An Equilibrium Model of Investment Under Uncertainty*

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Abstract

This paper analyzes the optimal investment decisions of heterogeneous firms in a competitive, uncertain environment. We characterize firms' optimal investment strategy explicitly, and derive a closed form solution for firm value. We show that in the strategic equilibrium real option premia are significant. As a result firms delay investment, choosing optimally not to undertake some positive NPV projects. The model predicts that firm returns vary over the business cycle, with returns negatively skewed during expansions but positively skewed in recessions.

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1 Introduction

Using real options theory to account for the value of future development is now standard in finance among both academics and practitioners. It provides a heuristic, intuitively appealing explanation of the real world's observed deviations from neoclassical, Tobin's Q theory of investment. The most notable of these deviations are market values that exceed book values, often greatly, and investment thresholds that may significantly exceed zero net present value. Real options theory, however, completely ignores the role of competition. The theory's conclusion that delaying investment can result in excess profits seems to be at odds with what we know about competition. Competition drives down profit opportunities, a fact known even before Cournot modeled the effect in 1838.

This paper presents an equilibrium model integrating the two disparate branches of the economics literature— real options theory and the theory of competition. Recent papers, most notably Grenadier (2002), have used the Cournot intuition to argue that competition erodes real option values and reduces investment delays. These results are difficult to reconcile, however, with important sectors of the economy. Option premia are significantly positive and firms delay investment in some highly competitive industries. Titman (1985) illustrates this with a simple example: empty lots in city centers. Real estate is a highly competitive industry with many players, yet owners sometimes choose not to build on property that could certainly be developed profitably. Instead, as Titman argues, the value of a lot derives from the option to develop, and owners sometimes delay building believing that the lot can be developed more profitably later.

The analysis presented in this paper shows that in a competitive industry firms can actually deviate more from neoclassical behavior than the standard real options analysis predicts. In particular, firms may delay irreversible investment longer, and invest only at significantly positive option premia.

Grenadier's results— that competition erodes option values and pushes firms back to the zero NPV investment rule— and similar results found in Leahy (1992) and Williams (1993), are a consequence of the type of industry they model, one in which the production technology is linear and incremental.[1] In these papers firms may add capacity in arbitrarily small increments without suffering any adjustment costs.

[1]Similar results may also be found in Spatt and Sterbenz (1985), in an economy in which investment by one firm forecloses other firms' investment opportunities.

Option values and behavior in the real estate market example, and in many other industries, differ from Grenadier's predictions because undertaking investment often entails opportunity costs, and because firms vary in scope and size. We show in this paper that in industries in which opportunity costs and heterogeneity are important real option values are significant, investment decisions are delayed, and investment is lumpy.

Since Hotelling's (1929) seminal paper it has been well understood in economics that demand side heterogeneity can reduce competition. The whole concept of horizontal product differentiation is predicated on the idea that variations in tastes allow firms to segment the market, compete less, and extract more of the consumer surplus. Heterogeneity can also provide a natural ordering to agents' actions. We do not all buy new computers, or cars, at the same time at least in part because those of us with older, obsolete models are more likely near term buyers than those with newer, contemporary models.

This paper shows that supply side heterogeneity can reduce competition as well. When heterogeneity extends to costs or profitability it is a wedge that breaks the idea of "perfect competition," even when firms are perfectly competitive. In the real estate example considered previously, a large number of firms compete vigorously, yet heterogeneity prevents them from all competing directly over any investment opportunity. When the owner of the empty lot considers putting up a thirty-story office tower she is not competing with the owner of the fifteen-story apartment complex next door. The opportunity costs to the owner of the fifteen-story apartment complex, which include walking away from the existing building, effectively preclude her from competing with the owner of the empty lot. The owner of the fifteen-story apartment complex however still possesses a valuable option to compete over investment opportunities in the future. If the city grows there may be demand in the future for a new sixty-story office tower. If at that time all the empty lots are developed the owner of the fifteen-story apartment complex may have low enough opportunity costs to compete.

While supply side heterogeneity results in reduced competition, the heterogeneity itself arises endogenously as a consequence of the fact that heterogeneous firms compete less. Firms naturally make investment decisions that differentiate them from others in a manner that reduces the amount of competition they face. The natural, stable cross-sectional distribution of firms is the one that minimizes intertemporal competition between firms.

We develop a model in which firms incur opportunity as well as direct investment costs to altering capacity, and face aggregate uncertainty regarding demand for their output, the price of which is determined endogenously and is a function of firms' investment decisions. We find that even with an infinite number of competitive heterogeneous firms, option pemia are significant and firms optimally delay irreversible investment, choosing not to undertake some positive NPV projects.

The Cournot intuition that competition should drive firms to invest earlier is compelling, so our result that competition can lead firms to delay investment longer is somewhat surprising. This result derives from the endogenously determined equilibrium price of firms' output. We show that the price of firms' output is negatively skewed because aggregate industry capacity responds asymmetrically to changing demand. Firms can add capacity quickly in response to rising demand, but cannot adjust capacity as quickly to falling demand due to investment irreversibility. As a result, increasing supply attenuates positive demand shocks, which are only partially translated into prices, while negative demand shocks are translated into prices more fully. As Dixit (1999) shows, negative skewness leads firms to delay investment. Firms have an incentive to delay investment when large drops in the price of firms' output are more likely. This additional incentive to delay investment, in conjunction with the significantly positive option premia, results in investment delays even longer than predicted by standard, partial equlibrium real options models.

Because these large drops in the price of firms' output result from firms adding capacity they are, *ipso facto*, most likely when firms choose to develop. In fact, firms always add capacity expecting prices to fall. That is, while capacity added immediately prior to large price drops might look, *ex post*, like overbuilding, its development was in fact *ex ante* optimal. We analytically construct forward curves for the price of firms' output to explore such implications of negative skewness in greater detail.

We also derive closed form expressions for firm value as a function of the price of firms' output and current aggregate industry capacity. This allows us to make predictions about stock returns. Stock returns are a combination of the returns to firms' ongoing projects and their growth options. Away from historic highs in the price of firms' output, aggregate capacity is unlikely to increase so the return to firms' output exhibits little skew. This translates into positive skew in stock returns because the option component of firm value is convex in the price of firm output. When the price of firms' output is high the effect of increasing aggregate capacity dominates and stock returns exhibit negative skew. This provides a hitherto untested empirical prediction: skewness in stock returns should vary over the business cycle. Stock returns should be negatively skewed during expansions, but positively skewed in recessions.

After reviewing the related literature, the remainder of the paper is organized as follows. Section 2 introduces the basic model. Section 3 demonstrates the equilibrium strategy. The section begins by discussing the intuition and the general form of the strategy before characterizing the strategy explicitly. Section 4 discusses implications of the analysis, including properties of the equilibrium price process. We also compare the equilibrium exercise strategy to the standard, partial equilibrium strategy in detail. Section 5 concludes.

1.1 Review of Related Literature

Several papers have considered the effects of competitive interactions on real options. Smets (1991), Grenadier (1996), Garlappi (1999), and Lambrecht and Perraudin (2002) have considered duopolistic settings; Leahy (1992) perfect competition; Spatt and Sterbenz (1985), Williams (1993) and Grenadier (2002) have considered the intermediate cases. In all of these papers competition leads agents to exercise options earlier than would a strategic monopolist who accounts for the price impact of her own option exercise strategy.

The essential intuition that equilibrium supply responses to demand shifts result in a price asymmetry that can lead firms to delay investment longer was recognized by Dixit (1999). Earlier Leahy (1992) noted that a competitive firm's free entry threshold can resemble a monopolist's option-value threshold. In both of these papers options premia are forced to zero by competition, however, and the delay in investment they consider is not a real options effect.

Irreversible investment in the presence of competition has been studied more extensively in industries with incremental, linear cost production technology. However this production technology essentially precludes positive option values or significant investment delays. Studies of this type include Leahy (1993), Caballero and Pindyck (1996), Kogan (2001), and Grenadier (2002).

Several authors have performed empirical tests of real options theory in the presence of competition. Real options theories are notoriously difficult to test empirically, however, because testable hypotheses generally require data that are largely unavailable. As a result, empirical tests are concentrated in industries where the theory predicts large deviations from neoclassical behavior, such as natural resource extraction and the commercial real estate market, both industries in which new investment entails high adjustment costs. Early tests include Paddock, Siegel and Smith's (1988) investigation of the pricing of offshore petroleum leases and Quigg's (1993) investigation of commercial real estate pricing. More recently Harchaoui and Lasserre (1999) have looked at copper mining, while Holland, Ott and Riddiough (2000), and Downing and Wallace (2001) have further investigated commercial real estate markets. The results of these papers are generally consistent with the predictions of real options theory: the authors find positive option premia and significant investment delays.

2 The Model

In this model the defining characteristic of a "firm" is ownership of productive capital. A firm's ongoing assets, or installed "capacity," costlessly produce a good (or service) flow. A firm is able to produce a flow of the good in proportion to its capacity. This good may then be sold in a competitive market at the instantaneous price P_t that clears the market. The total instantaneous cash flow to a site with capacity q, excluding development costs, is therefore simply $q \cdot P_t$.

Following the literature we assume that the market clearing price for firms' output satisfies an inverse demand function of a constant elasticity form,

$$P_t = X_t \cdot Q_t^{-1/\alpha}, \tag{1}$$

where Q_t is the instantaneous aggregate supply of the good, X_t is a multiplicative demand shock, and α is the price elasticity of demand.[2] This formulation is equivalent to assuming that prices are set by market clearing and demand is time varying but has constant elasticity with respect to price. That is, demand is given by

$$D_t = X_t^{\alpha} \cdot P_t^{-\alpha}, \tag{2}$$

where X_t^{α} is stochastic and may be thought of as demand in a world in which the good has unit price. The multiplicative demand shock is assumed to evolve as a geometric Brownian motion. That is,

[2] Because projects produce the good in proportion to their capacities, we will use Q_t to denote both the instantaneous aggregate supply of the good and the aggregate capacity to produce the good.

$$dX_t = \mu X_t dt + \sigma X_t dz_t \tag{3}$$

where μ and σ are constant.

At any time a firm may increase capacity by developing. Development, which may be undertaken repeatedly, entails two costs, the investment cost, which is the direct cost of development, and an opportunity cost.[3]

The investment is the cash outlay required to undertake the new project. For example, when a personal computer manufacturer retools a production line for a new model it incurs costs. Likewise, when the owner of a small tenement in Manhattan decides to redevelop her property she incurs the direct construction costs of building a new office tower.

The direct cost of development depends on the scale of the undertaking. Continuing the previous examples, it is more expensive to set up a production line to produce a million computers a year than it is to set it up to produce a hundred thousand computers; likewise, it is more expensive to build a sixty-story office tower than it is to build a thirty-story office tower.

We model the cost of investment as Cobb-Douglas with increasing costs-to-scale, multiplied by a unit construction cost which may be changing over time.[4] That is, the direct cost of developing capacity q^* is $c_t \cdot q^{*\gamma}$, where $\gamma > 1$ and the "construction cost" process c_t is stochastic. We will always use c_t as numeraire, in which case the

[3]While our focus is on the role of competition, allowing for firms that incur opportunity as well as investment costs to increase capacity repeatedly is itself a departure from the standard real options literature with important implications for optimal investment choice. Retaining development rights leads to higher option values, and to firms developing sooner but to lower capacities than they would if they were only able to develop once. For a more detailed discussion of the impact of retaining development rights see Novy-Marx (2002), available at http://faculty.haas.berkeley.edu/marx/, or Williams (1997).

[4]In a world with fixed capacity the earlier assumption of a geometric Brownian demand is equivalent to assuming a geometric Brownian price process. The joint use of geometric Brownian prices with a Cobb-Douglas cost of development has recently been criticized in Capozza and Li (2002), on the grounds that it always results in agents developing either immediately or never. Their argument is predicated implicitly on the additional assumption, inconsistent with the models they are critiquing, that development entails no additional costs. Models that have jointly used geometric Brownian prices and Cobb-Douglas costs have assumed additional adjustment or opportunity costs to development, and these models implicitly recognize that if adjustment costs go to zero agents build immediately or never (see, for example, Williams (1991)).

cost of developing capacity q^* is simply $q^{*\gamma}$.

Opportunity costs result because the new investment dammages the firm's ongoing business. In the examples above, for instance, undertaking the new project entails a significant loss in value of the ongoing assets. When the computer manufacturer introduces the new model it effectively kills demand for the old model. The manufacturer must consider this lost revenue in her investment decision. The owner of the Manhattan tenement is in much the same position. Before putting up the office tower she must first raze the tenement, foregoing future rents.

The opportunity cost to undertaking investment is largely independent of the scale of the undertaking. The computer manufacturer kills demand for the old model whether it produces a hundred thousand units or a million units of the new model, and the tenement owner must raze the existing building irregardless of the size of the new office tower.

The opportunity cost is modeled here as a fractional loss of the value of projects currently in place. We assume for convenience that the fractional loss from adjustment is one, i.e., that development entails abandonment of the ongoing project. This assumption simplifies the analysis, but with modification the analysis presented in this paper applies to any other choice.[5]

We assume, for the sake of simplicity, that cash flows are valued in a risk-neutral framework, discounted at a constant risk-free rate r.[6] Firms are then priced at the expected value of future revenues, less investment costs, all discounted appropriately for the time value of money. Firms are assumed to maximize value. That is, firms choose their investment strategies to maximizes the current expected value of all future cash flows, including development costs, discounted appropriately.

Finally, we would like to capture the fact that in reality firms vary greatly in scope and size. Even firms in the same industry differ tremendously. While Boeing and Airbus both manufacture planes, and presumably face similar investment costs to developing a new jumbo jet, it is almost inconceivable that Boeing would undertake such development before Airbus. Boeing, with a large existing business selling 747s, faces enormous opportunity costs to introducing a new jumbo, while Airbus makes no jumbos and consequently has low opportunity costs to such an undertaking.

[5]Assuming that development entails abandonment makes solving a fixed-point problem that arises in the course of the analysis particularly simple.

[6]The risk-neutral assumption is effectively equivalent to exogenously imposing a trivial pricing kernel on the economy.

One of the most important ways in which firms differ is in the size of their market shares. In the previous example the size of the firms' jumbo jet businesses determined their opportunity costs. It is along this dimension that we choose to model heterogeneity. Because opportunity costs are proportional to the scale of a firm's ongoing project, this is really an assumption of heterogeneous costs to adjusting capacity. That is, while firms are homogenous in the cost of investing, they differ with respect to the opportunity cost to undertaking investment. It is precisely this heterogeneity that limits competition.

We model the initial heterogeneity as a sort of natural variation in firm size. Firm size initially follows Zipf's law, with firms distributed uniformly with respect to log-capacity between the smallest and largest firms in the industry. Firm size, and consequently firm heterogeneity, then evolves endogenously as a result of firms' equilibrium investment decisions.

3 Determination of the Equilibrium Strategy

We would like to value firms, and to do so we must determine firms' optimal equilibrium investment strategy. A firms needs to account for the actions of its competitors when determining its optimal investment strategy. Firms produce a good that they sell in a competitive market at the market clearing price, which is determined by supply and demand. While demand is exogenous, supply is endogenous, resulting from firms' investment decisions.[7] Firms consequently must invest accounting for the investment strategy of other firms in the industry and the impact of other firms' investment decisions on prices.

3.1 Preliminaries

The equilibrium concept employed in this paper is rational expectations. An equilibrium consists of a consistent 1) price process for the industry good, and 2) set of investment strategies for all firms in the industry. That is, in equilibrium firms' investment decisions are both 1) consistent with the evolution of the price process, and 2) optimal given the price process. As a firm has no incentive to deviate from its

[7]Supply is determined purely by firms' investment decisions because in the model firms never idle capacity; as they are small (price takers) and able to produce the good costlessly, they operate at full capacity.

investment strategy readers more comfortable with Nash equilibrium may choose to interpret the equilibrium using that concept.

3.2 The Nonexistence of a Symmetric Equilibrium

Before outlining the equilibrium strategy argument we briefly note that the equilibrium we are looking for is *not* a symmetric equilibrium, because no such equilibrium exists. This is formalized in the following proposition. In an effort to avoid excessive digression the proof of this proposition, and of all further propositions, will be left for the appendix.

Proposition 3.1 *There is no symmetric equilibrium.*

3.3 Overview of the Equilibrium Strategy Argument

To demonstrate an equilibrium strategy we must establish two major elements. The first is to hypothesize a strategy and determine the resulting price process conditional on all firms following the strategy. The second is to show that conditional on the resultant price process the optimal strategy is the hypothesized strategy.

The fastest way to establish these elements is to simply start by hypothesizing the equilibrium price process. While starting in this manner would be sufficient to demonstrate the existence of an equilibrium strategy, it is more informative first to complete some preliminary work motivating the hypothesized strategy. The argument used to motivate the strategy will also establish a limited uniqueness of the development strategy. In particular, the argument will demonstrate that there is a unique equilibrium strategy for which the return process for the price of firms' output has some natural stationary properties. The argument yielding this equilibrium exercise strategy is briefly outlined below.

We start by showing that, for a class of processes for the price of firms' output, firms will only build at price maxima. Hypothesizing that the equilibrium price process is in this class, we decompose the value of an arbitrary project into two parts: the value of the cash flows until prices return to the historical maximum, and the present value of the firm on that date. Because firms do not develop below the price maximum the price of firms' output has the same evolution as the demand process, and we use this fact to calculate the first part explicitly. This reduces the valuation of any firm to the valuation of the firm at price maxima.

We then decompose the value of the firm at a price maximum into two parts: the value of the cash flows until the firm is ready to increase capacity, and the present value of the firm on that date. Because other firms add capacity before the firm in question the price of firms' output does not evolve like demand on the interval in question. Explicit valuation of the two parts requires, therefore, a further investigation of the price process. Using a general property of firm value, implied by cash flow considerations, we are able to restrict the class of investment strategies that may be optimal. Assuming that all firms follow a strategy in this class we determine the evolution for the price of firms' output. Using this conditional price process we then complete the decomposition of firm value into the value of the cash flows until the firm is ready to increase capacity, and the present value of the firm on that date. The calculation is complicated, however, by two factors: the equilibrium price process is not Itô, confounding attempts to directly apply standard solution techniques, and allowing firms to add capacity repeatedly introduces a further complexity for which the literature provides no guidance. We introduce techniques to solve both these problems.

