

# **Should Exact Index Numbers have Standard Errors?**

## ***Integrating the Stochastic and Economic Approaches***

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#### Abstract

In this paper we derive the standard error of a price index when both prices and tastes are treated as stochastic. Changing tastes (or technology) are a reason for the *weights* in the price index to be treated as stochastic, which can interact with the stochastic price themselves. Our rationale for using stochastic tastes comes from the economic approach to index numbers, whereby an *exact* price index reflects the ratio of expenditure needed to obtain a fixed level of utility at two different prices. This ratio of expenditures depends on the tastes of the consumer, and if the taste parameters are treated as stochastic, then so is the exact index number. We derive results for the constant elasticity of substitution expenditure function (with Satio-Vartia price index), and also the translog function (with Törnqvist price index), which proves to be more general and easier to implement.

## 1. Introduction

In the stochastic approach to index numbers prices are viewed as draws from some distribution, and the price index is viewed as a measure the trend change in prices, with an estimable standard error. The most comprehensive treatment of this problem is by Selvanathan and Rao (1994), but the idea dates back to Keynes (1909) and earlier writers. Keynes points out that the price changes should be modeled as including both a common trend (i.e. inflation) and a commodity-specific component,<sup>1</sup> which makes the trend difficult to identify. Selvanathan and Rao (1994, pp. 61-67) solve this problem using purely statistical techniques, as we describe in section 2, and the standard error of their price index reflects the precision of the estimate of the trend. Keynes does not offer a solution, but elsewhere he observes that changes in the purchasing power of money (and therefore, the trend and commodity-specific component of price changes) can occur for three distinct reasons: “The first of these reasons we may classify as a change in tastes, the second as a change in environment, and the third as a change in relative prices.”<sup>2</sup>

The first factor identified by Keynes – changing tastes – can be expected to affect the weights in a price index and not just the prices. Accordingly, we will derive the standard error of a price index when both prices and tastes are treated as stochastic. The rationale for our treatment of stochastic tastes (or technology) comes from the economic approach to index numbers, (e.g. Diewert, 1976), which shows that certain price indexes, known as *exact* indexes, equal the ratio of expenditure needed to obtain a fixed level of utility at two different prices.

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<sup>1</sup> In Keynes (1909) essay on “Index Numbers,” section VIII deals with the “Measurement of General Exchange Value by Probabilities,” which is the stochastic approach. He writes: “We may regard price changes, therefore, as partly due to causes arising from the commodities themselves raising some, lowering others, and all different in degree, and, superimposed upon the changes due to these heterogeneous causes, a further change *affecting all in the same ratio* arising out of change on the side of money. This uniform ratio is the object of our investigations.” (from *The Collected Writing of John Maynard Keynes*, volume XI, p. 106).

<sup>2</sup> Keynes (1930), cited from *The Collected Writing of John Maynard Keynes*, volume V, p. 85.

This ratio of expenditures depends on the tastes of the consumer, so if the taste parameters are stochastic, then the exact index number is also. By allowing for *both* random prices and random tastes, we are integrating the stochastic and economic approaches to index numbers.

We first deal with the case of a constant elasticity of substitution (CES) expenditure function (for a consumer) or cost function (for a firm). In section 3 we suppose that the taste parameters are random, and obtain a simple specification for demand that depends on these random parameters and on prices. We show how the estimated error from this demand equation can be used to infer the standard error of the exact price index. Inverting the demand equation, we also obtain a simple specification in which price changes depend on a trend (the price index) and on a commodity-specific component (demand shares). The CES case therefore provides a good comparison to the specification used by Selvanathan and Rao (1994). In section 4 we extend our treatment of the CES case to deal with both random prices and random tastes.

The method used in the CES case involves estimating the corresponding demand system, and using the error in this regression to infer the standard error of the exact price index. This methodology applies equally well to other functional forms for the expenditure or cost function, and in sections 5 and 6 we consider the translog case, first with random tastes and then also with random prices. The demand equations estimated are the familiar translog share equations, and again, the error in this system of regressions is used to infer the standard error of the exact price index. The fact that the translog system is linear in taste parameters makes it computationally easier to implement than the non-linear CES system, and we recommend this system for future use. The benefit of the CES system is that it provides a particularly clear comparison with the conventional stochastic approach.

In Feenstra and Reinsdorf (2003) we provide an application of our results to Asian productivity growth, and in particular, productivity growth in Singapore. Young (1992, 1995) argued that the growth in some East Asian countries is mainly due to capital accumulation rather than productivity, and in fact, found negative productivity growth in Singapore. Recently, however, Hsieh (2002) has re-examined the productivity performance of several East Asia countries using *dual* measures of total factor productivity, and for Singapore finds positive productivity growth, contrary to Young. We compare these differing estimates of productivity growth, and find that for any single year in the sample, TFP growth in Singapore from Hsieh's data is insignificantly different from zero. The same holds true when estimating cumulative TFP growth over any five-year or ten-year period of the sample. For the 15-year period, however, we find that cumulative TFP growth in Singapore is significantly positive. Thus, the estimates of Hsieh (2002) are indeed statistically different from those of Young (1992, 1995), provided that cumulative TFP over a long enough time period is considered.

## 2. The Stochastic Approach to Index Numbers

An example of the stochastic approach to price indexes is a model where price changes satisfy the equation:

$$\ln(p_{it}/p_{it-1}) = \pi_t + e_{it}, \quad i=1, \dots, N, \quad (1)$$

where the errors are independent and heteroskedastic satisfying  $E(e_{it}) = 0$  and  $\text{var}(e_{it}) = \sigma^2/w_i$ ,

where  $w_i$  are some exogenous values that sum to unity,  $\sum_{i=1}^N w_i = 1$ . Under these conditions, an

unbiased and efficient estimate of trend change in prices  $\pi_t$  is,

$$\hat{\pi}_t = \sum_{i=1}^N w_i \ln(p_{it}/p_{it-1}), \quad (2)$$

which can be obtained by running weighted least squares (WLS) on (1) with the weights  $\sqrt{w_i}$ .

