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Durable Goods and Aggregate Fluctuations

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Abstract

We consider an economy with two goods, a durable and a non-durable good, populated with a continuum of infinitely lived agents who differ in their income level and interact through borrowing and lending. Each agent chooses the consumption path for the durable and the non-durable good. We further assume that durables may depreciate in an abrupt, discontinuous manner. The last assumption gives rise to the optimal decision rules that are not of (s, S) type: consumers switch between two replacement frequencies at irregular intervals. As it turns out, consumers who follow these rules would spend *all* of their windfall income on durables. This feature of our model helps explain several empirical regularities in aggregate consumption behavior.

1 Introduction

Durable goods are purchased repeatedly because they depreciate over time. Apart from physical decay, one major reason for depreciation is obsolescence. New and better models periodically appear and make the old ones obsolete. Over the post-war period in the US, the price of consumer durables relative to non-durables and services has been falling at an average of 2 percent a year. However, obsolescence is by no means a continuous process. For example, new car models are introduced once a year only. Home appliances, electronics and furniture also follow yearly cycles. Sometimes new models are drastically different from previous ones, causing obsolescence to be abrupt. It turns out that irregular obsolescence is important in explaining the consumption patterns for durable and non-durables goods.

We consider an economy with two goods, a durable and a non-durable good, populated with a continuum of infinitely lived agents who differ in their income level and interact through borrowing and lending. Each agent chooses the consumption path for the durable and the non-durable good. The agent's state variables are prices, her liquid wealth and her stock of durable. The assumption of no resale makes durable purchases infrequent, because the service from the current stock of durable good acts as a fixed cost of adjustment. We characterize the optimal consumption rules and the distributions of durable goods holdings that give rise to price-stationary equilibria.

In equilibrium, consumers endogenously partition into classes, with each class following a distinct durable replacement rule. Two types of rules are optimal, depending on the consumer's class. One type, which we term a "fixed" rule, is similar to the familiar (s, S) rule with a constant replacement frequency that rises with agent's income. The other type is a "flexible rule", where the individual switches between two different fixed rules at irregular intervals. Flexible rules arise as a way to smooth durable consumption in the presence of abrupt obsolescence. A key difference between the two types of rules is how the agents who follow them react to wealth shocks. Consumers who follow a fixed rule adjust *only* their non-durable consumption in response to a small wealth shock. By contrast, consumers who follow flexible rules adjust *only* their durable consumption. This dichotomy in response to shocks allows our model to match several empirical observations on aggregate consumption behavior.

It is a robust empirical observation that aggregate durable consumption is more volatile than aggregate non-durable consumption. Our model can match this observed excess volatility of durables because consumers from flexible rule classes adjust only their durable consumption and leave non-durable consumption unchanged. In addition, the relative size of flexible rule classes is negatively related to the depreciation rate of the durable. Therefore, the more "durable" the good, the more volatile its aggregate consumption.

Another empirical regularity is that lower income consumers have a substantially higher cross-sectional volatility of durable purchases. This feature arises in our model because consumers who belong to the same class have close income levels. Low income classes replace durables infrequently and thus have low average per period durable expenditures. Then, consumers who do purchase durables in the current period spend much more than the average for their class, implying higher volatility.

According to Caballero (1990), aggregate durable purchases are slow to respond to aggregate wealth shocks. This effect is present in our model, because a positive shock induces few consumers to replace their durables immediately. Most consumers save the extra wealth and switch to a higher replacement frequency after a delay. Thus the aggregate expenditure adjusts slowly.

Beyond this, the model's predictions match, for example, the observation in Adda and Cooper (2000) that the impulse response of auto sales to price and income shocks exhibits dampened oscillations. This feature of the model is due to consumer heterogeneity and fixed costs of adjustment. As stated, an aggregate shock induces some, but not all, agents to change their durable consumption. This, in turn, affects the future distribution of durable goods holdings and introduces echo effects that make future sales of durables oscillate. The oscillating impulse response is inconsistent with Mankiw's (1982) $ARMA\,(1,1)$ specification for durable expenditure process that is based on a representative agent framework. Our results thus warn that empirical tests of permanent income hypothesis that ignore agent heterogeneity may produce misleading results that falsely reject the hypothesis.

The related recent literature is vast, but there are some gaps in it that this paper fills. Many macroeconomic models that are used for studying consumption of durables and non-durables feature continuous adjustment of the durable stock as an equilibrium outcome. This feature arises because adjustment is either frictionless, as it would be in a presence of a perfect resale market, (e.g. Mankiw, 1982) or the costs of adjustments are convex (e.g. Bernanke, 1985). Such models tend to only partially account for the "excess smoothness" of durable consumption. On the other hand, there is another large body of literature that models the replacement problems for durables in either representative agent setting (Grossman and Laroque, 1990, Eberly, 1994) or with heterogenous agents (Adda and Cooper, 2000, Caballero and Engel, 1999, Caplin and Leahy, 1999, Fisher and Hornstein, 2000). This class of models either feature only one good (the durable) or abstract from general equilibrium considerations, or both. A third body of literature (e.g. Caballero, 1993, Bar-Ilan and Blinder, 1992, Attanasio, 2000) does not feature optimization at the microeconomic level and *assumes* that the optimal replacement policy for the durable is an (s, S) rule.

Our contribution to the literature is that we develop a model that brings infrequent adjustment and agent heterogeneity into a two-good general equilibrium framework. We also show that the optimality of (s, S)-type replacement rules for durables is sensitive to assumptions about depreciation.

2 Model

We consider an economy with two goods, a durable and a non-durable good, and a continuum of agents of mass 1, indexed by $\omega \in [0, 1]$. The durable good is indivisible and each unit has a useful life span of $T \geq 2$ periods. It is produced by a constant returns to scale technology that uses p_0 units of the non-durable good for each unit of the durable good. The consumers are infinitely-lived and have a (common) separable utility function $v(\alpha, c) = x_\alpha + u(c)$, where $\alpha \in \{0, 1, \ldots, T\}$ denotes the age of the durable good consumed, $x_0 > x_1 > \cdots x_{T-1} > x_T = 0$, and c is the amount of the non-durable good consumed. Durable goods of any age less than T are perfect substitutes and each agent consumes at most one durable good (additional units of the good provide no utility). We think of the non-durable good as money for the consumption of other goods, and of u as an indirect utility function. The consumers and the monopolist discount future payoffs by $\beta \in (0, 1)$. Thus, if an agent consumes in each period t a durable good of age α_t and an amount c_t of the non-durable good, his total discounted payoff is

$$\sum_{t=0}^{\infty} \beta^t [x_{\alpha_t} + u(c_t)].$$

We assume that $u' > 0$, $u'' < 0$, and $u'(0) = \infty$. A consumer "type" $\omega \in [0, 1]$ is endowed with $y(\omega)$ units of the non-durable good in every period. Without loss of generality, we assume that the function $y : [0, 1] \to \mathbf{R}_+$ is strictly increasing. The consumers can borrow and lend. Let r_t denote the interest rate in period t.

