4}$$

By moving closer to the origin, the agent can reduce his transportation costs but this gain must be balanced by an increase in rent. This implies that the equilibrium rent function is decreasing. Combining (3) and (4) gives

$$\frac{q'(x)}{q(x)} = \frac{-t_2(w\lambda, x)}{(1-\alpha)\,[w\lambda - t(w\lambda, x)]}. \tag{5}$$

By the Implicit Function Theorem, if $q(\cdot)$ is twice continuously differentiable, then $x_b(\lambda)$ exists and is continuously differentiable. The following lemma states the condition under which $x_b(\lambda)$ is strictly increasing.

Lemma 1. *Given any twice continuously differentiable function $q(\cdot)$, the optimal location choice function $x_b(\lambda)$ is strictly increasing in λ if and only if the income elasticity of housing demand exceeds the income elasticity of marginal commuting cost, or*

$$\frac{w\lambda t_{12}(w\lambda, x)}{t_2(w\lambda, x)} < \frac{w\lambda\,[1 - t_1(w\lambda, x)]}{w\lambda - t(w\lambda, x)}. \tag{6}$$

Proof. See Appendix. □

Since the rich (those with high ability) have a higher land demand than the poor (those with low ability) at any location, the former benefit by living further away from the city center where the rent is lower. However, the rich also have a higher time cost than the poor. This induces them to live closer to the city center. The location choice function is monotonically increasing if and only if the first effect dominates. Given the log utility function, the two elasticities are determined by the transportation costs functions alone. For instance, if $t(w\lambda, x)$ is multiplicatively separable in w and x, then the slope of the location choice function is solely determined by the income elasticity of $t(w\lambda, x)$.

Lemma 2. *If $t(w\lambda, x) = \delta(x)\,\kappa(w\lambda) + \eta$, where η is a nonnegative constant, then $x_b(\lambda)$ is strictly increasing if and only if*

$$(w\lambda - \eta)\,\kappa'(w\lambda) < \kappa(w\lambda). \tag{7}$$

Proof. See Appendix. □

Notice that condition (7) is satisfied if $\kappa(w)$ is linear and passes through the origin. If $x_b(\lambda)$ is strictly monotonic, then any location $x = x_b(\lambda)$ will be inhabited by bus-users with ability λ alone. Positive consumption requires that

$$w\lambda \geq t(w\lambda, x_b(\lambda)).$$

[11]The nonnegativity constraint on location is binding for a measure 0 of agents since each location $x \geq 0$ has a measure 0 of land. Hence only interior solutions exist.

2.3 Car-owner's Problem

If an agent with ability, λ, chooses to own a car, then he solves the static problem (P2), taking as given the rent function $q(\cdot)$.

$$V^c(\lambda) = \max_{c,l,x} \{\alpha \ln c + (1-\alpha)\ln l\} \tag{P2}$$

subject to

$$c + q(x)\, l = w\lambda - \tau(w\lambda, x) - p_c,$$

$$x \geq 0,$$

where p_c denotes the price of a car, $\tau(w\lambda, x)$ is the cost of traveling distance x by car. The transportation cost function $\tau(w\lambda, x)$ is assumed to share the same properties as $t(w\lambda, x)$. When comparing to the bus, a car takes less time to travel the same distance, i.e. $\tau_2 < t_2$, for all $(w\lambda, x)$, but costs more to use, so that

$$\tau(w\lambda, 0) + p_c > t(w\lambda, 0) \tag{8}$$

holds for all $w\lambda$.

Given (8), bus-users will always reside closer to the origin than car-owners. To see this consider an agent with ability λ who lives at the origin. If he owns a car, his transportation expenses are $\tau(w\lambda, 0) + p_c$, which are higher than $t(w\lambda, 0)$, the cost of taking a bus. This means the agent would have a higher net income (net of transportation expenses) by switching to take a bus. Since it is never optimal to locate at the origin *and* to own a car, car-owners will location at locations $x > 0$.

Conditional on any given location $x > 0$, expenditures on goods and land are given by

$$c_c(\lambda, x) = \alpha\left[w\lambda - \tau(w\lambda, x) - p_c\right], \tag{9}$$

$$q(x)\, l_c(\lambda, x) = (1-\alpha)\left[w\lambda - \tau(w\lambda, x) - p_c\right]. \tag{10}$$

The first-order condition for location choice is

$$\frac{q'(x)}{q(x)} = \frac{-\tau_2(w\lambda, x)}{(1-\alpha)\left[w\lambda - \tau(w\lambda, x) - p_c\right]}. \tag{11}$$

Given any rental function $q(\cdot)$, (11) determines the optimal location choice of a car-owner with ability λ, $x_c(\lambda)$. Similar to $x_b(\lambda)$, if $q(\cdot)$ is twice continuously differentiable, then $x_c(\lambda)$ exists and is continuously differentiable. If we define $\widetilde{t}(w\lambda, x) \equiv \tau(w\lambda, x) + p_c$, the function $\widetilde{t}(w\lambda, x)$ would share the same properties as $t(w\lambda, x)$, and the car-owners' problem would then be isomorphic to the bus-users' problem (P1). This implies that the previous two lemmas still hold for the car-owners' problem.

If $x_c(\lambda)$ is the optimal location choice for car-owners, then it must yield positive values for consumption and land services, or

$$w\lambda - \tau(w\lambda, x_c(\lambda)) > p_c.$$

2.4 Car-ownership Decision

An agent with ability λ will choose to own a car if and only if $V^c(\lambda) > V^b(\lambda)$. The car-ownership decision can be characterized by the function $\Omega(\lambda)$ where

$$\Omega(\lambda) = \begin{cases} 1 & \text{if } V^c(\lambda) > V^b(\lambda) \\ 0 & \text{if } V^c(\lambda) \leq V^b(\lambda), \end{cases} \tag{12}$$

for $\lambda \in [\lambda_{\min}, \lambda_{\max}]$.

Substituting (9) and (10) into the utility function gives the value function of a car-owner,

$$V^c(\lambda) = \widetilde{\alpha} + \ln\{w\lambda - \tau[w\lambda, x_c(\lambda)] - p_c\} - (1-\alpha)\ln q[x_c(\lambda)], \tag{13}$$

where $\widetilde{\alpha} \equiv \ln \alpha^{\alpha}(1-\alpha)^{1-\alpha}$. Similarly, the value function of a bus-user is

$$V^b(\lambda) = \widetilde{\alpha} + \ln\{w - t[w\lambda, x_b(\lambda)]\} - (1-\alpha)\ln q[x_b(\lambda)]. \tag{14}$$

Define a critical ability level $\overline{\lambda}$ such that $V^c(\overline{\lambda}) = V^b(\overline{\lambda})$. Then $\overline{\lambda}$ must satisfy

$$\frac{w\overline{\lambda} - \tau\left[w\overline{\lambda}, x_c\left(\overline{\lambda}\right)\right] - p_c}{w\overline{\lambda} - t\left[w\overline{\lambda}, x_b\left(\overline{\lambda}\right)\right]} = \left\{\frac{q\left[x_c\left(\overline{\lambda}\right)\right]}{q\left[x_b\left(\overline{\lambda}\right)\right]}\right\}^{1-\alpha}. \tag{15}$$

This equation shows how location choices and car-ownership decisions are interdependent. If there does not exist any λ in $[\lambda_{\min}, \lambda_{\max}]$ that satisfies (15), then the economy is said to have no car-owner.

Suppose condition (6) is satisfied for both $t(w\lambda, x)$ and $\tau(w\lambda, x)$, then $x_b(\lambda)$ and $x_c(\lambda)$ are both monotonically increasing. This implies that no bus-user will live further from the origin than $x_b\left(\overline{\lambda}\right)$ and no car-owner will live closer to the origin than $x_c\left(\overline{\lambda}\right)$. In equilibrium, $x_c\left(\overline{\lambda}\right) \leq x_b\left(\overline{\lambda}\right)$ since, as discussed below, land rent will adjust so that agents are distributed continuously over the city. $x_c\left(\overline{\lambda}\right) < x_b\left(\overline{\lambda}\right)$ implies that the location choices of car-owners overlap with those of bus-users. The following lemma shows that this is not possible.

Lemma 3. *In equilibrium, no car-owner and bus-user will live at the same point. Hence,* $x_c\left(\overline{\lambda}\right) = x_b\left(\overline{\lambda}\right) \equiv \overline{x}$ *holds.*

Proof. See Appendix. □

2.5 Production

The aggregate output of final goods, Y, and automobiles, A, are produced using labor alone,

$$Y = \eta L_Y, \tag{16}$$

and

$$A = z L_A, \tag{17}$$

where L_Y and L_A denote the aggregate labor inputs devoted to the goods sector and the automobile sector, respectively[12]. The variables η and z capture the TFP in the production of final goods and automobiles.

All markets are assumed to be competitive. Agents can choose to work in any one of the sectors. The market wage for an efficiency unit of labor is given by

$$w = \eta = p_c z. \tag{18}$$

3 Competitive Equilibrium

In this section an equilibrium is defined. To simplify the analysis, the transportation costs functions are specified as

$$t(w\lambda, x) = \tau_b w\lambda x + \gamma_b, \tag{19}$$

and

$$\tau(w\lambda, x) = \tau_c w\lambda x + \gamma_c, \tag{20}$$

where $\tau_b > \tau_c > 0$ and $\gamma_c + p_c > \gamma_b > 0$. Under this specification, the critical ability, $\overline{\lambda}$, and the corresponding location, $x_c(\overline{\lambda}) = x_b(\overline{\lambda}) \equiv \overline{x}$, must satisfy

$$\overline{x} = \frac{\gamma_c + p_c - \gamma_b}{(\tau_b - \tau_c)\, w\overline{\lambda}}. \tag{21}$$

The critical ability level, if it exists, is unique, with $V^c(\lambda) > V^b(\lambda)$ if and only if $\lambda > \overline{\lambda}$. Moreover, condition (6) stated in Lemma 1 is satisfied under this specification. Hence, $x_b(\lambda)$ and $x_c(\lambda)$ are both strictly increasing. The optimal location choice function for any agent is then given by

$$x(\lambda) = x_c(\lambda)\,\Omega(\lambda) + x_b(\lambda)\,[1 - \Omega(\lambda)], \tag{22}$$

where $x_b(\lambda)$ is implicitly defined by (5) and $x_c(\lambda)$ is determined by (11). The function is continuous by continuity of $x_b(\lambda)$ and $x_c(\lambda)$ and Lemma 3. Strict monotonicity of $x_b(\lambda)$ and $x_c(\lambda)$ implies that $x(\lambda)$ is also strictly monotonic. Thus, no one will reside beyond $x_c(\lambda_{\max})$, where $\lambda_{\max}$ is the maximum ability in the ability distribution (assumed to be finite). Define $\tilde{x} \equiv x_c(\lambda_{\max})$, then the size of the city is given by

$$\mathcal{C} = [0,\ \tilde{x}].$$

In equilibrium, every point in $\mathcal{C}$ must be occupied. Otherwise, any rational land-owner would lower the rent at the empty point so as to induce someone to move in. Hence, $x(\lambda)$ should be continuous over the range of abilities, $[\lambda_{\min}, \lambda_{\max}]$.

[12] Alternatively, the production function for automobiles could be specified as

$$A = zM^{\varepsilon} L_A^{1-\varepsilon},$$

where M is the total quantity of intermediate inputs, and $\varepsilon \in (0, 1)$. Here, one unit of final goods would be transformed costlessly into one unit of intermediate goods. Both specifications would yield the same set of major results.

Land beyond the boundary of the city is used for agriculture. Let q_A denote the agricultural rent. If $q(\tilde{x}) < q_A$, then any rational land-owner would choose to rent out the land at $\tilde{x}$ to agricultural users. If $q(\tilde{x}) > q_A$, then households at the boundary would be strictly better off by moving slightly further away from the city center. The reason is, if the movement is sufficiently small, then transportation costs would only go up marginally but would be compensated by a significant reduction in rent, $[q(\tilde{x}) - q_A]$. Thus, in equilibrium,

$$q(\tilde{x}) = q_A. \tag{23}$$

3.1 Population Density

Let $f(\lambda)$ be the density function governing the ability distribution. To derive the population density function, consider an ability, λ, and a small neighborhood of length $d\lambda$ around it. The fraction of population within this neighborhood is $f(\lambda)\,d\lambda$. Since $x(\lambda)$ is strictly monotonic and hence one-to-one, one can find a neighborhood dx around $x = x(\lambda)$ such that all agents with ability $\lambda \in d\lambda$ are located in this interval. Population density at x is, hence, given by

$$\mu(x) = \frac{f(\lambda)\,d\lambda}{dx}$$

or

$$\mu[x(\lambda)]\,x'(\lambda) = f(\lambda), \tag{24}$$

for any $\lambda \in [\lambda_{\min}, \lambda_{\max}]$. Alternatively, (24) can be derived using the transformation of variable technique. The equilibrium population density function over $\mathcal{C}$ is then defined as

$$\mu(x) = \begin{cases} \mu[x_b(\lambda)] & \text{if } x = x_b(\lambda) \in [0, \bar{x}], \\ \mu[x_c(\lambda)] & \text{if } x = x_c(\lambda) > \bar{x}. \end{cases} \tag{25}$$

3.2 Market Clearing

In equilibrium, the land demand function for any type-λ agent is given by

$$\widetilde{l}(\lambda) = \begin{cases} l_b[x_b(\lambda), \lambda] & \text{for } \lambda \leq \overline{\lambda} \\ l_c[x_c(\lambda), \lambda] & \text{for } \lambda > \overline{\lambda}. \end{cases} \tag{26}$$

Consider any subinterval $[\lambda_1, \lambda_2] \subseteq [\lambda_{\min}, \lambda_{\max}]$. Since $x(\lambda)$ is strictly increasing, there exists a unique interval $\mathcal{C}' = [x(\lambda_1), x(\lambda_2)]$ in $\mathcal{C}$ that contains all the agents in $[\lambda_1, \lambda_2]$. Land markets in $\mathcal{C}'$ clear when the total demand for land equals the total supply

$$\begin{aligned} \int_{\lambda_1}^{\lambda_2} \widetilde{l}(\lambda)\,f(\lambda)\,d\lambda &= \int_{x(\lambda_1)}^{x(\lambda_2)} dx \\ &= \int_{\lambda_1}^{\lambda_2} x'(\lambda)\,d\lambda. \end{aligned} \tag{27}$$

Since (27) has to hold for all $[\lambda_1, \lambda_2]$, the land market clearing condition at any $x = x(\lambda)$ can be restated as

$$\widetilde{l}(\lambda) f(\lambda) = x'(\lambda), \tag{28}$$

for $\lambda \in [\lambda_{\min}, \lambda_{\max}]$.