An unbiased estimate for the variance of  $\hat{\pi}_t$  is,

$$s_{\pi}^2 = \frac{s_p^2}{N-1}, \quad (3)$$

where  $s_p^2 = \sum_{i=1}^N w_i (\Delta \ln p_{it} - \hat{\pi}_t)^2$ .

Diewert (1995) criticizes the stochastic approach and argues that: (a) the common trend  $\pi_t$  in (1) is limiting; (b) the variance assumption  $\text{var}(e_{it}) = \sigma^2/w_i$  is unrealistic; (c) some choices of  $w_i$  (such as budget shares) will not be exogenous. In assessing these criticisms, we believe that a distinction should be made between *lower-level* and *higher-level* aggregation. At the lowest level of aggregation (e.g. chocolate bars across various stores), the assumption of a common trend  $\pi_t$  will often be realistic, and the expenditure shares needed to compute the weights  $w_i$  may be unavailable so that  $w_{it} = 1/N$  is used. In that case, the stochastic approach in (1)-(3) amounts to using a simple average of the log-change in prices, and its variance. At the lowest level of aggregation, this is a convenient way to form price averages used at the next stage, in a higher-level price index (e.g. categories of food).

At this higher-level, however, Diewert's (1995) criticisms become more apt, and the pricing equation in (2) is unrealistic. At a minimum, we should follow the observation of Keynes and allow for commodity-specific changes in prices. Selvanathan and Rao (1994, pp. 61-67) proceed by extending an equation like (1) to include commodity-specific effects  $\delta_i$ :

$$\ln(p_{it}/p_{it-1}) = \pi_t + \delta_i + e_{it}, \quad i=1, \dots, N, \quad (1')$$

where again we assume that the errors are independent and satisfy  $E(e_{it}) = 0$  and  $\text{var}(e_{it}) = \sigma^2/w_i$ , with  $\sum_{i=1}^N w_i = 1$ . In the absence of some additional assumptions on  $\delta_i$ , the trend change in prices  $\pi_t$  are not identified, so let us assume:

$$\sum_{i=1}^N w_i \delta_i = 0. \quad (4)$$

Selvanathan and Rao derive the maximum likelihood estimate of  $\pi_t$  and  $\delta_i$  under the constraint (4), and show that the estimate of  $\pi_t$  is still given by (2). This follows intuitively because when weighting (1') by  $\sqrt{w_i}$ , we see from (4) that the commodity-specific terms  $\delta_i \sqrt{w_i}$  are *orthogonal* to the trend  $\pi_t \sqrt{w_i}$ ; therefore, the estimate of the trend is unaffected by whether these commodity effects are included or not. The standard error of  $\hat{\pi}_t$  when the commodity effects are used is somewhat less than that in (3), since the residual error of the pricing equation is reduced.

### 3. The Exact Index for the CES Function

#### A. The Sato-Vartia Index

Missing from the stochastic approach is an economic justification for the pricing equation in (1) or (1'), as well as the constraint in (4). It turns out that this can be obtained by using a CES utility or production function, which is given by,

$$f(\mathbf{x}_t, \mathbf{a}_t) = \left( \sum_{i=1}^N a_{it} x_{it}^{(\eta-1)/\eta} \right)^{\frac{\eta}{(\eta-1)}},$$

where  $\mathbf{x}_t = (x_{1t}, \dots, x_{Nt})$  is the vector of quantities, and  $\mathbf{a}_t = (a_{1t}, \dots, a_{Nt})$  are technology or taste parameters that we will allow to vary over time, as described below. The elasticity of substitution  $\eta > 0$  is assumed to be constant.

We will assume that the quantities  $\mathbf{x}_t$  are optimally chosen to minimize  $\sum_{i=1}^N p_{it} x_{it}$ , subject to achieving  $f(\mathbf{x}_t, \mathbf{a}_t) = 1$ . The solution to this optimization problem gives us the corresponding unit-cost function ,

$$c(\mathbf{p}_t, \mathbf{b}_t) = \left( \sum_{i=1}^N b_{it} p_{it}^{1-\eta} \right)^{\frac{1}{(1-\eta)}}, \quad (5)$$

where  $\mathbf{p}_t = (p_{1t}, \dots, p_{Nt})$  is the vector of prices, and  $b_{it} \equiv a_{it}^\eta > 0$ , with  $\mathbf{b}_t = (b_{1t}, \dots, b_{Nt})$ .

Differentiating (5) provides the expenditure shares  $s_{it}$  implied by the taste parameters  $\mathbf{b}_t$ :

$$s_{it} = \partial \ln c(\mathbf{p}_t, \mathbf{b}_t) / \partial \ln p_{it} = c(\mathbf{p}_t, \mathbf{b}_t)^{\eta-1} b_{it} p_{it}^{1-\eta}. \quad (6)$$

Diewert (1976) defines a price index formula whose weights are functions of the expenditure shares  $s_{it-1}$  and  $s_{it}$  as *exact* if it equals the ratio of unit-costs. For the CES unit-cost function with *constant*  $\mathbf{b}_t$ , the price index due to Sato (1976) and Vartia (1976) has this property. The Sato-Vartia price index equals the geometric mean of the price ratios with weights  $w_i$ :

$$\prod_{i=1}^N \left( \frac{p_{it}}{p_{it-1}} \right)^{w_i} = \frac{c(\mathbf{p}_t, \mathbf{b}_t)}{c(\mathbf{p}_{t-1}, \mathbf{b}_t)}, \quad (7)$$



where the weights  $w_i$  are defined as:

$$w_i = \frac{(s_{it} - s_{it-1}) / (\ln s_{it} - \ln s_{it-1})}{\sum_{j=1}^N (s_{it} - s_{it-1}) / (\ln s_{it} - \ln s_{it-1})} . \quad (8)$$