An interest rate sequence $\{r_t\}_{t \geq 0}$ is a Walrasian equilibrium price if the aggregate non-durable demand in every period is equal to the total supply:

$$Y = \int_0^1 y(\omega) d\omega.$$

THE CONSUMER'S PROBLEM: A consumer must choose the periods when he purchases a (new) unit of the durable good, and the amount of the non-durable good he consumes in every period. A *durable purchasing policy* $\delta = \{\delta_t\}_{t \geq 0}$ specifies in which periods the agent buys a new unit ($\delta_t = 1$) or keeps the old unit he has ($\delta_t = 0$). For any $i, j \in \mathbf{N}$, let

$i \oplus j = \min\{i+j, T\}$ and $i \ominus j = \max\{i-j, 0\}$. Given an initial unit of age α_{-1}, a purchasing policy determines the age of the unit consumed in every period $t \geq 0$ recursively as follows: $\alpha_t = 0$ if $\delta_t = 1$ and $\alpha_t = \alpha_{t-1} \oplus 1$ if $\delta_t = 0$.

Given the interest rates $\{r_t\}_{t \geq 0}$, it is convenient to study the consumer's problem immediately after he makes his durable good decision in period 0 but before he decides on the non-durable consumption for period 0, and the problem before he makes any decision in period 0. In the former problem, δ_0 is not a decision variable (has already been chosen). The *total initial wealth* of an agent type ω is

$$w_0(\omega) = y(\omega) \sum_{t \geq 0} \left[\prod_{\tau=0}^{t-1} \frac{1}{1+r_\tau} \right],$$

and becomes $w_0(\omega) - \delta_0 p_0$ after the durable good decision is made in period 0. Below, whenever it is clear from context, we omit the type ω.

We say that an agent is in state (α_0, w_0) when he has wealth w_0 and a durable of age α_0. The problem of an agent with initial state (α_0, w_0), after the durable good decision δ_0 has been made in period 0, is

$$\begin{aligned} U(\alpha_0, w_0) &= \max_{(c,\delta,w)} \sum_{t \geq 0} \beta^t [x_{\alpha_t} + u(c_t)] \qquad (1) \\ \text{s.t.} \quad w_{t+1} &= (1+r_t)[w_t - \delta_t p_0 - c_t] \\ \alpha_{t+1} &= (1-\delta_{t+1})[\alpha_t \oplus 1] \\ c_t &\geq 0, \quad \delta_t \in \{0,1\}, \quad w_{t+1} \geq 0. \end{aligned}$$

Note that $\alpha_{t+1} = 0$ if $\delta_{t+1} = 1$ and $\alpha_{t+1} = \alpha_t \oplus 1$ otherwise. For convenience, we will often write $U_\alpha(w)$ instead of $U(\alpha, w)$.

The problem of a consumer with initial state (α_0, w_0) before any decision is made in period 0 is

$$J(\alpha_0, w_0) = \max_{\delta_0 \in \{0,1\}} U((1-\delta_0)\alpha_0, w_0 - \delta_0 p_0).$$

Again, for convenience, we will often write $V_\alpha(w)$ instead of $V(\alpha, w)$.

Fix a durable purchasing policy δ (optimal or otherwise). Then, the consumer's problem reduces to

$$\begin{aligned} & \max_{(c,w)} \sum_{t \geq 0} \beta^t u(c_t) \\ \text{s.t.} \quad & w_{t+1} = (1+r_t)[w_t - \delta_t p_0 - c_t] \\ & c_t \geq 0, \quad w_{t+1} \geq 0. \end{aligned}$$

The assumptions on u imply that the following first order conditions (FOC) are necessary and sufficient:

$$\beta^t u'(c_t) = \lambda_0 / [\prod_{\tau=0}^{t-1} (1+r_\tau)], \qquad (2)$$

where $\lambda_0 > 0$ is a Lagrange multiplier.

DEFINITION: An equilibrium is price-stationary if there exists $r > 0$ such that $r_t = r$ for all $t \geq 0$.

The next theorem states that the only interest rate that is compatible with a price-stationary equilibrium is $r = (1-\beta)/\beta$.

Theorem 1. In a price-stationary equilibrium with interest rate r, $\beta = (1+r)^{-1}$ and every consumer of type $\omega \in [0,1]$ consumes a fixed amount $c(\omega)$ of the non-durable in every period.

Proof: The FOC $\beta^t u'(c_t) = \lambda_0/(1+r)^t$ imply that $u'(c_t)$ is increasing (decreasing) if $\beta(1+r) < 1$ (> 1). Since $u'' < 0$, this implies that c_t is decreasing (increasing) if $\beta(1+r) < 1$ (> 1). This is the case for every consumer of every type ω. But in equilibrium we must have that

$$\int_0^1 c_t(\omega)d\omega = Y \quad \text{for all } t \geq 0.$$

Thus, in equilibrium, $\beta(1+r) = 1$ and $c_t(\omega) = c(\omega)$ for all t, where $u'(c(\omega)) = \lambda_0(\omega)$. ■

Theorem 1 says that any optimal solution to (1) has constant c every period. Therefore, we can transform the problem (1) by letting the consumer choose c once and for all and then use the rest of his wealth $b = w - \frac{c}{1-\beta}$ to optimally purchase durables. Then, with $(1+r_t)^{-1} = \beta$, (1) becomes

$$U(\alpha, w) = \max_{c\in[0,w]} \left(\frac{u(c)}{1-\beta} + V_\alpha \left(w - \frac{c}{1-\beta} \right) \right), \tag{3}$$

where

$$\begin{aligned} V_\alpha(b) &= \max_{\delta_t \in \{0,1\}} \sum_{t\geq 0} \beta^t x_{\alpha_t} \\ \text{s.t. } b &= p_0 \sum_{t=0} \beta^t \delta_t \\ \alpha_{t+1} &= (1-\delta_{t+1})[\alpha_t \oplus 1] \end{aligned} \tag{4}$$

is the optimal value function of a consumer who has budget b to spend on durables. This problem can be transformed into a Bellman equation. Consumer starts with a budget b and a durable of age $\alpha = 1, ..., T$ and decides whether to keep or replace his current durable. If consumer keeps the current good, he gets the service flow x_α this period, and enters next period with budget $\frac{b}{\beta}$ and the good of age $\alpha \oplus 1$. On the other hand, if the consumer decides to replace his durable, he acts as if his current good is useless and gets $V_T(b)$ - the present value of having a useless good. Therefore, the optimal value function must solve

$$V_\alpha(b) = \max \left\{ x_\alpha + \beta V_{\alpha\oplus 1}\left(\frac{b}{\beta}\right), V_T(b) \right\}, \ \alpha = 1, ..., T. \tag{5}$$

We will now turn to solving the problem (5), and will then determine the optimal non-durable consumption from (3) given $V_\alpha(b)$.

CYCLICAL POLICIES: Consider the following two rules.

- **R-fixed rule:** replace the durable good each time it reaches age R.

• **R-flexible rule:** buy durables in such a way that each durable is replaced when it reaches age R or age $R+1$ for $R<T$. The T-flexible rule is to replace each durable at age that is greater or equal to T.

There are $T+1$ fixed rules ($R=1,2,\ldots,T+1$), and T flexible rules ($R=1,2,\ldots,T$). The $(T+1)$-fixed rule is to never replace the durable (and eventually consume no durables).

