Limit-Order Markets and Floor Exchanges: An Irrelevance Proposition

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Abstract

This paper compares limit-order markets and floor exchanges. Floor exchanges are modelled as markets in which liquidity providers compete to fill a buy or sell order after observing the size. The order is assumed to be transacted at a single price; thus floor exchanges are modelled as uniform-price markets. We allow for different order sizes and endogenize the flow of market orders, assuming risk-neutral competitive liquidity providers. Optimization by informed and uninformed market-order traders implies that the two markets are equivalent.

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The goal of this paper is to compare alternative market designs. Relative to the previous literature on this topic, our primary innovation is to analyze a dynamic model, in which traders can choose different order sizes and different frequencies of trading in response to the differences in the trading protocols. Our primary result, which is distinct from any in the previous literature, is an irrelevance proposition: under certain assumptions (the key assumption being perfect competition in liquidity provision) limit-order markets are equivalent in all important respects to uniform-price markets. We argue that our result is important for understanding the relation between limit-order markets and floor exchanges.

Floor exchanges can be and are organized in a variety of ways. Any floor exchange has numerous rules that are not precisely captured by our model. However, the essence of such an exchange is exposing each market order to the trading crowd, so that competition between the members of the crowd can produce the best available price for the order.[1] We assume the trading crowd consists of two or more risk-neutral liquidity providers, who maximize expected profits from trade. Bertrand competition between the liquidity providers implies that the price obtained by any market order is the expectation of the asset value conditional on past information and on the size of the order. Thus, our model of a floor exchange implies that each market order executes at a single price. Our model most closely characterizes exchanges such as the CBOE, in which the trading crowd consists in part of market makers trading on their own accounts. Of course, the risk neutrality assumption may be counterfactual, but the assumption that the trading crowd competes to fill orders after observing the size of each order seems reasonable for such exchanges.[2]

Both floor exchanges and limit-order markets typically provide quotes for small orders. The distinction we make between the two types of markets has to do with how large orders are handled. In a limit-order market, a large order will "walk up the book," transacting at multiple prices. A floor exchange may also include a limit order book (like the NYSE) but we assume that a large order on such an exchange is exposed to the trading crowd before walking up the book. Under our assumption about liquidity provision, adverse selection imposed by floor traders on the limit order book (as discussed in Rock (1990)) implies a wide quoted spread and the effective spread is determined by the floor traders, with each market order executing at a single price.[3]

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[1] For example, NYSE rules state that market orders request execution "at the most advantageous price obtainable after the order is represented in the Trading Crowd" (cited by Hasbrouck, Sofianos and Sosebee (1993)).

[2] It is more problematic to apply our results to the NYSE. The traders on the floor of the NYSE, other than the specialist, are floor brokers. While it is commplace and sensible to regard these members of the trading crowd as providing liquidity to market orders (for example, speaking recently of proposed rule changes, John Thain, the CEO of the NYSE, stated "Floor brokers and specialists will continue to provide liquidity and make markets ..."), they are executing orders from off-exchange traders, which should affect the prices at which they are willing to trade.

[3] It is interesting to note that, in the data analyzed by Balcidore, Ross and Sofianos (1999), 97% of NYSE market orders executed at a single price.

Limit-order traders face a well-understood adverse selection problem that stems from discriminatory pricing. By "discriminatory pricing," we mean as usual that each limit order transacts at its limit price (rather than at, for example, the marginal price paid/received by a market order that executes at multiple prices). If large market orders are more likely to be informed than small market orders, then transacting against large orders will be costly and these costs must be recovered on average from transactions against small orders. As a result, limit-order traders will not offer "fair" prices to small market orders. For example, Glosten (1994) shows that even with an infinite number of risk-neutral traders submitting limit orders, there will be a nonzero bid-ask spread for infinitesimally small (essentially uninformed) orders. In contrast, with risk-neutral competitive liquidity providers, prices in a uniform-price market are always efficient in the semi-strong-form sense. In particular, prices for small market orders would seem to be better in uniform-price markets than in discriminatory limit-order markets.

The essence of our irrelevance proposition is that the above intuition is incorrect. The reason it is incorrect is that the frequency of orders of different sizes will depend on the market structure. We show that large orders in limit-order markets will be "worked" in uniform-price markets, i.e., submitted as a series of small orders. Thus, the information that is present in large orders in limit-order markets will be manifested in small orders in uniform-price markets, leading to prices for small orders that are the same as the prices for small orders in a limit-order market. The penalty for small market orders inherent in a limit-order market is simply unavoidable — in a uniform-price market, it occurs not through the market design but through equilibrium behavior, specifically the pooling of small orders with large orders that are worked as a series of small orders. Or, to put this another way, the "penalty" is not really a penalty at all; it is an incentive compatibility condition necessary to separate large orders from small orders in a limit-order market.

There are certainly important distinctions between limit order markets and floor exchanges that our model does not capture. These include the extent to which orders must be exposed (and hence subject to front-running) and the anonymity of orders (and hence the adverse selection faced by traders). Our irrelevance proposition obviously does not imply that those features are irrelevant, but it provides a benchmark that might be gainfully employed to evaluate those features in future research.

Most of the literature on market microstructure assumes uniform pricing. In particular, this is true of all of the dynamic models of market microstructure of which we are aware. The Kyle (1985) model and all of its variations assume uniform pricing. Most of the literature that applies the Glosten-Milgrom (1985) framework assumes a single order size, so there is no distinction between uniform pricing and discriminatory pricing. Two notable exceptions that discuss optimal order sizes— Easley and O'Hara (1987) and Seppi (1990)— assume uniform pricing. Via the equivalence we establish between limit-order markets and uniform-price markets, it may be possible to apply many of the results of this literature to limit-order markets.