Notice that these weights are proportional to  $(s_{it} - s_{it-1}) / (\ln s_{it} - \ln s_{it-1})$ , which is the logarithmic means of  $s_{it}$  and  $s_{it-1}$ , and are normalized to sum to unity.<sup>3</sup> Provided that the expenditure shares are computed with *constant* taste parameters  $\mathbf{b}_t$ , then the Sato-Vartia index on the left of (8) equals the ratio of unit-costs on the right, also computed with constant  $\mathbf{b}_t$ .

### ***B. Effect of Random Preferences***

As discussed in the introduction, we want to allow for random technology or taste parameters  $\mathbf{b}_{it}$ , and derive the standard error of the exact price index due to this uncertainty. First, we must generalize the concept of an exact price index to allow for the case where the parameters  $\mathbf{b}_t$  change over time, as follows:<sup>4</sup>

#### **Proposition 1**

*Given  $\mathbf{b}_{t-1} \neq \mathbf{b}_t$ , let  $s_{i\tau}$  denote the optimally chosen shares as in (6) for these taste parameters,  $\tau = t-1, t$ , and define  $\bar{b}_{i\tau} \equiv b_{i\tau} / \prod_{i=1}^N b_{i\tau}^{w_i}$ , using the weights  $w_i$  computed as in (8). Then there exists  $\tilde{b}_i$  between  $\bar{b}_{it-1}$  and  $\bar{b}_{it}$  such that,*

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<sup>3</sup> The logarithmic mean of the expenditure shares in the two periods approximately equals an average of their arithmetic and geometric means with a 2/3 weight on the geometric mean. If  $s_{it-1}$  equals  $s_{it}$ , then the logarithmic means is defined as  $s_{it}$ .

<sup>4</sup> The proofs of all Propositions are in an Appendix, at: <http://www.econ.ucdavis.edu/faculty/fzfeens/papers.html>.

$$\prod_{i=1}^N \left( \frac{p_{it}}{p_{it-1}} \right)^{w_i} = \frac{c(\mathbf{p}_t, \tilde{\mathbf{b}})}{c(\mathbf{p}_{t-1}, \tilde{\mathbf{b}})} . \quad (9)$$

This result shows that the Sato-Vartia index equals the ratio of unit-costs evaluated with parameters  $\tilde{b}_i$  that lie *between* the normalized values of the  $b_{it}$  in each period. Therefore, the Sato-Vartia index is exact for this *particular* value  $\tilde{\mathbf{b}}$  of the parameter vector, but the index will change as the random parameters  $\mathbf{b}_{t-1}$  and  $\mathbf{b}_t$  change. The standard error of the index should reflect this variation. The following proposition shows how the variance of the log Sato-Vartia index is related to the variance of the  $\ln b_{it}$ , denoted by  $\sigma_{\beta}^2$ :

**Proposition 2**

*Suppose that  $\ln b_{i\tau}$  are independently and identically distributed with variance  $\sigma_{\beta}^2$  for  $i = 1, \dots, N$  and  $\tau = t-1$  or  $t$ . Using the weights  $w_i$  as in (8), denote the log Sato-Vartia index by  $\pi_{sv} \equiv \sum_{i=1}^N w_i \Delta \ln p_{it}$ . Then conditional on prices, its variance can be approximated as:*

$$\text{var } \pi_{sv} \approx \frac{1}{2} \sigma_{\beta}^2 \sum_{i=1}^N w_i^2 (\Delta \ln p_{it} - \pi_{sv})^2 = \frac{1}{2} \sigma_{\beta}^2 s_p^2 \overline{\overline{w}} , \quad (10)$$

*where  $\overline{\overline{w}} \equiv \sum_{i=1}^N w_i^2 (\Delta \ln p_{it} - \pi_{sv})^2 / \sum_{i=1}^N w_i (\Delta \ln p_{it} - \pi_{sv})^2$  is a weighted average of the  $w_i$  with the weights proportional to  $w_i (\Delta \ln p_{it} - \pi_{sv})^2$ , and  $s_p^2 \equiv \sum_{i=1}^N w_i (\Delta \ln p_{it} - \pi_{sv})^2$  is the weighted variance of prices.*

Equation (10) shows how the variance of the Sato-Vartia index reflects the underlying randomness of the taste parameters  $\mathbf{b}_t$  and  $\mathbf{b}_{t-1}$  and is *conditional on* the observed prices. In effect, we are computing the variance in the price index from the randomness in the weights  $w_i$  in (8) rather than from the randomness in prices used in the conventional stochastic approach.

To compare (10) to (3), note that  $\bar{w}$  equals  $1/N$  when the weights  $w_i$  also equal  $1/N$ . In that case, we see that the main difference between our formula (10) for the variance of the price index and formula (3) used in the conventional stochastic approach is the presence of the term  $\frac{1}{2}\sigma_\beta^2$ . This term reflects the extent to which we are uncertain about the parameters of the underlying CES function that the exact price index is intended to measure. It is entirely absent from the conventional stochastic approach. We next show how to obtain a value for this variance.

