An Examination of Heterogeneous Beliefs with a Short-Sale Constraint*

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Abstract

We study the effects of a short-sale constraint on stock prices in a dynamic general equilibrium economy populated by investors with heterogeneous beliefs. To clear both the stock and bond markets when the constraint binds, the most optimistic investor's equity demand must decrease relative to the unconstrained economy. The market price of risk must drop to reduce the optimist's demand for stock. The expected return on the investors' portfolios decrease as a result of the short-sale constraint. If the substitution effect is stronger than the income effect, then aggregate consumption demand must drop for the current consumption market to clear. In this case, the stock price drops as a consequence of imposing the short-sale constraint. If the substitution effect is weaker than the income effect, then aggregate demand must rise for the current consumption market to clear. In this case, the stock price rises as a consequence of imposing the short-sale constraint. In parameterized examples, we find that this resulting price rise is small. In all cases, the pessimistic investors believe the stock to be overvalued in the constrained economy.

Keywords: heterogeneous beliefs, learning, short-sale constraints, general equilibrium, stock price, stock volatility
JEL Codes: D51, G11, G12, G14

1 Introduction

Equilibrium stock prices reflect investors' beliefs. If investors cannot short-sell stock, then equilibrium stock prices may not reflect the beliefs of pessimistic investors. Stock prices may therefore be higher when short-sales are not allowed than when short-sales are allowed. Lintner (1969), Miller (1977), Jarrow (1980), and Chen et al. (2002) study one period models with heterogeneous beliefs and an exogenous interest rate. They show that in the presence of a short-sale constraint and heterogeneous beliefs, stock prices prices may be higher than otherwise if pessimistic investors face short-sale constraints.

Does a binding short-sale constraint lead to an increase in stock prices in a dynamic economy with endogenous interest rates? We examine the effects of a short-sale constraint on asset prices in a dynamic general equilibrium economy where investors have heterogeneous beliefs about stock returns. For tractability, we study a continuous-time economy. The economy is made up of optimistic and pessimistic investors who disagree about the expected growth rate of aggregate output. The investors learn about the expected growth rate by observing realized output growth. We interpret the price of a security paying out aggregate output as the price of the stock market. The stock price can fall or rise when a short-sale constraint is imposed in an economy with heterogeneous beliefs.

Suppose that the pessimistic investors short-sell the stock in an unconstrained economy, and a short-sale constraint is now imposed. In order for securities markets to clear with the constraint imposed, security prices must change so that the optimistic investors reduce their stock demand. Two changes can lead to a reduction in the optimistic investors stock demand. The risk premium on the stock can drop, and the cost of borrowing at the riskfree rate to purchase the stock can rise. Both changes can occur in our model. In both cases, imposing the short-sale constraint reduces the expected return on the optimistic investors' portfolios relative to the unconstrained economy. As a consequence, the optimistic investors' demand for savings and current consumption changes. The resulting substitution effect leads the optimistic investors to decrease their demand for savings and to increase their demand for current consumption. The resulting income effect leads the optimistic investors to increase their demand for savings and to decrease their demand for consumption.

If the income effect is stronger than the substitution effect, the demand for savings rises and the demand for current consumption falls as a consequence of the short-sale constraint. Given the aggregate supply of current consumption is fixed, the demand for current consumption must rise to clear the current consumption market. In order for the demand for current consumption to rise, aggregate wealth must rise. Given the value of aggregate wealth is equal to the value of stock market, the price must therefore rise when the short-sale constraint is imposed.

If the substitution effect is stronger than the income effect, the demand for current consumption rises as a consequence of the short-sale constraint. In order for the demand for current consumption to drop, the value of aggregate wealth must fall. The stock market must therefore fall when the short-sale constraint is imposed.

The strength of the substitution and income effects depend on the investors' intertemporal elasticity of substitution. If the elasticity is greater than one, the substitution effect dominates and the stock price falls when the short-sale constraint is imposed. If the elasticity is less than one, the income effect dominates and the stock price rises when the short-sale constraint is imposed. If the elasticity is equal to one, neither effect dominates and the stock price does not change when the short-sale constraint is imposed.

Consider an economy where all investors have homogeneous beliefs. Now suppose that some of the investors become more optimistic about future output growth. The increased optimism will only increase the stock market price if the substitution effect is stronger than the income effect — exactly the same situation when the short-sale constraint reduces stock prices.

Although the investors in our model have expected utility preferences, our results would be robust to allowing for preferences that separate risk-aversion from the intertemporal elasticity of substitution. Epstein (1988) shows that the stock market price rises when the distribution of aggregate dividends improves only when the elasticity of intertemporal substitution is greater than one if the representative agent has Epstein and Zin (1989) preferences — the substitution effect must be larger than the income effect for an increase in expected output to increase the stock price. Bansal and Yaron (2003) show that risky long-run growth prospects and time-varying economic uncertainty can generate many of the observed features of asset market data if the representative investor's intertemporal elasticity of substitution is greater than one.

In any equilibrium where the short-sale constraint binds with positive probability, the stock price is strictly higher than the pessimistic investor's valuation for the asset. The result holds irrespective of the investors' intertemporal elasticity of substitution. But the overvaluation does not necessarily mean that the stock price would fall if the short-sale constraint were removed from the economy.

We provide numerical examples of the asset pricing effects of the short-sale constraint. When the optimist's elasticity of intertemporal substitution is greater than one, the market price of risk drops dramatically when the short-sale constraint is imposed, and the pessimist may believe the stock to be overvalued by as much as 29%. In all cases, the stock price would rise if the short-sale constraint were removed from the economy. When the optimist's elasticity of intertemporal substitution is less than one, the market price of risk drops by much less when the short-constraint is imposed, and the pessimist may believe the stock to be overvalued by a much smaller amount. In this case, the stock price would fall by a small amount when the constraint is removed from the economy.

2 Literature Review

We use recently developed techniques to analyze our dynamic general equilibrium economy with a short-sale constraint, heterogeneous beliefs, and learning. Basak (2000) analyzes a general equilibrium economy with heterogeneous beliefs about sunspots in the presence of zero net supply securities, showing that two factor asset pricing results. Basak and Croitoru (2000) study a dynamic general equilibrium economy with heterogeneous beliefs and portfolio constraints, and demonstrate that in equilibrium, zero net-supply derivative securities can be mispriced.

Zapatero (1998) allows for heterogeneous beliefs in an economy populated with investors with logarithmic utility, showing that heterogeneous beliefs tend to increase interest rate volatility. Gallmeyer (2002) studies the effects of heterogeneous beliefs and learning on stock market volatility, showing that heterogeneous beliefs lead to significant effects on stock volatility in the presence of learning.

Harrison and Kreps (1978), Morris (1996), and Scheinkman and Xiong (2003) show that in partial equilibrium economies with short-sale constraints, the stock price can be above the expected present value of dividends under even the most optimistic investor's beliefs. Detemple and Murthy (1997) analyze general equilibrium continuous-time economies with portfolio and collateral constraints, and heterogeneous beliefs. Equilibrium stock prices can be decomposed into a consumption value, a speculative value premium, and a collateral value. All the models feature a resale premium in the equilibrium price of the stock; investors know that they may not always have the highest valuation for the asset, and so are willing to pay above their estimate of the present value of dividends for the option to resell the asset to another agent in the future. Duffie et al. (2002) study a model with heterogeneous beliefs in a search environment, showing that the stock price can be above the highest possible valuation for the dividends, because of an endogenous shorting fee.

The effects of constraints on stock prices have also been studied in models with homogeneous beliefs. Heaton and Lucas (1996) compute numerical solutions to an economy with heterogeneous preferences, idiosyncratic income shocks, short-sale constraints, and transactions costs, finding small effects on the equity premium, unless the transactions costs are large. Coen-Pirani (2004) studies the effects of introducing margin requirements in an economy with Epstein and Zin (1989) preferences and unitary elasticity of substitution, finding small stock price effects and large interest rate effects from introducing margin requirements.

3 The Model

We study a continuous-time pure exchange economy with heterogeneous investors. The economy has a finite horizon equal to $T < \infty$. All uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$.

The exogenous aggregate output $\delta(t)$ follows the process

$$d\delta(t) = \delta(t)\left[\mu_\delta dt + \sigma_\delta dw(t)\right], \quad \delta(0) > 0, \tag{1}$$

where $w(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. The diffusion coefficient is constant with $\sigma_\delta > 0$.

Realized output follows a geometric Brownian motion. Our characterization of the equilibrium does not depend on the geometric Brownian motion assumption. Instead, all we require is that the diffusion coefficient on output growth is $\mathcal{F}_t^\delta$-progressively measurable, where $\mathcal{F}_t^\delta$ is the augmented filtration generated by $\delta(t)$. Such a mild assumption allows for a variety of models for the expected output growth process, such as a mean-reverting process or a regime-switching process.

The investors do not observe the expected growth rate of output. Instead, they only observe the continuous record of realized output. All investors agree that the evolution of output is governed by equation (1), but they do not know the true expected output growth. Heterogeneous beliefs are modeled with investor-specific priors about expected output growth. Since output is observed continuously, all investors can perfectly estimate its volatility by computing the output process' quadratic variation.