Important theoretical papers on limit-order markets include the following:

- Rock (1990) describes the adverse selection imposed by floor traders on limit-order traders.
- Glosten (1994) demonstrates the robustness of limit-order markets against competition from other markets, assuming perfect competition in liquidity provision.
- Bernhardt and Hughson (1997) show that strategic competition between a finite number of liquidity providers using limit orders must result in positive profits for liquidity providers.
- Seppi (1997) compares a specialist market (in which the specialist faces competition in liquidity provision from floor traders) to a limit order market, assuming perfect competition in liquidity provision.
- Parlour and Seppi (1998) extend the analysis of Seppi (1997) to analyze competition between exchanges.
- Ready (1999) extends Rock's (1990) analysis of the adverse selection imposed by a specialist on a limit-order book, assuming a single order size. His model is dynamic but limit-order traders move first and cannot kill orders that are unexecuted.[4]
- Biais, Martimort and Rochet (2000) extend Bernhardt and Hughson (1997), providing additional analysis of strategic competition between liquidity providers in limit-order markets.
- Viswanathan and Wang (2002) compare limit-order markets and uniform-price markets, assuming strategic competition between liquidity providers. In their model of a uniform-price market, a finite number of "dealers" submit demand-supply schedules before the size of the market order is known.
- Glosten (2003) compares limit-order markets and uniform-price markets, assuming perfect competition in liquidity provision. He endows market order traders with preferences and derives an equilibrium with optimizing market order traders as well as optimizing liquidity providers.

[4] In contrast, we assume limit order traders continuously monitor the market, killing and resubmitting orders instantaneously. The truth obviously lies somewhere between. Interesting evidence on this point is provided by Hasbrouck and Saar (2002), who show that more than one-fourth of the limit orders on the Island ECN are killed within two seconds or less, though at least some such orders may be from liquidity demanders rather than liquidity providers.

Of these papers, the ones most closely related to this paper are Seppi (1997), Viswanathan and Wang (2002) and Glosten (2003). Each of these compares limit-order markets to uniform-price markets. The key distinction between their analyses and the analysis in this paper is that we endogenize informed trading (and, to a certain extent, uninformed trading) in a dynamic model. Seppi (1997) and Viswanathan and Wang (2002) analyze the market structures at a point in time, taking the market order flow as given and assuming it is the same in both types of markets. Glosten (2003) points out that the market order flow should depend on the market structure, and he endogenizes it but still within a static model. Thus, the previous literature does not capture the option of working large orders as a series of small orders. Modelling this option is the main novelty of our paper and is at the heart of our irrelevance proposition. As mentioned before, we show that in a uniform-price market (abstracting from commissions) only small market orders should be observed in equilibrium. Moreover, the resulting market, in which only small trades are observed in equilibrium, is equivalent in all other respects to a limit-order market, in which multiple trade sizes are observed in equilibrium. To model the endogeneity of market orders in a dynamic model, we use a generalization of the model of Back and Baruch (2004), which is essentially a continuous-time Glosten-Milgrom (1985) model with endogenous informed trading.

We do not address the optimality of using market orders versus limit orders for traders that are motivated to trade either due to informational reasons or for liquidity reasons. Papers that address the choice between market orders and limit orders include Kumar and Seppi (1993), Chakravarty and Holden (1995), Handa and Schwartz (1996), Harris (1998), Parlour (1998), Foucault (1999), Foucalt, Kadan and Kandel (2003) and Goettler, Parlour and Rajan (2003, 2004). We also do not analyze competition between exchanges, which is considered by Glosten (1994), Parlour and Seppi (1998), Hendershott and Mendelson (2000, 2004), and Viswanathan and Wang (2002).

The plan of the paper is as follows. We describe the basic elements of the model and how we model the different market structures in the following section. Section II describes the updating rules of liquidity providers. Section III describes the optimization problem of the informed trader and in particular the first-order conditions for a mixed strategy to be optimal. The first-order conditions imply there must be more information in large market orders than in small orders. Section IV shows that it is strictly optimal for uninformed market-order traders "liquidity traders") to work large orders as a series of small orders in a uniform-price market. Section V presents the irrelevance proposition. Section VI concludes the paper.

I Model

Our model is perhaps the simplest possible model of endogenous informed trading with multiple order sizes. It is a generalization of the model of Back and Baruch (2004). We consider a continuous-time market for a risky asset and one risk-free asset with interest rate set to zero.[5] A public release of information takes place at a random time τ, distributed as an exponential random variable with parameter r. After the public announcement has been made, the value of the risky asset, denoted by $\tilde{v}$, will be either zero or one, and all positions are liquidated at that price. There is a single informed trader who knows $\tilde{v}$ at date 0. All trades are anonymous.

There is no minimum tick size—any real number is a feasible transaction price. There is an infinite number of risk-neutral limit-order traders, as in Rock (1990), Glosten (1994), and Seppi (1997). Market orders are submitted by the informed trader and by "liquidity" traders, whose behavior we do not model, except that we do not allow them to use dominated order-size strategies (this is the topic of Section V). The specialist is assumed to be risk neutral.

There are two possible order sizes $\Delta_s < \Delta_\ell$, which we call small and large. We assume $\Delta_\ell = 2\Delta_s$. As in Easley, Kiefer, O'Hara and Paperman (1996), uninformed buy and sell orders arrive as Poisson processes with constant arrival intensities. The arrival intensities for buy and sell orders are assumed for simplicity to be the same. For small orders, the arrival intensity of buys and sells is denoted by β_s. For large orders, the arrival intensity is denoted by β_ℓ.

As is discussed in Back and Baruch (2004), there cannot be a strategy equilibrium in this market. Instead, the informed trader must follow a mixed strategy, randomizing between trading (either buying or selling or both) and waiting at each instant. Formally, his equilibrium strategy will consist of counting processes (counting the number of buys/sells through each date) with random arrival dates, much like the Poisson processes assumed for the liquidity traders.

At a point in time, pricing in each of the two market types is as follows. We focus on the ask side, the bid side being symmetric. Let a_s denote the expected value of the asset conditional on a small buy order, a_ℓ the expected value conditional on a large buy order, and a the expected value conditional on a buy order but unconditional on the size. We will verify that $a_s < a < a_\ell$; i.e., large orders have more information content in equilibrium. Of course, the conditional expectatons depend on the trading strategies; hence, they will vary

[5] Obviously, a one-period model would be simpler, but such a model cannot capture the ability of a trader to transact a large quantity by submitting a series of small orders. This is an important issue in the choice of order size, and it requires a dynamic model. The standard dynamic models (Kyle (1985), Glosten and Milgrom (1985)) are inadequate for our purposes, the Kyle model because it imposes a uniform-price market, and the Glosten-Milgrom model because the choice of who gets to trade at each date is made randomly rather than being determined endogenously.

across the two market types. The transaction prices in the two markets will be as follows:

Limit-Order Market There will be a limit order of size Δ_s at price a and a limit order of size Δ_s at price a_ℓ. Small market buy orders will transact at price a and large market buy orders will transact at an average price of $(a + a_\ell)/2$.