### C. Estimator for $\sigma_\beta^2$

To use equation (10) to estimate the variance of a Sato-Vartia price index we need an estimate of  $\sigma_\beta^2$ . Changes in expenditure shares not accounted for by the CES model can be used to estimate this variance. After taking logarithms, write equation (6) in first-difference form as:

$$\Delta \ln s_{it} = (\eta - 1)\Delta \ln c_t - (\eta - 1)\Delta \ln p_{it} + \Delta \ln b_{it}, \quad (11)$$

where  $\Delta \ln c_t \equiv \ln[c(\mathbf{p}_t, \mathbf{b}_t)/c(\mathbf{p}_{t-1}, \mathbf{b}_{t-1})]$ . Next, eliminate the term involving  $\Delta \ln c_t$  by subtracting the weighted mean (over all the  $i$ ) of each side of equation (11) from that side. Then using the fact that  $\sum_{i=1}^N w_i \Delta \ln s_{it} = 0$ , equation (11) becomes,

$$\Delta \ln s_{it} = \alpha_t - (\eta - 1) \Delta \ln p_{it} + \varepsilon_{it}, \quad i = 1, \dots, N, \quad (12)$$

where,

$$\alpha_t = (\eta - 1) \sum_{i=1}^N w_i \Delta \ln p_{it}, \quad (13)$$

and,

$$\varepsilon_{it} = \Delta \ln b_{it} - \sum_{i=1}^N w_i \Delta \ln b_{it}. \quad (14)$$

Equation (12) may be regarded as a regression of the change in shares on the change in prices, with the intercept given by (13) and the errors in (14). These errors indicate the extent of taste change, and are related to the underlying variance of the Sato-Vartia index.

Denote by  $\hat{\alpha}_t$  and  $\hat{\eta}$  the estimated coefficients from running weighted least-squares on (12) over  $i = 1, \dots, N$ , using the weights  $w_i$ . Unless the supply curve is horizontal, the estimate of  $\eta$  may well be *biased* because of a covariance between the error term (reflecting changes preferences and shifts in the demand curve) and the log prices. We ignore this in the next result, however, because we treat the prices as non-stochastic so there is no correlation between them and the errors  $\varepsilon_{it}$ . We return to the issue of stochastic prices at the end of this section, and in the next.

The weighted mean squared error of regression (12) is useful in computing the variance of the Sato-Vartia index, as shown by the following result:

### **Proposition 3**

Define  $\bar{w} = \sum_{i=1}^N w_i^2$  as the weighted average of the  $w_i$ , using  $w_i$  from (8) as weights, with  $\bar{w}$  as in Proposition 2. Also, denote the mean squared error of regression (12) by  $s_\varepsilon^2 = \sum_{i=1}^N w_i \hat{\varepsilon}_{it}^2$ , with  $\hat{\varepsilon}_{it} = \Delta \ln s_{it} - \hat{\alpha}_t + (\hat{\eta} - 1) \Delta \ln p_{it}$ . Then an unbiased estimator for  $\sigma_\beta^2$  is:

$$s_{\beta}^2 = \frac{s_{\varepsilon}^2}{2(1 - \bar{w} - \overline{\bar{w}})}. \quad (15)$$

To motivate this result, notice that the regression errors  $\varepsilon_{it}$  in (14) depend on the *changes* in  $\ln b_{it}$  minus their weighted mean. We have assumed that the  $\ln b_{it}$  are independently and identically distributed with variance  $\sigma_{\beta}^2$ . This means that the variance of  $\Delta \ln b_{it} = \ln b_{it} - \ln b_{it-1}$  equals  $2\sigma_{\beta}^2$ . By extension, the mean squared error of regression (12) is approximately twice the variance of the taste parameters, so the variance of the taste parameters is about one-half of the mean squared error, with the degrees of freedom adjustment in the denominator of (15) coming from the weighting scheme.

Substituting (15) into equation (10) yields a convenient expression for the variance of the Sato-Vartia index,

$$\text{var } \pi_{sv} \approx \frac{s_{\varepsilon}^2 s_p^2 \overline{\bar{w}}}{4(1 - \bar{w} - \overline{\bar{w}})}. \quad (16)$$

If, for example, each  $w_i$  equals  $1/N$ , the expression for  $\text{var } \pi_{sv}$  becomes  $s_{\varepsilon}^2 s_p^2 / 4(N - 2)$ . By comparison, the conventional stochastic approach resulted in the index variance  $s_p^2 / (N - 1)$  in (3), which can be greater or less than that in (16). In particular, when  $s_{\varepsilon}^2 < 4(N - 2)/(N - 1)$  then conventional stochastic approach gives a standard error of the index that is too high, as will occur if the fit of the share equation is good.

To further compare the conventional stochastic approach with our CES case, let us write regression (12) in reverse form as,

$$\Delta \ln p_{it} = \pi_{sv} - (\eta - 1)^{-1} \Delta \ln s_{it} + (\eta - 1)^{-1} \varepsilon_{it}, \quad (12')$$

where,

$$\pi_{sv} = \sum_{i=1}^N w_i \Delta \ln p_{it}, \quad (13')$$

and the errors are defined as in (14). Thus, the change in prices equals a trend (the Sato-Vartia index), plus a commodity-specific term reflecting the change in shares, plus a random error reflecting changing tastes. Notice the similarity between (12') and the specification of the pricing equation in (1'), where the change in share is playing the role of the commodity-specific terms  $\delta_i$ . Indeed, the constraint in (4) that the weighted commodity-specific effect sum to zero is automatically satisfied when we use (12') and the Sato-Vartia weight  $w_i$  in (8), because then  $\sum_{i=1}^N w_i \Delta \ln s_{it} = 0$ . Thus, the CES specification provides an economic justification for the pricing equation (1') used in the stochastic approach.

