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An Examination of Heterogeneous Beliefs with a Short-Sale
Constraint*

Michael Gallmeyer
GSIA and Center for Computational Finance
Tech and Frew Streets
Carnegie Mellon University
Pittsburgh, PA 15213-3890
mg47@andrew.cmu.edu

Burton Hollifield
GSIA and Center for Computational Finance
Tech and Frew Streets
Carnegie Mellon University
Pittsburgh, PA 15213-3890
burtonh@andrew.cmu.edu

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Abstract

We use a calibrated dynamic general equilibrium economy with heterogeneous beliefs to study the effects of a short-sale constraint on stock prices, stock price volatility, and interest rates. We find large stock price and volatility effects from heterogeneous beliefs. When we introduce a stock short-sale constraint, we find small additional stock price valuation effects because of large and offsetting effects on interest rates and Sharpe ratios.

Keywords: heterogeneous beliefs, learning, short-sale constraints, general equilibrium, stock price, stock volatility

JEL Codes: D51, G11, G12, G14

1 Introduction

During the past few years, the U.S. stock market has risen dramatically and stock returns have become much more volatile than in the past. Ofek and Richardson (2001) argue that an important feature of the experience is that investors have heterogeneous beliefs about the underlying values of assets.

Lintner (1969), Miller (1977) Jarrow (1980), Diamond and Verrecchia (1987) and Chen et al. (2001) show that in economies with a short-sale constraints, stock prices may not completely reflect the beliefs of all investors. Specifically, the stock price may be higher than otherwise, if pessimistic investors are constrained from shorting the stock. Diether et al. (2002) provide empirical evidence that individual stock prices tend to increase with high dispersion of analysts' beliefs; stock prices may not reflect the most pessimistic investors' beliefs. Chen et al. (2001), and Jones and Lamont (2001) provide empirical evidence consistent with overpricing of individual stocks when short-sale constraints bind.

What are the quantitative effects of imposing short-sale constraints with heterogeneous beliefs? We examine the effects of short-sale constraints on asset prices and volatility in a dynamic general equilibrium economy, where investors have heterogeneous beliefs about economic fundamentals. For tractability, we analyze a continuous time economy. The economy is made up of optimistic and pessimistic investors who disagree about the expected growth rate of aggregate output. The investors learn about the expected growth rate through observing realized output growth. We interpret the value of a security paying out aggregate output as the value of the stock market in the economy.

We find large stock price and volatility effects from heterogeneous beliefs in parameterized versions of the economy without a short-sale constraint. Introducing a short-sale constraint on the pessimist tends to depress the initial stock price relative to the economy without the short-sale constraint.

Consider a date where the short-sale constraint is binding. Only the optimist holds the stock. As a consequence, the stock risk premium is lower than in an economy without a short-sale constraint, since the risk premium only reflects the optimist's beliefs. In the absence of the constraint, the pessimist would short the stock, and use some of the proceeds to invest in riskless bonds. In the

presence of the constraint, the pessimist cannot short the stock and so reduces his demand for riskless bonds relative to his demand in an economy without a short-sale constraint. The reduction in demand for the bond leads to an increase in the equilibrium riskless interest rate relative to an economy without the short-sale constraint. We find that the interest rate effect is larger than the risk premium effect; the stock price would rise if the short-sale constraint were removed from the economy.

We use recently developed techniques to analyze our dynamic general equilibrium economy with a short-sale constraint, heterogeneous beliefs, and learning. Basak (2000) analyzes a general equilibrium economy with heterogeneous beliefs about sunspots, learning and zero net supply securities, showing that two factor asset pricing results. Basak and Croitoru (2000) study a dynamic general equilibrium economy with heterogeneous beliefs and portfolio constraints, and demonstrate that mispricing can result in with zero net-supply derivative assets.

Zapatero (1998) allows for heterogeneous beliefs in an economy populated with investors who have logarithmic utility, showing that heterogeneous beliefs tend to increase interest rate volatility. Gallmeyer (2000) studies the effects of heterogeneous beliefs and learning on stock market volatility, showing that heterogeneous beliefs leads to significant effects on stock volatility in the presence of learning.

Our model features both heterogeneous preferences and heterogeneous beliefs. With homogeneous beliefs, our model is similar to the economies studied by Dumas (1989) and Wang (1996). If the investors have homogeneous beliefs and learn about the growth rate of the aggregate endowment, then our model corresponds to a heterogeneous investor version of the learning economies studied by Brennan and Xia (1999), Veronesi (1999), Veronesi (2000) and Yan (1998).

Harrison and Kreps (1978), Morris (1996) and Scheinkman and Xiong (2002) show that in economies with short-sale constraints and heterogeneous expectations, stock prices can be above the expected present value of dividends under the most optimistic investor's beliefs. Detemple and Murthy (1997) analyze general equilibrium continuous-time economies with portfolio and collateral constraints, and heterogeneous beliefs. Equilibrium stock prices can be decomposed into a consumption value, a speculative value premium, and a collateral value. All the models feature a resale premium in the equilibrium price of the stock; investors know that they may not always

have the highest valuation for the asset, and so are willing to pay above their estimate of the present value of dividends for the option to resell the asset to another agent in the future. Duffie et al. (2002) study a model with heterogeneous beliefs in a search environment, showing that the stock price can be above the highest possible valuation for the dividends, because of an endogenous shorting fee.

In our model, the stock price can be strictly higher than the pessimistic investor's valuation for the asset. The pessimistic investor may hold a positive amount of the stock in his portfolio, even when the stock price is strictly greater than his valuation for the stock. The pessimistic investor holds the stock when he believes it is mispriced because there is positive probability that he can sell the stock to a more optimistic investor in the future. There is a resale premium from the point of view of the pessimistic investor. Even when the stock is overvalued from the point of view of the pessimistic investor, the stock price would rise if the short-sale constraint were removed.

The effects of constraints on stock prices have also been studied in models with homogeneous beliefs. Heaton and Lucas (1996) compute numerical solutions to an economy with heterogeneous preferences, idiosyncratic income shocks, short-sale constraints, and transactions costs, finding small effects on the equity premium, unless the transactions costs are large. Coen-Pirani (2001) studies the effects of introducing margin requirements in an economy with Epstein-Zin preferences and unitary elasticity of substitution, finding small stock price effects and large interest rate effects from introducing margin requirements.

2 The Economy

We study a continuous-time pure exchange economy with heterogeneous investors. The economy has a finite horizon equal to $T < \infty$. All uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$.

The exogenous aggregate output $\delta(t)$ follows the process

$$d\delta(t) = \delta(t) [\mu_\delta(t)dt + \sigma_\delta dw(t)], \quad \delta(0) > 0, \quad (1)$$

where $w(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. The diffusion coefficient is

constant with $\sigma_\delta > 0$. More general forms of dependence in the diffusion coefficient are possible. The only requirement is that the diffusion coefficient is \mathcal{F}_t^δ -progressively measurable, where \mathcal{F}_t^δ is the augmented filtration generated by $\delta(t)$.

We allow two processes for the expected growth rate of output, $\mu_\delta(t)$. In the first process, expected output growth is constant; output follows geometric Brownian motion,

$$\mu_\delta(t) = \mu_\delta(0) \quad \forall t \in [0, T]. \quad (2)$$

In the second process, the expected output growth follows a regime switching process. Specifically, $\mu_\delta(t)$ is a N state continuous time Markov chain,

$$\mu_\delta(t) \in \{\mu_1, \mu_2, \dots, \mu_N\}, \quad (3)$$

with $\mu_\delta(0)$ the initial value of expected output growth. The characterization of the equilibrium in the next section does not depend on the functional form the expected output process. Our two different models for $\mu_\delta(t)$ are used in numerical analysis in Section 4.

Investors do not observe the expected growth rate of output, they only observe realized output. All investors agree that the evolution of output is governed by the model in equation (1), and agree on the process governing the evolution of expected output growth. Heterogeneous beliefs are modeled by endowing different investors with different priors about the initial value of output growth, $\mu_\delta(0)$.

The economy is populated by two classes of investors: optimists and pessimists. The optimists all have the same preferences, endowments, beliefs and constraints, and so can be summarized by a representative optimist. Similarly, the pessimists can be summarized by a representative pessimist. Optimists and pessimists differ in their preferences, endowments, beliefs and constraints. The superscript o refers to the optimists and the superscript p refers to the pessimists.

For investor $i = \{o, p\}$, uncertainty in the economy is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t^\delta\}, \mathcal{P}^i)$. Beliefs about expected output growth are given by the conditional expected

tations

$$\mu_\delta^o(t) = E^i \left[\mu_\delta(t) \middle| \mathcal{F}_t^\delta \right], \quad i = \{o, p\}, \quad (4)$$

where the superscript i on the expectation operator denotes taking expectations using investor i 's beliefs, \mathcal{P}^i . Optimists have weakly higher expectations about initial output growth than the pessimists,

$$\mu_\delta^o(0) \geq \mu_\delta^p(0). \quad (5)$$

Given the two investors start with different priors, they will continue to disagree about the expected output growth over the life of the economy as they update their beliefs using the commonly observed output process. Following standard filtering theory as in Liptser and Shiriyayev (1977), define the investor-specific innovation processes,

$$\begin{aligned} dw^i(t) &= \frac{1}{\sigma_\delta} \left[\frac{d\delta(t)}{\delta(t)} - \mu_\delta^i(t) dt \right] \\ &= \frac{\mu_\delta(t) - \mu_\delta^i(t)}{\sigma_\delta} dt + dw(t), \quad i = \{o, p\}. \end{aligned} \quad (6)$$

From equation (6), the innovation processes across investors are related by

$$dw^p(t) - dw^o(t) = \bar{\mu}(t) dt, \quad (7)$$

where

$$\bar{\mu}(t) \equiv \frac{\mu_\delta^o(t) - \mu_\delta^p(t)}{\sigma_\delta}. \quad (8)$$

The process $\bar{\mu}(t)$ measures investors' differences of opinion; we call it the disagreement process. The disagreement process is given exogenously, since all learning about expected output growth comes from observing realized output.

Lemma 1 *Under each investor's beliefs, the aggregate output process is*

$$d\delta(t) = \delta(t) \left[\mu_\delta^i(t) dt + \sigma_\delta dw^i(t) \right], \quad \delta(0) > 0, \quad i \in \{o, p\}. \quad (9)$$

Each investor's beliefs about the expected growth of aggregate output follows the process

$$d\mu_\delta^i(t) = m^i(t) dt + \eta^i(t) dw^i(t), \quad i \in \{o, p\}, \quad (10)$$

where $m^i(t)$ and $\eta^i(t)$ are \mathcal{F}_t^δ -measurable, and $w^i(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^\delta\}, \mathcal{P}^i)$, for $i = \{o, p\}$.

As long as (9) has a strong solution, the investors' information and innovation filtrations coincide. Each investor acts as an investor with complete information, who faces uncertainty in the economy generated by $w^i(t)$. The processes $\delta(t), \mu^o(t), \mu^p(t), m^i(t)$, and $\eta^i(t)$, for $i \in \{o, p\}$ need not be Markovian.

Two securities are traded, a locally risk less bond and a stock with dividends equal to aggregate output. The stock is interpreted as the market portfolio. The bond is in zero net supply with price $B(t)$. Posited bond price dynamics are

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1, \quad (11)$$

with $r(t)$ the instantaneously risk less interest rate in the economy. The stock is in fixed net supply of one share, paying a continuous dividend rate equal to aggregate output, $\delta(t)$. The posited gains process for the stock is

$$dS(t) + \delta(t)dt = S(t) [\mu(t)dt + \sigma(t)dw(t)], \quad S(T) = 0. \quad (12)$$

Equation (12) gives the dynamics of the stock under the true probability beliefs.

Under each investor's probability beliefs, the stock gains process is

$$dS(t) + \delta(t)dt = S(t) [\mu^i(t)dt + \sigma(t)dw^i(t)], \quad S(T) = 0, \quad i = \{o, p\}. \quad (13)$$

Here $\mu^i(t)$ is investor i 's beliefs about the expected stock gains. All investors agree on the diffusion

coefficient of the stock gains process. Stock price agreement requires that

$$\frac{\mu^o(t) - \mu^p(t)}{\sigma(t)} = \bar{\mu}(t). \quad (14)$$

All investors choose consumption and portfolio strategies to maximize lifetime utility. The optimist's portfolios are unconstrained; they face a complete financial market. The pessimist's portfolios are constrained from shorting the stock; they face an incomplete financial market.

Each investor's lifetime utility function is time and state separable and all investors have the same time-preference parameter equal to ρ . Optimists have constant relative risk averse period utility functions and pessimists have logarithmic period utility functions.

Assuming that the short-sale constrained pessimists have logarithmic preferences allows us to obtain closed form solutions for the pessimists portfolio and consumption decisions. Assuming that the optimists have non-logarithmic preferences leads to both stock and interest rate price effects from heterogeneous beliefs. Heterogeneous beliefs do not change stock prices if both investors have logarithmic preferences. The result is shown by Zapatero (1998) without portfolio constraints and Detemple and Murthy (1994) with collateral and short-sale constraints.

Turning now to the optimist's consumption portfolio problem, the optimist's wealth $X^o(t)$ evolves as

$$dX^o(t) = X^o(t)r(t)dt - c^o(t)dt + \theta^o(t)S(t) [(\mu^o(t) - r(t))dt + \sigma(t)dw^o(t)], \quad X^o(0) = x^oS(0), \quad (15)$$

with x^0 be the optimist's initial holdings of shares, $\theta^o(t)$ the optimist's shareholding at t , $\alpha^o(t)$ the optimist's bond holdings at t , and $c^o(t)$ the optimist's consumption rate at t . Following Dybvig and Huang (1988), we impose a non-negative wealth constraint, $X^o(t) \geq 0$, to rule out doubling strategies.

The optimist faces complete markets. We apply martingale techniques to characterize the optimal consumption and trading strategy, as in Cox and Huang (1989) and Karatzas et al. (1987). Using martingale techniques, we can solve the optimist's problem while allowing equilibrium securities prices to be non-Markovian.