Uniform-Price Auction Small market buy orders will transact at price a_s and large market buy orders will transact at price a_ℓ.

The fact that a (rather than a_s) is the best ask quote in the limit-order market is analogous to (though obviously simpler than) Glosten's (1994) result that there is a nontrivial spread even for arbitrarily small orders. A limit order at a price below a would lose money on average, because it transacts against both small and large market orders, meaning that the expected value conditional on execution is a. Competition between limit-order traders enforces a zero-expected-profit condition, implying that the book is as described.

Based on this description of the markets, one might conclude that the uniform-price market offers the best execution for small orders and that large orders receive the best execution in the limit-order market. However, this would neglect the fact that the expected values a_s, a and a_ℓ should vary depending on the market design, because they depend on the relative probabilities of informed and uninformed orders. We will address this by solving for the optimal strategy of the informed trader in each market.

II Dynamics of Conditional Expectations

We will demonstrate that there is an equilibrium in each market type in which the conditional expectation of the asset value is a Markov process. Thus, the expectation at any date t is a sufficient statistic for the information in the order flow up to and including date t. We let m_t denote the conditional expectation at date t and as is customary define $m_{t-} = \lim_{s \uparrow t} m_s$. The interpretation of m_{t-} is that it is the conditional expectation just before observing whether an order is submitted at date t. This will be the state variable for the Bayesian updating of liquidity providers and the optimization of the informed trader at date t. Of course, because the asset value is either zero or one, m_t also denotes the conditional probability that the asset value is one.

We will allow for the possibility that the high-type informed trader may want to sell the asset and/or the low type buy the asset for strategic reasons. We define θ_{TDQ} to be the intensity of informed orders of type $T \in \{H, L\}$ in direction $D \in \{B, S\}$ and of quantity $Q \in \{s, \ell\}$, where H denotes the high-type informed trader and L the low type, and B stands for "buy" and S for "sell." We will show that in equilibrium each intensity at date t is a function of m_{t-}. Given $m_{t-} = m$, the probability, conditional on the information in the prior order flow, of an order in direction $D \in \{S, B\}$ of quantity $Q \in \{s, \ell\}$ in an instant

dt is therefore $\pi_{DQ}(m)\,dt$, where we define

$$\pi_{DQ}(m) = m\theta_{HDQ}(m) + (1-m)\theta_{LDQ}(m) + \beta_Q. \tag{1}$$

A simple Bayes rule calculation (provided in the appendix) yields the following:

Proposition 1. *The expected value of the asset conditional on a buy order of size $Q \in \{s, \ell\}$ at date t and given $m_{t-} = m$ is*

$$a_Q(m) = \frac{m\theta_{HBQ}(m) + m\beta_Q}{\pi_{BQ}(m)}. \tag{2}$$

Similarly, the expected value conditional on a sell order of size Q is

$$b_Q(m) = \frac{m\theta_{HSQ}(m) + m\beta_Q}{\pi_{SQ}(m)}. \tag{3}$$

Conditional on a buy order, but unconditional on the order size, the expected value of the asset is

$$a(m) = \frac{\pi_{Bs}(m)a_s(m) + \pi_{B\ell}(m)a_\ell(m)}{\pi_{Bs}(m) + \pi_{B\ell}(m)}. \tag{4}$$

Likewise, the expected value conditional on a sell order but unconditional on the size is

$$b(m) = \frac{\pi_{Ss}(m)b_s(m) + \pi_{S\ell}(m)b_\ell(m)}{\pi_{Ss}(m) + \pi_{S\ell}(m)}. \tag{5}$$

To describe the evolution of the conditional expectation m_t through time, we need to introduce notation for the counting processes of buys and sells. The counting processes for liquidity trades are denoted by Z, the counting processes for informed trades by X, and the counting process for total trades by Y ($Y = X + Z$). We use subscripts BQ to denote counting processes for buy orders of size $Q \in \{s, \ell\}$ and SQ to denote counting processes for sell orders of size Q. Thus, for example, Z_{BQt} denotes the total number of buy orders of size Q by liquidity traders through time t. It jumps up by one each time a liquidity buy order of size Q arrives. We are assuming that Z_{BQ} and Z_{SQ} are Poisson processes with intensities β_Q. Likewise, X_{BQ} counts the buy orders of size Q from the informed trader, and $Y_{BQ} = X_{BQ} + Z_{BQ}$ counts the total buy orders of size Q. Without loss of generality (given risk neutrality), we assume the informed trader has no initial position in the risky asset, so $X_{st}\Delta_s + X_{\ell t}\Delta_\ell$ denotes the number of shares the informed trader owns at date t.

We define $Z_Q = Z_{BQ} - Z_{SQ}$, $X_Q = X_{BQ} - X_{SQ}$, and $Y_Q = Y_{BQ} - Y_{SQ}$. For example, the process Y_Q jumps up by one when any buy order of size Q arrives and jumps down by one when any sell order of size Q arrives. The process $Y = (Y_s, Y_\ell)$ reveals the complete history of anonymous trades.

The conditional expectation m_t will perhaps evolve between transactions and will certainly jump up or down when an order arrives. We can write its dynamics as

$$dm_t = f(m_{t-})\,dt + \sum_{Q\in\{s,\ell\}} [a_Q(m_{t-}) - m_{t-}]\,dY_{BQt} + \sum_{Q\in\{s,\ell\}} [b_Q(m_{t-}) - m_{t-}]\,dY_{SQt}. \tag{6}$$

This simply means that the conditional expectation jumps up to a_Q when there is a buy order of size Q (so $dY_{BQt} = 1$) and jumps down to b_Q when there is a sell order of size Q (when $dY_{SQt} = 1$) and that between transactions it evolves as $dm_t = f(m_{t-})\,dt$. We take 0 and 1 to be absorbing points for m, because further information cannot change beliefs that put probability one on the asset value being low or probability one on the asset value being high.

Proposition 2. *In equilibrium,*

$$f(m) = m(1-m) \sum_{Q\in\{s,\ell\}} [\theta_{LBQ}(m) + \theta_{LSQ}(m) - \theta_{HBQ}(m) - \theta_{HSQ}(m)]. \tag{7}$$

PROOF: See the appendix.