If we run WLS on regression (12') with the Sato-Vartia weights  $\sqrt{w_i}$ , the estimate of the trend term is exactly the log Sato-Vartia index,  $\hat{\pi}_{sv} = \sum_{i=1}^N w_i \Delta \ln p_{it}$ . If this regression is run without the share terms in (12'), then the standard error of the trend is given by equation (3) as modified to allow for heteroskedasticity by replacing  $N$  with  $1/\bar{w}$ . The standard error is somewhat lower if the shares are included. Either of these can be used as the standard error of the Sato-Vartia index under the conventional stochastic approach.

By comparison, in formula (16) we are using *both* the mean squared error  $s_e^2$  of the “direct” regression (12), and the mean squared error  $s_p^2$  of the “reverse” regression (12') (run without the share terms). The product of these is used to obtain the standard error of the Sato-Vartia index, as in (16). Recall that we are assuming in this section that the taste parameters are

stochastic, but not prices. Then why does the variance of prices enter (16)? This occurs because with the weight  $w_i$  varying randomly, the Sato-Vartia index  $\pi_{sv} = \sum_{i=1}^N w_i \ln(p_{it} / p_{it-1})$  will also vary if and only if the price ratios  $(p_{it}/p_{it-1})$  differ from each other. Thus, the variance of the Sato-Vartia index must depend on the *product* of the taste variance, estimated from the “direct” regression (12), and the price variance, estimated from the “reverse” regression (12 $\hat{}$ ) without the share terms.

We should emphasize that in our discussion so far, regressions (12) or (12 $\hat{}$ ) are run across commodities  $i = 1, \dots, N$ , but for a given  $t$ . In the appendix, we show how to generalize Proposition 3 to the case where the share regression (12) is estimated across goods  $i = 1, \dots, N$  and time periods  $t = 1, \dots, T$ . In that case, the mean squared error that appears in the numerator of (15) is formed by taking the weighted sum across goods and periods. But the degrees of freedom adjustment in the denominator of (15) is modified to take into account the fact that the weights  $w_i$  in (8) are correlated over time (since they depend on the expenditure shares in period  $t-1$  and  $t$ ). With this modification, the variance of the taste parameters is still relatively easy to compute from the mean squared error of regression (12), and this information may be used in (10) to obtain the variance of the Sato-Vartia index computed between any two periods.

Another extension of Proposition 3 would be to allow for stochastic prices as well as stochastic taste parameters. This assumption is introduced in the next section under the condition that the prices and taste parameters are *independent*. But what if they are not, as in a supply-demand framework where shocks to the demand curve influence equilibrium prices: then how is Proposition 3 affected? Using the mean squared error  $s_e^2$  of regression (12) in equation (15) to compute the variance of the taste disturbances, or in equation (16) to compute the variance of the

index, will give a lower-bound estimate. The reason is that running WLS on regression (12) will result in a downward biased estimate of the variance of tastes:  $E(s_{\epsilon}^2 / 2(1 - \bar{w} - \bar{\bar{w}})) \leq s_p^2$  if taste shocks affect prices, because the presence of taste information in prices artificially inflates the explanatory power of the regression.

#### 4. Index Variance for the CES Function with Stochastic Prices

Often the prices of goods or services used to calculate a price index are averages of individual price quotes. In these cases, differences in rates of change between the price quotes for a good or service imply the existence of a sampling variance. That is, for any item  $i$ ,  $\Delta \ln p_{it}$  will have a variance of  $\sigma_i^2$ , which can be estimated by  $s_i^2$ , the sample variance of the rates of change of the various quotes for item  $i$ . Then we are interested in extending Proposition 3 to encompass this “lower level” sampling error in the prices.

Note that we do not necessarily assume that the various commodity prices share a common trend, as in the conventional stochastic approach (1). Consider, however, the special case where the commodities in the index are homogeneous enough to justify the assumption of a common trend, and in addition, the variance of the error term is the same for every price,

$\sigma_i^2 = \sigma_j^2$ . Then an unbiased estimator for this variance is  $s_p^2 / (1 - \bar{w})$ , where  $s_p^2$  is defined in Proposition 2.<sup>5</sup> We will use this special case to obtain results that are most related to the conventional stochastic approach.

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<sup>5</sup> The denominator is a degrees of freedom correction derived as follows. Assume for simplicity that  $E[\Delta \ln p_{it} - E(\Delta \ln p_{it})]^2 = 1$ . Then  $E(\Delta \ln p_{it} - \pi_t)^2 = E(\Delta \ln p_{it} - \sum w_j \Delta \ln p_{jt})^2 = E[(1 - w_i) \Delta \ln p_{it} - \sum_{j \neq i} w_j \Delta \ln p_{jt}]^2 = (1 - w_i)^2 + \sum_{j \neq i} w_j^2 = 1 - 2w_i + \sum_j w_j^2$ . The weighted average of these terms,  $\sum_i w_i [1 - 2w_i + \sum_j w_j^2]$ , is  $1 - \sum_i w_i^2 = 1 - \bar{w}$ .