We demonstrate the optimal investment strategy for a firm that takes, as given, the conditional evolution of the price of firms' output. This strategy is in the class we assumed to derive the conditional evolution of prices, and is, therefore, an equilibrium strategy. If all other firms follow the strategy then following the same strategy is a firms optimal response.

The following sections develop the argument more thoroughly, and are arranged as follows. Section 3.4 makes mild assumptions about the equilibrium price process, which are verified later, and shows that under these assumptions firms will only add capacity when the price of the good they produce is high. Section 3.5 decomposes the value of an arbitrary firm into two parts: the value of cash flows until the price of firms' output returns to its historical maximum and the value of the cash flows after that time. It then further decomposes the value of a firm at a maximum in the price of the good it produces into two parts: the value of cash flows until the firms is ready to add capacity, and the value of the cash flows after that time. Section 3.6 uses standard techniques and the functional form for firm value derived in section 3.5 to demonstrates explicitly firms' optimal equilibrium investment strategy. It also provides a closed form solution for the value of an arbitrary firm.

3.4 Restricting the Scope of Study

We begin the analysis by noting that under some mild, reasonable assumptions about the equilibrium evolution of the price of firms' output, firms will only choose to add capacity when the price of the good they produce is "high." In fact, firms will only increase production when the price of their output is at historically high levels. This finding is generally consistent with what we observe in real industries: high output prices spur investment while low output prices discourage investment.

The intuition underlying this finding is quite natural. A firm will not choose to add capacity when the price of its output has dropped because any investment it considers undertaking at this time could have been undertaken more profitably when prices were higher. Because the firm optimally chose not to invest at the earlier, more profitable date it will not invest now that it is less profitable. The next proposition formalizes this concept. The statement of the proposition is simplified by the following definition. We will call a process P_t "semi-Markov" if the instantaneous evolution of the process depends only on P_t whenever $P_t < \overline{P}_t$, where $\overline{P}_t$ denotes the maximum of the process up to time t. That is, a process is semi-Markov if, away from its maxima, its instantaneous evolution depends only on its level.

Proposition 3.2 *Suppose the price of firms' output is continuous and semi-Markov. Then a firm will only develop at a price maximum.*

If no development occurs below maxima in the price of firms' output, then the price of firms' output follows a geometric Brownian motion below the price maximum. While no development occurs aggregate capacity is fixed. The price response to changing demand in a fixed capacity economy is a standard problem. In the absence of a supply response, demand shocks are translated directly into price shocks. We have, therefore, the following corollary to Proposition 3.2, under the assumption that the conditions of the previous proposition hold, which we will verify later.

Corollary 3.1 *The price of firms' output evolves as a geometric Brownian motion away from its maxima.*

3.5 Firm Value Decomposition

We can think of firm value as consisting of two parts: the value of the cash flows received up until the time that the price of firms' output returns to its historical maximum, and the value of the cash flows received after this time. The second part, the value of the cash flows received after prices return to their historical maximum, is the present value of receiving the firm at that date. This second part may be conceptualized in terms of financial instruments. It is a contract for forward delivery of the whole firm on the day the price of firms' output returns to its historical maximum, delivered at a price of zero.

Thinking of firm value as consisting of these two parts, we can write the value of a firm as follows. Let $V(q, P_t, \overline{P}_t)$ denote the value of a firm with capacity q when price of firms' output is P_t and the all-time price high is $\overline{P}_t$. Then

$$V(q, P_t, \overline{P}_t) = \mathsf{E}_t \left[\int_t^{\tau_{\overline{P}_t}} e^{-r(s-t)} q P_s ds + e^{-r(\tau_{\overline{P}_t} - t)} W(q, \overline{P}_t) \right], \tag{4}$$

where $\tau_{\overline{P}_t}$ denotes the stopping time for the first passage of the price process back to its historical maximum, and $W(q, \overline{P}_t)$ is shorthand for the value of a firm with capacity q when the price of firms' output is at its historical maximum, which is $\overline{P}_t$. That is, $\tau_{\overline{P}_t} \equiv min\{s \geq t \,|\, P_s = \overline{P}_t\}$ and $W(q, \overline{P}_t) \equiv V(q, \overline{P}_t, \overline{P}_t)$. We will refer to $W(q, P)$ as the "value-at-max" function.

The first term in the expectation on the right hand side of the previous equation, $\mathsf{E}_t \left[\int_t^{\tau_{\overline{P}_t}} e^{-r(s-t)} q P_s ds \right]$, is the value of the cash flows received up until the price of firms' output returns to its historical maximum. The second term, $\mathsf{E}_t \left[e^{-r(\tau_{\overline{P}_t} - t)} \right]$ $W(q, \overline{P}_t)$, is the present expected value of receiving the firm on that date but not receiving any of the intervening cash flows, where we have used the fact that $W(q, \overline{P}_t)$ is nonstochastic to take it outside the expectation.

Evaluating the right hand side of equation (4) is standard in the literature, at least under the restriction $\mu < r$. This is a technical issue. While we make no such restriction here conceptually the valuation is unchanged, and is provided by the following proposition:

Proposition 3.3 *The value of a firm with capacity q when the price of firms' output is P_t and the all-time high in the price of firms' output is $\overline{P}_t$ is given by*

$$V(q, P_t, \overline{P}_t) = \left(\pi q P_t - \left(\frac{P_t}{\overline{P}_t} \right)^{\beta} \pi q \overline{P}_t \right) + \left(\frac{P_t}{\overline{P}_t} \right)^{\beta} W(q, \overline{P}_t), \tag{5}$$

where $\pi = \frac{1}{r - \mu}$ *and* $\beta = \sqrt{(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} - (\frac{\mu}{\sigma^2} - \frac{1}{2})$.

The first term on the right hand side is, again, the value of the cash flows received up until the price of firms' output returns to its historical maximum; the second term is the value of receiving the firm on that date, but not receiving any of the intervening cash flows. Note that firm value is expressed in terms of known quantities, except for $W(q, \overline{P}_t)$, which is unknown. We now turn our attention to the calculation of $W(q, \overline{P}_t)$, the value of the firm at the time the price of firms' output returns to its historical maximum.

3.5.1 Firm Value at a Maximum in the Price of Firms' Output

We will now decompose the value of the firm at a historic high in the price of firms' output in exactly the same way we previously decomposed the value of a firm at an arbitrary price level. At a maximum in the price of firms' output the value of the firm consists of two parts: the value of the cash flows received up until the moment before the firm increases capacity, and the value of all the cash flows after this time. The second part, the value of receiving all cash flows from the moment the firm decides to increase capacity, is the present value of receiving the firm at that time. Again, in terms of financial instruments, the second part is a contract for forward delivery of the firm on the day the firm will be developing new capacity, at a delivery price of zero.

Thinking of firm value at a historic high in output prices as consisting of these two parts, we can write the value of a firm as follows:

$$W\left(q, P_t\right) = \mathsf{E}_t\left[\int_t^{\tau_{P_q^*}} e^{-r(s-t)} q P_s ds + e^{-r\left(\tau_{P_q^*}-t\right)} W\left(q, P_q^*\right)\right], \tag{6}$$

where P_q^* denotes the price level at which the firm optimally increases capacity, and $\tau_{P_q^*}$ denotes the first passage of the price process to this exercise boundary, $\tau_{P_q^*} \equiv min\{s \geq t \,|\, P_s = P_q^*\}$.

At this point we encounter a complication: we do not know how the price of firms' output evolves. Earlier, when we were considering the value of a firm as the value of cash flows before and after the price of firms' output returned to the historical maximum, we could easily evaluate the two parts. Below the price maximum capacity was fixed, so we knew prices evolved, like demand, as a geometric Brownian process. Knowing the evolution of prices allowed us to calculate values explicitly.

At maxima in the price of firms' output, however, aggregate capacity in not fixed, so we do not know how the price of firms' output evolves. Without knowing this evolution it is impossible to evaluate equation (6). We must determine the equilibrium evolution of the price of firms' output, therefore, before attempting the evaluation.

3.5.2 The Price of Firms' Output

At the price maxima aggregate supply responds to positive demand shocks, so prices do not evolve as does demand. Because of this supply effect we need to consider changes to supply as well as demand to determine the evolution of prices.

The inverse pricing function relates the price of firms' output to supply and demand i.e., relates prices P_t to aggregate industry capacity Q_t and the multiplicative demand shock X_t:

$$P_t = X_t \cdot Q_t^{-1/\alpha}.$$

The demand multiple, X_t, and the price elasticity of demand, α, are known, but we are still left with an underdetermined system: a single equation with two unknowns. We need another condition that relates the price at which firms can sell their output to aggregate industry capacity. We are able to derive another condition relating the price of firms' output to industry capacity by considering the following choice.

Suppose the rental rate for office space in downtown Manhattan and downtown Houston follow the same process, but that rents are twice as high in Manhattan. Suppose further that, as a result of the complexities involved in constructing a large building on a small plot of land, the cost of building is convex in the size of development. Let us assume, for the sake of simplicity, that the cost of building is quadratic in size. The choice we need to consider is the following: should one prefer to own a single twenty-story office tower in Manhattan, or four ten-story office towers in Houston?

The two sets of properties produce identical current cash flows. Forty stories in Houston generate the same rental income as twenty stories in Manhattan, where rents are twice as high. Because the current cash flows are the same, the choice is determined by the value of future development opportunities.

Now suppose that at some future time the owner of the Manhattan property optimally redevelops the property to forty stories. At that time the owner of the Houston properties can redevelop all four properties to twenty stories at the same cost. Because costs are quadratic in size, developing to half the size costs one quarter as much. Developing four properties each at one quarter the costs results in the same total cost of development. The situation is also symmetric. An analogous argument shows that the owner of the Manhattan property can replicate the cash flows of the Houston properties exactly, including the cost of new development, at all times.

One should be completely indifferent, therefore, between the Manhattan property and the Houston properties. Because the owner of one set of properties can exactly replicate whatever cash flows the owner of the other set of properties receives, the two sets of properties must have the same value. The following proposition formalizes this idea.

Proposition 3.4 *Suppose that the evolution of the log-price process only depends on the ratio of the price to the price maximum, i.e., that the instantaneous return depends only on $P/\overline{P}$. Then the value-at-max function has the following scaling property:*

$$W(q, P) = q^{\gamma} W(1, q^{(1-\gamma)} P). \tag{7}$$

There was nothing special, however, about our choice of Houston. You can think of Houston as New York in the past, when rents were half of what they are today. In otherwords, we should also be indifferent between holding a single building at today's high rents, or multiple smaller buildings at lower rents some years in the past.

This yields a dynamic relation between yesterday's investment decisions and investment today, and between today's investment decision and investment tomorrow. Consider the example above, but replace the owner in today's Houston with an owner in "old New York." The owner of the twenty-story building in today's New York optimally redevelops her property to forty stories when rents reach twice the level at which the owner of the ten-story properties in "old New York" optimally redeveloped to twenty stories. Now the two owners, one in "old New York" and one in New York today, could actually be the same owner at two different points in time. That is, the owner who initially developed from ten stories to twenty stories redevelops again to forty stories, optimally, when rents double. Following the same line of reasoning we can show that she will optimally redevelop in the future, again to twice the capacity, when rents double again.

The scaling condition on the value function imposes a general form on the optimal strategy. In particular, firms will develop to a fixed multiple of existing capacity whenever the price of their output reaches some fixed multiple of the price at which they last undertook development. The following proposition formalizes this concept.

Proposition 3.5 *Suppose the price process is continuous and that the evolution of the log-price of firms' output only depends on the ratio of the price to the price maximum, i.e., that the instantaneous return depends only on $P/\overline{P}$. Then a firm that owns a project with existing capacity q will optimally redevelop to capacity $q \cdot q_1^*$ when prices reach $q^{(\gamma-1)} \cdot P_1^*$, for some $q_1^* > 1$ and $P_1^* > 0$ that are independent of q.*

The strategy in Proposition 3.5 entails developing to a fixed multiple q_1^* times existing capacity at price multiples of $q_1^{*(\gamma-1)}$. Proposition 3.5 also reduces the explicit characterization of the optimal investment strategy of any firm to solving for the capacity to which a firm with existing capacity one optimally develops. A firm with existing capacity one optimally develops to capacity q_1^* when the price level reaches P_1^*, which motivates the notation.[8]

Recall that we were looking for a second equation relating aggregate industry capacity to the price of firms' output. The fact, provided by Proposition 3.5, that firms develop to a fixed multiple q_1^* times existing capacity at price multiples of $q_1^{*(\gamma-1)}$, allows us to generate the additional relation we need.

We do not yet know q_1^* explicitly, but we can consider the evolution of aggregate industry capacity conditional on all firms following a strategy of the general form suggested by Proposition 3.5. That is, we will assume firms develop to some fixed, arbitrary multiple κ times existing capacity when the price of firms' output reaches $\kappa^{(\gamma-1)}$ times the price at which they last increased capacity.

The intuition underlying the aggregate capacity process can be described as follows. Firms that have added capacity recently are less likely to undertaken near-term investment than firms that have not added capacity recently. Smaller capacity projects were developed farther in the past, while larger capacity projects were developed more recently. Because the opportunity costs to developing are lowest to the firm with the smallest ongoing project, this firm will be the next to add capacity. At the time it does so it leap-frogs all other firms in the size of existing capacity, becoming the largest firm in the industry. Because only the low-cost firm adds capacity at new price highs, aggregate capacity is incremental even though the investment of individual firms is lumpy. While aggregate capacity is incremental, it is not smooth:

[8] In Proposition 3.5 q_1^* is unitless. In an abuse of notation, made for the sake of convenience, we have also used q_1^* to denote the capacity to which a firm with existing capacity one optimally develops. We will use q_1^* to denote both the unitless development multiple and, when convenient, "the capacity 1 times the unitless factor q_1^*." The meaning will always be clear from the context.

most of the time aggregate capacity is unchanging, but occasionally bursts in investment activity cause aggregate capacity to adjust very quickly. The aggregate capacity process is given explicitly in the following proposition.

Proposition 3.6 *Suppose firms follow a strategy of developing to a fixed multiple κ times existing capacity when the price of firms' output reaches $\kappa^{(\gamma-1)}$ times the level at which they previously developed capacity. Further suppose that the initial distribution of firm sizes follows Zipf's law, and that the prices at which firms developed to their current sizes is consistent with the development rule in Proposition 3.5.*[9] *Then aggregate capacity is related to the price of firms' output shock by the following equation:*

$$Q_t = \left(\frac{\overline{P}_t}{\overline{P}_0}\right)^{\left(\frac{1}{\gamma-1}\right)} Q_0.^{10} \tag{8}$$

Proposition 3.6 provides a second equation relating the price of firms' output to capacity and, in conjunction with the inverse pricing equation, allows us to solve for the evolution of the equilibrium price process. Taking the inverse pricing equation at a maximum in the price of firms' output, and substituting into equation (8), we can determine explicitly the price of the firms' output as a function of the demand process. The price process is given explicitly in the following proposition.

Proposition 3.7 *Suppose that the conditions from Proposition 3.6 hold. Then the price of firms' output is related to the multiplicative demand shock by the following equation:*

$$ln\, P_t = ln\, P_0 + ln\, X_t - \frac{1}{1+\alpha(\gamma-1)}\, ln\, \overline{X}_t. \tag{9}$$

[9]That is, we are supposing that the initial distribution of capacities and development prices are log-uniform: initially distributed $((\xi/\kappa)\cdot q_0^{max}, (\xi/\kappa)^{(\gamma-1)}\cdot P_0)$ where $ln_\kappa\, \xi$ is distributed uniformly on (0,1]. (See appendix.)

[10]That aggregate capacity evolves in a manner *independent* of the specific choice of κ is quite remarkable. This independence is particular to our modeling opportunity costs as abandonment of ongoing assets. If we choose to model opportunity costs as some other fractional loss f of the value of assets in place, we would need to replace $\frac{1}{\gamma-1}$ in the exponent of equation (8) with $\frac{1}{\gamma-1}(1+ln_\kappa(1+\frac{1-f}{\kappa}))$. The rest of the analysis presented in this paper does not depend on the choice of abandonment, but this choice simplifies a fixed point problem we need to solve in the course of the analysis.

The first term on the right hand side of equation(9) calibrates the level of prices at time zero. The second term expresses the direct effect of demand on prices, and is directly proportional to the current level of the multiplicative demand shock. The third term expresses the effect of supply, and aggregate capacity is related to the highest level demand has reached previously.

The log-price process follows a sort of "attenuated geometric Brownian motion." Below historic highs in the price of firms' output capacity is fixed, so the instantaneous evolution of the price process is the same as the evolution of the multiplicative demand shock. Below price highs the only term that changes in the right hand side of equation (9) in response to a demand shock is $ln\, X_t$, so prices change in exactly the same way as demand. At historic highs, however, positive demand shocks result in a supply response that mitigates the effect on prices. Because $X_t = \overline{X}_t$ at these times the $-\frac{1}{1+\alpha(\gamma-1)}\, ln\, \overline{X}_t$ term in equation (9) disappears, canceling $\frac{1}{1+\alpha(\gamma-1)}\, ln\, X_t$ of the $ln\, X_t$ term, and positive demand shocks are translated into upward movements in the log-price process attenuated by the factor $\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}$, which is less than one.