Summarizing the stock and bond price system, the optimist's state price density, $\xi^o(t)$ has

dynamics given by

$$d\xi^o(t) = -\xi^o(t) [r(t)dt + \kappa^o(t)dw^o(t)], \quad \xi^o(0) = 1. \quad (16)$$

Here $\kappa^o(t)$ is the market price of risk, or instantaneous Sharpe ratio, under the optimist's beliefs,

$$\kappa^o(t) = \frac{\mu^o(t) - r(t)}{\sigma(t)}. \quad (17)$$

The optimist's lifetime consumption portfolio problem is

$$\max_{\{c^o(t)\}} E^o \left[\int_0^T e^{-\rho t} \frac{c^o(t)^{1-\gamma} - 1}{1-\gamma} dt \right], \quad (18)$$

subject to:

$$E^o \left[\int_0^T \xi^o(t) c^o(t) dt \right] \leq x^o S(0). \quad (19)$$

The optimal consumption function is

$$\hat{c}^o(y^o \xi^o(t), t) = \left(\frac{e^{-\rho t}}{y^o \xi^o(t)} \right)^{\frac{1}{\gamma}}, \quad (20)$$

and the multiplier y^o is

$$y^o = \left(\frac{E^o \left[\int_0^T e^{-\rho t} \xi^o(t)^{\frac{\gamma-1}{\gamma}} dt \right]}{x^o S(0)} \right)^{\gamma}. \quad (21)$$

The optimist's optimal wealth process, $\hat{X}^o(t)$, satisfies

$$\hat{X}^o(t) = \frac{1}{\xi^o(t)} E^o \left[\int_t^T \xi^o(s) \hat{c}^o(y^o \xi^o(s), s) ds \middle| \mathcal{F}_t^{\delta} \right]. \quad (22)$$

The optimal portfolio is found by solving for the trading strategy that replicates the optimal wealth process.

Turning now to the pessimist's consumption portfolio problem, his wealth $X^p(t)$ evolves as

$$dX^p(t) = X^p(t)r(t)dt - c^p(t)dt + \theta^p(t)S(t) [(\mu^p(t) - r(t))dt + \sigma(t)dw^p(t)], \quad X^p(0) = x^p S(0). \quad (23)$$

Here, x^p is the pessimist's initial holdings of shares, $\theta^p(t)$ is the pessimist's shareholding at t , $\alpha^p(t)$ is the pessimist's bond holdings at t , $\theta^p(t)$ and $c^p(t)$ is the pessimist's consumption rate at t . The pessimist's dynamic budget constraint is The short-sale constraint is

$$\theta^p(t) \geq 0, \quad \forall t. \quad (24)$$

Since the pessimist faces a short-sale constraint, he faces an incomplete market. We use the approach developed by He and Pearson (1991) and Karatzas et al. (1991) to solve the pessimist's problem. We embed the constrained pessimist's problem in a fictitious, unconstrained complete markets problem. With the appropriate choice of state prices in the fictitious market, the solution to the problem in the fictitious market is the solution to the original constrained problem. With logarithmic preferences for the constrained pessimist, we can analytically compute the fictitious market price system with general underlying price systems. With non-logarithmic preferences, we cannot analytically compute the fictitious price system without making strong restrictions on the underlying price system.

The market price of risk under the pessimist's beliefs is defined as

$$\kappa^p(t) = \frac{\mu^p(t) - r(t)}{\sigma(t)}. \quad (25)$$

Since the investors agree on the interest rate and stock price processes, the optimist's and pessimist's price of risk are linked through the disagreement process by

$$\kappa^o(t) - \kappa^p(t) = \bar{\mu}(t). \quad (26)$$

If the pessimist does not face a short-sale constraint his optimal portfolio is the log-optimal portfolio. The log-optimal portfolio is short in the stock only when the pessimist's instantaneous market price of risk is negative. The short selling constraint is therefore binding in the original lifetime consumption portfolio problem only when $\kappa^p(t) < 0$. From equation (26), $\kappa^p(t) < 0$ when

$$\bar{\mu}(t) > \kappa^o(t). \quad (27)$$

The constrained pessimist's state price density in the fictitious market is given by $\xi^p(t)$, and follows the process

$$d\xi^p(t) = -\xi^p(t) [r(t)dt + \max(\kappa^p(t), 0) dw^p(t)], \quad \xi^p(0) = 1. \quad (28)$$

Equation (28) implies that the state price density in the fictitious market is locally deterministic when the short-sale constraint is binding.

The pessimist's lifetime consumption portfolio choice problem in the fictitious unconstrained economy is given by

$$\max_{\{c^p(t)\}} E^p \left[\int_0^T e^{-\rho t} \ln(c^p(t)) dt \right], \quad (29)$$

subject to:

$$E^p \left[\int_0^T \xi^p(t) c^p(t) dt \right] \leq x^p S(0). \quad (30)$$

The optimal consumption function is

$$\hat{c}^p(y^p \xi^p(t), t) = \frac{e^{-\rho t}}{y^p \xi^p(t)}, \quad (31)$$

where the multiplier y^p is

$$y^p = \frac{1 - e^{-\rho T}}{\rho x^p S(0)}. \quad (32)$$

The optimal consumption function in equation (31) and the dynamics of the pessimist's fictitious state price density (28) imply that the pessimist's optimal consumption is locally deterministic when the constraint is binding.

The optimal wealth process of the pessimist, $\hat{X}^p(t)$, satisfies

$$\hat{X}^p(t) = \frac{1}{\xi^p(t)} E_t^p \left[\int_t^T \xi^p(s) \hat{c}^p(y^p \xi^p(s), s) ds \middle| \mathcal{F}_t^\delta \right]. \quad (33)$$

The optimal portfolio strategy is to fully invest in the bond when the market price of risk is negative and hold the log-optimal portfolio otherwise.

3 Characterization of Equilibrium

Equilibrium is defined as follows.

Definition 1 *Given preferences, endowments, and beliefs, an equilibrium is a collection of allocations $((\hat{c}^o, \hat{\alpha}^o, \hat{\theta}^o), (\hat{c}^p, \hat{\alpha}^p, \hat{\theta}^p))$ and a price system $(r, \mu^o, \mu^p, \sigma)$ such that $(\hat{c}^i, \hat{\alpha}^i, \hat{\theta}^i)$ is an optimal solution to investor i 's consumption-portfolio problem given his perceived price processes, security prices are consistent across investors, and all markets clear for $t \in [0, T]$:*

$$\begin{aligned} \hat{c}^o(t) + \hat{c}^p(t) &= \delta(t), & \text{Consumption Market,} \\ \hat{\theta}^o(t) + \hat{\theta}^p(t) &= 1, & \text{Stock Market,} \\ \hat{\alpha}^o(t) + \hat{\alpha}^p(t) &= 0, & \text{Bond Market.} \end{aligned} \tag{34}$$

The martingale formulation of each investor's consumption-portfolio problem provides a direct method to construct the equilibrium. To determine the equilibrium price system, it suffices to find the two state-price density processes that clear the consumption market,

$$\hat{c}^o(y^o \xi^o(t), t) + \hat{c}^p(y^p \xi^p(t), t) = \delta(t). \tag{35}$$

Once the two state price processes are determined from equation (35), they uniquely determine the equilibrium stock and bond prices. If the state price processes clear the consumption markets, the associated optimal portfolio strategies also clear the stock and bond markets.

Rather than working directly with both state price processes in the consumption market clearing condition above, it is convenient to define the stochastic weighting process $\lambda(t)$ given by

$$\lambda(t) = \frac{y^o \xi^o(t)}{y^p \xi^p(t)}, \tag{36}$$

where $\lambda(0) = \frac{y^o}{y^p}$, since $\xi^o(0) = \xi^p(0) = 1$.

The process conveniently summarizes the differences in the investor's opportunity sets from heterogeneous beliefs and the short-sale constraint by directly modeling the ratio of marginal utilities across the investors. Using the dynamics of the state price densities in equations (16) and (28), the

dynamics of the stochastic weighting process are

$$d\lambda(t) = -\lambda(t) \min(\bar{\mu}(t), \kappa^0(t)) dw^o(t). \quad (37)$$

When the short-sale constraint is not binding, the volatility of the weighting process depends only on the exogenous disagreement process. When the short-sale constraint is binding, the pessimists state prices are locally deterministic; the volatility of the ratio of state prices depends only on the optimist's market price of risk, $\kappa^o(t)$. The initial level of the weighting process, $\lambda(0)$, is determined by the initial endowment structure in the economy. As a result, the dynamics of the weighting process are completely determined by initial endowments, the exogenous disagreement process and the optimist's market price risk.

Using the weighting process, the consumption market clearing condition is

$$\hat{c}^o(y^o \xi^o(t), t) + \hat{c}^p(y^o \xi^o(t)/\lambda(t), t) = \delta(t). \quad (38)$$

Equation (38) can be used to uniquely solve for the optimist's state price density in terms of aggregate output, $\delta(t)$ and the level of the stochastic weighting process, $\lambda(t)$. See, for example Karatzas et al. (1990). The resulting solution can be interpreted in terms of the marginal utility of a state-dependent representative agent,

$$y^o \xi^o(t) = e^{-\rho t} U_c(\delta(t), \lambda(t)), \quad y^o = U_c(\delta(0), \lambda(0)). \quad (39)$$

The function $U_c(\delta(t), \lambda(t))$ is the marginal utility of a state-dependent representative agent,

$$U(c, \lambda) \equiv \max_{c^o, c^p} \frac{(c^o)^{1-\gamma} - 1}{1-\gamma} + \lambda \ln(c^p), \quad \text{subject to: } c^o + c^p = c. \quad (40)$$

Such a weak notion of aggregation was introduced by Cuoco and He (1994) and has been used in a variety of models with incompleteness. See for example Basak and Cuoco (1998); Basak (2000); Detemple and Serrat (1998); Basak and Gallmeyer (2000).

Proposition 1 *In equilibrium, the state price density for the optimist and the state price density*

in the fictitious market for the pessimist are

$$\xi^o(t) = e^{-\rho t} \frac{U_c(\delta(t), \lambda(t))}{U_c(\delta(0), \lambda(0))}, \quad \xi^p(t) = \frac{\lambda(0)}{\lambda(t)} \xi^o(t). \quad (41)$$

Defining

$$\sigma_\lambda(t) = \min \left(\bar{\mu}(t), \left(\frac{\gamma}{\hat{c}^o(\delta(t), \lambda(t), t)} \right) \delta(t) \sigma_\delta \right), \quad (42)$$

the stochastic weighting process $\lambda(t)$ has dynamics

$$d\lambda(t) = -\lambda(t) \sigma_\lambda(t) dw^o(t), \quad (43)$$

and $\lambda(0)$ is solved from the pessimists budget constraint,

$$\frac{U_c(\delta(0), \lambda(0))}{\lambda(0)} = \frac{1 - e^{-\rho T}}{\rho x^p S(0)}. \quad (44)$$

The stock price process $S(t)$ is

$$S(t) = E^o \left[\int_t^T e^{-\rho s} \frac{U_c(\delta(s), \lambda(s))}{U_c(\delta(t), \lambda(t))} \delta(s) ds \middle| \mathcal{F}_t^\delta \right]. \quad (45)$$

Equilibrium consumption allocations are

$$\hat{c}^o(\delta(t), \lambda(t)) = \left(\frac{1}{U_c(\delta(t), \lambda(t))} \right)^{\frac{1}{\gamma}}, \quad \hat{c}^p(\delta(t), \lambda(t)) = \frac{\lambda(t)}{U_c(\delta(t), \lambda(t))}. \quad (46)$$

Conversely, if there exists processes $\xi^o(t)$, $\xi^p(t)$, $\lambda(t)$, and $S(t)$ satisfying equations (41), (43), (44) and (45), the associated optimal consumption-wealth-portfolio policies clear all markets.

Existence of an equilibrium is shown by establishing that a solution to the stochastic weighting process in equations (43) and (44) exist and all resulting stochastic processes are well-defined.

The initial value of the weighting process is set to satisfy the budget constraint of the pessimist. The disagreement process enters the equilibrium through the dynamics of the weighting process. The optimist is unconstrained in stock holdings; the stock price is determined by the optimist's marginal utility. In general, the equilibrium is non-Markovian, since the disagreement process and

aggregate output are not restricted to be Markovian.

Two regions of the equilibrium exist. In the first region, the disagreement process is large enough so that the pessimist desires to short the stock. Here, the short-sale constraint is binding. The dynamics of the weighting process in the first region depend only on the optimist's market price of risk, as in the limited participation equilibrium in Basak and Cuoco (1998). In the second region, the disagreement process is small enough so that the pessimist desires to go long the stock. Here, the short-sale constraint is not binding. The dynamics of the weighting process in the second region depend only on the disagreement process as in the heterogeneous beliefs equilibrium in Gallmeyer (2000).

Proposition 2 *The price of risk for the optimist is given by*

$$\kappa^o(t) = \begin{cases} A^o(t)\delta(t)\sigma_\delta, & \text{constraint binds,} \\ A(t)\delta(t)\sigma_\delta + \frac{A(t)}{A^p(t)}\bar{\mu}(t), & \text{else,} \end{cases} \quad (47)$$

where $A^i(t)$, $i = \{o, p\}$ is the absolute risk aversion coefficient for each investor and $A(t)$ is the absolute risk aversion coefficient for the state-dependent representative agent.

$$A^o(t) \equiv \frac{\gamma}{\hat{c}^o(t)}, \quad A^p(t) \equiv \frac{1}{\hat{c}^p(t)}, \quad A(t) \equiv \frac{A^o(t)A^p(t)}{A^o(t) + A^p(t)}. \quad (48)$$

The instantaneous interest rate is

$$r(t) = \begin{cases} \rho + A(t)\delta(t)\mu^o(t) - \frac{1}{2}A(t)B^o(t)\delta(t)^2\sigma_\delta^2, & \text{constraint binds,} \\ \rho + A(t)\delta(t)\left(\frac{A(t)}{A^o(t)}\mu^o(t) + \frac{A(t)}{A^p(t)}\mu^p(t)\right) - \frac{1}{2}A(t)B(t)\delta(t)^2\sigma_\delta^2 & \text{else,} \\ -\frac{A(t)^2}{A^o(t)A^p(t)}\left(\frac{1}{2}\frac{B^o(t)+B^p(t)}{A^o(t)+A^p(t)} - 1\right)\bar{\mu}(t)^2 \\ -\frac{A(t)^3}{A^o(t)A^p(t)}\left(\frac{B^o(t)}{A^o(t)} - \frac{B^p(t)}{A^p(t)}\right)\delta(t)\sigma_\delta\bar{\mu}(t), \end{cases} \quad (49)$$

where $B^i(t)$, $i = \{o, p\}$ is the absolute prudence coefficient for each agent and $B(t)$ is the absolute prudence coefficient of the state-dependent representative agent,

$$B^o(t) \equiv \frac{\gamma + 1}{\hat{c}^o(t)}, \quad B^p(t) \equiv \frac{2}{\hat{c}^p(t)}, \quad B(t) \equiv \left(\frac{A(t)}{A^o(t)}\right)^2 B^o(t) + \left(\frac{A(t)}{A^p(t)}\right)^2 B^p(t). \quad (50)$$

Our model is consistent with the one period models with heterogeneous beliefs and short-selling constraints in Lintner (1969), Miller (1977) and Jarrow (1980). The market price of risk only depends on the optimist's beliefs when the constraint is binding, and on both the optimist's beliefs and pessimist's beliefs when the constraint is not binding. Unlike in the one period models where the interest rate is exogenous, the interest rate in our model depends only on the optimist's beliefs when the constraint is binding and on both the optimist's beliefs and the pessimist's beliefs when the constraint is not binding.

