Financing Constraints, Irreversibility, and Investment Dynamics*

Andrea Caggese
Pompeu Fabra University

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Abstract

We develop a structural model of an industry with many entrepreneurial firms in order to investigate the cyclical behaviour of aggregate fixed investment, variable capital investment and output. In particular, we consider an environment in which the entrepreneurs cannot borrow unless the debt is secured by collateral and cannot sell fixed capital without liquidating their whole business. We show that when these entrepreneurs experience persistent idiosyncratic and aggregate shocks, the interplay between financing constraints and irreversibility of fixed capital helps to explain several common observations: it explains why inventory investment is very volatile and procyclical, especially during recessions, and it explains why the output and inventories of small firms are more volatile and more cyclical than that of large firms. The model is also consistent with the observations that inventory investment leads the business cycle, and that both fixed and inventory investment are sensitive to the net worth of firms, even when marginal productivity of capital is taken into account.

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1 Introduction

In this paper we develop a structural dynamic model of an entrepreneurial firm which is subject to both borrowing constraints and irreversibility of fixed capital. We use the model to simulate industry dynamics with heterogeneous entrepreneurs who experience both idiosyncratic and aggregate uncertainty, and we illustrate how the irreversibility of fixed capital amplifies the effect of financing constraints on variable capital investment. Such amplification effect is both quantitatively and qualitatively important. In particular, it helps to explain why aggregate inventory investment is very volatile and procyclical, and why the drop in inventories accounts for a large part of the decline in business spending during recessions.

This paper is motivated by a body of theoretical literature which has shown that asymmetric information and contract incompleteness may prevent firms from accessing external finance, and thus make them unable to fund profitable investment opportunities[1]. However, are these imperfections relevant for the investment decisions of firms? To what extent do they affect aggregate investment and production dynamics over the business cycle? Most of the existing empirical literature[2], following the seminal paper by Fazzari, Hubbard and Petersen (1988), addresses only the first question. It shows that investment is significantly correlated with cash flow, especially for firms likely to face capital markets imperfections. But the relevance of this finding has been seriously questioned. Kaplan and Zingales (1997 and 2000) find that the investment-cash flow correlation is stronger for firms which are financially very wealthy and surely not financially constrained[3]. Gomes (2001) as well as Ericson and Whited (2001) show that measurement errors are the most likely cause of the positive correlation between investment and cash flow.

Instead in this paper we follow a different approach. We develop a structural model of firm behaviour under financing and irreversibility constraints, and we use the model to address both questions regarding the effects of financing constraints at firm level and the consequences for aggregate investment and output dynamics. The model has three distinctive features. First, output is produced by an entrepreneur who operates a concave risky technology using two complementary factors of production: fixed and variable capital. Both factors take one period to produce output. Fixed capital cannot be disinvested unless the whole business is sold. Second, the entrepreneur's only source of external finance is debt secured by collateral asset. Therefore her borrowing capacity depends on the value of her assets. Third, the entrepreneur is risk averse and discounts future consumption at a rate higher than the market interest rate. This implies that she never accumulates enough financial wealth to eliminate the chances to face borrowing constraints.

It is well known that future expected financing constraints may affect current consumption decisions[4]. In this paper we show that, for an entrepreneurial firm, future expected

[1]Stiglitz and Weiss (1981), Besanko and Thakor (1986), Milde and Riley (1988), Hart and Moore (1998), Albuquerque and Hopenhayn (2000).

[2]See Hubbard (1998) for a review of this literature.

[3]Similar evidence is produced by Cleary (1999), who studies a larger sample of 1317 US firms.

[4]See Zeldes (1989) and Carroll (2001).

financing constraints significantly affect also investment decisions. This is because the entrepreneur, anticipating a risk of binding financing constraints in the future, reduces the investment spending in the risky technology and keeps some financial assets (or spare borrowing capacity) as a precautionary saving motive.

More importantly, we show that the effects of current and future expected financing constraints on variable capital are amplified when fixed capital is irreversible. Consider for example an entrepreneur who faces a persistent negative productivity shock. She would like to reduce the amount of wealth invested in the risky technology, which is likely to have a low return for a while. Since fixed capital is irreversible she cannot reduce it, and as a consequence she expects a lower return on her assets due to the inefficiently high level of fixed capital. If the productivity remains low for some time, her financial wealth will decrease rapidly until she may become financially constrained. Anticipating this possibility she tries to reduce the exposure to the risky technology and to increase the share of her wealth invested in risk free assets. The problem is that she can do this only by reducing her variable capital investment. Hence following a negative productivity shock, variable capital drops a lot, not only because it is the only reversible asset but also because the irreversibility of fixed capital amplifies the precautionary behaviour of the entrepreneur. This overshooting of variable capital is very large when, after a particularly big or prolonged negative shock, both constraints are contemporaneously binding. In this case, not only is fixed capital inefficiently high but, due to a binding financing constraint, variable capital is inefficiently low. Consequently expected returns decrease dramatically due to the unbalanced use of factors of production, and this increases the probability that the entrepreneur will face financing constraints also in the future. When the bad period ends and productivity starts to rise, the entrepreneur is very cautious about investing in fixed capital, fearing the consequences of the combination of irreversibility and financing constraints in case of a new negative productivity shock in the future. Thus she mainly invests in variable capital which overshoots upwards.

This discussion illustrates that the combination of irreversibility and financing constraints greatly amplifies the volatility of variable capital relative to fixed capital and hence also relative to output. Such amplification effect is more severe for smaller firms, which are more likely to suffer from both irreversibility and financing constraints[5]. The other important implication of the model is that the fluctuations in investment, especially in variable capital, are partly driven by net worth fluctuations, even when expected productivity is properly accounted for[6].

In this paper we quantify the implications of these effects for firm dynamics by simulating an artificial economy with many heterogeneous entrepreneurs. The cross-sectional distribution of net worth and fixed capital among entrepreneurs is determined by both idiosyncratic and aggregate uncertainty, and it affects the way aggregate output and in-

[5]In fact a large multi-establishment firm could avoid the irreversibility of fixed capital simply by shutting down some of the establishments, and it is also less likely to face the informational and contractual problems that prevent a smaller firm to access financial markets.

[6]In reality investment, productivity and net worth are highly correlated in the business cycles, even in the absence of financing imperfections. Hence we mean that financing imperfections in our model imply that investment is very sensitive to net worth even conditional on the productivity shock.

vestment react to aggregate shocks. Although the model is relatively stylised, we can use the results of the simulations to explain the observed cyclical fluctuations of aggregate inventory investment, fixed investment and output. In fact, because of the time lag for the installed capital to produce output, the stock of variable capital at the end of one period can be interpreted as input inventories, such as raw materials and work in progress. Likewise the change in the stock of variable capital can be interpreted as inventory investment.

We calibrate the model in order to match the long run average investment and output of a four digit US industrial sector. Also we use the information about the total factor productivity growth for the same sector to directly estimate the moments of the distribution of the aggregate productivity shock. With the calibrated model we simulate several artificial economies over many periods, with and without financing imperfections and irreversibility of fixed capital.

The comparison of the simulated data with the empirical data shows that the interaction between financing and irreversibility constraints is able to explain why aggregate inventory investment is very volatile and procyclical, and why during downturns inventory investment accounts for a large part of the drop in business spending. This is an important result, since the high volatility and procyclicality of inventories is a stylised fact that has recently received considerable attention. This is because the drop in inventories accounts for a large part of the GDP decline in recessions[7] (Ramey and West, 1999). The choice of explaining these stylised facts focusing on input rather than output inventories is supported by Ramey (1989), who shows that input inventories are larger and much more volatile than finished goods inventories during US recessions.

Moreover, the model implies that investment, conditional on a certain productivity shock, is sensitive to internal finance. It also implies that this sensitivity is stronger for inventory investment than for fixed investment, especially for entrepreneurs with smaller businesses. This result is consistent with a study which shows that "*inventory investment for small firms absorb from 15% to 40% of cash flow fluctuations*" (Carpenter, Fazzari and Petersen, 1998). The model is also able to confirm the observed fact that inventory investment leads the fluctuations in output (Stock and Watson, 1998), and that small firms are more procyclical than large ones in terms of inventories and output. This is because in the model entrepreneurs with smaller businesses tend to have smaller net worth relative to output, and are on average more financially constrained. This result is consistent with Kashap, Lamont and Stein (1994), Gertler and Gilchrist (1994) and Oliner and Rudebusch (1996), who compare the behaviour of small versus large manufacturing firms after Romer dates, that is, episodes of tight monetary policy resulting in a recession[8].

As a robustness check of the model's ability to explain investment dynamics, we also simulate an artificial economy using the sequence of aggregate shocks estimated from the

[7]Stock and Watson (1998) report that "*Changes in business inventories, which constitute but a small fraction of total GDP, account for one-fourth of the cyclical movements in GDP*". In addition of being procyclical at business cycle frequencies, Hornstein (1998) shows that fluctuations of GDP at frequency shorter than business cycle are almost entirely driven by changes in inventories.

[8]Moreover Bernanke Gertler and Gilchrist (1996) show that one third of aggregate fluctuations in the US manufacturing sector can be accounted for by the difference between small and large firms.

chosen US manufacturing sector for the 1962-1986 period. We find that the presence of both financing and irreversibility constraints improves the ability of the model to explain aggregate and output fluctuations, in particular the large drop in inventories and output during the recession in the beginning of the 1980s.

Finally, in the last section of the paper we briefly illustrate some empirical evidence in favour of the assumption of financing constraints at firm level. We describe the results obtained by Caggese (2003), who analyses a data-set of small and medium sized Italian manufacturing firms, which includes both balance sheet data and qualitative response data about entrepreneurs' financing problems. Caggese (2003) shows that the premium of productivity of variable capital over the cost of capital, which our model predicts to be the best indicator of the intensity of financing constraints, is strongly positively correlated with the likelihood that the entrepreneurs state problems in obtaining external funds to finance new investment projects.

Three recent papers adopt a similar approach to our paper, and analyse an economy with heterogenous entrepreneurs where financing constraints are binding for a fraction of them in equilibrium: Cooley and Quadrini (2001), Cooley, Marimon and Quadrini (2002) and Gomes (2001). Cooley and Quadrini (2001) show that financing imperfections in a model of industry dynamics explain a stylised fact regarding growth dynamics of firms which is not explained by models based only on technological shocks. Cooley, Quadrini and Marimon (2002) focus on financing imperfections in the context of long term contracts between firms and banks. They show that imperfect enforceability makes the diffusion of new technologies sluggish and amplifies their impact on aggregate output. Gomes (2001) builds a model with heterogeneous firms and financing constraints that replicates some stylised facts about industry dynamics and shows that cash flow is not significant in reduced form investment regressions when average Q is properly measured. Moreover our paper is also related to the real option theory literature, and in particular to Bertola and Caballero (1994).

Yet, our paper is substantially different from all those above. We focus on the interactions between financing and irreversibility constraints as well as on business cycle dynamics rather than growth dynamics of firms. Moreover, we analyse a multifactor technology and use the model to explain the empirical evidence of both fixed and inventory investment. Thus our paper adds to the existing literature in several important ways: we model theoretically and quantify with simulations the effects of precautionary saving on the risky investment of an entrepreneurial firm. We model theoretically an amplification effect between irreversibility and financing constraints and show that such effect is essential for explaining several stylised facts regarding business cycle dynamics of aggregate investment. Thus we complement and extend the findings of Bertola and Caballero (1994) who only consider the effects of irreversibility at firm level on the behaviour of aggregate fixed investment.

The paper is organised as follows: section 2 illustrates the theoretical model; section 3 describes the solution method and simulation results; section 4 presents the conclusions.

2 The model

We consider an economy populated by many infinitely lived entrepreneurs and many competitive banks. We assume that each entrepreneur (henceforth E) chooses consumption and investment in order to maximise the expected value of her lifetime utility function. All entrepreneurs have same preferences and have access to the same technology. Utility from consumption is measured by a concave function $U(x_t)$, where x_t is consumption at time t.

$$x_t \geq 0; U'(.) > 0;\ U''(.) < 0;\ U'(0) = \infty;\ U'(\infty) = 0 \tag{1}$$

$E's$ subjective discount rate β is such that:

$$\beta R < 1$$

where $R = 1 + r$, and r is the lending/borrowing interest rate. This implies that E is impatient[9], and that she chooses her optimal capital structure by balancing her desire to borrow in order to anticipate consumption with her desire to save in order to avoid future borrowing constraints.

