

Security Design With Investor Private Information *

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Abstract

I argue that an important friction in the issuance of financial securities is that potential investors may be privately informed about the value of the underlying assets. I show how security design can help overcome this friction. In the single asset case, I show that debt is often an optimal security when the number of potential investors is small, while equity becomes optimal as the degree of competition increases. In the multiple asset case, debt backed by a pool of assets is optimal if the number of assets is large relative to the degree of competition, while equity backed by individual assets is optimal when the number of assets is small relative to the degree of competition. I use the theory to interpret security design choices in financial markets.

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1. Introduction

This paper studies the security design problem of a firm (or any seller of financial assets) faced with investors (bidders) who have private information about the value of the assets being sold. Because of their private information, the investors are able to extract rents, leading to underpricing of the seller's securities. To minimize this underpricing, should a seller issue debt or equity, and if he has access to several assets or projects, should he issue securities backed by a pool of assets or backed by individual assets? I provide answers to these questions and show how the security design is affected by the degree of competition in financial markets, the number of assets being sold, and the precision of investors' information.

In a corporate finance context, most information-based theories of security design make the plausible assumption that the issuer rather than the investors has private information about the prospects of a firm.¹ This leads to a signalling problem for the issuer in which the securities are chosen to minimize mispricing. The importance of the signalling problem notwithstanding, this paper argues that the problem created by *investors* having private information may also be important in many instances. For example, a start-up company seeking financing usually has to raise money from professional intermediaries such as venture capitalists or banks. These investors, based on their industry expertise and long experience in financing, may be better at evaluating the likely success of the firm than the entrepreneur.² As I will show, once the implications for security design from this screening problem are taken seriously, predictions can be made along dimensions on which the signalling literature is silent. One important such dimension is how the degree of competition among investors (as measured by the number of informed investors) affects the choice of securities.³

The theory developed here may also be relevant for understanding security design patterns that have emerged in other financial markets. One example that fits well with the assumptions of the model is the disposition of assets performed by the Resolution Trust Corporation (RTC). The RTC was set up as a government agency in 1989 under the Financial Institutions Reform, Recovery, and

¹See for example Myers and Majluf [23] and Nachman and Noe [24].

²Even at the initial public offering stage, there is evidence consistent with the existence of investor private information. Firstly, investors do capture rents through the persistent underpricing in IPOs. Secondly, a major consideration in designing the issue appears to be to elicit the privately held valuation of potential buyers. For example, in the bookbuilding procedure, investors indicate their willingness to buy at different prices. More informative bids are rewarded with a higher allocation (see Cornelli and Goldreich [8]).

³Since investors have no private information in a signalling model, it is enough with two investors competing for the security to make the price perfectly competitive. This would be the outcome, for example, in Bertrand competition or in any standard auction format. Therefore, the price is independent of the degree of competition.

Enforcement Act to dispose of the assets of failed savings and loans institutions. It was clear that the RTC had very little expertise in valuing these assets, as opposed to the eventual buyers who were sometimes the original owners of the assets. Initially, the RTC used mostly individual sales of assets, generating very little volume. Through the use of pooled asset auctions and securitized issues, the RTC had sold \$455 billion worth of assets in 1995 (see Vandell and Riddiough [27] and Watkins [28]).

More generally, the model may be applicable to mortgage-backed and other asset-backed securities markets. Although not as immediately apparent as in the RTC example, one can still argue that there exist some sources of investor private information in these markets. Buyers are generally large investment banks, brokerage firms, and institutional investors, who have substantial expertise in valuing the securities issued. Even if participants do not have private knowledge about the actual quality of the assets being sold, there can be private information stemming either from inside knowledge of secondary market conditions based on customer relations, or from a better capability of valuing securities based on proprietary valuation models.⁴ I show that the information-theoretic perspective taken in this paper has implications both for the optimal level of “aggregation” in the security design and the shape the asset-backed security should take.

In modelling this problem, I start out analyzing the single asset case. I assume that a single seller is endowed with an asset (or a project) that he wants to sell (or finance). He faces a limited number of potential buyers who receive privately observed signals about the value of the asset. I assume that the seller can retain part of the cash flow, but that he values immediate consumption higher than the bidders. This may be either because he has to raise financing for the projects he is about to undertake, or because he is more liquidity constrained than bidders for some other reason. The seller will trade off the liquidity cost of retaining cash flow against the expected underpricing. The seller uses a standard auction procedure (second price auctions) to sell securities backed by the asset. Since bidders are imperfectly competitive and have private information, they will typically be able to extract some surplus in the sale, leading to underpricing. It is the goal of the security design to minimize this underpricing.

In signalling models, the typical strategy for minimizing underpricing is to issue a security with

⁴Bernardo and Cornell [6], in analyzing data from an auction of collateralized mortgage obligations in which the participants were major investment firms, found that the dispersion in bids in the auction was surprisingly high. This indicates that investors had significant differences in valuation ex ante, and Bernardo and Cornell provide evidence that these differences were mainly attributable to private information regarding valuation rather than, for example, differences in the utility of holding the assets.

a value that is as insensitive as possible to the private information of the seller. I call this the *immunization* strategy, and it typically leads to debt as an optimal security. Equity, on the other hand, features high underpricing since the value of equity is very sensitive to issuer information.

In the situation where investors have private information, the information sensitivity of a security also plays a crucial role, but in a more subtle way. I show that what causes underpricing is the sensitivity of the security valuation to the *winner's* private information, *after* conditioning on the information contained in the price. The information contained in the price reduces the informational advantage of the winner over the issuer, and it is only the remaining private information that leads to underpricing.

There are two necessary conditions for the underpricing to be high given a certain realization of the price. First, it should not be the case that the price completely reveals the value of the security. The security must feature what I call *residual information sensitivity*, meaning that observing an extra signal on top of the price can change the conditional expectation of the security. Second, for the winner to be able to exploit this information sensitivity, it should not be the case that the price reveals what the winner's signal is. Observing the price turns out to be equivalent to observing the realization of the second highest signal. The higher this realization is, the more precisely one can predict what the winner's signal must have been. Therefore, residual information sensitivity tends to be more harmful for low than for high realizations of the price setting signal.

This last characteristic will make the choice of security design less straight forward than in the signalling case. I identify two opposite strategies for the issuer to pursue. The first is to follow the immunization strategy and issue a security featuring low *unconditional* information sensitivity, such as debt. Such a security will also feature low average *residual* information sensitivity. The disadvantage of this strategy is that, since debt has a fixed pay off in good scenarios but a variable pay off over the default region, the residual information sensitivity will be relatively more concentrated to low realizations of the price setting signal where it is more harmful.

A second strategy is to go to the opposite extreme and issue a security featuring high unconditional information sensitivity, such as levered equity. Such a security will have high average residual information sensitivity. In contrast to debt, however, since the pay off of levered equity is flat over the default region, the residual information sensitivity tends to be relatively more concentrated to high realizations of the price setting signal, where it is less harmful. I call this the *sensitization* strategy.

Whether the immunization strategy or the sensitization strategy is more effective will depend

on the type of information conveyed by realizations of the price setting signals that are not high enough to reveal the value of the winner's signal. If such a realization conveys very bad news, so that the asset value is very likely to be in the default region, the sensitization strategy typically works better. When such a price setting signal is observed, levered equity is revealed to be worthless with high probability regardless of the winner's information. The value of debt, on the other hand, is still uncertain and will be sensitive to the winner's information. If a high price setting signal is observed, the value of levered equity is of course much more uncertain than the value of debt. However, the winner is no better equipped to resolve this uncertainty than someone observing just the price setting signal, since the price setting signal reveals the value of the winner's signal. Therefore, the underpricing of equity is low.

If the number of bidders is relatively small, price setting signals can end up in the lower range quite frequently even if the asset is not in default. Such a realization, then, does not convey extremely bad news. This in turn implies that the residual information sensitivity of levered equity will be high over the whole range of price setting signals, leading to a high degree of underpricing. For a limited number of bidders and an underlying signal distribution that is not very skewed to the top, the immunization strategy is therefore typically more effective.

As the number of bidders in the auction grows, however, the probability that the price setting signal comes from the top of the signal distribution grows. This leads to an increased effectiveness of the sensitization strategy, since a price setting signal below the top conveys increasingly bad news. Therefore, as the degree of competition in the auction grows, levered equity becomes the optimal security design. For intermediate cases, some combination of levered equity and debt with a retention of junior debt is typically optimal.

I go on to study the multiple asset problem. I assume that each bidder gets a signal about each asset. When the seller has multiple assets or projects, the security design problem involves an extra stage: Before deciding on the shape of the security, the seller must decide whether a security should be backed by a single asset or a pool of assets. The problem of how and when to pool common-value assets in auctions with no retention and no security design was studied in Axelson [3]. In that paper, it was shown that a pooled sale is optimal if the number of assets is large enough for a fixed number of bidders, while separate sales becomes optimal as the number of bidders grows. Pooling assets will change the underlying signal distribution in the auction. The signal a seller observes about a pool of assets can be viewed as the average of the signals he observes about each single asset. Pooling therefore increases the central tendency of the signal distribution, while decreasing

the thickness of the tails: Average signals become more common and extreme signals less common. If the number of assets is large enough, and if the assets are not too correlated, this “diversification” effect of pooling will make the informational differences between all bidders very small, since all bidders will be expected to draw average signals. Therefore, the underpricing per asset goes to zero as the number of assets in the pool goes to infinity, holding the number of bidders fixed. However, as the number of bidders grows, the price setting signal is increasingly likely to be drawn from the upper tail of the signal distribution. Since the underpricing is small when the price setting bidder draws a signal from the top of the distribution, a fatter tail leads to lower underpricing. Therefore, individual sales become optimal as the degree of competition grows, holding the number of assets fixed.

As is shown in this paper, seller utility can be substantially improved when retention and security design is allowed on top of the pooling decision. Furthermore, the choice of security design will be intimately linked to the pooling decision. Pooling will increase the effectiveness of the immunization strategy, since debt backed by a pool of assets is less information sensitive than debt backed by a single asset. At the same time, pooling will make the sensitization strategy less effective, as the price setting signal is less likely to come from the top of the signal distribution. Therefore, as the number of assets the seller has access to is increased, debt backed by a pool of assets becomes the optimal security design. On the other hand, as the number of bidders is increased, equity backed by individual assets becomes optimal.

The results above were developed under the restriction that the pay off of a security must be monotonically increasing in the value of the underlying asset, a common assumption in the security design literature that can be justified through an appeal to moral hazard issues. I go on to show that when this restriction is removed, non-monotonic securities will typically be used as part of an optimal security design. In fact, I show that it is sometimes possible to construct risky non-monotonic securities that do not feature any underpricing at all. The fact that we seldom observe these type of security designs empirically, despite their apparent attractiveness, may be explained both by the moral hazard issues they introduce and by the fact that the design is not robust to small variations in assumptions about the underlying signal distribution.

In a further extension of the model, I endogenize the information acquisition of bidders. It turns out that all results are robust to this extension, once I reinterpret the effect of increased competition as an effect of lower information acquisition costs. A lower information acquisition cost leads to a higher degree of competition in the market, which in turn leads the seller to perform

security design reflecting this higher degree of competition.

Several empirical predictions emerge from the theory. First, debt should be more commonly observed when the degree of competition among investors is low, or when information acquisition costs are high. Second, the degree of aggregation of a security should be negatively related to the degree of competition. Third, debt and pooling should be observed in combination, while equity should be more common when there are few assets backing the security. Although undoubtedly there are many factors outside the model that, depending on the institutional setting, are of first-order importance for explaining security design choices, I view the results as being largely consistent with observed phenomena. For example, the prediction that a large asset base should be sold through debt issues backed by a pool is consistent with securitization patterns in the asset-backed securities markets, while equity issues are more prevalent when the asset base is more focused (as is the case for individual firms). I discuss in the conclusion how the results developed here may also be helpful in understanding life cycle patterns of financing for a firm, the difference in financing between bank oriented and market oriented systems, and the role of financial intermediation.

The rest of the paper is organized as follows. The next subsection discusses related literature. In Section 2 the model assumptions and fundamentals are described. Section 3 describes equilibria for second-price auctions and the determination of underpricing in these auctions. The security design results for the single asset case are developed in Section 4, and for the multiple asset case in Section 5. The extensions of the model are discussed in Section 6. Section 7 concludes. All proofs are in Appendix B.

1.1. Related literature

Comprehensive surveys of the security design literature in corporate finance can be found in Harris and Raviv [18] and, more recently, in Allen and Winton [2]. Neither of these surveys discuss the investor private information problem, indicating the scarcity of research on this topic. The idea that outside investors may have private information is not completely new to the corporate finance literature, however. As opposed to the view in this paper, the focus has mostly been on the positive role of informed investors. Allen [1] and Habib and Johnsen [17], for example, emphasize the benefits of security price information in guiding investment decisions of the firm. Many papers discuss the importance of having an informed investor who monitors the firm to solve moral hazard problems. Somewhat more closely related to this paper, Boot and Thakor [7] and Fulghieri and Larkin [12] extend the signalling literature to allow some investors to acquire information about the value of

assets. The security design motive in these papers ultimately stems from signalling considerations, however. Good firms want to separate themselves from bad firms, and might therefore issue equity securities to encourage information production about their assets. None of these papers discuss the role of security design in screening investors.

Closest to my paper in a corporate finance context is Garmaise [13], who looks at a situation in which two informed investors offer securities to a firm in a first price auction. The results are driven by the fact that investors view the entrepreneur as being either overly optimistic or overly pessimistic, leading to debt in the first case and equity in the second case. There is no analysis of how the degree of competition affects the security design. In another paper, Garmaise [14] does look at the effect of increasing the number of bidders in a single asset situation. Although his focus is different, he derives the result that debt is optimal in common value auctions because it serves to minimize the difference of opinions between bidders.

In a trading context, Gorton and Pennachi [16] and Subrahmanyam [26] use Kyle-type models to explain the existence of basket securities in stock markets. They show that liquidity traders can avoid getting picked off by informed traders by trading in pooled securities like stock-index futures. Even though there is no issuer doing any security design in their models, the intuition for these results is similar to the intuition for why pooling may help in reducing underpricing in my model. Similarly, Gorton and Pennachi [15] show that firms or financial intermediaries have an incentive to split cash flows into debt and equity so that uninformed traders can protect themselves against losses to informed traders. None of these papers study the impact of an increased degree of competition on the security design.

Finally, it is interesting to contrast the results in my paper with the results developed by DeMarzo [10], who studies the same security design problem when the seller is endowed with private information. He also derives pooling backed by debt as an optimal security design when the number of assets is large enough. However, equity is never an optimal security design, and the degree of competition plays no role in his analysis.

2. Model Set-Up

I use a standard common value auction set-up (see, for example, Milgrom and Weber [22]), with the exception that the securities sold are endogenous, and that there are possibly multiple assets backing the securities.