In the unconstrained region of the equilibrium, the first three terms in the interest rate are of the same form as in a common belief economy. In the heterogeneous beliefs economy, a risk-tolerance weighted average of the investor's beliefs about output growth replaces the investor's common beliefs in the common belief economy. The final two terms capture the effects of heterogeneous beliefs and heterogeneous preferences on the interest rate arising from the risk shifting induced by the disagreement process. The final two terms are both negative when the optimist is less risk-averse than the pessimist.

Given a large enough level of disagreement, the short-sale constraint binds.

Corollary 1 *The constraint is binding when*

$$\bar{\mu}(t) \geq A^o(t)\delta(t)\sigma_\delta. \quad (51)$$

Corollary 2 *If the short-selling constraint does not bind, then the optimist's price of risk is increasing in the disagreement process. If the short-selling constraint binds, then the optimist's price of risk and equilibrium interest rate are independent of the level of disagreement.*

We now compare the risk premium and instantaneous interest rate a constrained and unconstrained economy, with same level level of current output and stochastic weighting process.

Lemma 2 *Consider two economies at time t with the same time t consumption allocations and same level of the disagreement process. In one of the economies, the short-sale constraint is binding while in the other economy there is no constraint. In the short-sale constrained economy, the optimist's market price of risk is lower than in the unconstrained economy. A bound $\gamma^* > 1$*

exists such that for $\gamma \leq \gamma^*$, the instantaneous interest rate is higher in the constrained than in the unconstrained economy.

When the optimist's relative risk aversion coefficient is small enough, there are two offsetting effects on the stock price caused by a binding constraint relative to an unconstrained economy. Instantaneous riskless interest rates increase, and equity risk premium decrease. The net effect of heterogeneous beliefs on stock prices with a short-selling constraint is therefore ambiguous.

In addition to the stock price, we are interested in the effect of heterogeneous beliefs and the short-sale constraint on the stock's volatility. We use the Clark-Ocone to compute the stock's volatility. The Clark-Ocone formula is used by Ocone and Karatzas (1991) and Detemple et al. (2002) to compute trading strategies in an optimal investment problem and by Fournié et al. (1999) to compute hedging parameters in option pricing applications.

The following proposition provides a decomposition of the stock's volatility. It is an extension of the representation in Gallmeyer (2000).

Proposition 3 *Stock volatility is given by*

$$\begin{aligned} \sigma(t) = & A(t)\delta(t)\sigma_\delta + \frac{E^\circ \left[\int_t^T \xi^\circ(s)(1 - A(s)\delta(s))\mathcal{D}_t\delta(s)ds \middle| \mathcal{F}_t^\delta \right]}{E^\circ \left[\int_t^T \xi^\circ(s)\delta(s)ds \middle| \mathcal{F}_t^\delta \right]} \\ & + \frac{A^\circ(t)}{A^\circ(t) + A^p(t)}\sigma_\lambda(t) + \frac{E^\circ \left[\int_t^T \xi^\circ(s) \frac{A^\circ(s)}{A^\circ(s) + A^p(s)} \frac{\delta(s)}{\lambda(s)} \mathcal{D}_t\lambda(s)ds \middle| \mathcal{F}_t^\delta \right]}{E^\circ \left[\int_t^T \xi^\circ(s)\delta(s)ds \middle| \mathcal{F}_t^\delta \right]}. \end{aligned} \quad (52)$$

where $\mathcal{D}_t\delta(s)$ is the the Malliavin derivative for the output process, $\delta(u)$, $\mathcal{D}_t\lambda(u)$ is the the Malliavin derivative for the weighting process, $\lambda(u)$.

The only stochastic shock in the economy is the shock in the output process. There are two state variables relevant to the current stock price that react to the shock; the output process itself and the stochastic weighting process, $\lambda(t)$.

The terms in the first row of equation (52) measure the effect of a shock to output on the stock price holding $\lambda(t)$ constant. The first term is the effect on today's state price density. The second term is the effect on the variability of future cash flows from a shock to today's output, holding the future path of $\lambda(t)$ fixed. The Malliavin derivative $\mathcal{D}_t\delta(s)$ measures how perturbing $w^\circ(t)$ affects

the instantaneous change of $\delta(s)$ for $s \geq t$.

The terms in the second line equation (52) capture the effect of the shock through $\lambda(t)$ on the stock price, holding the output fixed. When beliefs are identical and the short sale constraint does not bind the weighting process is constant; the terms in the second line are zero. The first term in the second line is the effect of the shock through $\lambda(t)$ on today's state price density. The second term is the contribution of the stock's volatility from future shifts in $\lambda(s)$. The Malliavin derivative $\mathcal{D}_t \lambda(s)$ measures how perturbing $w^o(t)$ affects the instantaneous change of $\lambda(s)$ for $s \geq t$.

Define the pessimist's marginal valuation for the stock at time t as the price that the pessimist would be willing to pay for a small quantity of the stock at time t if he were required to hold it forever. If the pessimist did not face a short-selling constraint, the equilibrium price of the stock would equal the pessimist's marginal valuation for the stock. The fictitious state-price density process used to solve the pessimist's consumption-portfolio problem provides a way to calculate the pessimist's marginal valuation for the stock. The pessimist's marginal valuation for the stock has dynamics,

$$dS^p(t) + \delta(t)dt = S^p(t) [\max(\mu^p(t), r(t))dt + \sigma(t)dw^p(t)], \quad S^p(T) = 0, \quad (53)$$

and solution

$$S^p(t) = E^p \left[\int_t^T e^{-\rho s} \frac{u_c^p(\hat{c}^p(\delta(s), \lambda(s)))}{u_c^p(\hat{c}^p(\delta(t), \lambda(t)))} \delta(s) ds \middle| \mathcal{F}_t^\delta \right], \quad (54)$$

with $u_c^p(\delta(s), \lambda(s))$ the pessimist's marginal utility.

Lemma 3 *The equilibrium stock price and marginal valuation processes satisfy*

$$\begin{aligned} S(t) &> S^p(t), \quad \text{if the constraint binds with positive probability for some } t' \geq t, \\ S(t) &= S^p(t), \quad \text{else.} \end{aligned} \quad (55)$$

If the constraint is not binding for the pessimist at time t , the pessimist holds the log-optimal portfolio, which is long the stock. If there is positive probability that the short-selling constraint will bind at some time in the future, the stock price is higher than the pessimist's marginal valuation for the stock. The difference between the pessimist's marginal valuation and the stock price can be

interpreted as a speculative premium. The speculative premium arises because the pessimist will sell stock to the optimist at a price higher than the pessimist's valuation, when the constraint next binds. Harrison and Kreps (1978), Morris (1996), Scheinkman and Xiong (2002) and Detemple and Murthy (1997) analyze models where all investors face a short-sale constraint; the market price is higher than all investors' marginal valuations for the stock.

4 Calibration Results

When the short sale constraint binds, the market price of risk for the optimist is lower than in an unconstrained heterogeneous beliefs equilibrium. For low enough risk aversion for the optimist, the interest rate is higher than in an unconstrained heterogeneous beliefs equilibrium. As a result, it is unclear how the stock price is influenced by the short sale constraint.

We examine parameterized versions of the model to study how the stock price is influenced by imposing the short-sale constraint. We assume that optimist has square root preferences: $\gamma = 0.5$. Both investor's rate of time preference ρ parameters are set equal to 0.03. Both the optimist's and the pessimist's initial endowment is 50% of the market portfolio and zero bonds. The economy has a 20 year horizon.

We assume that the optimist is more risk-averse than the pessimist for two reasons. First, the more risk-averse the optimist is, the lower his demand for stock. If the optimist were more risk-averse than the pessimist, the optimist's demand for stock would generally be lower than the pessimist's and the short-sale constraint would bind only for extreme levels of disagreement. Second, if the optimist were more risk-averse than the pessimist, then representative agent would be more risk-averse than log. In this case, equilibrium stock prices would be decreasing in expected output growth. Good news about expected output growth would then decrease the equilibrium stock price.

Two models for expected output growth are considered. In the first model, expected output growth is constant; output follows geometric Brownian motion. In the second model, expected output growth is regime switching; it follows a continuous time, discrete state Markov chain.

In the geometric Brownian motion case, heterogeneous beliefs arise by assuming each investor

has a different Gaussian prior about the drift of the output process $\mu_\delta(0)$.

$$\mu_\delta^i \sim \mathcal{N}(\mu_\delta^i(0), \nu^i(0)), \quad i \in \{o, p\}, \quad (56)$$

where $\mu_\delta^i(0)$ is investor i 's initial belief about the dividend process' expected return with $\nu^i(0)$ the initial variance of his beliefs.

Each investor's optimal filtering problem is given by a standard Kalman-Bucy filter. Investor i 's prediction $\mu_\delta^i(t)$ has dynamics:

$$d\mu_\delta^i(t) = \frac{\nu^i(t)}{\sigma_\delta} dw^i(t), \quad \nu^i(t) = \frac{\nu^i(0)\sigma_\delta^2}{\nu^i(0)t + \sigma_\delta^2}, \quad i \in \{o, p\}. \quad (57)$$

Details about the optimal filtering problem can be found in Liptser and Shiriyayev (1977). As t gets large, both investors learn the true expected growth of output.

Under the optimist's beliefs, the disagreement process $\bar{\mu}(t)$ has the dynamics,

$$d\bar{\mu}(t) = -\frac{\nu(t)^p}{\sigma_\delta^2} \bar{\mu}(t) dt + \frac{\nu^o(t) - \nu^p(t)}{\sigma_\delta^2} dw^o(t). \quad (58)$$

At t gets large, the disagreement process converges to zero. If both investors beliefs have the same initial precision, the disagreement process is deterministic under all beliefs. If the optimist's prior beliefs have higher volatility than the pessimist's prior beliefs, the disagreement has a positive instantaneous correlation with output growth. If the optimist's prior beliefs have lower initial variance than the pessimists prior beliefs, the disagreement process has a negative instantaneous correlation with output growth. When both investor's prior beliefs have zero volatility, the disagreement process is a constraint. Here, investors are so confident in their priors that they completely ignore any information from the endowment process.

Proposition 1 provides a method to construct the equilibrium numerically. From equation (43), the weighting process dynamics do not depend on stock price dynamics. Conditional on an initial guess for $\lambda(0)$, we solve the weighting process separately from the stock price, and solve for the stock price from a Monte-Carlo simulation of equation (45). We then iterate to find a $\lambda(0)$ solving equation (44). Stock volatility is constructed through a Monte-Carlo simulation of equation (52).

The Malliavin derivatives needed to compute stock volatility is provided in Appendix B. Technical details of the simulations are in Appendix C.

Table 1 reports the results for benchmark economies where the investors have homogeneous beliefs. Gordon (2000) reports that the average growth rate in GNP in the U.S. from 1995 to 1999 was 4.90%. We consider expected output growth of 4%, 5% and 6%. The volatility of output growth is 2%. For each level of expected output growth, we set initial belief volatilities to 0%, 1% and 3%. With zero initial belief volatilities, investors do not need to learn about expected output growth. Increasing initial belief volatilities increase the amount of learning in the economy.

For each economy, we consider three parameterizations for the investor's preferences. In the log investor economy, all investors have logarithmic utility functions. In the mixed investor economy, one half of the aggregate endowment is held by logarithmic investors and one half of the aggregate endowment is held by square root investors. In the square root investor economy, all investors have square root utility functions. The top panel of the table reports properties of equilibrium prices and interest rates. The bottom panel reports properties of the equilibrium allocations.

In the log investor economy, the initial stock price and stock volatility do not depend upon the expected growth rate output, nor do they depend upon the precision of the investor's initial beliefs. All asset pricing effects are through the interest rate. As expected output growth increases, the instantaneous interest rate increases.

For all economies, the initial stock price is higher in the mixed investor economy than in the log investor economy, because the representative investor's relative risk aversion is lower in the mixed economy than in the log investor economy. In the mixed investor economy, the initial stock price and stock volatility are increasing in the expected output growth and the volatility of the initial beliefs.

The first line of the second panel reports the bond open interest, equal to the value of the long positions in the bond. The second line reports the square root investor's initial equity holdings. The square root investor's portfolio is a levered position in equity. The third line of the second panel reports the square root investor's hedging demands measured as a percentage of the square root investor's stock position. The hedging demand is computed as the difference between square root investor's equilibrium demand and his demand if returns were *i.i.d.* at their initial mean and

standard deviation.

With zero initial belief volatility, there is no learning in the economy. Increasing initial belief volatility increases learning in the economy leading the square root investor to increase his hedging demand for the stock. For example, with expected output growth equal to 4% and no learning, the square root investor's hedging demand is -2% of his stock holdings. With a 3% belief volatility, hedging demand is 91% of the square root investor's stock holdings. The initial stock price is increasing in the amount of learning in the economy because of the effects of learning on stock demand. The instantaneous risk free interest rate depends upon the instantaneous drift and diffusion coefficient of output growth, which are both independent of the level of learning.

In the power investor economy, stock prices rise relative to the mixed investor economy since the relative risk aversion of the representative agent is lower than in the mixed investor economy. As in the mixed investor economy, increasing learning increases the square root investor's hedging demand for the stock, and so increases the stock price.