Regarding the technology, we assume that E can invest in the risky technology of the business she owns and manages. k_t and l_t are respectively the stock of fixed and variable capital, installed at or before time $t-1$, which will generate output at time t. Variable capital represents variable inputs such as raw materials and work in progress, while fixed capital represents fixed inputs such as plant and equipment. The "time to build" assumption implies that borrowing constraints are relevant when E needs additional funds to exploit new investment opportunities. It also allows us to interpret l_t as end of period $t-1$ inventories of variable capital. Output y_t is produced according to a Cobb-Douglas production function:

$$y_t = e^{\theta_t} k_t^{\alpha} \left(l_t + l^E\right)^{\kappa} \text{ with } \alpha > 0; \kappa > 0; \alpha + \kappa < 1 \tag{2}$$

l^E is a small fixed amount of variable capital supplied[10] each period by E. θ_t is the productivity shock. All prices are assumed constant and normalised to 1. Regarding the factors of production, variable capital is non-durable, while fixed capital is durable:

$$1 = \delta_l > \delta_k \tag{3}$$

δ_l and δ_k are respectively the depreciation factors of variable and fixed capital[11]. Moreover variable capital is reversible, while fixed capital is irreversible and can only be disinvested if E sells her whole business. In this case she cannot start a new business and must retire[12]. Irreversibility of fixed capital is justified by the fact that in many industries plant and equipment do not have a secondary market because they cannot be easily converted to other productions. Yet we allow fixed capital to be used as collateral by assuming that

[9] Kiyotaki and Moore (2002) show that such a feature arise endogenously in a general equilibrium model with heterogenous agents and financing imperfections.

[10] It can be interpreted as E's effortless labour supply.

[11] Full depreciation of variable capital is not necessary, but it simplifies the exposition.

[12] Once retired, E simply lives forever of her financial assets.

such conversion is easier if the whole of the assets is sold. The assumption that fixed capital is irreversible conditional on the continuation of the activity is consistent with the empirical evidence from a very large sample of US manufacturing plants analysed by Caballero, Engel and Haltiwanger (1995). Therefore, conditional on continuation, E is subject to the following constraints:

$$k_{t+1} \geq (1 - \delta_k) k_t \tag{4}$$

$$l_{t+1} \geq 0 \tag{5}$$

Equation (4) is the irreversibility constraint. The productivity shock θ_t is assumed to follow a stationary autoregressive stochastic process:

$$\begin{aligned} \theta_t &= \overline{\theta} + \rho\theta_{t-1} + \zeta_t; \text{ with } 0 \leq \rho < 1 \\ \zeta_t &\sim iid\left(0, \sigma_\zeta^2\right) \end{aligned} \tag{6}$$

In order to introduce financial markets imperfections in the model we assume that equity finance and risky debt are not available. At time t E can borrow from (and lend to) the banks one period debt, with face value b_{t+1}, at the market riskless rate r. A positive (negative) b_{t+1} indicates that E is a net borrower (lender). Banks only lend secured debt, and the only collateral they accept is the next period´s residual value of physical capital. Therefore at the time t the amount of borrowing is limited by the following constraint[13]:

$$b_{t+1} \leq \tau_k k_{t+1} \tag{7}$$

τ_k is the share of fixed capital value that can be used as collateral[14]:

$$\tau_k \leq 1 - \delta_k \tag{8}$$

From equations (3) and (8) it follows that $\tau_l = 0$. The timing of the model is represented in figure 1: E inherits from time $t-1$ the stocks of fixed and variable capital k_t and l_t, and the stock of debt b_t. Then θ_t is realised, y_t is produced and b_t repaid. Residual wealth w_t is:

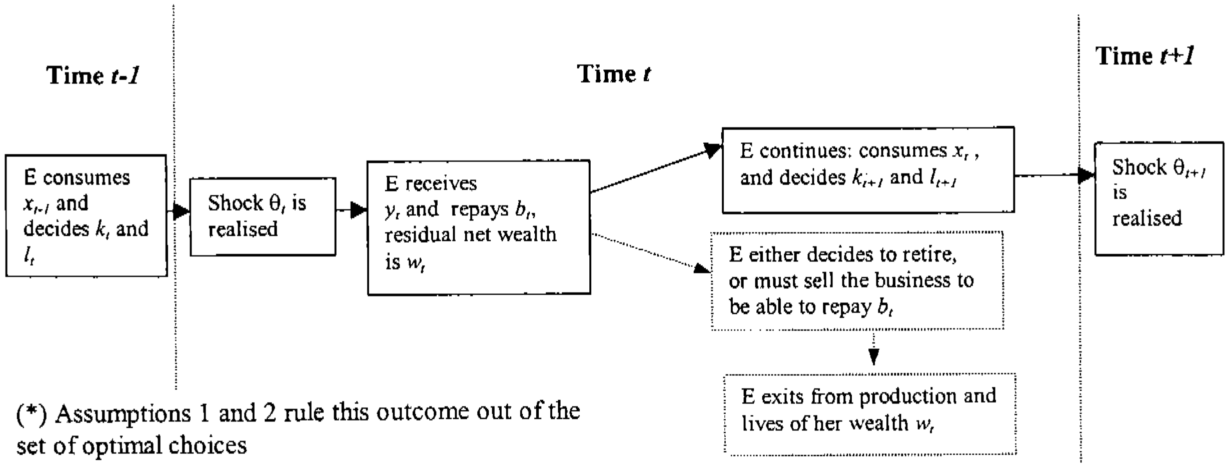
$$w_t = y_t + (1 - \delta_k)k_t - b_t \tag{9}$$

After producing E either retires or continues activity. The interaction between financing and irreversibility constraints implies that both forced and voluntary retirement can happen in equilibrium. This is because after a negative shock E may be forced to sell the fixed assets to repay the debt. But even if she manages to repay the debt without being forced to sell the assets and retire, she may be left with no funds to invest in variable

[13] The rationale for this collateral constraint is that E can hide the revenues from the production. Being unable to observe such revenues the lenders can only claim, as repayment of the debt, the value of E's physical assets (Hart and Moore, 1998). Therefore E can only lend or borrow one period secured debt at the market interest rate r offered by the banks. We also implicitly assume that in any default and renegotiation of the debt with the bank E has all the bargaining power. Otherwise the bank could use the threat of liquidation of fixed capital to enforce the repayment of uncollateralised debt.

[14] $\tau_k < 1 - \delta_k$ can be motivated by assuming that E can 'steal' a $1 - \tau_k$ fraction of the residual value of capital.

Figure 1: The timing of the model



capital, and hence be unable to generate output. If the negative shock is persistent, then the expected return from the risky technology may be lower than the expected return from selling the assets and retire. While this is an interesting intuition to explore in future research, it goes beyond the scope of this paper. Therefore in order to simplify the analysis[15] we restrict the set of parameters so that forced and voluntary retirement never happen in equilibrium. More specifically, we assume the following:

Assumption 1: $l^E \geq l^E_{\min}(\Theta)$

Assumption 2: *w_0 and Θ are such that: i) the net present value of the expected utility from consumption is always higher conditional on continuing activity than on retiring; ii) constraint (5) is never binding with equality.*

Θ is the vector of parameters: $\Theta' = \left\{\overline{\theta}, \rho, \sigma^2_{\zeta}, \beta, R, \delta_k, \tau_k, l^E\right\}$. Assumptions *1* and *2* imply that E does not retire voluntarily, is never forced to retire, and that it is never optimal to sell part of l^E rather than using it in her own production (proof in Appendix A). $l^E_{\min}$ is defined in equation (44) of Appendix A. It represents the minimum amount of variable input that always allows E to generate enough revenues to repay the debt without liquidating k_t. Therefore she is never forced to retire. Assumption *2* ensures that E never voluntary retires. It is important to note that assumptions *1* and *2* do not affect the qualitative results of the model. In fact, they rule out the extreme outcomes that would increase the expected cost of irreversibility and financing constraints, and therefore would strengthen rather than weaken the results of the model.

In order to continue activity E borrows one period debt with face value b_{t+1}, receiving the discounted value b_{t+1}/R. The net worth w_t plus the new borrowing b_{t+1}/R are allocated between consumption and investment. Therefore the budget constraint faced by E is the following:

$$x_t + l_{t+1} + k_{t+1} = w_t + b_{t+1}/R \tag{10}$$

[15]The presence of endogenous retirement would imply that the discount factor of the problem becomes endogenous. This would make the dynamic maximisation problem extremely difficult to solve, even numerically.

We denote the expected lifetime utility at time t of E, after θ_t is realised, by $V_t(w_t, \theta_t, k_t)$, where w_t, θ_t and k_t are the three state variables of the problem.

$$V_0(w_0, \theta_0, k_0) = \underset{\begin{pmatrix} k_{t+1} = k(w_t, \theta_t, k_t) \\ l_{t+1} = l(w_t, \theta_t, k_t) \\ b_{t+1} = b(w_t, \theta_t, k_t) \end{pmatrix}_{t=0,1,\ldots\infty}}{MAX} E_0 \left\{ \sum_{t=0}^{\infty} \beta^t U(x_t) \right\} \tag{11}$$

The problem is defined by equation (11) subject to constraints (4), (7) and (10). These constraints define a compact and convex feasibility set for l_{t+1}, k_{t+1}, b_{t+1} and x_t, and the law of motion of w_{t+1} conditional on w_t, k_t and θ_t is continuous. Therefore, given the assumptions on θ_t and the concavity of the production function, a unique solution to the problem exists[16]. Let μ_t and λ_t be the Lagrangian multipliers associated to the constraints (4) and (7). Taking the first order conditions of equation (11) with respect to b_{t+1}, l_{t+1} and k_{t+1} it is possible to show that the solution of the problem is given by the optimal sequence of $\{k_{t+1}, l_{t+1}, x_t, \lambda_t, \mu_t \mid k_t, w_t, \theta_t, \Theta\}_{t=0}^{\infty}$ which satisfies equations (4), (12), (13), (14) and (15), plus the standard complementary slackness conditions on λ_t and μ_t:

$$U'(x_t) = \beta R E_t [U'(x_{t+1})] + R\lambda_t \tag{12}$$

$$U'(x_t) = \beta E_t [U'(x_{t+1})(MPK_{t+1} + 1 - \delta_k)] + \mu_t + \lambda_t \tau_k - \beta(1-\delta_k) E_t(\mu_{t+1}) \tag{13}$$

$$U'(x_t) = \beta E_t [U'(x_{t+1}) MPL_{t+1}] \tag{14}$$

$$D_k k_{t+1} + l_{t+1} + x_t \leq w_t \tag{15}$$

$MPK_{t+1} = \frac{\partial y_{t+1}}{\partial k_{t+1}}$ and $MPL_{t+1} = \frac{\partial y_{t+1}}{\partial l_{t+1}}$ are the marginal productivities of fixed and variable capital respectively. $D_k = 1 - \frac{\tau_k}{R}$ is the downpayment required to purchase one additional unit of fixed capital. Equation (15) combines the budget constraint (10) and the collateral constraint (7) and implies that the downpayment necessary to buy k_{t+1}, l_{t+1} and x_t must be lower than E's net wealth[17]. λ_t is positive when the financing constraint is binding, and is equal to zero otherwise. The term $(1-\delta)\beta E_t(\mu_{t+1})$ is the cost of future expected irreversibility constraints. μ_t is positive when the irreversibility constraint is binding, and is equal to zero otherwise. Since it is not possible to obtain an analytical solution to this problem, we will solve it in the next section using a numerical method. In the remaining part of this section we will describe the main qualitative features of the model. We first analyse the solution without financing problems, then we analyse the solution without irreversibility problems, and finally we explain how the two problems interact with each other. In order to illustrate the intuition of the model, let us use equation (12) to rewrite equations (13) and (14) in the following way:

[16] See Lucas and Stokey (1989), chapter 9.2.