Although allowing  $\Delta \ln p_{it}$  to have a non-zero covariance with the  $w_i$  is appealing if positive shocks to  $b_{it}$  are thought to raise equilibrium market prices, this makes the expression for  $\text{var } \pi_{sv}$  quite complicated. Hence, for the sake of simplicity, we will assume that the shocks to prices are independent of the error term for preferences. We then obtain:

#### **Proposition 4**

*Let prices and weights  $w_i$  in (8) have independent distributions, and let  $s_i^2$  be an estimate of the variance of  $\Delta \ln p_{it}$ ,  $i = 1, \dots, N$ . Then the variance of the Sato-Vartia price index can be approximated by:*

$$\text{var } \pi_{sv} \approx \sum_{i=1}^N w_i^2 s_i^2 + \frac{1}{2} s_\beta^2 s_p^2 \bar{w} + \frac{1}{2} s_\beta^2 \sum_{i=1}^N s_i^2 w_i^2 (1 - w_i)^2, \quad (18)$$

where  $s_\beta^2$  is estimated from the mean squared error of the regression (12) as in (15). In the special case when every price variance may be estimated by  $s_p^2 / (1 - \bar{w})$ , then (18) becomes,

$$\text{var } \pi_{sv} \approx s_p^2 \bar{w} / (1 - \bar{w}) + \frac{1}{2} s_\beta^2 s_p^2 \bar{w} + \frac{1}{2} s_\beta^2 [s_p^2 / (1 - \bar{w})] \sum_{i=1}^N w_i^2 (1 - w_i)^2. \quad (18')$$

This proposition shows that the approximation for the variance of the price index is the sum of three components: one that reflects the variance of prices, another that reflects the variance of preferences and holds prices constant, and a third that reflects the interaction of the price variance and the taste variance. The first term in (18) or (18') is similar to the conventional stochastic approach. For example, if each  $w_i$  equals  $1/N$ , then the first term in (18') becomes  $s_p^2 \bar{w} / (1 - \bar{w}) = s_p^2 / (N - 1)$ , just as in (3). The second term in (18) or (18') is the same as what we

derived from stochastic tastes, in Proposition 2. The third term is analogous to the interaction term that appears in the expected value of a product of random variables (Mood, Graybill and Boes, 1974, p. 180, Corollary to Theorem 3). The presence of this term means that the interaction of random prices and tastes tends to raise the standard error of the index.

## 5. Translog Function

We next consider a translog unit-cost or expenditure function, which is given by:

$$\ln c(\mathbf{p}_t, \boldsymbol{\alpha}_t) = \alpha_0 + \sum_{i=1}^N \alpha_{it} \ln p_{it} + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N \gamma_{ij} \ln p_{it} \ln p_{jt}, \quad (19)$$

where we assume without loss of generality that  $\gamma_{ij} = \gamma_{ji}$ . In order for this function to be linearly homogeneous in prices we must have  $\sum_{i=1}^N \alpha_{it} = 1$  and  $\sum_{i=1}^N \gamma_{ij} = 0$ . The corresponding share equations are,

$$s_{it} = \alpha_{it} + \sum_{j=1}^N \gamma_{ij} \ln p_{jt}, \quad i=1, \dots, N. \quad (20)$$

We will treat the taste or technology parameters  $\alpha_{it}$  as random variables, but assume that the  $\gamma_{ij}$  are fixed. Suppose that  $\alpha_{it} = \alpha_i + \varepsilon_{it}$ , where the constant coefficients  $\alpha_i$  satisfy  $\sum_{i=1}^N \alpha_i = 1$ , while the random errors  $\varepsilon_{it}$  satisfy  $\sum_{i=1}^N \varepsilon_{it} = 0$ . Using this specification, the share equations are,

$$s_{it} = \alpha_i + \sum_{j=1}^N \gamma_{ij} \ln p_{jt} + \varepsilon_{it}, \quad i=1, \dots, N. \quad (21)$$

We assume that  $\varepsilon_{it}$  is identically distributed with  $E(\varepsilon_{it})=0$  for each equation  $i$ , though it will be correlated across equations (since the errors sum to zero), and may also be correlated over time.

Since the errors sum to zero the autocorrelation must be identical across equations. We will denote the covariance matrix of the errors by  $E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t') = \boldsymbol{\Omega}$ , and their autocorrelation is then

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_{t-1}') = \rho \boldsymbol{\Omega}.$$

With this stochastic specification of preferences, the question again arises as to what a price index should measure. In the economic approach, with  $\alpha_{it}$  constant over time, the ratio of unit-costs is measured by a Törnqvist (1936) price index. The following result shows how this generalizes to the case where  $\alpha_{it}$  changes:

### **Proposition 5**

*Defining  $\bar{\alpha}_i = (\alpha_{it-1} + \alpha_{it})/2$ , and  $w_{it} = (s_{it-1} + s_{it})/2$ , where the shares  $s_{it}$  are given by (20)*

*for the parameters  $\alpha_{it}$ ,  $\tau = t-1, t$ , then,*

$$\prod_{i=1}^N \left( \frac{p_{it}}{p_{it-1}} \right)^{w_{it}} = \frac{c(\mathbf{p}_t, \bar{\boldsymbol{\alpha}})}{c(\mathbf{p}_{t-1}, \bar{\boldsymbol{\alpha}})}. \quad (22)$$

The expression on the left of (22) is the Törnqvist price index, which measures the ratio of unit-costs evaluated at an *average value* of the taste parameters  $\alpha_{it}$ . This result is suggested by Caves, Christensen and Diewert (1982), and shows that the Törnqvist index is still meaningful when the first-order parameters  $\alpha_i$  of the translog function are changing over time.