Table 2 reports results from economies where investors have heterogeneous beliefs. Both investors have Gaussian priors about expected output growth with the same initial belief volatilities. The disagreement process therefore is deterministic and always positive. Belief volatilities are either 0%, 1% or 3%. The pessimist in the heterogeneous beliefs economy corresponds to the log investor in the homogeneous beliefs economy and the optimist in the heterogeneous beliefs economy corresponds to the square root investor in the homogeneous beliefs economy. The pessimist's initial prior mean is 4%, and the optimist has either a 5% or 6% prior mean. True expected output growth is 4% – the pessimist's beliefs are correct on average. Under the objective probabilities, the economy corresponds to the mixed investor economy with 4% output growth and belief volatility of zero, whose properties are reported in the third column of Table 1.

We report properties of the equilibrium for unconstrained economies and short-sale constrained economies. The top panel reports properties equilibrium prices, the middle panel reports properties of equilibrium allocations and the bottom panel reports properties of the short-sale constraint in the constrained economy.

The stock price rises in the unconstrained heterogeneous beliefs economy relative to the homogeneous beliefs economy. For example, in the homogeneous beliefs economy with a common

expected output growth of 5% and zero initial belief volatility, the initial stock price is 17.45 in the mixed investor economy. Results for the unconstrained heterogeneous beliefs economy where the optimist has a 6% initial expected growth rate; the pessimist has a 4% initial expected growth rate; and no learning, are reported in the seventh column of Table 2. Here, the initial price dividend ratio is 19.6. The initial interest rate drops from 6.5% in the homogeneous beliefs economy to -0.3% in the unconstrained heterogeneous beliefs economy, and the optimist's risk premium rises from 0.04 in the homogeneous beliefs economy to 0.557 in the unconstrained heterogeneous beliefs economy. Stock volatility rises from 2.07% in the homogeneous agents economy to 6.8% in the unconstrained heterogeneous agents economy.

Bond open interest increases from 16% of aggregate wealth in the homogeneous beliefs economy to 272% in the unconstrained heterogeneous beliefs economy. The optimist holds 132% of his wealth in stocks in the homogeneous expectations economy and 690% in the unconstrained heterogeneous beliefs economy. The optimistic investor borrows at the risk free rate to invest in stocks for two reasons in the unconstrained heterogeneous beliefs economy: he is less risk averse than the pessimist and is more optimistic about stock returns in the future. The pessimist holds 68% of his wealth in stocks in the homogeneous beliefs economy and shorts 477% in the unconstrained heterogeneous beliefs economy.

The eighth column of Table 2 reports the results from the constrained heterogeneous beliefs economy with the same parameters as above. The initial stock price-dividend ratio drops from 19.6 in the unconstrained economy to 18.0 in the constrained economy. The second row reports the pessimist's marginal valuation for the stock. Even though the pessimist believes the stock price is overvalued by 29%, the stock price would rise if the constraint were removed from the economy.

Initial stock volatility drops from 6.8% in the unconstrained economy to 2.3% in the constrained economy. The weighting process itself is more volatile in the unconstrained economy than in the constrained economy – the weighting process accounts for 69.7 % of stock volatility in the unconstrained case and 9.8% in the constrained economy.

The optimist's price of risk drops from 0.557 in the unconstrained economy to 0.025 in the constrained economy and the initial interest rate rises from -0.3% in the unconstrained economy to 7.2% in the constrained economy.

In the unconstrained economy, the stock has a 3.5% initial expected return and initial equity premium of 3.8% under the optimist's beliefs; the stock has a -3.0% expected return and the equity premium of -4.8% under the pessimist's beliefs. In the constrained economy, the initial expected return on the stock is 7.2% with the equity premium approximately zero under the optimist's beliefs; the initial expected return on the stock is 4.9% with the equity premium -2.3% under the pessimist's beliefs.

In the constrained economy, the optimist holds all the stock. Since the bond is in zero net supply, aggregate wealth is equal to the total value of the stock. Both investors start out with 50% of the aggregate wealth, and so the optimist initially borrows 50% of aggregate wealth to purchase the pessimist's stock.

The final panel of the table reports information on the constraint. The constraint always binds in the unconstrained economy.

In all cases reported in Table 2, the stock price would rise in the short-sale constrained economy if the short-sale constraint were removed. Increasing the volatility of the investor's beliefs increases the amount of learning in the economy. Increasing learning decreases the initial stock price, increase the initial risk free rate, increase stock volatility in both the unconstrained and constrained economies. Increasing learning also decreases pessimist's stock over-valuation.

Figures 1, 2 and 3 provide information about the long-run properties of the equilibrium. In all cases, the pessimist has a prior mean of 4% and the optimist has a prior mean of 6%. We compute the long-run properties of the economy under three sets of beliefs: the optimists, the pessimists and the objective beliefs conditional upon the true output drift. Each row in the figures corresponds to a different initial belief volatility, with zero in the first row, and belief volatility increasing as we move down the rows. The first column provides the limiting relative risk aversion distribution in an economy without the short-sale constraint, the second column gives the limiting risk aversion distribution imposing the constraint, and the final column provides the distribution of the time that the constraint stops binding. Once the constraint stops binding, it never binds again. The distributions are computed through a simulation. The pessimistic investor has logarithmic preferences, and the optimistic investor has a relative risk aversion coefficient equal to 0.5, and so if the long run distribution of the representative agent's relative risk aversion skews toward one,

then the pessimist has most of the wealth in the long run, and vice-versa if the distribution skews toward 0.5.

When the economy is unconstrained and belief volatility is low, each investor believes that they will hold most of the wealth in the economy after twenty years. The result is shown in the upper left hand plot in Figure 1 where the long run distribution is plotted using the optimist's beliefs and Figure 2 where it is plotted with the pessimist's beliefs. Under the objective beliefs, the pessimist ends up holding all the wealth in the unconstrained economy, since the pessimist has correct beliefs about expected output growth.

The relative risk-aversion in the long-run with the constraint skews slightly toward the optimist under all beliefs, skewing more as learning increases. The skew occurs because the constraint reduces the leverage in the optimist's portfolio. Increasing learning in the economy speeds up the convergence of the investors' beliefs; leading to a faster reduction in leverage in the optimist's portfolio.

We also examine equilibrium prices and allocations in economies where the constraint does not bind initially, but may bind in the future. Table 3 reports results from economies where both the optimist and pessimist have prior means equal of 4% and 6%. The pessimist's prior has zero volatility; the pessimist does not learn from observing realized output growth. The optimist's prior has volatility of 1% or 3%; the optimist learns about expected output growth from observing realized output growth. If output growth is high, then the optimist revises his beliefs upward and if output growth is low, he revises his beliefs downward. The optimist reacts more from realized output with 3% belief volatility than with 1% belief volatility.

Since both investors start out with the same expected output growth, the disagreement process starts out at zero: the constraint does not bind initially. But if there are enough good realizations of output growth, the optimist's beliefs will rise enough for the constraint to bind in the future.

With the optimist's prior belief volatility at 1%, the initial stock price-dividend ratio in the unconstrained economy is 17.2, higher than the homogeneous beliefs economy with both investors having zero or 1% prior belief volatility reported in the second and fifth columns of Table 1. Stock volatility is higher in the unconstrained heterogeneous beliefs economy than the corresponding homogeneous investor economies, and the initial interest rate is higher in the unconstrained het-

erogeneous beliefs economy. The stock's initial expected return is also higher in the unconstrained heterogeneous beliefs economy than in the corresponding homogeneous beliefs economies. Although initial expected returns are higher in the unconstrained heterogeneous investor economy than the corresponding homogeneous beliefs economy, future expected returns are lower in the unconstrained heterogeneous investor economy because there is some chance that that constraint will bind in the future. When the constraint binds in the future, the stock price is lower than in an otherwise identical unconstrained economy.

The pessimist's marginal valuation for the stock in the constrained economy is lower than the initial stock price. But the short-sale constraint does not bind initially; the pessimist holds the stock even though the price is higher than his marginal valuation. The instantaneous risk premium for the stock is high enough so that the optimist finds it worthwhile to hold the stock. For example, with a 1% optimist volatility, the pessimist invests 37.4% of his wealth in the stock in the unconstrained economy and 43.5% in the constrained economy because stock volatility is lower in the constrained economy. The bottom panel of the table reports the fraction of the time that the constraint binds and the expected time until the constraint first starts binding. In all economies considered, the constraint binds about 25% of the time, and starts binding in about one year.

In the second model for the investors' beliefs, expected output growth is regime switching, and follows a finite-state continuous-time Markov process. Specifically, expected output growth can assume one of three possible values,

$$\mu_{\delta}(t) \in \{\mu_{OE}, \mu_{TE}, \mu_{NE}\}. \quad (59)$$

The three values capture three different states for the output process: Old Economy, Transition Economy, and New Economy. In the transition economy, it is possible for the economy to become either an old economy or a new economy. Once expected output growth enters either the old economy or new economy state, it stays in that state forever. Such a stark transition is chosen for simplicity and to aid in interpreting the results. As a result, the infinitesimal transition matrix of the expected output growth, with states ordered as old economy; transition economy; and new

economy, is given by

$$\begin{bmatrix} 0 & 0 & 0 \\ \psi_{TE,OE} & -(\psi_{TE,OE} + \psi_{TE,NE}) & \psi_{TE,NE} \\ 0 & 0 & 0 \end{bmatrix}, \quad (60)$$

where $\psi_{TE,OE}$ is the instantaneous transition intensity from transition economy to old economy and $\psi_{TE,NE}$ from transition economy to new economy.

All the investors agree on the possible states of the expected output process as well as the infinitesimal transition matrix given by (60). Investors do not observe the path of $\mu_\delta(t)$, they infer expected output by observing the realized output process $\delta(t)$. The three regime formulation allows expected output growth to be higher in the new economy regime and that it is difficult to discern whether the economy is in the old economy or the transition economy,

$$\mu_{NE} > \mu_{TE} = \mu_{OE}. \quad (61)$$

Each investor has a different prior about the initial state of the expected output growth process, leading to heterogeneous beliefs. Define investor i 's conditional probability that $\mu_\delta(t)$ is in state ω by

$$\pi_\omega^i(t) = P^i(\mu_\delta(t) = \omega | \mathcal{F}_t^\delta), \quad (62)$$

where $\omega \in \{\mu_{OE}, \mu_{TE}, \mu_{NE}\}$ and $i \in \{o, p\}$. Investor i 's time- t prediction about the expected output growth is given by

$$\mu_\delta^i(t) = \sum_\omega \mu_\omega \pi_\omega^i(t). \quad (63)$$

We provide the dynamics of the conditional probabilities in Appendix B.

The optimist believes that there is some chance that the economy starts out in the transition economy: $\pi_{TE}^o(0) \neq 0$. The pessimist believes that the economy starts out in the old economy: $\pi_{OE}^p(0) = 1$, $\pi_{TE}^p(0) = 0$, and $\pi_{NE}^p(0) = 0$. Since old economy is an absorbing state, the pessimist believes he is will always be in the old economy: $\pi_{OE}^p(t) = 1$; $\pi_{TE}^p(t) = 0$; and $\pi_{NE}^p(t) = 0$, $\forall t$.

Table 3 reports results from the Markov-switching economy. Expected output growth in the old economy and transition economy is 4%, and output growth in the new economy is either 6% or

8%.

Tables 4 and 5 reports result from the Markov-switching economy. Expected output growth in the old economy and transition economy is 4%, and expected output growth in the new economy is either 6% or 8%. The volatility of output growth is 2%.

In Table 4, the optimist's initial expected output growth is either 5% or 6%. Since the pessimist believes that the economy starts out in the old economy state, he believes that expected output growth is equal to 4% and that it will not change in the future. Under the objective probabilities, the economy starts out in the old economy, with expected output growth of 4%. Since old economy is an absorbing state, there is no chance that output growth will every change under the objective probabilities. The constraint binds initially.

In the economies reported in the first six columns, the optimist's initial expected growth is 5%: the optimist puts prior probability of 0.5 on the new economy and either 0.5, 0.25 or 0.0 on the transition economy. The lower the optimist's prior probability on the transition economy, the lower the optimist's prior probability that the economy will eventually switch to the new economy in the future, and the lower the optimist's prior expectations about output growth in the long run.

Decreasing the optimist's prior probability of the transition economy decreases the initial stock price in the unconstrained economy. In all cases, the initial stock price in the constrained economy is lower than in the unconstrained economy. The pessimist's marginal valuation for the stock is between 7 and 8.6% higher than the initial stock price. But removing the constraint from the economy would increase the stock price. As in the Gaussian case, the short-sale constraint causes initial interest rates to rise and the initial stock risk premium to fall. Imposing the constraint leads to a reduction in stock volatility, since the constraint reduces the volatility of the weighting process.

The second panel of the table reports the equilibrium allocations. In the unconstrained economies, the optimist's initial portfolio is a levered position in the stock; he borrows approximately twice the value of all the stock in the economy, investing the proceeds in stock. The pessimist shorts approximately 300% of his wealth in stock. Decreasing the optimist's prior probability that the economy starts out in the transition economy state reduces leverage in the optimist's portfolio, and reduces the optimist's hedging demand for the stock. When the constraint is imposed, the optimist holds all the stock in the economy, financing it by borrowing from the pessimist.

The third panel of the table reports properties of the constraint. In all economies considered, the constraint typically always binds.

Table 5 reports outcomes in regime switching economies where the constraint does not initially bind. Both the optimist and the pessimist believe that initial expected output growth is 4%. The pessimist believes that the economy starts out in the old economy state, and the optimist believes that the economy starts out with some probability in the transition economy state and some probability in the old economy state. The optimist believes that the economy can switch to a permanently higher growth rate in the future, either 6% or 8%. Under the objective beliefs, the economy starts out in the old economy, and so will have an expected output growth of 4% forever.

Moving across the first six columns, the optimist's prior that the economy starts out in the transition economy decrease; the optimist's expectations of long term output growth drop. Decreasing the optimist's expectations of long term output growth decreases the initial stock price in the unconstrained economy. In all cases, imposing the short-sale constraint reduces the initial stock price relative to the unconstrained economy. Initial stock volatility, interest rates and risk-premium all change by a small amount with the constraint. The constraint decreases stock volatility, increases interest rates and and decreases stock risk-premiums in the future, when the constraint first binds. Initial allocations are also relatively insensitive to imposing the constraint.