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III Equilibrium Informed Trading

The intensities θ_{TDQ} in the definitions of a_Q, b_Q and f represent the intensities conjectured by the liquidity providers to be played by the informed trader in equilibrium. The actual trading strategy chosen by the informed trader cannot affect the functions a_Q, b_Q and f. So, the informed trader takes them as given and maximizes his expected profit subject to the dynamics (6) for the state variable m_t. This means that he understands a buy order of size Q will cause the conditional expectation to jump up to a_Q and a sell order of size Q will cause the conditional expectation to jump down to b_Q.

In each market, the transaction price for a buy order of size Q is a particular function of $a_s(m)$, $a(m)$ and $a_\ell(m)$ as described in Section I. We let $p_{BQ}(m)$ denote this transaction price. Similarly, we let $p_{SQ}(m)$ denote the transaction price for a sell order of size Q, which will be a function of $b_s(m)$, $b(m)$ and $b_\ell(m)$ defined symmetrically.

The informed trader chooses a trading strategy X to maximize

$$E\int_0^\tau \sum_{Q\in\{s,\ell\}} \left\{[\tilde{v} - p_{BQ}(m_{t-})]\Delta_Q\,dX_{BQt} + [p_{SQ}(m_{t-}) - \tilde{v}]\Delta_Q\,dX_{SQt}\right\} \tag{8}$$

conditional on knowing the value $\tilde{v}$ of the asset. The integral

$$\int_0^\tau [\tilde{v} - p_{BQ}(m_{t-})]\Delta_Q\,dX_{BQt} \tag{9}$$

is the sum until the announcement date τ of the profit per trade $[\tilde{v} - p_{BQ}(m_{t-})]\Delta_Q$ on each buy order of size Q, summed over the various dates at which the informed trader submits buy orders of size Q (the dates at which $dX_{BQt} = 1$). Likewise, the integral

$$\int_0^\tau [p_{SQ}(m_{t-}) - \tilde{v}]\Delta_Q \, dX_{SQt} \tag{10}$$

is the total profits earned from sell orders of size Q.

In this maximization problem, we allow the informed trader to choose arbitrary counting processes. However, as remarked before, we will show that there is an equilibrium in which the counting processes have intensities θ_{TDQ} that are functions of m_{t-}. By this we mean that for each direction $D \in \{B, S\}$, the stochastic process

$$X_{DQu} - \tilde{v} \int_0^u \theta_{HDQ}(m_{t-}) \, dt - (1 - \tilde{v}) \int_0^u \theta_{LDQ}(m_{t-}) \, dt \tag{11}$$

is a martingale, relative to the informed trader's information. Thus, for example, when $\tilde{v} = 1$,

$$\text{prob}(dX_{DQt} = 1) = E[dX_{DQt}] = \theta_{HDQ}(m_{t-}) \, dt. \tag{12}$$

This probability is known both to the informed trader and the liquidity providers. Thus, even given the informed trader's information and the information in the order flow, the decision by the informed trader to trade is made randomly, as in any mixed strategy equilibrium.

The informed trader must play a mixed strategy in equilibrium. To see this, suppose to the contrary that the equilibrium strategy of the high-type trader calls for him to buy an amount Q of the asset at a known date t (knowable from the history of trades to that date). Then the high-type trader should refrain from the purchase, causing the market to presume that the information is low and to quote a price of 0 thereafter and leading hence to infinite profits for the informed trader. Moreover, the high-type trader must buy both quantities with positive probability for each value m of m_{t-} and the low-type trader must sell both quantities with positive probability. For example, if the high-type trader's equilibrium strategy were to avoid small buys at some value of m_{t-}, then there would be no price impact associated with small buys at that value of m_{t-}, and hence the trader could obtain unbounded profits from submitting a large number of small buy orders, one instantaneously after the other.

We therefore search for a mixed-strategy equilibrium in which $\theta_{HBQ}(m) > 0$ and $\theta_{LSQ}(m) > 0$ for each $m \in (0, 1)$ and each $Q \in \{s, \ell\}$. The first-order conditions for such a strategy to be optimal are straightforward. Because the objective function (8) is stationary, we can define the value at any date t as a function of the state variable m_{t-}. Let $V_T(m)$ denote the value function for the trader of type T. An equilibrium in which the high type submits buy orders of both sizes with positive probability and the low type submits sell orders of both sizes with positive probabilities must satisfy the following conditions for each $m \in (0,1)$ and each $Q \in \{s, \ell\}$:

$$V_H(m) = [1 - p_{BQ}(m)]\Delta_Q + V_H(a_Q(m)), \tag{13}$$

$$V_H(m) \geq [p_{SQ}(m) - 1]\Delta_Q + V_H(b_Q(m)), \text{ with equality when } \theta_{HSQ} > 0, \tag{14}$$

$$V_L(m) = p_{SQ}(m)\Delta_Q + V_L(b_Q(m)), \tag{15}$$

$$V_L(m) \geq -p_{BQ}(m)\Delta_Q + V_L(a_Q(m)), \text{ with equality when } \theta_{LBQ} > 0. \tag{16}$$

The first condition means that the optimal value for the high type can be realized by submitting a buy order of size Q. The effect of submitting such an order is an instantaneous profit[6] of $[1 - p_{BQ}(m)]\Delta_Q$ and a continuation value of $V_H(a_Q(m))$. The second condition means that submitting a sell order of size Q is not a strictly superior strategy for the high type and that it must be an optimal strategy when such an order is submitted with positive probability. The third and fourth conditions have analogous interpretations for the low type.

To have an equilibrium in which the informed trading strategy is characterized by intensities θ_{TDQ}, it must also be optimal for the informed trader to refrain from trading at each time. If he does so, then during an instant dt the announcement will occur with probability $r\,dt$, and, if the announcement occurs, the value function becomes 0. An uninformed buy order of size Q will arrive with probability $\beta_Q\,dt$, in which case m will jump to $a_Q(m_{t-})$ and the value function will jump (up or down) to $V_T(a_Q(m_{t-}))$. Similarly, with probability $\beta_Q\,dt$ an uninformed sell order of size Q will arrive and the value function will jump to $V_T(b_Q(m_{t-}))$. Finally in the absence of an announcement or an order, m will change by $f(m_{t-})\,dt$ and the value function will change by $V_T'(m_{t-})f(m_{t-})\,dt$. For it to be optimal for the informed trader to refrain from trading, all of these expected changes in the value function must cancel, which means that for each $m = m_{t-} \in (0,1)$ we must have

$$rV_T(m) = V_T'(m)f(m) + \sum_{Q\in\{s,\ell\}} \beta_Q\left[V_T(a_Q(m)) - V_T(m)\right] + \beta_Q\left[V_T(b_Q(m)) - V_T(m)\right]. \tag{17}$$

The natural boundary conditions are

$$V_L(0) = V_H(1) = 0 \quad \text{and} \quad V_L(1) = V_J(0) = \infty. \tag{18}$$

These conditions mean that each type would earn zero future profit if his type were detected and could earn infinite profit if the market were certain his type were the opposite of what it is.