There are two time periods, zero and one. A risk-neutral seller has access to a single asset or project that pays a random amount Z in period one. (The important extension to multiple assets is postponed until Section 5). The random variable $Z \in [0, 1]$ has a cumulative distribution function $G(z)$ with associated density $g(z)$.⁵ The seller, as opposed to bidders, is liquidity constrained and has preference for early consumption:

$$U(c_0, c_1) = c_0 + \lambda c_1$$

Here, c_t is consumption in period t and $\lambda \in [0, 1)$ is a discount parameter.⁶ Alternatively, c_0 can be interpreted as the investment amount needed in period zero to finance projects that pay off in period one, in which case the seller maximizes consumption in period one subject to raising a fixed amount of capital in period zero. To raise money for consumption or investment, the seller designs and sells a security $w(Z)$ backed by the cash flow of the underlying asset.⁷ The security is then sold using an exogenously given auction procedure (discussed below).

There are N bidders in the auction of the security. Bidders are risk neutral and have a discount parameter of one.⁸ Each bidder $n \in \{1, \dots, N\}$ draws a privately observed signal X_n which is informative about the value of the asset. Conditional on the realized value of the asset, signals are distributed identically and independently according to the probability density $f(x|z)$ on the support $[0, H]$ where H is an integer. I denote the associated cumulative distribution function by $F(x|z)$. Letting a be any integer from 0 to $H - 1$, I make the following additional assumptions about the signal distribution:

Discreteness: $f(x|z) = f(x'|z)$ if $x, x' \in [a, a + 1)$.

MLRP: $\frac{f(x|z)}{f(x'|z)} > \frac{f(x|z')}{f(x'|z')}$ if $x \in [a, a + 1)$ and $a > x'$ and $z > z'$.

The first assumption implies that signals within an interval $[a, a + 1)$ are equivalent in their information content about Z . In effect, one can think of the signals as being discrete but “ironed out” onto a continuous interval, where each unit subinterval represents one discrete signal. This

⁵Throughout, I denote random variables by capital letters and realizations of random variables with lower case letters, so that a realization of Z in period one is denoted by z .

⁶This way of modelling a reason for trade between risk neutral parties using a higher discount rate for the seller is also used by, for example, DeMarzo and Duffie [11] and Garmaise [14].

⁷Restricting the seller to issuing only one security when there is only one asset turns out to be without loss of generality, as is shown in Lemma 1 below. This is no longer true in the multiple asset case (see Section 5).

⁸In the parlance of auction theory, the asset (and, hence, any security backed by the asset) is a *common value good*, in the sense that all of the bidders derive the same utility from cash flow once the value is realized.

representation is useful because strategies turn out to be pure in the continuous signal X_n .⁹ The second assumption states that signals satisfy the monotone likelihood ratio property (MLRP), which roughly means that signals and values are correlated. Thus, signals are informative, and a higher signal leads to a more optimistic view of the value of the underlying asset. An important consequence of MLRP that I use further down is that it implies that $F(x|z)$ is decreasing in z , so that $F(x|z)$ first order stochastically dominates $F(x|z')$ for $z > z'$ (see Milgrom and Weber [22]).

I also add the following non-degeneracy assumption on $f(x|z)$ and $g(z)$:

Non-degeneracy: $f(x|z) \geq \varepsilon$ and $g(z) \geq \varepsilon$ for some $\varepsilon > 0$ for all x, z .

The non-degeneracy assumption assures that no value realization of Z can be completely ruled out, regardless of what signal is observed.

Following Nachman and Noe [24], I define a security $w(Z)$ to be *admissible* if it satisfies the following conditions:

Limited liability: $0 \leq w(z) \leq z$.

Monotonicity: $w(z)$ and $z - w(z)$ are nondecreasing.

These conditions are commonplace in the security design literature. The constraint $0 \leq w(z)$ is a condition of limited liability for bidders: Once the security is bought, they are not obliged to provide any additional cash flow in period one. The constraint $w(z) \leq z$ is a limited liability constraint on the seller reflecting his limited wealth outside of the cash flow generated by assets.

Less obvious is the restriction to monotone securities. It can be formally justified on grounds of moral hazard in period one (see Nachman and Noe [24]): Suppose $w(z)$ is decreasing on a region $a < z < b$, and that the underlying cash flow turns out to be $Z = a$. The seller then has an incentive to secretly borrow money from a third party and add it on to the aggregate cash flow to push it into the decreasing region, thereby reducing the payment to the security holder while still being able to pay back the third party. Similarly, if the seller's retained claim $z - w(z)$ is decreasing over some region $a < z \leq b$ and the realized cash flow is $Z = b$, the seller has a strong incentive to decrease

⁹See, for example, Pesendorfer and Swinkels [25] for a similar way of representing signals. When signals are discrete and there are more than two bidders, strategies will typically be mixed. The continuous representation of the signal space can be viewed as a discrete, integer valued signal plus an added noise term distributed uniformly on the unit interval. The noise term is a draw from the bidders mixing strategy, and strategies will therefore be pure in the augmented continuous signal.

the observed cash flow by engaging in some type of “money burning”.¹⁰ In Section 6.1 I allow the monotonicity requirement to be relaxed, and show that it is in fact a binding constraint in that non-monotonic securities will typically be part of an optimal security design if they are allowed.

The conditions of limited liability and monotonicity turn out to be equivalent to requiring that w is left-continuous and satisfies $w(0) = 0$ and $\frac{dw(z)}{dz} \in [0, 1]$ for all z .¹¹ I use these latter conditions in the operationalization of the problem.

Denote by $\pi(w(Z))$ the price received in the auction of security $w(Z)$. The seller’s expected utility from selling security w is then given by $U(w) = E(\pi(w(Z))) + \lambda(E(Z) - E(w(Z)))$. Noting that the term $\lambda E(Z)$ does not change with the choice of security, I can drop it from the objective function and write the maximization problem as

$$\begin{aligned} \max_w (1 - \lambda) E(w) - E(w - \pi(w)) & \quad (2.1) \\ \text{such that } w(0) = 0 \text{ and } \frac{dw(z)}{dz} \in [0, 1] \quad \forall z & \end{aligned}$$

The term $(1 - \lambda) E(w)$ denotes the benefits from transferring money from the buyer to the liquidity constrained seller in period zero in exchange for claims to asset cash flows in period one. The term $E(w - \pi(w))$ denotes the expected underpricing of the security, or the informational rents captured by the better informed bidders. This can be viewed as the cost of the transfer.

An important alternative interpretation of the set-up is one where the seller minimizes the expected underpricing $E(w - \pi(w))$, subject to raising a fixed amount of capital C in period zero to finance the investment that comes on line in period one.:

$$\begin{aligned} \min_w E(w - \pi(w)) & \quad (2.2) \\ \text{such that } w(0) = 0 \text{ and } \frac{dw(z)}{dz} \in [0, 1] \quad \forall z & \\ \text{and } E(\pi(w)) = C & \end{aligned}$$

¹⁰Obviously, if possible, the seller can also decrease the observed cash flow by appropriating it secretly for his own consumption. But allowing for this possibility would introduce as severe a moral hazard problem with monotonic securities, and would therefore make any sale of securities impossible.

¹¹Monotonicity together with the assumption of a continuous distribution for Z implies that $w(z)$ is continuous, since any jump down would violate monotonicity of $w(z)$, and any jump up would violate the monotonicity of $z - w(z)$. Thus, $\frac{dw(z)}{dz}$ exists. There may be kinks in $w(z)$ in which case $\frac{dw(z)}{dz}$ is taken to be the left derivative: $\frac{dw(z)}{dz} = \lim_{z' \uparrow z} \frac{w(z) - w(z')}{z - z'}$. Given the existence of $\frac{dw(z)}{dz}$, monotonicity is then equivalent to requiring $\frac{dw(z)}{dz} \in [0, 1]$. Limited liability implies $w(0) = 0$. Given the bound on $\frac{dw(z)}{dz}$, limited liability is then guaranteed for all z .

I refer to this interpretation as the *capital raising problem* in the text below.¹² Denoting by $w^*(\lambda)$ the optimal solution to Problem 2.1 and by $w^*(C)$ the optimal solution to Problem 2.2, it is straight forward to show that the solution set $W^{*'} \equiv \{w^{*'}(C) : C \in [0, \pi(Z)]\}$ for the capital raising problem is equivalent to the solution set $W^* \equiv \{w^*(\lambda) : \lambda \in [0, 1]\}$ for problem 2.1, and that there is a one-to-one mapping $\lambda(C)$ from $[0, \pi(Z)]$ to $[0, 1]$ such that $w^*(\lambda(C)) = w^{*'}(C)$. Therefore, the two problems are identical for our purposes.

3. Equilibrium and Underpricing in the Auction

I assume that securities are sold using second price auctions. In second price auctions bidders submit sealed bids, the highest bidder gets the asset, and pays the bid of the second highest bidder. In case of a tie, the winner is selected by a fair lottery among the tying bidders.

The choice of second price auctions as the selling format is made for expository reasons. All major qualitative results in the paper go through for other standard auction formats like first price or ascending price auctions.¹³ However, I do not solve the full mechanism design problem, which would involve specifying the rules of the auction in conjunction with the security design. Instead, I solve the security design problem taking an auction format as given. This is not an innocuous restriction. With pure common values, you can typically design a mechanism that extracts all bidder surplus (see, for example, Crémer and McLean [9] and McAfee, McMillan, and Reny [19]).

There are at least three reasons why I still find it useful to concentrate on a standard auction format in this paper. First, one of the objectives of this paper is to show how standard auction procedures can be used in conjunction with security design to improve revenues for the seller, illustrating that security design can be a substitute for designing a more complicated auction procedure. It seems plausible that a seller often has more power over how to structure his securities than over how the particular market institution used for selling is designed. The second reason relies on casual observation and the usefulness of the model for empirical extensions: First price, second price, and ascending price auctions are the prevalent existing auction forms which suggests that they may have attractive features not modelled in the current paper. For empirical tests, we need to have a theory of how existing market institutions work. Third, the optimal mechanisms developed

¹²The reader may have spotted a problem with this interpretation. The problem is that the capital raising restriction is in terms of *expected* revenues - the seller cannot be assured of actually receiving C in each state of the world, since revenues from the auction are stochastic. One way to get around this is to assume that there exists an uninformed intermediary who is willing to pay the seller C in anticipation of auctioning off the securities himself at an expected revenue of $E(\pi(w))$.

¹³Proofs for first price and ascending price auctions are available upon request.

in the literature are often very complicated and involves unattractive features like payments from losers, ability to precommit ex ante from a buyer's point of view, the existence of side bets among the bidders, etcetera. They are typically not robust to small changes in the environment in the way that standard auction formats are (see Milgrom [21] for a discussion).

Equilibrium for the second price auction was characterized by Milgrom [20], and uniqueness of this equilibrium was proved by Pesendorfer and Swinkels [25]. I give a sketch of the derivation here. Since bidders are symmetric, I can without loss of generality analyze the situation from the perspective of bidder 1. Denote by $X \equiv X_1$ the signal of bidder one, and by $Y_1 \equiv \max_{n \neq 1} X_n$ the highest signal among his opponents. The equilibrium bid function $b(x)$ for a bidder observing signal $X = x$ is given in Proposition 1.

Proposition 1. *(Milgrom, Pesendorfer and Swinkels): The unique symmetric Nash equilibrium in the auction of security $w(Z)$ is given by*

$$b(x) = E(w|X = x, Y_1 = x) \tag{3.1}$$

$$= \int_0^1 w(z) \frac{f(x|z)^2 F(x|z)^{N-2} g(z)}{\int_0^1 f(x|u)^2 F(x|u)^{N-2} g(u) du} dz \tag{3.2}$$

where $b(x)$ is increasing in x .

The equilibrium in Proposition 1 states that everyone bids based on the hypothesis that they just marginally win the auction, which is when their signal is as high as the highest signal among the opponents. The intuition for the equilibrium can be developed as follows: Suppose bidder one has signal $X = x$ and the highest signal among his opponents takes on value $Y_1 = y$. If bidder one wins the auction, and all other bidders bid according to equation 3.1, he will pay a price $\pi(w) = E(w|X = y, Y_1 = y)$, since that will be the highest bid among his opponents. Upon observing this price, he can deduce that the value of Y_1 must have been y , so his own valuation of the security conditional on winning at price $\pi(w)$ will be given by $E(w|X = x, Y_1 = y)$. This gives him a conditional pay-off of $E(w|X = x, Y_1 = y) - E(w|X = y, Y_1 = y)$, which from the assumption of MLRP is positive if x is higher than y and negative if x is lower than y . He is therefore only interested in winning the auction when his signal is at least as high as the highest signal of his opponents, which is assured by bidding according to equation 3.1.

4. Security Design: The single asset case

I now turn to the solution of the security design problem 2.1. First, I show that there is no loss of generality from assuming that only one security is issued. This follows from two observations: Firstly, if a security is split into two monotonic parts, bidders will line up in exactly the same order when bidding for the parts as when bidding for the original security, since the bid function $b(x)$ in 3.1 is increasing in x for all monotonic securities.¹⁴ Therefore, the same bidder will be the price setter in all auctions. Secondly, since the bid function is a linear expectations operator on $w(Z)$, the sum of bids for the parts will be equal to the bid for the original security. Since the same bidder is the price setter in all auctions, the sum of the prices for the parts is therefore equal to the price of the original security.¹⁵ Thus, revenue is unaffected by the split-up, so that there is no loss from just issuing one security. This is stated in the following lemma:

Lemma 1. *Suppose an admissible security $w(Z)$ is split up into K admissible securities that are sold in simultaneous auctions. Then, the revenue generated from these auctions is identical to the revenue generated from selling $w(Z)$.*

One implication of Lemma 1 is that if the seller is very liquidity constrained and needs to sell claims to all cash flows ($\lambda = 0$), it is irrelevant how the securities are designed. The seller might as well auction the whole asset. This in turn implies that, if there is no retention of cash flow, the theory is silent as to what proportion of debt and equity a firm should have in its capital structure. There needs to be some retention by the seller for security design to matter.¹⁶

To further characterize the security design problem of the seller, it is convenient to decompose a security $w(Z)$ into its “smallest component” securities where each component security pays one if cash flow is above a certain threshold v and zero otherwise. I denote such a security by an indicator function $1_{Z \geq v}$. Note that any admissible security can be expressed as a portfolio of component

¹⁴That $b(x)$ is increasing in x for all monotonically increasing securities $w(z)$ follows from the assumption of MLRP and Theorem 5 of Milgrom and Weber [22].

¹⁵As is shown below, this result does not hold when we look at the decision of whether to sell assets separately or in a pool. Lemma 1 relies on the assumption that the signal space does not change when we split securities. This is true when the signal space is one dimensional and securities are split into monotonic components. However, it is *not* true when there are multiple assets about which bidders observe multiple (independent) signals. In that case, a different set of signals are used in the evaluation of a single asset and a pool of assets. It is also not true if we relax the monotonicity assumption (see Section 6.1 below), since monotonicity is crucial for preserving the ordering of signals in the bid function.