Let $b_{\alpha,R}$ be the budget that allows a consumer to follow the R-fixed rule from initial state $\alpha \leq R$

$$b_{\alpha,R}=\frac{\beta^{R-\alpha}}{1-\beta^{R}}p_0,\ \alpha\leq R,\ R=1,...,T \tag{6}$$

It is also convenient to define $b_{\alpha,T+1}=0$, for $\alpha\leq T$ and $b_{T+1,T+1}=p_0$. Let $v_{\alpha,R}$ be the present value of durable goods services resulting from following the R-fixed rule from state α:

$$v_{\alpha,R}=X_{\alpha,R}+\mathbf{1}_{[R\leq T]}\cdot\frac{\beta^{R\ominus\alpha}}{1-\beta^{R}}\cdot X_{0,R} \tag{7}$$

Given an initial durable of age α, the consumer can follow the R-flexible rule only if his wealth is in a certain range. The minimum feasible wealth to follow the rule corresponds to the case in which the consumer can only afford to replace each durable when this reaches age $R+1$, and the maximum feasible wealth corresponds to the case in which he can afford to replace his durable each time this reaches age R. Thus, the R-flexible rule is feasible only in the wealth interval $[b_{\alpha,R+1},b_{\alpha,R}]$.

Let A_R denote the marginal utility of wealth for a consumer who switches from an $R+1$-fixed rule to an R-fixed rule. :

$$A_R=\frac{v_{R,R}-v_{R+1,R+1}}{b_{R,R}-b_{R+1,R+1}}=\frac{1}{p_0}\frac{\frac{X_{0,R}}{1-\beta^{R}}-\frac{X_{0,R}+\beta^{R}x_R}{1-\beta^{R+1}}}{\frac{1}{1-\beta^{R}}-\frac{1}{1-\beta^{R+1}}}=\left(X_{0,R}-x_R\frac{1-\beta^{R}}{1-\beta}\right). \tag{8}$$

This number will also turn out to be equal to the marginal utility of wealth of a consumer who follows an R-flexible rule. The following lemma will establish that when the discount rate is sufficiently large, the optimal solution to (4) is always a flexible rule.

Lemma 1 The optimal value function for the replacement problem (4) when $\alpha=T$ is

$$V_T(b)=\begin{cases}A_T\cdot b, & b\in[0,b_{T,T}]\\ v_{R+1,R+1}+A_R\cdot(b-b_{R+1,R+1}), & b\in[b_{R+1,R+1},b_{R,R}],\ R=T-1,...,1\\ v_{1,1}, & b\geq b_{1,1}\end{cases}. \tag{9}$$

if and only if

$$\beta^{T-1}(1+\beta)>1. \tag{10}$$

For every wealth level $b\in[0,b_{1,1}]$, the optimal replacement policy is an R-flexible rule. That is, for every $k\geq 0$

$$\tau_{k+1}-\tau_k=\begin{cases}\{T,T+1,....\} & b\in[0,b_{T,T}]\\ \{R,R+1\} & b\in[b_{R+1,R+1},b_{R,R}],\ R=T-1,1\\ 1, & b\geq b_{1,1}\end{cases} \tag{11}$$

Proof: See Appendix

Knowing $V_T(b)$, we can use the Bellman equation (5) to recursively construct $V_\alpha(b)$ for all $\alpha < T$. In particular, we have

Lemma 2 The optimal value function that solves (5) is

$$V_\alpha(b) = \begin{cases} v_{\alpha,R+1} + A_R \cdot (b - b_{\alpha,R+1}), & b \in (b_{\alpha,R+1}, b_{\alpha,R}), \ R = T, ..., \alpha \\ V_T(b), & b \geq b_{\alpha,\alpha} \end{cases} \quad \text{for } \alpha = 1, .., T. \tag{12}$$

The corresponding optimal replacement policy is

$$\delta_\alpha(b) = \begin{cases} 0, & b < b_{\alpha+1,\alpha+1} \\ \{0,1\}, & b_{\alpha+1,\alpha+1} \leq b \leq b_{\alpha-1,\alpha} \\ 1, & b > b_{\alpha-1,\alpha} \end{cases} \quad \text{for } \alpha = 1, .., T. \tag{13}$$

Proof: See Appendix

Consumers who follow an R-flexible rule have a constant marginal utility of wealth equal to A_R. This implies that their non-durable consumption should be constant and equal to c_R, with

$$u'(c_R) = A_R.$$

Suppose that consumer initially has the good of age α. The wealthiest consumer who follows the R-flexible rule consumes c_R and replaces his good every R periods. His wealth is

$$w_{\alpha,R}(c_R) = \frac{c_R}{1-\beta} + b_{\alpha,R}.$$

On the other hand, the poorest consumer who follows the $R-1$ flexible rule also replaces his good every R periods, and therefore spends exactly the same amount $b_{\alpha,R}$ on durables. However, this individual must be strictly wealthier, since he consumes $c_{R-1} > c_R$:

$$w_{\alpha,R}(c_{R-1}) = \frac{c_R}{1-\beta} + b_{\alpha,R} > w_{\alpha,R}(c_R)$$

Now take consumers with wealth w between $w_{\alpha,R}(c_R)$ and $w_{\alpha,R}(c_{R-1})$. We will show that they consume different amount of non-durable, depending on w, but spend exactly $b_{\alpha,R}$ on durables, regardless of w. That is, all these individuals follow an R-fixed rule. The following lemma establishes this result formally.

For every α, R, define $w_{\alpha,R}(c)$ as the total wealth that is necessary to follow an R-fixed rule from the initial state α and consume c of non-durable every period.

$$w_{\alpha,R}(c) = \frac{c}{1-\beta} + b_{\alpha,R}$$

Let $c_{\alpha,R}(w)$ be the inverse function for $w_{\alpha,R}(c)$:

$$c_{\alpha,R}(w) = (1-\beta)(w - b_{\alpha,R}).$$

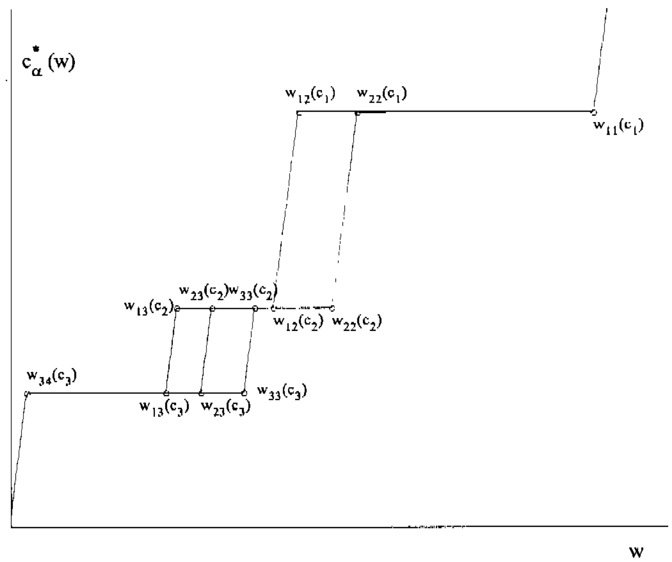

Figure 1: Optimal consumption of non-durables

Lemma 3

$$c_{\alpha}^{*}(w) = \arg\max_{c}\left(\frac{u(c)}{1-\beta} + V_{\alpha}\left(w - \frac{c}{1-\beta}\right)\right).$$