The economy is populated by two classes of investors: optimists and pessimists. All investors are price-takers allowing each class to be summarized by a single representative investor. All investor-specific quantities are labeled with a superscript o for an optimist or a superscript p for a pessimist.

For investor $i = \{o, p\}$, uncertainty in the economy is represented by the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}^\delta\}, \mathcal{P}^i\right)$. Investor-specific beliefs about expected output growth are given by the conditional expectations

$$\mu_\delta^i(t) \quad = \quad E^i\left[\mu_\delta \middle| \mathcal{F}_t^\delta\right], \quad i = \{o, p\}. \tag{2}$$

Optimists have weakly higher expectations about initial output growth than the pessimists,

$$\mu_\delta^o(0) \geq \mu_\delta^p(0). \tag{3}$$

Given the two investors start with different priors, they will continue to disagree about the expected output growth over the life of the economy as they update their beliefs using the commonly observed output process. Following standard filtering theory as in Liptser and Shiryayev (1977), the investor-specific innovation processes are defined as

$$\begin{aligned} dw^i(t) &= \frac{1}{\sigma_\delta}\left[\frac{d\delta(t)}{\delta(t)} - \mu_\delta^i(t)dt\right] \\ &= \frac{\mu_\delta - \mu_\delta^i(t)}{\sigma_\delta}dt + dw(t), \quad i = \{o,\, p\}. \end{aligned} \tag{4}$$

From equation (4), the innovation processes across investors are related by

$$dw^p(t) - dw^o(t) = \bar{\mu}(t)dt, \tag{5}$$

where

$$\bar{\mu}(t) \equiv \frac{\mu_\delta^o(t) - \mu_\delta^p(t)}{\sigma_\delta}. \tag{6}$$

The process $\bar{\mu}(t)$ measures investors' differences of opinion; we call it the disagreement process. The disagreement process depends on the initial priors and the path of realized aggregate output through the filtering problems of the two classes of investors. The disagreement process does not depend on equilibrium asset prices or allocations.

Lemma 1 *Under each investor's beliefs, the aggregate output process is*

$$d\delta(t) = \delta(t)\left[\mu_\delta^i(t)\, dt + \sigma_\delta\, dw^i(t)\right], \quad \delta(0) > 0, \quad i \in \{o,\, p\}. \tag{7}$$

Each investor's beliefs about the expected growth of aggregate output follows the process

$$d\mu_\delta^i(t) = m^i(t)\, dt + \eta^i(t)\, dw^i(t), \quad i \in \{o,\, p\}, \tag{8}$$

where $m^i(t)$ and $\eta^i(t)$ are $\mathcal{F}_t^\delta$-measurable, and $w^i(t)$ is a standard Brownian motion defined on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t^\delta\}, \mathcal{P}^i\right)$, for $i = \{o, p\}$.

Given (7) has a strong solution, the investors' information and innovation filtrations coincide. Each investor acts as an investor with complete information who faces uncertainty in the economy generated by $w^i(t)$.

A locally riskless bond and a stock with dividends equal to aggregate output are continuously traded by the investors. The stock is interpreted as the market portfolio. The bond is in zero net supply with price $B(t)$. Posited bond price dynamics are

$$dB(t) = B(t)\, r(t)\, dt, \quad B(0) = 1, \tag{9}$$

with $r(t)$ the instantaneous riskless interest rate in the economy. The stock is in fixed net supply of one share, paying a continuous dividend rate equal to aggregate output, $\delta(t)$. The posited gains process for the stock under the true probability beliefs is

$$dS(t) + \delta(t)dt = S(t)\left[\mu(t)\, dt + \sigma(t)\, dw(t)\right], \quad S(T) = 0. \tag{10}$$

Under each investor's beliefs, the stock gains process is

$$dS(t) + \delta(t)dt = S(t)\left[\mu^i(t)\, dt + \sigma(t)\, dw^i(t)\right], \quad S(T) = 0, \quad i = \{o, p\}. \tag{11}$$

Here $\mu^i(t)$ is investor i's beliefs about the expected stock gains. All investors agree on the diffusion coefficient of the stock gains process. Agreement on the traded stock price requires that

$$\frac{\mu^o(t) - \mu^p(t)}{\sigma(t)} = \bar{\mu}(t). \tag{12}$$

All investors choose consumption and portfolio strategies to maximize lifetime utility. The optimists' portfolio strategies are unconstrained; they face a complete financial market even though they have incomplete information about the economy (Riedel (2001)). The pessimists' portfolio strategies are constrained from shorting the stock; they face an incomplete financial market.

Each investor's lifetime utility function is time and state separable and all investors have the same time-preference parameter equal to ρ. Pessimists have logarithmic period utility functions with coefficient of relative risk aversion equal to 1 and elasticity of intertemporal substitution equal to 1. Optimists have constant relative risk averse period utility functions, with coefficient of relative risk aversion γ and elasticity of intertemporal substitution equal to $\frac{1}{\gamma}$.

Assuming that the short-sale constrained pessimists have logarithmic preferences allows us to obtain closed-form solutions for the pessimists' portfolio and consumption decisions for general no-arbitrage price systems. Assuming that the optimists have non-logarithmic preferences leads to the possibility both stock price and interest rate effects from heterogeneous beliefs. Heterogeneous beliefs do not change stock prices if both investors have logarithmic preferences. The result is shown by Zapatero (1998) without portfolio constraints, and Detemple and Murthy (1994) with collateral and short-sale constraints.

We start with the optimist's consumption portfolio problem. The optimist's wealth $X^o(t)$ evolves as

$$dX^o(t) = X^o(t)r(t)dt - c^o(t)dt + \theta^o(t)S(t)\left[(\mu^o(t) - r(t))dt + \sigma(t)dw^o(t)\right], \quad X^o(0) = x^o S(0), \tag{13}$$

with x^o the optimist's initial stock endowment, $\theta^o(t)$ the optimist's shareholding, and $c^o(t)$ the optimist's consumption rate. Let $\alpha^o(t) = X^o(t) - \theta^o(t)S(t)$ denote the optimist's bond holdings. Following Dybvig and Huang (1988), we impose a non-negative wealth constraint, $X^o(t) \geq 0$, on the class of admissible trading strategies to rule out doubling strategies.

The optimist faces complete markets allowing us to apply standard martingale techniques to characterize the optimal consumption and trading strategy as in Cox and Huang (1989) and Karatzas et al. (1987). Using martingale techniques, we can solve the optimist's problem while allowing quite general equilibrium securities prices.

Summarizing the stock and bond price system, the optimist's state price density, $\xi^o(t)$ has dynamics given by

$$d\xi^o(t) = -\xi^o(t)\left[r(t)dt + \kappa^o(t)dw^o(t)\right], \quad \xi^o(0) = 1, \tag{14}$$

with $\kappa^o(t)$ the market price of risk, or instantaneous Sharpe ratio, under the optimist's beliefs defined as

$$\kappa^o(t) \equiv \frac{\mu^o(t) - r(t)}{\sigma(t)}. \tag{15}$$

The optimist's lifetime consumption portfolio problem is

$$\max_{\{c^o(t)\}} E^o\left[\int_0^T e^{-\rho t}\frac{c^o(t)^{1-\gamma}-1}{1-\gamma}dt\right], \tag{16}$$

subject to:

$$E^o\left[\int_0^T \xi^o(t)c^o(t)dt\right] \leq x^o S(0). \tag{17}$$

The optimal consumption function is

$$\hat{c}^o(y^0\xi^0(t),t) = \left(\frac{e^{-\rho t}}{y^o\xi^o(t)}\right)^{\frac{1}{\gamma}}, \tag{18}$$

where the Lagrange multiplier y^o for the static budget constraint (17) is

$$y^o \;=\; \left(\frac{E^o\left[\int_0^T e^{-\frac{\rho}{\gamma}t}\xi^o(t)^{\frac{\gamma-1}{\gamma}}dt\right]}{x^o S(0)}\right)^{\gamma}. \tag{19}$$

We now turn to the pessimist's consumption portfolio problem. The pessimist's wealth $X^p(t)$ evolves as

$$dX^p(t) = X^p(t)r(t)dt - c^p(t)dt + \theta^p(t)S(t)\left[(\mu^p(t)-r(t))dt + \sigma(t)dw^p(t)\right], \quad X^p(0) = x^p S(0), \tag{20}$$

with x^p the pessimist's initial stock endowment, $\theta^p(t)$ the pessimist's shareholdings, and $c^p(t)$ the pessimist's consumption rate. Let $\alpha^p(t) = X^p(t) - \theta^p(t)S(t)$ denote the pessimist's bond holdings. The short-sale constraint is

$$\theta^p(t) \geq 0, \quad \forall t. \tag{21}$$

Since the pessimist faces a short-sale constraint, he faces an incomplete financial market. We use the approach developed by He and Pearson (1991) and Karatzas et al. (1991) to solve the pessimist's problem. The constrained pessimist's problem is embedded in a fictitious, unconstrained complete markets problem. With the appropriate choice of state prices in the fictitious market, the solution to the problem in the fictitious market is the solution to the original constrained problem. With logarithmic preferences for the constrained pessimist, we can analytically compute the fictitious state prices for general no-arbitrage price systems. With non-logarithmic preferences, we cannot analytically compute the fictitious state prices without making strong restrictions on the underlying economy.

