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Directed Search On the Job and Wage Ladder

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Abstract

In this paper we characterize the equilibrium in a labor market where employed workers search on the job and firms direct workers' search by posting wages and selection criteria. All workers/jobs are homogeneous and there is free entry of firms to create jobs. The equilibrium features a wage ladder, with a finite number of rungs. Workers on each particular rung of the ladder choose (optimally) to apply to only the jobs at one level above their current wage, despite that they observe all higher wage offers. Workers choose not to leap several rungs at a time on the wage ladder because the jobs at one level above their current wage provide a significantly higher employment probability, and hence a higher expected surplus, than the jobs at two or more levels above. At all equilibrium wage levels, recruiting firms earn the same expected profit, with a higher wage level being compensated by a higher probability of hiring a worker successfully. Furthermore, the gap between two adjacent wages on the ladder shrinks as a worker climbs up the ladder, the density of offer (posted) wages decreases with the wage, and the density of employed wages can also be a decreasing function.

Keywords: Directed Search, Wage Dispersion.

Preliminary; Comments welcome.

1. Introduction

We study a large labor market where employed workers search on the job and firms direct the search process intentionally. There are a large fixed number of homogeneous workers, while the number of jobs (also homogeneous) is determined by free entry. All firms simultaneously post a wage offer and a selection rule of applicants. Workers, including unemployed ones, first observe all offers and then decide which job to apply to. After receiving the applicants, a firm selects a worker for the job according to the posted rule and pays the posted wage. Search is *directed*, in the sense that firms explicitly take into account how their offers will affect workers' application strategies. The game repeats without memory. Recruiting generates endogenous transitions of workers between jobs, while exogenous separation sends workers into unemployment. We characterize the stationary equilibrium in this market and study its properties.

Table 1. Search models

type of search	whether employed workers are allowed to search on the job: no	whether employed workers are allowed to search on the job: yes
undirected	Diamond (1982) Mortensen (1982) Pissarides (1990)	Burdett and Judd (1983) Burdett and Mortensen (1998) Pissarides (1994)
directed	Peters (1991) Montgomery (1991) Burdett et al. (2001)	This paper

Directed search on the job is a realistic feature of the labor market.[1] However, the search literature (see Table 1 for a guide) offers very little knowledge about the equilibrium with directed search on the job. The popular search models assume that search is *undirected*, e.g., Diamond (1982), Mortensen (1982) and Pissarides (1990). That is, firms ignore how their offers affect the number of matches they will receive and workers do not choose which job to apply to ex ante, although workers do choose whether to accept a job ex post. These popular models also rule out on-the-job search by assuming that unemployed workers are the only workers who search. The subsequent research has relaxed these two assumptions separately but not simultaneously. For example, Burdett and Mortensen (1998) and Pissarides (1994) examine undirected search on the job, while Peters (1991), Montgomery (1991) and Burdett et al. (2001) examine directed search without on-the-job search.[2] Our focus in this paper is to characterize the equilibrium with the combined feature of directed and on-the-job search.

[1]Reviewing the evidence by Blanchard and Diamond (1989), Pissarides (1994) concludes that about 20% of the monthly hires in the US are direct job-to-job movements. He argues that this fraction is about 40% in UK.

[2]Burdett and Judd (1983) analyzed sequential asearch in the goods market and did not frame their model as on-the-job search. However, the key element in their model, that some agents each have two or more price quotes while others have only one, is borrowed by Burdett and Mortensen (1998) to generate wage dispersion with on-the-job search. Similarly, Peters (1991) analyzed directed search in the goods market, but his analysis can be readily adapted to the labor market. Other examples of directed search models are Acemoglu and Shimer (1999), Cao and

Our study is also motivated by the fact that existing search models fail to capture some important aspects of the wage distribution among homogeneous workers with homogeneous jobs. First, with undirected search on the job, the density of the wage distribution is a strictly increasing function, but the empirical density is hump-shaped with the hump occurring at a low wage level (see Kiefer and Neumann (1993)).[3] Second, workers seem to climb up the wage ladder over time in reality, which existing search models fail to produce. In particular, wage dispersion in models with undirected search on the job does not constitute a wage ladder, because all workers can ascend immediately to the end of the wage spectrum with positive probability.