Then

$$c_{\alpha}^{*}(w) = \begin{cases} c_{\alpha,R}(w), & \text{if } w \in \left[w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})\right],\ R = T+1, ..., 1 \\ c_R, & \text{if } w \in \left[w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R)\right],\ R = T, ..., 1 \end{cases}. \tag{14}$$

The optimal consumption rule for durables as a function of total wealth is

$$\delta_{\alpha}^{*}(w) = \begin{cases} 0, & w < w_{\alpha+1,\alpha+1}(c_{\alpha}) \\ \{0,1\}, & w_{\alpha+1,\alpha+1}(c_{\alpha}) \leq w \leq w_{\alpha-1,\alpha}(c_{\alpha}) \\ 1, & w > w_{\alpha-1,\alpha}(c_{\alpha}) \end{cases} \quad \text{for } \alpha = 1, .., T. \tag{15}$$

Proof: See Appendix.

3 Stationary Equilibrium

If all the consumers follow a R-fixed or (R, c_R)-flexible rule, they separate by their state into disjoint closed classes. For simplicity, we only consider here the case $T = 3$, so we can refer to Figure 1. Let $c_0 = c_{11}(\overline{w})$ and $c_4 = c_{34}(\underline{w})$ so that $w_{\alpha 1}(c_0) = \overline{w}$ and $w_{\alpha 4}(c_4) = \underline{w}$ for each α. Then define

$$\begin{aligned} W_R^{\alpha} &= [w_{\alpha R}(c_R), w_{\alpha R}(c_{R-1})], \quad \alpha = 1, \ldots, R \text{ and } R = 1, \ldots, 4 \\ W_{RR+1}^{\alpha} &= (w_{\alpha R+1}(c_R), w_{\alpha R}(c_R)), \quad \alpha = 1, \ldots, R+1 \text{ and } R = 1, \ldots, 3 \end{aligned}$$

(Recall that $w_{\alpha R}(c) = w_{RR}(c)$ for all $\alpha > R$ and $c \geq 0$, and that $w_{\alpha 4}(c_3) = w_{34}(c_3)$ for all α.) Note that for each α, $\{W_R^\alpha\}_{R=1}^4 \cup \{W_{RR+1}^\alpha\}_{R=1}^3$ forms a partition of $[\underline{w}, \overline{w}]$. At the beginning of every period, agents with state in $C_R = \bigcup_{\alpha=1}^R [\{\alpha\} \times W_R^\alpha]$ follow the R-fixed rule, and with state in $C_{RR+1} = \bigcup_{\alpha=1}^R [\{\alpha\} \times W_{RR+1}^\alpha]$ follow the R-flexible rule. The classes C_R and C_{RR+1} are closed: if an agent follows the R-fixed rule, for example, and his initial state is in C_R, then his state remains in C_R forever. In the long run, there are no consumers in state $(2, w)$ with $w \geq w_{11}(c_1)$, in state $(3, w)$ with $w \geq w_{22}(c_2)$, or in state (α, w) with $\alpha < 3$ and $w < w_{34}(c_3)$.

We construct a stationary equilibrium in which each class represents an economy in autharchy: it does not borrow from or lend to agents in any other class. To achieve this, we construct specific wealth distributions for each class. The classes that follow flexible rules require a more delicate balance and the construction of their distributions is more complex.

A wealth distribution for C_R is a vector $F = (F_1, \ldots, F_R)$, where for each α, $F_\alpha : W_R^\alpha \to \mathbf{R}_+$ is a non-decreasing function such that $F_\alpha(w_{\alpha R}(c_R)) = 0$. The total mass of agents in C_R with durable of age α is $F_\alpha(w_{\alpha R}(c_{R-1}))$. Similarly, a wealth distribution for C_{RR+1} is a vector $F = (F_1, \ldots, F_{R+1})$, where for each α, $F_\alpha : W_{RR+1}^\alpha \to \mathbf{R}_+$ is a non-decreasing function such that $F_\alpha(w_{\alpha R+1}(c_R)) = 0$.

When an agent follows the (R, c_R)-flexible rule, his next period wealth is $k_R(w) = (w - c_R)/\beta$ if he keeps the durable and $r_R(w) = (w - c_R - p_0)/\beta$ if he replaces it. Similarly, when a consumer with initial state (α, w), $w \in [w_{\alpha R}(c_R), w_{\alpha R}(c_{R-1})]$, follows the R-fixed rule, his next period wealth is $w + \Delta b_{\alpha R}$, where the constant $\Delta b_{\alpha,R}$ does not depend on w (for $R = 2$, for example, $\Delta b_{1,2} = -\Delta b_{2,2} = p_0/(1+\beta)$).

Consider the class of agents that follow the 1-flexible rule. If an agent has initial state $(1, w)$ with $w \in (w_{22}(c_1), w_{01}(c_1)) \subset W_{12}^1$, he can arbitrarily choose between replacing and keeping the durable. Assume that for an arbitrary $m \in (w_{22}(c_1), w_{01}(c_1))$ (the same for all agents), the agent keeps the durable when $w \in (w_{22}(c_1), m]$ and replaces it when $w \in (m, w_{01}(c_1))$. Let

$$\mathbf{B} = \{F = (F_1, F_2) \mid F_\alpha : W_{12}^\alpha \to \mathbf{R} \text{ is bounded and continuous}, \alpha = 1, 2\}$$

with the norm $||F|| = \max\{||F_1||, ||F_2||\}$, where $||F_\alpha|| = \sup\{|F_\alpha(w)| \mid w \in W_{12}^\alpha\}$. For any given population mass $M_{12} \geq 0$, let

$$K = \{F \in \mathbf{B} \mid F \text{ is a wealth distribution for } C_{12} : F_1(w_{11}(c_1)) + F_2(w_{11}(c_1)) = M_{12}\}.$$

If today's wealth distribution for C_{12} is given by $F \in \mathbf{B}$, tomorrow's wealth distribution is given by $\Psi(F) = (\Psi_1(F), \Psi_2(F))$, where $\Psi_2(F)(w) = F_1(k_1^{-1}(w))$, $w \in W_{12}^2$, and

$$\Psi_1(F)(w) = \begin{cases} F_2(r_1^{-1}(w)), & \text{if } r_1^{-1}(w) \leq m \\ F_2(r_1^{-1}(w)) + \left[F_1\left(r_1^{-1}(w)\right) - F_1(m)\right] & \text{if } r_1^{-1}(w) > m \end{cases}.$$

Since K is a closed, bounded and convex subset of the Banach space $\mathbf{B}$, and $\Psi : K \to K$ is continuous, Schauder's fixed point theorem guarantees the existence of an $F \in K$ such that $F = \Psi(F)$.