Since the investors agree on the interest rate and stock price processes, the optimists' and pessimists' expected excess returns for the stock are linked by the disagreement process:

$$\mu^o(t) - \mu^p(t) = \bar{\mu}(t)\sigma(t). \tag{22}$$

If $\bar{\mu}(t)\sigma(t) > 0$, the optimist believes the equity market has a higher expected excess return than the pessimist.

The market price of risk under the pessimist's beliefs is defined as

$$\kappa^p(t) \equiv \frac{\mu^p(t) - r(t)}{\sigma(t)}. \tag{23}$$

Since the optimists and pessimists agree on the interest rate, equation (22), implies that the optimist's and pessimist's price of risk are linked through the disagreement process:

$$\kappa^o(t) - \kappa^p(t) = \bar{\mu}(t). \tag{24}$$

If the pessimist does not face a short-sale constraint his optimal portfolio is the log-optimal portfolio. The log-optimal portfolio is short in the stock only when the pessimist's instantaneous market price of risk is negative. The short selling constraint is therefore binding in the original lifetime consumption portfolio problem only when $\mu^p(t) < r(t)$. From equation (22), $\mu^p(t) < r(t)$ when

$$\bar{\mu}(t) > \kappa^o(t), \text{ and } \sigma(t) > 0. \tag{25}$$

We will only consider equilibria where $\sigma(t) > 0$ — a positive shock to aggregate output increases stock prices, and the pessimist is short-sale constrained only when he believes the expected return on the stock is low enough. Since $\sigma(t)$ is endogenous, the condition puts a restriction on the parameters of the economy. In our numerical work, we verify that the condition holds.

The constrained pessimist's state price density in the fictitious market is given by $\xi^p(t)$, and follows the process

$$d\xi^p(t) = -\xi^p(t) \left[r(t)dt + \max\left(\kappa^p(t), 0\right) dw^p(t) \right], \quad \xi^p(0) = 1. \tag{26}$$

Equation (26) implies that the state price density in the fictitious market is locally deterministic when the short-sale constraint is binding.

The pessimist's lifetime consumption portfolio choice problem in the fictitious unconstrained economy is given by

$$\max_{\{c^p(t)\}} E^p \left[\int_0^T e^{-\rho t} \ln(c^p(t)) dt \right], \tag{27}$$

subject to:

$$E^p \left[\int_0^T \xi^p(t) c^p(t) dt \right] \leq x^p S(0). \tag{28}$$

The optimal consumption function is

$$\hat{c}^p(y^p \xi^p(t), t) = \frac{e^{-\rho t}}{y^p \xi^p(t)}, \tag{29}$$

where the Lagrange multiplier y^p from the static budget constraint (28) is

$$y^p = \frac{1 - e^{-\rho T}}{\rho \ x^p S(0)}. \tag{30}$$

The optimal consumption function in equation (29) and the dynamics of the pessimist's fictitious state price density (26) imply that the pessimist's optimal consumption is locally deterministic when the constraint is binding. The optimal portfolio strategy is to fully invest in the bond when the market price of risk is negative and hold the log-optimal portfolio otherwise.

4 Characterization of Equilibrium

Definition 1 *Given preferences, endowments, and beliefs, an equilibrium is a collection of allocations* $((\hat{c}^o, \hat{\alpha}^o, \hat{\theta}^o), (\hat{c}^p, \hat{\alpha}^p, \hat{\theta}^p))$ *and a price system* $(r, \mu^o, \mu^p, \sigma)$ *such that* $(\hat{c}^i, \hat{\alpha}^i, \hat{\theta}^i)$ *is an optimal* *solution to investor i's consumption-portfolio problem given his perceived price processes, security prices are consistent across investors, and all markets clear for $t \in [0, T]$:*

$$\begin{aligned} &\hat{c}^o(t) + \hat{c}^p(t) = \delta(t), && \textit{Consumption Market,} \\ &\hat{\theta}^o(t) + \hat{\theta}^p(t) = 1, && \textit{Stock Market,} \\ &\hat{\alpha}^o(t) + \hat{\alpha}^p(t) = 0, && \textit{Bond Market.} \end{aligned} \tag{31}$$

The martingale formulation of each investor's consumption-portfolio problem provides a direct method to construct the equilibrium. To determine the equilibrium price system, it suffices to find the two state-price density processes that clear the consumption market,

$$\hat{c}^o(y^o \xi^o(t), t) + \hat{c}^p(y^p \xi^p(t), t) = \delta(t). \tag{32}$$

Once the two state price processes are determined from equation (32), they in conjunction with the output process uniquely determine the equilibrium stock and bond prices. If the state price processes clear the consumption market, the associated optimal portfolio strategies also clear the stock and bond markets.

Rather than working directly with both state price processes in the consumption market clearing condition above, it is convenient to define the stochastic weighting process $\lambda(t)$ given by

$$\lambda(t) = \frac{y^o \xi^o(t)}{y^p \xi^p(t)}, \tag{33}$$

where $\lambda(0) = \frac{y^o}{y^p}$, since $\xi^o(0) = \xi^p(0) = 1$. The stochastic weighting process $\lambda(t)$ summarizes the differences in the investor's opportunity sets from heterogeneous beliefs and the short-sale constraint.

Using the stochastic weighting process, the consumption market clearing condition is

$$\hat{c}^o(y^o \xi^o(t), t) + \hat{c}^p(y^o \xi^o(t)/\lambda(t), t) = \delta(t). \tag{34}$$

Market clearing can be used to uniquely solve for the optimist's state price density in terms of aggregate output, $\delta(t)$, and the level of the stochastic weighting process, $\lambda(t)$,

$$y^o \xi^o(t) = e^{-\rho t} U_c(\delta(t), \lambda(t)). \tag{35}$$

Details are given in Karatzas et al. (1990) for example. The function $U_c(\delta(t), \lambda(t))$ can be interpreted as the marginal utility of a state-dependent representative agent defined by

$$U(c, \lambda) \equiv \max_{c^o + c^p = c} \frac{(c^o)^{1-\gamma} - 1}{1 - \gamma} + \lambda \ln(c^p). \tag{36}$$

Such a weak notion of aggregation was introduced by Cuoco and He (1994) and has been used in a variety of models with incompleteness. See for example Basak and Cuoco (1998); Basak (2000); Detemple and Serrat (2003); Basak and Gallmeyer (2003).

Using the dynamics of the state price densities in equations (14) and (26), the dynamics of the stochastic weighting process are

$$d\lambda(t) = -\lambda(t)\sigma_\lambda(t) dw^o(t), \tag{37}$$

with

$$\sigma_\lambda(t) \equiv \min\left(\bar{\mu}(t), \kappa^o(t)\right). \tag{38}$$

If the optimist has a higher expected growth rate than the pessimist, $\bar{\mu}(t) > 0$, the diffusion coefficient on $\lambda(t)$ is negative — a positive shock to output leads to a decrease in $\lambda(t)$. From the construction of the representative agent in equation (36), a decrease in $\lambda(t)$ reduces the pessimist's share of aggregate output and increases the optimist's share of output. A positive shock to aggregate output therefore increases the optimist's share of output and decreases the pessimist's share of output.

In an unconstrained economy, the diffusion coefficient on the stochastic weighting process is $\bar{\mu}(t)$. The short-sale constraint binds when $\bar{\mu}(t) \geq \kappa^o(t)$; a binding short-sale constraint reduces the sensitivity of the optimist's consumption to output shocks relative to an unconstrained economy.

Imposing the short-sale constraint reduces the optimist's ability to consume more in good states and less in bad states — the short-sale constraint reduces the optimist's ability to trade on his beliefs.

Proposition 1 provides a construction of the equilibrium.

Proposition 1 *Assume that an equilibrium exists with $\sigma(t) > 0$. In equilibrium, the state price density for the optimist and the state price density in the fictitious market for the pessimist are*

$$\xi^o(t) = e^{-\rho t}\frac{U_c(\delta(t), \lambda(t))}{U_c(\delta(0), \lambda(0))}, \quad \xi^p(t) = \frac{\lambda(0)}{\lambda(t)}\xi^o(t). \tag{39}$$

The stock price process $S(t)$ is priced from the marginal utility of the optimist,

$$S(t) \quad = \quad E^o\left[\int_t^T e^{-\rho s}\frac{U_c(\delta(s), \lambda(s))}{U_c(\delta(t), \lambda(t))}\delta(s)ds\middle|\mathcal{F}_t^\delta\right]. \tag{40}$$

The equilibrium consumption allocations are

$$\hat{c}^o(\delta(t), \lambda(t)) = \left(\frac{1}{U_c(\delta(t), \lambda(t))}\right)^{\frac{1}{\gamma}}, \quad \hat{c}^p(\delta(t), \lambda(t)) = \frac{\lambda(t)}{U_c(\delta(t), \lambda(t))}. \tag{41}$$

The stochastic weighting process $\lambda(t)$ has dynamics

$$d\lambda(t) \quad = \quad -\lambda(t)\sigma_\lambda(t)dw^o(t), \tag{42}$$

where $\lambda(0)$ solves the pessimist's optimal time zero consumption demands:

$$\hat{c}^p(\delta(0), \lambda(0)) = \frac{\rho}{1-e^{-\rho T}}x^p S(0). \tag{43}$$

Conversely, if there exists processes $\xi^o(t)$, $\xi^p(t)$, $\lambda(t)$, and $S(t)$ satisfying equations (39)-(43), with $\sigma(t) > 0$, the associated optimal consumption-wealth-portfolio policies clear all markets.