¹⁶This is also true in signalling models of capital structure. Signalling is done by retaining part of the cash flow. How the cash flow that is sold is split over different securities is irrelevant.

securities as follows:

$$w(Z) = \int_0^1 1_{Z \geq v} \frac{dw(v)}{dv} dv$$

Therefore, using Lemma 1, I can express the auction of security w as a collection of infinitesimal simultaneous auctions, each involving a component security $1_{Z \geq v}$ with weight $\frac{dw(v)}{dv} dv$. The security design problem is then reduced to choosing the weights $\frac{dw(v)}{dv}$ on these auctions. The maximization problem in 2.1 can be written as:

$$\max_{\frac{dw(v)}{dv} \in [0,1]} \int_0^1 [(1 - \lambda) E(1_{Z \geq v}) - E(1_{Z \geq v} - \pi(1_{Z \geq v}))] \frac{dw(v)}{dv} dv$$

where $\pi(1_{Z \geq v})$ denotes the price in the auction of security $1_{Z \geq v}$. The problem can thus be viewed as one of deciding which component securities to auction off. The solution is obvious: It is worth including $1_{Z \geq v}$ in the security design if the expected benefit $(1 - \lambda) E(1_{Z \geq v})$ is larger than the expected underpricing $E(1_{Z \geq v} - \pi(1_{Z \geq v}))$, or if the normalized underpricing $\phi(1_{Z \geq v}) \equiv E(1_{Z \geq v} - \pi(1_{Z \geq v})) / E(1_{Z \geq v})$ is smaller than $1 - \lambda$. If so, set $\frac{dw(v)}{dv}$ to one. Otherwise, set $\frac{dw(v)}{dv}$ to zero.

Denote by $h(y)$ the probability density function for the price setting signal Y , which is the second order statistic in the sample of signals:

$$h(y) = \int_0^1 N(N-1)(1-F(y|z))f(y|z)F(y|z)^{N-2}g(z)dz$$

(see, for example, Balakrishnan and Cohen [4]). Using this notation, the following proposition summarizes the solution to the security design problem and derives an expression for the normalized underpricing.

Proposition 2. *The optimal admissible security design is given by*

$$w(Z) = \int_0^1 1_{\phi(1_{Z \geq v}) \leq 1 - \lambda} * 1_{Z \geq v} dv \tag{4.1}$$

where $\phi(1_{Z \geq v}) \equiv \frac{E(1_{Z \geq v} - \pi(1_{Z \geq v}))}{E(1_{Z \geq v})}$ is the normalized underpricing for component security $1_{Z \geq v}$.

The function $\phi(1_{Z \geq v})$ is given by:

$$\begin{aligned}\phi(1_{Z \geq v}) &= \int_{y=0}^H \frac{P(Z \geq v | X \geq y, Y_1 = y) - P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v)} h(y) dy \quad (4.2) \\ &= \int_{y=0}^H \phi(1_{Z \geq v} | y) h(y) dy\end{aligned}$$

where $\phi(1_{Z \geq v} | y)$ is the normalized underpricing conditional on observing a price setting signal $Y = y$.

From Proposition 2, it is clear that the key to deriving the optimal security design is to understand how the normalized underpricing for a component security $1_{Z \geq v}$ varies with v . For example, if $\phi(1_{Z \geq v})$ is increasing over the entire range of v , low cash flows have relatively low underpricing. Equation 4.1 then says that all component securities $1_{Z \geq v}$ with v below some certain cut off level \bar{v} defined by $\phi(\bar{v}) = 1 - \lambda$ should be included. This corresponds to a debt security $w(Z) = \min(Z, \bar{v})$. On the other hand, if $\phi(1_{Z \geq v})$ is decreasing over the entire range, high cash flows have lower underpricing, and all component securities $1_{Z \geq v}$ with v above some certain cut off level \underline{v} should be included. This corresponds to a levered equity security $w(Z) = \max(0, Z - \underline{v})$. In general, $\phi(1_{Z \geq v})$ does not have to be either monotonically increasing or decreasing.

The degree of underpricing in the auction of a component security, which will determine whether the component security should be included or not, is decided as follows. Given a certain realization y of the price setting signal, the price $\pi(1_{Z \geq v})$ in the auction will be given by

$$\begin{aligned}\pi(1_{Z \geq v}) &= E(1_{Z \geq v} | X = y, Y_1 = y) \\ &= P(Z \geq v | X = y, Y_1 = y)\end{aligned}$$

Upon observing this price, the seller can deduce what the price setting signal must have been, and can also deduce that the winner's signal must have been at least as high. Therefore, the seller's valuation conditional on observing the price is

$$\begin{aligned}E(w | \pi(w)) &= E(1_{Z \geq v} | X \geq y, Y_1 = y) \\ &= P(Z \geq v | X \geq y, Y_1 = y)\end{aligned}$$

so that the normalized conditional underpricing $\phi(1_{Z \geq v} | y)$ is given as the integrand in expression 4.2. This is the normalized average difference in valuation between a bidder who correctly observes

the highest signal x and the second highest signal y (as measured by $P(Z \geq v | X = x, Y_1 = y)$), and a bidder who correctly observes the second highest signal but wrongly assumes that the highest signal is the same (as measured by $P(Z \geq v | X = y, Y_1 = y)$). I call $\phi(1_{Z \geq v} | y)$ the *informational distance*, as it depends on how much more information about the value of the security there is in correctly observing the winner's signal. For an arbitrary security $w(Z)$, I write the informational distance $\phi(w(Z) | y)$ as

$$\phi(w(Z) | y) \equiv \frac{E(w(Z) | X \geq y, Y_1 = y) - E(w(Z) | X = y, Y_1 = y)}{E(w(Z))}$$

Two separate criteria decide whether the informational distance given a certain realization of the price setting signal is large or not. First, it must be the case that there is still some uncertainty left about the value of the security after observing the price, and that observing an extra signal helps reduce that uncertainty. The security must feature what I call *residual information sensitivity*. I define the residual information sensitivity for a security $w(Z)$ as

$$r(w(Z) | y) \equiv \frac{E(w | X = H, Y_1 = y) - E(w | X = 0, Y_1 = y)}{E(w)}$$

This is the difference in valuation (conditional on the information already contained in the price) resulting from a draw of either a high or a low extra signal. Note that the residual information sensitivity is always at least as high as the informational distance, and therefore provides an upper bound for $\phi(w(Z) | y)$:

$$r(w(Z) | y) \geq \phi(w(Z) | y)$$

This follows since the valuation $E(w | X = x, Y_1 = y)$ is increasing in x (see Lemma 4). If the residual information sensitivity is low, the valuation of the security is almost the same regardless of the information in an extra signal. Therefore, the informational distance between the highest and second highest bidder is small, and there is little underpricing.

However, high residual information sensitivity does not automatically translate into high underpricing. The second criterion is that the winner's signal is not revealed by the price setting signal, so that there is some room for the winner to take advantage of the residual information sensitivity. Upon observing $Y_1 = y$, the winner's signal is revealed to be in the interval $[y, H)$. The informational distance is therefore limited if y is large, since the winner's signal cannot be much

higher. In particular, if y is in the top equivalence interval $[H - 1, H)$ the winner's signal must also be in the top equivalence interval. In that case, there is no informational distance and the underpricing is zero. These results are established in the following lemma.

Lemma 2. *The informational distance (underpricing) $\phi(w(Z) | y)$ for a security $w(Z)$ conditional on observing a price setting signal y is zero if and only if one of the following conditions hold:*

- 1) $r(w(Z) | y) = 0$ (Zero residual information sensitivity)
- 2) $y \in [H - 1, H)$ (Winner's signal revealed)

For price setting signals $y \in [0, H - 1)$, the informational distance $\phi(w(Z) | y)$ is bounded above by the residual information sensitivity $r(w(Z) | y)$.

A security then features little underpricing if it has minimal residual information sensitivity for realizations of the price setting signal below the top of the signal distribution. This points to two opposite strategies for reducing the expected underpricing. The first, maybe most intuitive, is to include component securities that are very low-risk and hence have low *unconditional* information sensitivity, in that a bidder's valuation is not very dependent on his signal. Such a security is also expected to feature low residual information sensitivity, and hence low underpricing, for most realizations of the price setting signal. I call this strategy the *immunization* strategy, as it aims to immunize the value of the security against the signal received by a bidder. Define the unconditional information sensitivity $r(1_{Z \geq v})$ as

$$r(1_{Z \geq v}) = \frac{P(Z \geq v | X = H) - P(Z \geq v | X = 0)}{P(Z \geq v)}$$

It is straight-forward to show that the unconditional information sensitivity as well as the variance of a component security is increasing in the cut-off level v . For example, the component security $1_{Z \geq 0}$ is completely immune to private information, and therefore has zero underpricing. Increasing the cut-off level v slightly above zero increases the risk of a zero pay-off slightly, but the residual information sensitivity is still practically zero except for extremely pessimistic realizations of the price setting signals. For higher realizations, a zero pay-off can be ruled out with reasonable confidence, so that the value of the security is relatively precisely revealed by the price. The information contained in the winner's signal is then superfluous, and there is little underpricing. In fact, the following proposition shows that there always exists a $\bar{v} > 0$ such that all component securities $1_{Z \geq v}$ with $v < \bar{v}$ should be included in the security design as long as $\lambda < 1$. This

corresponds to a debt security with face value \bar{v} .¹⁷

Proposition 3. *There exists a $\bar{v} > 0$ such that $\phi(1_{Z \geq v}) \leq 1 - \lambda$ for $v \in [0, \bar{v}]$ and $\lambda < 1$. Thus, the optimal security design always contains a debt component $w(Z) = \min(Z, \bar{v})$.*

An opposite strategy, which I call the *sensitization* strategy, is to include component securities with very high cut-off values v . An example of such a security is levered equity, which includes all component securities above a certain cut-off level v . Such securities are high-risk and have high unconditional information sensitivity. However, if most of the *residual* information sensitivity is concentrated to high realizations of the price setting signal, such a security will feature little underpricing since the informational distance is small over the top range of price setting signals. This will typically be the case when price setting signals below the top of the signal distribution convey very negative information about the value of the underlying asset. In such a situation, a price setting signal below the top reveals that the asset value is almost surely in the default region. Therefore, the levered equity security is revealed to be almost surely worthless, regardless of the realization of the winner's signal. Debt, on the other hand, features considerable residual information sensitivity exactly when default is likely, since the value of debt in the default region varies one-for-one with the value of the underlying asset. Of course, after observing a price setting signal close to the top, the residual information sensitivity of equity is much higher than the residual information sensitivity of debt. However, the winner is no better equipped to resolve the remaining uncertainty than someone who just observes the price, since such an observer will be able to deduce exactly what the winner's signal must be. Thus, in this region, the informational distance is small for levered equity as well as debt, and the total effect is that equity will feature lower expected underpricing than debt.

The sensitization strategy will therefore be more effective than the immunization strategy in situations when a price setting signal below the top conveys very bad information. For few bidders, this only happens when the underlying signal distribution is very skewed to the top so that high signals are very likely except in bad circumstances. For less extreme signal distributions, the immunization strategy will typically be more effective, leading to debt as a better security design. As the number of bidders in the auction grows, however, the probability that the second highest signal comes from the top of the signal distribution also grows, regardless of the shape of the

¹⁷It should be pointed out that this result depends on the assumption that $G(v)$ has no mass at zero. If there is a positive probability of a zero cash flow realization, the corollary would no longer be true, since a security $1_{Z \geq 0}$ is no longer riskless.

underlying signal distribution. This leads to an increased effectiveness of the sensitization strategy, since a price setting signal below the top is increasingly bad news. Therefore, as the degree of competition in the auction grows, levered equity becomes the optimal security design. This is established in the following proposition.

Proposition 4. *For any $v > 0$, the normalized underpricing $\phi(1_{Z \geq v})$ goes to zero with N at the same rate as $F(H - 1|v)^N$, the probability that the winner is not in the top equivalence interval. Since $F(H - 1|v)$ is strictly decreasing in v , $\phi(1_{Z \geq v})$ goes to zero faster for higher v , and so becomes decreasing in v over $[\underline{v}(N), 1]$ where $\underline{v}(N) \rightarrow 0$. Therefore, the proportion of levered equity in the optimal security design goes to one as N goes to infinity.*

It should be noted that the underpricing for *all* component securities goes to zero as N goes to infinity. However, the underpricing for high cash flows goes to zero at a faster rate, which leads to the result in Proposition 4. Of course, for a fixed liquidity cost $1 - \lambda$, the optimal security design with a very high degree of competition will be to sell the whole asset, since the underpricing for each component security will be below $1 - \lambda$ eventually. Proposition 4 should then be interpreted as saying that on the path towards selling the whole asset, the security design looks more and more like equity. In the capital raising problem, it is no longer true that more and more cash flows are sold out as the degree of competition increases. The parameter $\lambda(C)$ will be increasing with the number of bidders, since the seller can raise C by issuing securities with less and less underpricing. In the capital raising problem, the result in Proposition 4 is therefore that in the limit, the optimal security design is levered equity with retention of lower cash flows.^{18,19}

Propositions 3 and 4 show that the component securities that are most likely to feature low underpricing are the ones with either very low or very high cut-off values v . If $\phi(1_{Z \geq v})$ is increasing over the entire range, pure debt is optimal, and if it is decreasing over the entire range, levered equity is optimal. There are also cases in which $\phi(1_{Z \geq v})$ is increasing over the bottom range and decreasing over the top range, in which case the optimal security design is some combination

¹⁸Of course, Proposition 3 states that there will always be *some* debt in the optimal security design. However, as the number of bidders grows, this tranche of debt becomes vanishingly small (and, as was alluded to in a previous footnote, would disappear completely if we allowed for some strictly positive probability of having $Z = 0$.)

¹⁹The result in Proposition 4 does not rely on the assumption of discrete signals (or, equivalently, equivalence intervals of positive mass). However, it does rely on the fact that the support of the signal distribution is bounded, so that a very high price setting signal reveals the winner's signal with considerable accuracy. For signal distributions with unbounded support such as the normal or the exponential distribution this may no longer be the case. I have not been able to characterize the limiting behavior of the underpricing function in such cases, but it is entirely possible that there exist signal distributions with unbounded support for which debt is optimal regardless of the number of bidders. I leave further investigation of this issue for future research.

of levered equity and debt with a retention of junior debt. It is over the middle range, where neither the immunization strategy nor the sensitization strategy can be utilized effectively, that the underpricing is the highest. Although I have not been able to prove that a combination of debt and levered equity (possibly with zero weight on either) is generally optimal, the parametric example below does have this feature.

4.1. Example

Suppose that signals and asset values are distributed as follows:

Distribution D1 Cash flow Z is uniformly distributed on the unit interval, and signals are Bernoulli random variables with $P(1|z) = \frac{1-k}{2} + kz$, $P(0|z) = 1 - P(1|z) = \frac{1-k}{2} + k(1-z)$ where $k \in [0, 1]$ is a constant.

Adding a noise term, and using the notation developed above, the signal distribution can be represented using a continuous random variable with density $f(x|z)$ given by

$$f(x|z) = \begin{cases} \frac{1-k}{2} + kz & 1 \leq X < 2 \\ \frac{1-k}{2} + k(1-z) & 0 < X < 1 \end{cases}$$

The parameter k measures how informative signals are. When $k = 0$, signals are pure noise, and as k increases, signals become increasingly more informative about the asset value.