Agents supply their non-commuting hours to market work so labor supply depends on their transportation mode and location choices. The labor market clears if the following holds:

$$L_A + L_Y = \int_{\lambda_{\min}}^{\lambda_{\max}} \{1 - \tau_b x_b(\lambda)[1 - \Omega(\lambda)] - \tau_c x_c(\lambda)\Omega(\lambda)\} \lambda dF(\lambda). \tag{29}$$

Aggregate demand for automobiles is given by the fraction of population that choose to own a car. Hence, the auto market clears when

$$A = \int_{\bar{\lambda}}^{\lambda_{max}} dF(\lambda). \tag{30}$$

Final goods produced in this economy, net of those dissipated in commuting, are available for consumption. Aggregate demand for consumption is the sum of the demand by agents and that by the landowner. Since all the rents collected by the landowner are spent on consumption, the demand for consumption by the landowner is

$$Q = \int_{\lambda_{\min}}^{\lambda_{\max}} \{q[x_b(\lambda)] l_b(\lambda)[1 - \Omega(\lambda)] + q[x_c(\lambda)] l_c(\lambda)\Omega(\lambda)\} dF(\lambda). \tag{31}$$

Total demand for consumption goods by the agents is

$$C = \int_{\lambda_{\min}}^{\lambda_{\max}} \{c_b(\lambda)[1 - \Omega(\lambda)] + c_c(\lambda)\Omega(\lambda)\} dF(\lambda). \tag{32}$$

The final goods market clear when

$$Y - \left[\gamma_b \int_{\lambda_{\min}}^{\overline{\lambda}} dF(\lambda) + \gamma_c \int_{\bar{\lambda}}^{\lambda_{max}} dF(\lambda)\right] = Q + C. \tag{33}$$

3.3 Definition

Define the equilibrium of this economy as follows:

Definition 1. *Given a distribution of abilities, $F(\lambda)$, an equilibrium of this economy consists of a set of decision rules for car-owners, $\{c_c(\lambda), l_c(\lambda), x_c(\lambda)\}$, a set of decision rules for bus-users, $\{c_b(\lambda), l_b(\lambda), x_b(\lambda)\}$, labor inputs, $\{L_A, L_Y\}$, a car-ownership decision rule, $\Omega(\lambda)$, a population density function, $\mu(x)$, a critical ability level, $\bar{\lambda}$, a rental function, $q(\cdot)$, and prices, $\{p_c, w\}$ such that*

1. *Given $q(\cdot)$, p_c, and w, $\{c_b(w), l_b(w), x_b(\lambda)\}$ solves (P1).*

2. *Given $q(\cdot)$, p_c, and w, $\{c_c(w), l_c(w), x_c(w)\}$ solves (P2).*

3. *The prices, p_c and w, satisfy* (18).

4. *The car-ownership decision rule, $\Omega(\lambda)$, is given by* (12).

5. *The population density function, $\mu(x)$, defined by* (25), *satisfies*

$$\int_{\mathcal{C}} \mu(x)\, dx = 1. \tag{34}$$

6. *The rent at the boundary of the city equals the agricultural rent, or* (23) *holds.*

7. *All markets clear:*

   (a) *The land market at every x clears, or* (28) *hold.*

   (b) *The auto market clears or* (30) *holds.*

   (c) *The final good market clears or* (33) *holds.*

3.4 Equilibrium Rent

In equilibrium, the rent function must be continuous over the range of the city. Given a continuous distribution of abilities and a continuous transportation cost function, it is immediate to see that $q(x)$ is continuous over the region where there are bus-users (or car-owners) alone. Hence $q(x)$ is continuous over $\mathcal{C} \backslash \{\overline{x}\}$. Suppose it was discontinuous at $\overline{x}$, and

$$q(\bar{x}) > \lim_{x \to \bar{x}^+} q(x).$$

Consider an agent with the critical ability, $\bar{\lambda}$. This agent lives at $\bar{x}$. Given the gap in rent at $\overline{x}$, he can benefit by moving slightly further out to a location $\bar{x} + \varepsilon$. Since ε can be made arbitrarily small, a new location can always be found such that the agent's additional transportation costs are less than his discrete gain from savings in rent. This creates an incentive to move and hence cannot be an equilibrium. By a similar argument, one can rule out the case with $\lim_{x \to \bar{x}^-} q(x) > q(\bar{x})$. Hence, $q(x)$ is continuous over $\mathcal{C}$.

3.5 Characterization of Equilibrium

Since it is possible that no car-owner exists in equilibrium, it is important to determine the conditions under which this will occur. First, consider the case in which car-owners exist. Then an equilibrium, as defined above, is made up of three parts: the bus-user's problem, the car-owner's problem and a critical ability level $\overline{\lambda}$ that connects the two. The bus-user's problem is characterized by

$$\frac{q'[x_b(\lambda)]}{q[x_b(\lambda)]} = \frac{-w\lambda\tau_b}{(1-\alpha)\{w\lambda[1-\tau_b x_b(\lambda)] - \gamma_b\}}, \tag{35}$$

$$x_b'(\lambda) = \frac{(1-\alpha)\{w\lambda[1-\tau_b x_b(\lambda)] - \gamma_b\} f(\lambda)}{q[x_b(\lambda)]}, \tag{36}$$

for $\lambda \in \left[\lambda_{\min}, \overline{\lambda}\right]$, and the boundary conditions:

$$\begin{aligned} x_b(\lambda_{\min}) &= 0, \\ x_b(\bar{\lambda}) &= \bar{x}. \end{aligned} \tag{37}$$

The car-owner's problem is characterized by

$$\frac{q'\left[x_c\left(\lambda\right)\right]}{q\left[x_c\left(\lambda\right)\right]} = \frac{-w\lambda\tau_c}{\left(1-\alpha\right)\left\{w\lambda\left[1-\tau_c x_c\left(\lambda\right)\right]-\left(\gamma_c+p_c\right)\right\}}, \tag{38}$$

$$x_c'\left(\lambda\right) = \frac{\left(1-\alpha\right)\left\{w\lambda\left[1-\tau_c x_c\left(\lambda\right)\right]-\left(\gamma_c+p_c\right)\right\}f\left(\lambda\right)}{q\left[x_c\left(\lambda\right)\right]}, \tag{39}$$

for $\lambda \in \left(\overline{\lambda}, \lambda_{\max}\right]$, and by Lemma 3 and equation (23) the boundary conditions:

$$\begin{aligned} x_c(\bar{\lambda}) &= \bar{x} \\ q\left[x_c\left(\lambda_{\max}\right)\right] &= q_A, \end{aligned} \tag{40}$$

In equilibrium the critical ability level is determined by (15). Since the equilibrium rent function is continuous at $\bar{x}$, it follows that

$$q(\bar{x}) = q\left[x_b(\bar{\lambda})\right] = q\left[x_c(\bar{\lambda})\right] \tag{41}$$

and (15) becomes

$$\bar{x} = \frac{\gamma_c + p_c - \gamma_b}{\left(\tau_b - \tau_c\right)w\overline{\lambda}}. \tag{42}$$

Define the composite functions $q_b(\lambda) \equiv q\left[x_b(\lambda)\right]$ and $\mu_b(\lambda) = \mu\left[x_b(\lambda)\right]$. Combining (35) and (36) gives

$$x_b'(\lambda) = \frac{\left(1-\alpha\right)\left\{w\lambda\left[1-\tau_b x_b(\lambda)\right]-\gamma_b\right\}f(\lambda)}{q_b(\lambda)}, \tag{43}$$

$$q_b'(\lambda) = -w\lambda\tau_b f(\lambda). \tag{44}$$

Similarly for the equations corresponding to the car-owner's problem, combining (38) and (39) yields

$$x_c'(\lambda) = \frac{\left(1-\alpha\right)\left\{w\lambda\left[1-\tau_c x_c(\lambda)\right]-\left(\gamma_c+p_c\right)\right\}f(\lambda)}{q_c(\lambda)}, \tag{45}$$

$$q_c'(\lambda) = -w\lambda\tau_c f(\lambda). \tag{46}$$

Notice that $q_b'(\lambda)$ and $q_c'(\lambda)$ are independent of $q(\lambda)$ and $x(\lambda)$. This occurs in this particular setting because (i) land supply is fixed and constant at each location and (ii) given $w\lambda$, the costs of commuting one more unit of distance [i.e. $t_2\left(w\lambda, x\right)$ and $\tau_2\left(w\lambda, x\right)$] are identical for all locations. Integrating both sides of (44) and (46), and using the boundary conditions give

$$q_b(\lambda) = \psi - \tau_b w \int_{\lambda_{\min}}^{\lambda} u f(u) du, \tag{47}$$

for $\lambda \in \left[\lambda_{\min}, \overline{\lambda}\right]$, where ψ is the integration constant, and

$$q_c(\lambda) = q_A + \tau_c w \int_{\lambda}^{\lambda_{\max}} u f(u) du, \tag{48}$$

for $\lambda \in \left[\overline{\lambda}, \lambda_{\max}\right]$.

If there is no car-owner in equilibrium, then the equilibrium is characterized by (35) and (36) with $\overline{\lambda}$ replaced by $\lambda_{\max}$, and the boundary conditions

$$\begin{aligned} x_b(\lambda_{\min}) &= 0, \\ q_b(\lambda_{\max}) &= q_A. \end{aligned} \tag{49}$$

The following theorem states the necessary and sufficient condition under which an equilibrium with car-owners exists.

Theorem 4. *There exists a unique $q^* > 0$ such that an equilibrium with car-owners exists if and only if*

$$\lambda_{\max} > \frac{\tau_b \left(p_c + \gamma_c\right) - \tau_c \gamma_b}{\left(\tau_b - \tau_c\right) w} \quad \textit{and} \quad q_A \in (0, q^*) .$$

Proof. See Appendix. □

The intuition of the theorem is as follows. Since $V^c(\lambda)$ and $V^b(\lambda)$ are strictly increasing in λ, it follows that if the highest ability agents do not want to own a car then neither does any other agent. The highest ability agents cannot afford a car if their ability, and hence income, is below a certain level. Holding other things constant, the barrier, $\frac{\tau_b(p_c+\gamma_c)-\tau_c\gamma_b}{(\tau_b-\tau_c)w}$, is lowered when (i) the fixed costs of owning a car declines; or (ii) the fixed costs of taking the bus rises. By the monotonicity of $x(\lambda)$, the highest ability agents will live at the right endpoint of the city and pay rent q_A. When q_A rises, the agents face a higher rent. To compensate for this, they will reduce their commuting costs by living closer to the origin. This reduces the need for a car. Hence, no car-owners will exist if q_A is "too" high.

3.6 Computation of Equilibrium

The equilibrium is completely characterized by the optimal location choice function, $x(\lambda)$, the equilibrium rent function, $q(\lambda)$, and the critical ability level, $\bar{\lambda}$. The equilibrium can be computed numerically by utilizing (41), (42) and the initial value problems arising from plugging (47) and (48) into (43) and (45) along with initial conditions equal to the former boundary conditions for $x_b(\lambda)$ and $x_c(\lambda)$. The algorithm employed here is outlined in the Appendix.

4 Calibration and Estimation

Can the model account for the decreasing population density gradients and rising level of car ownership? To see how well the model can do at explaining suburbanization and car ownership consider the following experiment. Compute the model for a series of steady states. The steady states represent an average U.S. city during the decennial years from 1910 to 1970. Then, compare the model's prediction for the density gradients and car ownership with the mean density gradients of forty-one US cities, presented in Table 4, and the percentages of car-owners in the U.S., approximated by the number of registered vehicles per person age twenty to sixty-four, given in Table 5. Before the steady states can be computed, a parametrization must be imposed on the model. This is done through a combination of calibration and estimation.[13]

4.1 Parameters to Set

Parameters that can be pinned-down from the U.S. data will be calibrated accordingly. Some parameters cannot be pinned-down, and will be chosen in order to minimize the "distance" between the model's outcome and the data. Let one period of the model equal five years, approximately the average median age of passenger cars during the period 1950-70.[14] The parameters which need to be set are:

Preferences: The parameter, α measures the relative weight on consumption in the utility function. Set α to 0.89 using data on consumer expenditures from Lebergott (1996).

Technology: The parameters, η_t and z_t, which capture the TFP in production of final goods and automobiles, are chosen such that w_t equals the mean income level for each steady state, and $p_{c,t}$ at each steady state matches the data on the price of a new car.[15] The mean income level and automobile prices are based on the data on wages and car prices presented in Section 1.2.

Abilities: The distribution of abilities is approximated by a doubly-truncated lognormal. The standard deviation is calibrated so that the distribution of income ($w\lambda$) matches the distribution of income in the U.S[16]. Given the mean and standard deviation, the truncation points, λ_{min} and λ_{max}, are chosen so as to encompass 95% of the area of the underlying non-truncated distribution, omitting 2.5% from each side.

Agricultural Rent: The rental rate of agricultural land, $q_{A,t}$, is set to the rent that would have to be paid for an average single-family-sized lot of farmland at each date[17].

[13]Note that the 'estimation' procedure done here is not based on any statistical model and hence is not the basis for any inference.

[14]The average median age of passenger cars in the U.S. during the period 1950-70 is 5.1 years. Source: Ward's Motor Vehicle Facts & Figures (1999).

[15]Subscript 't' implies that the parameter is not assumed to be constant across the steady states.

[16]According to Gottschalk and Smeeding (1997), Table 3, the adjusted disposable personal income of a household at the 80th percentile is 2.7 times higher than one at the 20th percentile. This implies that, if the ln of income is normally distributed, then the standard deviation is 0.59.

[17]The rental rate for an average single-family-sized lot of farmland is computed by dividing the gross rent paid for an acre of farmland, taken from the United States Department of Agriculture Economic Research Service and is based on various sources including the *Census of Agriculture*, the *Farm Costs and Returns Survey*, and the *Farm Finance*

Transportation Costs: The parameters related to transportation costs, namely, the fixed and time costs for taking a bus, $\gamma_{b,t}$ and $\tau_{b,t}$, and those for commuting by car, $\gamma_{c,t}$ and $\tau_{c,t}$, are the most difficult to pin down. These parameters must capture all the costs associated with commuting. For car-owners this includes all costs associated with operating a car, plus the time costs of commuting. For bus-users, it includes bus fares and time costs. The time cost of taking the bus, as opposed to commuting by car, includes not only the time spent on the bus but also all the inconveniences associated with using public transportation. Since it is difficult to measure all of these costs and their changes over time, a combination of calibration and estimation is used to pin down these parameters.