Overall, our numerical analysis of the model provides the following results. Increasing investor's uncertainty about the fundamentals in the economy tends to increase stock prices and volatility, both with homogeneous and heterogeneous expectations. Adding belief heterogeneity to the economy tends to increase stock prices, volatility, interest rates and leverage in the economy relative to the homogeneous expectations case. When we introduce a short-sale constraint in the economy with heterogeneous expectations, initial stock price, stock volatility and open interest fall. Stock prices falls because interest rates rise to clear the bond market, offsetting the effects from decreased risk premiums. The effects are robust across different choices of the initial beliefs and processes for expected output growth.

5 Conclusions

We study the effects of heterogeneous beliefs and short-sale constraints in a continuous time economy with optimistic and pessimistic investors. We compute the stock price, stock volatility, and interest rate effects from imposing a short-sale constraint on the pessimistic investor. Introducing heterogeneous beliefs into the economy tends to increase the initial stock price and volatility relative to economies with homogeneous expectations. Introducing a short-sale constraint tends to reduce the stock price relative to that in the economy without the constraint, because of large and offsetting effects on interest rates and the market price of risk.

We study an economy with only one risky asset equal to a claim on aggregate output; we study the effects of heterogeneity and a short-sale constraint on the valuation of the market as a whole. Chen et al. (2001), Diether et al. (2002) Jones and Lamont (2001) and Ofek and Richardson (2001) present evidence of mispricing of individual stocks. What are the effects of increased heterogeneity about the future prospects of one asset in a dynamic multi-asset economy in the presence of a short-sale constraint? We leave that question for future work.

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A Proofs

Proof of Lemma 1 The result follows from Theorem 8.1 of Liptser and Shiriyayev (1977) for the geometric Brownian motion case and Theorem 12.1 of Liptser and Shiriyayev (1977) for the continuous time Markov process case. ■

Proof of Proposition 1 The proof follows Karatzas et al. (1990) with the appropriate modifications taken to accommodate for investors facing different state prices. From clearing in the consumption good market, equations (41) and (46) follow from the representative agent construction. The stock price (45) follows by substituting (41) into the stock price valued using the state price density process for the optimistic investor. The valuation equation exists from the existence of a state price density. To compute the dynamics of the weighting process, apply Itô's lemma to $\lambda(t) = \frac{y^p \xi^o(t)}{y^p \xi^p(t)}$ using the dynamics of the state price densities in the constrained and unconstrained regions.

To show the converse, assume that there exists ξ^o , ξ^p , λ , and S satisfying (41) - (45). Clearing in the consumption good market (35) follows from the solutions to the investors' consumption-portfolio problems with (41) substituted. To show clearing in the bond market, note that $\xi^o(t)(X^o(t) + X^p(t))$ is a martingale since the pessimist's optimal wealth process is a tradeable strategy, and the optimist and the pessimist must agree upon the prices of all tradeable strategies.

$$X^o(t) + X^p(t) = \frac{1}{\xi^o(t)} E^o \left[\int_t^T \xi^o(s) (\hat{c}^o(s) + \hat{c}^p(s)) ds \middle| \mathcal{F}_t^\delta \right]. \quad (\text{A1})$$

By substituting clearing in the consumption good market, clearing in the bond market results. Finally, to show clearing in the stock market, compare the diffusion dynamics of $\xi^o(t)S(t)$ with $\xi^o(t)(X^o(t) + X^p(t))$ using clearing in both the good market and the bond market. Clearing in the stock market then results. ■

Proof of Proposition 2 Applying Itô's lemma to each agent's first order conditions, $u_c^i(\hat{c}^i(t)) = y^i \xi^i(t)$, and matching diffusion terms,

$$\sigma_{c^i}(t) = \frac{\kappa^i(t)}{A^i(t)}, \quad i \in \{o, p\}, \quad (\text{A2})$$

where $\hat{c}^i(t)$ satisfies $d\hat{c}^i(t) = \left(\mu_{c^i}(t) + \frac{\mu_\delta(t) - \mu_\delta^i(t)}{\sigma_\delta(t)} \right) dt + \sigma_{c^i}(t) dW^n(t)$. From market clearing in the consumption good market, $\sigma_{c^o}(t) + \sigma_{c^p}(t) = \delta(t)\sigma_\delta(t)$ and $1/A(t) = 1/A^o(t) + 1/A^p(t)$, and this implies equations (47). The constraint binds when $\kappa^o \geq \bar{\mu}$, which implies equation (51).

The interest rate, follows from applying Itô's lemma to the state price density for each agent, matching deterministic terms, and rearranging. ■

Proof of Corollary 1 The constraint is binding when $\kappa^p(t) = \kappa^o(t) - \bar{\mu}(t) < 0$. The result then follows from equation (47). ■

Proof of Corollary 2 Monotonicity of the risk premium in the disagreement process when the constraint does not bind follows from equation 47, since $A(t) > 0$ and $A^p(t) > 0$. The second statement follows directly from equations (47) and (49) ■

Proof of Lemma 2 Consider two economies with identical time t consumption allocations and the same level of the disagreement process. In one of the economies, the short-sale constraint is binding. In the other economy, no constraint is present. Note that for ease of exposition all time dependency is suppressed.

Denote the market price of risk for the optimist in the constrained economy as κ_c^o and in the unconstrained economy as κ_u^o . Calculating $\kappa_u^o - \kappa_c^o$, we have

$$\begin{aligned}\kappa_u^o - \kappa_c^o &= \frac{A^o A^p}{A^o + A^p} \delta \sigma_\delta + \frac{A^o}{A^o + A^p} \bar{\mu} - A^o \delta \sigma_\delta, \\ &= \frac{A^o}{A^o + A^p} \bar{\mu} - A^o \delta \sigma_\delta \left(1 - \frac{A^p}{A^o + A^p}\right), \\ &= \frac{A^o}{A^o + A^p} (\bar{\mu} - A^o \delta \sigma_\delta) \geq 0,\end{aligned}$$

where the inequality is established since the constraint binds in the constrained economy.

For the equilibrium interest rate, denote r_c as the rate in the constrained economy and r_u as the rate in the unconstrained economy. The equilibrium interest rates are given by

$$\begin{aligned}r_c &= \rho + A \delta \mu^o \delta - \frac{1}{2} A B^o \left(\frac{\kappa_c^o}{A^o}\right)^2, \\ r_u &= \rho + A \delta \mu^o \delta - \frac{A}{A^p} \kappa_u^p \bar{\mu} - \frac{1}{2} A B^o \left(\frac{\kappa_u^o}{A^o}\right)^2 - \frac{1}{2} A B^p \left(\frac{\kappa_u^p}{A^p}\right)^2,\end{aligned}$$

where κ_u^p is the pessimist's market price of risk in the unconstrained economy. From these two interest rate expressions, $r_c \geq r_u$ if and only if

$$\frac{\kappa_u^p}{A^p} \bar{\mu} + \frac{1}{2} B^o \left(\left(\frac{\kappa_u^o}{A^o}\right)^2 - \left(\frac{\kappa_c^o}{A^o}\right)^2 \right) + \frac{1}{2} B^p \left(\frac{\kappa_u^p}{A^p}\right)^2 \geq 0.$$

Substituting our preference assumptions, this expression is equivalent to showing that

$$\begin{aligned}((1 - \gamma)(\delta - \hat{c}_o)(\mu_\delta^o - \mu_\delta^p) + 2(1 - \gamma^2) \delta \sigma_\delta^2)(\mu_\delta^o - \mu_\delta^p) \hat{c}_o^2 \\ + \gamma((\gamma^2 + \gamma - 2)\sigma_\delta^2 + 2\gamma(\mu_\delta^o - \mu_\delta^p)) \hat{c}_o \delta^2 \sigma_\delta^2 - \gamma^2(1 + \gamma) \delta^3 \sigma_\delta^4\end{aligned} \quad (\text{A3})$$

is non-negative.

When $\gamma \leq 1$, equation (A3) is verified to be non-negative over all consumption allocations where the short-sale constraint binds after tedious, but straightforward algebra.

Evaluating equation (A3) at $\gamma = 1$ leads to the expression equaling $2\delta^2 \sigma_\delta^3 (\bar{\mu} \hat{c}_o - \delta \sigma_\delta)$ which is strictly positive across all consumption allocations except when $\hat{c}_o = \frac{\delta \sigma_\delta}{\bar{\mu}}$. This is the minimum consumption for the optimist such that the short-sale constraint binds. Hence, by continuity of (A3), a $\gamma > 1$ exists such that $r_c \geq r_u$ for all consumption allocations when the short-sale constraint binds. ■

Proof of Proposition 3 From Proposition 1, the stock is priced with respect to the optimist:

$$S(t) = \frac{1}{U_c(\delta(t), \lambda(t))} E^o \left[\int_t^T e^{-\rho s} U_c(\delta(s), \lambda(s)) \delta(s) ds \middle| \mathcal{F}_t^\delta \right].$$

Define the $\mathcal{L}^2(\mathcal{P}^o)$ martingale $M(t)$

$$M(t) = E^o \left[\int_0^T e^{-\rho s} U_c(\delta(s), \lambda(s), t) \delta(s) ds \middle| \mathcal{F}_t^\delta \right]. \quad (\text{A4})$$

By the martingale representation theorem, there exists a process ϕ such that $E^o \left[\int_0^T |\phi(t)|^2 dt \right] < \infty$ where $M(t) = M(0) + \int_0^t \phi(s) dw^o(s)$.

The value of the stock can be rewritten as

$$S(t) = \frac{1}{U_c(\delta(t), \lambda(t))} \left[M(t) - \int_0^t e^{-\rho s} U_c(\delta(s), \lambda(s), t) \delta(s) ds \middle| \mathcal{F}_t^\delta \right]. \quad (\text{A5})$$

By applying Itô's lemma to (A5) and matching diffusion coefficients with the dynamics of S given by (12),

$$\begin{aligned} \sigma(t) = & -\frac{U_{cc}(\delta(t), \lambda(t))}{U_c(\delta(t), \lambda(t))} \delta(t) \sigma_\delta(t) + \frac{U_{c\lambda}(\delta(t), \lambda(t))}{U_c(\delta(t), \lambda(t))} \sigma_\lambda(t) \\ & + \frac{\phi(t)}{E^o \left[\int_t^T U_c(\delta(s), \lambda(s), t) \delta(s) ds \middle| \mathcal{F}_t^\delta \right]}. \end{aligned} \quad (\text{A6})$$

Equation (A6) gives a representation of the stock volatility in terms of the representative agent's utility function; the processes δ and λ ; and the process ϕ from the martingale representation theorem.

To complete the characterization of the stock's volatility, we need to relate the dynamics of the martingale $M(t)$ to the dynamics of the processes δ and λ . From Lemma 4 below, the dynamics of $M(t)$ are,

$$\begin{aligned} \phi(t) &= E^o \left[\int_t^T e^{-\rho s} U_{c\lambda}(\delta(s), \lambda(s)) \delta(s) \mathcal{D}_t \lambda(s) ds \middle| \mathcal{F}_t^\delta \right] \\ &= E^o \left[\int_t^T e^{-\rho s} (U_{cc}(\delta(s), \lambda(s)) \delta(s) + U_c(\delta(s), \lambda(s))) \mathcal{D}_t \delta(s) ds \middle| \mathcal{F}_t^\delta \right]. \end{aligned} \quad (\text{A7})$$

The Malliavin derivatives $\mathcal{D}_t \lambda(s)$ and $\mathcal{D}_t \delta(s)$ are

$$\mathcal{D}_t \delta(s) = \delta(s) \left[\int_t^s \mathcal{D}_t \mu^o(z) dz + \sigma_\delta \right], \quad (\text{A8})$$

$$\mathcal{D}_t \lambda(s) = \lambda(s) \left[\int_t^s \sigma_\lambda(z) \mathcal{D}_t \sigma_\lambda(z) dz - \int_t^s \mathcal{D}_t \sigma_\lambda(z) dw^o(z) - \bar{\mu}(t) \right]. \quad (\text{A9})$$

The results are computed by using the chain rule for Malliavin derivatives. See Nualart (1995) for a full exposition.

From the representative agent construction,

$$U_{c\lambda}(\delta(s), \lambda(s)) = \frac{U_c(\delta(s), \lambda(s))}{\lambda(s)} \frac{A(s)}{A^p(s)} = \frac{U_c(\delta(s), \lambda(s))}{\lambda(s)} \frac{A^o(s)}{A^o(s) + A^p(s)}.$$

Substituting gives the result. ■

Proof of Lemma 3: The market stock price under the pessimist's beliefs given by (13) and the pessimist's marginal valuation for the stock given by (53) follow backward stochastic differential equations (BSDEs) with $S(T) = 0$. The comparison theorem presented in El Karoui et al. (1997) is directly applicable leading to $S(t) \geq S^p(t)$ almost surely for any time t .