I will derive the optimal security design for this example further down, but suppose first the seller splits the cash flow of the asset over a debt security $D(Z) = \min(Z, F)$ and a levered equity security $L(Z) = \max(0, Z - F)$, and has to decide which security to issue and which to retain. Suppose he sets the face value F of the debt such that both securities have the same expected value, so that the decision will be based on which security features the lowest expected underpricing. It is easy to check that the face value will be $F = 1 - \frac{1}{\sqrt{2}}$, or about 0.3, and the expected value of each security will be 0.25, which is half of the expected value of the underlying asset.²⁰ Also, suppose first that $k = \frac{1}{2}$. It is straight-forward to verify that the unconditional information sensitivity is

²⁰More detailed calculations for this example can be found in Appendix A.

higher for the equity security than the debt security:

$$\frac{E(L(Z)|X \geq 1) - E(L(Z)|X < 1)}{E(L(Z))} \approx .53$$

$$\frac{E(D(Z)|X \geq 1) - E(D(Z)|X < 1)}{E(D(Z))} \approx .14$$

In the auction of a security, the price setting signal Y will be either in the lower equivalence interval $[0, 1)$ or in the top equivalence interval $[1, 2)$. The top left panel of Figure 4.1 depicts the prices $\pi(D(Z))$ and $\pi(E(Z))$ for the two securities as a function of the price-setting signal for the case of two bidders. (In the case of two bidders, strategies are pure in the discrete signal, which is why the price just depends on what equivalence interval the price setting signal is in.) The price differential between the lower and upper equivalence intervals is larger for equity, reflecting the greater unconditional information sensitivity. Also, the expected values of the securities conditional on the prices (given by $E(w(Z)|\pi(w))$) are plotted. The difference between this expectation and the price is the conditional underpricing. Over the top equivalence interval, there is no underpricing, since the price reveals the winner's signal and is therefore not biased downwards. All the underpricing occurs over the bottom equivalence interval. It can be seen that debt features lower underpricing than equity, and is therefore the security that should be issued.

The reason for this can be gleaned from the lower left panels in Figure 4.1, which depict the pay-off of each security as a function of Z together with the probability density of Z conditional on the price-setting signal being in a certain equivalence interval. The middle panel shows $g(z|Y < 1)$, the probability density of Z conditional on Y being in the lower equivalence. This is the information which is available to someone observing the price. There is still considerable residual uncertainty about the value of the securities since the posterior is not very precise, and in this case the residual information sensitivity of equity is the highest, since it is more sensitive to movements in the underlying asset value. This is what leads to the higher underpricing. The bottom panel shows $g(z|Y \geq 1)$, which is the conditional density of someone observing a price from the top equivalence interval. Obviously, since this density is skewed to the top, the information sensitivity of equity is even higher. The debt security, however, is relatively likely to pay the face value in this scenario, and so features less information sensitivity. Of course, the information sensitivity over the top range is inconsequential for the underpricing.

The pictures in the right column of Figure 4.1 show the same plots for the case of $N = 15$. With an increased number of bidders, the price setting signal is less likely to come from the bottom

equivalence interval, which means that such an event is interpreted as very bad news. This is reflected in the fact that the prices for both debt and equity are depressed over this region. The effect is extra strong for levered equity, and the price is practically zero. The expected value of equity conditional on the price coming from the bottom equivalence interval is also close to zero, so that the absolute magnitude of the underpricing is small. For debt, the underpricing is higher, leading to equity as the preferred security. Again, the explanation for this lies in the amount of residual information sensitivity over the bottom interval. As can be seen from the middle right panel, the conditional probability density $g(z | Y < 1)$ is so skewed to the left that a price in this region practically reveals that default will occur. This means that the value of equity is revealed to be worthless, while the value of debt is still uncertain. Of course, if the price setting signal comes from the top interval, equity is more information sensitive, but again this information sensitivity has no impact on the underpricing

I go on to derive the optimal security design. In the Appendix, I derive the relative underpricing $\phi(1_{Z \geq v})$ when auctioning a security $1_{Z \geq v}$ as

$$\phi(1_{Z \geq v}) = \frac{P(0|v)^N - \left(\frac{1-k}{2}\right)^N - \frac{\left(\frac{1+k}{2}\right)^N - \left(\frac{1-k}{2}\right)^N}{\left(\frac{1+k}{2}\right)^{N+1} - \left(\frac{1-k}{2}\right)^{N+1}} \left(P(0|v)^{N+1} - \left(\frac{1-k}{2}\right)^{N+1} \right)}{k(1-v)} \quad (4.3)$$

The left panel of Figure 4.2 plots the underpricing function $\phi(1_{Z \geq v})$ for different values of k when there are two bidders. When k is very small, signals are very noisy and hence the asymmetry of information is small. When k is zero, there is no asymmetric information and the underpricing disappears completely for all component securities. As k is increased, the unconditional information sensitivity at all levels of v goes up except at $v = 0$. There is no underpricing in issuing a security $1_{Z \geq 0}$, as it is risk free and completely insensitive to information at all levels of k . For $v > 0$, underpricing initially goes up as k is increased. This effect is stronger for higher levels of v , so that debt is an optimal security design. This shows that the immunization strategy is more effective for low values of k , since a low price setting signal does not rule out high values of z when the precision of information is moderate. A risky component security therefore features high residual information sensitivity even for low price setting signals.

However, as k gets even larger, the sensitization strategy starts to become effective over the top range. This is because as signals become more informative, price setting signals below one constitute increasingly bad news for highly levered cash flows. For example, when $k = 1$ the

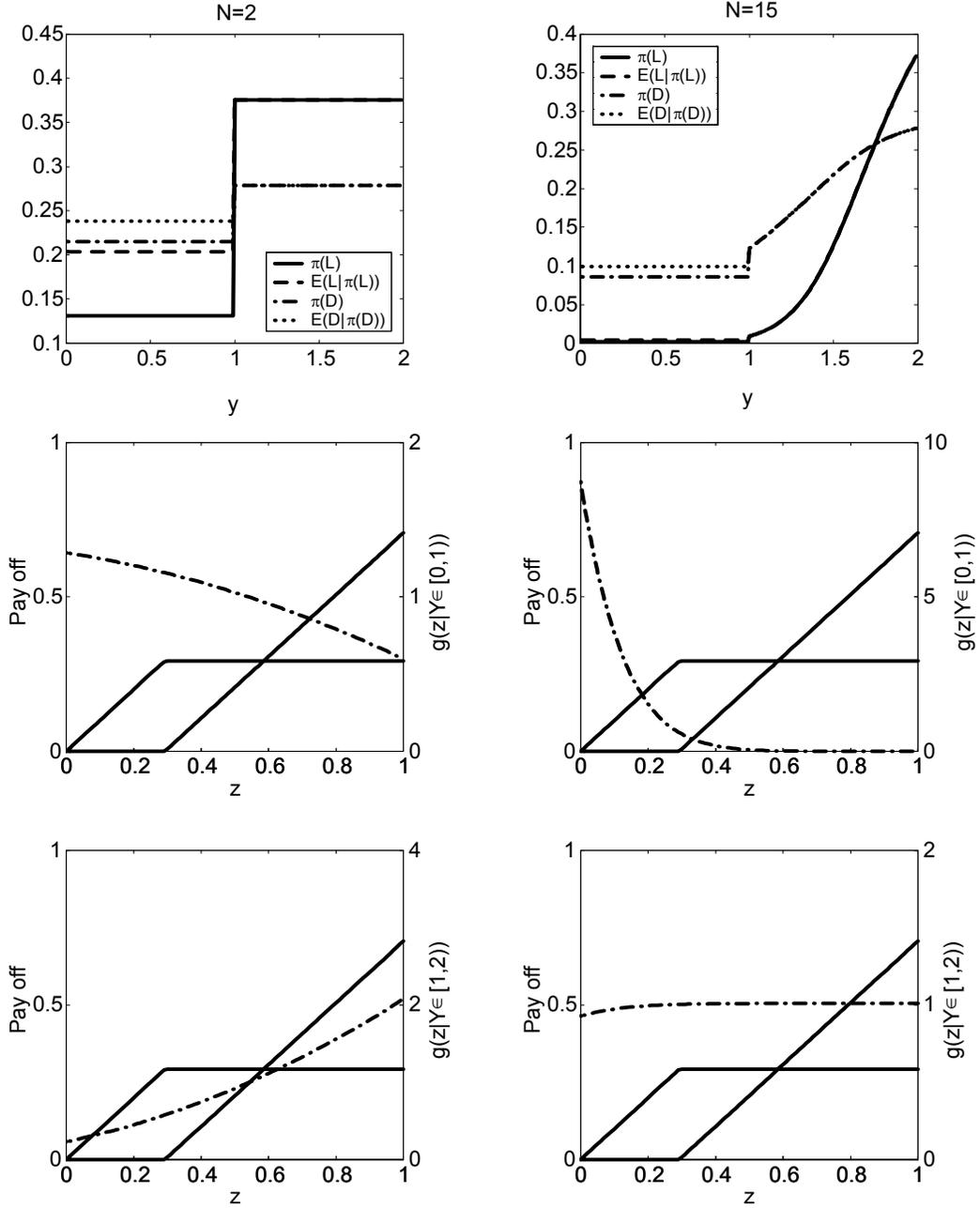


Figure 4.1: The top two panels show prices $\pi(w(Z))$ for levered equity $L(Z)$ and debt $D(Z)$ with face value $1 - 1/\sqrt{2}$ as a function of the price setting signal y . Also, the expected value of the securities conditional on observing a price $E(w(Z)|\pi(w(Z)))$ are shown. The underpricing conditional on the price is given by the difference $E(w(Z)|\pi(w(Z))) - \pi(w(Z))$. The middle two panels show the pay off of debt and equity as a function of the asset value Z (left axis), and the probability density of the asset value $g(z|Y \in [0,1])$ conditional on observing a price setting signal in the lower equivalence interval. The bottom two panels show the security pay offs together with the probability density of the asset value $g(z|Y \in [1,2])$ conditional on observing a price setting signal in the upper equivalence interval.

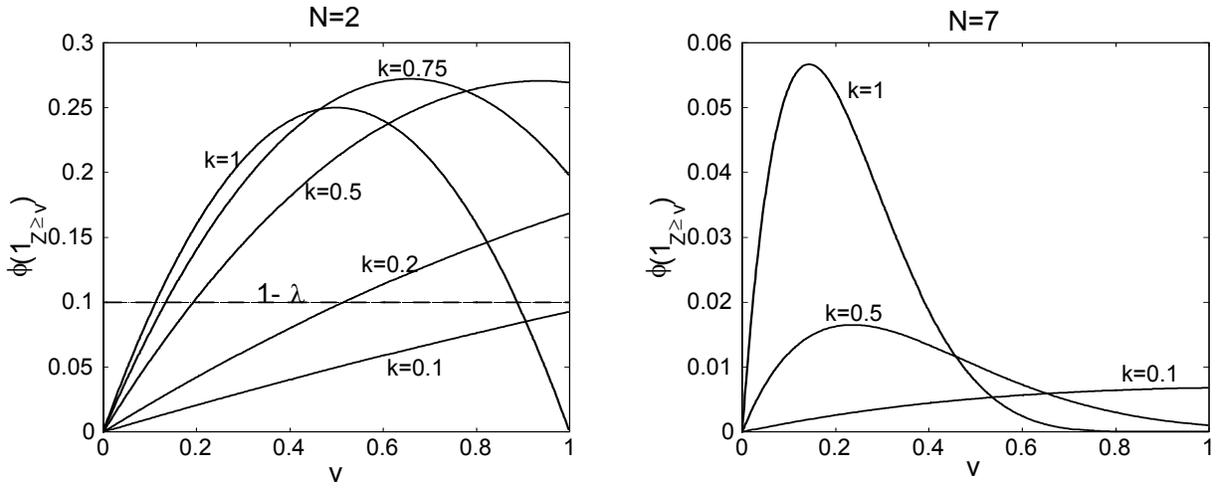


Figure 4.2: Relative underpricing $\phi(1_{Z \geq v})$ for $N = 2$ (left panel) and $N = 7$ (right panel). Assets are uniformly distributed on $[0, 1]$, and signals are given by $P(X = 0 | v) = \frac{1-k}{2} + k(1-v)$, $P(X = 1 | v) = \frac{1-k}{2} + kv$.

probability of observing a low signal if the asset value is above v is below $1 - v$. For a component security $1_{Z \geq v}$ with v close to one, a price setting signal below one therefore signals that the security is almost certainly worthless, and there is no residual information sensitivity.

The optimal security design is easily obtained by aggregating all composite securities such that $\phi(1_{Z \geq v}) \leq 1 - \lambda$. It is straight-forward to show that $\phi(1_{Z \geq v})$ is quasi-concave in v for all values of k and N .²¹ This implies that some combination of debt and levered equity will always be optimal, as $\phi(1_{Z \geq v})$ is either increasing or decreasing over the whole range, or initially increasing and later decreasing. The choice of security design is illustrated in the left panel of Figure 4.2 for $\lambda = .9$. When $k = .1$, it is optimal to sell the whole asset, as $\phi(1_{Z \geq v})$ never crosses $1 - \lambda$. When $k = .2$, the optimal security is debt with face value .5. When $k = 1$, the optimal security is debt with face value .11 and equity that starts paying off at $v = .89$.²²

The right panel of Figure 4.2 shows $\phi(1_{Z \geq v})$ for $N = 7$. This illustrates the effect of increasing the number of bidders. Underpricing in general goes down as the number of bidders increases, since the degree of competition goes up. Consistent with Proposition 4, the decrease in underpricing is

²¹A function $f(x)$ is quasiconcave if the derivative changes sign at most once.

²²In the capital raising problem, the optimal security design is found by finding the $\lambda(C)$ such that the sale of the security made up of the component securities with $\phi(1_{Z \geq v}) \leq 1 - \lambda(C)$ raise an expected amount of capital C . This $\lambda(C)$ will be lower the lower the general level of $\phi(1_{Z \geq v})$. For example, in Figure 4.2, the optimal security design for $\lambda = .1$ when $k = .2$ is debt with face value .5. The same amount of capital can be raised when $k = .1$ by issuing debt with a slightly lower face value, corresponding to a $\lambda(C)$ below .05.

most dramatic for high cash flows. Therefore, levered equity becomes a more dominant component of the security design. For example, for $k = .5$, equity is now part of any optimal security design while debt is always optimal when $N = 2$. The effect is stronger the larger the level of initial information sensitivity as captured by the level of k . For $k = 1$, the underpricing has all but disappeared for all cash flows higher than $.7$. Note that, in fact, the effect is so strong that it is now cheaper to raise low levels of cash for $k = 1$ than for $k = .1$. This is so since underpricing for debt securities - the optimal security when $k = .1$ - disappears slower than underpricing for equity securities. As N is increased even more, equity becomes optimal for low levels of k as well.

5. Security Design: The multiple asset case

The previous analysis assumed that bidders only have private information about one asset. I now turn to the important generalization where the seller is endowed with multiple assets or projects, about each of which bidders can have private information. This introduces an extra dimension to the security design problem in that a seller now has to decide not only on how a security should depend on the underlying cash flow, but also on whether the underlying cash flow should come from a pool of assets or an individual asset.