[17] The optimal choices of k_{t+1}, l_{t+1} and x_t determine the optimal level of net financial wealth but do not determine how such wealth is allocated between debt and financial assets. In fact given that lending and borrowing rates are the same, when the borrowing constraint is not binding E is indifferent between borrowing up to the limit and investing in financial assets and keeping some spare borrowing capacity.

$$E_t\left(MPK_{t+1}\right) = UK + \frac{1}{E_t\left[U'\left(x_{t+1}\right)\right]}\left\{\frac{1}{\beta}\left[\left(R-\tau_k\right)\lambda_t - \mu_t\right] - cov_{t+1}^{xk} + \left(1-\delta_k\right)E_t\left(\mu_{t+1}\right)\right\} \tag{16}$$

$$E_t\left(MPL_{t+1}\right) = UL + \frac{1}{E_t\left[U'\left(x_{t+1}\right)\right]}\left(\frac{1}{\beta}R\lambda_t - cov_{t+1}^{xl}\right) \tag{17}$$

UK and UL are the user costs of fixed and variable capital respectively:

$$UK = R - \left(1-\delta_k\right); UL = R \tag{18}$$

The remaining terms on the right hand side represent current and future expected costs of financing and irreversibility constraints. cov_{t+1}^{xk} and cov_{t+1}^{xl} are the following covariances:

$$cov_{t+1}^{xk} = cov\left[U'\left(x_{t+1}\right), MPK_{t+1}\right] \tag{19}$$

$$cov_{t+1}^{xl} = cov\left[U'\left(x_{t+1}\right), MPL_{t+1}\right] \tag{20}$$

2.1 Solution with the irreversibility problem only.

In this subsection we rule out current and future expected financing constraints by assuming that the entrepreneur can borrow upfront future expected earnings. Moreover we assume that the utility function is linear in consumption, and that E discounts the future at the market interest rate:

$$U\left(x_t\right) = x_t;\ R\beta = 1 \tag{21}$$

Therefore the following transversality condition is also necessary:

$$\lim_{t\to\infty} \beta^t b_t = 0 \tag{22}$$

Linear utility and perfect markets imply that we are considering a standard profit maximising problem where E is never financially constrained and consumption and financing decisions are irrelevant for investment choices. Therefore $\lambda_t = cov_{t+1}^{xk} = cov_{t+1}^{xl} = 0$ and $E_t\left[U'\left(x_{t+1}\right)\right] = 1$ for any t. The first order conditions (16) and (17) can be reduced to the following equations:

$$E_t\left(MPK_{t+1}\right) = UK + \left(1-\delta_k\right)E_t\left(\mu_{t+1}\right) - R\mu_t \tag{23}$$

$$E_t\left(MPL_{t+1}\right) = UL \tag{24}$$

Equations (4) , (23) and (24) determine μ_t, k_{t+1} and l_{t+1}. They describe the solution to a version of a well known irreversible investment problem (e.g. see Bertola and Caballero, 1994). The main difference from the irreversible investment literature is that we allow for a reversible factor of production to be used in conjunction with the irreversible one. The intuitive consequence is that l_{t+1}, the reversible factor, is more volatile than k_{t+1}, the irreversible factor, both after a positive and a negative shock[18]. This does not necessarily imply that variable capital is volatile also in absolute terms. In fact the more the two factors of production are complementary, the more the irreversibility of fixed capital also reduces variable capital volatility.

2.2 Solution with financing constraints only

In this section we rule out current and future expected irreversibility constraints by assuming that both variable capital and fixed capital are reversible. Hence the irreversibility constraint (4) no longer applies and $\mu_t = E_t(\mu_{t+1}) = 0$ for any t. Let us first consider the case without financing constraints as well. In this case equations (16) and (17) are simplified to:

$$E_t(MPK_{t+1}) = UK \tag{25}$$

$$E_t(MPL_{t+1}) = UL \tag{26}$$

Equations (25) and (26) determine the profit maximising levels of investment k_{t+1} and l_{t+1} when E faces no constraints.

Let us now consider the case with financing constraints. It is easy to show that there exists a minimum level of net worth $\underline{w}(\theta_t, k_t)$ for which E is not currently financially constrained. Therefore there are two possibilities: if $w_t < \underline{w}(\theta_t, k_t)$ then E is financially constrained at time t, because she borrows up to the limit without exhausting all profitable investment opportunities. Equation (15) is binding with equality, which means that one additional unit of wealth allows E to increase either k_{t+1} by $1/D_k$ units, or l_{t+1} and x_t by 1 unit. In this case equations (12), (15), (16) and (17) evaluated for $\mu_t = E_t(\mu_{t+1}) = 0$ determine λ_t, x_t, k_{t+1} and l_{t+1}. λ_t is greater than zero and it represents the shadow cost of not being able to increase investment because of the lack of additional borrowing. Therefore financing constraints affect both investment and consumption decisions. More precisely, by substituting recursively in equation (12) we obtain:

$$U'(x_t) = R \sum_{j=0}^{\infty} \beta^j E_t(\lambda_{t+j}) \tag{27}$$

Equation (27) shows that the expected marginal utility from consumption is increasing in the shadow cost of current and future expected borrowing constraints.

[18] This is evident from the comparison between equations (23) and (24). After a negative productivity shock at time t that reduces the marginal productivity of capital, E reduces l_{t+1}, to ensure that equation (24) is satisfied. When constraint (4) is binding, k_{t+1} cannot be reduced, and as a consequence marginal productivity of capital is lower than the user cost UK. This is compensated by a positive μ_t on the right hand side of equation (23). Instead after a positive productivity shock E wants to invest more in both factors. Therefore constraint (4) is not binding and $\mu_t = 0$. Instead $E_t(\mu_{t+1}) > 0$ because, applying the same reasoning made before, constraint (4) can be binding at time $t+1$ conditional on a future negative shock. The positive $E_t(\mu_{t+1})$ represents the cost associated to future expected irreversibility. Such cost increases the required marginal productivity of fixed capital $E_t(MPK_{t+1})$, thereby reducing k_{t+1}. Therefore k_{t+1} increases less than l_{t+1} after a positive shock.

The other possibility is that $w_t \geq \underline{w}(\theta_t, k_t)$. In this case E has enough resources to invest and consume and $\lambda_t = 0$. Since in this section we assume that also $\mu_t = E_t(\mu_{t+1}) = 0$, equations (16) and (17) become:

$$E_t(MPK_{t+1}) = UK - \frac{cov_{t+1}^{xk}}{E_t[U'(x_{t+1})]} \tag{28}$$

$$E_t(MPL_{t+1}) = UL - \frac{cov_{t+1}^{xl}}{E_t[U'(x_{t+1})]} \tag{29}$$

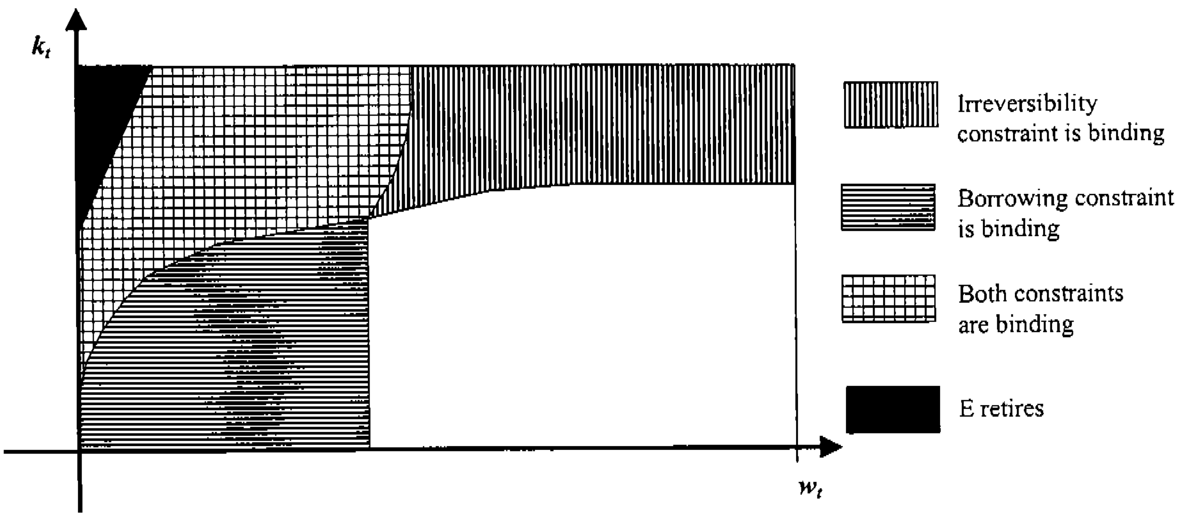
Equations (28) and (29) determine the optimal investment k_{t+1} and l_{t+1} when E is unconstrained today but could face borrowing constraints in the future. The covariances cov_{t+1}^{xk} and cov_{t+1}^{xl} between marginal productivities and marginal utility of consumption (see equations 19 and 20) are negative. This would be the case also in the absence of financial imperfections, because a positive shock which increases marginal productivity also increases permanent wealth and consumption. But in the case of borrowing constraints such covariances become much larger in absolute value, especially when w_t is small. This is because the closer w_t is to $\underline{w}(\theta_t, k_t)$, the more likely it is that the borrowing constraint will be binding in the future, conditional on a negative productivity shock. Hence such shock at the same time reduces marginal productivity of capital and increases marginal utility of consumption which is sensitive to financing constraints (see equation 27). Equations (28) and (29) show that the negative covariances cov_{t+1}^{xk} and cov_{t+1}^{xl} increase the required return on the investment in the risky technology, thus lowering the optimal investment levels k_{t+1} and l_{t+1}. Therefore future expected financing constraints reduce the optimal investment choices of a risk averse E. It is well known in consumption literature that financing constraints may reduce current consumption because of a precautionary saving motive (see Zeldes, 1989 and Carroll, 2001). This is to our knowledge the first paper to show the same effect for investment decisions. In section 3 we will show that, for realistic parameter choices, this precautionary saving effect on investment is quantitatively important.

It is finally worth mentioning that financing constraints do not alter the optimal mix between fixed and variable capital, when next period's residual fixed capital is fully collateralisable (see a formal proof in appendix B). The intuition is that the advantage of fixed capital over variable capital for an unconstrained E (lower user cost) is identical to the advantage for a constrained E (collateral value). This result means that current and future expected financing problems affect the trade off between safe and risky investment but not necessarily the mix between inputs in the risky technology. This result is reversed when financing and irreversibility constraints coexist, as it is shown in the simulations in section 3.

2.3 Solution with the financing and the irreversibility constraints.

We now consider the solution of the problem with both constraints. Figure 2 summarises the different types of optimal policy functions $k_{t+1}(w_t, k_t \mid \theta_t)$ and $l_{t+1}(w_t, k_t \mid \theta_t)$ in the $\{k_t, w_t\}$ space, conditional on a certain productivity shock. The black area on the top left corresponds to a situation in which E is either forced to sell the firm to repay the debt,

Figure 2: Policy functions for a given productivity shock



or finds convenient to retire. This area is ruled out by assumptions *1* and *2*. Instead of describing in detail such solution, we focus only on the most interesting feature: the fact that irreversibility and financing constraints interact and amplify each other. When both constraints are binding, μ_t is determined by equation (16). By substituting recursively we obtain:

$$\mu_t = \sum_{j=0}^{\infty} (1-\delta)^j \left[\Gamma_{t+1+j} + \frac{1}{\beta} (R - \tau_k) E_t (\lambda_{t+j}) \right] \tag{30}$$

Equation (30) shows that μ_t, the shadow cost of the irreversibility of fixed capital, increases with the present and expected costs of financing constraints. The term Γ_{t+1+j} on the right hand side is the following:

$$\Gamma_{t+1+j} = E_t [UK - MPK_{t+1+j}] E_t [U'(x_{t+1+j})] - cov_{t+j}^{xk} \tag{31}$$

From equations (28) and (31) it follows that Γ_{t+1+j} is equal to zero if no constraint is binding, and is positive if fixed capital is inefficiently high due to the irreversibility constraint. Hence Γ_{t+1+j} represents the expected cost of overinvestment. The term $\frac{1}{\beta}(R - \tau_k) E_t (\lambda_{t+j})$ represents an additional[19] cost of overinvestment caused by current and future expected financing constraints. Therefore financing constraints increase the cost of irreversibility μ_t. But this implies that they also increase the value of $E_t (\mu_{t+1})$ when the irreversibility constraint is not binding, and amplify the cautious investment effect on fixed capital. More importantly, the reverse is also true: the irreversibility constraint increases the chances of facing financing constraints now, because fixed capital cannot be liquidated, and also in the future, because the unbalanced use of factors leads to lower wealth accumulation. A quantification of this amplification effect is computed in the next section in figure 7.