In order to compute the variance of the Törnqvist index, we can make use of the right of (19), expressed in logs. Conditional on prices, the variance of this unit-cost ratio will depend on the variance of  $\bar{\alpha}_i = (\alpha_{it-1} + \alpha_{it})/2 = \alpha_i + (\varepsilon_{it-1} + \varepsilon_{it})/2$ . It follows that the covariance of these

taste parameters is  $E(\bar{\alpha} - \alpha)(\bar{\alpha} - \alpha)' = (1 + \rho)\mathbf{\Omega}/2$ . Then taking the log of (22) and substituting from equation (19), we find that the coefficients  $\bar{\alpha}_i$  multiply the log prices. This leads to the next result:

**Proposition 6**

*Suppose the parameters  $\alpha_{it}$  distributed as  $\alpha_{it} = \alpha_i + \varepsilon_{it}$ , with  $E(\varepsilon_t \varepsilon_t') = \mathbf{\Omega}$  and  $E(\varepsilon_t \varepsilon_{t-1}') = \rho \mathbf{\Omega}$ .*

*Using the weights  $w_{it} = (s_{it-1} + s_{it})/2$ , denote the log Törnqvist index by  $\pi_t \equiv \sum_{i=1}^N w_{it} \Delta \ln p_{it}$ .*

*Then conditional on prices, its variance is:*

$$\text{var } \pi_t = \frac{1}{2}(1 + \rho) \Delta \ln \mathbf{p}_t' \mathbf{\Omega} \Delta \ln \mathbf{p}_t. \quad (23)$$

It is worth noting that since the errors of the share equations in (18) sum to zero, the covariance matrix  $\mathbf{\Omega}$  is singular, with  $\mathbf{\Omega} \mathbf{1} = 0$  where  $\mathbf{1}$  is a  $(N \times 1)$  vector of one's. It follows that for any weights  $w_{it}$  with  $\sum_{i=1}^N w_{it} = 1$ , the variance in (23) is equal to,

$$\frac{1}{2}(1 + \rho) [\Delta \ln \mathbf{p}_t - \sum_{i=1}^N w_{it} \Delta \ln p_{it}]' \mathbf{\Omega} [\Delta \ln \mathbf{p}_t - \sum_{i=1}^N w_{it} \Delta \ln p_{it}]. \quad (24)$$

Thus, the variance of the Törnqvist index will approach zero as the prices approach a common growth rate, and this property also holds for the variance of the Sato-Vartia index in Proposition 2 and the conventional stochastic approach in (3). But unlike the stochastic approach in (3), the variance of the Törnqvist index will depend on the fit of the share equations. Our formula for the variance of the Törnqvist index is more general than what we obtained for the Sato-Vartia index, because in the latter case we had assumed the randomness in preferences was

*identical* across goods and uncorrelated over time, whereas this assumption is not used in Proposition 6.

The fit of the share equations will depend on how many time periods we pool over, and this brings us to the heart of the distinction between the stochastic and economic approaches. Suppose we estimated (21) over just two periods,  $t-1$  and  $t$ . Then it is readily verified that there are enough free parameters  $\alpha_i$  and  $\gamma_{ij}$  to obtain a *perfect fit* to the share equations. In other words, the translog system is flexible enough to give a perfect fit for the share equation (21) at two points. In the economic approach to index, such flexibility is regarded as a virtue: Diewert (1976) defines an index to be *superlative* if it is exact (i.e. measure the ratio of unit-costs) for an aggregator function that is flexible.<sup>6</sup> But from an econometric point of view, it is clear that we are dealing with zero degrees of freedom when estimating the share equations over two periods, so that the covariance matrix  $\mathbf{\Omega}$  is not even well-defined. How are we to resolve this apparent conflict between the economic and stochastic approaches?

We believe that a faithful application of the economic approach requires that we pool observations over *more than two periods* when estimating (21). In the economic approach, Christensen and Diewert (1982a,b) allow the first-order parameters  $\alpha_{it}$  of the translog unit-cost function to vary over time (as we do above), but strictly maintain the assumption that the second-order parameters  $\gamma_{ij}$  are constant (as we also assume). Suppose the researcher has data over three (or more) periods. If the share equations are estimated over periods one and two, and then again

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<sup>6</sup> Diewert (1976) defines an aggregator function to be flexible if it provides a second-order approximation to an arbitrary function at one point, i.e. if the parameters can be chosen such that the value of the aggregator function, and its first and second derivatives, equal those of an arbitrary function at one point. We are using a slightly different definition of flexibility: if the ratio of the aggregator function, and the value of its first and second derivatives, equal those of an arbitrary function at two points.

over periods two and three, this would clearly *violate* the assumption that  $\gamma_{ij}$  are constant. The way to maintain the constancy of  $\gamma_{ij}$  is to pool over multiple periods. Since this is an essential assumption of the economic approach, there is every reason to use it in our stochastic approach, in which case the covariance matrix  $\mathbf{\Omega}$  and the standard error of the Törnqvist index can be computed. Pooling over multiple periods is also recommended for the CES share equations in (12), to satisfy the maintained assumption that  $\eta$  is constant, even though in the CES case we do not obtain a perfect fit if (12) is estimated over a single cross-section (provided that  $N > 2$ ).

Once we pool the share equations over multiple periods, it makes sense to consider more general specifications of the random parameters  $\alpha_{it}$ . In particular, we can use  $\alpha_{it} = \alpha_i + t\beta_i + \varepsilon_{it}$ , where the coefficients  $\beta_i$  on the time trend satisfy  $\sum_{i=1}^N \beta_i = 0$ . Then the share equations become,

$$s_{it} = \alpha_i + \beta_i t + \sum_{j=1}^N \gamma_{ij} \ln p_{jt} + \varepsilon_{it} \quad , \quad i=1, \dots, N; t=1, \dots, T. \quad (25)$$

We make the same assumptions as before on the errors  $\varepsilon_{it}$ . Proposition 5 continues to hold as stated, but now the  $\bar{\alpha}_i$  are calculated as  $\bar{\alpha}_i = (\alpha_{it-1} + \alpha_{it})/2 = \alpha_i + \beta_i/2 + (\varepsilon_{it-1} + \varepsilon_{it})/2$ . The variance of these is identical to that calculated above, so Proposition 6 continues to hold as well. Thus, including time trends in the share equations does not affect the variance of the Törnqvist index.