There are two regions of the equilibrium. In the first region, the disagreement process is large enough so that the pessimist desires to short the stock. Here, the short-sale constraint is binding. The dynamics of the weighting process in the first region depend only on the optimist's market price of risk, as in the limited participation equilibrium in Basak and Cuoco (1998). In the second region, the disagreement process is small enough so that the pessimist desires to go long the stock. Here, the short-sale constraint is not binding. The dynamics of the weighting process in the second region depend only on the disagreement process as in the heterogeneous beliefs equilibrium in Gallmeyer (2002).

The next proposition reports the equilibrium price of risk and the instantaneous interest rate.

Proposition 2 *The price of risk for the optimist is given by*

$$\kappa^o(t) = \begin{cases} A^o(t)\delta(t)\sigma_\delta, & \textit{constraint binds,} \\ A(t)\delta(t)\sigma_\delta + \frac{A(t)}{A^p(t)}\bar{\mu}(t), & \textit{else,} \end{cases} \tag{44}$$

where $A^i(t),\ \ i = \{o, p\}$ *is the absolute risk aversion coefficient for each investor and* $A(t)$ *is the absolute risk aversion coefficient for the state-dependent representative agent,*

$$A^o(t) \equiv \frac{\gamma}{\hat{c}^o(t)},\ \ A^p(t) \equiv \frac{1}{\hat{c}^p(t)}, \tag{45}$$

$$A(t) \equiv \left(\frac{1}{A^o(t)} + \frac{1}{A^p(t)}\right)^{-1} = \left(\delta(t) + \hat{c}^o(t)\left(\frac{1-\gamma}{\gamma}\right)\right)^{-1}. \tag{46}$$

The instantaneous interest rate is

$$r(t) = \begin{cases} \rho + A(t)\delta(t)\mu_\delta^o(t) - \frac{1}{2}A(t)B^o(t)\delta(t)^2\sigma_\delta^2, & \textit{constraint binds,} \\ \\ \rho + A(t)\delta(t)\left(\frac{A(t)}{A^o(t)}\mu_\delta^o(t) + \frac{A(t)}{A^p(t)}\mu_\delta^p(t)\right) - \frac{1}{2}A(t)B(t)\delta(t)^2\sigma_\delta^2 & \textit{else,} \\ \quad -\frac{A(t)^2}{A^o(t)A^p(t)}\left(\frac{1}{2}\frac{B^o(t)+B^p(t)}{A^o(t)+A^p(t)} - 1\right)\bar{\mu}(t)^2 & \\ \quad -\frac{A(t)^3}{A^o(t)A^p(t)}\left(\frac{B^o(t)}{A^o(t)} - \frac{B^p(t)}{A^p(t)}\right)\delta(t)\sigma_\delta\bar{\mu}(t), & \end{cases} \tag{47}$$

where $B^i(t),\ \ i = \{o, p\}$ *is the absolute prudence coefficient for each agent and* $B(t)$ *is the absolute prudence coefficient of the state-dependent representative agent,*

$$B^o(t) \equiv \frac{\gamma+1}{\hat{c}^o(t)},\ \ B^p(t) \equiv \frac{2}{\hat{c}^p(t)},\ \ B(t) \equiv \left(\frac{A(t)}{A^o(t)}\right)^2 B^o(t) + \left(\frac{A(t)}{A^p(t)}\right)^2 B^p(t). \tag{48}$$

The market price of risk in our model is consistent with the one period models with heterogeneous beliefs and short-sale constraints in Lintner (1969), Miller (1977), and Jarrow (1980). The market price of risk only depends on the optimist's beliefs when the constraint is binding, and on both the optimist's beliefs and pessimist's beliefs when the constraint is not binding. But unlike in one period models where the interest rate is exogenous, our model's interest rate depends on the optimist's beliefs when the constraint is binding and on both investors' beliefs when the constraint is not binding.

In the unconstrained region of the equilibrium, the first three terms in the interest rate are of the same form as in a common belief economy. The first term is a risk-tolerance weighted average of the investor's beliefs about output growth. The final two terms capture the effects of heterogeneous beliefs and heterogeneous preferences on the interest rate arising from the risk shifting induced by the disagreement process. The final two terms are both negative when the optimist is less risk-averse than the pessimist.

The stock's risk premium depends on the price of risk and the stock's volatility. Proposition 2 provides the price of risk and we use the Clark-Ocone formula to compute the stock's volatility. The Clark-Ocone formula is used by Ocone and Karatzas (1991) and Detemple et al. (2003) to compute trading strategies in an optimal investment problem and by Fournié et al. (1999) to compute hedging parameters in option pricing applications.

Proposition 3 *The stock's volatility is given by*

$$\begin{aligned} \sigma(t) \quad = \quad & A(t)\delta(t)\sigma_\delta + \frac{E^o\left[\int_t^T \xi^o(s)(1 - A(s)\delta(s))\mathcal{D}_t\delta(s)ds \middle| \mathcal{F}_t^\delta\right]}{E^o\left[\int_t^T \xi^o(s)\delta(s)ds \middle| \mathcal{F}_t^\delta\right]} \\ & + \frac{A^o(t)}{A^o(t) + A^p(t)}\sigma_\lambda(t) + \frac{E^o\left[\int_t^T \xi^o(s)\frac{A^o(s)}{A^o(s)+A^p(s)}\frac{\delta(s)}{\lambda(s)}\mathcal{D}_t\lambda(s)ds \middle| \mathcal{F}_t^\delta\right]}{E^o\left[\int_t^T \xi^o(s)\delta(s)ds \middle| \mathcal{F}_t^\delta\right]}. \end{aligned} \tag{49}$$

where $\mathcal{D}_t\delta(s)$ is the Malliavin derivative for the output process, $\delta(u)$, and $\mathcal{D}_t\lambda(u)$ is the Malliavin derivative for the weighting process, $\lambda(u)$.

The only stochastic shock in the economy is the output process shock. The stock price depends on two state variables: the current level of the output process and the stochastic weighting process. Both of them react to the output shock.

The terms in the first row of equation (49) measure the effect of a shock to output on the stock price holding $\lambda(t)$ constant. The first term is the effect on today's state price density. The second term is the effect on the variability of future cash flows from a shock to today's output, holding the future path of $\lambda(t)$ fixed. The Malliavin derivative $\mathcal{D}_t\delta(s)$ measures how perturbing $w^o(t)$ affects the instantaneous change of $\delta(s)$ for $s \geq t$.

The terms in the second line of equation (49) capture the effect of the shock through $\lambda(t)$ on the stock price, holding the output fixed. The first term in the second line of equation (49) is the effect of the shock through $\lambda(t)$ on today's state price density. The second term in the second line of equation (49) is the contribution to the stock's volatility from future shifts in $\lambda(s)$. The Malliavin derivative $\mathcal{D}_t\lambda(s)$ measures how perturbing $w^o(t)$ affects the instantaneous change of $\lambda(s)$ for $s \geq t$. When beliefs are identical and the short sale constraint does not bind, the weighting process is constant and both the terms in the second line are zero.

Given the expectations in equation (49) cannot be computed in closed-form, it is difficult to establish the effect of the short-sale constraint on the stock's volatility. However, the expectations in equation (49) can be computed through a Monte-Carlo simulation.

We are interested in the effects of imposing the constraint on the initial stock price. Imposing the short-sale constraint reduces the initial price if the optimist has a higher elasticity of intertemporal substitution than one and increases the initial price if the optimist has a lower elasticity of intertemporal substitution than one.

Proposition 4 *Consider two economies with identical beliefs, preferences, and endowments, one economy short-sale constrained and one economy unconstrained.*

- *If the constraint binds with positive probability and $\gamma < 1$, then the initial stock price in the short-sale constrained economy is lower than the initial stock price in the unconstrained economy.*
- *If the constraint binds with positive probability and $\gamma > 1$, then the initial stock price in the constrained economy is higher than the initial stock price in the unconstrained economy.*

Although the initial stock price can either rise or fall when the constraint is imposed, the pessimist always finds the stock overpriced if the constraint ever binds. The next proposition establishes the result formally.

The pessimist's marginal valuation for the stock at time t is the price that the pessimist would be willing to pay for a small quantity of the stock at time t if he were required to hold the stock forever. If the pessimist did not face a short-sale constraint, the equilibrium price of the stock would equal the pessimist's marginal valuation for the stock. The pessimist's marginal valuation can be computed with the fictitious state-price density process used to solve the pessimist's consumption-portfolio problem. The pessimist's marginal valuation for the stock has dynamics,

$$dS^p(t) + \delta(t)dt \quad = \quad S^p(t) \left[\max\left(\mu^p(t), r(t)\right) dt + \sigma(t) dw^p(t) \right], \quad S^p(T) = 0, \tag{50}$$

and solution

$$S^p(t) \quad = \quad E^p \left[\int_t^T e^{-\rho s} \frac{u_c^p(\hat{c}^p(\delta(s), \lambda(s)))}{u_c^p(\hat{c}^p(\delta(t), \lambda(t)))} \delta(s) ds \middle| \mathcal{F}_t^\delta \right], \tag{51}$$

with $u_c^p(\delta(s), \lambda(s))$ the pessimist's marginal utility.