Automobile: The time cost of commuting by car depends on, among other things, the quantity and quality of roads and highways. Throughout the period from 1910 to 1970 the U.S. government consistently invested in roads and highways. The stock of U.S. highways and roads per capita rose at an average rate of 2% during the period from 1925 to 1994, and at an average rate of 2.3% over the period 1925-70. Figure 9 plots the stock per capita over the period from 1925 to 1994. Assume that the time cost of commuting is inversely related to the highway and road stock per capita. Then $\tau_{c,t}$ can be approximated by the following:

$$\tau_{c,t} = \Phi h_t^{-\kappa},$$

where h_t is the capital stock of highways and streets per capita at time t. The parameters Φ and κ are chosen through the estimation procedure discussed in Section 4.2. The value for the highway and street stock for the periods 1910 and 1920 is obtained by extending the trend line out to these dates.

The fixed cost, $\gamma_{c,t}$ is calibrated using data on consumer car-related expenditures from Lebergott (1996). The data includes expenditures on tires, gas, oil, automobile repairs, automobile insurance, and tolls. The expenditure per registered vehicle is computed. Then $\gamma_{c,t}$ is set to 30% of period t's value. 30% is the percentage of total miles that are driven to or from work. Since this percentage was 26.8 for the period 1951-58 and 33.7 in 1969, 30% is a reasonable value.[18]

Bus: The parameters $\tau_{b,t}$ and $\gamma_{b,t}$ are difficult to calibrate since they must capture the costs associated with the inconveniences of using public transportation. Hence these parameters are derived through an estimation procedure.

Similarly to the time cost of commuting by car, it is intuitive that the time cost of travelling by bus is inversely related to the quantity and quality of public transit equipment and structures

Survey, by the number of average-sized lots in an acre. To compute the number of average-sized lots in an acre, data on the average lot size for a single-family home is taken from the National Association of Realtors, available on the web at http://www.realtor.org/SG3.nsf/files/landuse.pdf/$FILE/landuse.pdf.

[18]Source: For 1951-58, MVMA Motor Vehicle Facts & Figures (1978). For 1969, U.S. Census Bureau, Statistical Abstract of the United States (2001).

per capita. Figure 10 shows the gross and net capital stock per capita of equipment and structures for intercity and local passenger transit for the period 1947-89. Both the gross and net stock have been steadily declining since 1947 at an annual rate of approximately 6% over the period 1947-70. Figure 11 shows the trend in transit ridership per capita over the period 1910-70. According to the graph, transit ridership rates peaked sometime during the 1920's. Table 8 shows the trend in net capital expenditures on transit equipment and structures for the period 1890-1950. The table shows that disinvestment in public transportation started during the 1920's. It appears that the capital stock in public transit has been declining since, at least, 1930. As the rate of transit ridership and the stock of transit capital declines, the cost associated with using public transportation should increase. With less riders it is logical to assume that buses, subways, etc. ran less frequently and that their routes covered less portions of cities. The declining capital stock implies that the number of buses, etc. actually declined. Together these factors indicate that the cost of commuting by public transit should be rising since, at least, 1930 due to the increasing inconvenience and unavailability costs bus-users faced. As for the time cost of commuting by car, the time cost of commuting by bus is assumed to be inversely related to this capital stock, but there is no data on this stock for years preceding 1947. So, instead, consider the following formulation:

$$\tau_{b,t} = \begin{cases} \tau_{b,0}(g_{\tau_b})^{(\frac{t-1910}{10})}, & \text{for } 1910 \leq t \leq 1930, \\ \Omega p_t^{-\rho}, & \text{for } t \geq 1930. \end{cases} \tag{50}$$

Here, $\tau_{b,t}$ is a function of the capital stock, p_t, for the years 1930 to 1970 and is assumed to grow at a constant rate for the years 1910 to 1930. For the years 1930-47, the capital stock is computed by extending the trend line out to these dates. The trend is not extended to the years 1910-30 because it is questionable whether the capital stock was declining during these early years. Hence $\tau_{b,t}$ at each time t will be determined by choosing $\Omega, \rho, \tau_{b,0}$, and g_{τ_b} through the estimation procedure described below subject to the constraint:

$$\tau_{b,0}(g_{\tau_b})^{(\frac{1930-1910}{10})} = \Omega p_{1930}^{-\rho},$$

or, in the year 1930, the time cost implied by assuming a constant growth rate of $\tau_{b,t}$ must be equal to the one implied by assuming that $\tau_{b,t}$ is inversely related to the public transit stock per capita.

The fixed cost of commuting by bus, $\gamma_{b,t}$, should capture costs such as bus fares and the inconvenience cost of taking the bus relative to commuting with a car, but data on average bus fares and other costs of taking the bus in the U.S. for this time period are unavailable. Hence this parameter is derived through the estimation procedure. To enforce some structure on this cost, it is assumed to grow at some constant rate over the period from 1910 to 1970. If g_{γ_b} is

γ_b's growth factor, and $\gamma_{b,0}$ is the initial 1910 value then

$$\gamma_{b,t} = \gamma_{b,0} g_{\gamma_b}^{\left(\frac{t-1910}{10}\right)}.$$

$\gamma_{b,0}$ and g_{γ_b} are chosen through the estimation procedure described below.

4.2 Estimation

Denote by v_t the percentage of car-owners in the U.S. at date t. v_t can be approximated by the number of registered vehicles per person age twenty to sixty-four, given in Table 5. Likewise, let d_t be the average density gradient for an American city. d_t is given by the density gradients in Table 4.

Define the following vector of unknown parameters:

$$\theta = \left(\Phi, \kappa, \tau_{b,0}, g_{\tau_b}, \Omega, \rho, \gamma_{b,0}, g_{\gamma_b}\right).$$

Given θ, a mean wage $\bar{w}_t$, and a price for cars, $p_{c,t}$, the model's prediction for the percentage of car-owners at date t is denoted by

$$V_t(\theta; \bar{w}_t, p_{c,t}).$$

Likewise, the model's predicted density gradient is

$$D_t(\theta; \bar{w}_t, p_{c,t}).$$

To compute $D_t(\theta; \bar{w}_t, p_{c,t})$, the density is calculated at 1000 points from zero to the end of the city. The density gradients are then computed by regressing the sample on an exponential function. In all the experiments below, the R-squared's are always above 0.80.

The exercise, now, consists of two steps: First, θ, is chosen to minimize the sum of the distances between the model's output and the U.S. economy at a particular set of steady states corresponding to the decennial years from 1910 to 1970. Formally:

$$\hat{\theta} = \arg\min_{\theta} \sum_{t=1910}^{1970} \left\{ \frac{1}{2}\left(v_t - V_t(\theta; \bar{w}_t, p_{c,t})\right)^2 + \frac{1}{2}\left(d_t - D_t(\theta; \bar{w}_t, p_{c,t})\right)^2 \right\}.$$

Second, the model's predictions, $V_t(\hat{\theta}; \bar{w}_t, p_{c,t})$ and $D_t(\hat{\theta}; \bar{w}_t, p_{c,t})$, for $t = 1910, \ldots, 1970$, is computed using $\hat{\theta}$.

5 The Baseline Economy

The baseline parametrization is presented in Table 6. The table also displays statistics on the costs of commuting, implied by the parameters. The statistics given are the fixed, variable, and total costs of commuting for a bus-user at the average distance from the center and a car-owner at the average distance from the center. These will be discussed in Section 5.1.

The results are presented in Table 7. The table shows the model's predictions for the percentage of car-owners, the population density gradient, and the change in the population density gradient over the period from 1910 to 1970. In addition, the table provides the distance at which the furthest bus-user lives (*End of Bus-Users*) and that at which the furthest car-owner lives (*Boundary of City*) in each steady state.

5.1 Analysis

In this section the baseline model and its ability to match the data is assessed. To begin consider the transportation costs, specifically, consider those associated with taking public transit, or "riding the bus." The fixed and time cost of riding the bus are rising over the period from 1910 to 1970. Since bus-users move closer and closer to the origin and bus-users who switch to car ownership are always those with the highest incomes, the average variable cost of travelling by bus does not rise over the period, it declines from 1910-40, then rises between 1940-50 and then declines from 1950-70. The average total cost declines at an annual rate of 3.7% from 1910-30 and then rises at an annual rate of 2.9% between 1930-70. The minimization procedure generated results that are consistent with the assumption that the cost of commuting by public transit is inversely related to the capital stock of public transit equipment and structures.

The fixed costs of commuting by car was already calibrated to match the data. The time cost is declining, as was expected given the rising capital stock of highways and roads. Since the model predicts zero car-owners in 1910, it is not possible to compute the variable and fixed costs for this year. The average variable cost falls between 1920 and 1930 but rises from 1930 to 1960 then falls slightly in 1970. The average total cost follows a similar trend to the variable cost. This is because average variable and total costs depend on both the income levels and distance from the center. For the car-owners both of these variables are rising over the period. Taken into account the rising fixed costs, the fact that average total costs for car-owners are increasing does not seem unreasonable.

Now, let's approach these costs at a different, more quantitative angle. There is significantly more data available for later years. Therefore, for the moment, consider comparing the transportation costs associated with commuting by public transit and by automobile in the steady state corresponding to the year 1970 with the data. The goal is to determine if the magnitude of these costs are reasonable. Focus on the average total costs. In 1970 this cost for a bus-user is \$9.25 per day. Based on the 1975 Census, the mean travel time to work in 1975 by those who take public transportation is 40.1 minutes. This implies a total daily travel time of 80.2 minutes a day. The average wage of a bus-user in 1970 is \$6.15 an hour.[19] This implies the dollar cost of time spent commuting is \$8.22 a day. Hence, considering the additional cost of bus-fare, \$9.25 a day is a very reasonable value. For the

[19]This is the wage of the average bus-user and is not the same as the wage of the bus-user who is at the average distance which is given in Table 6.

average car-owner the total cost of commuting in 1970 is \$29.88. Again from the 1980 Census, the mean travel time to work in 1980 of people who commute by car is 21.6 minutes. This implies that the total commute time is 43.2 minutes. The average wage of a car-owner in 1970 is \$23.22 an hour. Hence the value of time spent commuting by car is \$16.72 a day. The distance from the center of this average car-owner is 7.77 miles.[20] The total cost of operating a vehicle in 1975 was approximately 58.59 cents/mile[21]. Thus, the total 'car-related' cost for the average car-owner in the model is \$4.55 a day. Summing the car-owner's costs suggests that his total costs of commuting by car are \$21.27 a day. So a total cost of \$23.22 a day resulting from the minimization is very reasonable.

How are the results? Qualitatively, the model is able to predict a persistent increase in car-ownership accompanied by increased suburbanization. The latter is evident from the decline in the population density gradient and the expansion of the city as shown in Table 7. The model is able to match the data on car-ownership for earlier years but has some difficulty reaching the level of car-ownership in later years. One reason for this is that the model abstracts from other uses of cars. In 1969, about 32% of personal trips by car involved work travel, this number decreases to 23.8% by 1995[22]. Meanwhile, vehicle trips that involve shopping and, social and recreational activities experience an opposite trend. By 1995, about 40% of personal trips by car are for these purposes. By including additional uses of car into the model, the demand for automobiles could be raised. In terms of the population density gradient, the model is able to match the trend but underestimates the density gradient for later years. The model has an especially difficult time matching the suburbanization trend from 1940-50. Figure 12 displays the density functions predicted by the model for the years 1910, 1940, and 1970. The graph shows how the agents spread out over time.

The average distance between home and work in the US in 1955 was 6.4 miles.[23] The average distance in 1969 was 9.4 miles.[24] In the model the average distance of a car-owner in 1950 is 6.19 miles, in 1960 it is 6.95 miles, and in 1970 it is 8.13 miles. It seems that the distances in the model are within a reasonable magnitude.

5.2 Counterfactual Experiments

Table 9 provides the results of a series of counterfactual experiments. Each experiment consists in shutting down one or some combination of the factors that lead to suburbanization and car ownership: rising real wages, falling prices of cars, declining cost of commuting by car, and rising

[20]Again, this is not the same average as in Table 6.

[21]Source: American Automobile Association. (1993) "Your Driving Costs." the total cost includes fixed costs–depreciation, insurance, finance charge, and license fee, and variable costs–gas, oil, maintenance, and tires.

[22]Source: U.S. Federal Highway Administration, *Summary of Travel Trends, 1995 National Personal Transportation Survey*.

[23]Source: MVMA Motor Vehicles Facts & Figures (1978).

[24]Source: U.S. Census Bureau, Statistical Abstract of the United States (2001).

costs of commuting via public transportation. The counterfactual experiments can be used to assess the role that various factors play in generating the increasing trend in car-ownership and decreasing density gradient.

The baseline model is able to account for 86% of the car-ownership in 1970 and 77% of the difference between the density gradient in 1910 and that in 1970. When both prices and wages remain fixed at their 1910 values (Experiment 3a), the model accounts for less than 1% of car-ownership. The density gradient in this case actually increases. This is because agents face a high unchanging price for cars and bus-users face rising costs. Hence the agents move towards the origin instead of away from it. Removing the rising costs of public transit (Experiment 3b) reduces the amount of centralization. If only the car price remains at the 1910 value (Experiment 4a) then the model can account for 51% of car-ownership in 1970, and 60% of the change in the density gradient. Without rising costs of public transit (Experiment 4b), the model only accounts for 13% of car-ownership and 65% of suburbanization. When the wage distribution remains as in 1910 (Experiment 5a), then 21% of car-ownership in 1970 is accounted for and the density gradient rises, but removing the rising time costs of public transit (Experiment 5b) reduces the incentive to switch to car, resulting in no car-owners and again less centralization.

Experiment 6 in the table shows how the results change when the highway and street stock per capita is kept fixed at its 1910 value. Since the stock is not rising in this experiment, the time cost of commuting by car remains constant and the cost to car-owners of spreading out is high. The percentage of car-owners still rises over time but only reaches 74% of the percentage in the baseline model. Despite the significant percentage of car-owners, 54% in 1970, there is no suburbanization trend, in fact agents become more concentrated around the city center. This result occurs because the cost of commuting by bus is rising, pushing bus-users closer to the center, while the cost of commuting by car remains constant, removing the incentive for car-owners to spread further out. In experiment 7 the time and fixed costs of commuting by bus are kept constant at their 1910 values. Notice that, without a rising cost associated with taking the bus, the percentage of agents who switch to car ownership is much lower than in the baseline results. Under the parametrization of experiment 7 only 31% of car-ownership can be explained, but the suburbanization trend, while not matching the overall trend in the data better, is actually stronger than the trend predicted by the baseline model. These results occur because without rising costs of traveling by public transit, bus-users are less concentrated around the center and, with falling cost of commuting by automobile, the car-owners (which are 26% of the population) spread out, as they do in the baseline.