[6]The profit is actually realized at the announcement date τ but we have normalized the interest rate to zero.

Intuitively, the conditions (2)–(7), and (13)–(18) are necessary for an equilibrium. The following, which is Theorem 2 of Back-Baruch (2004) applied to the present model, states that they are also sufficient conditions.[7]

Theorem 1. *Let a_Q, b_Q, a, b, f, V_T, θ_{TDQ} and the process m satisfy* (2)–(7)*, and* (13)–(18)*. Assume V_H is a nonincreasing function and V_L is a nondecreasing function. Assume the functions $m \mapsto \theta_{TDQ}(m)$ are continuously differentiable on $(0,1)$. Then the informed trading strategy is optimal and, for each t, m_t is the expectation of $\tilde{v}$ conditional on observing $Y = (Y_s, Y_\ell)$ through date t.*

One consequence of the first-order conditions (13)–(16) is that, in each market, large orders must have a greater impact on beliefs than small orders.

Proposition 3. *If V_H is a nonincreasing function and V_L is a nondecreasing function, and the first-order conditions* (13)–(16) *are satisfied, then it must be the case that there is more information content in large orders than in small orders; i.e., $a_\ell(m) > a_s(m)$ and $b_\ell(m) < b_s(m)$.*

IV Working Large Orders

Even though we do not model the behavior of liquidity traders, it is important to recognize that a trader with a large liquidity shock has the discretion to work his or her desired trade via a series of small orders. Recall that the uniform-price market appears to give better execution to small orders (small buys execute at a_s in the uniform-price market and at a in the limit-order market) and that it appears to give worse execution to large orders (large buys execute at a_ℓ in the uniform-price market and at a weighted average of a and a_ℓ in the limit-order market). Therefore, if there is an advantage in either market to submitting multiple small orders rather than a single large order, one would expect that this would occur in the uniform-price market. This is precisely the case: we can show that it is better for liquidity traders to submit two small orders in the uniform-price market but it is better for liquidity traders to submit a single large order in the limit-order market. In both cases, the result is a consequence of the fact that the informed trader must be indifferent about the order size he uses, because he must randomize over both sizes. The distinction between the preferences of the informed trader and the preferences of liquidity traders is that the informed trader cares about the impact of trades on beliefs, because they determine the remaining profit he can obtain, whereas liquidity traders care only about the execution cost.

[7] We have omitted here a very mild technical condition. To ensure expected profits are well-defined, a strategy is defined to be admissible in Back-Baruch (2004) if it does not incur infinite expected losses. We need to add to Theorem 1 the assumption that the trading strategy is admissible. The proof of the theorem follows the same lines as the proof of Theorem 2 in Back-Baruch (2004), so it is omitted.

In both markets submission of a small buy order changes the expected value of the asset to $a_s(m)$. The transaction price on a second small buy order can be made arbitrarily close to $a_s(a_s(m))$ in the uniform-price market and arbitrarily close to $a(a_s(m))$ in the limit-order market, with probability arbitrarily close to one, by submitting it a very short period of time after the first. Working a large buy order is therefore optimal for liquidity traders in the uniform-price market if

$$a_s(m)\Delta_s + a_s(a_s(m))\Delta_s < a_\ell(m)\Delta_\ell. \tag{19}$$

It is optimal in the limit-order market if

$$a(m)\Delta_s + a(a_s(m))\Delta_s < a(m)\Delta_s + a_\ell(m)[\Delta_\ell - \Delta_s],$$

or, equivalently,

$$a(a_s(m)) < a_\ell(m). \tag{20}$$

Inequality (19) is true, but inequality (20) is not. These facts, established in the following propositions, are the key technical results of the paper, because they lead immediately to the irrelevance proposition.

Proposition 4. *Consider an equilibrium of a uniform-price market. Assume V_H is a decreasing function, V_L is an increasing function, and the first-order conditions* (13)–(16) *are satisfied. Then submitting a pair of small orders is strictly better for liquidity traders than submitting a large order; i.e., for each $m \in (0,1)$,*

$$a_s(m)\Delta_s + a_s(a_s(m))\Delta_s < a_\ell(m)\Delta_\ell, \tag{21}$$

and

$$b_s(m)\Delta_s + b_s(b_s(m))\Delta_s > b_\ell(m)\Delta_\ell. \tag{22}$$

Proposition 5. *Consider an equilibrium of a limit-order market. Assume V_H is a decreasing function, V_L is an increasing function, and the first-order conditions* (13)–(16) *are satisfied. Then submitting a large order is strictly better for liquidity traders than submitting a pair of small orders one instantaneously after the other; i.e., for each $m \in (0,1)$, $a_\ell(m) < a(a_s(m))$ and $b_\ell(m) > b(b_s(m))$.*

V The Irrelevance Result

We have not ascribed particular preferences to the liquidity traders; however, we view them as caring about speed and cost. The results of the previous section show that there is no equilibrium in a uniform-price market in which liquidity traders submit large orders: as long as they care at all about cost, they will find submitting two small orders close together to be preferable to submitting a single large order.

The irrelevance proposition is actually independent of the preferences of liquidity traders, provided they care about speed and cost.[8] This is because the two markets are essentially strategically equivalent. This equivalence is a result of assuming that trading occurs in continuous time. Because we are working in continuous time, there is no fixed amount of time that must elapse between submission of two successive orders; thus, we will allow orders to be submitted instantaneously one after the other—i.e., with zero elapsed time. Mathematically, this means introducing a third type of order, a worked large order, consisting of two small orders submitted by the same trader, arriving at the same time but executed at different prices.