Lemma 4 (*Proposition 1.3.5 (Clark-Ocone) - Nualart (1995)*)

Let $F \in D^{1,2}$, where the space $D^{1,2}$ is the closure of the class of smooth random variables S with respect to the norm

$$\|F\|_{1,2} = \left[E(|F|^2) + E(\|\mathcal{D}F\|_{L^2(T)}^2) \right]^{1/2}.$$

Suppose that W is a one-dimensional Brownian motion. Then

$$F = E(F) + \int_0^T E(\mathcal{D}_s F | \mathcal{F}_s) dW(s), \quad (\text{A10})$$

Taking conditional expectations of (A10),

$$E(F | \mathcal{F}_t) = E(F) + \int_0^t E(\mathcal{D}_s F | \mathcal{F}_s) dW(s). \quad (\text{A11})$$

Proof of Lemma 4: See page 42 of Nualart (1995). ■

B Optimal Filtering Problem and Malliavin Derivatives under Each Output Growth Assumption

B.1 Output Follows a Geometric Brownian Motion

When the exogenous dividend process follows a geometric Brownian motion, the Malliavin derivatives $\mathcal{D}_t \delta(s)$ and $\mathcal{D}_t \lambda(s)$ are given by:

$$\mathcal{D}_t \delta(s) = \delta(s) \left[\sigma_\delta + \frac{z(t)}{\sigma_\delta} (s - t) \right] > 0, \quad (\text{A12})$$

$$\mathcal{D}_t \lambda(s) = -\lambda(s) \left[\sigma_\lambda(t) + \int_t^s \sigma_\lambda(z) \mathcal{D}_t \sigma_\lambda(z) dz + \int_t^s \mathcal{D}_t \sigma_\lambda(z) dW^o(z) \right], \quad (\text{A13})$$

where $\sigma_\lambda(t) = \min(\bar{\mu}(t), \kappa_c^o(t))$ and $\mathcal{D}_t \sigma_\lambda(z)$ is given by

$$\mathcal{D}_t \sigma_\lambda(z) = \begin{cases} \mathcal{D}_t \bar{\mu}(z) & \text{if } \sigma_\lambda(z) = \bar{\mu}(z), \\ \frac{\partial \kappa_c^o(z)}{\partial \delta(z)} \mathcal{D}_t \delta(z) + \frac{\partial \kappa_c^o(z)}{\partial \lambda(z)} \mathcal{D}_t \lambda(z) & \text{if } \sigma_\lambda(z) = \kappa_c^o(z). \end{cases}$$

B.2 Output Growth Follows a Regime Switching Process

We report the dynamics of the investor's conditional probabilities of being in state ω when the infinitesimal transition matrix given by (60). Further details of the optimal filtering problem are

discussed in chapter 9 of Liptser and Shirayev (1977). The conditional probabilities evolve as

$$\begin{aligned}
\pi_{OE}^i(t) &= \pi_{OE}^i(0) + \int_0^t \phi_{TE,OE} \pi_{TE}^i(u) du \\
&\quad + \int_0^t \pi_{OE}^i(u) \frac{\mu_{OE} - \mu_{\delta}^i(u)}{\sigma_{\delta}} dw^o(u), \\
\pi_{TE}^i(t) &= \pi_{TE}^i(0) - \int_0^t (\phi_{TE,OE} + \phi_{TE,NE}) \pi_{TE}^i(u) du \\
&\quad + \int_0^t \pi_{TE}^i(u) \frac{\mu_{TE} - \mu_{\delta}^i(u)}{\sigma_{\delta}} dw^o(u), \\
\pi_{NE}^i(t) &= \pi_{NE}^i(0) + \int_0^t \phi_{TE,NE} \pi_{TE}^i(u) du \\
&\quad + \int_0^t \pi_{NE}^i(u) \frac{\mu_{NE} - \mu_{\delta}^i(u)}{\sigma_{\delta}} dw^o(u),
\end{aligned} \tag{A14}$$

The Malliavin derivatives for $\mu_{\delta}^o(s)$ and $\mu_{\delta}^p(s)$ are needed to compute the stock's volatility. For $\mu_{\delta}^o(s)$, $\mathcal{D}_t \mu_{\delta}^o(s)$ is given by

$$\mathcal{D}_t \mu_{\delta}^o(s) = (\mu_{OE} - \mu_{TE}) \mathcal{D}_t \pi_{OE}^o(s) + (\mu_{NE} - \mu_{TE}) \mathcal{D}_t \pi_{NE}^o(s), \tag{A15}$$

where

$$\begin{aligned}
\mathcal{D}_t \pi_{OE}^o(s) &= \pi_{OE}^o(t) \frac{\mu_{OE} - \mu_{\delta}^o(t)}{\sigma_{\delta}} - \int_t^s \phi_{TE,OE} (\mathcal{D}_t \pi_{OE}^o(u) + \mathcal{D}_t \pi_{NE}^o(u)) du \\
&\quad + \int_t^s \left(\frac{\mu_{OE} - \mu_{\delta}^o(u)}{\sigma_{\delta}} \mathcal{D}_t \pi_{OE}^o(u) - \frac{\pi_{OE}^o(s)}{\sigma_{\delta}} \mathcal{D}_t \mu_{\delta}^o(u) \right) dw^o(u),
\end{aligned} \tag{A16}$$

$$\begin{aligned}
\mathcal{D}_t \pi_{NE}^o(s) &= \pi_{NE}^o(t) \frac{\mu_{NE} - \mu_{\delta}^o(t)}{\sigma_{\delta}} - \int_t^s \phi_{TE,NE} (\mathcal{D}_t \pi_{OE}^o(u) + \mathcal{D}_t \pi_{NE}^o(u)) du \\
&\quad + \int_t^s \left(\frac{\mu_{NE} - \mu_{\delta}^o(u)}{\sigma_{\delta}} \mathcal{D}_t \pi_{NE}^o(u) - \frac{\pi_{NE}^o(s)}{\sigma_{\delta}} \mathcal{D}_t \mu_{\delta}^o(u) \right) dw^o(u).
\end{aligned} \tag{A17}$$

Since the pessimist's beliefs do not change, $\mathcal{D}_t \mu_{\delta}^p(s) = 0$.

C Numerical Appendix

The equilibrium stock price (45) and stock volatility (52) can be written as conditional expectations under the optimist's beliefs that are driven by the stochastic processes $\delta(s)$, $\lambda(s)$, $\mu_\delta^o(s)$, $\mu_\delta^p(s)$, $\mathcal{D}_t\lambda(s)$, and $\mathcal{D}_t\delta(s)$.

Since the output and weighting processes are strictly positive processes, we express them in exponential form:

$$\begin{aligned}\delta(s) &= \delta(0) \exp\left(\int_0^s \mu_\delta^o(t) - \frac{1}{2}\sigma_\delta^2 dt + \int_0^s \sigma_\delta dw^o(t)\right), \\ \lambda(s) &= \lambda(0) \exp\left(-\int_0^s \frac{1}{2}\sigma_\lambda^2(t) dt - \int_0^s \sigma_\lambda(t) dw^o(t)\right).\end{aligned}$$

The stochastic processes are then simulated forward along with the processes $\mu_\delta^o(s)$, $\mu_\delta^p(s)$, $\mathcal{D}_t\lambda(s)$, and $\mathcal{D}_t\delta(s)$ using a Milstein scheme¹ with a time-step of 0.0005 years.

The pathwise integrals in the stock price (45) and stock volatility (52) are calculated using the trapezoid rule with 80,000 paths used to calculate each expectation.

¹For the processes $\delta(s)$ and $\mu_\delta^o(s)$ the Milstein scheme collapses to an Euler scheme.

Equilibrium Prices and Allocations
Geometric Brownian Motion Output
Each Investor Class Endowed with 50% of Aggregate Wealth

	Common Output Growth = 4%												Common Output Growth = 5%												Common Output Growth = 6%																	
	Common Belief Vol. = 0.00				Common Belief Vol. = 0.01				Common Belief Vol. = 0.03				Common Belief Vol. = 0.00				Common Belief Vol. = 0.01				Common Belief Vol. = 0.03				Common Belief Vol. = 0.00				Common Belief Vol. = 0.01				Common Belief Vol. = 0.03									
	Log. Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Log. Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Log. Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Log. Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Log. Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Log. Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.	Mixed Investor Econ.	Sq. Root Investor Econ.										
Equilibrium Prices:																																										
Price/Dividend Ratio	15.04	16.94	18.12	16.96	18.15	17.11	18.38	15.04	17.45	19.02	17.47	19.05	17.62	19.30	15.04	17.97	19.99	20.02	18.14	20.29	15.04	17.45	19.02	17.47	19.05	17.62	19.30	15.04	17.97	19.99	20.02	18.14	20.29	15.04	17.97	19.99	20.02	18.14	20.29			
Stock Volatility	2.00%	2.08%	2.00%	3.66%	4.42%	16.73%	23.91%	2.00%	2.07%	2.00%	3.73%	4.46%	17.01%	24.25%	2.00%	2.09%	2.00%	3.77%	4.50%	17.25%	24.87%	2.00%	2.07%	2.00%	3.73%	4.46%	17.01%	24.25%	2.00%	2.09%	2.00%	3.77%	4.50%	17.25%	24.87%	2.00%	2.09%	2.00%	3.77%	4.50%	17.25%	24.87%
Equilibrium Interest Rate	6.96%	5.76%	4.95%	5.76%	4.99%	5.77%	4.99%	7.96%	6.50%	5.49%	6.50%	5.49%	6.51%	5.49%	8.96%	7.25%	5.99%	7.25%	5.99%	7.27%	5.99%	6.96%	6.50%	5.49%	6.50%	5.49%	6.51%	5.49%	8.96%	7.25%	5.99%	7.25%	5.99%	7.27%	5.99%	6.96%	7.25%	5.99%	7.25%	5.99%	7.27%	5.99%
Common Market Price of Risk	0.020	0.014	0.010	0.014	0.010	0.014	0.010	0.020	0.014	0.010	0.014	0.010	0.014	0.010	0.020	0.014	0.010	0.014	0.010	0.014	0.010	0.020	0.014	0.010	0.014	0.010	0.014	0.010	0.020	0.014	0.010	0.014	0.010	0.014	0.010	0.020	0.014	0.010	0.014	0.010	0.014	0.010
Common Stock Instantaneous Expected Return	7.00%	5.79%	5.01%	5.81%	5.03%	6.00%	5.22%	8.00%	6.53%	5.51%	6.55%	5.53%	6.75%	5.79%	9.00%	7.28%	6.00%	7.31%	6.00%	7.52%	6.23%	7.00%	6.53%	5.51%	6.55%	5.53%	6.75%	5.79%	9.00%	7.28%	6.00%	7.31%	6.00%	7.52%	6.23%	7.00%	7.28%	6.00%	7.31%	6.00%	7.52%	6.23%
Common Stock Risk Premium	0.04%	0.03%	0.02%	0.05%	0.04%	0.23%	0.24%	0.04%	0.03%	0.02%	0.05%	0.04%	0.04%	0.24%	0.04%	0.03%	0.02%	0.05%	0.04%	0.05%	0.25%	0.04%	0.03%	0.02%	0.05%	0.04%	0.04%	0.24%	0.04%	0.03%	0.02%	0.05%	0.04%	0.05%	0.25%	0.04%	0.03%	0.02%	0.05%	0.04%	0.05%	0.25%
Equilibrium Allocations:																																										
Bond Open Interest (% Agg. Wealth)	-	16%	-	31%	-	46%	-	-	16%	-	31%	-	46%	-	-	16%	-	31%	-	46%	-	16%	-	31%	-	46%	-	-	16%	-	31%	-	46%	-	16%	-	31%	-	46%			
Sq. Root Investor Stock Position (% Investor Wealth)	-	132%	100%	182%	100%	192%	100%	-	132%	100%	162%	100%	192%	100%	-	132%	100%	162%	100%	192%	100%	-	132%	100%	162%	100%	192%	100%	-	132%	100%	162%	100%	192%	100%	-	132%	100%	162%	100%	192%	100%
Sq. Root Investor Hedging Against Delta (% of Stock Position)	-	-2%	0%	53%	55%	91%	92%	-	-3%	0%	53%	55%	91%	92%	-	-4%	0%	53%	55%	91%	92%	-	-2%	0%	53%	55%	91%	92%	-	-4%	0%	53%	55%	91%	92%	-	-4%	0%	53%	55%	91%	92%
Log Investor Stock Position (% Inv. Wealth)	100%	86%	-	38%	-	8%	-	100%	68%	-	38%	-	8%	-	100%	68%	-	38%	-	8%	-	100%	68%	-	38%	-	8%	-	100%	68%	-	38%	-	8%	-	100%	68%	-	38%	-	8%	-

Table 1: Equilibrium Prices and Allocations for Benchmark Economies. The table reports the equilibrium prices and allocations in the benchmark cases with homogeneous beliefs for all investors and a geometric Brownian motion output process. All economies are unconstrained. The table's three vertical panels present equilibrium quantities for common expected output growths of 4%, 5%, and 6%. Within each common output growth panel, the equilibrium is presented for economies populated by all logarithmic investors, by logarithmic and square root investors with equal endowments, and by all square root investors. For all economies other than the logarithmic investor economy, results are presented for common belief volatilities of 0.00, 0.01, and 0.03. Note that the all logarithmic investor economy prices and allocations are invariant to the belief volatility. The largest Monte Carlo 95% confidence interval for the price/dividend ratios and the stock volatilities in the table has a relative size smaller than 0.3%. Parameter choice common across all benchmarks: $\delta(t) = 1.0$, $\mu_\delta = 0.04$, $\sigma_\delta = 0.02$, $\rho = 0.03$, and $T - t = 20$.

Equilibrium Prices and Allocations
 Geometric Brownian Motion Output When the Short Sale Constraint Initially Binds
 Each Investor Class Endowed with 50% of Aggregate Wealth
 Pessimist Output Growth 4%