What do these results imply about the relationship between car-ownership and suburbanization? Consider the following question: Can we observe a suburbanization trend without simultaneously observing a rising percentage of car-owners? In experiment 8, the time cost of travelling by car is kept at its 1910 value and, in addition, the time and fixed cost of commuting by bus are kept at their 1910 values. Under this parametrization the bus is as attractive as possible, relative to the car, without violating the assumption that the car is a form of transportation which has a lower time cost than the bus.[25] Here no agent becomes a car-owner and, in addition, despite the agents rising real wages, we do not observe any suburbanization trend. Hence the experiments suggest that:

(1) The suburbanization trend is caused by a combination of rising real wages and the diffusion of a transportation technology that, both, becomes more affordable and improves in efficiency over time. In other words, rising real wages and falling prices of automobiles alone are not enough to generate a suburbanization trend (Experiments 6 and 8), nor is a declining time cost associated with automobiles alone able to generate a suburbanization trend (Experiment 3).

(2) Without any car-owners, and assuming no rising costs of using public transit, there is no suburbanization trend (Experiments 3b, 5b, and 8). Yet, there can be a suburbanization trend with a much smaller percentage of car-owners than what is observed in the data (Experiment 7).

(3) A high percentage of car-owners can exist without a suburbanization trend if public transportation becomes increasingly unattractive due to rising time costs but there is no technological progress in automobile use. (Experiment 6).

What does this say about the relationship between car-ownership and suburbanization? The results suggest that if the adoption of the automobile (i.e. a fall in transit ridership per capita) and investment in highways and streets as opposed to public transportation equipment had no impact on the costs of using public transportation, then only about 30% of car-ownership is linked to suburbanization (Experiment 7). But if increasing use of automobiles for commuting combined with consistent investment in automobile use impacted or is interrelated with rising costs of using public transit, a relationship which is supported by the data, then the link between car-ownership and suburbanization is much stronger (Baseline Model).

6 Conclusion

A general equilibrium model of car-ownership and location choice is constructed. An agent, in the model, can choose his residential location and decide whether or not to own a car. Under the given specification, it is shown that wealthy agents (those with high abilities) tend to own a car and live further away from the city center, while poor agents tend to travel by bus and stay close to the city center. The model is then calibrated using U.S. data. It is able to predict the rising trends in both car-ownership and suburbanization. Despite the fact that a highly stylized framework is used, the

[25] In this model, decreasing the time cost of commuting by bus over time starting from the 1910 value would violate the definition of the bus as a form of transportation which has a higher time cost relative to the car.

model is able to explain 86% of car-ownership in 1970 and roughly 77% of suburbanization between 1910 and 1970 as reported in the data.

There are some important features of car-ownership that are not addressed by the model. The model abstracts from other uses of cars. By including additional uses of cars into the model, the demand for automobiles could be raised. This might help to improve the model's prediction on car-ownership. In the U.S., multiple car-ownership increases significantly during the latter half of the twentieth century. For instance, less than 10% of American households owned more than one car in the early 1950s, yet by 1995, 60% of them did. The same period also recorded a rapid increase in female labor-force participation rate. Data suggest that female participation and car-ownership decisions might be closely related. Two-income households are more likely to own a car than households in which only the man works. Moreover, the proportion of families with more than one car is rising with the wife's earnings. To account for these facts, the model can be extended to include female workers. By having an additional worker in the household, there is an extra source of income accompanied by an additional demand for a faster mode of transportation. This creates the need for a second car.

Given the static nature of the model, car-ownership decision is based solely on current prices and income level. In the presence of declining car price and rising income, agents might delay their car-ownership decisions. To explore the consequences of this type of forward-looking behavior, a dynamic framework is needed.

Future versions of the model would also benefit from modeling the development of roads and highways more explicitly. Clearly, the role that declining transportation costs for car-owners and rising transportation costs for bus-users play is a significant one in determining the relationship between car-ownership and suburbanization. Potentially adding an urban developer or a government to the model who creates roads and/or public transportation would be helpful.

Table 1: Coefficients for the Negative Exponential function, $d = ae^{-bx}$ for Chicago, 1900-1950.

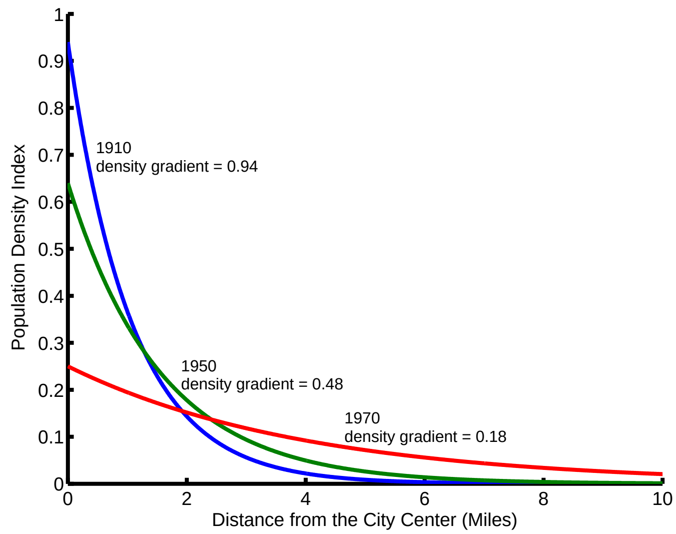
Year	a	b	% Change
1900	87400	0.386	
1910	92900	0.344	-10.9
1920	90400	0.297	-13.7
1930	84700	0.256	-13.8
1940	78850	0.233	-9.0
1950	69700	0.201	-13.7

Source: Edmonston (1975) p. 51.

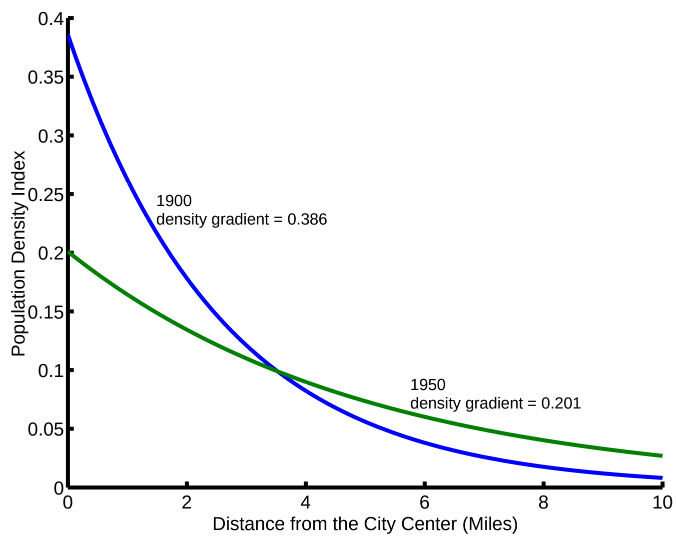
Table 2: Coefficients for the Negative Exponential function, $d = ae^{-bx}$ for Atlanta, 1900-1970.

Year	a	b	% Change
1900	14000	0.79	
1910	29000	0.94	19.0
1920	35000	0.89	-5.3
1930	33000	0.73	-18.0
1940	32000	0.64	-12.3
1950	25000	0.48	-25.0
1960	10000	0.25	-47.9
1970	8000	0.18	-28.0

Source: Edmonston (1975) p. 50.



(a) Atlanta: 1910, 1950, 1970.



(b) Chicago: 1900, 1950.

Figure 1: Population density distributions for Atlanta and Chicago.

Table 3: Average Density Gradients for Four Metropolitan Areas–Baltimore, Milwaukee, Philadelphia, and Rochester–for the Decennial years from 1900 through 1970.

Year	b	% Change
1900	1.0	
1910	0.96	-4.0
1920	0.86	-10.4
1930	0.62	-27.9
1940	0.57	-8.1
1950	0.45	-21.1
1960	0.34	-24.4
1970	0.28	-17.6

Source: Edmonston (1975) p. 67.

Table 4: Mean Gradients for Forty-One Cities That Were Metropolitan Districts in 1900.

Year	b	% Change
1900	0.82	
1910	0.83	1.2
1920	0.79	-4.8
1930	0.66	-16.5
1940	0.61	-7.6
1950	0.39	-36.1
1960	0.31	-20.5
1970	0.23	-25.8

Source: Edmonston (1975) p. 68.

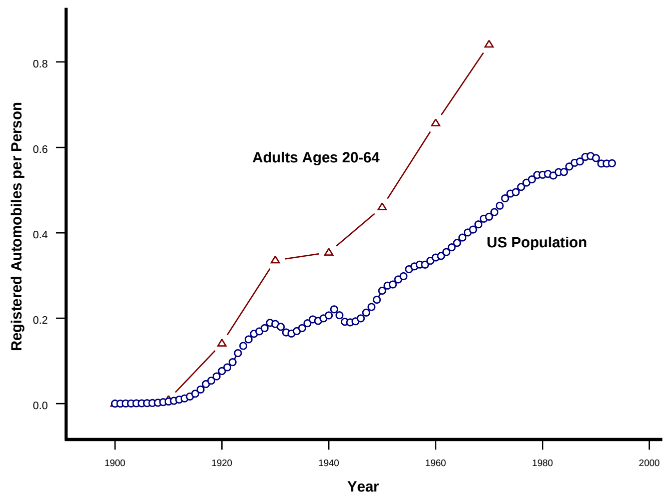


Figure 2: Number of Registered Automobiles per Capita : 1900-1993. Source: Highway Statistics, Annual., U.S. Federal Highway Administration.

Table 5: Number of Registered Vehicles Per Person 20-64 Years of Age, 1900-1970.

Year	Vehicle Per 100 Persons
1900	0.02
1910	0.93
1920	14.12
1930	33.57
1940	35.39
1950	46.02
1960	65.66
1970	84.11

Source: Registered Vehicles: U.S. Federal Highway Administration. Population: U.S. Census.

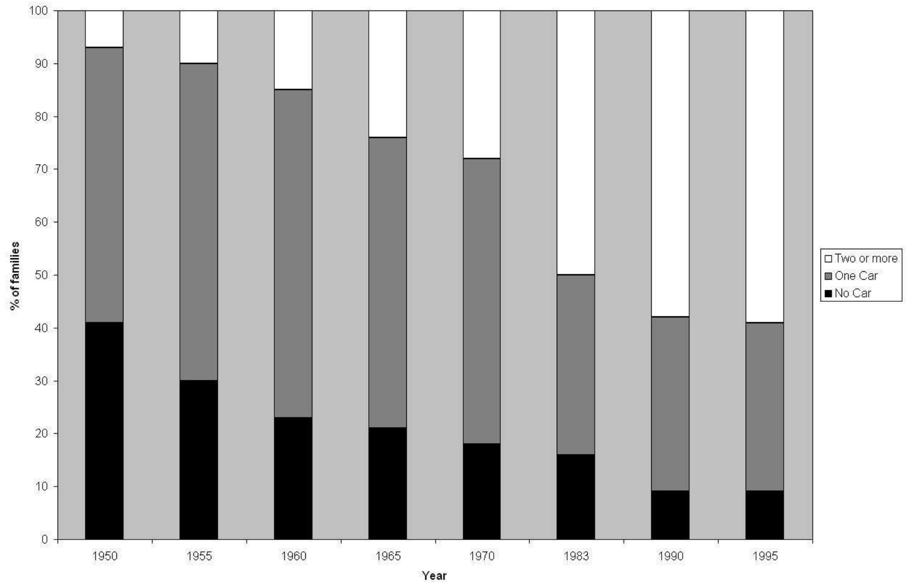


Figure 3: Percent of Families who Own a Car by Number of Cars Owned : 1950 - 1995. Source: 1950-70 Survey of Consumer Finances, University of Michigan, 1983-95 National Personal Transportation Survey, Summary of Travel Trends, Federal Highway Administration.

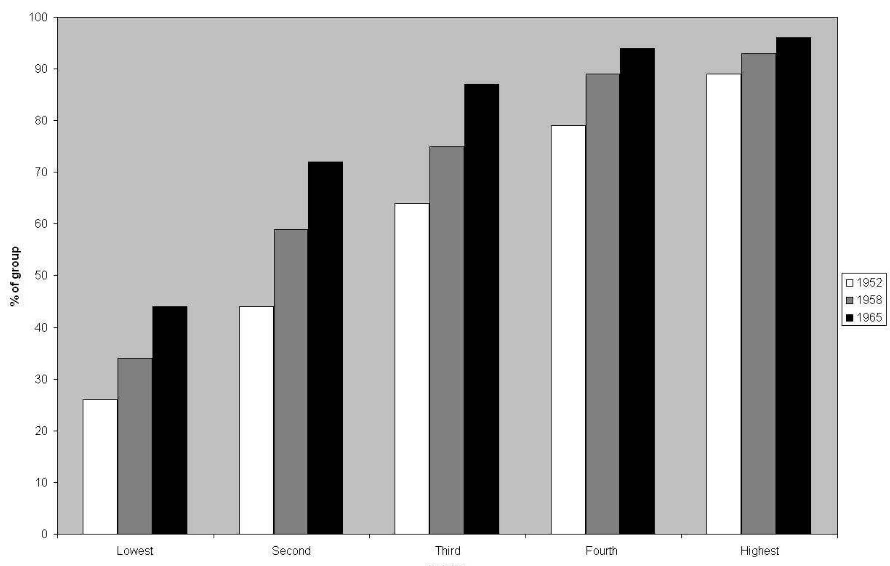
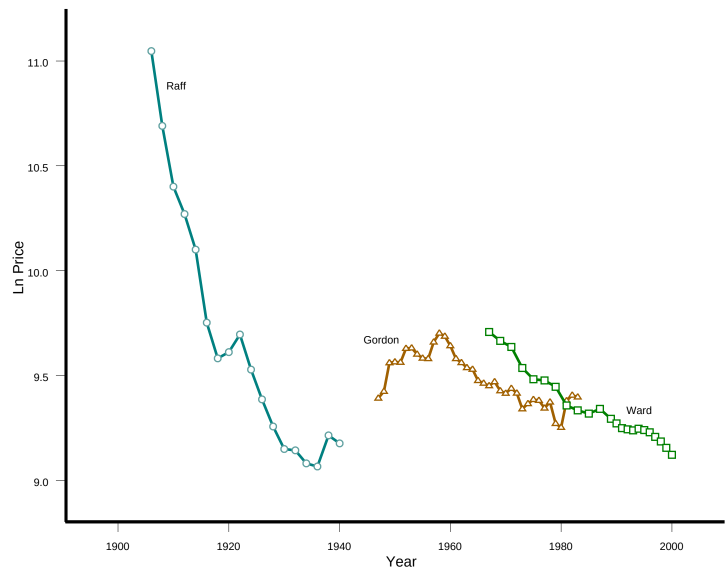
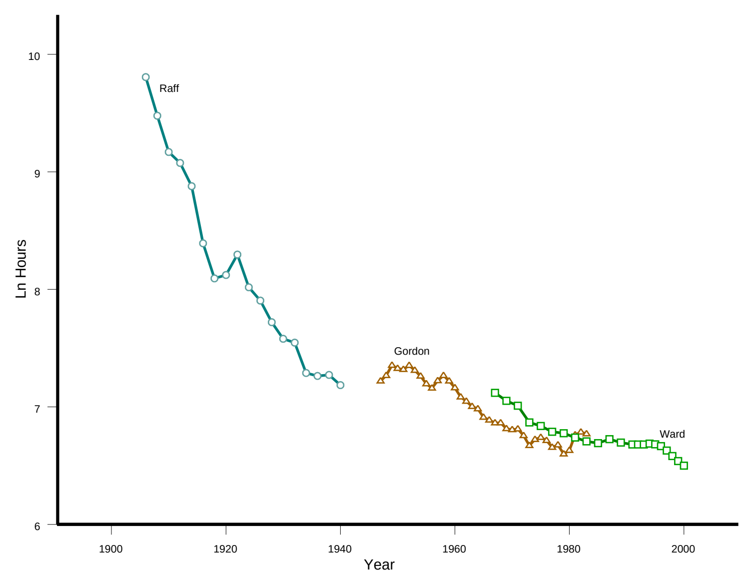


Figure 4: Car Ownership by Income Quintile : 1952 - 1965. Source: Survey of Consumer Finances, University of Michigan.