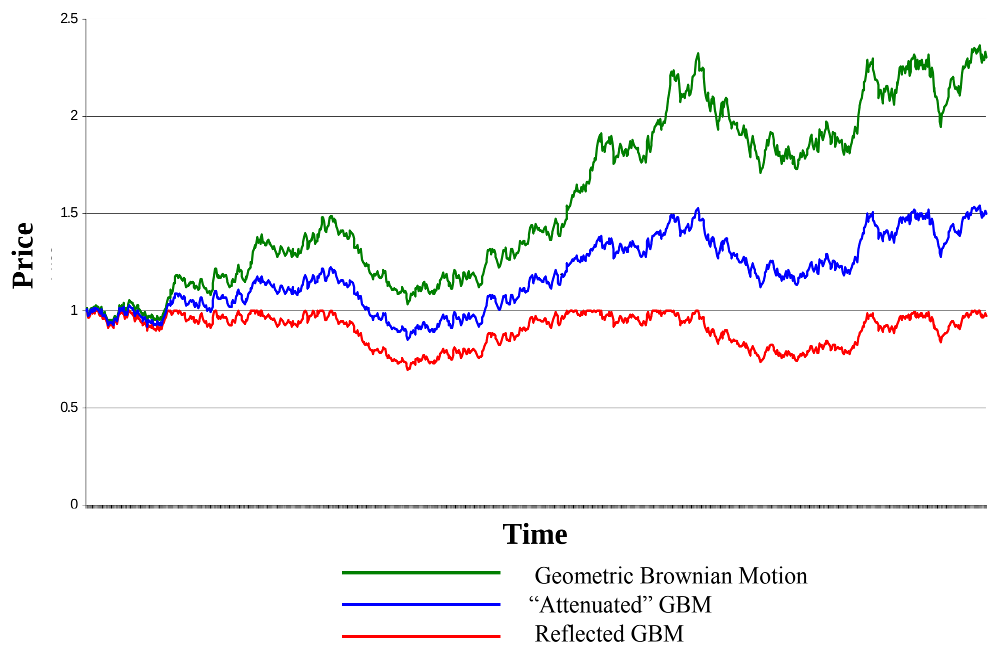

Figure 1. The top path is a standard, partial equilibrium, geometric Brownian price process. The bottom path follows reflected geometric Brownian motion. The middle path is the equilibrium price process, and follows "attenuated geometric Brownian motion." The degree of attenuation depends on the cost-to-scale of adding new capacity and the price elasticity with respect to new supply.

In Figure 1, above, this "attenuated geometric Brownian" equilibrium price process is shown for a particular realization of the evolution of the multiplicative demand shock. Also plotted, for the same realization of the demand shock and starting at the same initial price level, are the standard geometric Brownian motion typical of partial equilibrium models (top path), and a reflected geometric Brownian motion typical of linear, incremental investment models (bottom path, reflecting barrier at one). The degree of "attenuation" in the equilibrium price process depends on the cost-to-scale of adding new capacity, and on the elasticity of prices with respect to supply.

Now that we know the price process we are ready to reconsider the decomposition of firm value into the value of cash flows received up until the time at which the firm is ready to increase capacity and the value of the cash flows from that time on. Before doing so, however, we must note an important feature of the price process provided by equation (9).

The price of firms' output satisfies the conditions of Proposition 3.5: it is continuous and the distribution of instantaneous returns only depends on the ratio of the price to the price maximum.[11] We derived this process for price of firms' output under these same assumptions. As a corollary of Proposition 3.7 we have, therefore, the general form of the equilibrium strategy.

Corollary 3.2 *For some* $q_1^* > 1$ *and* $P_1^* > 0$ *the strategy from Proposition 3.5 of redeveloping existing capacity* q *to* $q \cdot q_1^*$ *when prices reach* $q^{(\gamma-1)} \cdot P_1^*$ *is an* **equilibrium strategy**. *If all firms follow the strategy, no firm has an incentive to deviate.*

We would like, of course, an explicit characterization of the strategy. That is, we would like to calculate the specific q_1^* and P_1^* in Proposition 3.5 for the price process given by equation (9). We will be able to calculate these explicitly, but to do so actually requires that we finish our decomposition of firm value at a maximum in the price of firms' output. We return now to this calculation.

3.5.3 Firm Value at a Maximum in the Price of Firms' Output, Redux

Recall that the value of a firm, when the price of firms' output is at a historical high, may be written as the expected value of cash flows received up until the date of the next development, plus the present expected value of receiving the firm at that date, but none of the intervening cash flows. That is, we can write firm value as

[11] The evolution of the log-price process depends on the ratio of price to price maximum in a simple, binary way. It only depends on whether the ratio is one or less than one.

$$W\left(q, P_t\right) = \mathsf{E}_t\left[\int_t^{\tau_{P_q^*}} e^{-r(s-t)} q P_s\, ds\right] + \mathsf{E}_t\left[e^{-r\left(\tau_{P_q^*}-t\right)}\right] W\left(q, P_q^*\right), \tag{10}$$

where P_q^* denotes the price level at which the firm optimally increases capacity and $\tau_{P_q^*}$ denotes the first passage of the price process to this exercise boundary, $\tau_{P_q^*} \equiv min\{s \geq t \,|\, P_s = P_q^*\}$, and we have used the fact that P_q^* is non-stochastic to take $W\left(q, P_q^*\right)$ out of the expectation.

While we now know the evolution of the price of firms' output, calculating the value of each of the parts is still non-trivial, because the price of firms' output is not an Itô process. This prevents us from directly applying the standard machinery of continuous time finance as we could with the first price decomposition. Nevertheless, we are able to find closed form solutions for the value of each of the parts.

Calculation of the first term is simplified by thinking of the intermediate cash flows to the firm as resulting from a portfolio of two perpetual cash flows, one held long and the other held short. Suppose you owned a portfolio that was 1) long a perpetual cash flow equal to the revenue generated by a project of fixed and immutable capacity q, (i.e., a project of capacity q that may not be expanded) and 2) short a contract for delivery of these same cash flows, deliverable at the moment the actual firm in question is ready to increase capacity and at a delivery price of zero. Because the actual firm in question is not going to alter capacity over the time frame in question, the portfolio described generates the same cash flows as the firm up to the moment of development. After development the portfolio delivers zero cash flows, as the long and short positions cancel. This is precisely the cash flow we are trying to value.

We can derive an expression for the value of the perpetual cash flows from a project of fixed and immutable capacity q using the following observation: the evolution of the log-price of firms' output is independent of the level of prices. The value of the perpetual cash flow on the day that prices first double today's prices is, therefore, double the value of the perpetual cash flow today. Using this fact we can write the intermediate cash flow part of the firm's value as

$$\begin{aligned} \mathsf{E}_t\left[\int_t^{\tau_{P_q^*}} e^{-r(s-t)} q P_s\, ds\right] &= \mathsf{E}_t\left[\int_t^{\infty} e^{-r(s-t)} q P_s\, ds\right] \\ &\quad - \mathsf{E}_t\left[e^{-r\left(\tau_{P_q^*}-t\right)}\right]\left(\frac{P_q^*}{P_t}\right) \mathsf{E}_t\left[\int_t^{\infty} e^{-r(s-t)} q P_s\, ds\right]. \end{aligned} \tag{11}$$

The first term on the right is the value of a project with capacity q that may not be expanded, the second term is the present value of those same cash flows starting the day the actual firm in question decides to increase capacity.

We can simplify the previous equation with the following definition. Let Π be the dollar price of a unit cash flow, at today's price, derived from a flow of the firm's output. That is, $\Pi \equiv \mathsf{E}_t \left[\int_t^\infty e^{-r(s-t)} \frac{P_s}{P_t}\, ds\right]$. Using this definition we can rewrite the previous equation as

$$\mathsf{E}_t \left[\int_t^{\tau_{P_q^*}} e^{-r(s-t)} q P_s\, ds\right] = \Pi\, q\, P_t - \mathsf{E}_t \left[e^{-r(\tau_{P_q^*}-t)}\right] \Pi\, q\, P_q^*. \tag{12}$$

We can see from equation (12) that valuing of the firm's cash flows up until the time the firm increases capacity, i.e., the first term in the price decomposition equation (equation (10)), reduces to calculating $\mathsf{E}_t \left[e^{-r(\tau_{P_q^*}-t)}\right]$, the value of the Arrow-Debreu security that pays a dollar when the price of firms' output first reaches P_q^*, and Π, the dollar price of a unit cash flow derived from a flow of the good.

First we will calculate $\mathsf{E}_t \left[e^{-r(\tau_{P_q^*}-t)}\right]$, the value of the Arrow-Debreu security that pays a dollar when the price of firms' output first reaches P_q^*. While we cannot apply the standard machinery directly to price $\mathsf{E}_t \left[e^{-r(\tau_{P_q^*}-t)}\right]$, because the price process is not Itô, we can bring the power of the standard machinery to bear on the problem indirectly.

We know, by assumption, that the firm is going to increase capacity when the price of its output first reaches P_q^*. The equilibrium price process derived in Proposition 3.7, equation (9), allows us to relate arbitrary prices for firms' output to the level of the demand process. This provides us with a solution to the problem that the price process is not Itô. While the fact that prices are not Itô prevents us from using the standard machinery to price assets that depend on prices, we can apply the standard machinery to problems that involve pricing assets that depend on demand, which is Itô. In particular, by determining the demand that will result in prices reaching the development price threshold P_q^* we can determine $\mathsf{E}_t \left[e^{-r(\tau_{P_q^*}-t)}\right]$.

We can determine the demand that will result in prices reaching the development price threshold P_q^* quite easily, because we know the evolution of the price of firms' output as a function of demand. Inspection of the price–demand relation derived in Proposition 3.7 yields

$$P_{\tau_{P_q^*}} = \overline{X}_{\tau_{P_q^*}}^{\left(\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\right)} \left(\frac{X_{\tau_{P_q^*}}}{\overline{X}_{\tau_{P_q^*}}}\right) P_t. \tag{13}$$

Using the fact that $P_{\tau_{P_q^*}} = P_q^*$, and P_q^* is a price maximum so $X_{\tau_{P_q^*}} = \overline{X}_{\tau_{P_q^*}}$, we have

$$X_{\tau_{P_q^*}} = \left(\frac{P_q^*}{P_t}\right)^{\left(1+\frac{1}{\alpha(\gamma-1)}\right)} X_t. \tag{14}$$

Using this price-demand relation we then have that $\tau_{P_q^*} \equiv min\{s \geq t \,|\, P_s = P_q^*\} = min\{s \geq t \,|\, X_s = \left(\frac{P_q^*}{P_t}\right)^{\left(1+\frac{1}{\alpha(\gamma-1)}\right)} X_t\}$. Applying the standard machinery we then have

$$\mathsf{E}_t\left[e^{-r(\tau_{P_q^*}-t)}\right] = \mathsf{E}_t\left[e^{-r(\tau_{X_{\tau_{P_q^*}}}-t)}\right] = \left(\frac{X_t}{X_{\tau_{P_q^*}}}\right)^{\beta} = \left(\frac{P_t}{P_q^*}\right)^{\left(1+\frac{1}{\alpha(\gamma-1)}\right)\beta}. \tag{15}$$

The second value we need, Π, the dollar price of a unit cash flow derived from a flow of the good, follows directly from the following proposition:

Proposition 3.8 *Suppose Y_t^δ follows "attenuated geometric Brownian motion," where $1-\delta$ denotes the degree of attenuation. That is, $Y_t^\delta = Exp(X_t^\delta)$, where $X_t^\delta = \delta\overline{X}_t - (\overline{X}_t - X_t)$, X_t is a drifted Brownian motion, $\overline{X}_t$ is the maximum of the Brownian process up to time t, and $0 \leq \delta \leq 1$.[12] Then the dollar price of a unit cash flow proportional to the process is given by*

$$\pi_\delta = \mathsf{E}\left[\int_0^\infty e^{-rt}\frac{Y_t^\delta}{Y_0^\delta}dt\right] = \left(\frac{\beta-1}{\beta-\delta}\right)\pi, \tag{16}$$

subject to the parameter restriction $\mu < \frac{r}{\delta} + (1-\delta)\frac{\sigma^2}{2}$, which ensures π_δ is finite.

The equilibrium price of firms' output follows attenuated geometric Brownian motion with $\delta = \frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}$, so as a corollary we have the dollar price of a unit cash flow derived from a flow of firms' output, and in equilibrium.

[12] We call X_t^δ attenuated Brownian motion, as up moves from historical maximum are "attenuated" by the factor δ. The process has full down moves (deficits from the highs). We call $Y_t^\delta = exp(X_t^\delta)$ attenuated geometric Brownian motion.

Corollary 3.3 *In equilibrium the dollar price of a unit cash flow derived from a flow of the good is given by*

$$\Pi = \pi_{\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}} = \frac{\pi}{1 + \frac{1}{(1+\alpha(\gamma-1))(\beta-1)}}. \tag{17}$$

We can now express the value of receiving the firm's cash flows up until the next time the firm increases capacity exactly. Substituting for $\mathsf{E}_t\left[e^{-r(\tau_{P_q^*}-t)}\right]$ and Π in equation (12) we have

$$\begin{aligned}\mathsf{E}_t\left[\int_t^{\tau_{P_q^*}} e^{-r(s-t)} q P_s \, ds\right] &= \Pi \, q \, P_t - \mathsf{E}_t\left[e^{-r(\tau_{P_q^*}-t)}\right] \Pi \, q \, P_q^* \\ &= \frac{\pi \, q \, P_t}{1 + \frac{1}{(1+\alpha(\gamma-1))(\beta-1)}} \left(1 - \left(\frac{P_t}{P_q^*}\right)^{\eta-1}\right), \end{aligned} \tag{18}$$

where $\eta \equiv \left(1 + \frac{1}{\alpha(\gamma-1)}\right)\beta$.[13] We will sometimes refer to η as the "convexity" of the value-at-max function.

Having already calculated $\mathsf{E}_t\left[e^{-r(\tau_{P_q^*}-t)}\right]$, valuing the cash flows from the firm starting at the moment prior to the next time the firm increases capacity is trivial. Substituting for $\mathsf{E}_t\left[e^{-r(\tau_{P_q^*}-t)}\right]$ we have

$$\mathsf{E}_t\left[e^{-r(\tau_{P_q^*}-t)}\right] W\left(q, P_q^*\right) = \left(\frac{P_t}{P_q^*}\right)^{\eta} W\left(q, P_q^*\right). \tag{19}$$

Taking the two parts together gives us the value of the firm at a maximum in the price of firms' output. Adding the right hand sides of equations (18) and (19) yields

$$W\left(q, P_t\right) = \Pi \, q \, P_t + \left(\frac{P_t}{P_q^*}\right)^{\eta} \left(W\left(q, P_q^*\right) - \Pi \, q \, P_q^*\right), \tag{20}$$

where $\Pi = \frac{\pi}{1+\frac{1}{(1+\alpha(\gamma-1))(\beta-1)}}$ and $\eta \equiv \left(1 + \frac{1}{\alpha(\gamma-1)}\right)\beta$.

The first term on the right hand side of equation (20) is the value of the future cash flows to the firm's existing assets, or the firm's "intrinsic value." The second

[13] Again, if adjusting capacity entails the loss of a fraction f of the value of projects in place then η is not independent of the constrained firms' strategy. In particular, $\eta = (1 + \lambda(\kappa)/\alpha)\beta$ where $\lambda(\kappa) = \frac{1}{\gamma-1}(1 + ln_\kappa(1 + \frac{1-f}{\kappa}))$.

term is the value of the firms future development opportunities, or the firm's "option value."

We still do not have a complete, explicit function for the value of the firm. The value of the firm in equation (20) is still given in terms of the unknown value of the firm at some future date. Moreover, this unknown, future firm value depends on the specific redevelopment strategy employed by the firm.

We can derive a complete, closed form solution for the firm's value, but doing so requires that we explicitly determine the firm's optimal investment strategy.

3.6 Explicit Characterization of the Optimal Strategy

At this point determining the details of the optimal strategy, i.e., the calculation of q_q^* and P_q^*, is a standard problem. Because Proposition 3.5 characterizes any firm's optimal strategy in terms of the optimal strategy for the firm with unit capacity, the problem reduces to the calculation of q_1^* and P_1^*.

We need to maximize the value of the firm with unit capacity over the choice of strategy, which requires that we solve a free-boundary problem. The standard procedure, value matching and smooth pasting at the time of development, requires that the following equations are satisfied:

$$W^1(1, P_1^*) = W^2(q_1^*, P_1^*) - q_1^{*\gamma} \tag{21}$$

and

$$W_P^1(1, P_1^*) = W_P^2(q_1^*, P_1^*) \tag{22}$$

where W^1 and W^2 are the value-at-max functions before and after development, respectively.[14]

[14]It is not necessarily obvious that equation (22) is the proper optimality condition. While it looks like the standard condition, it is really short hand for $V_P^1(1, P_1^*, P_1^*) + V_{\overline{P}}^1(1, P_1^*, P_1^*) = V_P^2(q_1^*, P_1^*, P_1^*) + V_{\overline{P}}^2(q_1^*, P_1^*, P_1^*)$, which is decidedly non-standard. Nevertheless, as stated by Dixit and Pindyck (1994), "the conditions applicable to free boundary problems are specific to each application and must come from economic considerations." In this application, economic considerations dictate the form of the value-at-max function. From equation (5) it is clear that the value of the project is maximized when the value-at-max function is maximized. The fact that (22) is indeed the proper optimality condition then follows directly from the fact that the convexity of the value-at-max function prior to development is greater than the convexity of the function after development (i.e., the left hand side of equation (21) is more convex than the right hand side).

Solving equation (21) and equation (22) requires the functional form of the value-at-max process, which is provided by inspection of equation (20). While we do not know the value of $(W\left(q, P_q^*\right) - \Pi\, q\, P_q^*)/P_q^{*\eta}$, it is a constant conditional on P_q^*. Letting $a_q = (W\left(q, P_q^*\right) - \Pi\, q\, P_q^*)/P_q^{*\eta}$, we have that

$$W(q, P) = \Pi\, q\, P + a_q\, P^{\eta}. \tag{23}$$

Using the functional form provided by equation (23), we see that equations (21) and (22) form a system of two equations with three unknowns: a_1, a_q, and P_1^* (q_1^* is a choice variable, not an unknown). The system is underdetermined. We need an additional constraining equation to fully specify the optimal strategy.

The standard additional constraint is not available to us. The common assumption in the literature, that a project may be developed one time only, is implicitly an assumption that $a_q = 0$. This is clearly inappropriate here.

The appropriate constraint is provided by the scaling condition on the value function, given in Proposition 3.4. Because development rights are retained at exercise, the scaling condition holds *across development boundaries*. Using this condition we have

$$q^{\gamma} W^1\left(1, q^{(1-\gamma)} P^*\right) = W^2\left(q, P^*\right). \tag{24}$$

Again using the functional form for W^1 and W^2 given by equation 23, we have

$$q^{\gamma}\left(\Pi\left(q^{(1-\gamma)} P^*\right) + a_1\left(q^{(1-\gamma)} P^*\right)^{\eta}\right) = \Pi q P^* + a_q P^{*\eta}. \tag{25}$$

Solving for a_q yields

$$a_q = q^{\gamma + (1-\gamma)\eta} a_1. \tag{26}$$

The value matching and smooth pasting equations, (21) and (22), in conjunction with this additional constraint implied by the scaling condition, allow us to determine firm value and the optimal investment strategy explicitly. The three equations are sufficient to pin down the three free variables in terms of the choice variable, the multiple of current capacity to which firms develop.