Proposition 5 *The equilibrium stock price and marginal valuation processes satisfy*

$$\begin{aligned} &S(t) > S^p(t), \quad \textit{if the constraint binds with positive probability for some } t' \geq t, \\ &S(t) = S^p(t), \quad \textit{else.} \end{aligned} \tag{52}$$

If the constraint is not binding for the pessimist at time t, the pessimist holds the log-optimal portfolio, which is long the stock. If there is positive probability that the short-sale constraint will bind at some time in the future, the stock price is higher than the pessimist's marginal valuation for the stock. The difference between the pessimist's marginal valuation and the stock price can be interpreted as a speculative premium. The speculative premium arises because the pessimist will sell stock to the optimist at a price higher than the pessimist's valuation, when the constraint next binds.

5 Numerical Results

Here, we provide numerical examples of the effects of imposing a short-sale constraint. Proposition 1 provides a method to construct the equilibrium numerically. From equation (42), the weighting process dynamics do not depend on stock price dynamics. Conditional on an initial guess for $\lambda(0)$, we solve the weighting process separately from the stock price, and solve for the stock price by Monte-Carlo simulating equation (40). We then iterate to find a $\lambda(0)$ solving the pessimist's budget constraint. Stock volatility is constructed through a Monte-Carlo simulation of equation (49). The Malliavin derivatives needed to compute stock volatility are provided in Appendix B. Technical details of the simulations are in Appendix C.

Heterogeneous beliefs arise by assuming each investor has a different Gaussian prior about the drift of the output process, μ_δ,

$$\mu_\delta^i \sim \mathcal{N}(\mu_\delta^i(0), \nu^i(0)), \quad i \in \{o, p\}, \tag{53}$$

where $\mu_\delta^i(0)$ is investor i's initial belief about the output's expected growth rate with $\nu^i(0)$ the initial variance of his beliefs. Typically, we refer to the volatility of investor i's beliefs, $\sqrt{\nu^i(0)}$.

Each investor's optimal filtering problem is given by a standard Kalman-Bucy filter. Investor i's prediction of expected output growth $\mu_\delta^i(t)$ has dynamics:

$$d\mu_\delta^i(t) = \frac{\nu^i(t)}{\sigma_\delta} dw^i(t), \quad \nu^i(t) = \frac{\nu^i(0)\sigma_\delta^2}{\nu^i(0)t + \sigma_\delta^2}, \quad i \in \{o, p\}. \tag{54}$$

As t grows large, both investors learn the true expected growth rate of output. Details about the optimal filtering problem can be found in Liptser and Shiryayev (1977).

Under the optimist's beliefs, the disagreement process $\bar{\mu}(t)$ has dynamics:

$$d\bar{\mu}(t) = -\frac{\nu^p(t)}{\sigma_\delta^2}\bar{\mu}(t)dt + \frac{\nu^o(t) - \nu^p(t)}{\sigma_\delta^2} dw^o(t). \tag{55}$$

If both investors do not have zero belief variances ($\nu^o(0) \neq 0$, $\nu^p(0) \neq 0$), the disagreement process converges to zero as t grows large. For identical belief volatilities across investors, the disagreement process is deterministic under all beliefs. In particular, if one investor is strictly more optimistic than the other initially ($\mu_\delta^o(0) > \mu_\delta^p(0)$), he will remain more optimistic in the future since $\bar{\mu}(t) > 0$, $\forall t$. If the optimist's prior beliefs have a higher volatility than the pessimist's prior beliefs, the disagreement process has a positive instantaneous correlation with output growth. When the optimist's beliefs have a lower volatility than the pessimist's, the opposite is true. When both prior beliefs have zero volatility, the disagreement process is a constant — investors are so confident in their priors that they completely ignore any information from the output process.

To parameterize the output process (1) and heterogeneous beliefs, we appeal to Gordon (2000) who reports that the average growth rate in GNP in the U.S. from 1995 to 1999 was 4.90%. We consider expected output growth rates of $\mu_\delta = 4\%$, 5%, and 6%. In most economies studied, the "true" output growth rate when needed is taken to be 4%. To parameterize heterogeneous beliefs, we consider possible belief volatilities for optimists and pessimists of 0% and 1%. With a zero initial belief variance, an investor does not use the observed output process to update his prior about the expected output growth. Increasing initial belief variances increases the amount of learning for that investor. The output growth rate prior for each investor is taken to be 4%, 5%, or 6% depending on the context.

5.1 Elasticity of intertemporal substitution greater than one

In our first numerical examples, we assume that the optimist has square root preferences: $\gamma = 0.5$, with elasticity of intertemporal substitution $\frac{1}{\gamma} = 2$. The rate of the time preference ρ for both investors is set equal to 0.03. Both the optimist's and the pessimist's initial endowment is 50% of the market portfolio with no initial bond position. The economy has a 20 year horizon. Here, we set the volatility of output growth to $\sigma_\delta = 2\%$.

Table 1 reports results from economies with equal endowments across investors where the short-sale constraint initially binds and investors learn from the output process at the same rate. This entails endowing optimists and pessimists with different initial beliefs about the expected output growth ($\mu_\delta^o(0) > \mu_\delta^p(0)$) while keeping a common belief volatility across investors ($\nu^o(0) = \nu^p(0)$). The common belief volatility assumption implies all investors learn from the output process at the same rate guaranteeing that the disagreement process $\bar{\mu}$ is deterministic and always non-negative.

Belief volatilities are either 0% or 1%. The pessimist's initial prior mean is 4%, and the optimist has either a 5% or 6% prior mean. True expected output growth is 4% – the pessimist's beliefs are correct on average.

We report properties of the equilibrium for unconstrained and constrained economies. The top panel reports properties of equilibrium prices, the middle panel reports properties of equilibrium allocations, and the bottom panel reports properties of the short-sale constraint in the constrained economy. The columns labeled "Unconst." correspond to economies without a short-sale constraint, while "Const." columns correspond to short-sale constrained economies.

The numerical parameterizations highlight the opposite effects on the riskless rate and the optimist's market price of risk when the short-sale constraint is imposed. For now we will focus on the parameterization when the optimist initially believes output growth is 6% and all investors have a common belief volatility of 0%. Here the equilibrium interest rate rises (−0.3% to 7.2%) while the optimist's market price of risk falls (0.557 to 0.025) when moving from the unconstrained to the constrained economy. The increase in the interest rate in the constrained economy significantly reduces the amount of leverage used to finance the optimist's and pessimist's equity positions; the open interest in the bond falls from 386% to 50% of the economy's aggregate wealth. The net effect of this shift in interest rates and market prices of risk is that the stock's price-dividend ratio falls from 19.6 in the unconstrained economy to 18.0 in the constrained economy. The increase in the interest rate dominates the fall in the optimist's market price of risk leading to a lower equity price in the presence of the constraint. Even though the pessimist believes the stock price is over-valued by 29%, the stock price would rise if the constraint were removed from the economy.

The stock's volatility is also significantly different in the presence of the short-sale constraint. Initial stock volatility drops from 6.8% in the unconstrained economy to 2.3% in the constrained economy. In the common beliefs benchmark, the stock's volatility was approximately 2%. The additional stock volatility in the unconstrained economy is driven by the weighting process being more volatile – the weighting process accounts for 69.7% of the stock's volatility in the unconstrained case and 9.8% in the constrained economy.

In all cases reported in Table 1, the stock price would rise in the short-sale constrained economy if the short-sale constraint were removed. Increasing the volatility of the investor's beliefs increases the amount of learning in the economy. Now the realization of the output process influences investor beliefs more. Increasing learning decreases the initial stock price, increases the initial riskless rate, and increases stock volatility in both the unconstrained and constrained economies. Increasing learning also decreases the pessimist's stock over-valuation. Since both investors rely on the data more, they put less trust in their initial priors mitigating the heterogeneous beliefs-induced valuation effects.

From the bottom panel of Table 1, the short-sale constraint binds over the lifetime of the economy unless the initial difference in beliefs narrows and belief volatility increases.

5.2 Elasticity of intertemporal substitution less than one

In our numerical example, we assume that the optimist's risk aversion $\gamma = 2$ with elasticity of intertemporal substitution $\frac{1}{\gamma} = 0.5$. The rate of the time preference ρ for both investors is set equal to 0.03. Both the optimist's and the pessimist's initial endowment is 50% of the market portfolio with no initial bond position. The economy has a 20 year horizon. Here, we set the volatility of the output process to be 5%. We chose such a large volatility in order for the short-sale constraint to bind.

Since the optimist is not myopic, his stock demand consists of a static mean-variance demand and a hedging demand. The mean-variance demand is decreasing in risk-aversion. The hedging demand is to hedge the stochastic weighting process, $\lambda(t)$. With the volatility of the output growth equal 2%, the hedging demand leads the short-sale constraint not to bind. Increasing the volatility of the output process reduces $\bar{\mu}(0)$ and from equation (37), such a change reduces the volatility of $\lambda(t)$. As a consequence, hedging demand is reduced and the optimist's demand for stock increases. Setting the volatility of the output process to 5% reduces the hedging demand enough for the constraint to bind initially.