Of course, one can introduce heterogeneity among workers or jobs into existing models to make their predictions more realistic. To generate the hump-shaped wage distribution, for example, one may argue that workers differ in ability or that jobs differ in productivity. To generate a wage ladder, one may argue that workers' abilities are gradually observed by their employers, or there is match-specific productivity, or there is learning-by-doing on the job. These ex ante and ex post sources of heterogeneity are realistic, but they do not diminish the need to construct a model without these features. Since there is sufficient wage dispersion among similar workers with similar jobs, the amount of heterogeneity that those models need in order to match the actual wage distribution may be more than what the data exhibit. More importantly, a main contribution of introducing search frictions into economic models is to use such models to explain labor market phenomena *without* resorting to heterogeneity. When there is enough heterogeneity, a frictionless model may even be able to generate wage dispersion.

We show that the equilibrium with directed search on the job is a wage ladder, which comprises of a finite number of wage levels. Firms are indifferent between posting these equilibrium wages. A firm offering a higher wage is compensated by a higher probability of hiring a worker successfully. The expected surplus for a recruiting firm (i.e., the product of the hiring probability and the ex post surplus) is the same for all equilibrium wages. This expected surplus exactly covers the vacancy cost, and so all recruiting firms earn zero net expected profit. In contrast, workers are *not* indifferent between the different wages in equilibrium. A worker applies to only such jobs that offer the worker the maximum expected surplus which is the (employment) probability of getting the job times the ex post gain from the job. These jobs lie one level about the worker's current wage on the ladder. That is, workers choose to climb up the wage ladder over time, one rung at a time, rather than leap on the ladder.

The wage ladder occurs here without any of the familiar assumptions that induce gradual wage increases. In particular, (i) there is no gradual revelation of workers' productivity, learning-by-

Shi (2000), Julien et al (2000), Peters and Siow (forthcoming), Shi (2002, 2001, 2000, forthcoming), and Shimer (2001).

[3] In directed search models without on-the-job search, a wage differential can arise among homogeneous workers who work for homogeneous jobs, but some additional elements must be introduced. In Shi (2002), firms are different in size (i.e., the number of employees), where large firms pay higher wages than small firms do. In Shi (forthcoming) and Shimer (2001), the presence of high-skill workers induces partial sorting and generates a wage differential among low-skill workers.

doing or match-specific productivity, as all workers have the same productivity which is observable before match; (ii) there is no differential information among employed workers regarding job openings, as all employed applicants observe all job openings before they apply; and (iii) firms do not discriminate employed workers according to their current wages, as each firm selects all applicants it receives with the same probability in equilibrium. Rather, a worker chooses to apply to only the jobs one level about his current wage because such jobs offer a significantly higher employment probability, and hence higher expected surplus for the worker, than other jobs do.

To understand this result better, imagine two groups of applicants who are currently employed at, respectively, a low wage and a high wage. These two groups of applicants differ in the trade-off between the wage and the employment probability, because their outside options (i.e., their current wages) are different. A high-wage applicant cares more about the wage level and less about that employment probability than a low-wage applicant does. More precisely, since the same wage offer yields a lower ex post gain to a high-wage applicant than to a low-wage applicant, the same amount of increase in the wage offer represents a larger proportional increase in the expected surplus to a high-wage applicant than to a low-wage applicant. Exploring this difference in the trade-off, firms can separate the groups of applicants by offering high-wage applicants a high wage with a low employment probability and low-wage applicants a low wage with a high employment probability. In a stationary equilibrium, the separation produces the wage ladder. Clearly, directed search and on-the-job search are both important for supporting the wage ladder as an equilibrium.