After pinning down the free variables we can substitute them back into the functional form for firm value. This allows us to express firm value in terms of the multiple of current capacity to which firms develop. The optimal strategy is the value maximizing strategy. Maximizing firm value over the choice variable yields the optimal strategy.

This explicit characterization of the optimal strategy, and the closed form solution for firm value, are provided in the following proposition.

Proposition 3.9 *The optimal strategy for a firm with existing capacity q is to redevelop to capacity $q \cdot q_1^*$ when prices reach $P_q^* = q^{(\gamma-1)} \cdot P_1^*$, where*

$$q_1^* = argmax_q \left\{ \frac{(q-1)^\eta}{q^\eta (q^{\gamma(\eta-1)-\eta} - 1)} \right\}, \tag{27}$$

$$P_1^* = \frac{\eta}{(\eta-1)\Pi} \frac{q_1^{*\gamma}}{q_1^* - 1}, \tag{28}$$

and η and Π are given by equations (23) and (17), respectively.[15]

The value-at-max function may then be written, for $P < P_q^*$, as

$$W(q,P) = \Pi\, q\, P + A q^\gamma \left(\frac{P}{P_q^*} \right)^\eta \tag{29}$$

where

$$A = \frac{q_1^{*\gamma}}{(\eta-1)(1 - q_1^{*\gamma+(1-\gamma)\eta})},$$

subject to the parameter restriction $\mu < (1 - \frac{\alpha-1}{\alpha\gamma})r - (\frac{\alpha-1}{1+\alpha(\gamma-1)})\frac{\sigma^2}{2}$, which ensures that firms have finite value.

Note that q_1^* may also be thought of as a root of a simple polynomial associated with the economy.

[15] If adjustment entails the loss of a fraction $f \neq 1$ of projects in place, then the $(q_1^* - 1)$ in the denominator of P_1^* must be replaced by $(q_1^* - f)$. Also, q_1^* in equation (27) has κ dependency through η, and the equilibrium strategy, q_1^{**}, is defined implicitly as the solution to $q_1^*(\kappa) = \kappa$. There is also an implication for the dollar price of a unit cash flow at a price maximum, $\Pi = (1 + \frac{\lambda^*}{(\alpha+\lambda^*)(\beta-1)})^{-1}\pi$, where here $\lambda^* = \lambda(q_1^{**})$.

Proposition 3.10 *The capacity q_1^* to which a firm will optimally redevelop a project with existing capacity one is the unique solution greater than one to*

$$(\gamma\eta - \gamma - \eta)x^{(\gamma\eta-\gamma-\eta+1)} - (\gamma\eta - \gamma)x^{(\gamma\eta-\gamma-\eta)} + \eta = 0. \tag{30}$$

Furthermore, $1 + \frac{\eta-1}{\gamma\eta-\gamma-\eta+1} < q_1^* < 1 + \frac{\eta}{\gamma\eta-\gamma-\eta}$.

Finally, combining the results of Propositions 3.3 and 3.9 allows us to express the value of an arbitrary firm explicitly.

Proposition 3.11 *The value of a firm with existing capacity q when the price of the firm's output is P_t and the historical maximum for prices is $\overline{P}_t < P_q^*$ is given by*

$$V(q, P_t, \overline{P}_t) = \pi\, q\, P_t + \left(\frac{P_t}{\overline{P}_t}\right)^{\beta} \left((\Pi - \pi)\, q\, \overline{P}_t + A q^{\gamma} \left(\frac{\overline{P}_t}{P_q^*}\right)^{\eta}\right), \tag{31}$$

where $\pi = \frac{1}{r-\mu}$ *and* $\Pi = \frac{\pi}{1+\frac{1}{(1+\alpha(\gamma-1))(\beta-1)}}$, $\beta = \sqrt{(\frac{\mu}{\sigma^2} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}} - (\frac{\mu}{\sigma^2} - \frac{1}{2})$ *and* $\eta = \left(1 + \frac{1}{\alpha(\gamma-1)}\right)\beta$, $A = \frac{q_1^{*\gamma}}{(\eta-1)(1-q_1^{*\gamma+(1-\gamma)\eta})}$, $P_q^* = q^{(\gamma-1)} \frac{\eta}{(\eta-1)\Pi} \frac{q_1^{*\gamma}}{q_1^*-1}$, *and q_1^* is the unique root of equation (30) between* $\frac{\gamma\eta-\gamma}{\gamma\eta-\gamma-\eta+1}$ *and* $\frac{\gamma\eta-\gamma}{\gamma\eta-\gamma-\eta}$.

The first term in equation (31), $\pi\, q\, P_t$, is the value of the cash flows from the current project, ignoring supply effects on prices. The second term, $\left(\frac{P_t}{\overline{P}_t}\right)^{\beta} (\Pi-\pi)\, q\, \overline{P}_t$, corrects for these supply effects and is negative reflecting the fact that new capacity puts downward pressure on the price of firms' output. The last term, $\left(\frac{P_t^{\beta}\, \overline{P}_t^{(\eta-\beta)}}{P_q^{*\eta}}\right) A q^{\gamma}$, is the option value of the project, which derives from the right to increase capacity in the future.

4 Implications

The preceding analysis yields several positive implications that we will explore in this section. These concern 1) properties of the equilibrium price of firms' output, 2) the evolution of firm value, i.e., stock returns, 3) equilibrium real option premia, and 4) firms' optimal investment strategy, especially as it compares to the predictions of the standard, partial equilibrium analysis.

With respect to the price of firms' output, the model predicts negative skewness, resulting from endogenous increases in capacity. Because capacity is added at price maxima, this skewness is especially pronounced when prices are high. In fact, at maxima in the price of firms' output we always expect prices to drop in the short term, and for some parameterizations to remain low far into the future. As firms only add capacity when the price of their output is high this means that firms always develop expecting prices to drop.

The skewness in the price of firms' output is translated into stock prices, but only partially. Firm value is convex in the price of the firm's output, and this tends to skew stock prices *positively*. When the price of firms' output is low output prices are essentially unskewed, and the effect of convexity dominates. At these times stock prices are positively skewed. When the price of firms' output is high, however, the effect of the strong negative skewness in the price of firms' output dominates. At these times stock prices are also negatively skewed. That is, in recessions we expect to see positive skewness in stock returns, while during expansions we expect to see negative skewness.

The analysis also demonstrates that accounting for the price impact of development leads firms to develop *later*. That is, in equilibrium firms delay capital improvements longer than the standard, partial equilibrium analysis prescribes. Finally, we show that competition does not erode option premia. Even after properly accounting for the impact of competition, future development rights can contribute just as large a fraction of overall project value as they do in the standard analysis that ignores competition.

4.1 The Price of Firms' Output

The negatively skewed equilibrium return to firms' output is itself a major prediction of the model and provides testable implications. Because we characterize the equilibrium price process analytically we can make specific predictions regarding properties of the skewness. The model predicts the degree of skewness to be a function of both industry cost structure and industry history. Cross sectionally, we would expect more pronounced price skewness in industries where the supply elasticity of demand is high, i.e., prices should be more skewed in industries where the costs of adding capacity are low. We would expect more pronounced price skewness when the demand for firms' output is elastic. In the time series, we expect to see less price skewness at short horizons. The model predicts supply to be more responsive when prices are near historic highs. At lower prices, when firms are less likely to add capacity, skewness should be less pronounced, and return skewness should tend to zero as the horizon becomes very short. At price maxima, on the other hand, skewness in prices should be especially pronounced.

Forward prices provide the most convenient method for studying the price process in greater detail. The forward price is an unbiased estimator of the risk adjusted future spot price. The closed form expression for the forward price of firms' output is provided in the next proposition. It is somewhat complicated, but studying the behavior of some of its asymptotic properties proves illuminating.

Proposition 4.1 *The equilibrium instantaneous t-ahead forward price of the good (i.e., the expected future spot price) is given by*

$$\begin{aligned} F_{t_0+t} &= P_{t_0} e^{\mu t} \left(N\left(\frac{ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right) - (\mu + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \right) \right. \\ &\quad + \theta N\left(\frac{-ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right) - (\mu + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}} \right) \left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{(1+\frac{2\mu}{\sigma^2})} \\ &\quad + (1-\theta) e^{-\left(\frac{1}{1+\alpha(\gamma-1)}\right)\left(\mu + \frac{\sigma^2}{2} - \frac{1}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}\right)t} \\ &\quad \left. \cdot\ N\left(\frac{-ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right) + (\mu + \frac{\sigma^2}{2} - \frac{\sigma^2}{1+\alpha(\gamma-1)})t}{\sigma\sqrt{t}} \right) \left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\left(\frac{1}{1+\alpha(\gamma-1)}\right)} \right). \quad ¥ \end{aligned} \tag{32}$$

where $\theta = \frac{1}{(1+\alpha(\gamma-1))(1+\frac{2\mu}{\sigma^2})-1}$.

An important special case of equation (32) results from considering forward prices when the spot is at a maximum, i.e., when $P_{t_0} = \overline{P}_{t_0}$. In this case we have

$$\begin{aligned} F_{t_o+t} &= P_{t_0} e^{\mu t} \left[(1+\theta) N\left(-\left(\frac{\mu}{\sigma} + \frac{\sigma}{2}\right)\sqrt{t} \right) \right. \\ &+ \left. (1-\theta) N\left(\left[\mu + \left(\frac{\alpha(\gamma-1)-1}{\alpha(\gamma-1)+1}\right)\frac{\sigma^2}{2}\right] \frac{\sqrt{t}}{\sigma} \right) e^{\frac{-1}{1+\alpha(\gamma-1)}(\mu + (\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)})\frac{\sigma^2}{2})t} \right]. \end{aligned} \tag{33}$$

Figure 2 depicts forward prices for a period of six years, for three different costs-to-scale of development. In all three cases, prices are at a historical maximum. Note that prices in the short term are *expected to decrease*. That is, firms develop optimally expecting prices to drop. As would be expected, prices are expected to rise faster in the future when the cost to scale of development is higher and supply is thus less responsive to prices.

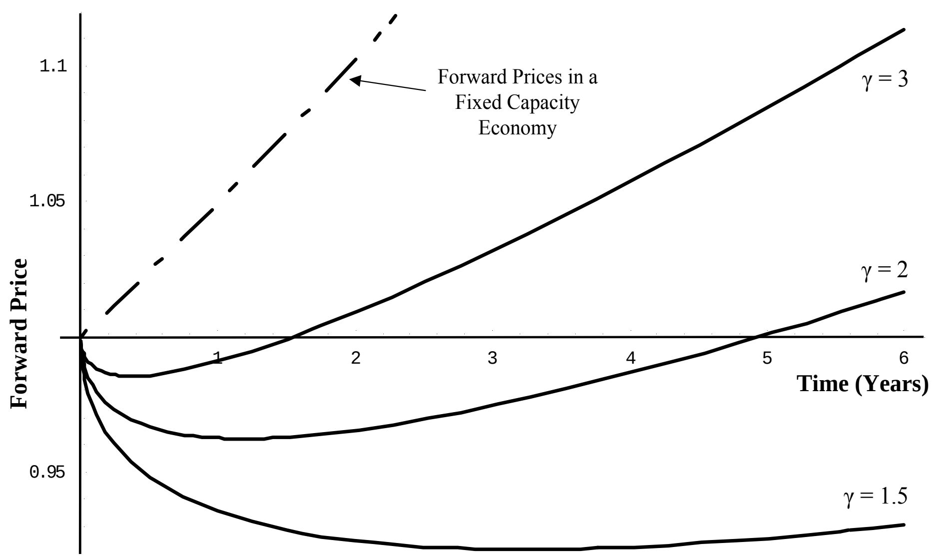

Figure 2.
Equilibrium Forward Prices for Three Costs-to-Scale of Development.
The equilibrium forward price is the unbiased estimator of the future spot price. In all three curves the initial spot price is one, which is the historical maximum. Assuming a price elasticity of one, the cost-to-scale of development is three for the top curve; for the middle curve it is two; and for the bottom curve it is one and a half. Other parameters are $\mu = 0.05$, $\sigma = 0.18$. The dashed line is the forward price in an economy in which total capacity to produce the good is fixed.

Figure 2 illustrates one extreme for forward process, the case when current prices are at the historical maximum. At the other extreme, when the previous maximum of the price process becomes very large, equation (32) reduces to $P_{t_0} e^{\mu t}$. This results because no building occurs below the maximum. When prices are low it is virtually certain that no development will occur for a significant amount of time and prices are then essentially drifted geometric Brownian motion.

The intermediate cases are more interesting. Figure 3 demonstrates the importance of the price to price-high ratio on forward prices. In the figure we plot three possible relations between prices and the historical price high: 1) spot prices at the high; 2) spot prices one-sixth below the high; and 3) spot prices one-third below the high. Forward prices are shown at horizons out to two years. In the figure the cost-to-scale of development is quadratic.

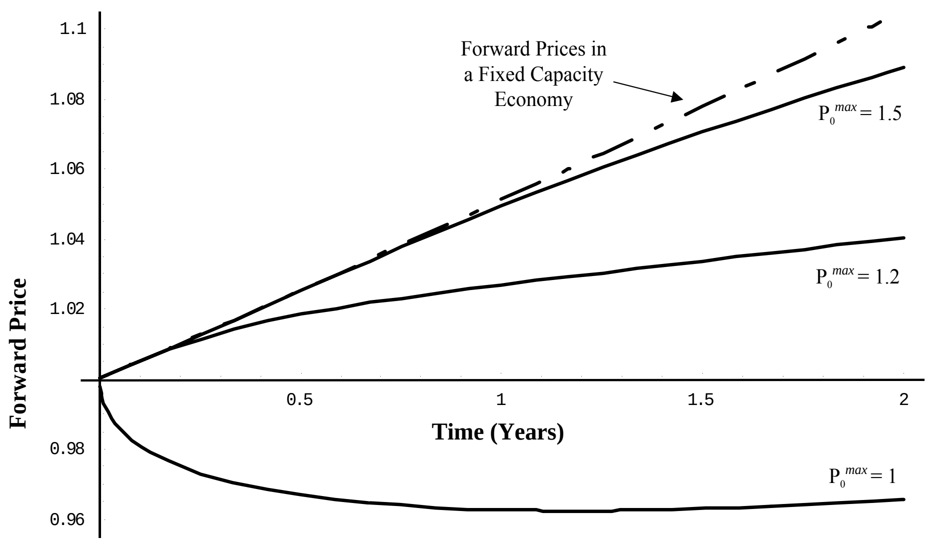

Figure 3.
Equilibrium Forward Prices for Three Ratios of Price to Price Maximum.
The equilibrium forward price is the unbiased estimator of the future spot price. Here the cost to scale of development is moderate, $\gamma = 2$. In all three curves, the initial spot price is 1, i.e., $P_0 = 1$. In the top curve, the spot price is one-third below the historical maximum, $P_0^{max} = 1.5$. In the middle curve, the spot price is one-sixth below the historical maximum, $P_0^{max} = 1.2$. In the bottom curve, the spot price is at the historical maximum, $P_0^{max} = 1$. Other parameter values are $\mu = 0.05$, $\sigma = 0.18$, and $\alpha = 1$. The dashed line is the forward price in an economy in which total capacity to produce the good is fixed.

The basic shapes of the forward price curves in Figure 3 are quite similar to those that we observe in commodity markets. When the price of firms' output is at or near historical highs, and increases in aggregate capacity are likely, the term structure of forward prices is downward sloping. When near term increases in capacity are unlikely the forward price curve will be upward sloping. In the parlance of commodity markets, forward prices may be either "in backwardization," with prices for future delivery of the good declining with time-to-delivery, or "in contango," with prices increasing with time-to-delivery.

It is also useful to consider some of the other asymptotic properties of the equilibrium price process. As the costs-to-scale of building becomes high equation (32) again reduces to $P_{t_0}e^{\mu t}$. That is, when it is too expensive to build the price process simply becomes drifted geometric Brownian motion.

As the cost of building goes to linear the forward prices go to

$$\left(N\left(\frac{ln(\frac{\overline{P}_{t_0}}{P_{t_0}}) - (\mu + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right) + \frac{\sigma^2}{2\mu} N\left(\frac{-ln(\frac{\overline{P}_{t_0}}{P_{t_0}}) - (\mu + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right)\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{(1+\frac{2\mu}{\sigma^2})}\right) P_{t_0}e^{\mu t}$$
$$+ \left(1 - \frac{\sigma^2}{2\mu}\right) N\left(\frac{-ln(\frac{\overline{P}_{t_0}}{P_{t_0}}) + (\mu - \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right)\overline{P}_{t_0}. \tag{34}$$

That is, the process begins at P_{t_0}. In the short run, prices evolve as a geometric Brownian motion, if $P_{t_0} < \overline{P}_{t_0}$, but in the long term they reach the steady state expectation of $(1 - \frac{\sigma^2}{2\mu})\overline{P}_{t_0}$ if $\mu > \frac{\sigma^2}{2}$, or zero if $\mu < \frac{\sigma^2}{2}$. When the cost to scale is linear, $\overline{P}_{t_0}$ is a reflecting barrier on prices. The factor $(1 - \frac{\sigma^2}{2\mu})$ captures the tension between the upward drift in the process and the variance. If the upward drift is large or the volatility is small, then the price process remains close to the barrier. On the other hand, when the drift is small or the volatility is large then prices, after bouncing off the barrier, may fall a long way prior to recovering.

Finally, we will consider more generally the long run average growth of the spot price, obtained by dividing the log of forward prices from equation (32) by t and letting t become large. Doing so in the interesting case, when μ is not too small, we find that the average long term rate of expected price growth is

$$\left(1 - \frac{1}{1+\alpha(\gamma-1)}\right)\left(\mu - \frac{1}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}\right). \tag{35}$$

4.2 Stock Returns

Our closed form expression for firm value as a function of the price of firms' output and aggregate industry capacity allows us to study the distribution of stock returns. Stock returns are a combination of the returns to firms' ongoing projects and their growth options. Away from historic highs in the price of firms' output, aggregate capacity is unlikely to increase so the return to firms' output exhibits little skew. This translates into positive skew in stock returns because the option component of firm value is convex in the price of firm output. When the price of firms' output is high the effect of increasing aggregate capacity dominates and stock returns exhibit negative skew. This provides a hitherto untested empirical prediction: skewness in stock returns should vary over the business cycle. Stock returns should be negatively skewed during expansions, but positively skewed in recessions.