[19]An indirect cost of future expected financing constraints is already in Γ_{t+1+j}, because $E_t [U'(x_{t+1+j})]$ increases in λ_{t+j+1}.

Figure 3: Summary statistics

	Output	Fixed Inputs	Variable Inputs
Empirical Data*	4695	2319	3320
Simulated Data**	4722	2259	3310

* Average (1962-1995) values for the average firm in the Industrial Machinery Sector. Values in thousands of US$ at constant 1987 prices; ** average values for an entrepreneur in the simulated economy.
Empirical variables are measured as follows: output is the value of industry shipments; fixed inputs are plant and equipment; variable inputs are total cost of materials and labour plus changes in inventories.

3 Numerical Solution and simulation

3.1 Model's solution

We solve the dynamic nonlinear system of equations defined by (4), (12), (13), (14) and (15) using a numerical method (see appendix C). Adding the subscript i to indicate the *i-th* entrepreneur, the solution consists of the optimal policy functions $k_{i,t+1}(w_{i,t}, \theta_{i,t}, k_{i,t})$ and $l_{i,t+1}(w_{i,t}, \theta_{i,t}, k_{i,t})$, the associated Lagrange multipliers $\lambda_{i,t}(w_{i,t}, \theta_{i,t}, k_{i,t})$ and $\mu_{i,t}(w_{i,t}, \theta_{i,t}, k_{i,t})$ and the value function $V_{i,t}(w_{i,t}, \theta_{i,t}, k_{i,t})$. We model preferences with a logarithmic utility function:

$$U(x_{i,t}) = \log(x_{i,t}) \tag{32}$$

Uncertainty regarding the output is modeled in the following way:

$$y_{i,t} = e^{\theta_{i,t}} k_{i,t}^{\alpha} \left(l_{i,t} + l^E\right)^{\kappa} \tag{33}$$

Where:

$$\theta_{i,t} = \theta_{i,t}^{f} \varepsilon_t \tag{34}$$

$\theta_{i,t}^{f}$ is the idiosyncratic productivity shock, and ε_t is the economy-wide shock common to all entrepreneurs. Both are first order autoregressive stochastic processes with autocorrelation coefficients ρ_θ and ρ_ε and variances σ_θ^2 and σ_ε^2 respectively. In practice, $\theta_{i,t}^{f}$ and ε_t are modeled as a two states and a 6 states symmetric Markow process respectively. We calibrate the model to match the aggregate yearly data on output and capital stock for the "Industrial Machinery Sector" in the US from 1962 to 1995 (source: NBER-CES manufacturing industry database). Therefore in this section the ability of the model to explain the cyclical fluctuations of aggregate output and investment will be measured by comparing the simulated statistics from the artificial economy with the corresponding empirical data for the US "Industrial Machinery Sector". This sector has been chosen as a generic representative one, but the theoretical model can be applied to any other sector where productive units use a combination of reversible and irreversible factors of production[20] and can be subject to borrowing constraints. The main advantage of using a single four digits sector, rather than the whole manufacturing industry, is that we have

[20]It is not necessary for the result that the collateral is provided by the irreversible factor. Moreover the results are quantitatively significant also when the depreciation rate of the irreversible factor is high, as is the case, for example, of technology spending.

Figure 4: Summary of parameters

$\alpha = 0.09$	$\sigma_\varepsilon = 0.024$
$\kappa = 0.86$	$\rho_\theta = 0.45$
$\delta = 0.15$	$\sigma_\theta = 0.11$
$\overline{\theta}^f = 0.643$	$\beta = 0.99/(1+r)$
$r = 0.03$	$\tau = 0.7(1\text{-}\delta)$
$\rho_\varepsilon = 0.91$	$l^E = 510$

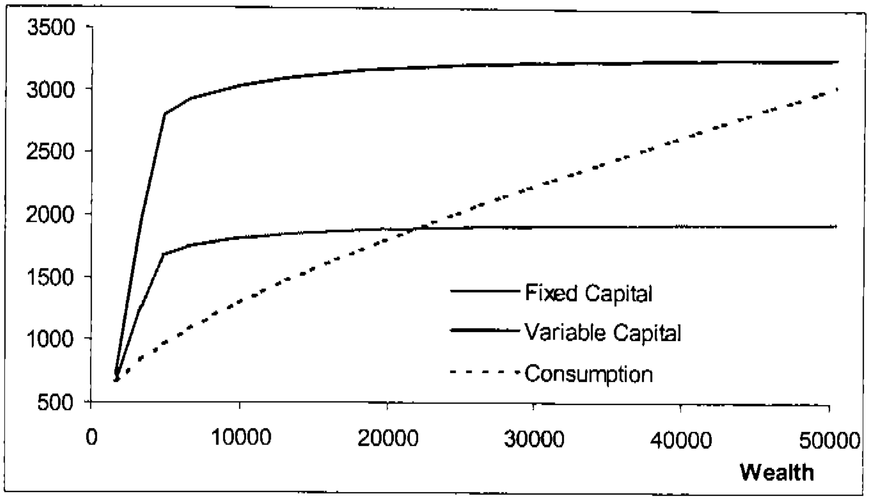
less aggregation problems in the empirical data. Moreover the chosen sector is populated by many firms, 4963 according to the 1997 census of the manufacturing industry, the majority of which are small firms: 63% of all the establishments have less than 20 employees, and more than 95% of the firms have only one establishment. Therefore it is reasonable to assume that a large share of these firms may be affected by borrowing and irreversibility constraints, like the entrepreneurial businesses in the simulated artificial economy.

The chosen parameter values are reported in figure (4). The technological parameters α, κ, δ and $\overline{\theta}^f$, given the chosen annual real interest rate $r = 3\%$, imply that the steady state level of output, fixed capital and variable capital for an entrepreneur in the simulated economy match the same statistics for the average firm in the US industrial machinery sector, as is shown in figure 3. The parameters of the aggregate shock ρ_ε and σ_ε are directly estimated using the information, included in the NBER dataset, about the total factor productivity growth for this sector. The parameters of the firm specific shock ρ_θ and σ_θ are chosen conservatively with respect to the parameters adopted in similar studies which analyse industry dynamics, like Cooley and Quadrini (2001) and Gomes (2001)[21]. Among the remaining parameters the value of l^E implies that the variable inputs provided by the entrepreneur matter for less than 2% of the average value of external inputs during expansion periods. The value of β implies that $\beta(1+r) = 0.99 < 1$, so that agents are impatient in their consumption decisions. Finally, the value of τ implies that 70% of the residual fixed capital is collateralisable.

Figures 5, 6 and 7 illustrate the policy functions for selected values of $\theta^f_{i,t}, \varepsilon_t$ and $k_{i,t}$. In figure 5 we plot $k_{i,t+1}(w_{i,t})$ and $l_{i,t+1}(w_{i,t})$, which are the investment decisions as a function of financial wealth, for a productive firm ($\theta^f_{i,t}$ is high) in a recessive economy (ε_t is the lowest among the six states of the world), for $k_{i,t} = 0$ (irreversibility is not binding). For very small wealth levels the financing constraint is currently binding and the policy functions of fixed and variable capital are very steep. The kink in these two functions represents the wealth level for which the borrowing constraint is no longer binding. After the kink, optimal capital is still increasing in wealth because the increase in wealth decreases future expected financing problems. This "precautionary saving effect" is quantitatively important. Consider two entrepreneurs who are identical in everything

[21]The comparison with these two studies is relevant given that both concern the same topic of our work and both calibrate industry dynamics using yearly data. Gomes 2001 chooses the values of $\rho_f = 0.62$ and $\delta_f = 0.15$, while Cooley and Quadrini 2001 choose higher standard deviation ($\delta_f = 0.28$) but no persistency ($\rho_f = 0$). The results we obtain in this section are however robust to a wide range of such parameters.

Figure 5: Policy functions: high idiosyncratic shock and lowest aggregate shock.



except their financial wealth, and who are both currently not financially constrained. The richer entrepreneur invests in the risky technology up to 18% more than the poorer entrepreneur, the difference being due to precautionary saving[22].

Figure 6 represents the policy functions for a productive firm ($\theta_{i,t}^{f}$ is high) in a booming economy (ε_t is the highest) with $k_{i,t} = 0$. The optimal unconstrained level of capital in this situation is six times larger than in figure 5. Therefore the region with a binding borrowing constraint is larger, given the high investment needs of E. But when the borrowing constraint is not binding, E has few future expected financing problems. As a result the richer unconstrained entrepreneur invests only 11% more in the risky technology with respect to the poorer one.

The left hand side of figure 7 represents an entrepreneur with the same productivity shock as in figure 5 but with $k_{i,t} = 9875$. This entrepreneur has too much fixed capital given the current state of the economy, and $k_{i,t+1}$ is therefore a flat line equal to $(1-\delta)\, k_{i,t}$, which indicates that the irreversibility constraint is binding. Consequently the region in which the borrowing constraint is binding is now larger than in figure 5. This is because fixed capital is too high, and less wealth is left for investment in variable capital. Moreover when the borrowing constraint is not binding the inefficiently high level of fixed capital makes it optimal to also increase the investment in variable capital. This is why l_{t+1} is three times higher in figure 7 than in figure 5, despite the entrepreneur is subject to the same productivity shock in both figures. This high investment makes the entrepreneur more vulnerable to future productivity shocks. For example if there is a sequence of negative aggregate shocks, financial wealth decreases more rapidly towards the binding constraints region, and variable capital declines faster as well, moving leftward along the policy function schedule. The consequence is that variable capital is more volatile than it is the case when fixed capital is reversible.

[22]Such precautionary saving is mainly due to future expected financing constraints, but not entirely. Also in the absence of financing imperfections there would be some precautionary saving of the poorer entrepreneur. But, for the set of parameters chosen in the paper, this would be only around 3-4%.

Figure 6: Policy functions: high idiosyncratic shock and highest aggregate shock.