## 6. Translog Case with Stochastic Prices

As in the CES case, we would like to extend the formula for the standard error of the price index to include randomness in prices as well as taste parameters. For each commodity  $i$  we suppose that  $\Delta \ln p_{it}$  is random with variance of  $\sigma_i^2$ , which can be estimated by  $s_i^2$ , the sample

variance of the rates of change of the various quotes for item  $i$ . These “lower level” sampling errors are assumed to be independent across commodities, and are also independent of error  $\varepsilon_{it}$  in the taste or technology parameters,  $\alpha_{it} = \alpha_i + \varepsilon_{it}$ . Then the standard error in Proposition 6 is extended as:

**Proposition 7**

*Let  $\Delta \ln p_{it}$ ,  $i=1, \dots, N$ , be independently distributed with mean 0 and variances estimated by  $s_i^2$ , and also independent of the parameters  $\alpha_{it} = \alpha_i + \varepsilon_{it}$ . Using the weights  $w_{it} = (s_{it-1} + s_{it})/2$ , the variance of the log Törnqvist index is approximated by:*

$$\begin{aligned} \text{var}(\pi) \approx & \sum w_{it}^2 s_i^2 + \frac{1}{4} \sum_i \sum_j (\Delta \ln p_{it})(\Delta \ln p_{jt}) [\sum_k \hat{\gamma}_{ik} \hat{\gamma}_{jk} s_k^2] \\ & + \frac{1}{2} (1 + \hat{\rho})(\Delta \ln \mathbf{p}_t)' \hat{\mathbf{\Omega}} (\Delta \ln \mathbf{p}_t) + \frac{1}{4} \sum_i (s_i^2) [\sum_j \hat{\gamma}_{ij}^2 s_j^2] + \frac{1}{2} (1 + \hat{\rho}) [\sum_i \hat{\Omega}_{ii} s_i^2]. \end{aligned} \quad (26)$$

*In the case where all prices have the same trend and variance, we can estimate  $\sigma_i^2$  by  $s_p^2/(1 - \bar{w})$  for all  $i$  and (26) becomes:*

$$\begin{aligned} \text{var}(\pi) \approx & s_p^2 \bar{w}/(1 - \bar{w}) + \frac{1}{4} [s_p^2/(1 - \bar{w})] \{ \sum_i \sum_j (\Delta \ln p_{it})(\Delta \ln p_{jt}) [\sum_k \hat{\gamma}_{ik} \hat{\gamma}_{jk}] \} \\ & + \frac{1}{2} (1 + \hat{\rho})(\Delta \ln \mathbf{p}_t)' \hat{\mathbf{\Omega}} (\Delta \ln \mathbf{p}_t) \\ & + \frac{1}{4} [s_p^2/(1 - \bar{w})]^2 [\sum_i \sum_j \hat{\gamma}_{ij}^2] + \frac{1}{2} [s_p^2/(1 - \bar{w})] (1 + \hat{\rho}) [\sum_i \hat{\Omega}_{ii}] \end{aligned} \quad (26')$$

Thus, with stochastic prices the variance of the Törnqvist index includes five terms. The first term in (26) contains the product of the squared weights times the price variances, the second term reflects the effect of the price variance on the weights, the third term reflects the

variance of the weights that comes from the taste shocks, the fourth term reflects the interaction of the price variance component and the component of the weight variance that comes from the price shocks, and the fifth term reflects the interaction between the price variance and the component of the weight variance that comes from the taste shocks. The expression for the variance of the Sato-Vartia index in equation (18) also includes the analogous expressions for the first, third and fifth terms in (26). The terms in (26) reflecting the effect of the price shocks on the weight variance are new, however, and arise because the linearity of the Törnqvist index allow use to include them; they were omitted for the sake of simplicity in the CES case.

## 7. Conclusions

The problem of finding a standard error for index numbers is an old one, and in this paper we have proposed what we hope is a useful solution. We have extended the stochastic approach to include both stochastic *prices* and *tastes*. The latter affect the weights in the price index formula, and the variance of the taste parameters is obtained by estimating a demand system. Our proposed method to obtain the standard error of prices indexes therefore involves two steps: estimating the demand system, and using the standard error of that regression (or system), combined with sampling error in the prices themselves, to infer the variance of the price index.

While our methods extend the stochastic approach, they also extend the economic approach to index numbers, while integrating the two. It is worth asking why the standard error of an index did not arise in the economic approach. Consider, for example, the problem of estimating a cost of living index. We could estimate the parameters of a model of preferences from data on expenditure patterns and then use these estimates to calculate a cost of living index. Yet, if the data fit the model perfectly, the cost of living index calculated from the parameter



estimates would have the same value as an exact index formula that uses the data on expenditure patterns directly. Moreover, Diewert's (1976) paper showed that the types of preferences or technology that can be accommodated using the exact index approach are quite general. As a result, econometric modeling was no longer thought to be necessary to estimate economic index numbers.

A consequence of the lack of econometric modeling is that estimates of economic index numbers are not accompanied by standard errors. Nevertheless, if the model that underlies an exact index number formula has positive degrees of freedom, an error term will usually need to be appended to the model to get it to fit the data perfectly. This will certainly be the case if the consumption or production model is estimated over a panel data set with multiple commodities and years, which is our presumption. Indeed, we would argue that the assumption of the economic approach that taste parameters are constant between two years, when applied consistently over a time series, means that the parameters are constant over *all years* of the panel. This will certainly mean that the demand system must have an error appended, and as a result, the taste parameters and exact price index are also measured with error. We have derived the formula for this error in the CES and translog cases, but the general approach we have suggested can be applied to any other functional form for demand or costs.

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