Suppose the seller has I assets denoted by Z_1, \dots, Z_I to sell. Assets are distributed independently of each other.²³ Each bidder draws a set of signals X_{n1}, \dots, X_{nI} , one for each asset. The security design involves two steps: First, the seller partitions the assets into K pools $\{J_1, \dots, J_K\}$, where $K \in \{1, \dots, I\}$. Denote by $|J_k|$ the number of assets in pool J_k and define a new variable $Z_k \equiv \frac{\sum_{i \in J_k} Z_i}{|J_k|}$. The variable Z_k is the normalized aggregate cash flow in pool J_k . Then, for each pool J_k the seller issues $|J_k|$ identical asset-backed securities $w_k(Z_k)$ where w_k is a function of the normalized aggregate cash flow Z_k and is admissible with respect to Z_k .²⁴ All I securities are then sold in simultaneously run auctions.²⁵

²³The assumption of independence is less restrictive than it seems. The Z_i 's can alternatively be viewed as orthogonal factors that bidders get signals about. An asset can then be a linear combination of these factors, and the pooling choice is whether securities should depend on several factors or not.

²⁴Equivalently, the seller could have issued a single security $w_k\left(\sum_{i \in J_k} Z_i\right)$ that is admissible with respect to the total aggregate cash flow $\sum_{i \in J_k} Z_i$. The bid function $b(x)$ is homogenous of degree 1 in the sense that a bid $b'(x)$ for a security $w' = \frac{w}{k}$ is given by $b'(x) = \frac{b(x)}{k}$, where $b(x)$ is the bid for security w . Hence, the identity of the winning bidder and the revenue raised is unchanged if k simultaneous auctions of identical securities w' are run instead of one auction for the security w .

²⁵The restriction that auctions are simultaneous is important. With sequential sales, bidders might have an incentive to hide their information in earlier stages. The equilibrium analysis becomes much more complicated and is beyond the scope of this paper.

The optimization problem of the seller can now be written as:

$$\max_{\{J_1, \dots, J_K\}} \left\{ \sum_{k=1}^K |J_k| \max_{w_k} \{(1 - \lambda) E(w_k) - E(w_k - \pi(w_k))\} \right\} \quad (5.1)$$

such that w_k is admissible $\forall k$

The outer optimization is the pooling problem, and the inner the security design problem studied in Section 4, with the exception that the security now depends on the value of all assets in the pool. Hence, strategies in the bidding game will depend on the vector of signals $\{X_{ni}\}_{i \in J_k}$. In general, handling multidimensional signals in auction games is highly complicated (see Axelson [3]). For the purposes of this paper, I circumvent these difficulties by using distributional assumptions on the signal space that allow me to represent the signal vector $\{X_{ni}\}_{i \in J_k}$ with a one-dimensional statistic X_n such that the equilibrium described in Proposition 1 is still valid.²⁶ As was shown in Axelson [3], one set of distributional assumptions that allows for this type of aggregation is when assets are i.i.d. and signals about each asset are Bernoulli random variables:

Lemma 3. (Axelson [3]) *Suppose assets are independently identically distributed, and each signal X_{ni} is either 1 or 0 with $P(X_{ni} = 1 | z_i)$ increasing in z_i . Define the variables $\{X_n\}_{n=1}^N$ as $X_n \equiv \sum_{i \in J_k} X_{ni} + S_n$, where S_n is a uniform noise variable distributed on $[0, 1)$. Then, equilibrium in the auction of an admissible security $w(Z_k)$ is given as in Proposition 1, using X_1 as the one dimensional signal observed by bidder one.*

One set of distributional assumptions that satisfy the conditions in Lemma 3 is distribution D1 used in the example above. I will assume for the remainder of this section that the assumptions in Lemma 3 hold.

The pooling problem with no retention and no security design was studied in Axelson [3]. In that paper, it was shown that a pooled sale is optimal if the number of assets is large enough for a fixed number of bidders, while separate sales becomes optimal as the number of bidders grows. The intuition again has to do with minimizing the informational distance. Pooling assets will change the underlying signal distribution in the auction. The signal a seller observes about a pool of assets

²⁶This restriction is not without loss of generality. With a discrete signal space, it is always possible to construct a one-dimensional sufficient statistic X_n for the vector of signals observed by bidders. However, for the equilibrium given in Proposition 1 to be valid using the set of statistics $\{X_1, \dots, X_N\}$ as signals, it is also required that the variables $\{X_1, \dots, X_N, \{Z_i\}_{i \in J_k}\}$ are *affiliated* (see Milgrom and Weber [22]). The existence of statistics that are both sufficient and satisfy affiliation is not generally guaranteed. When such statistics cannot be found, the equilibrium analysis is much more complicated.

can be viewed as the average of the signals he observes about each single asset. Pooling therefore increases the central tendency of the signal distribution, while decreasing the thickness of the tails: Average signals become more common and extreme signals less common. If the number of assets is large enough, and if the assets are not too correlated, this “diversification” effect of pooling will make the informational distance between all bidders very small, since all bidders will be expected to draw average signals. Therefore, the underpricing per asset goes to zero as the number of assets in the pool goes to infinity, holding the number of bidders fixed. However, as the number of bidders grows, the price setting signal is increasingly likely to be drawn from the upper tail of the signal distribution. Since the informational distance is small when the price setting bidder draws a signal from the top of the distribution, a fatter tail leads to lower underpricing. Therefore, individual sales become optimal as the degree of competition grows, holding the number of assets fixed.

Propositions 5 and 6 below show that these results continue to hold when retention and security design is allowed on top of the pooling decision, but that seller utility can be improved through the security design. Furthermore, the optimal security design will be intimately linked to the optimal pooling decision.

Proposition 5. *Suppose the seller has a capital raising requirement of $C < E(Z_i)$ per asset. Then, as the number of assets I goes to infinity, the optimal security design is debt with face value $F = C$ backed by the normalized pool cash flow $Z = \frac{\sum_{i=1}^I Z_i}{I}$.*

The result in Proposition 5 can be understood as follows. As the number of assets in the pool grows, the normalized cash flow Z becomes increasingly centered around the expected value $E(Z_i)$. Therefore, the riskiness of debt with face value $F < E(Z_i)$ goes to zero, as does the information sensitivity. Pooling thus gives an extra kick to the immunization strategy. The expected value of such a debt security goes to D , as the debt becomes less and less risky. Hence, the seller can raise arbitrarily close to the maximum (ex ante) expected capital $E(Z)$ by issuing debt with face value close to $E(Z)$.

Levered equity, on the other hand, becomes increasingly undesirable relative to debt as the number of assets in the pool grows. The sensitization strategy does not work well with a signal distribution that is concentrated in the middle and has thin tails, since price setting signals below the top do not signal particularly unfavorable information. The higher unconditional information sensitivity of levered equity therefore translates into higher residual information sensitivity and higher underpricing for most price setting signals.

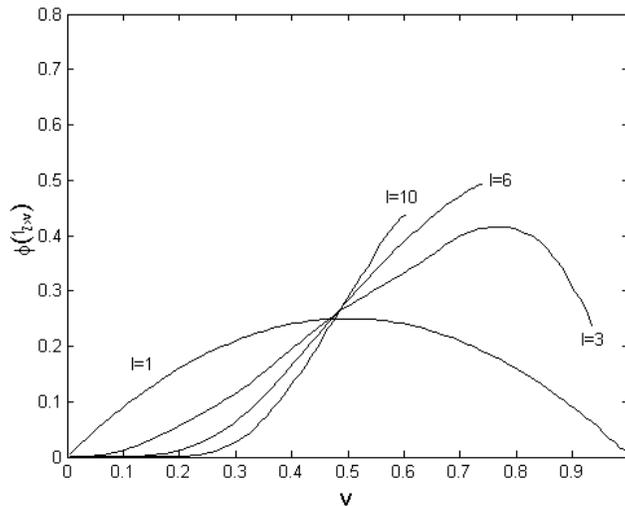


Figure 5.1: The figure shows relative underpricing $\phi(1_{Z \geq v})$ for pools $Z = \frac{\sum_{i=1}^I Z_i}{I}$ containing 1, 3, 6, and 10 assets. Signals and assets are distributed according to D1 with $N = 2$ and $k = 1$. The graph was produced using numerical simulations. As the number of assets in the pool increases, debt with face value less than .5 becomes cheaper and cheaper to issue, while equity becomes more and more expensive to issue.

This illustrates that in general, security design should go to one of two extremes - either make the security as informationally insensitive as possible, or make it as informationally sensitive as possible. Pooling pushes in the first direction by making the overall cash flow less information sensitive. Debt helps to enhance this effect. Equity, however, works in the opposite direction, so that the net effect is at neither extreme. Equity works best if each asset is sold individually. Therefore, we may very well have a situation where equity is optimal for a seller who has access to only one asset, while debt is optimal for a seller with several assets to pool. An illustration of how having access to a larger pool of assets affects the relative underpricing is given in Figure 5.1. For a single asset, some combination of debt and equity is optimal. As the number of assets grows, the cost of issuing debt goes down, while the cost of issuing equity goes up. For a fixed face value $F < E(Z_i)$ on the debt security, the seller raises more and more capital as the number of assets in the pool backing the debt increases. This is partly because the normalized underpricing goes down (as is illustrated in Figure 5.1), and partly because the expected value of the debt security goes up as default becomes less and less probable.

The analysis above shows that pooling gives extra leverage to the immunization strategy. However, as was shown in the single asset case, the sensitization strategy tends to become more impor-

tant when the degree of competition increases. This will also be true in the multiple asset case, and will lead to a preference for selling equity backed by single assets, as is shown in the following proposition.

Proposition 6. *As the number of bidders goes to infinity, equity backed by individual assets becomes the optimal security design.*

The results in Propositions 5 and 6 show that the optimal security design depends on the number of assets a seller has access to relative to the degree of competition in the market. When the number of assets is large relative to the degree of competition, the immunization strategy works best, leading to debt backed by pooling as the optimal security design. When the degree of competition is large relative to the number of assets, the sensitization strategy works best, leading to equity backed by individual assets as the optimal design. Note that Propositions 5 and 6 are limiting results, where first the number of assets and then the degree of competition is taken to the extreme. I have not given a full characterization of intermediate cases. The exact relation between the number of assets and the number of bidders will depend on the specifics of the signal distribution. For example, it is easy to find examples where separate sales dominates pooling even if there are only two bidders. However, the intuition developed above suggests that even in intermediate cases, the pooling decision and the security design decision should be linked.

6. Extensions

6.1. Non-monotonic security design

In this section, I show that the monotonicity requirement imposed throughout is not innocuous, and relaxing it can lead to substantial decreases in underpricing. I do this within the context of an example. The idea is the following: With monotonically increasing securities, any security which is not risk free will feature some expected underpricing. The immunization strategy can never be made completely effective, since even debt securities will have some information sensitivity left. However, if a non-monotonic security is introduced, it may be possible to make it completely insensitive to information even if it is not risk free. For example, imagine a security which is increasing in the value of the asset for low cash flow realizations, but decreasing for higher cash flow realizations. A bidder observing a high signal might have the same expectation about the value of such a security as a bidder observing a low signal. If that is the case, the information in their signal is less valuable,

and I show below that it is sometimes possible to construct a non-monotonic security which has *zero* underpricing.

Suppose there is one asset and that signals follow distribution D1 with parameter $k = 1$. Thus, there are two possible signal outcomes, $X = 1$ and $X = 0$, that occur with probability $P(1|Z = z) = z$ and $P(0|Z = z) = 1 - z$, and z is distributed uniformly between 0 and 1. Suppose we want to construct a security $w_1(Z)$ such that the expected value of the security is the same regardless of what signal you observe:

$$E(w_1(Z)|X = 1) = E(w_1(Z)|X = 0)$$

There are many ways of doing this, but one way is to set $w_1(Z) = Z$ for $Z \leq \bar{z}$, and $w_1(Z) = 0$ for $Z > \bar{z}$, where \bar{z} solves

$$\int_0^{\bar{z}} zg(z|X = 1) dz = \int_0^{\bar{z}} zg(z|X = 0) dz \quad (6.1)$$

Using the distributional assumptions above, you can show that the solution is $\bar{z} = \frac{3}{4}$. Notice that the expected value conditional on observing a signal is just the unconditional expectation of the security, since we have

$$\begin{aligned} & E(w_1(Z)) \\ &= P(X = 1) E(w_1(Z)|X = 1) + (1 - P(X = 1)) E(w_1(Z)|X = 0) \\ &= E(w_1(Z)|X = 1) \end{aligned}$$

Therefore, the security has zero unconditional information insensitive, since observing a signal does not change the valuation. I now claim that it is an equilibrium in the auction of $w_1(Z)$ for all bidders to bid the unconditional expectation, so that there is no expected underpricing. That this is an equilibrium is clear from the fact that conditional on winning, no further information about the security is gained, so that the expectation stays at $E(w_1(Z)|X = x) = E(w_1(Z))$. Deviating by bidding higher or lower gives zero expected profits: If you bid higher, you always win the auction, and so will pay $E(w_1(Z))$ which is your valuation. If you bid lower, you never win the auction.

Therefore, you can sell security $w_1(Z)$ without any underpricing, generating a certain revenue of $E(w_1(Z)) = \frac{9}{32}$, which is about 56% of the expected value of the underlying asset. For any capital raising requirement below this level, the seller captures the entire surplus, and is obviously

better off than in the case where he is restricted to issuing only monotonic securities. It is relatively straight-forward to show that for any capital raising requirement (or any liquidity parameter λ), the seller is strictly better off if allowed to issue non-monotonic securities relative to the case where only monotonic securities are allowed. Typically, he will issue at least one non-monotonic security similar to the one described above, plus possibly a “residual” monotonic security.

One implication of the above analysis is that when non-monotonicity is allowed, it is no longer the case that the seller is as well off when issuing a single security backed by a certain asset as when issuing multiple securities, a result that was established in Lemma 1. The irrelevance result in Lemma 1 hinged crucially on the assumption that a higher signal translates into a higher bid for each security issued. This no longer holds for non-monotonic securities, as the previous example showed. In general, the seller is made better off when issuing multiple securities. For example, even in the case of no retention, it is easy to see that seller revenues can be improved in the example above if he splits the cash flow of the asset over two securities $w_1(Z)$ (defined above) and $w_2(Z) = Z - w_1(Z)$.²⁷

Thus, non-monotonic securities, when they are allowed, are likely to be part of an optimal security design. However, it should be noted that the form of the optimal non-monotonic security design is typically more sensitive to assumptions about the underlying signal distribution than is the case for monotonic security designs. For example, to solve for the right cut off value \bar{z} in equation 6.1 above, the seller needs to know the exact shape of the posterior distributions given certain signal realizations. Even a small error will upset the equilibrium and can lead to lower utility than the simple alternative of selling the whole asset. This may be one of the reasons why such securities are not commonly observed.