Table 2 reports results from economies with this parameterization. Here, the initial stock price increases when the constraint is imposed. The market price of risk drops and the interest rate rises. But, the stock price does increase by much, nor does the market price of risk drop by much. According to the pessimist, the stock is over-valued at its current market price, but by much less than in Table 1.

5.3 Robustness

In all our numerical parameterizations in a heterogeneous beliefs economy, the introduction of a short-sale constraint leads to lower initial stock prices with the elasticity of intertemporal substitution greater than one. The introduction of a short-sale constraint leads to higher initial stock prices with the elasticity of intertemporal substitution less than one. These effects are more pronounced when the initial disagreement across investors is increased.

In addition to the heterogeneous belief economies considered in Tables 1, and 2, we considered numerous other belief parameterizations. We considered economies where the investors' priors had different precisions, so that the constraint did not initially bind, but could bind in the future. We also considered economies where the expected output process was not constant but followed a finite state regime-switching process. In all economies considered, we found the asset pricing effects to be robust. In particular, the short-sale constrained stock price was always lower than the unconstrained stock price with $\frac{1}{\gamma} > 1$ and higher with $\frac{1}{\gamma} < 1$. When the short-sale constraint did increase stock prices, the increase was small.

6 Conclusions

We study the effects of a short-sale constraint on prices and allocations in a dynamic economy with optimistic and pessimistic investors. We study the stock price, risk premium, and interest rate effects from imposing a short-sale constraint on pessimistic investors. In all cases, imposing the constraint decreases the risk-premium on holding the stock.

If the optimists' elasticity of intertemporal substitution is less than one, then imposing the short-sale constraint increases the equilibrium stock price. If the optimists' elasticity of intertemporal substitution is greater than one, then imposing the short-sale constraint decreases the equilibrium stock price.

We study an economy with one risky asset equal to a claim on aggregate output; we study the effects of heterogeneity and a short-sale constraint on the valuation of the market as a whole. But much of the empirical evidence on the effects of short-sale constraints deal with individual stocks.

Empirical evidence shows that average stock returns are lower with increased belief heterogeneity. For example, Diether et al. (2002) provide evidence that expected stock returns are higher for stocks with high dispersion of analysts' beliefs where they interpret the findings as evidence of binding short-sale constraints. Chen et al. (2002) and Jones and Lamont (2002) also provide empirical evidence of high expected stock returns when short-sale constraints bind.

Chen et al. (2002), Diether et al. (2002), Jones and Lamont (2002), and Ofek and Richardson (2003) present evidence of mispricing of individual stocks. Our results are driven by the effects of imposing the short-sale constraint on the demand for current consumption. Imposing the short-sale constraint leads to a reduction in the expected return on investor's portfolios — the effect of the short-sale constraint on the initial stock price therefore depends on the effect of the change in the portfolio expected return on current consumption demand. If current consumption demand rises as a result of the resulting reduction in portfolio returns, then the value of the stock market falls in equilibrium. If current consumption demand falls as a result of the resulting reduction in portfolio returns, then the value of the stock market rises in equilibrium.

In Jarrow (1980), a short-sale constraint increases the value of the constrained stock and also the value of the stock market as a whole. In this case, the substitution and income effects that drive our model would continue to hold. Overall stock prices would rise only if the income effect is stronger than the substitution effect on current consumption demand. We leave it to future work to determine the magnitude of the effects in a multi-asset economy.

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A Proofs

Proof of Lemma 1 The result follows from Theorem 8.1 of Liptser and Shiryayev (1977). ∎

Proof of Proposition 1 The proof follows Karatzas et al. (1990) with the appropriate modifications taken to accommodate for investors facing different state prices. From clearing in the consumption good market, equations (39) and (41) follow from the representative agent construction. The stock price (40) follows by substituting (39) into the stock price valued using the state price density process for the optimistic investor. The valuation equation exists from the existence of a state price density. To compute the dynamics of the weighting process, apply Itô's lemma to $\lambda(t) = \frac{y^p \xi^o(t)}{y^p \xi^p(t)}$ using the dynamics of the state price densities in the constrained and unconstrained regions.

To show the converse, assume that there exists ξ^o, ξ^p, λ, and S satisfying (39) - (40). Clearing in the consumption good market (32) follows from the solutions to the investors' consumption-portfolio problems with (39) substituted. To show clearing in the bond market, note that $\xi^o(t)(X^o(t)+X^p(t))$ is a martingale since the pessimist's optimal wealth process is a tradeable strategy, and the optimist and the pessimist must agree upon the prices of all tradeable strategies.

$$X^o(t) + X^p(t) = \frac{1}{\xi^o(t)} E^o \left[\int_t^T \xi^o(s)(\hat{c}^o(s) + \hat{c}^p(s))ds \middle| \mathcal{F}_t^\delta \right].$$

By substituting clearing in the consumption good market, clearing in the bond market results. Finally, to show clearing in the stock market, compare the diffusion dynamics of $\xi^o(t)S(t)$ with $\xi^o(t)(X^o(t)+X^p(t))$ using clearing in both the good market and the bond market. Clearing in the stock market then results. ∎

Proof of Proposition 2 Applying Itô's lemma to each agent's first order conditions, $u_c^i(\hat{c}^i(t)) = y^i\xi^i(t)$, and matching diffusion terms,

$$\sigma_{c^i}(t) = \frac{\kappa^i(t)}{A^i(t)}, \quad i \in \{o, p\},$$

where $\hat{c}^i(t)$ satisfies

$$d\hat{c}^i(t) = \left(\mu_{c^i}(t) + \frac{\mu_\delta(t) - \mu_\delta^i(t)}{\sigma_\delta(t)}\right) dt + \sigma_{c^i}(t)dw^i(t).$$

Market clearing in the consumption good market implies

$$\sigma_{c^o}(t) + \sigma_{c^p}(t) = \delta(t)\sigma_\delta(t)$$

and

$$1/A(t) = 1/A^o(t) + 1/A^p(t).$$

Note that $\sigma_{c^p}(t) = 0$ when the short-sale constraint binds. This implies equations (44).

The interest rate follows from applying Itô's lemma to the state price density for each agent, matching deterministic terms, and rearranging. ∎

Proof of Proposition 3 From Proposition 1, the stock is priced with respect to the optimist:

$$S(t) = \frac{1}{U_c(\delta(t), \lambda(t))} E^o \left[\int_t^T e^{-\rho s} U_c(\delta(s), \lambda(s)) \delta(s) ds \middle| \mathcal{F}_t^\delta \right].$$

Define the $\mathcal{L}^2(\mathcal{P}^o)$ martingale $M(t)$

$$M(t) = E^o \left[\int_0^T e^{-\rho s} U_c(\delta(s), \lambda(s), t) \delta(s) ds \middle| \mathcal{F}_t^\delta \right].$$

By the martingale representation theorem, there exists a process ϕ such that $E^o \left[\int_0^T |\phi(t)|^2 dt \right] < \infty$ where $M(t) = M(0) + \int_0^T \phi(t) dw^o(t)$.

The value of the stock can be rewritten as

$$S(t) = \frac{1}{U_c(\delta(t), \lambda(t))} \left[M(t) - \int_0^t e^{-\rho s} U_c(\delta(s), \lambda(s), t) \delta(s) ds \middle| \mathcal{F}_t^\delta \right]. \tag{A1}$$

By applying Itô's lemma to (A1) and matching diffusion coefficients with the dynamics of S given by (10),

$$\begin{aligned} \sigma(t) &= -\frac{U_{cc}(\delta(t), \lambda(t))}{U_c(\delta(t), \lambda(t))} \delta(t) \sigma_\delta(t) + \frac{U_{c\lambda}(\delta(t), \lambda(t)) \lambda(t)}{U_c(\delta(t), \lambda(t))} \sigma_\lambda(t) \\ &\quad + \frac{\phi(t)}{E^o \left[\int_t^T U_c(\delta(s), \lambda(s), t) \delta(s) ds | \mathcal{F}_t^\delta \right]}. \end{aligned} \tag{A2}$$

Equation (A2) gives a representation of the stock volatility in terms of the representative agent's utility function; the processes δ and λ; and the process ϕ from the martingale representation theorem.

To complete the characterization of the stock's volatility, we need to relate the dynamics of the martingale $M(t)$ to the dynamics of the processes δ and λ. From Lemma 2 below, the dynamics of $M(t)$ are

$$\begin{aligned} \phi(t) &= E^o \left[\int_t^T e^{-\rho s} U_{c\lambda}(\delta(s), \lambda(s)) \delta(s) \mathcal{D}_t \lambda(s) ds \middle| \mathcal{F}_t^\delta \right] \\ &+ E^o \left[\int_t^T e^{-\rho s} \left(U_{cc}(\delta(s), \lambda(s)) \delta(s) + U_c(\delta(s), \lambda(s)) \right) \mathcal{D}_t \delta(s) ds \middle| \mathcal{F}_t^\delta \right]. \end{aligned}$$

The Malliavin derivatives $\mathcal{D}_t \lambda(s)$ and $\mathcal{D}_t \delta(s)$ are

$$\begin{aligned} \mathcal{D}_t \delta(s) &= \delta(s) \left[\int_t^s \mathcal{D}_t \mu^o(z) dz + \sigma_\delta \right], \\ \mathcal{D}_t \lambda(s) &= \lambda(s) \left[\int_t^s \sigma_\lambda(z) \mathcal{D}_t \sigma_\lambda(z) dz - \int_t^s \mathcal{D}_t \sigma_\lambda(z) dw^o(z) - \bar{\mu}(t) \right]. \end{aligned}$$

The results are computed by using the chain rule for Malliavin derivatives. See Nualart (1995) for a full exposition.