(a) Ln Price of New Automobile.

Figure 5: Ln Price of New Automobiles: 1906 - 2000. Source: 1906-40 from Raff(1995), 1947-83 from Gordon (1990), and 1967-2000 from Ward(2002).



(a) Ln Time Cost of New Automobiles.

Figure 6: Ln Time Cost for New Automobiles : 1906 - 2000. (See footnote 7 for details.)

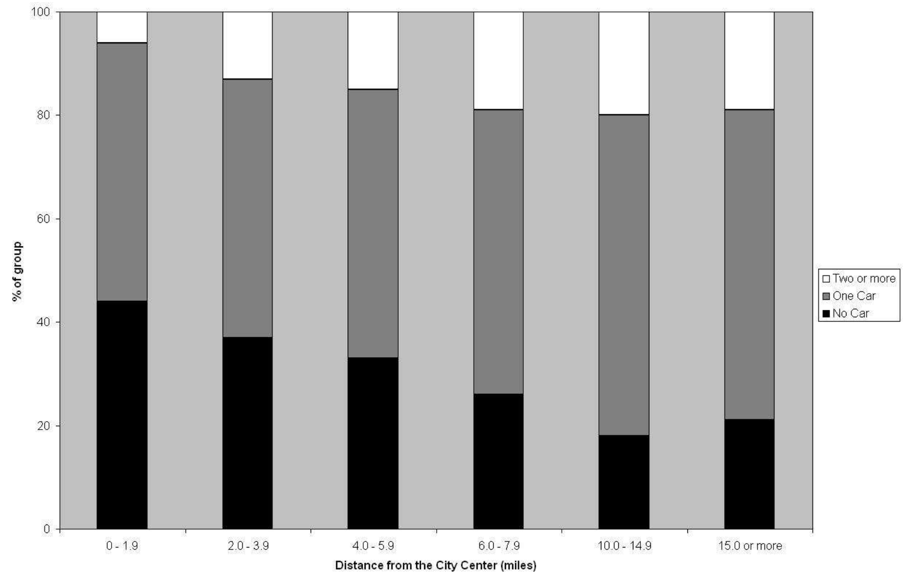


Figure 7: Car Ownership in 1962 by Distance from Center of Central City. Source: Survey of Consumer Finances, University of Michigan.

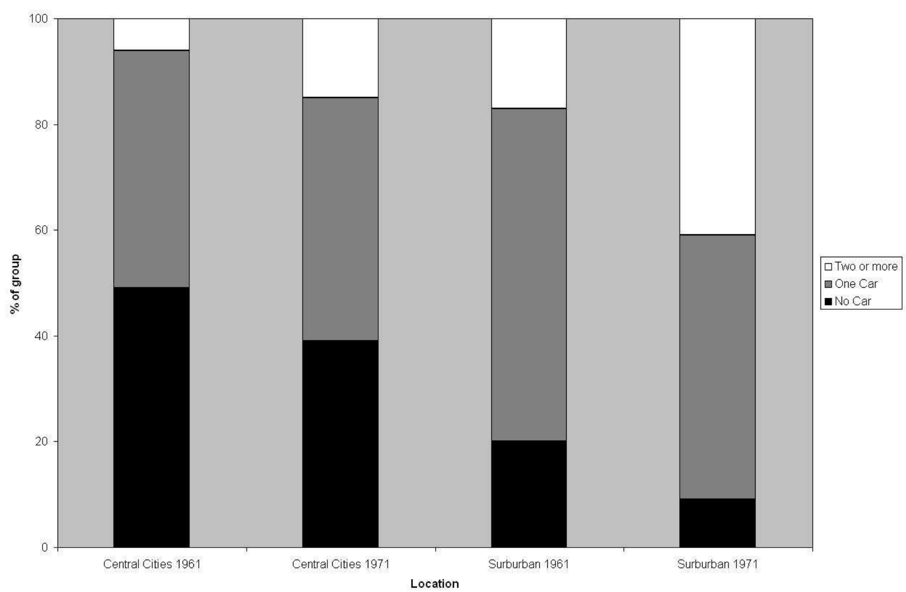


Figure 8: Car Ownership in 12 Largest SMSA's by Place of Residence. Source: Survey of Consumer Finances, University of Michigan.

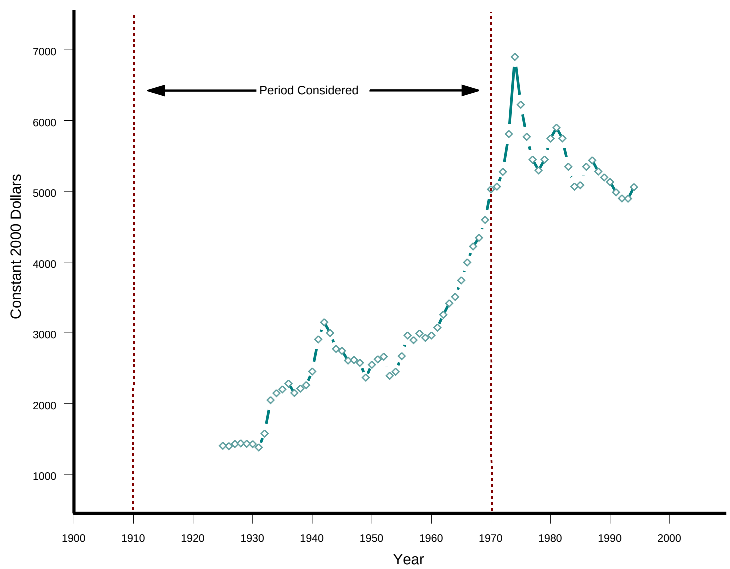


Figure 9: Highway and Street Stock per Capita. Source: Fixed Reproducible Tangible Wealth in the United States, 1925-94.

Table 6: Baseline Parametrization

	Parameter	1910	1920	1930	1940	1950	1960	1970
Income Distribution								
Mean ($/year)		7,134.40	9,214.40	10,004.80	15,225.60	19,468.80	24,835.20	28,246.40
Standard Deviation ($)		3,719.57	4,804.76	5,216.39	7,938.31	10,151.28	12,948.80	14,725.80
Minimum Income Level ($/year)		1,970.58	2,545.50	2,763.60	4,205.63	5,378.07	6,860.15	7,801.59
Maximum Income Level ($/year)		19,907.24	25,715.27	27,918.36	42,486.43	54,330.39	69,302.74	78,813.51
Price of Car ($)	$p_{c,t}$	32,880.00	14,934.00	9,412.00	9,664.00	14,253.00	15,399.00	12,271.00
Agricultural Land Rent ($/Lot[a])	$q_{A,t}$	7.29	4.86	3.21	5.23	13.89	11.23	19.55
Transportation Costs								
Bus								
Fixed Cost ($/day)	$\gamma_{b,t}$	0.05	0.11	0.25	0.58	1.33	3.05	6.97
Time Cost (hours/mile)	$\tau_{b,t}$	0.46	0.77	1.30	1.53	1.86	2.91	4.08
Average Distance (miles)		2.32	1.01	0.45	0.31	0.26	0.15	0.08
Wage at Average Distance ($/hour)		5.61	5.61	4.56	5.55	7.15	7.59	6.70
Average Variable Cost ($/day)		5.97	4.35	2.64	2.62	3.47	3.25	2.28
Average Total Cost ($/day)		6.02	4.46	2.90	3.20	4.81	6.30	9.25
Car								
Fixed Cost ($/day)	$\gamma_{c,t}$	5.88	2.44	4.06	4.87	6.69	7.19	8.11
Time Cost (hours/mile)	$\tau_{c,t}$	0.46	0.25	0.20	0.16	0.15	0.14	0.11
Average Distance (miles)		–	5.02	5.46	6.86	6.19	6.95	8.13
Wage at Average Distance ($/hour)		–	10.84	10.43	14.64	18.48	22.53	23.79
Average Variable Cost ($/day)		–	13.44	11.51	15.77	17.66	22.52	21.76
Average Total Cost ($/day)		–	15.88	15.57	20.64	24.35	29.71	29.88

[a]One lot is 12,910 square feet. (See footnote 17 for details.)

Table 7: Baseline Model Results

	1910	1920	1930	1940	1950	1960	1970
Car ownership (%)							
Data	0.93	14.12	33.57	35.39	46.02	65.66	84.11
Model	0	5.36	19.09	35.52	35.22	51.27	72.47
Population density gradient							
Data	0.83	0.79	0.66	0.61	0.39	0.31	0.23
Model	0.86	0.83	0.63	0.46	0.51	0.46	0.37
	(0.9829)[a]	(0.9219)	(0.8617)	(0.8644)	(0.8476)	(0.8492)	(0.8605)
% change in gradient							
Data		-4.8	-16.5	-7.6	-36.1	-20.5	-25.8
Model		-3.97	-24.16	-25.83	10.73	-11.46	-19.07
End of Bus Users (miles)	6.25	2.91	1.47	1.08	0.94	0.56	0.31
Boundary of City (miles)	6.25	7.98	11.69	15.34	12.70	15.83	17.90

[a] R-squared in parenthesis.

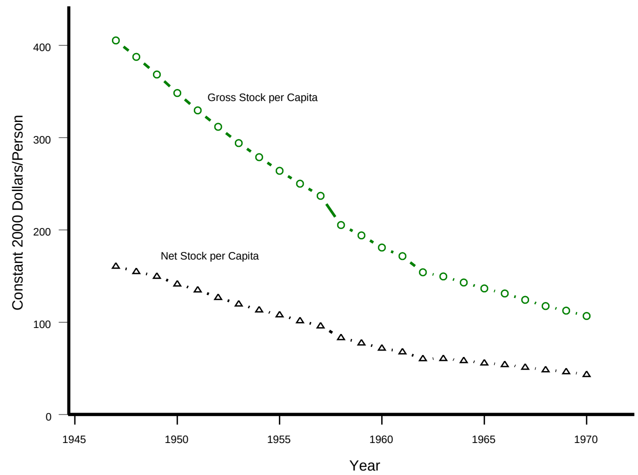


Figure 10: Capital Stock per Capita of Equipment and Structures for Intercity and Local Passenger Transit, 1947-1989. Source: Fixed Reproducible Tangible Wealth in the United States, 1925-89.

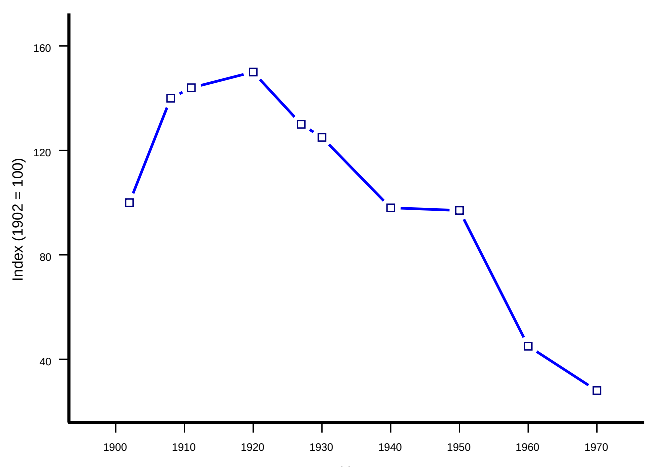


Figure 11: Public Transit Ridership per Capita, 1902-1970. Source: Jones (1985).

Table 8: Net Capital Expenditures of Urban Transit Properties, 1890-1950.

Year	Net Expenditures (millions in 1929 dollars)
1890	74.0
1895	176.2
1900	170.9
1905	229.8
1910	66.1
1915	15.2
1920	-128.5
1925	-105.4
1930	-85.3
1935	-60.7
1940	-10.4
1945	-58.4
1950	-53.5

Source: M. J. Ulmer, *Capital in Transportation, Communications, and Public Utilities.* Princeton, N.J.: Princeton University Press, 1960.