When a small order arrives, liquidity providers will need to estimate the likelihood that it is parted of a worked large order rather than a small order. If orders are worked by submitting one instantaneously after the other, this estimation is straightforward. With zero time elapsing between the two parts of a worked order, there is zero probability of another order arriving between the two parts. Thus, the second part of a worked order will be identified as such with probability one. Likewise, if a small order is not followed instantaneously by a second, then it will be identified as a "true" small order. However, we assume that this identification is too late to affect its transaction price.

Suppose now that small liquidity trades arrive with intensity β_s and worked liquidity trades arrive with intensity β_ℓ in a uniform-price market. Suppose further that the updating of beliefs when a large order arrives is the same as when a worked order arrives. Then both liquidity traders and the informed trader will prefer to submit worked orders, because the transaction prices are better and the effects on beliefs are the same. A small order in a uniform-price market will transact at a weighted average of the conditional expectations given small and large orders, because it could be either a small order or the first part of a worked order. This is exactly the same as in a limit-order market—the transaction price for a small order is a weighted average of the conditional expectations, because the limit order could execute against either a small or a large market order. Furthermore, the average price of a worked order in a uniform-price market is determined in exactly the same manner as the average price of a large order in a limit-order market. Thus, the "game" between

[8] We have to assume they care about speed, because otherwise it would be optimal in each of our models to wait until the announcement date to trade, given that trades have no price impacts after the announcement. More generally, the numerical results indicate that the transaction prices for buy orders are concave functions and the transaction prices for sell orders are convex functions of the conditional expectation m_{t-}. By Jensen's inequality and the fact that conditional expectations are martingales, this implies that transaction prices for buy orders are supermartingales, decreasing on average, and transaction prices for sell orders are submartingales, increasing on average. Thus, delaying submission of a market order will always improve the expected transaction price.

the informed trader and liquidity providers is the same in a uniform-price market as in a limit-order market. Furthermore an individual liquidity trader has the same alternatives in the two markets, given the assumption that other liquidity traders work large orders as we have described. Thus, the equivalence does not depend on the particular preferences of liquidity traders.

The two assumptions in the previous paragraph concern the arrival rates of liquidity trades and the updating of beliefs for large and worked orders. Given the latter assumption, the former seems justified, because the options available to liquidity traders from small and worked orders in a uniform-price market are the same as from small and large orders in a limit-order market; thus, the arrival intensities should be the same. Moreover, we have already established the following: given any equilibrium of a limit-order market, there is an equivalent equilibrium in a uniform-price market involving small and worked orders. In this equilibrium, large orders are out-of-equilibrium events. We need to specify beliefs conditional on these out-of-equilibrium events to specify the equilibrium fully. There are many beliefs that will support the equilibrium: if large orders have the same or larger impacts on beliefs as worked orders, then it is optimal for traders to use only small and worked orders, implementing the equilibrium of the limit-order market.

To establish the converse, we need to show that any equilibrium of a uniform-price market—in which the informed trader has the option to submit small, worked, and large orders—will involve only small and worked orders. This we will now do.

Based on the analogy to limit-order markets, we let $a(m)$ denote the expected value of the asset conditional on receipt of a small buy order in a uniform-price market but without conditioning on whether it is followed by a second. We let $a_s(m)$ denote the conditional expectation if a second order does not immediately follow, and we let $a_w(m)$ denote the conditional expectation if a second order does immediately follow. Thus, a small buy order or the first part of a worked buy order transacts at $a(m)$, and the second part of a worked buy order transacts at $a_w(m)$. Likewise, a small sell order or the first part of a worked sell order transacts at $b(m)$, and the second part of a worked sell order transacts at $b_w(m)$.

The first-order conditions (13)–(14) for the high-type informed trader in this model are:[9]

$$V_H(m) = [1 - a(m)]\Delta_s + V_H(a_s(m)), \tag{23}$$

$$V_H(m) \geq 2[1 - a_\ell(m)]\Delta_s + V_H(a_\ell(m)) \text{ with equality when } \theta_{HB\ell} > 0, \tag{24}$$

$$V_H(m) = [2 - a(m) - a_w(m)]\Delta_s + V_H(a_w(m)), \tag{25}$$

$$V_H(m) \geq [b(m) - 1]\Delta_s + V_H(b_s(m)), \text{ with equality when } \theta_{HSs} > 0, \tag{26}$$

$$V_H(m) \geq 2[b_\ell(m) - 1]\Delta_s + V_H(b_\ell(m)), \text{ with equality when } \theta_{HS\ell} > 0, \tag{27}$$

$$V_H(m) \geq [b(m) + b_w(m) - 2]\Delta_s + V_H(b_w(m)), \text{ with equality when } \theta_{HSw} > 0. \tag{28}$$

Because liquidity traders do not use large orders in this market, it is intuitive that the informed trader will not either. Such orders would be fully revealing unless they are used by both types (i.e., there is bluffing with large orders) and we can prove that this is not the case. Moreover, there must be more information content in worked orders than in small orders.

Proposition 6. *Assume small liquidity orders arrive with intensity β_s and worked liquidity orders arrive with intensity β_ℓ in a uniform-price market. Assume the first-order conditions* (23)–(28) *and analogous conditions for the low type are satisfied and the value functions are monotone (V_H nonincreasing and V_L nondecreasing). Then for each $m \in (0,1)$, $a_w(m) > a_s(m)$, $b_w(m) < b_s(m)$, and the informed trader does not submit large orders (he submits only small and worked orders).*

Hence, the relevant first-order conditions for the high-type trader are (23), (25)–(26) and (28). These are identical to the first-order conditions in a limit-order market. Furthermore, the fact that only small and worked orders are used in equilibrium means that any equilibrium can be implemented with small and large orders in a limit-order market.

Theorem 2. *Assume small liquidity orders arrive with intensity β_s and worked liquidity orders arrive with intensity β_ℓ in a uniform-price market. For each equilibrium in a limit-order market, there is an equivalent equilibrium in a uniform-price market. Conversely, for each equilibrium in a uniform-price market, there is an equivalent equilibrium in a limit-order market.*

VI Conclusion

We have shown that limit-order markets, which are discriminatory markets and hence impose adverse selection on limit orders, to the apparent detriment of small market orders,

[9] We require equality in the conditions for small and worked buy orders, for the same reason that we required equality in condition (13) for small and large orders in the previous section. Both small and worked buy orders must be used by the high-type informed trader in equilibrium (and small and worked sell orders by the low type), because, if they were not, then there would be no price impact and no belief revisions from such orders, which implies that the informed trader could make infinite profits by using them.

are equivalent to uniform-price markets. In uniform-price markets, large orders should be worked as a series of small orders. This pooling of order types in uniform-price markets leads to the same transaction prices as occur in limit-order markets. A floor exchange in which floor traders trade on their own accounts and compete to provide liquidity is an example of a uniform-price market.