	Optimist Output Growth = 5%												Optimist Output Growth = 6%											
	Belief Vol. = 0.00			Belief Vol. = 0.01			Belief Vol. = 0.03			Belief Vol. = 0.00			Belief Vol. = 0.01			Belief Vol. = 0.03								
	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.	No. Const. Econ.							
Equilibrium Prices:																								
Price/Dividend Ratio	18.6	17.5	17.9	17.5	17.7	17.6	19.6	18.0	19.2	18.0	18.5	18.2	18.6	17.5	17.9	17.5	17.7	17.6	19.6	18.0	19.2	18.0	18.5	18.2
Price/Dividend Ratio - Pessimist's Valuation	-	16.0	-	16.5	-	17.0	-	13.9	-	15.8	-	16.8	-	16.0	-	16.5	-	17.0	-	13.9	-	15.8	-	16.8
% Overvaluation According to Pessimist	-	8.9%	-	6.0%	-	3.5%	-	29.1%	-	13.6%	-	8.0%	-	8.9%	-	6.0%	-	3.5%	-	29.1%	-	13.6%	-	8.0%
Stock Volatility	3.8%	2.3%	6.1%	3.9%	20.2%	17.2%	6.8%	2.3%	8.5%	4.0%	24.1%	17.5%	3.8%	2.3%	6.1%	3.9%	20.2%	17.2%	6.8%	2.3%	8.5%	4.0%	24.1%	17.5%
Fraction of Volatility From Changes in Lambda	46.2%	7.9%	41.4%	4.6%	17.2%	1.1%	68.7%	9.8%	58.9%	5.8%	31.1%	1.5%	46.2%	7.9%	41.4%	4.6%	17.2%	1.1%	68.7%	9.8%	58.9%	5.8%	31.1%	1.5%
Equilibrium Interest Rate	4.7%	6.4%	4.8%	6.4%	4.8%	6.4%	-0.3%	7.2%	0.4%	7.2%	0.9%	7.2%	4.7%	6.4%	4.8%	6.4%	4.8%	6.4%	-0.3%	7.2%	0.4%	7.2%	0.9%	7.2%
Optimist's Market Price of Risk	0.242	0.024	0.226	0.024	0.223	0.024	0.557	0.025	0.493	0.025	0.456	0.025	0.242	0.024	0.226	0.024	0.223	0.024	0.557	0.025	0.493	0.025	0.456	0.025
Optimist's Stock Instantaneous Expected Return	5.6%	6.5%	6.2%	6.5%	9.3%	6.8%	3.5%	7.2%	4.6%	7.3%	11.8%	7.6%	5.6%	6.5%	6.2%	6.5%	9.3%	6.8%	3.5%	7.2%	4.6%	7.3%	11.8%	7.6%
Pessimist's Stock Instantaneous Expected Return	3.7%	5.3%	3.1%	4.6%	-0.6%	-1.7%	-3.4%	4.9%	-4.0%	3.3%	-12.2%	-9.9%	3.7%	5.3%	3.1%	4.6%	-0.6%	-1.7%	-3.4%	4.9%	-4.0%	3.3%	-12.2%	-9.9%
Optimist's Stock Risk Premium	0.9%	0.1%	1.4%	0.1%	4.5%	0.4%	3.8%	0.1%	4.2%	0.1%	11.0%	0.4%	0.9%	0.1%	1.4%	0.1%	4.5%	0.4%	3.8%	0.1%	4.2%	0.1%	11.0%	0.4%
Pessimist's Stock Risk Premium	-1.0%	-1.1%	-1.7%	-1.9%	-5.6%	-8.2%	-3.0%	-2.3%	-4.3%	-3.9%	-13.1%	-17.1%	-1.0%	-1.1%	-1.7%	-1.9%	-5.6%	-8.2%	-3.0%	-2.3%	-4.3%	-3.9%	-13.1%	-17.1%
Equilibrium Allocations:																								
Bond/Own Interest (% Agg. Wealth)	385%	50%	273%	50%	118%	50%	386%	50%	350%	50%	162%	50%	385%	50%	273%	50%	118%	50%	386%	50%	350%	50%	162%	50%
Optimist Stock Position (% Opt. Wealth)	901%	200%	648%	200%	337%	200%	937%	200%	808%	200%	424%	200%	901%	200%	648%	200%	337%	200%	937%	200%	808%	200%	424%	200%
Optimist Hedging Against Delta (% of Optimist Stock Position)	-42%	-6%	-14%	39%	0%	86%	-74%	-43%	0%	38%	11%	86%	-42%	-6%	-14%	39%	0%	86%	-74%	-43%	0%	38%	11%	86%
Optimist Hedging Against Lambda (% of Optimist Stock Position)	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Pessimist Stock Position (% Pess. Wealth)	-683%	0%	-445%	0%	-137%	0%	-647%	0%	-595%	0%	-226%	0%	-683%	0%	-445%	0%	-137%	0%	-647%	0%	-595%	0%	-226%	0%
Constraint Properties:																								
Fraction of Time Constraint Binding	-	100%	-	100%	-	55%	-	100%	-	100%	-	98%	-	100%	-	100%	-	55%	-	100%	-	100%	-	98%
Expected Time Until Constraint Stops Binding (Years)	-	20.0	-	20.0	-	10.9	-	20.0	-	20.0	-	19.6	-	20.0	-	20.0	-	10.9	-	20.0	-	20.0	-	19.6

Table 2: Equilibrium Prices and Allocations with Heterogeneous Beliefs and Common Belief Volatilities when Output is a Geometric Brownian Motion. The table reports the equilibrium prices and allocations when beliefs are heterogeneous and the short-sale constraint initially binds. The table's two vertical panels correspond to the optimists having initial output growth priors of 5% and 6% respectively. Within each optimist output growth panel, optimists and pessimists have common belief volatilities of 0%, 1%, and 3%. Columns labeled "No Const. Econ." correspond to equilibria with no short sale constraint imposed, while those labeled "Const. Econ." are equilibria where the pessimists are short sale constrained. The largest Monte Carlo 95% confidence interval for the price/dividend ratios and the stock volatilities in the table has a relative size smaller than 0.4%. Parameter choice: $x^o = x^p = 0.5$, $\delta(t) = 1.0$, $\mu_\delta = 0.04$, $\sigma_\delta = 0.02$, $\rho = 0.03$, and $T - t = 20$.

Equilibrium Prices and Allocations
 Geometric Brownian Motion Output when the Short Sale Constraint Initially Does Not Bind
 Each Investor Class Endowed with 50% of Aggregate Wealth
 Pessimist Belief Volatility = 0.0%

	Common Output Growth Belief = 4%				Common Output Growth Belief = 6%				
	Opt. Belief Vol. = 0.01		Opt. Belief Vol. = 0.03		Opt. Belief Vol. = 0.01		Opt. Belief Vol. = 0.03		
	No Const. Econ.	Const. Econ.	No Const. Econ.	Const. Econ.	No Const. Econ.	Const. Econ.	No Const. Econ.	Const. Econ.	
Equilibrium Prices:									
Price/Dividend Ratio	17.2	17.0	18.1	17.5	18.2	18.1	19.1	18.5	
Price/Dividend Ratio - Pessimist's Valuation	-	16.7	-	16.4	-	17.7	-	17.3	
% Overvaluation According to Pessimist	-	2.1%	-	6.3%	-	2.2%	-	6.8%	
Stock Volatility	3.7%	3.2%	16.0%	12.4%	3.8%	3.3%	16.5%	13.6%	
Fraction of Volatility From Changes in Lambda	3.5%	-13.9%	8.6%	-27.9%	3.4%	-12.7%	8.5%	-20.5%	
Equilibrium Interest Rate	5.8%	5.7%	5.8%	5.8%	7.3%	7.2%	7.4%	7.3%	
Investors' Market Price of Risk	0.014	0.014	0.014	0.014	0.014	0.014	0.015	0.014	
Investors' Instantaneous Expected Return for Stock	5.8%	5.8%	6.0%	6.0%	7.3%	7.3%	7.6%	7.5%	
Investors' Risk Premium for Stock	0.05%	0.04%	0.23%	0.18%	0.05%	0.05%	0.24%	0.20%	
Equilibrium Allocations:									
Bond Open Interest (% Agg. Wealth)	31%	26%	46%	47%	31%	28%	46%	40%	
Optimist Stock Position (% Opt. Wealth)	162%	157%	191%	193%	163%	156%	193%	181%	
Optimist Hedging Against Delta (% of Optimist Stock Position)	54%	44%	91%	88%	54%	45%	91%	88%	
Optimist Hedging Against Lambda (% of Optimist Stock Position)	0%	0%	0%	0%	0%	0%	0%	0%	
Pessimist Stock Position (% Pess. Wealth)	37.4%	43.5%	8.9%	11.3%	37.6%	43.2%	8.9%	10.6%	
Constraint Properties:									
Fraction of Time Constraint Binding	-	47.2%	-	46.6%	-	47.1%	-	46.6%	
Expected Time Until Constraint Starts Binding (Years)	-	1.2	-	0.8	-	1.3	-	0.8	

Table 3: Equilibrium Prices and Allocations with Common Initial Beliefs and Heterogeneous Belief Volatilities when Output is a Geometric Brownian Motion. The table reports the equilibrium prices and allocations when optimists and pessimists have common mean beliefs but heterogeneous belief volatilities. Given initial common beliefs, the short-sale constraint initially does not bind. The table's two vertical panels correspond to the investors having common initial output growth priors of 4% and 6% respectively. Within each common output growth panel, optimists have belief volatilities of 1% and 3%. Columns labeled "No Const. Econ." correspond to equilibria with no short sale constraint imposed, while those labeled "Const. Econ." are equilibria where the pessimists are short sale constrained. The largest Monte Carlo 95% confidence interval for the price/dividend ratios and the stock volatilities in the table has a relative size smaller than 0.4%. Parameter choice: $x^0 = x^P = 0.5$, $\delta(t) = 1.0$, $\sigma_\delta = 0.02$, $\nu^P(t) = 0.00$, $\rho = 0.03$, and $T - t = 20$.

Equilibrium Prices and Allocations
 Regime Switching Output Growth when the Short Sale Constraint Initially Binds
 Each Investor Class Endowed with 50% of Aggregate Wealth
 Old Economy Output Growth 4%

	New Economy Output Growth = 6% and Optimists Expected Output Growth = 5%						New Economy Output Growth = 8% and Optimists Expected Output Growth = 6%						
	Optimists Prob. in Transition Economy = 0.5			Optimists Prob. in Transition Economy = 0.25			Optimists Prob. in Transition Economy = 0.5			Optimists Prob. in Transition Economy = 0.25			
	No. Const. Econ.	Const. Econ.	No. Const. Econ.	No. Const. Econ.	Const. Econ.	No. Const. Econ.	No. Const. Econ.	Const. Econ.	No. Const. Econ.	Const. Econ.	No. Const. Econ.	Const. Econ.	
Equilibrium Prices:													
Price/Dividend Ratio	18.6	17.7	18.4	17.6	16.2	17.6	18.4	19.9	18.4	19.7	18.4	19.5	18.3
Price/Dividend Ratio - Pessimist's Valuation	-	16.3	-	16.3	-	16.4	-	16.2	-	16.2	-	16.3	-
% Overvaluation According to Pessimist	-	8.6%	-	7.9%	-	7.3%	-	13.4%	-	13.4%	-	12.7%	-
Stock Volatility	7.3%	3.4%	8.0%	3.6%	8.9%	3.9%	6.8%	13.8%	6.8%	15.5%	7.8%	17.5%	9.0%
Fraction of Volatility From Changes in Lambda	59.9%	5.6%	60.0%	5.1%	61.1%	4.7%	60.1%	3.9%	59.1%	3.3%	56.6%	2.8%	
Equilibrium Interest Rate	4.7%	6.4%	4.7%	6.4%	4.7%	6.4%	4.7%	6.4%	4.7%	6.4%	4.7%	6.4%	4.7%
Optimist's Market Price of Risk	0.238	0.024	0.235	0.024	0.232	0.024	0.505	0.026	0.500	0.026	0.491	0.026	
Optimist's Stock Instantaneous Expected Return	6.5%	6.5%	6.6%	6.5%	6.8%	6.5%	7.4%	6.0%	7.4%	6.0%	7.4%	7.4%	
Pessimist's Stock Instantaneous Expected Return	2.8%	4.8%	2.6%	4.7%	2.3%	4.6%	-6.6%	0.6%	-7.5%	-0.4%	-8.5%	-1.5%	
Optimist's Stock Risk Premium	1.7%	0.1%	1.9%	0.1%	2.1%	0.1%	7.0%	0.2%	7.7%	0.2%	8.6%	0.2%	
Pessimist's Stock Risk Premium	-1.9%	-1.6%	-2.1%	-1.7%	-2.4%	-1.8%	-6.8%	-6.6%	-7.7%	-7.5%	-8.9%	-8.7%	
Equilibrium Allocations:													
Bond Open Interest (% Agg. Wealth)	229%	50%	215%	50%	196%	50%	262%	50%	239%	50%	230%	50%	
Optimist Stock Position (% Opt. Wealth)	55%	200%	532%	200%	494%	200%	624%	200%	577%	200%	563%	200%	
Optimist Hedging Against Delta (% of Optimist Stock Position)	-16%	29%	-10%	34%	-5%	38%	-18%	62%	-12%	67%	0%	71%	
Optimist Hedging Against Lambda (% of Optimist Stock Position)	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	
Pessimist Stock Position (% Pess. Wealth)	-356%	0%	-331%	0%	-301%	0%	-359%	0%	-323%	0%	-291%	0%	
Constraint Properties:													
Fraction of Time Constraint Binding	-	99.6%	-	99.1%	-	97.5%	-	98.5%	-	97.9%	-	96.4%	-
Expected Time Until Constraint Stops Binding (Years)	-	19.6	-	19.4	-	18.6	-	19.2	-	18.8	-	18.1	-

Table 4: Equilibrium Prices and Allocations when the Short Sale Constraint Initially Binds - Expected Output Growth is Regime Switching. The table reports the equilibrium prices and allocations when beliefs are heterogeneous. The table's two vertical panels correspond to optimists having initial expected output growth priors of 5% and 6% respectively with the output growth in the new economy regime given by 6% and 8% respectively. Within each optimist expected output growth panel, the optimist's prior probability about being in the transition economy state is varied. Columns labeled "No Const. Econ." correspond to equilibria with no short sale constraint imposed, while those labeled "Const. Econ." are equilibria where the pessimists are short sale constrained. The largest Monte Carlo 95% confidence interval for the price/dividend ratios and the stock volatilities in the table has a relative size smaller than 0.3%. Parameter choice: $x^o = x^p = 0.5$, $\delta(t) = 1.0$, $\mu_\delta^p = 0.04$, $\sigma_\delta = 0.02$, $\pi_{NE}^o = 0.5$, $\pi_{OE}^p = 1.0$, $\rho = 0.03$, and $T - t = 20$.