Table 9: Counterfactual Experiments

		1910	1920	1930	1940	1950	1960	1970
1.	*Data*							
	Car-Owners	0.93	14.12	33.57	35.39	46.02	65.66	84.11
	Density Gradient	0.83	0.79	0.66	0.61	0.39	0.31	0.23
2.	*Baseline Model*							
	Car-Owners	0	5.36	19.09	35.52	35.22	51.27	72.47
	Density Gradient	0.86	0.83	0.63	0.46	0.51	0.46	0.37
3.	*Prices and Wages Remain at 1910 Values*							
	a) *With rising cost of public transportation*							
	Car-Owners	0	0	0.20	0.27	0	0.34	0.52
	Density Gradient	0.86	1.42	2.05	2.38	3.58	4.62	6.79
	b) *Without rising cost of public transportation*							
	Car-Owners	0	0	0	0	0	0	0
	Density Gradient	0.86	0.85	0.84	0.85	0.88	0.87	0.90
4.	*Price of Car Remains at 1910 Value*							
	a) *With rising cost of public transportation*							
	Car-Owners	0	0.19	2.43	9.93	15.88	29.65	43.09
	Density Gradient	0.86	1.29	1.13	0.68	0.67	0.55	0.47
	b) *Without rising cost of public transportation*							
	Car-Owners	0	0	0	2.22	3.57	7.23	11.20
	Density Gradient	0.86	0.85	0.84	0.60	0.59	0.47	0.40
5.	*Wage Distribution Remains as in 1910*							
	a) *With rising cost of public transportation*							
	Car-Owners	0	2.65	11.20	12.92	6.21	8.51	17.30
	Density Gradient	0.86	0.97	0.74	0.70	1.27	1.23	1.18
	b) *Without rising cost of public transportation*							
	Car-Owners	0	0	0	0	0	0	0
	Density Gradient	0.86	0.85	0.65	0.61	0.88	0.87	0.90
6.	*Stock of Highways and Roads per Capita Remains as in 1910*							
	Car-Owners	0	0.65	8.63	17.80	18.86	33.34	53.62
	Density Gradient	0.86	1.26	1.26	1.18	1.27	1.23	1.22
7.	*Time and Fixed Cost of Public Transit Remains at 1910 Value*							
	Car-Owners	0	1.10	5.52	14.44	12.06	16.92	26.19
	Density Gradient	0.86	0.73	0.56	0.42	0.46	0.39	0.32
8.	*Stock of Highways and Roads per Capita, and Cost of Using Public Transportation at 1910 Values*							
	Car-Owners	0	0	0	0	0	0	0
	Density Gradient	0.86	0.85	0.84	0.84	0.85	0.84	0.85

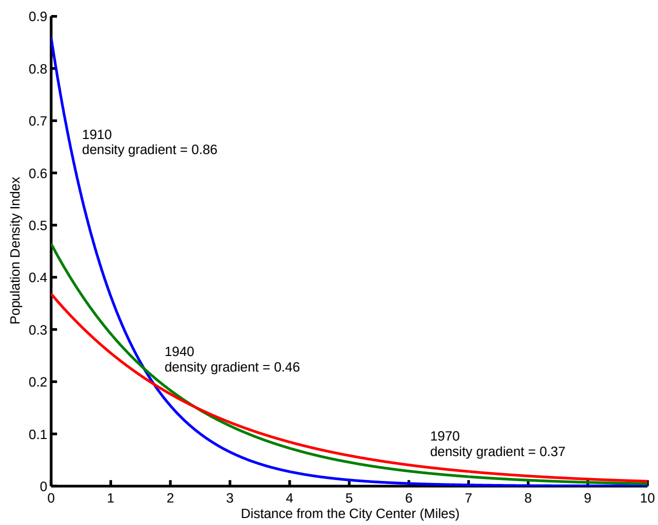


Figure 12: Estimated Population Density functions for the years 1910, 1940, and 1970 as predicted by the model.

7 Appendix

7.1 Proofs of Lemma 1 - 3

Proof of Lemma 1. To derive the slope of $x_b(\lambda)$, first totally differentiate equation (5)

$$\frac{q'(x)}{q(x)} = \frac{-t_2(w\lambda, x)}{(1-\alpha)[w\lambda - t(w\lambda, x)]}$$

with respect to x and λ. This gives

$$\left[\frac{q''}{q} - \left(\frac{q'}{q}\right)^2\right] dx = \frac{-[t_{22}dx + t_{12}wd\lambda]}{(1-\alpha)[w\lambda - t(w\lambda, x)]} - \frac{(-t_2)(1-t_1)wd\lambda + (-t_2)^2 dx}{(1-\alpha)[w\lambda - t(w\lambda, x)]^2}. \tag{51}$$

Consider the LHS of the above expression,

$$\begin{aligned} LHS &= \left\{\frac{-t_{22}}{(1-\alpha)[w\lambda - t(w\lambda, x)]} - \frac{(-t_2)^2}{(1-\alpha)[w\lambda - t(w\lambda, x)]^2}\right\} dx \\ &\quad + \left[\frac{wt_{12}}{t_2} - \frac{w(1-t_1)}{w\lambda - t(w\lambda, x)}\right] \frac{(-t_2)d\lambda}{(1-\alpha)[w\lambda - t(w\lambda, x)]} \\ &= \left[\frac{t_{22}}{t_2}\frac{q'}{q} - (1-\alpha)\left(\frac{q'}{q}\right)^2\right] dx + \left[\frac{wt_{12}}{t_2} - \frac{w(1-t_1)}{w\lambda - t(w\lambda, x)}\right]\left(\frac{q'}{q}\right) d\lambda. \end{aligned}$$

The second equality is obtained by using (5). Equality (51) can then be simplified as

$$\left[\frac{q''}{q} - \alpha\left(\frac{q'}{q}\right)^2 - \frac{t_{22}}{t_2}\frac{q'}{q}\right] dx = \left[\frac{w\lambda t_{12}}{t_2} - \frac{w\lambda(1-t_1)}{w\lambda - t(w\lambda, x)}\right]\left(\frac{q'}{q}\right)\frac{d\lambda}{\lambda}.$$

The desirable results can be obtained if the following inequality holds

$$\left[\frac{q''}{q} - \alpha\left(\frac{q'}{q}\right)^2 - \frac{t_{22}}{t_2}\frac{q'}{q}\right] > 0. \tag{52}$$

The reason is, since $q(x)$ is decreasing, $x_b(\lambda)$ is monotonically increasing if and only if

$$\frac{w\lambda t_{12}}{t_2} < \frac{w\lambda(1-t_1)}{w\lambda - t(w\lambda, x)}$$

for all positive $w\lambda$ and x. The RHS of the above inequality is the percentage increase in net income when labor income, $w\lambda$, increase by one percentage. Given the log utility, this coincides with the income elasticity of housing demand. Hence, what remains is to show that condition (52) holds. It turns out that (52) is a sufficient condition to ensure that $x_b(\lambda)$ is an interior solution for the maximization problem.

Given (2) and (3), the optimization problem (P1) is equivalent to

$$\max_x \; \{U(x) = \ln[w\lambda - t(w\lambda, x)] - (1-\alpha)\ln q(x)\}.$$

The first-order and second-order derivatives are given by

$$U_x = \frac{-t_2}{w\lambda - t(w\lambda, x)} - (1-\alpha)\frac{q'(x)}{q(x)},$$

$$U_{xx} = \frac{-t_{22}}{w\lambda - t(w\lambda, x)} - \frac{(-t_2)^2}{[w\lambda - t(w\lambda, x)]^2} - (1-\alpha)\left[\frac{q''}{q} - \left(\frac{q'}{q}\right)^2\right].$$

Given that $U_x = 0$, $U_{xx} < 0$ if and only if

$$(1-\alpha)\left[\frac{q''}{q} - \left(\frac{q'}{q}\right)^2\right] > \frac{-t_{22}}{w\lambda - t(w\lambda, x)} - \frac{(-t_2)^2}{[w\lambda - t(w\lambda, x)]^2}$$

$$\Leftrightarrow \quad (1-\alpha)\left[\frac{q''}{q} - \left(\frac{q'}{q}\right)^2\right] > \frac{-t_{22}}{w\lambda - t(w\lambda, x)} - (1-\alpha)^2\left(\frac{q'}{q}\right)^2$$

$$\Leftrightarrow \quad \left[\frac{q''}{q} - \alpha\left(\frac{q'}{q}\right)^2\right] > \frac{-t_{22}}{(1-\alpha)[w\lambda - t(w\lambda, x)]}.$$

□

Proof of Lemma 2. Given that $t(w\lambda, x) = \delta(x)\kappa(w\lambda) + \eta$,

$$\begin{aligned} & \frac{(1-t_1)}{w\lambda - t(w\lambda, x)} - \frac{t_{12}}{t_2} \\ = & \frac{1 - \delta(x)\kappa'(w\lambda)}{w\lambda - \delta(x)\kappa(w\lambda) - \eta} - \frac{\kappa'(w\lambda)}{\kappa(w\lambda)} \\ = & \frac{\kappa(w\lambda) - \kappa'(w\lambda)(w\lambda - \eta)}{[w\lambda - \delta(x)\kappa(w\lambda) - \eta]\kappa(w\lambda)}. \end{aligned}$$

The above expression is positive if and only if $\kappa(w\lambda) > \kappa'(w\lambda)(w\lambda - \eta)$. □

Proof of Lemma 3. Let $W^c(x;\lambda)$ denote the value function of a car-owner with ability λ who locates at x. By definition,

$$V^c(\lambda) \equiv W^c[x_c(\lambda);\lambda] \geq W^c(x;\lambda) \qquad \text{for all } x \geq 0.$$

Similarly, define $W^b(x;\lambda)$ for bus-users. Then,

$$V^b(\lambda) \equiv W^b[x_b(\lambda);\lambda] \geq W^b(x;\lambda) \qquad \text{for all } x \geq 0.$$

Suppose $\overline{\lambda}$ exists so that $V^c\left(\overline{\lambda}\right) = V^b\left(\overline{\lambda}\right)$. This implies

$$V^b\left(\overline{\lambda}\right) \geq W^c\left[x_b\left(\overline{\lambda}\right);\overline{\lambda}\right] \quad \text{and} \quad V^c\left(\overline{\lambda}\right) \geq W^b\left[x_c\left(\overline{\lambda}\right);\overline{\lambda}\right]. \tag{53}$$

The first inequality states that conditional on living in $x_b\left(\overline{\lambda}\right)$, the agent with ability $\overline{\lambda}$ has no incentive to switch to own a car. Since both car-owners and bus-users pay the same rent at $x_b\left(\overline{\lambda}\right)$, the inequality implies

$$w\overline{\lambda} - t\left[w\overline{\lambda}, x_b\left(\overline{\lambda}\right)\right] \geq w\overline{\lambda} - \tau\left[w\overline{\lambda}, x_b\left(\overline{\lambda}\right)\right] - p_c,$$

or

$$0 \geq t\left[w\overline{\lambda}, x_b\left(\overline{\lambda}\right)\right] - \tau\left[w\overline{\lambda}, x_b\left(\overline{\lambda}\right)\right] - p_c. \tag{54}$$

The second inequality in (53) can be interpreted similarly and implies

$$t\left[w\overline{\lambda}, x_c\left(\overline{\lambda}\right)\right] - \tau\left[w\overline{\lambda}, x_c\left(\overline{\lambda}\right)\right] - p_c \geq 0. \tag{55}$$

Define a function $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$g(x) = t\left(w\overline{\lambda}, x\right) - \tau\left(w\overline{\lambda}, x\right) - p_c.$$

Following the assumptions on $t(w\lambda, x)$ and $\tau(w\lambda, x)$, the function $g(\cdot)$ is differentiable with first-order derivative given by

$$g'(x) = t_2\left(w\overline{\lambda}, x\right) - \tau_2\left(w\overline{\lambda}, x\right) > 0 \tag{56}$$

for all $x \geq 0$. This, together with (54) and (55), implies $x_c\left(\overline{\lambda}\right) \geq x_b\left(\overline{\lambda}\right)$. For reasons stated in the text, $x_c\left(\overline{\lambda}\right) > x_b\left(\overline{\lambda}\right)$ cannot hold in equilibrium. Hence, $x_c\left(\overline{\lambda}\right) = x_b\left(\overline{\lambda}\right)$. □

7.2 Existence of Equilibrium

The objective of this section is to state the proof of Theorem 4. This involves a detailed characterization of the solution of the bus-user's initial value problem.

7.2.1 Bus-user's Initial Value Problem

For any $\psi_0 > 0$, form a candidate rent function for bus-users

$$q_b(\lambda; \psi_0) = \psi_0 - \tau_b w \int_{\lambda_{\min}}^{\lambda} u f(u)\, du, \tag{57}$$

for $\lambda \geq \lambda_{\min}$. The candidate rent function has to be strictly positive over its domain S. If $\psi_0 > \widetilde{\psi} \equiv \tau_b w \int_{\lambda_{\min}}^{\lambda_{\max}} u f(u)\, du$, then obviously $S = [\lambda_{\min}, \lambda_{\max}]$. If $\psi_0 \leq \widetilde{\psi}$, then there exists a unique $\widetilde{\lambda}(\psi_0) \in (\lambda_{\min}, \lambda_{\max}]$ such that $q_b\left[\widetilde{\lambda}(\psi_0); \psi_0\right] = 0$. This follows from the fact that $q_b(\lambda; \psi_0)$ is continuous and strictly decreasing in λ, $q_b(\lambda_{\min}; \psi_0) = \psi_0 > 0$ and $q_b(\lambda_{\max}; \psi_0) \leq 0$. In this case, $S = \left[\lambda_{\min}, \widetilde{\lambda}(\psi_0)\right)$. Also, $q_b(\lambda; \psi_0)$ is strictly increasing in ψ_0 for $\lambda \in S$.

Given the candidate rent function, one can form the initial value problem for bus-users

$$x_b'(\lambda) = (1-\alpha)\left\{w\lambda\left[1 - \tau_b x_b(\lambda)\right] - \gamma_b\right\} \frac{f(\lambda)}{q_b(\lambda; \psi_0)} \tag{IVP1}$$

and

$$x_b(\lambda_{\min}) = 0.$$

In equilibrium, positive consumption requires that

$$w\lambda\left[1 - \tau_b x_b(\lambda)\right] > \gamma_b. \tag{58}$$

Given the form of $x_b'(\lambda)$, condition (58) holds if and only if $x_b(\lambda)$ is strictly increasing. To show that a unique solution of (IVP1) exists, the following results in ordinary differential equations are needed.

Theorem 5. *Let $f(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ be defined and continuous on the rectangle*

$$R = \left\{(x,y) \mid |x - x_0| \leq a, \ \ |y - y_0| \leq b\right\},$$

where a and b are constants. Let $|f(x,y)| \leq M$, for $(x,y) \in R$, and $\alpha = \min\left\{a, \frac{b}{M}\right\}$. Then the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0, \tag{59}$$

has a unique solution $y(x)$ defined on $[x_0 - \alpha, x_0 + \alpha]$.

Theorem 6. *Let $f(x,y)$ be continuous on an open (x,y)-set E and let $y(x)$ be a solution of (59) on some interval. Then there exists $\varphi > 0$ such that $y(x)$ can be extended as a solution for $x \in [x_0, x_0 + \varphi)$ but not for $x \geq x_0 + \varphi$. Moreover, $y(x)$ tends to the boundary ∂E of E as $x \to x_0 + \varphi$.*

The boundary ∂E of E is defined as $\partial E = \overline{E} \sim E$, where $\overline{E}$ is the closure of E. The interval defined in Theorem 6 is called a right maximal interval of existence for y. Readers are referred to Hartman (1964, Ch.I) for the proof of these theorems. One immediate result of Theorem 6 is the following.

Corollary 7. *Let $f(x,y)$ be continuous on a strip $x_0 \leq x \leq x_0 + a, y \in \mathbb{R}$ arbitrary. Let $y = y(x)$ be a solution of (59) on a right maximal interval J. Then either $J = [x_0, x_0 + a]$ or $J = [x_0, \delta)$, $\delta \leq x_0 + a$ and $|y(x)| \to \infty$ as $x \to \delta$.*

With these results, the solution of (IVP1) is characterized in the next theorem.