We have assumed that limit orders are used exclusively by providers of liquidity and market orders exclusively by demanders of liquidity. The choice between market and limit orders for demanders of liquidity is a complicated problem, but attempting to extend our results to allow such a choice is another interesting direction for future research. The working of orders by floor brokers is an even more complicated issue that deserves further study.

Finally, we have not explored the issue of competition between exchanges. This has been studied in other papers with static models, but, as we have shown, the ability of traders to work large orders in a dynamic way can be critical to understanding the properties of the various market types.

A Proofs

Proof of Proposition 1

As noted before the proposition, the probability of a buy order of size Q arriving in an instant dt is

$$\pi_{DQ}(m)\,dt = m\theta_{HBQ}(m)\,dt + (1-m)\theta_{LBQ}(m)\,dt + \beta_Q\,dt, \tag{29}$$

where the three terms refer to the three possible sources of an order: the high-type informed trader, the low-type informed trader, and uninformed traders. The conditional expectation is the sum of the value-weighted conditional probabilities of the order coming from each of the three possible sources, namely,

$$\frac{m\theta_{HBQ}(m)}{\pi_{DQ}(m)} \times 1 + \frac{(1-m)\theta_{LBQ}(m)}{\pi_{DQ}(m)} \times 0 + \frac{\beta_Q}{\pi_{DQ}(m)} \times m. \tag{30}$$

This simplifies to equation (2). The calculation for equation (3) is similar, and these imply equations (4) and (5).

Proof of Proposition 2

Considering only the jumps to a_Q or b_Q, the expected change in m in an instant dt given the information in the order flow and given $m_{t-} = m$ is

$$\begin{aligned}
&\sum_{Q\in\{s,\ell\}} (a_Q(m)-m)\big[m\theta_{HBQ} + (1-m)\theta_{LBQ} + \beta_Q\big]\,dt \\
&\qquad + \sum_{Q\in\{s,\ell\}} (b_Q(p)-m)\big[m\theta_{HSQ} + (1-m)\theta_{LSQ} + \beta_Q\big]\,dt \\
&\qquad\qquad = m(1-m) \sum_{Q\in\{s,\ell\}} \big[\theta_{HBQ} + \theta_{HSQ} - \theta_{LBQ} - \theta_{LSQ}\big]\,dt.
\end{aligned}$$

This equation is a consequence of the formulas (2) and (3). The process m is a conditional expectation, hence a martingale, so this expected change must be canceled by the expected change in m between transactions. This implies the formula in the lemma.

Proof of Proposition 3

Consider first the specialist market. From (13), we have

$$\begin{aligned}
V_H(m) &= [1-a_\ell(m)]\Delta_s + V_H(a_s(m)), \\
V_H(m) &= [1-a_\ell(m)]\Delta_\ell + V_H(a_\ell(m)).
\end{aligned}$$

Therefore $V_H(a_\ell(m)) < V_H(a_s(m))$, which by the monotonicity of V_H implies $a_\ell(m) > a_s(m)$.

Now consider the uniform-price market. For this market, (13) is equivalent to

$$V_H(m) = [1 - a_s(m)]\Delta_s + V_H(a_s(m)),$$
$$V_H(m) = [1 - a_\ell(m)]\Delta_\ell + V_H(a_\ell(m)).$$

Given $\Delta_\ell = 2\Delta_s$, this implies

$$V_H(a_s(m)) - V_H(a_\ell(m)) = [1 - 2a_\ell(m) + a_s(m)]\Delta_s. \tag{31}$$

If this is positive, then the monotonicity of V_H implies again that $a_\ell(m) > a_s(m)$. Otherwise, we have

$$1 - 2a_\ell(m) + a_s(m) \le 0, \tag{32}$$

which implies

$$a_\ell(m) - a_s(m) \ge 1 - a_\ell(m) > 0. \tag{33}$$

The argument for the limit-order market is essentially the same. Applying (13) and using the fact that $p_{Bs} = a$ in this market, we obtain

$$V_H(a(m)) - V_H(a_\ell(m)) = [1 - 2a_\ell(m) - a_s(m)]\Delta_s. \tag{34}$$

If this is positive, then we have $a_\ell(m) > a(m)$ by the monotonicity of V_H. Since a is a weighted average of a_s and a_ℓ, this implies $a_\ell(m) > a_s(m)$. On the other hand, if it is nonpositive, then we have

$$a_\ell(m) - a_s(m) \ge 1 - a_\ell(m) > 0. \tag{35}$$

Proof of Proposition 4

We will prove the claim for the ask prices, the bid side being symmetric. From Proposition 3, we have $a_s(m) < a_\ell(m)$. So the claim is certainly true if $a_s(a_s(m)) \le a_\ell(m)$. Suppose the contrary; i.e., suppose $a_s(a_s(m)) > a_\ell(m)$. In this market, we have (by repeated application of (13))

$$\begin{aligned} V_H(m) &= [1 - a_s(m)]\Delta_s + V_H(a_s(m)) \\ &= [1 - a_s(m)]\Delta_s + [1 - a_s(a_s(m))]\Delta_s + V_H(a_s(a_s(m))) \end{aligned}$$

and (by $\Delta_\ell = 2\Delta_s$)

$$\begin{aligned} V_H(m) &= [1 - a_\ell(m)]\Delta_\ell + V_H(a_\ell(m)) \\ &= 2[1 - a_\ell(m)]\Delta_s + V_H(a_\ell(m)). \end{aligned}$$

Therefore

$$V_H(a_s(a_s(m))) - V_H(a_\ell(m)) = [a_s(m) + a_s(a_s(m)) - 2a_\ell(m)]\Delta_s. \tag{36}$$

The supposition $a_s(a_s(m)) > a_\ell(m)$ implies the left-hand side is nonpositive, by virtue of the fact that V_H is a nonincreasing function. Hence, the right-hand side is nonpositive, which is condition (21). Furthermore, if V_H is strictly monotone, then the right-hand side must be negative.