Proposition 4.2 *Away from maxima in the price of firms' output stock returns are positively skewed at sufficiently short horizons. At maxima in the price of firms' output stock returns are negatively skewed at short horizons.*

4.3 The Investment Strategy

At this point we can compare the optimal behavior of a firm in the equilibrium economy, where firms' development decisions affect aggregate capacity, to the optimal behavior of a firm in an economy in which firms' investment decisions somehow do not affect aggregate capacity, which is implicitly the standard assumption in the literature.[16]

The natural and instructive comparison is the case in which the expected long run average price growth and expected long run average price variances are the same in both economies.

An econometrician who calibrates a real options model under the assumption, standard in the literature, that the price of the firms' output follows geometric Brownian motion, will be surprised to find firms delaying irreversible investment even longer than his model predicts is optimal. This occurs because, in addition to the delay related to option effects, the equilibrium return to firms' output is actually negatively skewed.[17] How this leads firms to delay investment even longer can be understood in the following way. Option value derives from the ability to avoid some investments that would have been mistakes *ex post*. At the zero NPV threshold an investor may do two things: invest or delay. The choice to delay roughly reflects the fact that, at the current price level, the *ex post* benefit to having not invested if the economic environment takes a turn for the worse exceeds the *ex post* benefit to having invested earlier if the environment takes a turn for the better. Investment occurs when prices are just high enough that the loss-avoidance benefits to delaying balance the gains from

[16]Alternatively, we are comparing optimal behavior in the equilibrium economy to the behavior of a small, price-taking firm in an economy in which no other firms have the ability to develop.

[17]By return to firms' output we simply mean the log change in prices. There is no sense in which this is a holding period return to a real asset.

not delaying. Asymmetric downside risk exceeding upside potential induces firms to delay option exercise longer. The oversized downside risk results in the potential *ex post* benefits to delaying investment balancing the *ex post* benefit to not delaying investment at a higher price level. The results of this comparison are summarized in the next proposition.

Proposition 4.3 *In equilibrium a firm redevelops a project to the same capacity as she would in a fixed capacity economy with the same long run average price growth and variance. The development occurs later, however, in equilibrium.*

4.4 The Value of the Option to Delay Investment

Our analysis also shows that competition leaves option premiums unmitigated. Competition erodes option values, but also erodes the value of the cash flows from assets in place. As a result, the premia, or percent of total option value attributable to future development opportunities, is undiminished relative to an economy in which prices follow geometric Brownian motion and have the same average long-run price growth and variance. That is, option premia are unmitigated relative to the partial equilibrium analysis that ignores competition.

Refer to the first term in the value-at-max function as the "intrinsic value" of the project, denoted by $I(q,P) = \Pi qP$, and the second term as the "option value," denoted by $O(q,P) = Aq^{\gamma}(P/P_q^*)^{\eta}$.

Proposition 4.4 *The maximum ratio of option value to intrinsic value is given by*

$$max_P \left(\frac{O(q,P)}{I(q,P)} \right) = \frac{(q_1^* - 1)}{\eta(1 - q_1^{*(\gamma+\eta-\gamma\eta)})}. \tag{36}$$

This is strictly positive, implying that option values remain significant even when firms account for the impact of competition. Furthermore, for some parameter choices the ratio exceeds one, and the majority of a project's value may reside in the option.

We would like to compare the equilibrium economy to a fixed capacity economy with the same long run average price growth and long run average price variance. Noting that the fixed capacity economy is simply the limit of the equilibrium economy as the cost to scale of building becomes very large provides an easy way to do so. Let overscore tildes denote parameters relating to the fixed capacity economy, and choose $\widetilde{\mu}$ and $\widetilde{\sigma}$ such that this economy has the same long run average price growth and variance as the equilibrium economy. Then the maximal ratio of option value to intrinsic value in the fixed capacity economy is simply the right hand side of equation (36) with the η's replaced by $\widetilde{\beta} = \sqrt{(\frac{\widetilde{\mu}}{\widetilde{\sigma}^2} - \frac{1}{2})^2 + \frac{2r}{\widetilde{\sigma}^2}} - (\frac{\widetilde{\mu}}{\widetilde{\sigma}^2} - \frac{1}{2})$. Finally, using the fact that $\widetilde{\beta} = \eta$, proved in the appendix as part of proposition 4.3, yields the following corollary.

Corollary 4.1 *The maximal ratio of option value to intrinsic value in the equilibrium economy is the same as in a fixed capacity economy with the same long run average price growth and variance.*

5 Conclusion

We characterize firms' optimal investment strategy explicitly, and derive a closed form solution for firm value. We show that in the strategic equilibrium real option premia are significant. As a result firms delay investment, choosing optimally not to undertake some positive NPV projects. In fact, properly accounting for the effects of competition results in a negatively skewed equilibrium process for the price of firms' output, which leads firms to delay investment longer than predicted by the standard options analysis. Finally, the equilibrium analysis in this paper has implications for equity markets. The model presented here predicts that firm returns should vary over the business cycle. Firm returns should be negatively skewed during expansions, but positively skewed in recessions.

These conclusions differ from those in previous studies of competition on option exercise, which have tended to conclude that competition reduces option premiums and leads to earlier investment. Earlier studies have considered the effects of oligopolistic competition and have quite naturally, therefore, compared valuation and optimal timing of investment to the case of a hypothetical strategic monopolist. While their conclusions are valid in this context, they should not be used to draw conclusions with respect to the standard analysis, which ignores competition. The standard analysis is not a fundamentally better yardstick than the strategic monopolist, but it is the theory that has informed the intuition which is now widely held, and the fact that our methodology allows for comparison to the standard analysis is, therefore, a great advantage.

The results in this paper differ from those in earlier papers because we consider *het-*

erogeneous firms facing opportunity costs to investing. While it has long been known that demand side heterogeneity can reduce competition, in this paper we demonstrate that supply side heterogeneity can have the same effect, though for different reasons. Demand side heterogeneity— variable consumer preferences— allows firms to segment a market, through horizontal product differentiation, and extract more of the consumer surplus. Supply side heterogeneity— variable opportunity costs— provides a natural ordering to firms' investment decisions, allowing firms to act sometimes as local monopolists and extract more of the consumer surplus. The model predicts a "life-cycle" to firms investment decisions, because firms that have invested recently find it unprofitable to invest again for some time. This prevents them from competing over new opportunities, and significantly limits the role of competition.

A Appendix

Proof of Proposition 3.1

Homogeneous firms will not choose to undertake the same investment at the same time, because if all firms were to follow such a strategy an individual firm would find it more profitable to deviate either by developing sooner than other firms, or by delaying investment.

Suppose firms do follow a symmetric strategy. That is, suppose firms are homogeneous, possessing the same initial capacities q, and follow the same development strategy. That is, at some point all firms will develop the same new capacity at the same time. Until this time aggregate capacity is fixed, so the price of firms' output evolves like the multiplicative demand shock, as a geometric Brownian motion. The investment problem faced by firms is therefore Markovian up until the time of development, so the investment strategy may be characterized as a trigger strategy on the price of firms' output. That is, there exist some P^* such that the first time P_t, the price of firms' output, reaches P^* all firms add new capacity, developing to the new capacity q^*.

Because the cost of developing new capacity includes the opportunity cost of abandoning old capacity, individual firms do not undertake incremental investment, i.e., q^* is bounded away from q. Aggregate capacity would also be lumpy (discontinuous) in a symmetric equilibrium, so the price of firms' output immediately after development is strictly less than the trigger price, i.e. $P_{\tau^*+} < P^*$, where $\tau^* \equiv min\{t > 0 \,|\, P_t = P^*\}$.

Because the opportunity cost for firms is higher after they have developed to the new, larger capacity q^* no further development will happen until prices reach some price which is strictly higher than P^*. Letting $V_{s,t}$ denote the value of the firm at time s when development to capacity q^* will be undertaken at time t, we can write the value of a firm at the date of the initial development as the value of the expected, discounted cash flows from the new project up until prices return to P^* plus the expected value of receiving the project on the date price first return to P^*,

$$V_{\tau^*,\tau^*} = q^*\, P_{\tau^*+}\pi - C(q^*) + \mathsf{E}_{\tau^*}\left[e^{-r(\tau^{**}-\tau^*)}\right](V_{\tau^{**}} - q^*\, P^*\pi) \tag{37}$$

where $\pi = \frac{1}{r-\mu}$ is the standard geometric Brownian annuity factor, $\tau^{**} \equiv min\{t > \tau^* \,|\, P_t = P^*\}$ is the second passage time for the price process to P^*, and $V_{\tau^{**}}$ is the value of having capacity q^* at time τ^{**}.

The value, at time τ^*, of delaying development to capacity q^* until τ^{**} is just

$$V_{\tau^*,\tau^{**}} = q\,P_{\tau^*+}\pi + \mathsf{E}_{\tau^*}\left[e^{-r(\tau^{**}-\tau^*)}\right](V_{\tau^{**}} - C(q^*) - q\,P^*\pi)\,, \tag{38}$$

so the value of delaying investment is given by

$$\begin{aligned} V_{\tau^*,\tau^{**}} - V_{\tau^*,\tau^*} \quad &= \quad C(q^*)\left(1 - \mathsf{E}_{\tau^*}\left[e^{-r(\tau^{**}-\tau^*)}\right]\right) \\ &\quad - (q^* - q)\,P_{\tau^*+}\,\pi\left(1 - \mathsf{E}_{\tau^*}\left[e^{-r(\tau^{**}-\tau^*)}\right]\frac{P^*}{P_{\tau^*+}}\right). \end{aligned} \tag{39}$$

That is, delaying investment has the advantage of deferring the investment cost, but the disadvantage of loosing some intermediate cash flows. Whether the net benefit is positive or negative depends on the relative magnitudes of the two effects.

An identical argument shows that the value of investing early, to capacity q^* at time $\tau_* \equiv min\,\{t > 0\,|\,P_t = P_{\tau^*+}\}$, is given by

$$\begin{aligned} V_{\tau_*,\tau_*} - V_{\tau_*,\tau^*} \quad &= \quad (q^* - q)\,P_{\tau^*+}\,\pi\left(1 - \mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\frac{P^*}{P_{\tau^*+}}\right) \\ &\quad - C(q^*)\left(1 - \mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\right) \\ &= \quad - \;(V_{\tau^*,\tau^{**}} - V_{\tau^*,\tau^*}) \end{aligned} \tag{40}$$

because $\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right] = \mathsf{E}_{\tau^*}\left[e^{-r(\tau^{**}-\tau^*)}\right]$.

So unless an investor is indifferent between developing to capacity q^* at times τ_*, τ^* and τ^{**} she has a strict preference for either deviating from the symmetric strategy and developing either early or late.

If she is indifferent between developing at τ_* and τ^*, however, she has a strict preference for developing at $\tau_*^* \equiv min\,\{t > 0\,|\,P_t = \sqrt{P^*\,P_{\tau^*+}}\}$, as shown by the following argument.

If the firm is indifferent between developing at τ_* and τ^* then

$$\begin{aligned} 0 \quad &= \quad V_{\tau_*,\tau_*} - V_{\tau_*,\tau^*} \\ &= \quad (q^* - q)\,P_{\tau^*+}\,\pi\left(1 - \mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\frac{P^*}{P_{\tau^*+}}\right) - \;C(q^*)\left(1 - \mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\right) \\ &= \quad (q^* - q)\,P_{\tau^*+}\,\pi\left(1 - \sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\frac{P^*}{P_{\tau^*+}}}\right)\left(1 + \sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\frac{P^*}{P_{\tau^*+}}}\right) \\ &\quad - C(q^*)\left(1 - \sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}\right)\left(1 + \sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}\right) \end{aligned} \tag{41}$$

$$= \left(1+\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}\right)\left[(q^*-q)\,P_{\tau^*+}\,\pi\left(1-\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\frac{P^*}{P_{\tau^*+}}}\right)\right.$$

$$\left. - C(q^*)\left(1-\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}\right)\right]$$

$$+\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}\left(\sqrt{\frac{P^*}{P_{\tau^*+}}}-1\right)(q^*-q)\,P_{\tau^*+}\,\pi\left(1-\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\frac{P^*}{P_{\tau^*+}}}\right).$$

Now both $\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}\left(\sqrt{\frac{P^*}{P_{\tau^*+}}}-1\right)(q^*-q)\,P_{\tau^*+}\,\pi\left(1-\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]\frac{P^*}{P_{\tau^*+}}}\right)$ and $1+\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}$ are strictly positive, so equation (41) implies

$$(q^*-q)\,\pi\left(P_{\tau^*+}-\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]P^*\,P_{\tau^*+}}\right)-\;C(q^*)\left(1-\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}\right)<0. \qquad (42)$$

Now let $P_{gm}=\sqrt{P^*\,P_{\tau^*+}}$ be the geometric mean of the two price levels. Then using the fact that $\sqrt{\mathsf{E}_{\tau_*}\left[e^{-r(\tau^*-\tau_*)}\right]}=\mathsf{E}_{\tau_*}\left[e^{-r(\tau_*^*-\tau_*)}\right]$ we have

$$C(q^*)\left(1-\mathsf{E}_{\tau_*}\left[e^{-r(\tau_*^*-\tau_*)}\right]\right)-\;(q^*-q)\,\pi\,P_{\tau^*+}\left(1-\mathsf{E}_{\tau_*}\left[e^{-r(\tau_*^*-\tau_*)}\right]\frac{P_{gm}}{P_{\tau^*+}}\right)>0. \qquad (43)$$

The left hand side of equation (43) is, however, precisely the value of delaying development to capacity q^* from time τ_* to τ_*^*. That this is strictly positive implies that the firm strictly prefers investing at τ_*^* to investing at τ_*, and as the firm is indifferent between developing at τ_* to τ^* equation (43) also implies that the firm also strictly prefers investing at τ_*^* to investing at τ^*.

That is, if all agents develop to the same capacity at the same time an individual agent can behave more profitably by deviating either by developing earlier or later. Therefore no symmetric equilibrium exists. ¥

Proof of Proposition 3.2

Given any maximum of the price process, the expected value of building at any lower price, to any capacity, is less than it would have been at the time the maximum was achieved, so firms choose only build at price maxima.

More formally, because of time homogeneity it is sufficient to show that a firm that chooses to develop below the maximum would have prefered to develop at the time the maximum was achieved.

Now suppose a firms chooses to build at some price below the price maximum, and consider the last time she develops prior to price returning to the price maximum. Denote the price level at which this last development occurs by P_0 and the capacity by q.

The opportunity cost of developing below the maximum is at least as great as it would have been at the maximum because the opportunity costs are non-decreasing in the existing capacity, which could only have increased with intervening developments. The direct cost to building to q

at the maximum is the same as it is at P_0. It is therefore sufficient to show that the value of the project after development is greater at the price maximum than at P_0. That is, it is sufficient to show that $V(q, \overline{P}_0, \overline{P}_0) \geq V(q, P_0, \overline{P}_0)$.

Letting $\tau^* \equiv min\{t > 0 \,|\, P_t = \overline{P}_0\}$ we have that

$$V(q, P_0, \overline{P}_0) = \mathsf{E}_0 \left[\int_0^{\tau^*} e^{-rt}\, q\, P_t\, dt \,|\, P_0 = \overline{P}_0 \right] + \mathsf{E}_0 \left[e^{-r\tau^*} \right] V(q, \overline{P}_0, \overline{P}_0). \tag{44}$$

Now $V(q, \overline{P}_0, \overline{P}_0)$ is the expected discounted cash flows of the current development plus some option value, so is at least as great as the value of the development with no right to further development. That is,

$$V(q, \overline{P}_0, \overline{P}_0) \geq \mathsf{E}_0 \left[\int_0^{\infty} e^{-rt}\, q\, P_t\, dt \,|\, P_0 = \overline{P}_0 \right]. \tag{45}$$

At this point we will define an alternate process, $P_t^{'}$, which starts at $\overline{P}_0$ and evolves like the price process but with an upper reflecting barrier at $\overline{P}_0$. That is, below $\overline{P}_0$ the process $P_t^{'}$ evolves exactly like P_t, but at $\overline{P}_0$ the evolution is truncated so that the process can only go down or remain at $\overline{P}_0$. Also, let $\tau_h \equiv min\{t > 0 \,|\, P_t = P_t^{'}\}$. Figure A1, below, gives a graphical representation of the situation as described.

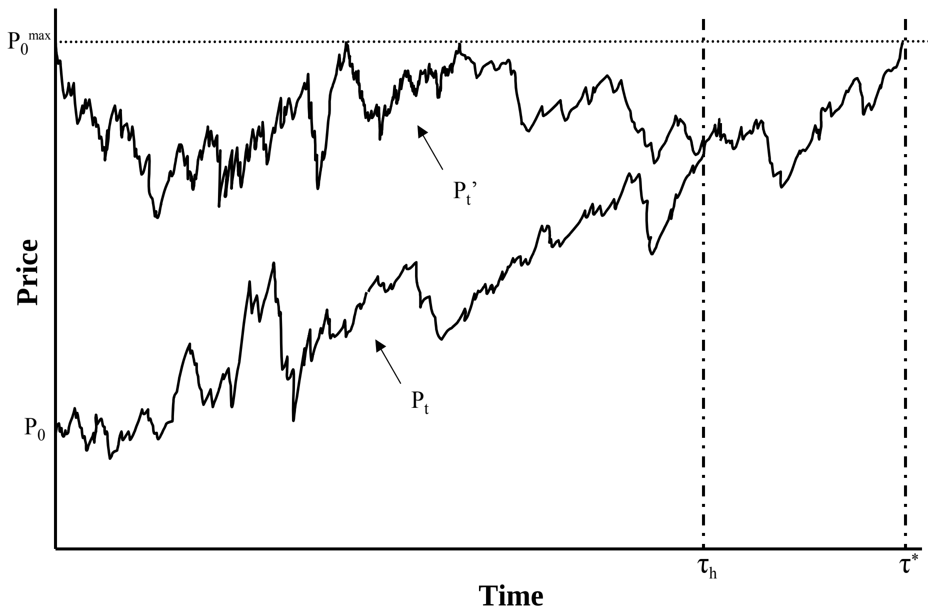

Figure A1. The lower price path is the actual price process, starting at P_0 when the price maximum is P_0^{max}. The upper path starts at P_0^{max} and evolves like the price process, except with a reflecting barrier at P_0^{max}. Because both processes evolve in the same manner on the interval from τ_h to τ^* the distribution of τ^* is the same for the two paths.