Suppose the initial wealth distribution of agents following the 1-flexible rule is given by the fixed point F. Then, the aggregate total net demand of this class

$$D = M_{12}c_1 + p_0 \int_m^{w_{11}(c_1)} dF_1(w) - (1-\beta)\left[\int w\,dF_1(w) + \int w\,dF_2(w)\right]$$

is constant over time. If $D > 0$, for example, then this class is a net borrower every period, and over time, their total discounted borrowing will amount to $D/(1-\beta)$. But each individual's total discounting borrowing is 0. This is a contradiction. Thus, D must be 0, and this class doesn't borrow from or lend to any other class.

A similar argument holds for the class of agents that follow the 2-flexible rule, where we can place an arbitrary population mass M_{23}.

An individual that follows the 1-fixed rule spends the same total amount in every period, and thus doesn't borrow or lend. Thus, the 1-fixed rule class is also authartic.

For the class C_R ($R = 2, 3$), let F be an arbitrary wealth distribution for C_R such that $F_\alpha(w_{\alpha R}(c_{R-1})) = M_R/R$ (here M_R denotes the total mass in class C_R). Then, the mass of agents with durable of age α is M_R/R for all α in every period. Since the aggregate durable consumption in any period is equal to the mass of agents with durable of age R in that period, the aggregate durable consumption is constant over time. Since each individual consumes a constant flow of non-durables, the aggregate non-durable consumption is also constant over time. Hence, as for the 1-flexible rule class, the net borrowing/lending of the C_R class is constant over time and therefore must be 0.

Finally, each agent that follows the 4-fixed rule never buys a new durable and consumes a constant flow of non-durables. Hence, he doesn't borrow or lend, and the 4-fixed rule is also authartic.

4 Properties of Equilibrium

The equilibrium described in Section 3 has a number of properties that are consistent with the evidence on aggregate durable purchases.

4.1 Slow adjustment of durable purchases to aggregate wealth shocks

It has been noted in the literature (e.g. Caballero, 1990) that the response of durable purchases to aggregate wealth shocks is slower than predicted by a frictionless adjustment model. The origins of this slowness is easy to understand within our framework. Suppose that every consumer in our economy receives a positive wealth shock. Assume that the shock is sufficiently small, so that we can ignore the consumers who change their wealth class as a result of the shock. Individuals who follow a fixed rule will exhibit no adjustment at all - they will spend all their extra wealth on *non-durable* consumption. In contrast, consumers who follow a flexible rule will change *only* their durable consumption. However, the only consumers who will increase their durable spending immediately are those who follow an R-flexible rule and have goods of age R. Consumers who follow the same rule but have goods of age $\alpha < R$ will the extra wealth to increase their durable consumption at some point in the future, but will leave their current durable and non-durable consumption unchanged.

4.2 Volatility of durable consumption

Volatility of durable consumption by income: Evidence shows that volatility of durable purchases decreases significantly with income. In our model, as it turns out, higher income consumers do have lower volatility of durable spending. This is because consumers with higher income replace their durables more frequently, and therefore each durable purchase represents a smaller deviation from average durable spending per period. To illustrate this phenomenon, take consumption class C_R of arbitrary mass M_R. For each individual in this class, the average per period spending on durables equals $b_{R,R}(1-\beta)$. In equilibrium, at any moment of time $\frac{R-1}{R}M_R$ individuals spend zero on durables, and the rest of the consumers, mass $\frac{1}{R}M_R$, spend p_0. Then the cross-sectional volatility of durable spending for class C_R can be defined as average squared deviation of durable spending divided by the squared mean spending within the class:

$$\sigma_R = \frac{\frac{R-1}{R}M_R \cdot [b_{R,R}(1-\beta)]^2 + \frac{1}{R}M_R \cdot [p_0 - b_{R,R}(1-\beta)]^2}{[b_{R,R}(1-\beta)]^2 M_R}. \tag{16}$$

Proposition 1 σ_R is an increasing function of R.
Proof: Rewrite (16) as

$$\sigma_R = 1 - \frac{1}{R} + \frac{1}{R}\left[\frac{p_0}{b_{R,R}(1-\beta)} - 1\right]^2 = 1 + \frac{1}{R}\left[\left(\frac{\beta-\beta^R}{1-\beta}\right)^2 - 1\right]$$

Volatility of durable consumption relative to non-durables: Another robust result in the empirical consumption literature is the higher volatility of durable consumption relative to non-durable. As stated above, in our model different consumers will react differently to a shock in wealth. Individuals who follow a fixed rule will adjust only their non-durable consumption, and individuals who follow a flexible rule will adjust only their durable consumption. The reaction of aggregate consumption to an aggregate wealth shock will depend on the mass of consumers in consumption classes C_R versus the consumption classes $C_{R,R+1}$. All other things being equal, if consumption classes C_R are relatively small, aggregate durable consumption will be more volatile. It turn out that the size of consumption class C_R is related to the difference between x_{R-1} and x_R,but the size of consumption class $C_{R,R+1}$ is not. Consumers who follow an R-fixed rule (are in class C_R) must have non-durable consumption level between c_R and c_{R-1}. The closer these two levels are to each other, the smaller is the consumption class C_R. As shown in the proposition below, faster depreciating durable goods imply larger feasible size for the class C_R.

Proposition 2 The size of the consumption class C_R, $\int_{w\in C_R} dw$, is an increasing function of $x_{R-1} - x_R$.

Proposition 2 implies that durable goods that have low depreciation and/or obsolescence rate will also have higher volatility of expenditure relative to non-durables.

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4.3 APPENDIX

Proof of Lemma 1: Any replacement sequence for durable goods can be represented as a sequence of purchase dates $\{\tau_k\}_{k=0}^{\infty}$, where $\tau_{k+1} - \tau_k \geq 1$. The present value of the service flow provided by the durables from the sequence $\{\tau_k\}$ equals

$$Z(\{\tau_k\}) = \sum_{k=0}^{\infty} \beta^{\tau_k} \cdot X_{0,\tau_{k+1}-\tau_k} \tag{17}$$

and the cost of sequence $\{\tau_k\}$ equals

$$b = p_0 \sum_{k=0}^{\infty} \beta^{\tau_k}$$

Consider first the case when $b > b_{1,1}$, when the consumer can afford to replace his durable every period. Since there is no other replacement sequence that gives a higher payoff, the consumer will replace every period and have the payoff equal to $v_{1,1}$.

Step 1 Prove that

$$V_T(b) \geq Z(\{\tau_k\}; b). \text{ for any } \{\tau_k\} \text{ and } b \in [0, b_{1,1}] \tag{18}$$

Suppose that the payoff from following the replacement sequence $\{\tau_k\}$ can be represented as a convex combination (with weights $\{\lambda_R(\{\tau_k\})\}_{R=1}^{T}$) of the payoffs from following different fixed rules:

$$Z(\{\tau_k\}; b) = \left(1 - \sum_{R=1}^{T} \lambda_R\right) \cdot V_T(0) + \sum_{R=1}^{T} \lambda_R \cdot V_T(b_{R,R}), \tag{19}$$

with

$$\sum_{R=1}^{T} \lambda_R \cdot b_{R,R} = b, \ \sum_{R=1}^{T} \lambda_R \leq 1$$

Since $A_1 > 0$ and $A_{R+1} - A_R > 0$ for every $R = 1, ..., T-1$, the value function $V_T(b)$ is concave.[1] Then (19) would imply (18) be definition of concavity.