Proof of Proposition 5

We will provide the proof for the buy side. Recall that the execution price of a small buy order is $p_{Bs} = a$ and the execution price for a large buy order is

$$p_{B\ell} = \frac{a\Delta_s + a_\ell(\Delta_\ell - \Delta_s)}{\Delta_\ell} = \frac{a + a_\ell}{2} \tag{37}$$

when $\Delta_\ell = 2\Delta_s$. From (13) we have

$$\begin{aligned} V_H(m) &= \left[1 - \frac{a(m) + a_\ell(m)}{2}\right]\Delta_\ell + V_H(a_\ell(m)) \\ &= [2 - a(m) - a_\ell(m)]\Delta_s + V_H(a_\ell(m)), \end{aligned}$$

and by repeated application of (13) we obtain

$$\begin{aligned} V_H(m) &= [1 - a(m)]\Delta_s + V_H(a_s(m)) \\ &= [1 - a(m)]\Delta_s + [1 - a(a_s(m))]\Delta_s + V_H(a_s(a_s(m))) \\ &= [2 - a(m) - a(a_s(m))]\Delta_s + V_H(a_s(a_s(m))). \end{aligned}$$

Therefore

$$-a_\ell(m)\Delta_s + V_H(a_\ell(m)) = -a(a_s(m))\Delta_s + V_H(a_s(a_s(m))). \tag{38}$$

If $a_\ell(m) \leq a_s(a_s(m))$, then by Proposition 3, we have $a_\ell(m) < a(a_s(m))$ as claimed, so suppose that $a_\ell(m) > a_s(a_s(m)$. Then the monotonicity of V_H implies $V_H(a_\ell(m) < V_H(a_s(a_s(m)))$. Hence, we must have

$$-a_\ell(m)\Delta_s > -a(a_s(m))\Delta_s, \tag{39}$$

which yields $a_\ell(m) < a(a_s(m))$ as claimed.

Proof of Proposition 6

First we will show that there is more information content in worked orders. Consider buy orders. From the first-order conditions(23) and (25) we obtain

$$V_H(a_s(m)) - V_H(a_w(m)) = [1 - a_w(m)]\Delta_s > 0. \tag{40}$$

which implies $a_s(m) < a_w(m)$ by the monotonicity of V_H.

Now we will show that the informed trader does not use large orders. Large orders would be fully revealing in equilibrium unless they are used by both types. Suppose that both types use large buy orders. We will derive a contradiction. When $\theta_{HB\ell} > 0$ and $\theta_{LB\ell} > 0$, we have from (24) and (25) that

$$V_H(a_\ell) - V_H(a_w) = (2a_\ell - a_s - a_w)\Delta_s, \tag{41}$$

and from the conditions for the low type analogous to (27) and (28) that

$$V_L(a_\ell) - V_L(a_w) \geq (2a_\ell - a_s - a_w)\Delta_s. \tag{42}$$

Thus,

$$V_L(a_\ell) - V_L(a_w) \geq V_H(a_\ell) - V_H(a_w). \tag{43}$$

Since V_L is nondecreasing and V_H is nonincreasing, this implies $a_\ell \geq a_w$. But then equation (41) implies $2a_\ell - a_s - a_w < 0$ which in conjunction with $a_w \leq a_\ell$ yields $a_w - a_s < 0$, contradicting the first part of the proposition.

REFERENCES

Back, K. and S. Baruch, 2004, Information in securities markets: Kyle meets Glosten and Milgrom, *Econometrica* 72, 433–465.

Bernhardt, D. and E. Hughson, 1997, Splitting orders, *Review of Financial Studies* 10, 69–101.

Chakravarty, S., and C. Holden, 1995, An integrated model of market and limit orders, *Journal of Financial Intermediation* 4, 213–241.

Easley, D., Kiefer, N., O'Hara, M., and J. Paperman, 1996, Liquidity, information, and infrequently traded stocks, *Journal of Finance* 51, 1405-1436.

Easley, D., and M. O'Hara, 1987, Price, trade size, and information in securities markets, *Journal of Financial Economics* 19, 69–90.

Easley, D., and M. O'Hara, 1992, Time and the process of security price adjustment, *Journal of Finance* 47, 577–606.

Foucault, T., 1999, Order flow composition and trading costs in a dynamic limit order market, *Journal of Financial Markets* 2, 99–134.

Foucault, T., Kadan, O., and E. Kandel, 2003, Limit order book as a market for liquidity, working paper.

Glosten, L., 1994, Is the electronic limit order book inevitable? *Journal of Finance* 49, 1127–1161.

Glosten, L., 2003, Discriminatory limit order books, uniform price clearing and optimality, working paper.

Glosten, L. and P. Milgrom, 1985, Bid, ask, and transaction prices in a specialist market with heterogeneously informed traders, *Journal of Financial Economics* 13, 71–100.

Goettler, R., Parlour, C., and U. Rajan, 2003, Equilibrium in a dynamic limit order market, working paper.

Hasbrouck, L. and G. Saar, 2002, Limit orders and volatility in a hybrid market: the Island

ECN, working paper.

Hendershott, T., and H. Mendelson, 2000, Crossing networks and dealer markets: competition and performance, *Journal of Finance* 55, 2071–2115.

Kumar, P. and D. Seppi, 1993, Limit orders and market orders with optimizing traders, working paper.

Kyle, A., 1985, Continuous markets and insider trading, *Econometrica* 53, 1315–1335.

Parlour, C., 1998, Price dynamics in limit order markets, *Review of Financial Studies* 11, 789–816.

Parlour, C., and D. Seppi, 2003, Liquidity-based competition for order flow, *Review of Financial Studies* 16, 301–343.

Ready, M., 1999, The specialist's discretion: stopped orders and price improvement, *Review of Financial Studies* 12, 1075–1112.

Rock, K., 1990, The specialist's order book and price anomalies, working paper.

Seppi, D., 1990, Equilibrium block trading and asymmetric information, *Journal of Finance* 45, 73–94.

Seppi, D., 1997, Liquidity provision with limit orders and a strategic specialist, *Review of Financial Studies* 10, 103–150.

Viswanathan, S. and J. Wang, 2002, Market architecture: limit-order books versus dealership markets, *Journal of Financial Markets* 5, 127–167.