Now the price process starting at $\overline{P}_0$ is clearly greater on average that the price process with the upper reflecting barrier and starting at the same level, so we have

$$\mathsf{E}_0\left[\int_0^\infty e^{-rt}\,q\,P_t\,dt\,|\,P_0=\overline{P}_0\right]\geq \mathsf{E}_0\left[\int_0^\infty e^{-rt}\,q\,P_t^{'}\,dt\,|\,P_0^{'}=\overline{P}_0\right]. \tag{46}$$

Because P_t and $P_t^{'}$ evolve in the same manner in the interval from τ_h to τ^* we have that

$$\left(1-\mathsf{E}_0\left[e^{-r\tau^*}\right]\right)\mathsf{E}_0\left[\int_0^\infty e^{-rt}\,q\,P_t^{'}\,dt\,|\,P_0^{'}=\overline{P}_0\right]=\mathsf{E}_0\left[\int_0^{\tau^*} e^{-rt}\,q\,P_t^{'}\,dt\,|\,P_0^{'}=\overline{P}_0\right]. \tag{47}$$

Finally, from figure A1 we can see that clearly

$$\mathsf{E}_0\left[\int_0^{\tau^*} e^{-rt}\,q\,P_t^{'}\,dt\,|\,P_0^{'}=\overline{P}_0\right]>\mathsf{E}_0\left[\int_0^{\tau^*} e^{-rt}\,q\,P_t\,dt\,|\,P_0\right]. \tag{48}$$

Putting it all together we have

$$\begin{aligned}
V(q,\overline{P}_0,\overline{P}_0) &= \left(1-\mathsf{E}_0\left[e^{-r\tau^*}\right]\right)V(q,\overline{P}_0,\overline{P}_0)+\mathsf{E}_0\left[e^{-r\tau^*}\right]V(q,\overline{P}_0,\overline{P}_0)\\
&\geq \left(1-\mathsf{E}_0\left[e^{-r\tau^*}\right]\right)\mathsf{E}_0\left[\int_0^\infty e^{-rt}\,q\,P_t\,dt\,|\,P_0=\overline{P}_0\right]+\mathsf{E}_0\left[e^{-r\tau^*}\right]V(q,\overline{P}_0,\overline{P}_0)\\
&\geq \left(1-\mathsf{E}_0\left[e^{-r\tau^*}\right]\right)\mathsf{E}_0\left[\int_0^\infty e^{-rt}\,q\,P_t^{'}\,dt\,|\,P_0^{'}=\overline{P}_0\right]+\mathsf{E}_0\left[e^{-r\tau^*}\right]V(q,\overline{P}_0,\overline{P}_0)\\
&= \mathsf{E}_0\left[\int_0^{\tau^*} e^{-rt}\,q\,P_t^{'}\,dt\,|\,P_0^{'}=\overline{P}_0\right]+\mathsf{E}_0\left[e^{-r\tau^*}\right]V(q,\overline{P}_0,\overline{P}_0) \qquad (49)\\
&> \mathsf{E}_0\left[\int_0^{\tau^*} e^{-rt}\,q\,P_t\,dt\,|\,P_0\right]+\mathsf{E}_0\left[e^{-r\tau^*}\right]V(q,\overline{P}_0,\overline{P}_0)\\
&= V(q,P_0,\overline{P}_0).
\end{aligned}$$

Because a firm would always prefer developing to any capacity at the price maximum no development occurs below the maxima. ¥

Proof of Proposition 3.3

Lemma. Let Y_t be a geometric Brownian motion with any drift, beginning at one. Then the value of cash flows proportional to the process and received until the first time the process reaches $\theta>1$ is given by

$$\mathsf{E}_0\left[\int_0^{\tau_\theta} e^{-rt}Y_t dt\right]=(1-\theta^{1-\beta})\pi. \tag{50}$$

Proof of lemma: We need to show that for all μ

$$\mathsf{E}_0\left[\int_0^{\tau_\theta} e^{-rt} e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t} dt\right] = \frac{1-\theta^{1-\beta}}{r-\mu}, \tag{51}$$

where $\tau_\theta \equiv min\{t>0 \,|\, e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t} = \theta\}$. That is, the value of a cash flow proportional to a drifted geometric Brownian motion and received up until the first passage time of the process to some fixed level θ is given by previous equation. In particular, given any finite stopping level, the value is finite for all drifts (and even decreasing in the drift when the drift is sufficiently large).

First, we will rewrite the equation in terms of a drifted Brownian motion with unit volatility,

$$\mathsf{E}_0\left[\int_0^{\tau_\theta} e^{-rt} e^{\sigma(B_t+(\frac{\mu}{\sigma}-\frac{\sigma}{2})t)} dt\right]. \tag{52}$$

Then changing measure, to demean the Brownian motion, and using the joint density for the value and the maximum of a standard Brownian, we may write the value as

$$\int_{t=0}^{\infty}\int_{m=0}^{\frac{ln(\theta)}{\sigma}}\int_{b=-\infty}^{m} e^{-rt}e^{\sigma b}\left(\sqrt{\frac{2}{\pi}}\frac{2m-b}{t\sqrt{t}}e^{\frac{-(2m-b)^2}{2t}}\right) e^{(\frac{\mu}{\sigma}-\frac{\sigma}{2})b-(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2\frac{t}{2}}\, db\, dm\, dt.^{18} \tag{53}$$

At this point we may proceed in two ways. The first is to restrict our attention to the case when $\mu < r$, in which case the valuation is simple, and use analytic continuation to argue that the resulting solution is valid for all μ. The second is to simply proceed with the integration, using a known result about Bessel functions.

First, arguing by analytic continuation, we have that when $\mu < r$ the value of the perpetual cash flows is finite and known. We may then write

$$\mathsf{E}_0\left[\int_0^{\tau_\theta} e^{-rt} e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}\, dt\right] = \mathsf{E}_0\left[\int_0^{\infty} e^{-rt} e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}\, dt\right] - \mathsf{E}_0\left[\int_{\tau_\theta}^{\infty} e^{-rt} e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}\, dt\right]. \tag{54}$$

For the second term on the right we have

$$\mathsf{E}_0\left[\int_{\tau_\theta}^{\infty} e^{-rt} e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t} dt\right] = \mathsf{E}_0\left[e^{-r\tau_\theta}\right]\theta\, \mathsf{E}_0\left[\int_{\tau_\theta}^{\infty} e^{-rt}\frac{e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}}{\theta} dt\right], \tag{55}$$

which, using the Markov property gives

$$\mathsf{E}_0\left[\int_{\tau_\theta}^{\infty} e^{-rt} e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t} dt\right] = \mathsf{E}_0\left[\int_0^{\infty} e^{-rt} e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}\, dt\right] = \frac{1}{r-\mu}. \tag{56}$$

using this in conjunction with the fact that $\mathsf{E}_0[e^{-r\tau_\theta}] = \theta^{-\beta}$ yields

[18] That the joint density for the value and the maximum of a standard Brownian is given by $\sqrt{2/\pi}((2m-b)/t\sqrt{t})e^{-((2m-b)^2/2t)}$ is a standard result in the probability literature that may be found in any good text on the subject; see, for example, Durrett (1996).

$$\mathsf{E}_0\left[\int_0^{\tau_\theta} e^{-rt}e^{(\mu-\frac{\sigma^2}{2})t+\sigma B_t}dt\right] = \frac{1-\theta^{1-\beta}}{r-\mu}. \tag{57}$$

The analytic continuation argument extending the result to all μ is valid because the integral of an analytic function (the integrand in (53)) is analytic.

Alternatively, we may simply proceed by integrating (53) directly. To do so requires the fact, from the literature on Bessel functions, that

$$\int_{t=0}^{\infty} e^{-\alpha t}\left(\sqrt{\frac{2}{\pi}}\frac{2m-b}{t\sqrt{t}}e^{\frac{-(2m-b)^2}{2t}}\right)dt = 2e^{-\sqrt{2\alpha}(2m-b)}. \tag{58}$$

Substituting into (53) with $\alpha = r + \frac{1}{2}(\frac{\mu}{\sigma} - \frac{\sigma}{2})^2$ we have that the value is given by

$$2\int_{m=0}^{\frac{ln(\theta)}{\sigma}} e^{-2\sqrt{2r+(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2}\,m}\left(\int_{b=-\infty}^{m} e^{\left((\frac{\mu}{\sigma}+\frac{\sigma}{2})+\sqrt{2r+(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2}\right)b}\,db\right)dm$$

$$\begin{aligned}
&= \frac{2}{(\frac{\mu}{\sigma}+\frac{\sigma}{2})+\sqrt{2r+(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2}}\int_{m=0}^{\frac{ln(\theta)}{\sigma}} e^{\left((\frac{\mu}{\sigma}+\frac{\sigma}{2})-\sqrt{2r+(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2}\right)m}\,dm \\
&= \frac{2}{(\frac{\mu}{\sigma}+\frac{\sigma}{2})^2-\left(2r+(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2\right)}\left(\theta^{\frac{1}{\sigma}\left((\frac{\mu}{\sigma}+\frac{\sigma}{2})-\sqrt{2r+(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2}\right)}-1\right) \\
&= \frac{1}{\mu-r}\left(\theta^{\frac{1}{\sigma}\left((\frac{\mu}{\sigma}+\frac{\sigma}{2})-\sqrt{2r+(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2}\right)}-1\right) \\
&= \frac{1-\theta^{1-\beta}}{r-\mu}. \quad ¥
\end{aligned} \tag{59}$$

Proof of the proposition: Intuitively, we think of the project as an asset which generates a stochastic cash flow that we will "trade in" for a "new project" when prices return to their historical maximum. We essentially value the project as a portfolio made by buying a perpetuity that pays a dividend proportional to the spot price of the good, selling the same perpetuity for forward delivery on the date price first return to their maximum, and buying a forward delivery contract on the project with the same delivery date. The last part is just the value of the project at the price maximum times the value of the Arrow-Debreu security that pays a dollar on the delivery date. The first two pieces net to identically zero cash flow after the delivery date, and up until that date, because no building occurs, prices have the same geometric Brownian evolution as the demand process and may be priced using the lemma.

More formally, because firms value projects based on discounted cash flows, we may relate the value function back to the value-at-max function by the following equation:

$$V(q, P_t, \overline{P}_t) = \mathsf{E}_t\left[\int_t^{\tau_{\overline{P}_t}} e^{-r(s-t)}qP_s ds + e^{-r(\tau_{\overline{P}_t}-t)}W(q, \overline{P}_t)\right], \tag{60}$$

where $\tau_{\overline{P}_t}$ denotes the stopping time for the first passage of the price process back to its previous maximum, $\overline{P}_t$. That is, $\tau_{\overline{P}_t} \equiv min\{s \geq t \,|\, P_s = \overline{P}_t\}$.

Proposition 3.2 guarantees that no development occurs between P_t and $\overline{P}_t$, so supply, Q_t, is fixed on the interval. As a consequence, the price process has the same evolution as the multiplicative demand shock up to time $\tau_{\overline{P}_t}$. That is, on the interval in question the price process is a geometric Brownian motion with drift μ and volatility σ. Then using the lemma, that $\mathsf{E}_0[\int_0^{\tau_\theta} e^{-rt} Y_t \, dt] = (1 - \theta^{1-\beta})\pi$, and $\mathsf{E}_0[e^{-r\tau_\theta}] = \theta^{-\beta}$ we have

$$\begin{aligned} V(q, P_t, \overline{P}_t) &= \mathsf{E}_t \left[\int_t^{\tau_{\overline{P}_t}} e^{-r(s-t)} q P_s ds + e^{-r(\tau_{\overline{P}_t} - t)} W(q, \overline{P}_t) \right] \\ &= q P_t \left(1 - \left(\frac{\overline{P}_t}{P_t} \right)^{1-\beta} \right) \pi + \left(\frac{\overline{P}_t}{P_t} \right)^{-\beta} W(q, \overline{P}_t) \qquad (61) \\ &= \pi q P_t + \left(\frac{P_t}{\overline{P}_t} \right)^{\beta} \left(W(q, \overline{P}_t) - \pi q \overline{P}_t \right). \quad ¥ \end{aligned}$$

Proof of Proposition 3.4

Cash flow considerations imply the structure. An owner should be indifferent between holding 1) one project at a given capacity, price and price-maximum, and 2) more properties developed to lower capacities when the price and price-maximum are somewhat lower. More formally,

$$V(\beta q, P_t, \overline{P}_t) = \beta^{\gamma} V(q, \beta^{(1-\gamma)} P_t, \beta^{(1-\gamma)} \overline{P}_t). \qquad (62)$$

This is because the instantaneous cash flows are the same at *all* times, and either firm can replicate the development decision of the other, at the same cost. Implicitly we are using that the evolution of $ln\, P_t$ is the same as the evolution of $ln\, (\beta^{(1-\gamma)} P_t)$, which follows from the fact that the scaling leaves the ratio of the price to the price maximum unchanged.[19]

Substituting the existing capacity for β yields

$$V(q, P_t, \overline{P}_t) = q^{\gamma} V(1, q^{(1-\gamma)} P_t, q^{(1-\gamma)} \overline{P}_t). \qquad (63)$$

To price any project it is, therefore, sufficient to price a project developed to unit capacity.

The scaling condition on the value-at-max function is inherited from equation (63). ¥

Proof of Proposition 3.5

The result follows from proposition 3.4, the scaling property of the value function.

Whatever the existing capacity, q_i, with redevelopment rights it is certainly optimal for the unconstrained firm to redevelop for sufficiently high prices. Denote the lowest price at which it is

[19]This argument is made in greater detail in Novy-Marx(2002). The article is available for download at http://faculty.haas.berkeley.edu/marx/.

optimal to redevelop $P^*_{q_i}$, and the optimal redevelopment capacity at this price by $q^*_{q_i}$. Then for any other capacity q_j we have

$$W(q_j, P) = \left(\frac{q_j}{q_i}\right)^{\gamma} W\left(q_i, \left(\frac{q_j}{q_i}\right)^{(1-\gamma)} P\right), \tag{64}$$

so

$$q^*_{q_j} = \left(\frac{q^*_{q_i}}{q_i}\right) q_j. \tag{65}$$

By symmetry, the ratio $q^*_{q_i}/q_i$ is independent of q_i. Denote the ratio q^*_1.[20] Then the previous equation says that firms optimally build to q^*_1 times existing capacity.

We also have

$$P^*_{q_j} = \left(\frac{q_j}{q_i}\right)^{(\gamma-1)} P^*_{q_i}. \tag{66}$$

In particular we have that

$$P^*_{q^*_{q_j}} = P^*_{q^*_1 \cdot q_j} = \left(\frac{q^*_1 \cdot q_j}{q_j}\right)^{(\gamma-1)} P^*_{q_i} = q_1^{*(\gamma-1)} P^*_{q_j}. \tag{67}$$

That is, an firm redevelops at $q_1^{*(\gamma-1)}$ times the price level of the last development. Letting $q_i = 1$ we also have

$$P^*_{q_j} = q_j^{(\gamma-1)} P^*_1. \tag{68}$$

That is, an firm develops a property with existing capacity q at a price level that is $q^{\gamma-1}$ times as high as the point at which a firm with existing capacity one develops. ¥

Proof of Proposition 3.6

Lemma. Suppose that the initial distribution follows Zipf's law: for any two capacities in the initial distribution, the relative likelihood of observing these capacities is inversely proportional to the capacities,

$$\frac{\nu(q_1)}{\nu(q_2)} = \frac{c/q_1}{c/q_2} = \frac{q_2}{q_1} \quad for \quad q_1, q_2 \in \left(q_0^{min}, q_0^{max}\right] \tag{69}$$

where q_0^{min} (respectively, q_0^{max}) is the infimum (respectively, maximum) of the capacities in the economy initially, and $\nu(q)$ is the probability density that a project picked at random has capacity q.

[20] The notation is motivated as follows: the ratio is independent of q_i, so letting $q_i = 1$ we have $q^*_1 = (q^*_1/1)$ where on the left "q^*_1" is a unitless ratio and on the right it is a density.

Then the initial distribution of capacities is log-uniform on the range: distributed $(\xi/\kappa)\,q_0^{max}$ where $ln_\kappa\,\xi$ is distributed uniformly on $(0,1]$ and $\kappa \equiv q_0^{max}/q_0^{min}$.

Proof of lemma: It follows directly from the fact that $d\,ln\,x = \frac{dx}{x}$. ¥

Lemma. Suppose that firms follow a strategy of building to a fixed multiple κ times existing capacity under a development timing rule such that the next development always occurs at the site of the smallest existing capacity. Then an initial distribution of capacities that follows Zipf's law is stationary: at all times the distribution of capacities in log-uniform.