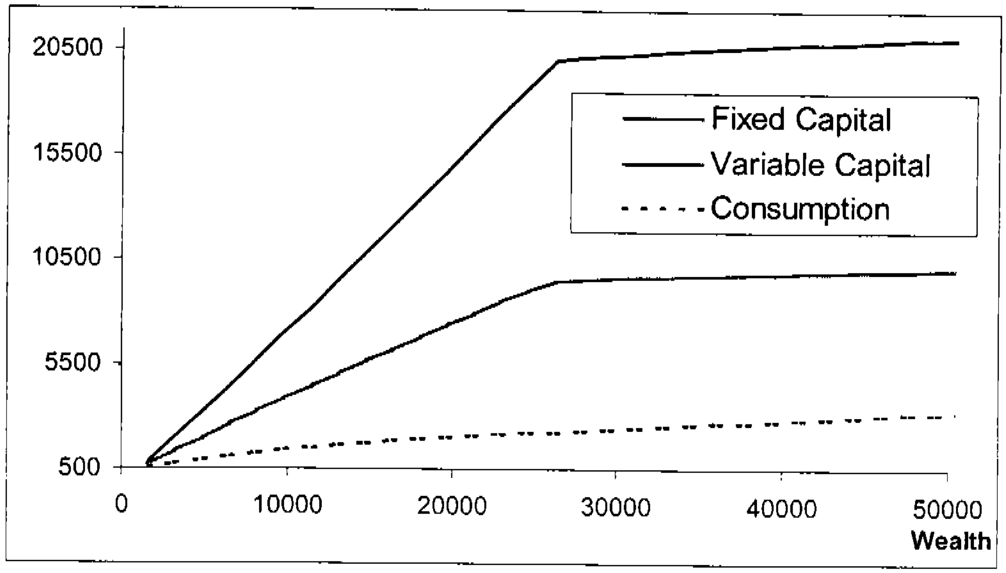
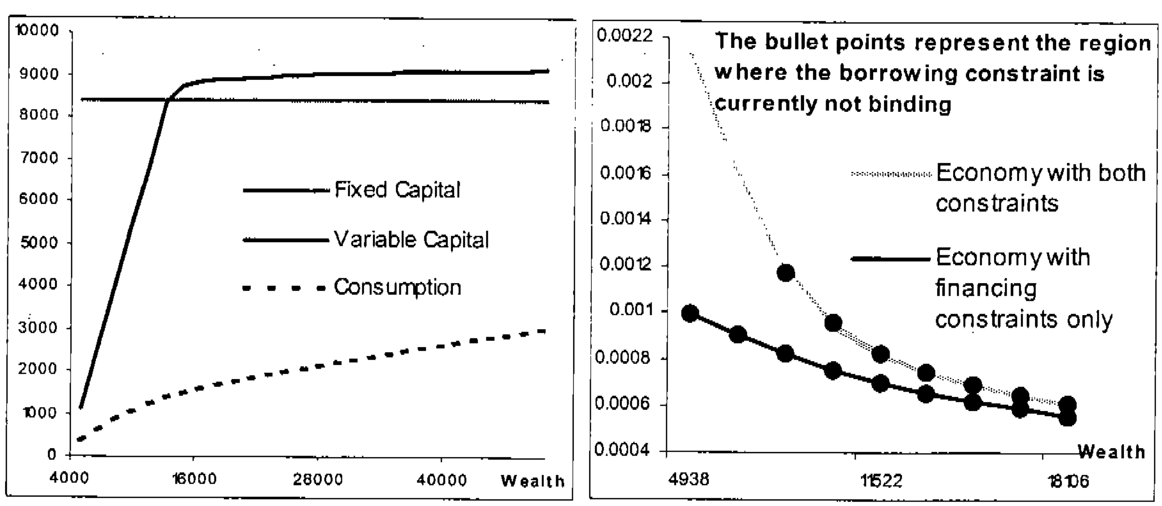


Figure 7: **On the left:** policy functions for high idiosyncratic shock, lowest aggregate shock and binding irreversibility constraint on fixed capital. **On the right:** present discounted value of the cost of future expected borrowing constraints



In order to quantify the amplification effect of irreversibility on borrowing constraints, it is useful to rearrange equation (27):

$$\sum_{j=1}^{\infty} \beta^j E_t (\lambda_{t+j}) = \frac{U'(x_{i,t}) - R\lambda_{i,t}}{R} \tag{35}$$

Using equation (35) we can compute $\sum_{j=1}^{\infty} \beta^j E_t (\lambda_{t+j})$, which is the discounted sum of the shadow cost of future expected financing constraints. Figure 7 illustrates the value of $\sum_{j=1}^{\infty} \beta^j E_t (\lambda_{t+j})$ as a function of financial wealth for an entrepreneur who suffers the worst possible productivity shock[23]. We compare two different economies with and without irreversibility of fixed capital. The figure shows that the cost of future expected financing constraints is much higher in the economy with irreversibility of fixed capital. More importantly, this is true also when the borrowing constraint is not currently binding. This means that the irreversibility of fixed capital amplifies not only current but also future expected borrowing constraints.

3.2 Dynamics of aggregate output and investment

We now use the model's solution to simulate the investment and production path of many heterogeneous entrepreneurs. The aim is to show how the combination of irreversibility and financing constraints generates a behaviour of aggregate investment and output consistent with the empirical evidence. In the simulated economy the behaviour of aggregate investment and production depends on the heterogeneity of the entrepreneurs in terms of the state variables. All entrepreneurs are identical ex ante, but each of them is subject to a different realisation of the idiosyncratic productivity shock $\theta_{i,t}^f$, which is uncorrelated across entrepreneurs and serially correlated for each entrepreneur. Therefore at time t entrepreneurs have different values of $w_{i,t}$ and $k_{i,t}$, depending on $\left\{\theta_{i,j}^f\right\}_{j=0}^t$. The distribution of $\{w_{i,t}, k_{i,t}\}$ depends on the parameters set Θ and on the history of aggregate shocks $\{\varepsilon_j\}_{j=0}^t$.

In this section we compare the empirical data from the US Industrial Machinery Sector with the simulated statistics generated by the simulation of an economy of 5000 entrepreneurs for 2000 periods. In each period the number of entrepreneurs is constant, since they are infinitely lived. Nevertheless, the model could be easily modified in order to allow finite lives and entry-exit dynamics. If we allowed new entrants to be less wealthy than the average existing entrepreneur we would increase the impact of borrowing constraints on aggregate investment dynamics. This would increase quantitatively the importance of some of the findings in this section, but would not modify them qualitatively. Therefore entry-exit dynamics are not necessary to generate the main results of this paper[24].

We simulate four different economies corresponding to the four versions of the model described in the previous section: without any constraint, with one of the two constraints only, and with both constraints. Figure 8 shows some statistics for the simulated

[23]the aggregate shock goes from the highest state to the lowest state and the firm specific shock goes from high to low. This entrepreneur has the same level of fixed capital than in the left hand side of figure 7.

[24]Simulation results for artificial economies with entry-exit dynamics are available upon request.

Figure 8: Aggregate volatilities: comparison between empirical data and simulated data

Variable	Empirical data	Simulated data: Both Constraints	Only financing constraint	Only irreversibility constraint	No constraints
Variable capital (l_t)	7.647[1]	11.836	13.482	22.862	35.366
Fixed capital (k_t)	1.982[2]	7.035	12.570	14.489	35.366
Consumption (x_t)	n.a.	1.492	1.270	n.a.	n.a.
Output (y_t)	6.926[3]	10.682	12.413	21.018	33.489
St. dev. of var. capital relative to output	**1.104**	**1.105**	**1.081**	**1.091**	**1.059**
Correlation between y_t-y_{t-1} and l_{t+1}- l_t	**0.229[4]** **0.367[5]**	**0.265**	**0.157**	**0.101**	**-0.056**

Note: volatility is measured as the standard deviation of percentage growth rates.

1. Cost of materials, excluded energy; 2. Real capital stock (plant + equipment); 3. Value of industry shipments; 4. l_{t+1}- l_t measured as change in cost of materials; 5. l_{t+1}- l_t measured as inventory investment.

economies compared with the empirical data of the US Industrial Machinery sector used to calibrate the model. For the empirical data we represent variable capital with the total cost of materials excluding energy. This is the empirical variable most similar to the definition of variable capital in our theoretical model[25]. However in the model the change in variable capital $l_{t+1} - l_t$ also coincides with inventory investment. The same is not true for our empirical measure of variable capital. Therefore in this section when we evaluate the ability of the model to explain inventory dynamics we compare the simulated statistics with two different empirical measures of inventory investment: one based on the changed in inventories, and one based on the change in materials expenditure. This happens for example in the last row of figure 8. Moreover in this section we do not report any simulated data about consumption in the economies without borrowing constraints because we impose, by construction, a separation between investment and consumption decisions in these economies.

The last column of figure 8 shows that in the economy without constraints aggregate capital and output, not surprisingly, have an implausibly large volatility. Such volatility is smaller in the economy with the irreversibility constraint only, but it is still very far from the empirical data[26]. On the contrary the volatilities of the economy simulated with both constraints are much closer to the empirical volatilities. Fixed capital is less volatile in comparison with the other simulated economies, while variable capital is relatively more volatile. In order to interpret this finding it is useful to remember that the irreversibility constraint induces a cautious behaviour in fixed capital investment: the entrepreneur is

[25]If we had the information about inventories in the different stages of fabrication we could have computed a more precise variable, which would have included only materials used in the period.

[26]This result is consistent with Bertola and Caballero (1994) who note that, when irreversibility of capital is the only friction to firm level investment, a model of industry dynamics can replicate observed investment volatility only with implausibly large level of the idiosyncratic shocks.

less willing to increase the size of her fixed assets because a future negative shock could make them inefficiently large. Conditional on such negative shock too much fixed capital lowers profits and cash flow, and this is particularly damaging when the entrepreneur also faces borrowing constraints. As a consequence the entrepreneur is even more cautious in investing in fixed capital when she faces both irreversibility and borrowing constraints. The residual difference in volatility between the empirical data and the simulated economy with both constraints is probably due to the partial equilibrium feature of the simulation. In the model both prices and interest rate are constant. But in reality they change in response to aggregate shock and dampen investment fluctuations.

The last two rows of figure 8 show that the economy with both constraints is the only one among the four simulated economies to be able to match the empirical data across two important dimensions: i) the volatility of variable capital relative to output; ii) the correlation between lagged changes in sales and inventory investment. Variable capital is relatively more volatile in the model with both constraints than in the other simulated models and in line with the empirical data. This is because in such economy, variable capital is the only reversible factor of production, and absorbs most of the fluctuations in financial wealth when the entrepreneur is financially constrained. Moreover financial wealth itself is more volatile, because the presence of both constraints causes an inefficient use of the factors of production, and higher volatility of profits[27].

Regarding the correlation between lagged changes in sales and inventory investment, empirical data show a positive correlation around 0.22-0.37. The correlation is negative in the economy without constraints because the productivity shock is mean reverting and affects current output more than variable capital investment, which is a forward looking variable. A positive correlation is present in the model with irreversibility only, because a positive aggregate shock reduces the number of firms with a binding irreversibility constraint, and therefore increases the sensitivity of investment to future shocks. In the model with financing constraints the correlation is also positive because a positive aggregate shock increases financial wealth and therefore increases investment by reducing current and future expected financing constraints. The sum of the two effects implies that in the economy with both constraints the correlation coefficient is higher and closer to the value observed in reality.

Figure 9 illustrates the main distributional features of the simulated economies. The economy with both constraints has an average ratio of constrained firms of 46.9%, of which 10.6% also have the irreversibility constraint binding. This value averages a very volatile ratio, which in some periods falls below 10%, when there are on average low investment needs, and in other periods rises up to 80%.

Figure 10 quantifies the effect of financing constraints on investment. We compute the percentage change in optimal capital if wealth increases by 1%, other variables kept constant. The calculation is reported only for productive entrepreneurs, defined as those with an high idiosyncratic shock in a given period. Therefore the figure shows the elasticity of investment to changes in financial wealth. For example an elasticity equal to 1.4

[27] The higher volatility of financial wealth is also reflected by the fact that consumption is more volatile in the economy with both constraints than in the economy with financing constraints only.

Figure 9: Distributional features

Variable	Simulated data: Both Constraints	Only financing constraint	Only irreversibility constraint	No constraints
Avg % of firms with a binding borrowing constraint	36.3	58.2	0	0
Avg % of firms with a binding irreversibility constraint	37.4	0	60.7	0
Avg % of firms with both constraints binding	10.6	0	0	0
Avg. ratio of debt over phisical assets	0.1115	0.1745	n.a.	n.a.
Avg. ratio of consumption over output	0.2602	0.2839	n.a.	n.a

Figure 10: Elasticity of investment to financial wealth for a productive entrepreneur**, conditional on the productivity shock and the level of fixed capital.

	variable investment		fixed investment	
	Both Constr.	Borr. const. only	Both Constr.	Borr. const. only
All entrepreneurs				
No constraint is currently binding	0.10	0.09	0.10	0.09
Borrowing constr. is currently binding	1.11	1.16	1.05	1.26
Both constraints are currently binding	1.62		0.95	
Smaller entrepreneurs				
No constraint is currently binding	0.11	n.a.*	0.13	n.a.*
Borrowing constr. is currently binding	1.23	1.31	1.19	1.40
Both constraints are currently binding	1.75		0.95	

*No entrepreneur is in this category; ** a entrepreneur with a positive idiosyncratic productivity shock.