²⁷The analysis above is made simple by the fact that there are only two possible signal realizations. Without this assumption, it is not generally possible to construct risky securities that feature zero underpricing. However, it is always possible to improve somewhat on the immunization strategy by issuing a non-monotonic security that pools some of the signals, thereby reducing the information sensitivity. A general proof of this result for the case of two bidders with an arbitrary number of signals is available from the author upon request. Also, although a full treatment is beyond the scope of this paper, it is possible to utilize the sensitization strategy more fully with non-monotonic securities. For example, by issuing securities that pay off only when the value of the underlying cash flow is below some low cut-off level $Z = z_L$, it may be possible to completely reverse the ranking of signals so that bidders observing lower signals submit higher bids for the security. In that case, the winner’s signal is revealed if the price setting signal is in the low range rather than the high range, so that residual information sensitivity over the low range is less costly. But such a security has its residual information sensitivity concentrated exactly over the low range of price setting signals, as it pays off only when the underlying asset value is low. As the number of bidders grows, it may therefore be utility improving to issue such a security in addition to a levered equity security.

6.2. Endogenous Information Acquisition

In this section, I endogenize the collection of information by introducing a cost θ of observing the signal vector X and solve for the equilibrium number of participants in the auction. I show that, for the particular setup I study, all major results still hold.

If information acquisition is done before the security design stage, the previous analysis remains intact since the security design does not affect the information acquisition. The interesting situation is when potential bidders can react to the choice of security design. I propose the following game-form:

1. The seller chooses the security design, which is observed by all potential bidders
2. Bidders decide whether to pay θ to get to observe the signal vector. I assume that they decide sequentially (in some arbitrary order) and that everyone can observe the choice of the previous players.²⁸
3. Bidders decide whether to participate in the auctions or not.
4. The auctions are run.

An informed bidder always has a positive expected surplus, so he always participates. When there are no informed bidders in the auction, everyone bids the ex ante expected value and there is no underpricing. When there are some informed bidders in the auction, all the uninformed bid as if they received the lowest possible signal, and receive zero expected profits in equilibrium. The informed ignore the uninformed and bid according to Proposition 1 where N is the number of informed in the auction. Since the expected surplus of an uninformed bidder in an auction is zero unless he is alone in the auction, I assume that all uninformed bidders stay out. If there is only one informed bidder, he gets the asset for free, and so the underpricing is $E(w(Z))$. I also assume that $\theta < E(w(Z))$, so that there will always be at least one informed bidder in the auction.

Given a security design $\omega \equiv \{w_k\}_{k=1}^K$, denote the expected underpricing by

$$U(\omega, N) \equiv \sum_{k=1}^K |J_k| E(w_k - \pi(w_k))$$

²⁸With sequential choices, we get the simple and intuitive outcome that the total acquisition cost should equal (or be just below) the expected surplus in the auctions. This outcome would also be an equilibrium if we assumed a simultaneous choice by participants, but in that case there would also be a range of equilibria in mixed strategies for which the analysis becomes harder. The qualitative features of the solution would not change.

where the dependence on the number of participating bidders N is made explicit. It is then an equilibrium for the N first bidders to acquire information and no one else where N is the smallest solution to

$$\begin{aligned} N\theta &\leq U(\omega, N) \\ (N+1)\theta &> U(\omega, N+1) \end{aligned} \tag{6.2}$$

Thus, the aggregate information acquisition cost should just be compensated by the expected surplus given to the bidders. The equilibrium number of bidders is unique when the expected revenue $E(\pi(w_k))$ is (weakly) increasing in N , which is the typical case. This follows since $U(\omega, N)$ is then decreasing in N , which ensures that there is a unique solution to 6.2. Denote this solution by $N(\theta, \omega)$. The sellers problem 2.2 can now be restated as

$$\begin{aligned} \min_{\{w_k\}_{k=1}^K \text{ admissible}} U(\omega, N(\theta, \omega)) \\ \text{such that } \sum_{k=1}^K |J_k| E(\pi(w_k)) \geq C \end{aligned} \tag{6.3}$$

The goal in this section is to show that all results in the paper still go through under this specification, with a re-interpretation of N in terms of θ . To show this, I make the assumption that expected revenues are increasing in the number of bidders for any admissible security. This is the typical case, although I do not supply a formal proof. Given this assumption, the reasoning is as follows: If we solve the security design problem above for all values of θ , there will be an equilibrium number of bidders $N^*(\theta)$ at the optimal security design. Suppose instead we are handed this function $N^*(\theta)$ at the outset, and are asked to solve the security design problem for a given value of θ . Then we get the right solution by treating $N^*(\theta)$ as the exogenously specified number of bidders (that is, treating N as not changing with the security design.)²⁹ Thus, Propositions 2 and 3 hold with $N = N^*(\theta)$ taken as exogenous. Also, by letting θ go to zero, we get the same security design results as in Propositions 4 and 6: As θ goes to zero, $N^*(\theta)$ goes to infinity, and equity backed by individual assets becomes the optimal security design.

²⁹As is shown in the proof of Proposition 7, there is a slight modification in the capital raising requirement when we treat $N^*(\theta)$ as exogenous: We now have the following capital raising constraint:

$$E(\pi) = \max(C, E(\omega) - N^*(\theta)\theta)$$

The extra term is a consequence of the discrete number of bidders.

To analyze the situation where the number of assets grows, assume that θ is the cost *per asset* of observing the signal vector. In Proposition 5, I let the number of assets grow and showed that debt backed by a pool of assets was the optimal security design, and that the underpricing per asset goes to zero. With endogenous information acquisition, a similar result holds, although with the caveat that the underpricing per asset must not be driven below 2θ . If the expected underpricing per asset for auctions with more than one informed bidder goes to zero, only one bidder will decide to get informed. This will result in the *maximal* underpricing $U(\omega, 1) = E(\omega)$. Hence, the seller must assure that at least two bidders participate, or that the underpricing per asset is equal to at least 2θ . For situations where the number of assets is not too large and the information acquisition cost θ is not too big, this lower bound on the underpricing is unlikely to be met. The seller should then minimize underpricing by issuing debt backed by the entire pool. However, as the number of assets grows very large, this will no longer be an optimal security design. The seller must issue a security that features enough information sensitivity to warrant information collection by two investors. He can do that by reducing the size of the pools, and by changing the shape of the security to make it more information sensitive. Although the optimal security design will involve some pooling in the limit, the actual shape of the security design and the size of the pool backing each security will be indeterminate. I gather these results in the following proposition.

Proposition 7. *Assume $\pi(w_k)$ is increasing in N for all admissible w_k . Then, $N^*(\theta)$ is decreasing in θ and $\lim_{\theta \downarrow 0} N^*(\theta) = \infty$. The solution to the security design problem taking $N^*(\theta)$ as exogenous for a given θ and using the capital raising condition $E(\pi) = \max(C, N^*(\theta)\theta)$ is the optimal security design. Propositions 2 and 3 hold with this modification. Propositions 4 and 6 hold when replacing $\lim_{N \rightarrow \infty}$ with $\lim_{\theta \rightarrow 0}$.*

Assume the acquisition cost θ is fixed per asset. Then, as the number of assets I goes to infinity, the optimal security design is to issue pooled securities such that the underpricing per asset is 2θ . The size of the pools will be finite, and the security design is indeterminate.

7. Conclusion

I have developed a theory of security design in which a seller chooses the form of the securities to minimize private information rents of potential investors. In the single asset case, I show that debt securities are often optimal when the number of bidders is small. As the degree of competition increases, levered equity becomes optimal. In the multiple asset case, I show that debt backed by

pooling is an optimal security design when the number of assets is large relative to the number of bidders, while equity backed by individual assets becomes optimal as the degree of competition increases. As is shown in an extension where information acquisition is endogenous, varying the degree of competition in the auction can alternatively be interpreted as varying the cost of acquiring information.

The results of the model are broadly consistent with observed patterns of security design. For example, the prediction that a large asset base should be sold through debt issues backed by a pool is consistent with securitization patterns in the asset-backed securities markets, while equity issues are more prevalent when the asset base is more focused (as is the case for individual firms).

The theory developed here may also be applied to analyze a variety of other phenomena. For example, the results have implications for the life cycle patterns in the financing of a firm. A feature of the theory is that it can provide an explanation for both debt and equity within the same framework, and provides conditions under which we would expect to observe one rather than the other. In the life cycle of a firm we might expect the degree of competition among investors and the availability of information about the company to be smaller at early stages, leading to a preference for debt securities. As the company grows older, information is more widely accessible, and more investors have their eyes on the company. At this stage it is possible for the company to issue equity through an IPO.

Similarly, it is possible to explain why different financial systems seem to favor different types of securities. In a bank oriented system with little capital market competition for the financing of companies, debt should be more prevalent. In a stock market based system, information acquisition is cheaper and the pool of investors larger, which makes it easier to issue equity. Improved information technology and increased investor participation in stock markets would also lead to an increased propensity of firms to issue equity at an earlier stage of the company life cycle. The increased fraction of young firms in stock markets, both in the US and in Europe, may be a reflection of this development, as is the decreased fraction of large diversified conglomerates.

Finally, the theory suggest a role for financial intermediation. As was shown in Proposition 5, having access to a large number of assets can reduce the underpricing per asset by allowing the seller to construct pooled securities. If a seller has access to only a single asset, there is room for a financial intermediary to step in and synthesize pooling by bringing together assets of many sellers and finance them through a pooled issue.³⁰ When information acquisition costs are high, we

³⁰If it is infeasible to find a sufficiently large number of assets at any single point in time for this type of pooling to

would expect such intermediation to arise endogenously. Similarly, the theory would predict that as information acquisition costs in the economy go down, we would expect to see *dis*intermediation as single asset sales become more beneficial. A more detailed investigation of these issues is left for future research.

be beneficial, it may be possible for an intermediary to engage in intertemporal pooling. For example, an underwriter who is involved in a large number of IPOs over time may be able to synthesize pooling by giving preferential treatment to investors who take allocations in each IPO, making it harder for investors to “cherry pick” (see Benveniste and Spindt [5] for a theoretical IPO model with this flavor and Cornelli and Goldreich [8] for empirical evidence).

APPENDICES

A. Derivation of underpricing in Example 4.1

This appendix delineates the calculations behind the example develop in Section 4.1. The expected value of debt and equity are calculated as

$$\begin{aligned} E(D(Z)) &= \int_0^F z dz + \int_F^1 F dz \\ E(L(Z)) &= \int_F^1 (z - F) dz \end{aligned}$$

Setting these equal gives $F = 1 - \frac{1}{\sqrt{2}}$.

The unconditional information sensitivity for equity is calculated as

$$\begin{aligned} &\frac{E(L(Z) | X \geq 1) - E(L(Z) | X < 1)}{E(L(Z))} \\ &= \frac{\int_F^1 (z - F) \left(\frac{1}{2} + z\right) dz - \int_F^1 (z - F) \left(\frac{1}{2} - (1 - z)\right) dz}{\frac{1}{4}} \approx .53 \end{aligned}$$

where $g(z | X \geq 1) = \frac{P(X \geq 1 | z)}{P(X \geq 1)} = \frac{1}{2} + z$ and $g(z | X < 1) = \frac{P(X < 1 | z)}{P(X < 1)} = \frac{1}{2} - (1 - z)$ are used. The unconditional information sensitivity for debt is calculated similarly.

The price for a security given a price setting signal is calculated using Equation 3.2. The expected value conditional on observing the price is calculated as

$$\begin{aligned} E(w(Z) | \pi(w)) &= E(w(Z) | X \geq y^{-1}(\pi(w)), Y_1 = y^{-1}(\pi(w))) \\ &= \int_0^1 w(Z) \frac{(1 - F(y|z)) f(y|z) F(y|z)^{N-2}}{\int_0^1 (1 - F(y|u)) f(y|u) F(y|u)^{N-2} du} dz \end{aligned}$$

for the cases where the inverse $y = y^{-1}(\pi(w))$ exists, which is when $N > 2$ and the price setting signal is in the top equivalence interval. For cases when the inverse does not exist, the price will only reveal what equivalence interval $[a, a + 1)$ the price signal came from. In that case, the expectation is calculated as

$$\begin{aligned} E(w(Z) | \pi(w)) &= E(w(Z) | X \geq a, Y_1 \in [a, a + 1)) \\ &= \int_0^1 w(Z) \frac{(1 - F(a|z)) P(Y_1 \in [a, a + 1) | z)}{\int_0^1 (1 - F(a|u)) P(Y_1 \in [a, a + 1) | u) du} dz \end{aligned}$$

The cost function $\phi(1_{Z \geq v})$ is given by

$$\phi(1_{Z \geq v}) = \int_0^1 \phi(1_{Z \geq v} | y) h(y) dy$$

(This follows since $\phi(1_{Z \geq v} | y) = 0$ for $y \geq 1$). We have that

$$\begin{aligned} \phi(1_{Z \geq v} | y) &= \frac{P(Z \geq v | X \geq y, Y_1 = y) - P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v)} \\ &= \frac{\int_v^1 \frac{h(y|z)g(z)}{h(y)} dz - \int_v^1 \frac{f(y|z)^2 F(y|z)^{N-2} g(y)}{\int_0^1 f(y|u) F(y|u)^{N-2} g(u) du} dz}{1 - G(v)} \end{aligned}$$

so that $\phi(1_{Z \geq v})$ becomes

$$\phi(1_{Z \geq v}) = \frac{1}{1-v} \int_0^1 \left[\int_v^1 h(y|z) dz - \int_v^1 \frac{f(y|z)^2 F(y|z)^{N-2}}{\int_0^1 f(y|u) F(y|u)^{N-2} du} \int_0^1 h(y|z) dz \right] dy$$

where I use $g(z) = 1$, $G(v) = v$, and $h(y) = \int_0^1 h(y|z) g(z) dz$. Note that we have, for $y < 1$,

$$\begin{aligned} f(y|z)^2 F(y|z)^{N-2} &= P(0|z)^2 (yP(0|z))^{N-2} \\ h(y|z) &= N(N-1) P(0|z) (1 - yP(0|z)) (yP(0|z))^{N-2} \end{aligned}$$

This follows from $f(y|z) = P(0|z)$ and $F(y|z) = yP(0|z)$ for $y < 1$. Plugging this into the expression for $\phi(1_{Z \geq v})$ and noting that all integrals over z are versions of $\int_v^1 P(0|z)^N dz$, where we have

$$\begin{aligned} \int_v^1 P(0|z)^N dz &= \int_v^1 \left(\frac{1-k}{2} + k(1-z) \right)^{N-1} dz \\ &= \frac{1}{k} \int_{\frac{1-k}{2}}^{P(0|v)} u^{N-1} du \\ &= \frac{1}{k} \left[\frac{P(0|v)^N}{N} - \frac{\left(\frac{1-k}{2}\right)^N}{N} \right] \end{aligned}$$

gives the result in 4.3 after integration over y and straight-forward manipulation.

B. Proofs

The following lemma, which is practically a restatement of Theorem 5 of Milgrom and Weber [22], is used repeatedly below.