Proof of lemma: That the distribution follows Zipf's law implies that the probability density that a project picked at random has capacity q is given by $\nu_0(q) = 1_{q\in\left(q_0^{min},q_0^{max}\right]}(c/q)$, where $c = ln\,\left(q_0^{max}/q_0^{min}\right)$ and $1_{q\in\left(q_0^{min},q_0^{max}\right]}$ is one if $q \in \left(q_0^{min}, q_0^{max}\right]$ and zero otherwise. When an infinitesimal of firms have developed the range of capacities goes from $\left(q_0^{min}, q_0^{max}\right]$ to $\left(q_0^{min} + \delta q, \kappa\left(q_0^{min} + \delta q\right)\right] = \left(q_0^{min} + \delta q, q_0^{max} + \kappa\delta q\right]$. The mass of firms in the newly developed region must equal the mass of firms that developed, so for δq small we have that $\kappa\delta q{\cdot}\nu_t(q_t^{max})$ is essentially equal to $\delta q{\cdot}\nu_0(q_0^{min}{+}\delta q)$. Using $q_t^{max} = \kappa q_0^{min} + \kappa\delta q$ we then have that $\nu_t(q_t^{max}) = \frac{1}{\kappa}\nu_0(q_t^{max}/\kappa) = \frac{1}{\kappa}\nu_t(q_t^{min})$, so new development preserves log-normality of the distribution. ¥

Proof of the proposition: The aggregate supply process is the integral of the individual firms' capacities,

$$Q_t = \int_{q_t^{min}}^{q_t^{max}} q\,d\nu_t(q). \tag{70}$$

From the previous lemma, the development rule imposed on almost all firms preserves the log-uniform distribution of capacities and build-prices. That is, at all times the distribution of capacities and build prices is $\left((\xi/\kappa)\cdot q_t^{max}, (\xi/\kappa)^{(\gamma-1)}\overline{P}_t\right)$, where $ln_\kappa\,\xi$ is distributed uniformly on $(0,1]$ and

$$\begin{aligned} q_t^{max} &= q_0^{max}\kappa^{ln_{\kappa^{(\gamma-1)}}(\overline{P}_t/\overline{P}_0)} \\ &= q_0^{max}(\overline{P}_t/\overline{P}_0)^{ln_{\kappa^{(\gamma-1)}}\kappa} \\ &= q_0^{max}(\overline{P}_t/\overline{P}_0)^{\left(\frac{1}{(\gamma-1)}\right)}, \end{aligned} \tag{71}$$

so

$$Q_t = q_0^{max}\left(\frac{\overline{P}_t}{\overline{P}_0}\right)^{\left(\frac{1}{(\gamma-1)}\right)}\int_0^\kappa \xi\,d\,\nu_t(\xi) = \left(\frac{\overline{P}_t}{\overline{P}_0}\right)^{\left(\frac{1}{\gamma-1}\right)}Q_0. \;¥ \tag{72}$$

Proof of Proposition 3.7

Proof of the proposition: Historic price highs correspond to historic highs in the multiplicative demand shock, and the inverse demand function is valid everywhere, so we have

$$\overline{P}_t = \overline{X}_t \cdot Q_t^{(-1/\alpha)}. \tag{73}$$

Now substituting this result into the result of the previous proposition, $Q_t = \left(\frac{\overline{P}_t}{\overline{P}_0}\right)^{\left(\frac{1}{\gamma-1}\right)} Q_0$, we have that

$$Q_t = \left(\frac{\overline{X}_t \cdot Q_t^{(-1/\alpha)}}{\overline{P}_0}\right)^{\left(\frac{1}{\gamma-1}\right)} Q_0, \tag{74}$$

which is equivalent to

$$Q_t = c \cdot \overline{X}_t^{(\alpha^{-1}+\gamma-1)^{-1}}, \tag{75}$$

where $c = (Q_0/P_0^{(\gamma-1)^{-1}})^{(1+\frac{1}{\alpha(\gamma-1)})^{-1}}$. That is, the supply elasticity of the aggregate demand maximum (i.e., the multiplicative shock maximum) is constant, and equal to $(\alpha^{-1}+\gamma-1)^{-1}$. Not surprisingly, supply is more responsive to demand when the cost to scale of building is low and when demand is inelastic with respect to prices. If the cost to scale of building is high, or if demand is elastic with respect to prices, then supply is less responsive to demand.

Finally, substituting into the inverse pricing function, we have

$$P_t = \overline{X}_t^{\left(\frac{-1}{1+\alpha(\gamma-1)}\right)} X_t \, P_0 \tag{76}$$

or

$$ln\, P_t = ln\, P_0 - \left(\frac{-1}{1+\alpha(\gamma-1)}\right) ln\, \overline{X}_t + ln\, X_t, \tag{77}$$

which proves the proposition. ¥

Proof of Proposition 3.8

The dollar price of a unit of cash flow at a price maximum is then given by

$$\pi_\delta = \mathsf{E}_0 \left[\int_0^\infty e^{-rt} Y_t^\delta \, dt\right]. \tag{78}$$

That this value is finite if and only if $\mu < \frac{r}{\delta} + (1-\delta)\frac{\sigma^2}{2}$ will be proved later.

Letting τ_θ^δ be the stopping time for the first time Y_t^δ hits $\theta > 1$, we then have

$$\begin{aligned} \pi_\delta &= \mathsf{E}_0 \left[\int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt\right] + \mathsf{E}_0 \left[\int_{\tau_\theta^\delta}^\infty e^{-rt} Y_t^\delta \, dt\right] \\ &= \mathsf{E}_0 \left[\int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt\right] + Y_{\tau_\theta^\delta}^\delta \, \mathsf{E}_0 \left[e^{-r\tau_\theta^\delta}\right] \, \mathsf{E}_0 \left[\int_{\tau_\theta^\delta}^\infty e^{-r(t-\tau_\theta^\delta)} \frac{Y_t^\delta}{Y_{\tau_\theta^\delta}^\delta} \, dt\right] \end{aligned} \tag{79}$$

$$= \quad \mathsf{E}_0\left[\int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt\right] + \theta\, \rho_\delta(1,\theta)\, \pi_\delta$$

where $\rho_\delta(1,\theta) = \rho(1,\theta)^{\frac{1}{\delta}} = \theta^{-(\beta/\delta)}$. Rearranging gives

$$\pi_\delta = \frac{\mathsf{E}_0\left[\int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt\right]}{1 - \theta^{1-(\beta/\delta)}}. \tag{80}$$

In the special case $\delta = 1$ we have, from the lemma in the proof of Proposition 3.3, that

$$\mathsf{E}_0\left[\int_0^{\tau_\theta^1} e^{-rt} Y_t^1 \, dt\right] = \left(1 - \theta^{1-\beta}\right)\pi. \tag{81}$$

We will now employ the fact that τ_θ^1 and $\tau_{\theta^\delta}^\delta$ have the same distribution. In particular, this implies τ_θ^δ is distributed the same as $\tau^1_{\sqrt[\delta]{\theta}}$. Now consider $\theta = 1 + \epsilon$, where $\epsilon \approx 0$ and positive. Then for all $t \le \tau_\theta^\delta$, $Y_t^\delta \approx Y_t^1$, and then

$$\mathsf{E}_0\left[\int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt\right] = (1 + o(\epsilon))\, \mathsf{E}_0\left[\int_0^{\tau^1_{\sqrt[\delta]{\theta}}} e^{-rt} Y_t^1 \, dt\right]. \tag{82}$$

Substituting into the right hand side of this equation using the preceeding equation gives

$$\begin{aligned} \mathsf{E}_0\left[\int_0^{\tau_\theta^\delta} e^{-rt} Y_t^\delta \, dt\right] &= (1 + o(\epsilon))\left(1 - \left(\theta^{(1/\delta)}\right)^{1-\beta}\right)\pi \\ &= \left(\frac{\beta - 1}{\delta}\right)\epsilon\pi + o(\epsilon^2). \end{aligned} \tag{83}$$

Substituting into equation (80) and taking the limit as ϵ goes to zero yields

$$\pi_\delta = lim_{\epsilon \to 0} \frac{\left(\frac{\beta-1}{\delta}\right)\epsilon\pi + o(\epsilon^2)}{(\frac{\beta}{\delta} - 1)\epsilon + o(\epsilon^2)} = \left(\frac{\beta - 1}{\beta - \delta}\right)\pi. \tag{84}$$

This completes the proof, except for the parameter restriction.

The parameter restriction comes from requiring that μ is small enough that $\pi_\delta < \infty$, which is equivalent to requiring that $\beta > \delta$. We then have

$$\begin{aligned} \beta > \delta \quad &\Leftrightarrow \quad \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2} > \left(\delta + \left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)\right)^2 \\ &\Leftrightarrow \quad \frac{2r}{\sigma^2} > \delta^2 + 2\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)\delta \\ &\Leftrightarrow \quad \mu < \frac{r}{\delta} + (1 - \delta)\frac{\sigma^2}{2}. \end{aligned} \tag{85}$$

Proof of Proposition 3.9

We will use the functional form for the value-at-max function provided by equation (23), $W(q,P) = \Pi\, q\, P + a_q\, P^{\eta}$, in the value matching and smooth pasting conditions, equations (21) and (22). When existing capacity is one and the redevelopment capacity is q (which is a *choice* variable) we have, after rearranging terms,

$$\Pi\,(q-1)\,P = (a_1 - a_q)\,P_1^{*\eta} + q^{\gamma} \tag{86}$$

and

$$\Pi\,(q-1) = \eta\,(a_1 - a_q)\,P_1^{*(\eta-1)}. \tag{87}$$

Solving these immediately gives

$$(a_1 - a_q) = \frac{q^{\gamma}}{(\eta-1)P_1^{*\eta}} \tag{88}$$

and

$$P_1^{*} = \frac{\eta}{(\eta-1)\Pi}\,\frac{q^{\gamma}}{(q-1)}. \tag{89}$$

We need one additional constraint to pin down the three variables, and one is implied by the scaling condition on the value function. Because development rights are retained at exercise the condition holds *across development boundaries.* Using this we have that

$$q^{\gamma} W^{1}\left(1, q^{(1-\gamma)}P^{*}\right) = W^{2}\left(q, P^{*}\right). \tag{90}$$

Again using the functional form for W^1 and W^2 given by equation (23), $W(q,P) = \Pi\, q\, P + a_q\, P^{\eta}$, we have

$$q^{\gamma}\left(\Pi\left(q^{(1-\gamma)}P^{*}\right) + a_1\left(q^{(1-\gamma)}P^{*}\right)^{\eta}\right) = \Pi q P^{*} + a_q P^{*\eta}. \tag{91}$$

Solving for a_q yields

$$a_q = q^{\gamma+(1-\gamma)\eta} a_1. \tag{92}$$

In conjunction with the equations for $(a_1 - a_q)$ and P_1^{*} this is sufficient to determine the value of a project in terms of the choice variable, q. In particular, we have that

$$\begin{aligned} a_1 &= \frac{q^{\gamma}}{(1-q^{\gamma+(1-\gamma)\eta})(\eta-1)P_1^{*\eta}} \\ &= \frac{\Pi^{\eta}}{(\eta-1)}\left(\frac{\eta}{\eta-1}\right)\frac{(q-1)^{\eta}}{q^{\eta}(q^{\gamma\eta-\gamma-\eta}-1)}. \end{aligned} \tag{93}$$

The optimal strategy is the one that maximizes the value. This occurs at the capacity that maximizes a_1, so

$$q_1^* = argmax_q \left\{ \frac{(q-1)^\eta}{q^\eta (q^{\gamma\eta-\gamma-\eta} - 1)} \right\}. \tag{94}$$

We then have that

$$\begin{aligned} W(q,P) &= q^\gamma W(1, q^{(1-\gamma)}P) \\ &= \Pi q P + \left(a_1 P_1^{*\eta}\right) q^\gamma \left(\frac{q^{(1-\gamma)}P}{P_1^*}\right)^\eta . \end{aligned} \tag{95}$$

The price of a unit cash flow at a price maximum, is computed by substituting for δ in Proposition 3.8. In our problem $\delta = \frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}$ so we have

$$\Pi = \pi_{\left(\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\right)} = \frac{\pi}{1 + \frac{1}{(1+\alpha(\gamma-1))(\beta-1)}}. \tag{96}$$

Substituting $\delta = \frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}$ into the parameter restriction in Proposition 3.8 also yields

$$\mu < \left(1 + \frac{1}{\alpha(\gamma-1)}\right) r + \left(\frac{1}{1+\alpha(\gamma-1)}\right) \frac{\sigma^2}{2}. \tag{97}$$

Finally, this parameter restriction is not sufficient to guarantee that *projects* have finite value. Requiring that μ is small enough such that they do, i.e., that $A < \infty$, is always more restrictive, and equivalent to demanding that $\gamma\eta - \gamma - \eta > 0$. Rearranging and using the definition of η results in $\beta > \frac{\alpha\gamma}{1+\alpha(\gamma-1)}$. Substituting $\delta = \frac{\alpha\gamma}{1+\alpha(\gamma-1)}$ yields the final parameter restriction:

$$\mu < \left(1 - \frac{\alpha-1}{\alpha(\gamma-1)}\right) r - \left(\frac{\alpha-1}{1+\alpha(\gamma-1)}\right) \frac{\sigma^2}{2}. \quad ¥ \tag{98}$$

Proof of Proposition 3.10

Let $F(x) = \frac{(x-1)^\eta}{x^\eta(x^{\gamma\eta-\gamma-\eta}-1)}$. Now q_1^* is greater than one, and we know it maximizes $F(x)$, so $F'(q_1^*) = 0$. Simple algebra shows that $F'(q_1^*) = 0$ if and only if the polynomial condition from Proposition 3.10 holds, $(\gamma\eta-\gamma-\eta)x^{\gamma\eta-\gamma-\eta+1} - (\gamma\eta-\gamma)x^{\gamma\eta-\gamma-\eta} + \eta = 0$. Now let $f(x) = (\gamma\eta-\gamma-\eta)x^{\gamma\eta-\gamma-\eta+1} - (\gamma\eta-\gamma)x^{\gamma\eta-\gamma-\eta} + \eta$, and note that the restriction on the drift implies that $\eta > 1$.

Because $f'(x) = 0$ if and only if $x = 0$ or $x = \frac{\gamma\eta-\gamma}{\gamma\eta-\gamma-\eta+1}$, which is strictly greater than one, $f(x)$ can have at most two positive solutions. Since $x = 1$ is clearly a solution, it can have at most one other. Since $f\left(\frac{\gamma\eta-\gamma}{\gamma\eta-\gamma-\eta+1}\right) = -\left(\frac{\gamma\eta-\gamma}{\gamma\eta-\gamma-\eta+1}\right)^{\gamma\eta-\gamma-\eta+1} - \eta < 0$, and $f(x)$ is clearly positive for sufficiently large x, $f(x)$ must have a root greater than $\frac{\gamma\eta-\gamma}{\gamma\eta-\gamma-\eta+1}$. That is, $f(x)$ has a unique root greater than one. This is the only value greater than one for which $F'(x) = 0$, so it must maximize $F(x)$.

Finally, for the upper bound we will compare the behavior here to that of an firm that lacks redevelopment rights making the development decision in the same economy. Lacking redevelopment rights results, mathematically, in solving the free-boundary problem of proposition 3.9 using the constraint $a_q = 0$ instead of the constraint implied by redevelopment rights, i.e. that implied by the scaling condition. Doing so, we get that the firm lacking redevelopment rights develops to a capacity $x_1^* = \frac{\gamma\eta-\gamma}{\gamma\eta-\gamma-\eta}$, where this comes from $argmax_x\{G(x)\}$ for $G(x) = \frac{(x-1)^\eta}{x^{\gamma\eta-\gamma}}$. Now letting $H(x) = \frac{1}{1-x^{\gamma\eta-\gamma-\eta}}$ we have $F(x) = G(x)H(x)$, so $F'(x) = G'(x)H(x) + G(x)H'(x)$. But if $x > 1$ then $G(x) > 1$, $H(x) > 1$ and $H'(x) < 1$, and if $x > x_1^*$ then $G'(x) < 1$. As a consequence, $F'(x) < 0$ for all $x > x_1^*$, so $q_1^* < x_1^*$. ¥

Proof of Proposition 3.11

Follows directly from Propositions 3.3, 3.9 and 3.10.

Proof of Proposition 4.1

Using the relationship between prices and the demand shock we have

$$lnP_{t_0+t} = \frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)} ln\overline{X}_{t_0+t} - (ln\overline{X}_{t_0+t} - lnX_{t_0+t}), \tag{99}$$

where we have assumed with out loss of generality that $X_{t_0} = 1$, but allowed for the possibility that $\overline{X}_{t_0} = \frac{\overline{P}_{t_0}}{P_{t_0}} > 1$.

The forward prices is just the expected future price, so is given by

$$F_{t_0+t} = \mathsf{E}_{t_0}\left[P_{t_0} X_{t_0+t}\left(\frac{\overline{X}_{t_0}}{\overline{X}_{t_0+t}}\right)^{\frac{1}{1+\alpha(\gamma-1)}}\right]. \tag{100}$$

Now X_{t_0+t} is just a drifted Brownian motion, so distributed $Exp[\mu t + \sigma\sqrt{t}\chi]$, where χ is the standard normal. Using this in conjunction with the joint density for the value and the maximum of a standard Brownian up to time t, and a change of measure, we have that the t ahead forward price is given by

$$\int_{m=0}^{\infty}\int_{b=-\infty}^{m} P_{t_0} e^{\sigma b}\left(\frac{\overline{P}_{t_0}}{max(\overline{P}_{t_0}, P_{t_0}e^{\sigma m})}\right)^{\frac{1}{1+\alpha(\gamma-1)}} e^{(\frac{\mu}{\sigma}-\frac{\sigma}{2})b-(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2\frac{t}{2}} \cdot \nu(b,m)\, db\, dm \tag{101}$$

where $\nu(b,m) = \left(\sqrt{\frac{2}{\pi}}\frac{2m-b}{t\sqrt{t}}e^{\frac{-(2m-b)^2}{2t}}\right)$ is the joint density for the value and the maximum of a standard Brownian. Rearranging the previous equation yields

$$P_{t_0}e^{-(\frac{\mu}{\sigma}-\frac{\sigma}{2})^2\frac{t}{2}}\int_{m=0}^{\infty}\left(\frac{\overline{P}_{t_0}}{max(\overline{P}_{t_0}, P_{t_0}e^{\sigma m})}\right)^{\frac{1}{1+\alpha(\gamma-1)}}\int_{b=-\infty}^{m} e^{(\frac{\mu}{\sigma}+\frac{\sigma}{2})b}\cdot\nu(b,m)\, db\, dm. \tag{102}$$

The interior integral, after completing the square in the exponential and simplifying, yields

$$e^{(\frac{\mu}{\sigma}+\frac{\sigma}{2})^2\frac{t}{2}} e^{2m(\frac{\mu}{\sigma}+\frac{\sigma}{2})} \int_{b=-\infty}^{m} \left(\sqrt{\frac{2}{\pi}}\frac{2m-b}{t\sqrt{t}} e^{\frac{-(b-(2m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t))^2}{2t}}\right) db. \tag{103}$$

We may do the integration using

$$\frac{2m-b}{t\sqrt{t}} = \frac{-1}{\sqrt{t}}\left(\frac{b-(2m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t)}{t}\right) - \frac{\frac{\mu}{\sigma}+\frac{\sigma}{2}}{\sqrt{t}}, \tag{104}$$

in which case we have

$$\begin{aligned}
&\int_{b=-\infty}^{m} \left(\sqrt{\frac{2}{\pi}}\frac{2m-b}{t\sqrt{t}} e^{\frac{-(b-(2m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t))^2}{2t}}\right) db \\
&= -\sqrt{\frac{2}{\pi t}} \int_{b=-\infty}^{m} \left(\frac{b-(2m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t)}{t}\right) e^{\frac{-(b-(2m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t))^2}{2t}} db \\
&\quad -2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right) \int_{b=-\infty}^{m} \frac{1}{\sqrt{2\pi t}} e^{\frac{-(b-(2m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t))^2}{2t}} db.
\end{aligned} \tag{105}$$