In order to show that representation (19) exists for any replacement sequence $\{\tau_k\}$, group together all terms in (??) that have the same replacement frequency:

$$Z(\{\tau_k\}; b) = \sum_{k=0}^{\infty} \beta^{\tau_k} \cdot X_{0,\tau_{k+1}-\tau_k} = \sum_{R=1}^{\infty} \left(\sum_{k \in \{k: \tau_{k+1}-\tau_k = R\}} \beta^{\tau_k} \right) X_{0,R} =$$

$$= \sum_{R=1}^{T} \left(\sum_{k \in \{k: \tau_{k+1}-\tau_k = R\}} \beta^{\tau_k} \right) X_{0,R} + \sum_{R=T+1}^{\infty} \left(\sum_{k \in \{k: \tau_{k+1}-\tau_k = R\}} \beta^{\tau_k} \right) X_{0,T}.$$

1

$$A_{R+1} - A_R = (X_{0,R+1} - X_{0,R}) - \frac{1}{1-\beta}\left(x_{R+1}\left(1-\beta^{R+1}\right) - x_R\left(1-\beta^R\right)\right) =$$

$$= \frac{\beta^R - \beta^{R+1}}{1-\beta} x_R - \frac{1-\beta^{R+1}}{1-\beta} x_{R+1} + x_R \frac{1-\beta^R}{1-\beta} = (x_R - x_{R+1}) \frac{1-\beta^{R+1}}{1-\beta} > 0,$$

The last line used the fact that for any $R > T$ $X_{0,R} = X_{0,T}$. Then, (??) can be rewritten as

$$Z\left(\{\tau_k\};b\right) = \sum_{R=1}^{T} \lambda_R V_T\left(b_{R,R}\right) = \sum_{R=1}^{T} \lambda_R V_T\left(b_{R,R}\right) + \left(1 - \sum_{R=1}^{T} \lambda_R\right) V_T\left(0\right),$$

where

$$\lambda_R = \begin{cases} \left(\sum_{k\in\{k:\tau_{k+1}-\tau_k=R\}} \beta^{\tau_k}\right)\left(1-\beta^R\right), & R = 1,...,T-1 \\ \left(\sum_{k\in\{k:\tau_{k+1}-\tau_k\geq T\}} \beta^{\tau_k}\right)\left(1-\beta^T\right), & R = T \end{cases}.$$

The above expression for λ_R implies that

$$\sum_{R=1}^{T} \lambda_R b_{R,R} = p_0 \cdot \sum_{R=1}^{T-1} \left(\sum_{k\in\{k:\tau_{k+1}-\tau_k=R\}} \beta^{\tau_k}\right) + p_0 \cdot \sum_{k\in\{k:\tau_{k+1}-\tau_k\geq T\}} \beta^{\tau_k} = p_0 \sum_{k=0}^{\infty} \beta^{\tau_k} = b.$$

It is left to show that

$$\sum_{R=1}^{T} \lambda_R \leq 1.$$

For this, denote $R_k = \tau_{k+1} - \tau_k$ and group the terms in $\sum_{R=1}^{T} \lambda_R$ in the order of replacement date τ_k. Then

$$\sum_{R=1}^{T} \lambda_R = \sum_{k=0}^{\infty} \beta^{\tau_k}\left(1-\beta^{\min\{T,R_k\}}\right) \leq \sum_{k=0}^{\infty} \beta^{\tau_k}\left(1-\beta^{R_k}\right) = \sum_{k=0}^{\infty} \beta^{\tau_k} - \sum_{k=0}^{\infty} \beta^{\tau_{k+1}} = \beta^{\tau_0} \leq 1. \tag{20}$$

We have established that $V_T(b)$ is an upper bound for $\max_{\{\tau_k\}} Z\left(\{\tau_k\};b\right)$. It is left to demonstrate that show that this upper bound is actually reached for $\{\tau_k\}$ that satisfy (11).

Step 2 Prove that

$$\beta^{T-1}(1+\beta) > 1 \Longleftrightarrow \exists \{\tau_k\} : V_T(b) = Z\left(\{\tau_k\};b\right)$$

Since $V_T(b)$ is piecewise linear, (9) implies that for any budget b

$$V_T(b) = Z\left(\{\tau_k\};b\right) \Longleftrightarrow \{\tau_k\} \text{ satisfies (11).}$$

This is true if and only if for any $b \in [b_{R+1,R+1}, b_{R,R}]$, $R = 1,...,T-1$ there exists a way to spend b replacing durables only at age R or at age $R+1$ and for any $b \in [0, b_{T,T}]$ there exists a way to spend b replacing durables only at age T or older. Let $\mathcal{B}_k$ denote the set of budgets that can be spent by replacing durables at age R or $R+1$, beginning date τ_k. We will demonstrate that

$$\beta^{T-1}(1+\beta) > 1 \Longleftrightarrow \mathcal{B}_k = [b_{R+1,R+1}, b_{R,R}] \text{ for all } k \geq 0 \text{ and } R = 1,...,T-1$$

Let $\mathcal{B}_{k+1} = [b_{R+1,R+1}, b_{R,R}]$. Then, replacement $k+1$ occurs either R periods or $R+1$ periods after replacement k, so

$$\mathcal{B}_k = \left[p_0 + \beta^{R+1} b_{R+1,R+1}, p_0 + \beta^{R+1} b_{R,R}\right] \cup \left[p_0 + \beta^R b_{R+1,R+1}, p_0 + \beta^R b_{R,R}\right] =$$

$$= \left[b_{R+1,R+1}, p_0 + \beta^{R+1} b_{R,R}\right] \cup \left[p_0 + \beta^R b_{R+1,R+1}, b_{R,R}\right].$$

Therefore,

$$\mathcal{B}_k = \mathcal{B}_{k+1} = [b_{R+1,R+1}, b_{R,R}] \Longleftrightarrow p_0 + \beta^{R+1} b_{R,R} > p_0 + \beta^R b_{R+1,R+1} \Longleftrightarrow$$

$$\Longleftrightarrow \frac{\beta}{1-\beta^R} > \frac{1}{1-\beta^{R+1}} \Longleftrightarrow \beta^R (1+\beta) > 1 \Longleftrightarrow \beta^{T-1} (1+\beta) > 1.$$

Finally, note that for $b \in [0, b_{T,T}]$, $\mathcal{B}_k = \mathcal{B}_{k+1} = [0, b_{T,T}]$ always holds, since

$$\mathcal{B}_k = \cup_{R=T}^{\infty} \left[p_0, p_0 + \beta^R b_{T,T}\right] = [0, b_{T,T}].$$

■

Proof of Lemma 2: First, we will establish that following the flexible rule is feasible from any initial state $\alpha = 1, ..., T$ if and only if (10) holds.