Lemma 4. *For any monotonically increasing security $w(Z)$ with $w(1) > w(0)$, $E(w|X = x, Y_1 = y)$ has the following properties:*

- 1) *It is increasing in x and y*
- 2) *It is constant in x over an equivalence interval $x \in [a, a + 1)$ for some integer a*
- 3) *For $x > x'$ and x not in the same equivalence interval as x' , $E(w|X = x, Y_1 = y) > E(w|X = x', Y_1 = y)$*

Proof. 1) and 3) follow from Theorem 5 of Milgrom and Weber [22]. To show 2), note that $E(w(Z)|X = x, Y_1 = y)$ can be derived as

$$\begin{aligned}
 E(w(Z)|X = x, Y_1 = y) &= \int_0^1 w(z) g(z|X = x, Y_1 = y) dz \\
 &= \int_0^1 w(z) \frac{f(x|z) f(Y_1 = y|z) g(z)}{\int_0^1 f(x|u) f(Y_1 = y|u) g(u) du} dz \\
 &= \int_0^1 w(z) \frac{f(x|z) f(y|z) F(y|z)^{N-2} g(z)}{\int_0^1 f(x|u) f(y|u) F(y|u)^{N-2} g(u) du} dz
 \end{aligned}$$

The first step follows from Bayes' law. I use the notation $f(Y_1 = y|z)$ to denote the density for the highest signal among $N-1$ bidders. This density is given by $f(Y_1 = y|z) = (N-1) f(y|z) F(y|z)^{N-2}$. Plugging this in gives the second step. But $f(x|z)$ is constant over an equivalence interval, which proves 2). ■

Proof of Proposition 1:

Equilibrium for the second price auction was characterized by Milgrom [20], and uniqueness of this equilibrium was proved by Pesendorfer and Swinkels [25]. The derivation of Equation 3.2 was done in the proof of Lemma 4, which also shows that $b(x)$ is increasing in x . ■

Proof of Lemma 1:

For any monotonically increasing security $w(Z)$, the bid $b(x) = E(w(Z)|X = x, Y_1 = x)$ will be an increasing function of the signal x (Proposition 1). Hence, the identity of the winner and the price setter in each auction will be the same. Denoting by $b_{w_i}(x)$ the bid function for security $w_i(Z)$ given a signal x we have

$$\begin{aligned}
b_w(x) &= E(w(Z) | X = x, Y_1 = x) = E\left(\sum_{k=1}^K w_k(Z) | X = x, Y_1 = x\right) \\
&= \sum_{k=1}^K E(w_k(Z) | X = x, Y_1 = x) = \sum_{k=1}^K b_{w_k(Z)}(x)
\end{aligned}$$

The price in the auction is given by the bid of the second highest bidder, and since he has the same identity in each auction, we have that $\pi(w(Z)) = \sum_{k=1}^K \pi(w_k(Z))$, so revenues are unchanged. ■

Proof of Proposition 2:

Expression 4.1 is proved in the text. The expected underpricing in Expression 4.2 can be derived as follows. For any security, conditional on observing a price $\pi(w) = E(w | X = y, Y_1 = y)$, the seller's expected underpricing is given by

$$E(w | \pi(w)) - \pi(w) = E(w | X \geq y, Y_1 = y) - E(w | X = y, Y_1 = y)$$

The expression for $\pi(w)$ follows from Proposition 1. The seller's expected value of the security is $E(w | X \geq y, Y_1 = y)$, since the seller can infer from observing the price that the second highest signal must have been y , and can also infer that the winner must have had a signal which is at least as high as y . Expression 4.2 then follows by integrating over outcomes for the price setting (second highest) signal, and from the fact that $E(1_{Z \geq v} | A) = P(Z \geq v | A)$ for some conditioning event A . ■

Proof of Lemma 2:

That $r(w(Z) | y) \geq \phi(w(Z) | y)$ follows since $E(w | X = x, Y_1 = y)$ is increasing in x (Lemma 4). This proves both that $r(w(Z) | y) = 0$ implies $\phi(w(Z) | y) = 0$ and that $\phi(w(Z) | y)$ is bounded above by $r(w(Z) | y)$. That $y \in [H - 1, H)$ implies $\phi(w(Z) | y) = 0$ follows from Lemma 4.

It remains to show that unless conditions 1) and 2) in the lemma hold, $\phi(w(Z) | y) > 0$. But from Lemma 4, $E(w(Z) | X = H, Y_1 = y)$ is strictly larger than $E(w(Z) | X = y, Y_1 = y)$ for $y < H - 1$ if $w(Z)$ is monotonically increasing and non-constant. Therefore, $E(w(Z) | X \geq y, Y_1 = y)$ is strictly larger than $E(w(Z) | X = y, Y_1 = y)$ unless $w(Z)$ is constant. But if $w(Z)$ is constant, the residual information sensitivity is also zero. ■

Proof of Proposition 3:

That $\phi(1_{Z \geq 0}) = 0$ is obvious. Also, for $v > 0$, it follows from Lemma 4 that $\phi(1_{Z \geq v})$ is strictly positive. If $\phi(1_{Z \geq v})$ is continuous in v , it follows that for any $\lambda < 1$, there exists a \bar{v} such that $\phi(1_{Z \geq v}) \leq 1 - \lambda$ for all $v \in [0, \bar{v}]$, so that all component securities with a cut-off level below \bar{v} should be included in the optimal security design. That $\phi(1_{Z \geq v})$ is continuous follows since

$P(Z \geq v | X = x, Y_1 = y) = \int_v^1 g(z | X = x, Y_1 = y) dz$, and hence $\phi(1_{Z \geq v} | y)$, is continuous in v . ■

Proof of Proposition 4:

Denote by Y the price setting (second highest) signal, with associated density $h(y)$. Since $\phi(1_{Z \geq v} | y) = 0$ for $y \geq H - 1$ (Lemma 2), I have that

$$\begin{aligned} \phi(1_{Z \geq v}) &= \int_{y=0}^{H-1} \frac{P(Z \geq v | X \geq y, Y_1 = y) - P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v)} h(y) dy \\ &= \int_{y=0}^{H-1} \frac{P(Z \geq v | X \geq y, Y_1 = y) - P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v | Y = y)} \frac{P(Z \geq v | Y = y)}{P(Z \geq v)} h(y) dy \\ &= \int_{y=0}^{H-1} \left(1 - \frac{P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v | Y = y)} \right) h(y | Z \geq v) dy \end{aligned}$$

The first equality follows from expression 4.2, and the third from Bayes' law.

Note that $\int_{y=0}^{H-1} h(y | Z \geq v) dy = P(Y \leq H - 1 | Z \geq v)$. Therefore, if I show that

$$\lim_{N \rightarrow \infty} \frac{P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v | Y = y)} < 1 - \varepsilon$$

for some strictly positive ε , we have that $P(Y \leq H - 1 | Z \geq v) > \phi(1_{Z \geq v}) > \varepsilon P(Y \leq H - 1 | Z \geq v)$

and hence $\phi(1_{Z \geq v})$ goes to zero with $P(Y \leq H - 1 | z \geq v)$. I show this as follows:

$$\begin{aligned} \frac{P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v | Y = y)} &= \frac{\int_v^1 g(z | X = y, Y_1 = y) dz}{\int_v^1 g(z | Y = y) dz} \\ &= \frac{\int_v^1 \frac{f(X=y, Y_1=y|z)g(z)}{f(X=y, Y_1=y)} dz}{\int_v^1 \frac{h(y|z)g(z)}{h(y)} dz} \\ &= \frac{\int_v^1 \frac{f(y|z)^2 F(y|z)^{N-2}}{\int_0^1 f(y|u)^2 F(y|u)^{N-2} g(u) du} g(z) dz}{\int_v^1 \frac{(1-F(y|z))f(y|z)F(y|z)^{N-2}}{\int_0^1 (1-F(y|u))f(y|u)F(y|u)^{N-2} g(u) du} g(z) dz} \\ &= \frac{A(y, v, N)}{A(y, 0, N)} \end{aligned}$$

where $A(y, v, N)$ is given by

$$A(y, v, N) \equiv \frac{\int_v^1 f(y|z)^2 F(y|z)^{N-2} g(z) dz}{\int_v^1 (1 - F(y|z)) f(y|z) F(y|z)^{N-2} g(z) dz}$$

Divide numerator and denominator of $A(y, v, N)$ by $F(y|v)^{N-2}$ and take the limit as $N \rightarrow \infty$ to get

$$\begin{aligned} \lim_{N \rightarrow \infty} A(y, v, N) &= \lim_{N \rightarrow \infty} \frac{\int_v^1 f(y|z)^2 \left(\frac{F(y|z)}{F(y|v)}\right)^{N-2} g(z) dz}{\int_v^1 (1 - F(y|z)) f(y|z) \left(\frac{F(y|z)}{F(y|v)}\right)^{N-2} g(z) dz} \\ &= \frac{f(y|v)}{1 - F(y|v)} \end{aligned}$$

This follows since $F(y|v)$ is strictly decreasing in v from MLRP. From the non-degeneracy assumption, $f(y|v)$ and $g(v)$ are bounded away from zero. Therefore, as $N \rightarrow \infty$, we have that

$$\frac{P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v | Y = y)} \rightarrow \frac{1 - F(y|0)}{f(y|0)} \frac{f(y|v)}{1 - F(y|v)}$$

The hazard rate $\frac{1-F(y|z)}{f(y|z)}$ is given by

$$\frac{1 - F(y|z)}{f(y|z)} = \int_y^H \frac{f(x|z)}{f(y|z)} dx$$

For $y < H - 1$, this is strictly increasing in z from the assumption of MLRP. Hence, we have that

$$\frac{1 - F(y|0)}{f(y|0)} \frac{f(y|v)}{1 - F(y|v)} < 1 \quad \forall v > 0$$

Hence, $\phi(1_{Z \geq v})$ goes to zero with $P(Y \leq H - 1 | Z \geq v)$.

I now derive the rate at which $P(Y \leq H - 1 | z \geq v)$ goes to zero. We can write $P(Y \leq H - 1 | Z \geq v)$ as

$$\begin{aligned} P(Y \leq H - 1 | Z \geq v) &= \int_v^1 P(Y \leq H - 1 | z) \frac{g(z)}{1 - G(v)} dz \\ &= \int_v^1 \left(F(H - 1 | z)^N + N(1 - F(H - 1 | z)) F(H - 1 | z)^{N-1} \right) \frac{g(z)}{1 - G(v)} dz \end{aligned}$$

Again, note that $F(x|z)$ is decreasing in z . Dividing the above by $F(H - 1 | v + \varepsilon)^{N-1}$ for some $\varepsilon > 0$ the integral goes to infinity as N goes to infinity. Dividing by $F(H - 1 | v - \varepsilon)^{N-1}$, the integral goes to zero. Since ε was arbitrary, we must have that $\phi(1_{Z \geq v})$ goes to zero at a rate proportional to $F(H - 1 | v)^{N-1}$. (Note that I have not pinpointed the exact convergence rate;

that is, a function $c(N)$ such that $\lim_{N \rightarrow \infty} \frac{P(Y \leq H-1 | Z \geq v)}{c(N)} = C > 0$ for some strictly positive constant C . All I need is that $\phi(1_{Z \geq v})$ goes to zero faster for a higher v). Since $F(H-1|v)^{N-1}$ is decreasing in v , we have that $\frac{\phi(1_{Z \geq v})}{\phi(1_{Z \geq v'})}$ goes to infinity with N for $v' > v > 0$. Hence, for any $\underline{v} > 0$, $\phi(1_{Z \geq v})$ becomes decreasing for all $v > \underline{v}$ as $N \rightarrow \infty$, and equity dominates except for a vanishingly small range around $v = 0$. ■

Proof of Proposition 5:

Divide the pool of assets $\sum_{i=1}^I Z_i$ into \sqrt{I} identical securities $Z = \frac{\sum_{i=1}^I Z_i}{\sqrt{I}}$. Asymptotically, the distribution of $Z - \sqrt{I}E(Z_i)$ is normal with zero mean and standard deviation $\sigma^2(Z_i)$. It is straight forward to show that the renormalized sum of signals $\frac{\sum_{i=1}^I X_{in}}{\sqrt{I}} - E\left(\frac{\sum_{i=1}^I X_{in}}{\sqrt{I}}\right)$ also reaches a non-degenerate asymptotic normal distribution, both unconditionally and conditional on Z . As I goes to infinity, the auction of a security $Z = \frac{\sum_{i=1}^I Z_i}{\sqrt{I}}$ will therefore be associated with a certain fixed amount of underpricing $U > 0$. When selling the whole asset pool, the underpricing therefore grows at rate \sqrt{I} with I (since the whole pool of assets consists of \sqrt{I} such securities with fixed underpricing). The underpricing per asset in the pool goes to zero (see Axelson [3]).

Imagine selling a debt claim backed by such an asset Z with face value $F = \sqrt{I}(E(Z_i) - \varepsilon)$ for some $\varepsilon > 0$. The cash flow to debt is given by $\min(Z, F)$. Since $P(Z < F) \rightarrow 0$ as I goes to infinity, the cash flow to debt goes to a degenerate random variable with expected value F and variance zero, a virtually risk free debt claim. In the limit, the expected underpricing in the auction of such a claim is therefore zero. This implies that the expected underpricing for the residual equity claim $\max(Z - F, 0)$ is equal to U . Thus, all underpricing is concentrated on the equity part. Also, note that per asset, the amount of capital raised through the issuance of these debt securities goes to $E(Z_i) - \varepsilon$, which can be made arbitrarily close to the upper bound $E(Z_i)$ by making ε small. Hence, for any capital raising requirement, debt strictly dominates. ■

Proof of Proposition 6:

The beginning of this proof is identical to the proof of Proposition 4 except for the derivation of the limit of $\frac{P(Z \geq v | X=y, Y_1=y)}{P(Z \geq v | Y=y)}$. I now have to condition on a vector of asset realizations $\{z_i\}$ instead of a scalar z , where $\{Z_i\} \equiv \{Z_1, \dots, Z_I\}$ is the vector of I assets that make up the pool $Z = \sum Z_i$. We now have that

$$\frac{P(Z \geq v | X = y, Y_1 = y)}{P(Z \geq v | Y = y)} = \frac{\int_v^1 g(z | X = y, Y_1 = y) dz}{\int_v^1 g(z | Y = y) dz}$$

$$\begin{aligned}
&= \frac{\int_{\{z_i\}: \sum z_i \geq v} \frac{f(y|\{z_i\})^2 F(y|\{z_i\})^{N-2} g(\{z_i\})}{\int_{\{u_i\}} f(y|\{u_i\})^2 F(y|\{u_i\})^{N-2} g(\{u_i\}) d\{u_i\}} d\{z_i\}}{\int_{\{z_i\}: \sum z_i \geq v} \frac{(1-F(y|\{z_i\})) f(y|\{z_i\}) F(y|\{z_i\})^{N-2} g(\{z_i\})}{\int_{\{u_i\}} (1-F(y|\{u_i\})) f(y|\{u_i\}) F(y|\{u_i\})^{N-2} g(\{u_i\}) d\{u_i\}} d\{z_i\}} \\
&= \frac{A(y, v, N)}{A(y, 0, N)}
\end{aligned}$$

where $A(y, v, N)$ is now defined as

$$A(y, v, N) \equiv \frac{\int_{\{z_i\}: \sum z_i \geq v} f(y|\{z_i\})^2 F(y|\{z_i\})^{N-2} g(\{z_i\}) d\{z_i\}}{\int_{\{z_i\}: \sum z_i \geq v} (1-F(y|\{z_i\})) f(y|\{z_i\}) F(y|\{z_i\})^{N-2} g(\{z_i\}) d\{z_i\}}$$