Completing the integrations yields

$$= \sqrt{\frac{2}{\pi t}} e^{\frac{-(m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t)^2}{2t}} - 2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right) N\left(\frac{-m-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)t}{\sqrt{t}}\right). \tag{106}$$

Substituting back into the integral over the maximum, and looking separately at the regions $m < \frac{1}{\sigma}\, ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)$ and $m > \frac{1}{\sigma}\, ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)$ gives forward price as

$$\begin{aligned}
&P_{t_0} e^{\mu t} \Bigg(\int_0^{\frac{1}{\sigma} ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)} e^{2m(\frac{\mu}{\sigma}+\frac{\sigma}{2})} \left(\sqrt{\frac{2}{\pi t}} e^{\frac{-(m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t)^2}{2t}}\right) dm \\
&- \int_0^{\frac{1}{\sigma} ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)} e^{2m(\frac{\mu}{\sigma}+\frac{\sigma}{2})} 2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right) N\left(\frac{-m-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)t}{\sqrt{t}}\right) dm \\
&+ \left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\frac{1}{1+\alpha(\gamma-1)}} \int_{\frac{1}{\sigma} ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)}^{\infty} e^{\left(2(\frac{\mu}{\sigma}+\frac{\sigma}{2})-\frac{\sigma}{1+\alpha(\gamma-1)}\right)m} \left(\sqrt{\frac{2}{\pi t}} e^{\frac{-(m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t)^2}{2t}}\right) dm \\
&- \left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\frac{1}{1+\alpha(\gamma-1)}} \int_{\frac{1}{\sigma} ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)}^{\infty} e^{\left(2(\frac{\mu}{\sigma}+\frac{\sigma}{2})-\frac{\sigma}{1+\alpha(\gamma-1)}\right)m} 2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right) N\left(\frac{-m-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)t}{\sqrt{t}}\right) dm \Bigg).
\end{aligned} \tag{107}$$

For the integral over the first region, the first term is just

$$2\left(N\left(\frac{ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)-(\mu+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right) - N\left(-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)\sqrt{t}\right)\right). \tag{108}$$

The second term may be done by parts, and is

$$\left[e^{2\left(\frac{\mu}{\sigma}-\frac{\sigma}{2}\right)m} N\left(\frac{-m-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)t}{\sqrt{t}}\right)\right]_{m=0}^{\frac{1}{\sigma}ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)} + \int_{m=0}^{\frac{1}{\sigma}ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)} \frac{1}{\sqrt{2\pi t}} e^{\frac{-\left(m-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)t\right)^2}{2t}} dm. \tag{109}$$

Substituting the previous two equations into the integral over the first region yields

$$N\left(\frac{ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)-(\mu+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right) - N\left(\frac{-ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)-(\mu+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right)\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\left(1+\frac{2\mu}{\sigma^2}\right)}. \tag{110}$$

For the other region, when $m > \frac{1}{\sigma}ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)$, the integral is

$$\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\frac{1}{1+\alpha(\gamma-1)}}\left(\int_{\frac{1}{\sigma}ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)}^{\infty} e^{\left(2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)-\frac{\sigma}{1+\alpha(\gamma-1)}\right)m}\sqrt{\frac{2}{\pi t}}e^{\frac{-(m+(\frac{\mu}{\sigma}+\frac{\sigma}{2})t)^2}{2t}} dm \right. \tag{111}$$
$$\left. + \int_{\frac{1}{\sigma}ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)}^{\infty} e^{\left(2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)-\frac{\sigma}{1+\alpha(\gamma-1)}\right)m} 2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right) N\left(\frac{-m-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)t}{\sqrt{t}}\right) dm\right).$$

The first integral is computed by completing the square in the exponent, and is

$$2e^{-\left(\frac{1}{1+\alpha(\gamma-1)}\right)\left(\mu+\frac{\sigma^2}{2}-\frac{1}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}\right)t}\left(1-N\left(\frac{ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)-(\mu+\frac{\sigma^2}{2}-\frac{\sigma^2}{1+\alpha(\gamma-1)})t}{\sigma\sqrt{t}}\right)\right). \tag{112}$$

The second term in the integral is, again, done by parts,

$$\frac{2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)}{2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)-\frac{\sigma}{1+\alpha(\gamma-1)}}\left(\left[N\left(\frac{-m-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)t}{\sqrt{t}}\right)e^{\left(2\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)-\frac{\sigma}{1+\alpha(\gamma-1)}\right)m}\right]_{\frac{1}{\sigma}ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)}^{\infty}\right.$$
$$\left. + \int_{\frac{1}{\sigma}ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)}^{\infty} e^{\frac{-\sigma}{1+\alpha(\gamma-1)}}\frac{1}{\sqrt{2\pi t}}e^{\frac{-\left(m-\left(\frac{\mu}{\sigma}+\frac{\sigma}{2}\right)t\right)^2}{2t}} dm\right). \tag{113}$$

Substituting the previous two equations into the integral over the second region yields

$$(1+\theta)N\left(\frac{-ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)-(\mu+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right)\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\left(1+\frac{2\mu}{\sigma^2}\right)}$$

$$+ (1-\theta)e^{-\left(\frac{1}{1+\alpha(\gamma-1)}\right)\left(\mu+\frac{\sigma^2}{2}-\frac{1}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}\right)t} \quad (114)$$

$$\cdot N\left(\frac{-ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)+(\mu+\frac{\sigma^2}{2}-\frac{\sigma^2}{1+\alpha(\gamma-1)})t}{\sigma\sqrt{t}}\right)\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\left(\frac{1}{1+\alpha(\gamma-1)}\right)},$$

where $\theta = \frac{1}{(1+\alpha(\gamma-1))(1+\frac{2\mu}{\sigma^2})-1}$.

Finally, summing over both regions yields

$$\begin{aligned} F_{t_0+t} &= P_{t_0}e^{\mu t}\Bigg(N\left(\frac{ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)-(\mu+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right) \\ &\quad + \theta N\left(\frac{-ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)-(\mu+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right)\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\left(1+\frac{2\mu}{\sigma^2}\right)} \\ &\quad + (1-\theta)e^{-\left(\frac{1}{1+\alpha(\gamma-1)}\right)\left(\mu+\frac{\sigma^2}{2}-\frac{1}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}\right)t} \\ &\quad \cdot N\left(\frac{-ln\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)+(\mu+\frac{\sigma^2}{2}-\frac{\sigma^2}{1+\alpha(\gamma-1)})t}{\sigma\sqrt{t}}\right)\left(\frac{\overline{P}_{t_0}}{P_{t_0}}\right)^{\left(\frac{1}{1+\alpha(\gamma-1)}\right)}\Bigg). \end{aligned} \quad (115)$$

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Proof of Proposition 4.2

We will show the result by perturbation analysis. We will consider the third center moment to returns as the time internal gets very small, and ignore dominated higher order terms.

Heuristically, away from maxima in the price of firms' output log-changes in the output price are normal at sufficiently short time intervals. As a result, returns to a firm, which has a price that is convex in the price of its output, is also convex. That is, if firm value is given by $V(P)$, where P is the price of the firms' output, then because V is convex so too is ln composed with V composed with *exp*. Firm return is given by $ln(V(P)) = ln(V(e^{lnP})$, so because lnP is normal large upward moves in $ln(V(e^{lnP}$ are more likely than large downward moves.

More formally, we will now consider the skewness in returns up to second order terms in dt and show that it is positive. If $\overline{P}_t > P_t$ then the probability that prices of firms' output returns to the maximum on the interval from t to $t+dt$ is given by

$$\begin{aligned} P(\overline{P}_{t+dt} > P_{t+dt}) &= P(\overline{X}_{t+dt} > X_{t+dt}) \\ &= 2 \cdot P(X_{t+dt} > X_{t+dt}) \\ &= o\left(N\left(\frac{X_t - \overline{X}_t}{\sigma\sqrt{dt}}\right)\right). \end{aligned} \quad (116)$$

and can be ignored in the second order expansion for sufficiently small dt. Because the instantaneous return is essentially normal we will without loss of generality ignore the mean when computing higher order centered moments. That is, for convenience we will assume zero drift in demand (it is of course not zero, but the drift does not effect the higher order centered moments).

By inspection of equation (5), the decomposition of the value of a firm into cash flows until price of firms' output returns to the historical maximum and the value of all cash flows after that, we have

$$V(P) = c_1 \, P + c_2 \, P^{\beta}. \tag{117}$$

where c_1, c_2 and β are all strictly positive.

Now firm returns are given by $d\,lnV(P_t)$, and $d\,lnV(P_t) = d\,ln(k\,V(P_t))$ for any constant k. So let us consider $f(x) = \frac{1}{1+c}x + \frac{c}{1+c}x^{\beta}$, for some c that makes f proportional to V. To second order expansion around 1 we have

$$f(x) \approx 1 + \left(\frac{1+\beta\,c}{1+c}\right) x + \left(\frac{\beta\,(\beta-1)\,c}{1+c}\right) \frac{(x-1)^2}{2}. \tag{118}$$

Using $e^y \approx 1 + y + \frac{y^2}{2}$ we then have, to second order, that

$$\begin{aligned} f(e^x) &\approx 1 + \left(\frac{1+\beta\,c}{1+c}\right)\left(x + \frac{x^2}{2}\right) + \left(\frac{\beta\,(\beta-1)\,c}{1+c}\right)\frac{x^2}{2} \\ &= 1 + \left(\frac{1+\beta\,c}{1+c}\right) x + \left(\frac{1+\beta\,c+\beta\,(\beta-1)\,c}{1+c}\right)\frac{x^2}{2}. \end{aligned} \tag{119}$$

Finally, using $ln\,(1+y) \approx y - \frac{y^2}{2}$ we have

$$\begin{aligned} ln\,f(e^x) &\approx \left(\frac{1+\beta\,c}{1+c}\right) x + \left(\frac{1+\beta\,c+\beta\,(\beta-1)\,c}{1+c}\right)\frac{x^2}{2} - \left(\frac{1+\beta\,c}{1+c}\right)^2 \frac{x^2}{2} \\ &= \left(\frac{1+\beta\,c}{1+c}\right) x + \left(\frac{(1+\beta^2 c)(1+c) - (1+\beta c)^2}{(1+c)^2}\right)\frac{x^2}{2} \\ &= \left(\frac{1+\beta\,c}{1+c}\right) x + \left(\frac{(\beta-1)^2 c}{(1+c)^2}\right)\frac{x^2}{2}. \end{aligned} \tag{120}$$

That is, for small x we have, ignoring terms higher than second order, that $ln\,f(e^x) = a\,x + b\,x^2$ where a and b are both strictly positive. Using $d\,ln\,P_t$ for x, and the fact that $d\,ln\,P_t$ is normally distributed with variance $\sigma^2\,dt$, we have that the third centered moment to stock returns is given by

$$\begin{aligned} \mathsf{E}\left[\left(ln\,f(e^x) - \mathsf{E}\left[ln\,f(e^x)\right]\right)^3\right] &= \mathsf{E}\left[\left(a\,x + b\,x^2 - \mathsf{E}\left[a\,x + b\,x^2\right]\right)^3\right] \\ &= \mathsf{E}\left[\left(a\,x + b\,(x^2 - \sigma^2 dt)\right)^3\right] \\ &= \mathsf{E}\left[b^3\,(x^2 - \sigma^2 dt)^3 + 3\,a^2 x^2 b(x^2 - \sigma^2 dt)\right] \\ &= 8\,b^3 \sigma^6 dt^3 + 18\,a^2 b dt^2. \end{aligned} \tag{121}$$

In the limit as dt gets very small stock returns' third centered moment goes to $18\,a^2bdt^2$. This is strictly positive because a and b are both strictly positive. We have, therefore, that if $P_t < \overline{P}_t$ then at sufficiently short horizons stock prices are positively skewed.

When $P_t = \overline{P}_t$ the negative skewness in stock returns is inherited directly from the negative skewness in price of their output. The price of firms' output is not Itô, and is negatively skewed even in the infinitesimal when $P_t = \overline{P}_t$. Because to order dt stock prices are linear in the price of firms's output, negative skewness in the returns to the the price of firms' output at short horizons (in the order dt term) translates directly to negative skewness in stock returns at short horizons.

Proof of Proposition 4.3

For the redevelopment density, we need to compare the q_1^* from the optimal strategy equation, equation (27)

$$q_1^* = argmax_q \left\{ \frac{(q-1)^{\eta}}{q^{\eta}(q^{\gamma(\eta-1)-\eta}-1)} \right\},$$

to the optimal strategy of a firm in the fixed capacity economy. Using the fact that the fixed capacity economy is just the limit of the equilibrium economy as the cost to scale of building gets very large, and letting overscore tildes denote parameters in the fixed capacity economy, we have that

$$\tilde{q}_1^* = argmax_q \left\{ \frac{(q-1)^{\tilde{\beta}}}{q^{\tilde{\beta}}(q^{\gamma(\tilde{\beta}-1)-\tilde{\beta}}-1)} \right\}. \tag{122}$$

The long run average price growth, $\left(\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\right)\left(\mu - \frac{1}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}\right)$, was given in equation (35). The long run average price variance, $\left(\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\right)^2\sigma^2$, is implied directly from the price process equation and the fact that a Brownian process with positive drift stays "close" to its maximum. Using these in the formula for $\tilde{\beta}$ we get

$$\begin{aligned}
\tilde{\beta} &= \sqrt{\left(\frac{\tilde{\mu}}{\tilde{\sigma}^2}-\frac{1}{2}\right)^2+\frac{2r}{\tilde{\sigma}^2}} - \left(\frac{\tilde{\mu}}{\tilde{\sigma}^2}-\frac{1}{2}\right) \\
&= \sqrt{\left(\frac{\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\mu - \frac{\alpha(\gamma-1)}{(1+\alpha(\gamma-1))^2}\sigma^2}{\left(\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\right)^2\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\left(\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\right)^2\sigma^2}} \\
&\quad - \left(\frac{\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\mu - \frac{\alpha(\gamma-1)}{(1+\alpha(\gamma-1))^2}\sigma^2}{\left(\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\right)^2\sigma^2}-\frac{1}{2}\right) \\
&= \sqrt{\left(\frac{\left(1+\frac{1}{\alpha(\gamma-1)}\right)\mu}{\sigma^2}-\frac{\left(1+\frac{1}{\alpha(\gamma-1)}\right)}{2}\right)^2+\left(1+\frac{1}{\alpha(\gamma-1)}\right)^2\frac{2r}{\sigma^2}}
\end{aligned} \tag{123}$$

$$
\begin{aligned}
& - \left(\frac{\left(1+\frac{1}{\alpha(\gamma-1)}\right)\mu}{\sigma^2} - \left(1+\frac{1}{\alpha(\gamma-1)}\right)\frac{1}{2} \right) \\
= & \left(1+\frac{1}{\alpha(\gamma-1)}\right)\left(\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}-\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)\right) \\
= & \eta.
\end{aligned}
$$

The fact that $\tilde{q_1^*} = q_1^*$ then follows directly from the fact that $\tilde{\beta} = \eta$.

For the timing of the redevelopment, we will again use the fact that the fixed capacity economy is just the limit of the equilibrium economy as the cost to scale of building gets very large. The optimal strategy equation, equation (28), in conjunction with $\tilde{\beta} = \eta$ and $\tilde{q_1^*} = q_1^*$ then yields

$$\tilde{P_1^*} = \frac{\Pi}{\tilde{\pi}} P_1^*. \tag{124}$$

That is, in equilibrium a firm redevelops later than it would in a fixed capacity economy with the same long run average price growth and the long run average price variance if and only if $\Pi < \tilde{\pi}$.

Using the formula for the price of a unit cash flow at a price maximum, and the formula for $\tilde{\pi}$, we have $\Pi < \tilde{\pi}$ if and only if

$$(r-\mu)\left(1+\frac{1}{(1+\alpha(\gamma-1))(\beta-1)}\right) > r - \left(\frac{\alpha(\gamma-1))}{1+\alpha(\gamma-1)}\mu - \frac{\alpha(\gamma-1))}{(1+\alpha(\gamma-1))^2}\frac{\sigma^2}{2}\right). \tag{125}$$

Simplifying and ignoring the strictly positive common denominator, this condition reduces to

$$r + \frac{\alpha(\gamma-1))}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2} > \beta\left(\mu + \frac{\alpha(\gamma-1))}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}\right), \tag{126}$$

or

$$\beta < \frac{r + \frac{\alpha(\gamma-1))}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}}{\mu + \frac{\alpha(\gamma-1))}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}}. \tag{127}$$

Using the definition of β we then have

$$\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2 + \frac{2r}{\sigma^2} < \left(\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right) + \frac{r + \frac{\alpha(\gamma-1))}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}}{\mu + \frac{\alpha(\gamma-1))}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}}\right)^2. \tag{128}$$

Simplifying and ignoring strictly positive common factors yields

$$r < \left(\frac{\alpha(\gamma-1)}{1+\alpha(\gamma-1)}\right)\left(\mu - \frac{1}{1+\alpha(\gamma-1)}\frac{\sigma^2}{2}\right). \tag{129}$$

This condition, that the discount rate simply exceeds the long run average price growth, is always satisfied by assumption. As a result, Π is always less than $\tilde{\pi}$, and we then have that $P_1^* > \tilde{P}_1^*$. ¥

Proof of Proposition 4.4

The option value is convex in prices, whereas the intrinsic value is linear, and both value components go to zero with prices. As a result, it must be that the maximum ratio of option value to intrinsic value occurs at the highest possible price, i.e., at the moment before redevelopment occurs. We then have

$$max_p \left(\frac{O(q,P)}{I(q,P)} \right) = \frac{O(q,P_q^*)}{I(q,P_q^*)} = \frac{Aq^{\gamma}}{q\Pi P_q^*}. \tag{130}$$

Substituting for A and P_q^* and simplifying results in

$$max_P \left(\frac{O(q,P)}{I(q,P)} \right) = \frac{(q_1^* - 1)}{\eta(1 - q_1^{*(\gamma+\eta-\gamma\eta)})}. \quad ¥ \tag{131}$$

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