Claim: for every $\alpha = 1, ..., T$

$$b_{\alpha+1,\alpha+1} < b_{\alpha-1,\alpha}$$

if and only if (10) is satisfied

Proof: For $\alpha \leq T-1$ we have

$$b_{\alpha+1,\alpha+1} < b_{\alpha-1,\alpha} \Longleftrightarrow \frac{1}{1-\beta^{\alpha+1}} < \frac{\beta}{1-\beta^{\alpha}} \Longleftrightarrow$$

$$\max_{\alpha} \left(\frac{1-\beta^{\alpha}}{1-\beta^{\alpha+1}}\right) < \beta \Longleftrightarrow$$

$$1 - \beta^{T-1} < \beta - \beta^{T+1} \Longleftrightarrow \beta^{T-1}\left(1-\beta^2\right) > (1-\beta) \Longleftrightarrow$$

$$\beta^{T-1}(1+\beta) > 1.$$

For $\alpha = T$

$$b_{T+1,T+1} < b_{T-1,T} \Longleftrightarrow 1 < \frac{\beta}{1-\beta^T} \Longleftrightarrow \beta^{T-1}(1+\beta) > 1.$$

■

The proof of Lemma 2 will be done by induction. First, set $\alpha = T$ and check that (5) holds and that (13) describes the optimal replacement decision. For $\alpha = T$ (5) becomes

$$V_T(b) = \max\left\{\beta V_T\left(\frac{b}{\beta}\right), V_T(b)\right\}.$$

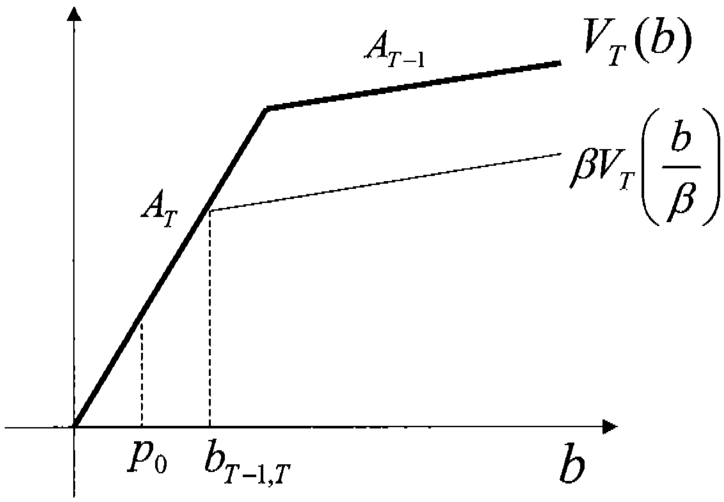
We can compute $\beta V_T\left(\frac{b}{\beta}\right)$ from (9) be noting that

$$\frac{b}{\beta} \in (0, b_{T,T}) \Longleftrightarrow b \in (0, b_{T-1,T})$$

Then, for all $b \in (0, b_{T-1,T})$

$$\beta V_T\left(\frac{b}{\beta}\right) = A_T \cdot b \equiv V_T(b)$$

At $b = b_{T-1,T}$, the slope of $\beta V_T\left(\frac{b}{\beta}\right)$ changes to A_{T-1}, but the slope of of $V_T(b)$ remains at A_T (as shown in the figure below). Besides, $V_T(b)$ is concave, so its slope is higher than that of $\beta V_T\left(\frac{b}{\beta}\right)$.



Therefore, for all $b > b_{T-1,T}$

$$V_T(b) > \beta V_T\left(\frac{b}{\beta}\right),$$

which implies that $\delta_T(b) = 1$. Also, if $b < p_0 \equiv b_{T+1,T+1}$, replacing the durable is not feasible, so $\delta_T(b)$, should equal zero on $[0, b_{T+1,T+1}]$. Finally, since $\beta V_T\left(\frac{b}{\beta}\right) \equiv V_T(b)$ on $[b_{T+1,T+1}, b_{T-1,T}]$, the consumer is indifferent between keeping and replacing good T, so $\delta_T(b) = \{0,1\}$.

Next, suppose that $V_{\alpha+1}(b)$, where $\alpha+1 \leq T$ is described by (12). We will then show that

$$V_\alpha(b) = \max\left\{x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right), V_T(b)\right\}$$

is described by (12) and that

$$\max\left\{x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right), V_T(b)\right\} = \begin{cases} x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right), & b < b_{\alpha+1,\alpha+1} \\ \text{either}, & b_{\alpha+1,\alpha+1} \leq b \leq b_{\alpha-1,\alpha} \\ V_T(b), & b > b_{\alpha-1,\alpha} \end{cases}. \tag{21}$$

To compute the expression for $x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right)$ from (12), note that

$$\frac{b}{\beta} \in (b_{\alpha+1,R+1}, b_{\alpha+1,R}) \Longleftrightarrow b \in (b_{\alpha,R+1}, b_{\alpha,R}), \ R = T, ..., \alpha+1,$$

$$\frac{b}{\beta} \in (b_{R+1,R+1}, b_{R,R}) \Longleftrightarrow b \in (b_{R,R+1}, b_{R-1,R}), \ R = \alpha, ..., 1$$

Substituting this into (12) and using the expression (9) for $V_T(b)$ on $(b_{\alpha+1,\alpha+1}, \infty)$ yields

$$x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right) = \begin{cases} x_\alpha + \beta v_{\alpha+1,R+1} + A_R \cdot (b - \beta b_{\alpha+1,R+1}), & b \in [b_{\alpha,R+1}, b_{\alpha,R}], \; R = T, ..., \alpha+1 \\ x_\alpha + \beta v_{\alpha+1,\alpha+1} + A_\alpha \cdot (b - \beta b_{\alpha+1,\alpha+1}), & b \in [b_{\alpha,\alpha+1}, b_{\alpha-1,\alpha}] \\ x_\alpha + \beta V_T\left(\frac{b}{\beta}\right), & b > b_{\alpha-1,\alpha} \end{cases} \tag{22}$$

We will now show that for all $b \in [b_{\alpha+1,\alpha+1}, b_{\alpha-1,\alpha}]$,

$$x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right) \equiv V_T(b)$$

Since $[b_{\alpha+1,\alpha+1}, b_{\alpha-1,\alpha}] \subset [b_{\alpha,\alpha+1}, b_{\alpha-1,\alpha}]$, using (22) yields

$$x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right) = x_\alpha + \beta v_{\alpha+1,\alpha+1} + A_\alpha \cdot (b - \beta b_{\alpha+1,\alpha+1}).$$

On the other hand, since $[b_{\alpha+1,\alpha+1}, b_{\alpha-1,\alpha}] \subset [b_{\alpha+1,\alpha+1}, b_{\alpha,\alpha}]$, using expression (9) for $R = \alpha$ yields

$$V_T(b) = v_{\alpha+1,\alpha+1} + A_\alpha \cdot (b - b_{\alpha+1,\alpha+1}).$$

These two expressions are identical if and only if

$$x_\alpha + A_\alpha (1-\beta) b_{\alpha+1,\alpha+1} = (1-\beta) v_{\alpha+1,\alpha+1}$$