From the representative agent construction and implicit differentiation,

$$U_{c\lambda}(\delta(s),\lambda(s)) = \frac{U_c(\delta(s),\lambda(s))}{\lambda(s)}\frac{A(s)}{A^p(s)} = \frac{U_c(\delta(s),\lambda(s))}{\lambda(s)}\frac{A^o(s)}{A^o(s)+A^p(s)}.$$

Substituting gives the result. ∎

Lemma 2 *(Proposition 1.3.5 (Clark-Ocone) - Nualart (1995))*
Let $F \in D^{1,2}$, where the space $D^{1,2}$ is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm

$$||F||_{1,2} = \left[E(|F|^2) + E(||\mathcal{D}F||^2_{L^2(T)})\right]^{1/2}.$$

Suppose that w is a one-dimensional Brownian motion. Then

$$F = E(F) + \int_0^T E(\mathcal{D}_s F|\mathcal{F}_s)dw(s),$$

Taking conditional expectations,

$$E(F|\mathcal{F}_t) = E(F) + \int_0^t E(\mathcal{D}_s F|\mathcal{F}_s)dw(s).$$

Proof of Lemma 2: See page 42 of Nualart (1995). ∎

We use the following lemmas to prove Proposition 4.

Lemma 3 *Let $\lambda(0)^u$ be the initial equilibrium value of $\lambda(0)$ in the unconstrained economy and $S(0)^u$ the initial stock price in the unconstrained economy. Let $\lambda(0)^c$ be the initial equilibrium value of $\lambda(0)$ in the constrained economy and $S(0)^c$ the initial stock price in the unconstrained economy. Then*

$$sign\left(S(0)^c - S(0)^u\right) = sign\left(\lambda(0)^c - \lambda(0)^u\right). \tag{A3}$$

Proof of Lemma 3. The result follows directly from equation (43) and since $c^p(\delta,\lambda)$ is increasing in λ. ∎

Lemma 4

$$sign\left(\frac{\partial E^o\left[\int_0^T e^{-\rho t}c^o(t)^{-\gamma}\delta(t)dt\right]}{\partial\lambda(0)}\right) > 0. \tag{A4}$$

$$sign\left(\frac{\partial E^o\left[\int_0^T e^{-\rho t}c^o(t)^{1-\gamma}dt\right]}{\partial\lambda(0)}\right) = \begin{cases} -, & \text{if } \gamma < 1. \\ +, & \text{if } \gamma > 1.\end{cases}. \tag{A5}$$

Proof of Lemma 4 From equation (42),

$$\lambda(t) = \lambda(0)e^{-\int_0^t 0.5\sigma_\lambda^2(s)ds - \int_0^t \sigma_\lambda(s)dw^o(s)}, \tag{A6}$$

and therefore

$$\frac{\partial\lambda(t)}{\partial\lambda(0)} > 0. \tag{A7}$$

The first-order condition for the representative agent problem is

$$\lambda(t) = c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t) - c^o(\delta(t), \lambda(t))^{1-\gamma}. \tag{A8}$$

Differentiating equation (A8)

$$\frac{\partial c^o(\delta(t), \lambda(t))}{\partial \lambda(t)} = -c^o(\delta(t), \lambda(t))^{\gamma} \left(\gamma \left(\frac{\delta(t)}{c^o(\delta(t), \lambda(t))} - 1\right) + 1\right)^{-1} < 0, \tag{A9}$$

since $\frac{\delta(t)}{c^o(\delta(t),\lambda(t))} \geq 1$. From the chain rule and equations (A7) and (A9),

$$\frac{\partial c^o(\delta(t), \lambda(t))}{\partial \lambda(0)} < 0. \tag{A10}$$

Applying the chain rule,

$$\begin{aligned} \frac{\partial c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)}{\partial \lambda(0)} &= -\gamma c^o(\delta(t), \lambda(t))^{-(\gamma+1)}\delta(t)\frac{\partial c^o(\delta(t), \lambda(t))}{\partial \lambda(0)} \\ &> 0, \end{aligned} \tag{A11}$$

and

$$\frac{\partial c^o(\delta(t), \lambda(t))^{1-\gamma}}{\partial \lambda(0)} = (1-\gamma)c^o(\delta(t), \lambda(t))^{-\gamma}\frac{\partial c^o(\delta(t), \lambda(0))}{\partial \lambda(0)}. \tag{A12}$$

Therefore,

$$\text{sign}\left(\frac{\partial c^o(\delta(t), \lambda(t))^{1-\gamma}}{\partial \lambda(0)}\right) = \begin{cases} - & \text{if } \gamma < 1, \\ + & \text{if } \gamma > 1. \end{cases} \tag{A13}$$

■

Lemma 5 *The function $c^o(\delta, \lambda)^{1-\gamma}$ is convex in λ if $\gamma < 1$, linear in λ if $\gamma = 1$, and concave in λ if $\gamma > 1$.*

Proof of Lemma 5

$$\frac{\partial^2 c^o(\delta(t), \lambda(t))^{1-\gamma}}{\partial \lambda(t)^2} = -(1-\gamma)\left(\gamma\left(\frac{\delta(t)}{c^o(\delta(t), \lambda(t))} - 1\right) + 1\right)^{-2} \gamma\frac{\delta(t)}{c^o(\delta(t), \lambda(t))^2}\frac{\partial c^o(\delta(t), \lambda(t))}{\partial \lambda(t)} \tag{A14}$$

■

Proof of Proposition 4   Equations (33) and (A8) together imply that

$$\frac{1-e^{-rT}}{\rho}\lambda(0) = E^o\left[\int_0^T e^{-\rho t}c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)dt\right] - E^o\left[\int_0^T e^{-\rho t}c^o(\delta(t), \lambda(t))^{1-\gamma}dt\right]. \tag{A15}$$

Differentiating with respect to $\lambda(0)$,

$$\frac{1-e^{-rT}}{\rho} = \frac{\partial E^o\left[\int_0^T e^{-\rho t}c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)dt\right]}{\partial \lambda(0)} - \frac{\partial E^o\left[\int_0^T e^{-\rho t}c^o(\delta(t), \lambda(t))^{1-\gamma}dt\right]}{\partial \lambda(0)}. \tag{A16}$$

The right-hand side of the equation is equal to the present value of the pessimist's consumption times the pessimist's initial consumption, $c^p(\delta(0), \lambda(0))$. In equilibrium, this is equal to the pessimist's endowment times $c^p(\delta(0), \lambda(0))$:

$$x^p E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)dt\right] = E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)dt\right] - E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{1-\gamma}dt\right]. \quad (A17)$$

with x^p the pessimist's endowment of shares.

Imposing the short-sale constraint reduces $\sigma_\lambda(t)$. If $\gamma < 1$, the convexity established in Lemma 5 and Jensen's inequality implies that

$$E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{1-\gamma}\right]$$

decreases when the short-sale constraint is imposed, holding $\lambda(0)$ at its original value. From equation (A15),

$$E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right]$$

decreases by the same amount. But then

$$x^p E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right] < E\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right] - E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{1-\gamma}\right]$$
$$= \frac{1 - e^{\rho T}}{\rho}\lambda(0)$$
$$\rightarrow x^p E\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right] - \frac{1 - e^{\rho T}}{\rho}\lambda(0) < 0. \quad (A18)$$

As a consequence, $\lambda(0)$ must adjust.

From equation (A16),

$$0 < \frac{\partial E^o\left[\int_0^T e^{-\rho t} c^o(\lambda(t), \delta(t))^{-\gamma}\delta(t)dt\right]}{\partial\lambda(0)} = \frac{1 - e^{-rT}}{\rho} + \frac{\partial E^o\left[\int_0^T e^{-\rho t} c^o(\lambda(t), \delta(t))^{1-\gamma}dt\right]}{\partial\lambda(0)}$$
$$< \frac{1 - e^{-rT}}{\rho}. \quad (A19)$$

From lemma, $E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right]$ increases in $\lambda(0)$ and from equation (A19)

$$\frac{\partial E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right] - \frac{1-e^{\rho T}}{\rho}\lambda(0)}{\partial\lambda(0)} < 0 \quad (A20)$$

Therefore, $\lambda(0)$ must decrease to satisfy equation (A17) and by Lemma 3 the initial stock price must drop.