Here, $\int_{\{z_i\}: \sum z_i \geq v}$ denotes a multiple integration over all asset vectors that sum up to at least v .³¹ Define the function $F_{\max}(y, v)$ as $F_{\max}(y, v) \equiv \arg \max_{\{z_i\}: \sum z_i = v} F(y|\{z_i\})$, with associated ‘‘density’’ $f_{\max}(y, v)$ defined as $f_{\max}(y, v) \equiv f(y|\arg \max(F_{\max}(y, v)))$. Since $F(y|\{z_i\})$ is decreasing in each z_i , $F_{\max}(y, v)$ is strictly decreasing in v . Also, from the non-degeneracy assumption, $g(\{z_i\})$ and $f(y|\{z_i\})$ are bounded away from zero. Divide numerator and denominator of the expression above by $F_{\max}(y, v)^{N-2}$ and take the limit as $N \rightarrow \infty$ to get:

$$\begin{aligned}
&\lim_{N \rightarrow \infty} A(y, v, N) = \\
&\lim_{N \rightarrow \infty} \frac{\int_{\{z_i\}: \sum z_i \geq v} f(y|\{z_i\})^2 \left(\frac{F(y|\{z_i\})}{F_{\max}(y, v)}\right)^{N-2} g(\{z_i\}) d\{z_i\}}{\int_{\{z_i\}: \sum z_i \geq v} f(y|\{z_i\}) (1-F(y|\{z_i\})) \left(\frac{F(y|\{z_i\})}{F_{\max}(y, v)}\right)^{N-2} g(\{z_i\}) d\{z_i\}} \\
&= \frac{f_{\max}(y, v)}{1 - F_{\max}(y, v)}
\end{aligned}$$

Therefore, as $N \rightarrow \infty$, we have that $\frac{P(Z \geq v | X=y, Y_1=y)}{P(Z \geq v | Y=y)} \rightarrow \frac{1-F(y|0)}{f(y|0)} \frac{f_{\max}(y, v)}{1-F_{\max}(y, v)}$. Since the hazard rate $\frac{1-F(y|\{z_i\})}{f(y|\{z_i\})}$ is strictly increasing in each z_i for $y < H - 1$, and since $\arg \max_{\{z_i\}: \sum z_i = v} F(y|\{z_i\}) > \{0\}$ for $v > 0$, this is strictly smaller than one for all $v > 0$. Hence, $\phi(1_{Z \geq v})$ again goes to zero with $P(Y \leq H - 1 | Z \geq v)$.

³¹I cannot write $h(y|z) = N(N-1)f(y|z)(1-F(y|z))F(y|z)^{N-2}$ anymore, as it is no longer necessarily true that the aggregate signal X for bidder n is independent of signal X' of bidder m conditional on the sum of values $Z = \sum Z_i$: $f(X=x, X'=x' | Z=z) \neq f(x|z)f(x'|z)$. Only if I condition on the whole vector $\{z_i\}$ does conditional independence hold.

I can now write $P(Y \leq H - 1 | Z \geq v)$ as

$$\begin{aligned}
& P(Y \leq H - 1 | Z \geq v) \\
&= \int_v^1 P(Y \leq H - 1 | Z = z) \frac{g(z)}{1 - G(v)} dz \\
&= \int_{\{z_i\}: \sum z_i \geq v}^1 \left(F(H - 1 | \{z_i\})^N + N f(H - 1) F(H - 1 | \{z_i\})^{N-1} \right) \frac{g(\{z_i\})}{1 - G(v)} dz
\end{aligned}$$

Again, note that $F(x | \{z_i\})$ is decreasing in each z_i . Dividing the above by $F_{\max}(H - 1, v + \varepsilon)^N$ for some $\varepsilon > 0$, the integral goes to infinity as N goes to infinity. Dividing by $F_{\max}(H - 1, v - \varepsilon)^N$, the integral goes to zero. Since ε was arbitrary, we must have that $\phi(1_{Z \geq v})$ goes to zero at a rate proportional to $F_{\max}(H - 1, v)^N$. Since $F_{\max}(H - 1, v)^N$ is strictly decreasing in v , we again have that $\frac{\phi(1_{Z \geq v})}{\phi(1_{Z \geq v'})}$ goes to infinity with N for $v' > v > 0$, and hence equity dominates just like in the single asset case (except for a vanishingly small range around $v = 0$). The rate of convergence of equity that starts paying off at v is proportional to $F_{\max}(H - 1, v)^N$. Note that the smaller $F_{\max}(H - 1, v)$, the faster the convergence. Therefore, the larger $f_{\max}(H, v)$, the faster the convergence.

I now show that if the seller needs to raise a certain amount of capital C per asset, individual sales dominate pooling as N goes to infinity. I show it for the case of a pool of two asset $Z = \frac{Z_1 + Z_2}{2}$ versus individual sales. The proof generalizes easily to the case of more than two assets. To raise C per asset when selling the pool Z , the optimal security in the limit as $N \rightarrow \infty$ is levered equity $L(Z) = \max(0, Z - v')$ with cut off value v' such that

$$\int_{v'}^1 (z - v') g\left(\frac{Z_1 + Z_2}{2} = z\right) dz = C \tag{B.1}$$

Similarly, for individual sale of asset Z_1 (and symmetrically for Z_2), the optimal security is equity with cut off value v such that

$$\int_v^1 (z_1 - v) g(z_1) dz_1 = C \tag{B.2}$$

We know that the underpricing of the pooled asset goes to zero at rate proportional to $f(H | \{\hat{z}_1, \hat{z}_2\}) = P(X_{n1} = 1 | Z_1 = \hat{z}_1) P(X_{n2} = 1 | Z_2 = \hat{z}_2)$ where $\{\hat{z}_1, \hat{z}_2\}$ minimize $f(H | \{z_1, z_2\})$ subject to $\frac{z_1 + z_2}{2} = v'$, while the underpricing for the individual asset goes to zero at rate $P(X_{n1} = 1 | Z_1 = v)$. If z_1 in $f(H | \{z_1, z_2\})$ can be set lower or equal to v while still having $\frac{z_1 + z_2}{2} = v'$, the result fol-

lows immediately since $P(X_{n1} = 1 | Z_1 = z_1)$ is increasing in z_1 . This is so since in that case we have $P(X_{n1} = 1 | Z_1 = \hat{z}_1) P(X_{n2} = 1 | Z_2 = \hat{z}_2) \leq P(X_{n1} = 1 | Z_1 = z_1) P(X_{n2} = 1 | Z_2 = z_2) < P(X_{n1} = 1 | Z_1 = v)$, so the underpricing for the individual asset goes to zero faster. For this not to hold, it is therefore necessary that $v' > \frac{v+1}{2}$. I show that this is not possible while preserving the capital raising conditions B.1 and B.2. Equivalently, I show that for $v' = \frac{v+1}{2}$, the expected value of the capital raised in the pool is smaller than the expected value of the capital raised in the individual sale, or that for all v , we have

$$\int_{\frac{v+1}{2}}^1 \left(z - \frac{v+1}{2} \right) g \left(\frac{Z_1 + Z_2}{2} = z \right) dz \leq \int_v^1 (z_1 - v) g(z_1) dz_1 \quad (\text{B.3})$$

To show this, rewrite the convolution density $g \left(\frac{Z_1 + Z_2}{2} = z \right)$ which, for the relevant case $z > \frac{1}{2}$, becomes

$$g \left(\frac{Z_1 + Z_2}{2} = z \right) = 2 \int_{2z-1}^1 g(z_1) g(2z - z_1) dz_1$$

Using Fubini's theorem and a change of variables, I can then rewrite the left-hand side of B.3 as

$$\begin{aligned} & \int_{\frac{v+1}{2}}^1 \left(z - \frac{v+1}{2} \right) \left(2 \int_{2z-1}^1 g(z_1) g(2z - z_1) dz_1 \right) dz \\ &= \int_v^1 \int_{\frac{v+1}{2}}^{\frac{z_1+1}{2}} 2 \left(z - \frac{v+1}{2} \right) g(2z - z_1) dz g(z_1) dz_1 \\ &= \int_v^1 \int_{v+1-z_1}^1 \frac{1}{2} (u - 1 + z_1 - v) g(u) du g(z_1) dz_1 \end{aligned}$$

so that B.3 becomes

$$\int_v^1 \left(\int_{v+1-z_1}^1 \frac{1}{2} (u - 1 + z_1 - v) g(u) du - (z_1 - v) \right) g(z_1) dz_1 \leq 0 \quad (\text{B.4})$$

But this always holds, since the expression in parenthesis is strictly negative for $v < 1$:

$$\begin{aligned} \int_{v+1-z_1}^1 \frac{1}{2} (u - 1 + z_1 - v) g(u) du &< \int_{v+1-z_1}^1 \frac{1}{2} (z_1 - v) g(u) du \\ &\leq (z_1 - v) \end{aligned}$$

The first relation follows from $u \leq 1$. This concludes the proof. ■

Proof of Proposition 7:

Denote the optimal solution to 6.3 by ω^* (with corresponding number of bidders $N^*(\theta)$). First, I show that

$$N^*(\theta)\theta \leq U(\omega^*, N^*(\theta)) < (N^*(\theta) + 1)\theta \quad (\text{B.5})$$

That $N^*(\theta)\theta \leq U(\omega^*, N^*(\theta))$ follows from inequality 6.2. Suppose $U(\omega^*, N^*(\theta)) \geq (N^*(\theta) + 1)\theta$, contrary to the claim in B.5. From 6.2 and the fact that $U(\omega, N)$ is decreasing in N , we also have that $(N^*(\theta) + 1)\theta > U(\omega^*, N^*(\theta) + 1)$ and that $U(\omega^*, N^*(\theta) + 1) < U(\omega^*, N^*(\theta))$. Note that, unless $\omega^* = Z$, we can always increase the amount of capital raised and the underpricing continuously by adding on more cash flow (more component securities) to the security design. Therefore, there exists a security design ω' that raises more capital than ω^* , and that features underpricing $U(\omega', N^*(\theta) + 1) = (N^*(\theta) + 1)\theta$. But then, it is better to issue security ω' , since $N(\omega', \theta) = (N^*(\theta) + 1)\theta \geq U(\omega^*, N^*(\theta))$. This contradicts the optimality of ω^* , and hence B.5 must hold. Expression B.5 also implies that the optimal security design minimizes $N^*(\theta)$ subject to the capital raising restriction.

Next, I show that $N^*(\theta)$ is decreasing in θ . Suppose ω_1 is the solution to 6.3 for $\theta = \theta_1$. This implies that $N(\theta_1, \omega)$ is minimized at ω_1 subject to raising enough capital, since $N(\theta, \omega)\theta \leq U(\omega, N(\theta, \omega)) < (N(\theta, \omega) + 1)\theta$. Now suppose that θ is decreased to $\theta = \theta_2 < \theta_1$. Suppose that ω_2 is the solution to 6.3 for $\theta = \theta_2$. Now suppose that $N(\theta_2, \omega_2) < N(\theta_1, \omega_1)$, contrary to the claim in the Proposition. But then, since $U(\omega, N)$ is decreasing in N , we must have $N(\theta_1, \omega_2) \leq N(\theta_2, \omega_2)$ to satisfy 6.2. But then $N(\theta_1, \omega_2) \leq N(\theta_2, \omega_2) < N(\theta_1, \omega_1)$, which contradicts that $N(\theta_1, \omega)$ is minimized at ω_1 .

Now suppose, contrary to the claim in the Proposition, that $N^*(\theta)$ does not go to infinity as θ goes to zero, but instead goes to some constant N_1 . We have that for any $C > 0$, and for any ω , $U(\omega, N_1) > \varepsilon$ for some $\varepsilon > 0$ (that is, if the security design is not completely risk free, and there is a limited number of bidders, there must be at least some underpricing). From 6.2, in the limit we must have $(N_1 + 1)\theta > U(\omega, N_1) > \varepsilon$. But since θ goes to zero, this cannot hold. Hence, $\lim_{\theta \downarrow 0} N^*(\theta) = \infty$.

I now show that the solution to the security design problem taking $N^*(\theta)$ as exogenous for a given θ is the optimal security design, with the modification that the capital raising constraint is $E(\pi(\omega)) = \max(C, E(\omega) - N^*(\theta)\theta)$. First, note that if $C < E(\omega) - N^*(\theta)\theta$, any security design

that raises an amount of capital $E(\pi(\omega)) \in [C, E(\omega) - N^*(\theta)\theta]$ will not satisfy the participation constraint in 6.2, since $U(\omega, N^*(\theta)) = E(\omega) - E(\pi(\omega)) < N^*(\theta)\theta$, which contradicts that $N^*(\theta)$ was the optimal solution.

Call the solution to the problem with this modified capital raising constraint ω_1 . Thus, ω_1 minimizes $U(\omega, N^*(\theta))$ subject to the capital raising constraint and admissibility, so that $U(\omega_1, N^*(\theta)) \leq U(\omega, N^*(\theta))$ for all ω . Now suppose ω_1 is not the solution to 6.3, so that there is an ω_2 with $U(\omega_2, N(\theta, \omega_2)) < U(\omega_1, N(\theta, \omega_1))$ and $N(\theta, \omega_2) = N^*(\theta) \leq N(\theta, \omega_1)$. But since $U(\omega_1, N)$ is decreasing in N , we then have $U(\omega_1, N(\theta, \omega_1)) \leq U(\omega_1, N^*(\theta)) \leq U(\omega_2, N^*(\theta))$, which is a contradiction. Hence, ω_1 must be the solution to 6.3.

This shows that Propositions 2 and 3 hold with N replaced by $N^*(\theta)$ and with the modified capital raising restriction. That 4 and 6 hold with the modification that we let θ go to zero instead of letting N go to infinity also follows, since for $N^*(\theta) \rightarrow \infty$ the optimal solution given any capital raising condition is equity backed by single assets.

Last, redefine the acquisition cost θ as being *per asset* and let the number of assets grow. Proposition 5 showed that for any fixed number of bidders $N \geq 2$, the underpricing per asset can be made arbitrarily small by issuing debt backed by the pool of assets. Note that, for any fixed θ , this will violate the participation constraint in 6.2. When the underpricing per asset goes below 2θ , there will be only one informed bidder, who will capture the entire surplus $E(\omega)$. Hence, the best the seller can do is drive the underpricing per asset down to 2θ . This will in general involve some amount of pooling, but not to the full degree. Also, the shape of the security backed by the pool of assets is indeterminate. It is easy to see that by an appropriate choice of number of pools backing each security and the security design, there is a continuum of security designs that raise C per asset subject to featuring the right amount of underpricing. ■

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