Equilibrium Prices and Allocations
 Regime Switching Output Growth when the Short Sale Constraint Initially Does Not Bind
 Each Investor Class Endowed with 50% of Aggregate Wealth
 Old Economy Output Growth 4%

	New Economy Output Growth = 5% and Optimist's Expected Output Growth = 4%						New Economy Output Growth = 8% and Optimist's Expected Output Growth = 4%								
	Optimists Prob. in Transition Economy = 1.0			Optimists Prob. in Transition Economy = 0.75			Optimists Prob. in Transition Economy = 1.0			Optimists Prob. in Transition Economy = 0.75					
	No. Const. Econ.	Const. Econ.	No. Const. Econ.	Const. Econ.	No. Const. Econ.	Const. Econ.	No. Const. Econ.	Const. Econ.	No. Const. Econ.	Const. Econ.	No. Const. Econ.	Const. Econ.			
Equilibrium Prices:															
Price/Dividend Ratio	17.6	17.3	17.5	17.3	17.2	17.3	17.3	17.2	17.8	18.3	17.8	18.1	17.7	17.9	17.5
Price/Dividend Ratio - Pessimist's Valuation	-	16.6	-	16.6	-	16.7	-	16.7	-	16.5	-	16.5	-	16.6	16.6
% Overvaluation According to Pessimist	-	4.5%	-	3.8%	-	3.0%	-	3.0%	-	7.3%	-	7.3%	-	6.4%	5.7%
Stock Volatility	2.0%	2.1%	2.0%	2.1%	2.1%	2.1%	2.1%	2.1%	2.0%	2.0%	2.1%	2.0%	2.0%	2.0%	2.1%
Fraction of Volatility From Changes in Lambda	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Equilibrium Interest Rate	5.8%	5.8%	5.8%	5.8%	5.8%	5.8%	5.8%	5.8%	5.9%	5.9%	5.8%	5.8%	5.8%	5.8%	5.8%
Investors' Market Price of Risk	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014	0.014
Investors' Instantaneous Expected Return for Stock	5.8%	5.8%	5.8%	5.8%	5.8%	5.8%	5.8%	5.8%	5.9%	5.9%	5.8%	5.9%	5.8%	5.8%	5.8%
Investors' Risk Premium for Stock	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%	0.03%
Equilibrium Allocations:															
Bond Open Interest (% Agg. Wealth)	15%	16%	16%	16%	16%	16%	16%	16%	15%	16%	16%	15%	16%	15%	16%
Optimist Stock Position (% Opt. Wealth)	131%	132%	131%	132%	131%	131%	131%	132%	130%	132%	132%	130%	132%	130%	132%
Optimist Hedging Against Delta (% of Optimist Stock Position)	-6%	-3%	-5%	-3%	-4%	-3%	-4%	-3%	-8%	-9%	-3%	-8%	-3%	-6%	-3%
Optimist Hedging Against Lambda (% of Optimist Stock Position)	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
Pessimist Stock Position (% Pess. Wealth)	69.2%	67.9%	68.6%	67.9%	68.4%	68.4%	68.4%	67.6%	70.6%	68.2%	68.2%	70.1%	68.1%	69.4%	68.0%
Constraint Properties:															
Fraction of Time Constraint Binding	-	48.9%	-	48.1%	-	46.0%	-	46.0%	-	48.2%	-	48.2%	-	47.3%	-
Expected Time Until Constraint Starts Binding (Years)	-	0.5	-	0.8	-	1.2	-	1.2	-	0.8	-	0.8	-	1.1	-

Table 5: Equilibrium Prices and Allocations when the Short Sale Constraint Initially Does Not Bind - Expected Output Growth is Regime Switching. The table reports the equilibrium prices and allocations when beliefs are heterogeneous. The table's two vertical panels correspond to the output growth in the new economy regime given by 6% and 8% respectively. Within each panel, the optimist's prior probability about being in the transition economy state is varied. Columns labeled "No Const. Econ." correspond to equilibria with no short sale constraint imposed, while those labeled "Const. Econ." are equilibria where the pessimists are short sale constrained. The largest Monte Carlo 95% confidence interval for the price/dividend ratios and the stock volatilities in the table has a relative size smaller than 0.3%. Parameter choice: $x^o = x^p = 0.5$, $\delta(t) = 1.0$, $\mu_\delta^p(t) = 0.04$, $\mu_\delta^o(t) = 0.04$, $\sigma_\delta = 0.02$, $\pi_{NE}^o = 0.0$, $\pi_{OE}^p = 1.0$, $\rho = 0.03$, and $T - t = 20$.

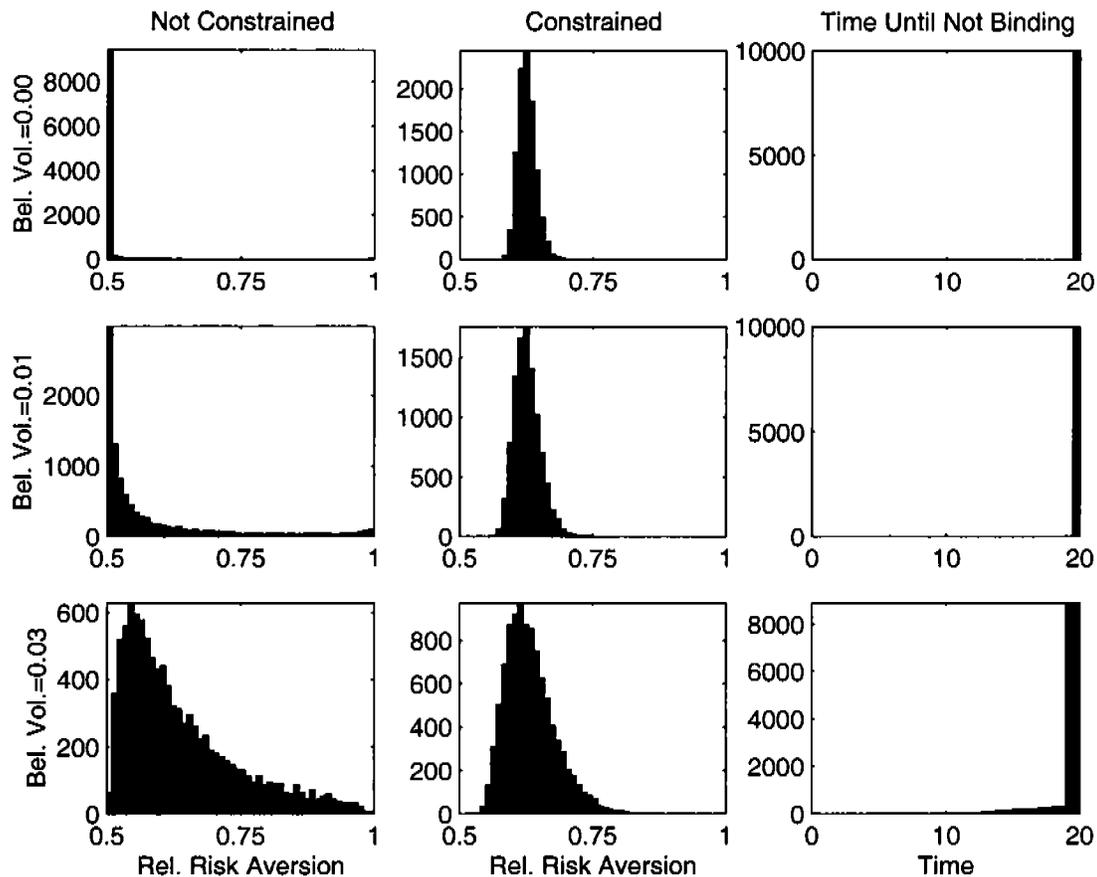


Figure 1: **Twenty Year Relative Risk Aversion Distributions under the Optimist's Beliefs.** The left-side (middle) of the figure denotes the distribution of the relative risk aversion in the economy with short sales allowed (not allowed). The right-side of the figure denotes the distribution of the time until the short-sale constraint no longer binds. Common belief uncertainty increases moving down the figure. Parameters: $x^o = x^p = 0.5$, $\delta(t) = 1.0$, $\mu_\delta^o = 0.06$, $\mu_\delta = \mu_\delta^p = 0.04$, $\sigma_\delta = 0.02$, $\rho = 0.03$, and $T - t = 20$.

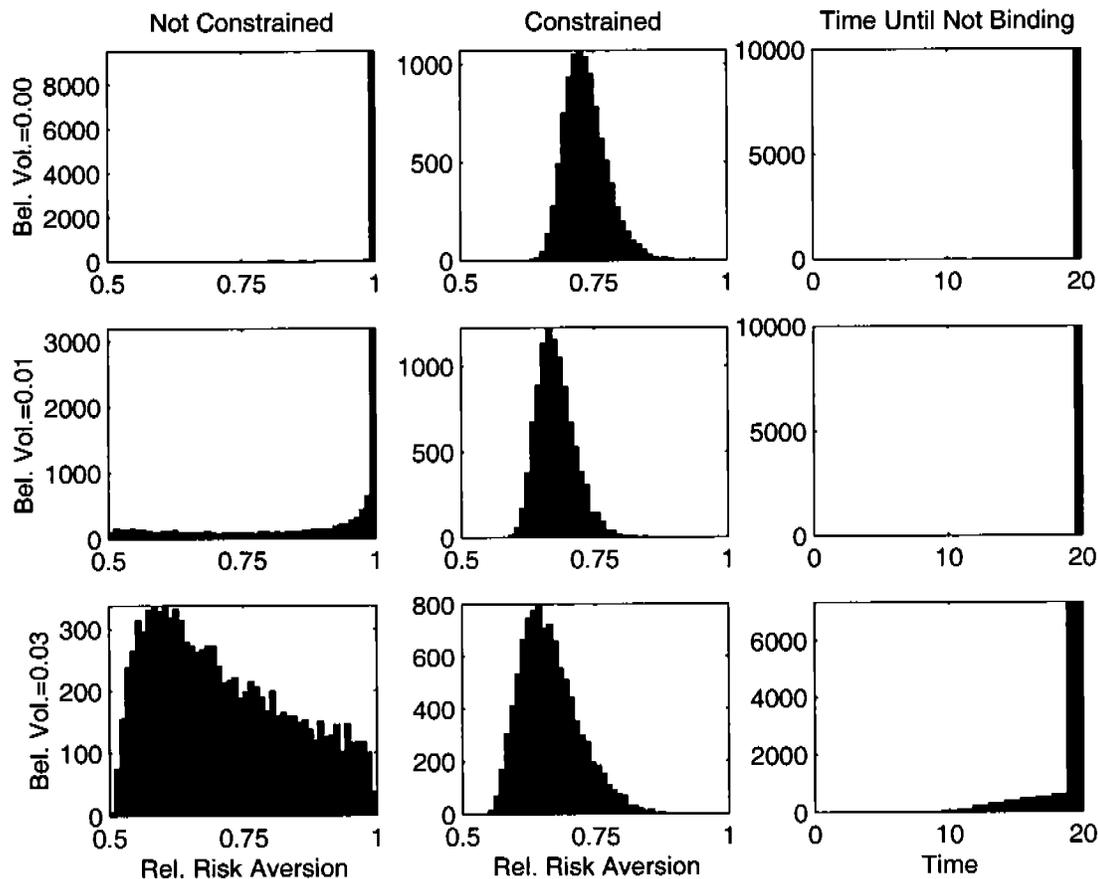


Figure 2: **Twenty Year Relative Risk Aversion Distributions under the Pessimist's Beliefs.** The left-side (middle) of the figure denotes the distribution of the relative risk aversion in the economy with short sales allowed (not allowed). The right-side of the figure denotes the distribution of the time until the short-sale constraint no longer binds. Common belief uncertainty increases moving down the figure. Parameters: $x^o = x^p = 0.5$, $\delta(t) = 1.0$, $\mu_\delta^o = 0.06$, $\mu_\delta = \mu_\delta^p = 0.04$, $\sigma_\delta = 0.02$, $\rho = 0.03$, and $T - t = 20$.

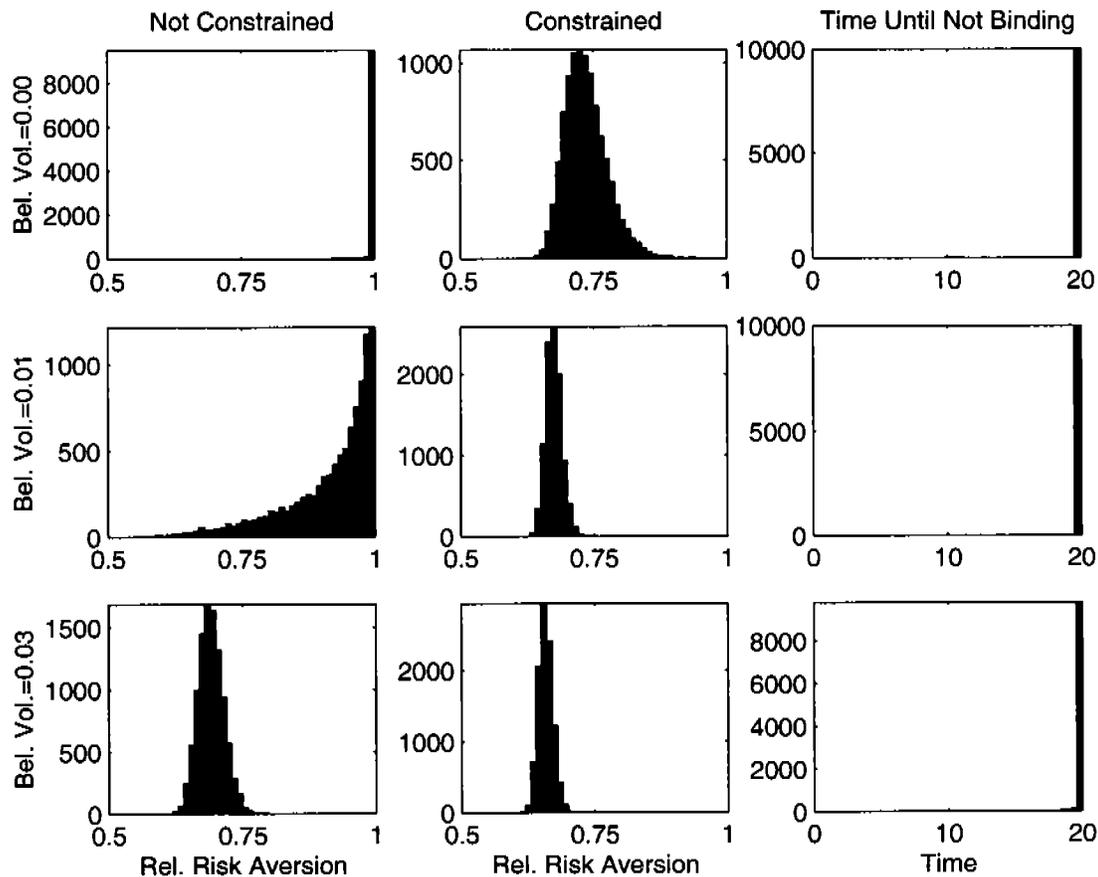


Figure 3: **Twenty Year Relative Risk Aversion Distributions under the True Beliefs.** The left-side (middle) of the figure denotes the distribution of the relative risk aversion in the economy with short sales allowed (not allowed). The right-side of the figure denotes the distribution of the time until the short-sale constraint no longer binds. Common belief uncertainty increases moving down the figure. Parameters: $x^o = x^p = 0.5$, $\delta(t) = 1.0$, $\mu_\delta^o = 0.06$, $\mu_\delta = \mu_\delta^p = 0.04$, $\sigma_\delta = 0.02$, $\rho = 0.03$, and $T - t = 20$.