Figure 11: Volatility and procyclicality of aggregate investment

	All entrepreneurs					**Small entrepreneurs**			
Volatilities relative to aggregate investment	Empirical data	Both constr.	Only irrev. constraint	Only borr. constraint	No constr.	Both constr.	Only irrev. constraint	Only borr. constraint	No constr.
Gross fixed investment	2.497	3.491	3.684	5.080	3.061	3.859	2.276	3.506	3.007
Inventory investment	14.779[2] 16.916[3]	12.859	6.931	10.149	4.040	10.650	3.585	4.902	4.024
Drop of inventories in recessions[1]	**86%[4]**	**82%**	**79%**	**65.8%**	**64.1%**	**83.2%**	**75.2%**	**65%**	**67.2%**

Note: relative volatility is measured as the ratio between the two following quantities. At the numerator, standard deviation of the changes in investment (respectively gross fixed investment and changes in inventories of variable capital) relative to the mean absolute investment. At the denominator, standard deviation of the changes in aggregate investment (sum of gross fixed and variable investment) relative to the mean absolute aggregate investment.

1. Absolute drop in inventories as a percentage of the absolute drop of aggregate capital (fixed+inventories) in recessions; 2. measured as change in cost of materials; 3. measured as inventory investment. 4. The empirical series are detrended, in order to make them comparable with the simulated series. Inventories are measured as total inventories.

means that if the wealth of all productive entrepreneurs increases by 1%, keeping constant expected productivity, the aggregate stock of capital increases by 1.4%. Such elasticity is equal to zero in the economies without financing imperfections, where investment decisions are not affected by capital structure, and therefore these economies are omitted. Figure 10 shows that aggregate investment is positively affected by changes in financial wealth in the economies with financing constraints. In particular, three results are worth mentioning: i) in the economy with both constraints variable capital is more sensitive to financial wealth than fixed capital[28]. This result is consistent with empirical evidence at firm level (Carpenter Fazzari and Petersen, 1998). ii) Entrepreneurs who are not currently financially constrained also show a positive elasticity around 0.1. This is due to the fact that the increase in wealth reduces future expected financing problems. This effect is small but not negligible, and it increases up to 0.2 at the beginning of expansion periods, when entrepreneurs engage in precautionary saving given the uncertainty about how long the expansion will be. iii) Entrepreneurs facing both constraints are those with the highest elasticity of variable capital investment with respect to wealth. This is especially true for smaller entrepreneurs. These very high values of the elasticity explain why variable capital is relatively more volatile in the economy with both constraints than in the other simulated economies.

Figure 11 summarises the aggregate consequences of financing and irreversibility constraints on investment volatility. As illustrated in the introduction, empirical evidence on US data provided by Ramey (1989) and by Blinder and Maccini (1991), and on G7 countries provided by Ramey and West (1999) show that inventories, especially in raw

[28]This result is obtained despite the distortion caused by the presence of the minimum level of variable capital l^E supplied by the entrepreneur, which biases downwards the elasticity of variable investment to financial wealth. This is why in the economy with only borrowing constraints fixed capital is more sensitive to financial wealth than variable capital.

materials and work in progress, are very volatile and procyclical. This is especially true during recessions, when they account for a large fraction of the drop in business spending. Moreover Bernanke Gertler and Gilchrist (1996) show that the drop in inventories during recessions is more pronounced for smaller firms.

Figure 11 shows that the model with both constraints is consistent with all these findings. The first two rows report the volatility of fixed and inventory investment relative to the volatility of aggregate investment. For the simulated data we measure inventory investment as the change in the stock of variable capital. For the empirical data we follow the strategy used in figure 8, and propose two alternative measures of investment in input inventories: i) the change in the cost of materials, which coincides with the theoretical variable and is also closely correlated to the unobservable investment in materials inventories; ii) the investment in total inventories. The figure shows that inventory investment is significantly more volatile in the economy with both constraints than in any of the other simulated economies and much closer to the empirical data.

Our simulations also provide a direct estimation of the relative importance of the drop in inventories during recessions. In our simulated data we define as recessions the episodes of several periods of decline in output which start when output is consistently above trend (greater than 0.75 of the standard deviation of the trend deviations of output) and end when output is below trend. We show not only that the aggregate drop in inventories during recessions is larger for the economy with both constraints, but also that the difference is driven by a larger drop in the inventories of smaller firms which is consistent with empirical evidence. A crucial factor for obtaining this result is the amplification effect between irreversibility and financing constraints. During recessions some entrepreneurs suffer from both constraints, and therefore they are forced to cut investment in variable capital dramatically. Small entrepreneurs are on average less rich and suffer more from this problem.

The comparison with empirical data shows that once again the economy with both constraints is the best performing one. However, it must be noted that the value of 86% computed for the empirical data is not very representative, being the average between only two values: 82% for the 1969-'71 period and 90% for the 1979-'83 period.

Figure 12 illustrates the correlation between the cumulative growth rates of aggregate output and aggregate investment. In the economies without irreversibility constraint (first 4 rows) output and investment are contemporaneously correlated. This is natural because the output directly depends on the inputs through the production function. The correlation is relatively low because inputs are decided one period in advance, based on the expected productivity shock $E_{t-1}\left(\theta_{i,t}^{f}\varepsilon_{t}\right)$, while output depends on the realised shock $\theta_{i,t}^{f}\varepsilon_{t}$.

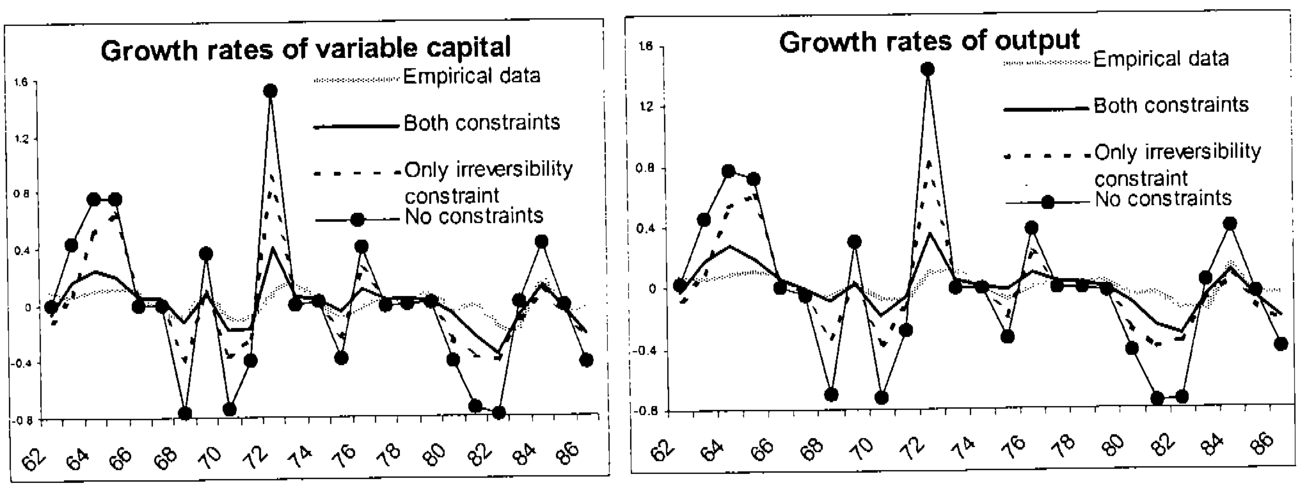
The two economies with the irreversibility constraint (last 4 rows) show that variable investment leads the fluctuations in output. This is because many entrepreneurs cannot immediately adjust fixed capital following an aggregate shock, and hence also the adjustment in output is a bit lagged with respect to the adjustment in variable capital. This feature, which is more evident in the model with both constraints, is consistent with empirical evidence from the US business cycles (Stock and Watson, 1998).

Figure 12: Cross correlations between investment and output

		Correlation$(input_t, output_{t+k})$				
		-2	-1	0	1	2
Economy without constraints	Fixed investment	-0.01	0.02	0.48	0.40	0.34
	Variable investment	0.22	0.23	0.25	0.21	0.18
Economy with borr. constr. only	Fixed investment	0.22	0.29	0.59	0.54	0.50
	Variable investment	-0.18	-0.13	0.20	0.19	0.17
Economy with irreversibility only	Fixed investment	0.23	0.38	0.67	0.63	0.57
	Variable investment	-0.22	-0.17	0.20	0.20	0.19
Economy with both constraints	Fixed investment	0.45	0.58	0.78	0.74	0.70
	Variable investment	-0.18	-0.11	0.18	0.20	0.19

Correlation between the cumulative growth rates of investment and output

Figure 13: Growth rates of variable capital and output, comparison between empirical data and simulated economies.



3.2.1 Comparison with empirical aggregate data.

In figures 13-16 we directly compare the aggregate time series of the US Industrial Machinery Sector (1962-1986) with the time series computed by using the simulated economies. We use the data about total factor productivity to estimate the aggregate productivity shock ε_t for the period 1962-1986. We then simulate the artificial economies feeding into the model the time series of the shocks estimated above. The aim is to compare the dynamics of output and investment predicted by the model with the ones observed in reality. If the interactions between financing and irreversibility constraints are important for explaining investment volatility, then we expect the economy with both constraints to be more consistent with empirical data, in particular during recessionary periods. We focus on the period from 1962 to 1986 which includes large changes in investment and output, as opposed to the less volatile period from 1987 to 1996.

Figure 13 compares the growth rate of output and variable capital for the empirical

Figure 14: Fit between empirical and simulated growth rates of aggregate capital and output

R^2 of the following OLS regressions: $\hat{\gamma}_t^x = \alpha + \beta\gamma_t^x + \varepsilon_t$

$\hat{\gamma}_t^x$ =empirical growth rate of variable x; γ_t^x =simulated growth rate of variable x

	Economy simulated with:			
Growth rates of:	Both Constraints	Only borr. constraint	Only irrev. constraint	No constraints
Fixed capital	0.220	0.030	0.130	0.001
Variable capital	0.475	0.439	0.379	0.387
Output	0.526	0.502	0.444	0.463

data and for three simulated economies: without constraints, with irreversibility only and with both constraints[29]. The economy with both constraints is closer to the empirical data. The other two economies are much more volatile, especially the one without constraints. This result confirms the statistics showed in figure (8).

Figure 14 quantifies the fit between empirical and simulated data. It reports the R^2 of several OLS regressions, with the observed growth rate of capital and output as the dependent variable and the corresponding simulated growth rate and a constant as the independent variables. The simulated economies with just one constraint (second and third column) do not explain the data very well. The economy with only the financing constraint has no predictive power regarding the growth rate of fixed capital, and the economy with only the irreversibility constraint has the lowest predictive power regarding the growth rate of variable capital and output. Yet surprisingly the model with both constraints has the best predictive power for all the three growth rates. This finding indicates that the interactions between the two constraints are important to understand the observed investment dynamics. Moreover if this is true then the model with both constraints should also be more successful in matching the empirical growth rates in years of large negative productivity shocks, when the two constraints are more likely to be binding at the same time. This intuition is confirmed by figure[30] 15, which compares empirical and simulated growth rates of variable capital. The comparison is facilitated by the fact that the growth rates are rescaled with respect to their standard deviations in the 1962-1986 period. The model with both constraints is better at replicating empirical data when growth rates are large, during both expansion and contraction phases. This is especially true for the recession of the beginning of the 1980s, which is underestimated by the model with irreversibility only, and for the expansion in the mid 1970s, which is overestimated by the model without constraints.

Unfortunately we do not have data available for small firms in the industrial sector

[29] The economy with borrowing constraints only is omitted in order to make these graphics more readable.

[30] The graphic about output growth rates shows similar results and is therefore omitted.