If $\gamma > 1$, the concavity established in Lemma 5 and Jensen's inequality imply that the expectation

$$E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{1-\gamma}\right]$$

increases when the short-sale constraint is imposed holding $\lambda(0)$ at its original value. From equation (A15), the term

$$E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right]$$

increases by the same amount. But then

$$x^p E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right] \quad > \quad E\frac{1-e^{\rho T}}{\rho}\lambda(0), \tag{A21}$$

or

$$E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{1-\gamma}\right] > (1-x^p)E^o\left[\int_0^T e^{-\rho t} c^o(\delta(t), \lambda(t))^{-\gamma}\delta(t)\right]. \tag{A22}$$

The left-hand side of equation (A22) is decreasing in $\lambda(0)$. Therefore, $\lambda(0)$ must rise to satisfy equation (A17). By Lemma 3, the initial stock price must rise. ∎

Proof of Proposition 5: The market stock price under the pessimist's beliefs given by (11) and the pessimist's marginal valuation for the stock given by (50) follow backward stochastic differential equations with $S(T) = 0$. The comparison theorem presented in El Karoui et al. (1997) is directly applicable leading to $S(t) \geq S^p(t)$ almost surely for any time t.

B The Malliavin Derivatives

When the exogenous output process follows a geometric Brownian motion, the Malliavin derivatives $\mathcal{D}_t\delta(s)$ and $\mathcal{D}_t\lambda(s)$ are given by:

$$\begin{aligned}
\mathcal{D}_t\delta(s) &= \delta(s)\left[\sigma_\delta + \frac{\nu^o(t)}{\sigma_\delta}(s-t)\right] > 0, \\
\mathcal{D}_t\lambda(s) &= -\lambda(s)\left[\sigma_\lambda(t) + \int_t^s \sigma_\lambda(z)\mathcal{D}_t\sigma_\lambda(z)dz + \int_t^s \mathcal{D}_t\sigma_\lambda(z)dw^o(z)\right],
\end{aligned}$$

where $\mathcal{D}_t\sigma_\lambda(z)$ is given by

$$\mathcal{D}_t\sigma_\lambda(z) = \begin{cases} \mathcal{D}_t\bar{\mu}(z) & \text{if } \sigma_\lambda(z) = \bar{\mu}(z), \\ \frac{\partial\kappa_c^0(z)}{\partial\delta(z)}\mathcal{D}_t\delta(z) + \frac{\partial\kappa_c^0(z)}{\partial\lambda(z)}\mathcal{D}_t\lambda(z) & \text{if } \sigma_\lambda(z) = \kappa_c^o(z). \end{cases}$$

C Numerical Appendix

The equilibrium stock price (40) and stock volatility (49) can be written as conditional expectations under the optimist's beliefs that are driven by the stochastic processes $\delta(s)$, $\lambda(s)$, $\mu_\delta^o(s)$, $\mu_\delta^p(s)$, $\mathcal{D}_t\lambda(s)$, and $\mathcal{D}_t\delta(s)$.

Since the output and weighting processes are strictly positive processes, we express them in exponential form:

$$\begin{aligned}\delta(s) &= \delta(0)\exp\left(\int_0^s \mu_\delta^o(t) - \frac{1}{2}\sigma_\delta^2 dt + \int_0^s \sigma_\delta dw^o(t)\right), \\ \lambda(s) &= \lambda(0)\exp\left(-\int_0^s \frac{1}{2}\sigma_\lambda^2(t)dt - \int_0^s \sigma_\lambda(t)dw^o(t)\right).\end{aligned}$$

The stochastic processes are then simulated forward along with the processes $\mu_\delta^o(s)$, $\mu_\delta^p(s)$, $\mathcal{D}_t\lambda(s)$, and $\mathcal{D}_t\delta(s)$ using a Milstein scheme[1] with a time-step of 0.0005 years.

The pathwise integrals in the stock price (40) and stock volatility (49) are calculated using the trapezoid rule with 80,000 paths used to calculate each expectation.

[1] For the processes $\delta(s)$ and $\mu_\delta^o(s)$ the Milstein scheme collapses to an Euler scheme.

Table 1: **Numerical example with optimist's intertemporal elasticity of substitution greater than one**

	5% Optimist Output Growth				6% Optimist Output Growth			
	0% Belief Vol.		1% Belief Vol.		0% Belief Vol.		1% Belief Vol.	
	Unconst.	Const.	Unconst.	Const.	Unconst.	Const.	Unconst.	Const.
Equilibrium Prices								
Price Dividend Ratio								
Equilibrium	18.6	17.5	17.9	17.5	19.6	18.0	19.2	18.0
Pessimist's Valuation	–	16.0	–	16.5	–	13.9	–	15.8
% Overvaluation	–	8.9	–	6.0	–	29.1	–	13.6
Stock Volatility	3.8	2.3	6.1	3.9	6.8	2.3	8.5	4.0
Fraction of volatility from lambda	46.2	7.9	41.4	4.6	69.7	9.8	58.9	5.8
Instantaneous interest rate	4.7	6.4	4.8	6.4	−0.3	7.2	0.4	7.2
Market price of risk	0.24	0.02	0.23	0.02	0.56	0.03	0.49	0.03
Stock expected return								
Optimist	5.6	6.5	6.2	6.5	3.5	7.2	4.6	7.3
Pessimist	3.7	5.3	3.1	4.6	−3.4	4.9	−4.0	3.3
Stock risk premium								
Optimist	0.9	0.1	1.4	0.1	3.8	0.1	4.2	0.1
Pessimist	−1.0	−1.1	−1.7	−1.9	−3.0	−2.3	−4.3	−3.9
Equilibrium Allocation								
Bond Open Interest	395	50	273	50	386	50	350	50
Stock Positions								
Optimist	901	200	648	200	937	200	808	200
Pessimist	−683	0	−445	0	−647	0	−595	0
Optimist Hedging								
Delta	−42	−6	−14	39	−74	−7	−43	38
Lambda	0	0	0	0	0	0	0	0
Constraint Properties								
Fraction of time Constraint Binding	–	100	–	100	–	100	–	100

The table reports the equilibrium prices and allocations when beliefs are heterogeneous and the short-sale constraint initially binds. The optimists have initial output growth priors of 5% and 6% respectively. Optimists and pessimists have common belief volatilities of 0% and 1%. Columns labeled "UnConst." correspond to equilibria with no short sale constraint imposed, while those labeled "Const." are equilibria where the pessimists are short sale constrained. The largest Monte Carlo 95% confidence interval for the price/dividend ratios and the stock volatilities in the table has a relative size smaller than 0.4%. Parameter choice: $x^o = x^p = 0.5$, $\delta(t) = 1.0$, $\sigma_\delta = 0.02$, $\rho = 0.03$, $\gamma = 0.5$, and $T - t = 20$.

Table 2: **Numerical example with optimist's elasticity of intertemporal substitution less than one**

	5% Optimist Output Growth				6% Optimist Output Growth			
	0% Belief Vol.		1% Belief Vol.		0% Belief Vol.		1% Belief Vol.	
	Unconst.	Const.	Unconst.	Const.	Unconst.	Const.	Unconst.	Const.
				Equilibrium Prices				
Price Dividend Ratio								
Equilibrium	12.9	13.0	12.9	13.0	12.3	12.6	12.4	12.6
Pessimist's Valuation	–	12.9	–	13.0	–	11.8	–	12.1
% Overvaluation	–	0.4	–	0.1	–	6.5	–	4.0
Stock Volatility	4.0	4.7	3.6	3.9	2.3	4.9	1.9	4.2
Fraction of volatility from lambda	−32	−13	−33	−23	−130	−10	−159	−15
Instantaneous interest rate	8.8	9.5	8.8	9.5	8.8	11.0	8.9	11.0
Market price of risk	0.19	0.18	0.19	0.18	0.31	0.17	0.31	0.17
Stock expected return								
Optimist	9.6	10.3	9.5	10.2	9.6	11.8	9.4	11.7
Pessimist	8.8	9.4	8.8	9.4	8.6	9.9	8.7	10.0
Stock risk premium								
Optimist	0.77	0.83	0.68	0.68	0.7	0.8	0.6	0.7
Pessimist	−0.0	−0.1	−0.0	−0.1	−0.2	−1.1	−0.2	−1.0
				Initial Equilibrium Allocation				
Bond Open Interest	62	50	65	50	248	50	300	50
Stock Positions								
Optimist	224	200	231	200	596	200	699	200
Pessimist	−24	0	−31	0	−396	0	−499	0
Optimist Hedging								
Delta	−310	−283	−378	−358	−307	−299	−349	−356
Lambda	0	0	0	0	0	0	0	0
				Constraint Properties				
Percentage of time Constraint Binding	–	64	–	15	–	100	–	91

The table reports the equilibrium prices and allocations when beliefs are heterogeneous and the short-sale constraint initially binds. The optimists have initial output growth priors of 5% and 6% respectively. Optimists and pessimists have common belief volatilities of 0% and 1%. Columns labeled "Unconst." correspond to equilibria with no short sale constraint imposed, while those labeled "Const." are equilibria where the pessimists are short sale constrained. The largest Monte Carlo 95% confidence interval for the price/dividend ratios and the stock volatilities in the table has a relative size smaller than 0.4%. Parameter choice: $x^o = x^p = 0.5$, $\delta(t) = 1.0$, $\sigma_\delta = 0.05$, $\rho = 0.03$, $\gamma = 2$, and $T - t = 20$.