Theorem 8. *For any $\psi_0 > 0$, the initial value problem (IVP1) has a unique solution $x_b(\lambda; \psi_0)$ defined on S. Moreover, the solution $x_b(\lambda; \psi_0)$ satisfies condition (58).*

Proof. Existence Fix $\psi_0 > 0$. Define a region $D = S \times \mathbb{R}_+ \subset \mathbb{R}_+^2$ and a function $h(\lambda, x; \psi_0)$ over D,

$$h(\lambda, x; \psi_0) = (1-\alpha)\left[w\lambda(1-\tau_b x) - \gamma_b\right] \frac{f(\lambda)}{q_b(\lambda; \psi_0)}.$$

Both h and $\frac{\partial h}{\partial x}$ are defined and continuous in D. Since $\frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_{\min}}\right) > 0$, a closed rectangle $R = \{(\lambda, x) \mid \lambda \in [\lambda_{\min}, \lambda_{\min} + a], x \in [0, b]\}$, contained in D, can be drawn for some a, b. It follows from Theorem 5 that (IVP1) has a solution $x_b(\lambda; \psi_0)$ defined on some neighborhood of $\lambda_{\min}$. Theorem 6 then guarantees the existence of a right maximal interval $J = [\lambda_{\min}, \varphi)$ on which $x(\lambda; \psi_0)$ is defined.

Boundedness Condition (58) is equivalent to having $x_b(\lambda; \psi_0)$ bounded above by the function $g(\lambda) = \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda}\right)$. Suppose the contrary that there exists $\lambda_1 \in [\lambda_{\min}, \varphi)$ such that

$$x_b(\lambda_1; \psi_0) > \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_1}\right). \tag{60}$$

Since $w\lambda_{\min} > \gamma_b$ and $x_b(\lambda_{\min}; \psi_0) = 0$, condition (58) is satisfied at $\lambda = \lambda_{\min}$. By the Intermediate Value Theorem, there exists $\lambda_2 \in (\lambda_{\min}, \lambda_1)$ such that $x_b(\lambda_2; \psi_0) = \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_2}\right)$. But for $\lambda \in (\lambda_2, \lambda_1]$, $x_b'(\lambda; \psi_0) < 0$ so that $x_b(\lambda; \psi_0) < x_b(\lambda_2; \psi_0) = \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_2}\right) < \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda}\right)$. This contradicts (60).

Next, it will be shown that $J = S$. First consider the case when $S = [\lambda_{\min}, \lambda_{\max}]$. By Corrollary 7, either $J = [\lambda_{\min}, \lambda_{\max}]$ or $J = [\lambda_{\min}, \delta)$, $\delta < \lambda_{\min}$ and $x_b(\lambda; \psi_0) \to \infty$ as $\lambda \to \delta$. But the boundedness result is violated in the second case, hence $J = [\lambda_{\min}, \lambda_{\max}]$. Similarly, when $S = \left[\lambda_{\min}, \widehat{\lambda}(\psi_0)\right)$, Theorem 6 guarantees that $x_b(\lambda; \psi_0) \to \partial D$ as λ approaches φ from the left. If $\varphi < \widehat{\lambda}(\psi_0)$, then $x_b(\lambda; \psi_0) \to \infty$ and contradicts the boundedness result.

Uniqueness Let $x_b(\lambda; \psi_0)$ and $z_b(\lambda; \psi_0)$ be solutions of (IVP1) on S. Define $\delta(\lambda) = [x_b(\lambda; \psi_0) - z_b(\lambda; \psi_0)]^2$ on S. Then $\delta(\lambda_{\min}) = 0$ and $\delta(\lambda) \geq 0$. The derivative of $\delta(\lambda)$ is given by

$$\begin{aligned}\delta'(\lambda) &= 2(x_b - z_b)(x_b' - z_b') \\ &= -2(1-\alpha)\tau_b(x_b - z_b)^2 \frac{\lambda f(\lambda)}{q_b(\lambda; \psi_0)}.\end{aligned}$$

If $x_b(\lambda; \psi_0) \neq z_b(\lambda; \psi_0)$ for $\lambda > \lambda_{\min}$, then $\delta'(\lambda) < 0$ and $\delta(\lambda) < 0$ for $\lambda > \lambda_{\min}$. This gives rise to a contradiction and hence, $x_b(\lambda; \psi_0) \equiv z_b(\lambda; \psi_0)$ on S. □

The next proposition establishes two useful properties of $x_b(\lambda; \psi_0)$ as a function of ψ_0, namely continuity and strict monotonicity.

Proposition 9. *For ψ_0', $\psi_0'' > 0$, let J be an interval over which both $x_b(\lambda; \psi_0')$ and $x_b(\lambda; \psi_0'')$ are defined. Then for any $\varepsilon > 0$, there exists an $\delta > 0$ such that*

$$\left|x_b(\lambda; \psi_0') - x_b(\lambda; \psi_0'')\right| \leq \varepsilon$$

for all $\lambda \in J$, whenever $\left|\psi_0' - \psi_0''\right| \leq \delta$. Moreover, if $\psi_0' < \psi_0''$, then $x_b(\lambda; \psi_0') > x_b(\lambda; \psi_0'')$ for all $\lambda \in J$.

Proof. Define $\phi(\lambda) \equiv x_b(\lambda; \psi_0')$ and $\mu(\lambda) \equiv x_b(\lambda; \psi_0'')$. Then, for $\lambda \in J$,

$$\phi(\lambda) \equiv \int_{\lambda_{\min}}^{\lambda} h\left[u, \phi(u); \psi_0'\right] du \tag{61}$$

and

$$\mu(\lambda) \equiv \int_{\lambda_{\min}}^{\lambda} h\left[u, \mu(u); \psi_0''\right] du. \tag{62}$$

Pick any $\widetilde{\varepsilon} > 0$. Without loss of generality, assume $\psi_0' < \psi_0''$. For $\lambda \in J$, the difference between (61) and (62) can be written as

$$\begin{aligned}\phi(\lambda) - \mu(\lambda) &= \int_{\lambda_{\min}}^{\lambda} \left\{h\left[u, \phi(u); \psi_0'\right] - h\left[u, \mu(u); \psi_0'\right]\right\} du \\ &\quad + \int_{\lambda_{\min}}^{\lambda} \left\{h\left[u, \mu(u); \psi_0'\right] - h\left[u, \mu(u); \psi_0''\right]\right\} du.\end{aligned}$$

Applying the triangular inequality gives

$$\begin{aligned}|\phi(\lambda) - \mu(\lambda)| &\leq \int_{\lambda_{\min}}^{\lambda} \left|h\left[u, \phi(u); \psi_0'\right] - h\left[u, \mu(u); \psi_0'\right]\right| du \\ &\quad + \int_{\lambda_{\min}}^{\lambda} \left|h\left[u, \mu(u); \psi_0'\right] - h\left[u, \mu(u); \psi_0''\right]\right| du. \end{aligned} \tag{63}$$

Let L be the bound of $\left|\frac{\partial h}{\partial x}\right|$ in D. Then

$$\left|h\left[u,\phi\left(u\right);\psi_0'\right]-h\left[u,\mu\left(u\right);\psi_0'\right]\right|\leq L\left|\phi\left(u\right)-\mu\left(u\right)\right|.$$

Hence, the first term in inequality (63) must satisfy

$$\int_{\lambda_{\min}}^{\lambda}\left|h\left[u,\phi\left(u\right);\psi_0'\right]-h\left[u,\mu\left(u\right);\psi_0'\right]\right|du\leq\int_{\lambda_{\min}}^{\lambda}L\left|\phi\left(u\right)-\mu\left(u\right)\right|du.$$

Since $h\left(\lambda,x;\psi_0''\right)$ is continuous in ψ_0'', there exists $\delta>0$ such that

$$\left|h\left[u,\mu\left(u\right);\psi_0'\right]-h\left[u,\mu\left(u\right);\psi_0''\right]\right|\leq\widetilde{\varepsilon},$$

whenever $\left|\psi_0'-\psi_0''\right|\leq\delta$. The second term thus satisfies

$$\int_{\lambda_{\min}}^{\lambda}\left|h\left[u,\mu\left(u\right);\psi_0'\right]-h\left[u,\mu\left(u\right);\psi_0''\right]\right|du\leq\widetilde{\varepsilon}\left(\lambda-\lambda_{\min}\right)$$

for ψ_0' and ψ_0'' close enough. Then for any $\psi_0'<\psi_0''$ such that $\left|\psi_0'-\psi_0''\right|\leq\delta$,

$$\left|\phi\left(\lambda\right)-\mu\left(\lambda\right)\right|\leq\widetilde{\varepsilon}\left(\lambda-\lambda_{\min}\right)+\int_{\lambda_{\min}}^{\lambda}L\left|\phi\left(u\right)-\mu\left(u\right)\right|du.$$

Applying Gronwall's inequality, this becomes

$$\left|\phi\left(\lambda\right)-\mu\left(\lambda\right)\right|\leq\widetilde{\varepsilon}\left(\lambda-\lambda_{\min}\right)e^{L\left(\lambda-\lambda_{\min}\right)}<\widetilde{\varepsilon}\left(\lambda_{\max}-\lambda_{\min}\right)e^{L\left(\lambda_{\max}-\lambda_{\min}\right)},$$

for $\lambda\in J$. Set $\varepsilon=\widetilde{\varepsilon}\left(\lambda_{\max}-\lambda_{\min}\right)e^{L\left(\lambda_{\max}-\lambda_{\min}\right)}$. Since $\widetilde{\varepsilon}$ is arbitrary, this establishes continuity.

Strictly Decreasing in ψ_0 For any $\psi_0',\psi_0''>0$, $\psi_0'<\psi_0''$, define $\phi\left(\lambda\right),\mu\left(\lambda\right)$ and J as before. Since $\phi\left(\lambda_{\min}\right)=\mu\left(\lambda_{\min}\right)=0$ and $h\left(\lambda_{\min},0;\psi_0'\right)>h\left(\lambda_{\min},0;\psi_0''\right)$, it follows that $\phi\left(\lambda\right)>\mu\left(\lambda\right)$ for λ close to $\lambda_{\min}$. Suppose the contrary that there exists $\lambda_1\in J$, $\lambda_1\neq\lambda_{\min}$, such that $\phi\left(\lambda_1\right)=\mu\left(\lambda_1\right)=x_1$, and $\mu\left(\lambda\right)>\phi\left(\lambda\right)$ for $\lambda>\lambda_1$. Then $\mu\left(\lambda\right)$ must cut $\phi\left(\lambda\right)$ from below at λ_1, or $\mu'\left(\lambda_1\right)>\phi'\left(\lambda_1\right)$. Since $q_b\left(\lambda_1;\psi_0'\right)<q_b\left(\lambda_1;\psi_0''\right)$,

$$\phi'\left(\lambda_1\right)=h\left(\lambda_1,x_1;\psi_0'\right)>h\left(\lambda_1,x_1;\psi_0''\right)=\mu'\left(\lambda_1\right).\tag{64}$$

This gives rise to a contradiction. Hence, $\phi\left(\lambda\right)>\mu\left(\lambda\right)$ for $\lambda\in J$. □

Proposition 10. *For any* K, $0<K<\frac{1}{\tau_b}\left(1-\frac{\gamma_b}{w\lambda_{\max}}\right)$, *there exists a unique* $\psi_0>0$ *such that* $x_b\left(\lambda_{\max};\psi_0\right)=K$.

Proof. For $\psi_0>\widetilde{\psi}\equiv\tau_b w\int_{\lambda_{\min}}^{\lambda_{\max}}uf\left(u\right)du$, $S=\left[\lambda_{\min},\lambda_{\max}\right]$ and hence $x_b\left(\lambda_{\max};\psi_0\right)$ is defined. It follows immediately from Proposition 9 that, $x_b\left(\lambda_{\max};\psi_0\right)$ as a function in ψ_0 defined over $\left(\widetilde{\psi},\infty\right)$, is continuous and strictly decreasing. Moreover, the range of $x_b\left(\lambda_{\max};\psi_0\right)$ is contained in the interval $\left[0,\frac{1}{\tau_b}\left(1-\frac{\gamma_b}{w\lambda_{\max}}\right)\right]$.

To prove the proposition, it suffices to show that $x_b\left(\lambda_{\max};\psi_0\right)$ tends to 0 as ψ_0 approaches infinity, and $x_b\left(\lambda_{\max};\psi_0\right)$ tends to $\frac{1}{\tau_b}\left(1-\frac{\gamma_b}{w\lambda_{\max}}\right)$ as ψ_0 approaches $\widetilde{\psi}$ from the left. With these limiting conditions and continuity of $x_b(\lambda_{\max};\psi_0)$ in ψ_0, existence of $\psi_0 > 0$ that solves $x_b(\lambda_{\max};\psi_0) = K$ is guaranteed by the Intermediate Value Theorem. Uniqueness is ensured by the strict monotonicity of $x_b(\lambda_{\max};\psi_0)$ in ψ_0.

As $\psi_0 \to \infty$, $q_b(\lambda;\psi_0) \to \infty$ for $\lambda \in S$. Hence, $h(\lambda, x;\psi_0) \to 0$ for all $(\lambda, x) \in D$. Thus $x_b(\lambda;\psi_0) \to 0$ for $\lambda \in S$.