Figure 15: Growth rates of variable capital (rescaled)

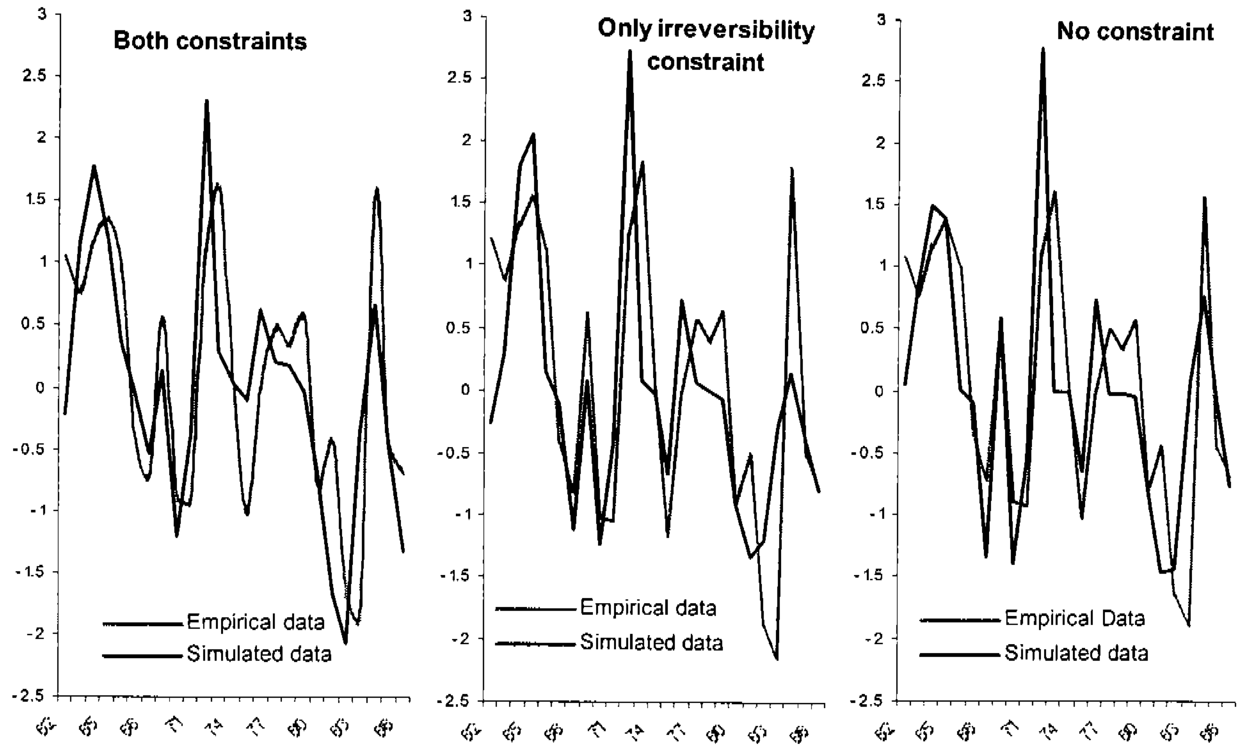
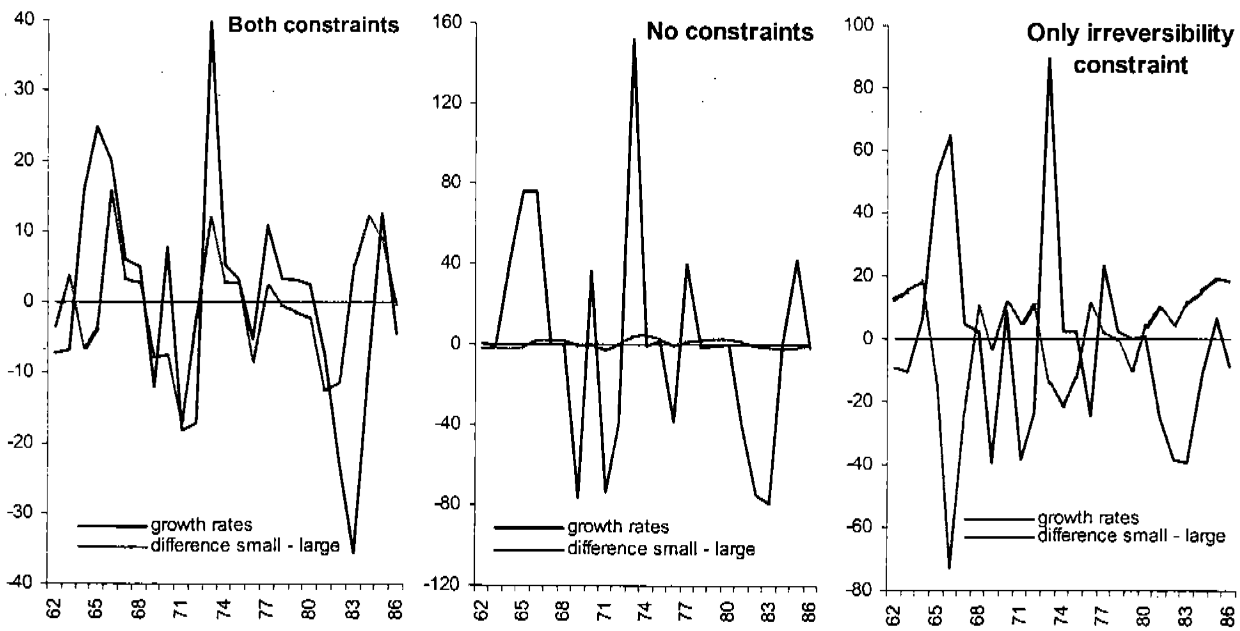


Figure 16: Simulated growth rate of aggregate variable capital and difference in the cumulative growth rates of small and large firms



which we used as a benchmark, and therefore we cannot directly evaluate the ability of the model to explain the behaviour of small versus large firms in the sample period. Nonetheless our simulation results can be related to the empirical literature on the US manufacturing sector. In particular it relates to Bernanke, Gertler and Gilchrist (1996) who show that small firms are more volatile and procyclical than large ones, both in terms of inventories and output. Figure 16, which is computed with the same sequence of shocks used above, is consistent with these findings. It shows that in the economy with both constraints small firms are more procyclical than large ones, both during downturns and upturns. The same result is not obtained in the other two simulated economies. The economy with only irreversibility has the opposite result, while the economy without constraints has no difference between small and large firms.

3.3 Empirical microeconomic evidence

In the above simulations we showed that the interaction between irreversibility and financing constraints explains a number of stylised facts about the dynamics of aggregate investment. An essential condition for these results is the presence of financing constraints on firm investment decisions. In this section we discuss some direct empirical evidence of such constraints. The idea is simple: because variable capital investment is reversible, the "premium" of expected marginal productivity over user cost of variable capital reflects the tightness of current and future expected financing constraints. In Caggese (2003) this premium is estimated using a unique dataset of 561 Italian manufacturing firms with both balance sheet and qualitative data. The sample contains the following information: i) 11 years of balance sheet data available from 1982 to 1992. This panel is a subset of the dataset produced by Centrale dei Bilanci, which is the largest and most reliable source of data for Italian firms. ii) Qualitative information from the First Mediocredito Centrale Survey on Small and Medium Italian Manufacturing Firms. Among the information in this survey there are the statements from the firms about the financing problems they faced in the 1989-1991 period. Firms were asked whether they had any of the following problems regarding investment financing: Q1) lack of collateral; Q2) lack of medium-long term financing; Q3) too high cost of banking debt.

The premium in expected productivity of variable capital, which constitutes a financing constraints indicator, is computed in Caggese (2003) as the difference between expected marginal productivity and user cost of variable capital. Using equation (17):

$$E_t\left(\Psi_{i,t+1}\right) = E_t\left(MPL_{i,t+1}\right) - UL_{i,t} \tag{36}$$

Where:

$$E_t\left(\Psi_{i,t+1}\right) = \frac{1}{E_t\left[U'\left(x_{t+1}\right)\right]}\left(\frac{1}{\gamma\beta}R\lambda_t - cov^{xl}_{t+1}\right) \tag{37}$$

$E_t\left(\Psi_{i,t+1}\right)$ as defined in equation (37) is an indicator of the intensity of current and future expected financing constraints. It is monotonously increasing in the shadow cost of being unable to borrow to finance new investment. Equation (36) implies that such indicator can be estimated as the difference between expected marginal productivity and user cost of capital. In Caggese (2003) the empirical counterparts of $UL_{i,t}$ and $E_t\left(MPL_{i,t+1}\right)$ are estimated using the panel of balance sheet data for Italian manufacturing firms, and an empirical counterpart of $E_t(\Psi_{i,t+1})$, called ${}_t\widehat{\Psi}^w_{i,t+1}$, is obtained. The validity of ${}_t\widehat{\Psi}^w_{i,t+1}$ as an indicator of financing constraints is tested using the direct information about financing problems available in the Mediocredito Centrale survey. Caggese (2003) considers the entrepreneurs that stated problems in accessing external finance in the 1989-91 period (questions Q1, Q2 and Q3). Such problems are directly related to the ${}_t\widehat{\Psi}^w_{i,t+1}$ variable. The bigger ${}_t\widehat{\Psi}^w_{i,t+1}$, the higher the shadow value of additional funding for the *i-th* entrepreneur and the higher the probability that she answers positively to one of the questions regarding financing constraints. Among the 561 firms considered, 21.6% of their entrepreneurs indicate one of the three problems in accessing bank credit during the 1989-1991 period. Using this information 4 dichotomous variables, $ration_i^j$ with $j = \{1, 2, 3, 4\}$, are constructed. They have value 0 if the $i-th$ entrepreneur does not state any financing problem, and 1 if she answers positively to questions Q1, Q2 and Q3 respectively ($j = 1, 2$ and 3) or states any of the three problems ($j = 4$). Figure 17 from Caggese (2003) shows the result of a regression of $ration_i^j$ on $\overline{\Psi}_i^w$, which is the average value of ${}_t\widehat{\Psi}^w_{i,t+1}$ in the period covered by the Mediocredito Centrale survey:

$$\mathbf{ration}_i^j = \alpha_0 + \alpha_1 \overline{\Psi}_i^w + \alpha_2 \dim_i \tag{38}$$

$\overline{\Psi}_i^w = \sum_{t=1989}^{1992} \left({}_t\widehat{\Psi}^w_{i,t+1}\right)$. The time interval used to compute $\overline{\Psi}_i^w$ includes 1989, 1990 and 1991, the period which the questions refer to, and 1992, the year in which the questionnaire has been compiled. $\dim_i$ is the number of employees of the $i-th$ firm, which is included to control for the effect of firms' size on the probability to state financing problems. The first column of figure 17 is relative to the whole sample and to $ration_i^4$ as the dependent variable. The coefficient relative to $\overline{\Psi}_i^w$ is positive and significant. The second and third columns repeat the same regression for larger (more than 300 employees, 19% of the sample) and smaller (less than 300 employees, 81% of the sample) firms.

In order to interpret this result, we note that this estimation is done under the assumption that the productivity shock is stationary plus the condition that $\alpha + \kappa < 1$. Hence firms are assumed to have different steady state sizes according to their specific levels of $\overline{\theta}^f$. Each firm evolves around such steady state according to the realisations[31] of its idiosyncratic shock. Therefore the result illustrated in figure 17 is consistent with the assumption that the higher the average size of a firm, the less likely the firm is to face the informational or contractual problems which cause the financing constraint (7). This assumption is realistic because large Italian firms usually have strong links with financial intermediaries and do not face tight borrowing constraints like the one represented by equation (7).

The strong correlation between $ration_i^j$ and $\overline{\Psi}_i^w$ for small and medium firms below 300 employees is confirmed by the probit regression results in figure 17. The last four columns show that the $\overline{\Psi}_i^w$ coefficient is positive and strongly significant, especially for the specification ($j = 4$) that pools together the three different questions. The result shows that ${}_t\widehat{\Psi}^w_{i,t+1}$ is a valid indicator of the intensity of financing constraints. This supports the

[31] This stationarity assumption is reasonable in this context, given that the time series is 11 years only.