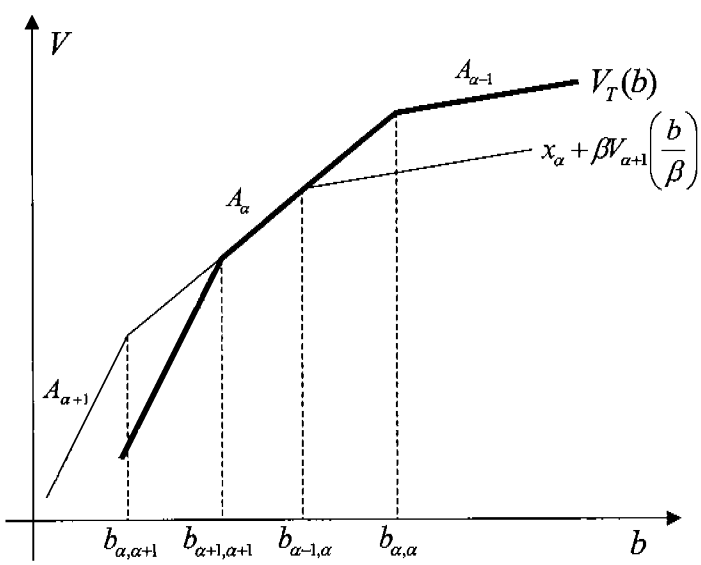
Using (6), (7) and (8) in the above expression, its left hand side becomes

$$x_\alpha + \left(X_{0,\alpha} - \frac{1-\beta^\alpha}{1-\beta} x_\alpha\right) \frac{(1-\beta)}{1-\beta^{\alpha+1}} = \frac{(1-\beta)(X_{0,\alpha} + \beta^\alpha x_\alpha)}{1-\beta^{\alpha+1}} = (1-\beta)\frac{X_{0,\alpha+1}}{1-\beta^{\alpha+1}} = (1-\beta) v_{\alpha+1,\alpha+1}.$$

In order to show (21), it suffices to demonstrate that the difference

$$V_T(b) - \beta V_{\alpha+1}\left(\frac{b}{\beta}\right)$$

is weakly increasing in b. We will show that for every b, $V_T(b)$ has a (weakly) larger slope. For $b \in [b_{\alpha+1,\alpha+1}, b_{\alpha-1,\alpha}]$ we know that both functions are equal and have the same slope A_α (see figure below).



Expression (22) says that for any $b > b_{\alpha-1,\alpha}$

$$x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right) \equiv x_\alpha + \beta V_T\left(\frac{b}{\beta}\right).$$

Then, since $V_T(b)$ is concave, its slope is larger than that of $\beta V_T\left(\frac{b}{\beta}\right)$.

On $[b_{\alpha,\alpha+1}, b_{\alpha+1,\alpha+1}]$, the slope of $V_T(b)$ equals $A_{\alpha+1}$, but the slope of $x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right)$ equals $A_\alpha < A_{\alpha+1}$. Finally, we must show that $V_T(b)$ has (weakly) higher slope than $x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right)$ on $[0, b_{\alpha,\alpha+1}]$. This interval can be represented as

$$[0, b_{\alpha,\alpha+1}] = \cup_{R=\alpha+1}^{T}[b_{\alpha,R+1}, b_{\alpha,R}]$$

According to (22), the slope of $x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right)$ equals A_R on $[b_{\alpha,R+1}, b_{\alpha,R}]$, $R = \alpha+1, ..., T$. If $x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right)$ has the slope no larger than that of $V_T(b)$ at the point $b = b_{\alpha,R}$, then it follows from concavity of $V_T(b)$ that $V_T(b)$ must have the higher slope on the whole interval $[b_{\alpha,R+1}, b_{\alpha,R}]$. To check this, simply note that $b_{\alpha,R} < b_{R,R}$, which implies that the slope of $V_T(b)$ at $b = b_{\alpha,R}$ is at least A_R. Thus we have established (21).

Combining (22), (9) and (21) yields

$$\max\left\{x_\alpha + \beta V_{\alpha+1}\left(\frac{b}{\beta}\right), V_T(b)\right\} =$$

$$= \begin{cases} v_{\alpha,R+1} + A_R \cdot (b - b_{\alpha,R+1}) & b \in [b_{\alpha,R+1}, b_{\alpha,R}],\ R = T, ..., \alpha \\ V_T(b) & b > b_{\alpha,\alpha} \end{cases} \equiv V_\alpha(b).$$

To conclude the proof, note that (13) follows trivially from (21). ■

Proof of Lemma 3: Since $u'(c)$ is strictly decreasing and the slope of $V_\alpha\left(w - \frac{c}{1-\beta}\right)$ is increasing in c for any w, there is a unique c satisfying the first order condition, and this point is a maximum. Taking the first order condition in (3) yields the following two cases:

Case 1 For $w \in [w_{\alpha,R+1}(c_R), w_{\alpha,R}(c_R)]$, $R = T, ..., 1$, the first order condition holds as equality

$$u'(c_\alpha^*(w)) = A_R,$$

which implies that $c_\alpha^*(w) = c_R$.

Case 2 The first order condition holds as inequality. Temporarily, define $A_{T+1} = \infty$, $A_0 = 0$, $c_{T+1} = 0$, and $c_0 = \infty$. Then, whenever

$$A_{R-1} < u'(c_\alpha^*(w)) < A_R,\ R = 1, ..., T+1 \tag{23}$$

it must be the case that

$$w - \frac{c_\alpha^*(w)}{1-\beta} = b_{\alpha,R} \Longleftrightarrow c_\alpha^*(w) = c_{\alpha,R}(w).$$

For (23) to hold when $c^*_\alpha(w) = c_{\alpha,R}(w)$, w must satisfy the inequality

$$c_R < (1-\beta)(w - b_{\alpha,R}) < c_{R-1} \Longleftrightarrow w \in [w_{\alpha,R}(c_R), w_{\alpha,R}(c_{R-1})], \ R = T+1, ..., 1.$$

To get $\delta^*_\alpha(w)$, observe that consumers who are indifferent between keeping and replacing a good of age α must follow an α-flexible rule. Therefore, their non-durable consumption equals c_α, and they act as if they follow the optimal policy (13) with a budget $b = w - \frac{c_\alpha}{1-\beta}$. This immediately implies (15) on $[w_{\alpha+1,\alpha+1}(c_\alpha), w_{\alpha-1,\alpha}(c_\alpha)]$. The budget allocated to durable purchases is an increasing function of the total wealth w. Therefore

$$b > b_{\alpha-1,\alpha} \Longleftrightarrow w > \frac{c^*_\alpha(b_{\alpha-1,\alpha})}{1-\beta} + b_{\alpha-1,\alpha} = w_{\alpha-1,\alpha}(c_\alpha)$$

and

$$b < b_{\alpha+1,\alpha+1} \Longleftrightarrow w < \frac{c^*_\alpha(b_{\alpha+1,\alpha+1})}{1-\beta} + b_{\alpha+1,\alpha+1} = w_{\alpha+1,\alpha+1}(c_\alpha).$$

Then (13) implies (15) for $w < w_{\alpha+1,\alpha+1}(c_\alpha)$ and $w > w_{\alpha-1,\alpha}(c_\alpha)$. ■