Next consider $\psi_0 = \widetilde{\psi}$, then $S = [\lambda_{\min}, \lambda_{\max})$. Let L be the limit of $x_b\left(\lambda;\widetilde{\psi}\right)$ as λ approaches $\lambda_{\max}$ from the left. Since $x_b\left(\lambda;\widetilde{\psi}\right)$ is strictly increasing in λ, this means $x_b\left(\lambda;\widetilde{\psi}\right)$ is bounded above by L. Suppose $L > \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_{\max}}\right)$, then for any λ sufficiently close to $\lambda_{\max}$, we have $x_b\left(\lambda;\widetilde{\psi}\right) > \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_{\max}}\right)$. This contradicts condition (60). Suppose $L < \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_{\max}}\right)$. For any $\lambda \in S$,

$$w\lambda\left[1 - \tau_b x_b\left(\lambda;\widetilde{\psi}\right)\right] > w\lambda\left(1 - \tau_b L\right),$$

which implies

$$h\left[\lambda, x_b\left(\lambda;\widetilde{\psi}\right);\widetilde{\psi}\right] > (1-\alpha)\left[w\lambda\left(1 - \tau_b L\right) - \gamma_b\right]\frac{f(\lambda)}{q_b\left(\lambda;\widetilde{\psi}\right)}. \tag{65}$$

Since the RHS of (65) tends to infinity as λ tends to $\lambda_{\max}$, the same is true for $x_b'\left(\lambda;\widetilde{\psi}\right) = h\left[\lambda, x_b\left(\lambda;\widetilde{\psi}\right);\widetilde{\psi}\right]$. This means for every $M > 0$, there exists $\delta > 0$ such that $x_b'\left(\lambda;\widetilde{\psi}\right) > M$ whenever $\lambda \in S \cap (\delta, \lambda_{\max})$. Construct sequences of $\{M_n\}$ and $\{\lambda_n\}$ as follows: Set $\lambda_0 = \lambda_{\min}$ and define $M_{n+1} = \frac{L - x_b\left(\lambda_n;\widetilde{\psi}\right)}{\lambda_{\max} - \lambda_n} > 0$, for $n = 0, 1, \ldots$. Then for each M_{n+1}, there exists $\delta_{n+1} > 0$ such that $\lambda \in S \cap (\delta_{n+1}, \lambda_{\max})$ implies $x_b'\left(\lambda;\widetilde{\psi}\right) > M_{n+1}$. Set $\lambda_{n+1} = \lambda_{\max} - \frac{1}{n}\left[\lambda_{\max} - \max\left(\delta_{n+1}, \lambda_n\right)\right]$, then for $\lambda \in (\lambda_{n+1}, \lambda_{\max})$,

$$L > x_b\left(\lambda;\widetilde{\psi}\right) > x_b\left(\lambda_{n+1};\widetilde{\psi}\right) + M_{n+1}\left(\lambda - \lambda_{n+1}\right). \tag{66}$$

This yields a sequence of λ_n that converges to $\lambda_{\max}$ and a sequence of M_n that approaches infinity. In the limit, (66) becomes

$$L > L + \lim_{n\to\infty} M_n\left(\lambda_{\max} - \lambda_n\right). \tag{67}$$

Since $\lim_{n\to\infty} M_n\left(\lambda_{\max} - \lambda_n\right) \geq 0$, a contradiction arises. Hence, $L = \frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_{\max}}\right)$. □

7.2.2 Proof of Theorem 4

From (42), if car-owners exist, then the location of the agent with $\overline{\lambda}$ is given by $x_c\left(\overline{\lambda}\right) = x_b\left(\overline{\lambda}\right) = \frac{\gamma_c + p_c - \gamma_b}{(\tau_b - \tau_c)w\overline{\lambda}}$. This means $\overline{\lambda}$ is determined by the point where $x_b(\lambda;\psi_0)$ intersects the curve

$$\rho(\lambda) = \frac{\gamma_c + p_c - \gamma_b}{(\tau_b - \tau_c)\, w\lambda}$$

in the interior of D. Let $W^b(x;\lambda)$ be the value function of a bus-user with ability λ who locates at x. Similarly, define $W^c(x;\lambda)$. Then $\rho(\lambda)$ is the location where $W^b[\rho(\lambda);\lambda] = W^c[\rho(\lambda);\lambda]$. Since $x_b(\lambda;\psi_0)$ is strictly increasing in λ, there can be at most one intersection. If intersection does not occur, or $x_b(\lambda;\psi_0) \leq \frac{\gamma_c + p_c - \gamma_b}{(\tau_b - \tau_c)w\lambda}$, for $\lambda \in S$, then there will be no car-owner.

Proof of Theorem 4. The proof consists of three main steps. First, it is shown that no car-owner exist if $\lambda_{\max} \leq \kappa$. Second, the critical value q^* is specified and it is shown that, given $\lambda_{\max} > \kappa$ and $q_A \in (0, q^*)$, a unique equilibrium with car-owners can be constructed. Finally, it is proved that no car-owner exists for any economy with $q_A > q^*$.

Since $\rho(\lambda)$ lies in the interior of D only for $\lambda \in (\kappa, \lambda_{\max})$, all possible values for $\overline{\lambda}$ must lie within this interval. But this interval is nonempty only if $\lambda_{\max} > \kappa$ holds. Hence, no car-owner exists if $\lambda_{\max} \leq \kappa$.

Suppose $\lambda_{\max} > \kappa$, this is equivalent to

$$\frac{1}{\tau_b}\left(1 - \frac{\gamma_b}{w\lambda_{\max}}\right) > \frac{\gamma_c + p_c - \gamma_b}{(\tau_b - \tau_c)\, w\lambda_{\max}}.$$

By Proposition 10, there exists a unique $\psi_0^* > \widetilde{\psi}$ such that

$$x_b(\lambda_{\max}; \psi_0^*) = \frac{\gamma_c + p_c - \gamma_b}{(\tau_b - \tau_c)\, w\lambda_{\max}}. \tag{68}$$

This corresponds to the case where $\overline{\lambda}(\psi_0^*) = \lambda_{\max}$. Before an equilibrium is constructed, first consider the following mapping,

$$\Psi(\psi_0) = \psi_0 - \widetilde{\psi} + (\tau_b - \tau_c)\, w \int_{\overline{\lambda}(\lambda_0)}^{\lambda_{\max}} u f(u)\, du.$$

For $\psi_0 > \psi_0^*$, $x_b(\lambda_{\max}; \psi_0) < x_b(\lambda_{\max}; \psi_0^*)$ so that $\overline{\lambda}(\lambda_0)$ is not defined. Hence $\Psi(\psi_0)$ is defined only for $\psi_0 \leq \psi_0^*$. Moreover, the function $\Psi(\psi_0)$ is continuous with $\Psi(\psi_0^*) > 0$. $\overline{\lambda}(\psi_0) \in (\lambda_{\min}, \lambda_{\max}]$ implies

$$\psi_0 - \tau_b w \int_{\lambda_{\min}}^{\lambda_{\max}} u f(u)\, du \leq \Psi(\psi_0) < \psi_0 - \tau_c w \int_{\lambda_{\min}}^{\lambda_{\max}} u f(u)\, du. \tag{69}$$

This means there exists ψ_0^{**}, $\tau_c w \int_{\lambda_{\min}}^{\lambda_{\max}} u f(u)\, du < \psi_0^{**} \leq \psi_0^*$, such that $\Psi(\psi_0^{**}) = 0$. ψ_0^{**} need not be unique in general. Let ψ_0^{**} denotes the smallest value that solves $\Psi(\psi_0) = 0$. Define $q^* = \max_{[\psi_0^{**}, \psi_0^*]} \{\Psi(\psi_0)\}$. Such a value exists by the contuity of $\Psi(\psi_0)$. It follows that, for any $q_A \in (0, q^*)$, there exists $\psi_0 \in (\psi_0^{**}, \psi_0^*)$ such that $\Psi(\psi_0) = q_A$.

Now we are ready to construct an equilibrium. For any $q_A \in (0, q^*)$, pick a ψ_0 such that $\Psi(\psi_0) = q_A$. $q_A < q^*$ implies $\psi_0 > \psi_0^*$. Using ψ_0, construct a candidate rent function $q_b(\lambda; \psi_0)$ as in (57) and solves the initial-value problem (IVP1). With the resulting solution, $x_b(\lambda; \psi_0)$, the corresponding critical ability level, $\overline{\lambda}(\psi_0)$, can be computed as described above. Such a value exists as $\psi_0 < \psi_0^*$. Then form the candidate rent function for car-owners,

$$q_c(\lambda; \psi_0) = q_A + \tau_c w \int_{\lambda}^{\lambda_{\max}} u f(u)\, du,$$

defined over the interval $\left[\overline{\lambda}(\psi_0), \lambda_{\max}\right]$. Car-owners' location choice function can be obtained by solving

$$x_c'(\lambda) = (1 - \alpha)\{w\lambda[1 - \tau_c x_c(\lambda)] - (\gamma_c + p_c)\} \frac{f(\lambda)}{q_c(\lambda; \psi_0)} \tag{IVP3}$$

and

$$x_c\left[\overline{\lambda}\left(\psi_0\right)\right]=\frac{\gamma_c+p_c-\gamma_b}{\left(\tau_b-\tau_c\right)w\overline{\lambda}\left(\psi_0\right)}.$$

By the same argument as in Theorem 8, (IVP3) has a unique solution $x_c\left(\lambda;\psi_0\right)$ defined on $\left[\overline{\lambda}\left(\psi_0\right),\lambda_{\max}\right]$. Moreover, $x_c\left(\lambda;\psi_0\right)$ satisfies the positive consumption condition for car-owners. The candidate rent functions $q_b\left(\lambda;\psi_0\right)$, $q_c\left(\lambda;\psi_0\right)$; the resulting location choice functions $x_b\left(\lambda;\psi_0\right)$, $x_c\left(\lambda;\psi_0\right)$; and the critical value, $\overline{\lambda}\left(\psi_0\right)$, can be supported as equilibrium if the rent function defined for all ability level is continuous at $\overline{\lambda}\left(\psi_0\right)$, i.e. $q_b\left[\overline{\lambda}\left(\psi_0\right);\psi_0\right]=q_c\left[\overline{\lambda}\left(\psi_0\right);\psi_0\right]$, or

$$\psi_0-\tau_b w\int_{\lambda_{\min}}^{\overline{\lambda}(\psi_0)}uf\left(u\right)du=q_A+\tau_c w\int_{\overline{\lambda}(\psi_0)}^{\lambda_{\max}}uf\left(u\right)du. \tag{70}$$

This is equivalent to $\Psi\left(\psi_0\right)=q_A$ which is ensured by the choice of ψ_0. This completes the construction of an equilibrium. Since $\Psi\left(\psi_0\right)$ is bounded above by q^*, there does not exist any $\psi_0>0$ such that $\Psi\left(\psi_0\right)=q_A$ holds for $q_A>q^*$. This means given $q_A>q^*$, (70) would not be satisfied for any $\psi_0>0$. Hence, an equilibrium with car-owners does not exist for $q_A>q^*$. □

7.3 Numerical Algorithm

The model's equilibrium is computed numerically using the algorithm outlined below. This particular algorithm is chosen for it's speed and stability.

1. Guess on a value of $\bar{\lambda}$. Compute $q(\lambda_{\min})$ using (41), (47), and (48).

2. Solve the bus-user's initial value problem for $x_b(\lambda)$ and $q_b(\lambda)$.

3. Compute $\bar{x}$ using (42).

4. Update guess on $\bar{\lambda}$ and iterate until $|\bar{x}-x_b(\bar{\lambda})|$ is less than desired tolerance.

5. To check the solution, solve the car owner's initial value problem with initial conditions:

$$\begin{aligned}x_c(\bar{\lambda}) &= \bar{x},\\ q_c(\bar{\lambda}) &= q_b(\bar{\lambda}),\end{aligned}$$

   for $x_c(\lambda)$ and $q_c(\lambda)$. Verify that $q_c(\lambda_{\max})$ 'equals' q_A.

References

[1] Clark, Colin. 1951. "Urban Population Densities." *Journal of the Royal Statistical Society. Series A.* V.114, No.4: 490-496.

[2] Cullen, Julie and Steven Levitt. 1999. "Crime, Urban Flight, and the Consequences for Cities." *Review of Economics and Statistics.* V.81, No.2: 159-169.

[3] Edmonston, Barry. 1975. *Population Distribution in American Cities.* D.C. Heath and Company.

[4] Glaeser, Edward, Matthew Kahn and Jordan Rappaport. 2000. "Why do the Poor Live in Cities?" *NBER working paper No. 7636.*

[5] Glaeser, Edward and Matthew Kahn. 2003. "Sprawl and Urban Growth." *NBER working paper No. 9733.*

[6] Gordon, Robert J. 1990. *The Measurement of Durable Goods Prices.* Chicago, IL: University of Chicago Press.

[7] Gottschalk, Peter, and Timothy M. Smeeding. 1997. "Cross-National Comparisons of Earnings and Income Inequality." *Journal of Economic Literature.* V.35, No.2: 633-687.

[8] Hartman, Philip. 1964. *Ordinary Differential Equations.* John Wiley & Sons, Inc.

[9] Jones, David W. 1985. *Urban Transit Policy: An Economic and Political History.* Englewood Cliffs, NJ: Prentice-Hall, Inc.

[10] Lebergott, Stanley. 1996. *Consumer Expenditures: New Measures and Old Motives.* Princeton, NJ: Princeton University Press.

[11] LeRoy, Stephen and Jon Sonstelie. 1983. "Paradise Lost and Regained: Transportation Innovation, Income, and Residential Location." *Journal of Urban Economics.* V.13, No.1: 67-89.

[12] Mieszkowski, Peter, and Edwin S. Mills. 1993. "The Causes of Metropolitan Suburbanization." *Journal of Economic Perspectives.* V.7, No. 3: 135-147.

[13] Margo, Robert A. 1992. "Explaining the Postwar Suburbanization of Population in the United States: The Role of Income." *Journal of Urban Economics.* V.31, pp. 301-310.

[14] McDonald, John F. 1989. "Econometric Studies of Urban Population Density: A Survey." *Journal of Urban Economics.* V.26, pp. 361-385.

[15] Mills, Edwin and Laun Sende Lubuele. 1997. "Inner Cities." *Journal of Economic Literature.* V.35, No.2: 727-756.

[16] Mills, Edwin and Jee Peng Tan. 1980. "A Comparison of Uran Population Density Functions in Developed and Developing Countries." *Urban Studies.* V.17, pp.33-321.

[17] Motor Vehicle Manufacturers Association (MVMA). 1978. Motor Vehicle Facts & Figures.

[18] Raff, Daniel M.G. and Manuel Trajtenberg. 1995. "Quality-Adjusted Prices for the American Automobile Industry: 1906–1940." *NBER Working Paper No. 5035.*

[19] Survey Research Center, University of Michigan. 1971. *Survey of Consumer Finances.*

[20] Taylor, George. 1996. "The Beginning of Mass Transportation in Urban America, Parts I and II." *Smithsonian Journal of History.* V.1: 35-50 and 51-54.

[21] Ulmer, Melville J. 1960. *Capital in Transportation, Communications, and Public Utilities: Its Formation and Financing.* Princeton, NJ: Princeton University Press.

[22] U.S. Department of Commerce, Economics and Statistics Administration, Bureau of Economic Analysis. *Fixed Reproducible Tangible Wealth in the United States, 1925-1994.*

[23] U.S. Federal Highway Administration. *Highway Statistics, annual.*

[24] Voith, Richard. 1999. "Does the Tax Treatment of Housing Create an Incentive for Exclusionary Zoning and Increased Decentralization?" *Federal Reserve Bank of Philadelphia Working Paper No. 99-22.*

[25] Ward, D. *Cities and Immigrants: A Geography of Change in Nineteenth-Century America.* 1971. New York, NY: Oxford University Press.

[26] *Ward's Automotive Yearbook.* 2002.

[27] Warner, S. B. Jr. 1962. *Streetcar Suburbs: The Process of Growth in Boston, 1870-1900.* Cambridge, MA: Harvard University Press.

[28] Williamson, Jeffrey J. 1995. "The Evolution of Global Labor Markets since 1830: Background Evidences and Hypotheses." *Explorations in Economic History.* V. 32 no. 2: 141–196.