Figure 17: Relation between stated financing problems and the financing constraints indicator

Probit regression: $ration_i^j = \alpha_0 + \alpha_1 \overline{\Psi}_i^W + \alpha_2 \dim_i$

	Whole sample	Larger firms[1]	Smaller firms[2]			
Dependent variable	*Ration*[4] (all problems)	*Ration*[4] (all problems)	*Ration*[4] (all problems)	*Ration*[1] (Low collateral)	*Ration*[2] (Lack of bank credit)	*Ration*[3] (High cost of debt)
α_0	-0.64*** (0.09)	-0.69 (0.42)	-0.78*** (0.16)	-2.05*** (0.31)	-1.48*** (0.20)	-1.04*** (0.17)
α_1	0.24** (0.11)	-0.16 (0.36)	0.30*** (0.12)	0.36* (0.20)	0.28** (0.14)	0.29** (0.13)
α_2	-0.0006* (0.0003)	-0.0007 (0.0007)	0.0005 (0.001)	0.001 (0.002)	0.002* (0.001)	0.0005 (0.001)
Obs with ration=0	341	70	271	341	310	296
Obs with ration=1	92	11	81	11	42	56
(% of total)	(21.2%)	(13.6%)	(23%)	(3.1%)	(11.9)	(15.9%)
Total obs	433	81	352	352	352	352

Standard error in parenthesis; 1: More than 300 employees; 2: Less than 300 employees; * Significant at 90% confidence level; ** significant at 95% confidence level; *** significant at 99% confidence level; $ration_i = 1$ if the entrepreneur stated financing constraints, and 0 otherwise; $\dim_i$ = dimension in number of employees; Ψ_i^W = premium in the expected productivity of variable capital;
Source: Caggese (2003)

validity of our theoretical model and rejects the view of efficient financial markets[32].

4 Conclusions

In this paper we illustrated a structural model of an entrepreneurial firm subject to both borrowing constraints and irreversibility of fixed capital. The solution of the optimal investment problem shows that not only expected productivity but also financing problems affect the investment and saving decisions of the entrepreneurs. In particular the proportion of wealth allocated to risky projects rather than safe assets is negatively affected by future expected financing constraints. This precautionary saving effect on investment is substantial. Consider two entrepreneurs identical in everything except their endowment of financial wealth. Our simulations show that the richer entrepreneur may invest up to 18% more in the risky technology than the poorer one, mainly because she expects less future financing constraints.

More importantly we showed that irreversibility and financing problems are complementary: the irreversibility of fixed capital amplifies the effects of financing constraints on variable capital both during upturns and downturns. By simulating an artificial economy

[32] This result is robust to possible biases induced by measurement errors for at least three reasons: i) the qualitative and quantitative information come from different sources (see Caggese 2002 for a description of the samples). This reduces the probability that those entrepreneurs that declare financing constraints also manipulate their balance sheets data to show that their investment is inefficiently low; ii) Caggese (2003) conditions for firms' size, thus ruling out the possibility that $\overline{\Psi}_i^w$ is on average higher for small firms, which are also more likely to state financing constraints; iii) the result is not driven by sectorial differences: in Caggese (2003) it is shown that financing constraints are equally distributed in the different industrial sectors.

with many heterogeneous entrepreneurs, we showed that this amplification effect, which has not yet been studied in the literature, is essential for explaining why aggregate inventory investment is very volatile and procyclical, and why the drop in inventories accounts for a large part of the decline in business spending in recessions. Our theory accounts also for the observed facts that most of the volatility in aggregate inventories and output is created by small firms, that inventory investment leads the business cycle, and that both fixed and inventory investment are sensitive to the net worth, even when marginal productivity of capital is taken into account.

Although all the simulations in the paper refer to one specific US four digits manufacturing sector, similar results could be obtained for any sector where productive units satisfy the following assumptions: i) both financing and irreversibility constraints are binding for a non negligible share of firms in equilibrium; ii) firms produce output using a combination of reversible and irreversible inputs. Thus the results do not depend on the fact that the irreversible factor of production is also the source of collateral. Nor is it necessary to assume that variable capital fully depreciates. Moreover, our parametrisation implies that on average around 45% of the entrepreneurs are financially constrained in every period, but the results would also hold for other parametrizations which would generate a lower fraction of financially constrained entrepreneurs. For example, we assume a logaritmic utility function with a unitary coefficient of risk aversion. Using a higher value of the coefficient of risk aversion would imply that entrepreneurs engage in more precautionary saving. This would lower the share of constrained entrepreneurs in equilibrium, but would not change the results because the precautionary saving effect has the same implication of a currently binding constraint for the quantitative results derived in this paper.

Finally, the model mainly applies to entrepreneurial firms. However, we think that this is not a necessary restriction, because the model could be easily modified to analyse similar issues pertaining to publicly owned companies. Moreover, we believe that the focus on entrepreneurial firms is one of the strong points of this paper. Hamilton (2000) notes that, as of 1997, business owners constituted approximately 13% of non agricultural employees in the Unites States. This large class of businesses is therefore responsible for both a large share of production and of the wealth accumulation in the economy. Gentry and Hubbard (2000) and Heaton and Lucas (2000) study the effect of entrepreneurial risk on aggregate saving and on the portfolio choices of the private sector. But this is the first study to link the issue of entrepreneurial risk to the issue of investment and output fluctuations of firms.

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Appendix A: proof of proposition 1

For simplicity we assume that θ_t is a symmetric two state stochastic Markov process, even though the proof can be generalised for more complicated processes:

$$\theta_t \in \{\theta_L, \theta_H\} \ \ with \ \theta_H > \theta_L; \qquad pr(\theta_{t+1} = \theta_t) = \epsilon > 0.5; \qquad pr(\theta_{t+1} \neq \theta_t) = 1 - \epsilon \tag{39}$$

Hence the first order autocorrelation coefficient is $\rho = 2\epsilon - 1 > 0$, and we have that:

$$E_t \left(\theta_{t+1} \mid \theta_t = \theta_H\right) > E_t \left(\theta_{t+1} \mid \theta_t = \theta_L\right) \tag{40}$$

We define D_t as the dichotomous variable that represents the retirement choice of E:

$$D_t = \begin{cases} 1 \text{ if E continues activity} \\ 0 \text{ if E retires} \end{cases}$$

the choice of D_t is subject to the following constraint:

$$\begin{array}{ll} D_t = 0 & if \ y_t + \frac{\tau_k}{R}\left(1 - \delta_k\right) k_t < b_t \\ D_t \in \{0, 1\} & otherwise \end{array} \tag{41}$$

D_t is constrained to be 0 if the inequality (41) is satisfied[33]. We first prove that assumption *1* implies that E is never forced to retire. We substitute equation (7) in equation (10) obtaining the following condition:

$$x_t + l_{t+1} + k_{t+1} \leq w_t + \frac{\tau_k}{R} k_{t+1} \tag{42}$$

the left hand side of equation (42) is constrained downwards by constraints (1), (4) and (5). Now let us substitute w_t in equation (42) using equation (9) and k_{t+1} and l_{t+1} from constraints (4) and (5) holding with equality. At the limit, for x_t that goes to zero equation (42) becomes:

$$y_t + \frac{\tau_k}{R}\left(1 - \delta_k\right) k_t \geq b_t \tag{43}$$

equation (43) is symmetric to the disequality in condition (41). It ensures that E is not forced to violate the irreversibility constraint to repay the debt. We determine $l^E_{\min}$, the constant level of variable capital supplied by E, as the level of l^E such that equation (43) is always satisfied for all the possible levels of state variables w_t, k_t and θ_t. The left hand side of equation (43) is monotonously decreasing in θ_t and l_t, therefore the worst possible situation is the one in which $l_t \to 0$, $\theta_t = \theta_L$ and E is already at her borrowing limit: $b_t = \tau_k k_t$. We substitute these three limit values in equation (43), and we substitute y_t using equation (2). We also define $\overline{k}$ as the maximum level[34] of k_t compatible with $\theta_t = \theta_L$. Solving equation (43) in terms of l^E yields the following:

$$l^E \geq \left[\frac{\tau_k \left(1 - \frac{1-\delta_k}{R}\right) \overline{k}^{1-\alpha}}{e^{\theta_L} A}\right]^{\frac{1}{\kappa}} \equiv l^E_{\min} \tag{44}$$

[33] The left hand side of this inequality represents the maximum amount of funds that can be liquidated to repay b_t without selling the residual value of fixed capital $(1 - \delta_k)k_t$. Hence if b_t is higher than this amount the only way to repay the debt is by selling all the assets and retiring.

[34] This would be the stock of capital chosen by an unconstrained entrepreneur.

$l^E_{\min}$ is the minimum level of l^E such that E s never forced to retire. Regarding voluntary retirement, the value function for the entrepreneur at time t before the continuation decision is the following:

$$V_t(w_t,\theta_t,k_t) = D_t\left[\underset{x_t,l_{t+1},k_{t+1}}{MAX}\ U(x_t)+\beta E_t(V_{t+1})\right]+(1-D_t)\overline{V}_t(w_t) \tag{45}$$

The problem is defined by equation (45) subject to constraints (4) (5), (7) and (10). $\overline{V}_t(w_t)$, the value function after retiring at time t, is the following:

$$\overline{V}_t(w_t) = \underset{x_t}{MAX}\ U(x_t)+\beta\overline{V}_{t+1} \tag{46}$$

Such that:

$$x_t = w_t - \frac{w_{t+1}}{R} \tag{47}$$

It is then clear that the exit decision is the following:

$$D_t = \left\{\begin{array}{l} 1 \text{ if } \underset{x_t,l_{t+1},k_{t+1}}{MAX}\ \ln x_t+\beta E_t(V_{t+1}) > \overline{V}_t(w_t) \\ 0 \text{ otherwise}\end{array}\right\} \tag{48}$$

Assumption *a2* simply states that the parameters of the model are chosen such that: i) continuation ($D_t = 1$) is always the optimal choice of a utility maximising entrepreneur. ii) constraint (5) is never binding with equality. There is no analytical proof of this result, because there is no analytical solution of the problem. We rather ensure that the numerical solution of the model satisfy these conditions for all possible values of the state variables. ■

Appendix B

In section 2.2 we argue that financing constraints do not alter the optimal mix between fixed and variable capital when fixed capital is fully collateralisable ($\tau_k = 1-\delta_k$). To see this, let's divide (16) by (17) and evaluate such ratio for $\mu_t = E_t(\mu_{t+1}) = 0$:

$$\frac{E_t(MPK_{t+1})}{E_t(MPL_{t+1})} = \frac{E_t[U'(x_{t+1})]UK+\left\{\frac{1}{\beta}[(R-\tau_k)\lambda_t]-cov_{xk}\right\}}{E_t[U'(x_{t+1})]UL+\left(\frac{1}{\beta}R\lambda_t-cov_{xl}\right)} \tag{49}$$

Lets consider the two extreme cases: if there are no current and future expected financing constraints then (49) becomes:

$$\frac{E_t(MPK_{t+1})}{E_t(MPL_{t+1})} = \frac{UK}{UL} \tag{50}$$

On the other extreme, if current financing constraints are very strong and λ_t becomes very large, (49) converges to:

$$\lim_{\lambda_t\to\infty}\frac{E_t(MPK_{t+1})}{E_t(MPL_{t+1})} = D_k \tag{51}$$

But when $\tau_k = 1-\delta_k$ then $D_k = \frac{R-1+\delta_k}{R} = \frac{UK}{UL}$, and the optimal mix between fixed and variable capital is the same.

Appendix C: numerical solution

In order to obtain a numerical solution of the dynamic nonlinear system of equations defined by (4), (12), (13), (14) and (15), we discretise the state space as follows: $k_{i,t}$ and $w_{i,t}$ are discretised in 70 elements each, while θ_t is discretised in 12 elements, which correspond to the six states of the aggregate shock and the two states of the idiosyncratic shock. First, we formulate an initial guess of the forward variables $E_t\left[U'\left(x_{t+1}\right)\right]$, $E_t\left[U'\left(x_{t+1}\right)MPL_{t+1}\right]$, $E_t\left[U'\left(x_{t+1}\right)MPK_{t+1}\right]$ and $E_t\left(\mu_{t+1}\right)$. Second, we solve the static optimisation problem conditional on this guess, for each discrete value of the state variables $w_{i,t}$, $k_{i,t}$ and $\theta_{i,t}$. This static optimisation problem is nonlinear as well. Conditional on the value of $\theta_{i,t}$ the solution falls into four possible categories, depending on the values of the couple $\{w_{i,t}, k_{i,t}\}$. These categories correspond to the four areas in figure 2. Third, we update the guess of $E_t\left[U'\left(x_{t+1}\right)\right]$, $E_t\left[U'\left(x_{t+1}\right)MPL_{t+1}\right]$, $E_t\left[U'\left(x_{t+1}\right)MPK_{t+1}\right]$ and $E_t\left(\mu_{t+1}\right)$. We repeat these steps until the value function converges.