The wage ladder has strong and novel implications on equilibrium wages and worker flows. First, the gap between two adjacent rungs on the ladder becomes smaller and smaller as wage increases. This implies that the marginal gain from climbing up on the ladder diminishes. Second, as wage increases on the ladder, the probability of getting a higher wage falls. This implies that the quit rate decreases with wages. Third, the density of offer (posted) wages is a strictly decreasing function; i.e, there are more firms recruiting at a low wage than at a high wage. This occurs despite the fact that a high wage makes hiring more successful than a low wage. Finally, the density of employed wages can decrease with wages, but not always so. Although there are more vacancies at a low wage than at a high wage, which generates a large flow of worker into a low wage, there are also more workers quitting low-wage jobs. Depending on which of these two flows dominates, the density of employed wages can be decreasing, increasing, or hump-shaped. In particular, when a low-wage job has a sufficiently higher inflow of workers than a high-wage job, the density of employed wages is a decreasing function.[4]

[4]With the wage ladder, workers employed at low wages do not apply to very high wages. This contributes in part to the decreasing density at high wage levels. As a comparison, the undirected search model of Burdett and Mortensen (1998) produces the result that workers at all wage levels apply to the highest wage with positive probability, which generates a much higher density of workers employed at the highest wage than at lower wages.

2. A Model of Directed Search On the Job

2.1. The Labor Market and Equilibrium Definition

Time is continuous. A labor market is populated by a large number, L, of risk neutral and infinitely-lived workers. All workers are homogeneous. There are also a large number of firms, determined endogenously by free entry, each of which has one job to offer. All jobs are the same, each yielding a flow of output, y. The flow cost of a vacancy is $C > 0$. Workers and firms both discount future with a rate of time preference $r > 0$. Although all workers are identical, we refer to a worker's current wage as the worker's type and call a worker at wage w a w-worker. For convenience, we sometimes refer to the unemployment benefit, denoted b, as an unemployed worker's "wage" and denote it w_0.

Employed workers search on the job as follows. Each employed worker receives an opportunity of job application at rate $\lambda(w) > 0$ according to the Poisson process, which allows the worker to observe the job descriptions offered by all firms. A job description consists of a wage offer and the firm's selection rule (described later). All recruiting firms post (and commit to) the job descriptions simultaneously before workers apply. After observing the offers, all applicants choose the application strategies simultaneously. Each applicant can apply to only one job and must incur a small fixed cost $S > 0$ for the application.[5] The application strategy can be mixed over the jobs. Because the applicants observe firms' offers before the application, a firm can choose the offer intentionally to attract particular applicants, i.e., to direct workers' search.

Unemployed workers' search is also directed. Each unemployed worker receive the job application opportunity at rate λ_0, which may or may not be different from λ. The number of unemployed workers is U and the unemployment rate is $u = U/L$, which are endogenous variables.

Once employed, a worker produces and is paid the posted wage until the job separates, either endogenously or exogenously. A worker separates from a job endogenously when he quits one job to accept another job. In addition, each worker-job pair separates exogenously at rate $\sigma > 0$ according to the Poisson process, after which the worker returns to the unemployment pool.

The most important feature of the directed search process is that each applicant observes many offers before application. This creates ex ante competition between recruiting firms that is absent in sequential search models (e.g., Burdett and Mortensen 1998), where a positive fraction of the applicants each observe only one offer at a time.[6] Nevertheless, the directed search process has the following familiar frictions. First, the job application opportunity is not abundant, in the sense that λ and λ_0 are finite. This is a proxy for the cost of gathering the information about jobs. Second, each applicant can apply to only a small number of jobs at a time, which is set to be one in our paper. This is a proxy for the physical cost of attending job interviews. Third,

[5]The small cost S in needed here to help the existence of an equilibrium. See subsection 4.2.

[6]We emphasize the fixed cost, rather than the variable cost, of this information-gathering process. The assumption that each applicant observes all firms' offers after getting a job applicantion opportunity simplifies the analysis, but it is not necessary for the analytical results. For ex ante competition to occur, it is sufficient to assume that each applicant observes two offers that are randomly drawn from all recruiting firms' offers.

agents cannot coordinate their decisions, which leads to the possibility of unmatched agents.

We focus on stationary equilibria in this market. Let Ω be the equilibrium support of wages, where the lowest wage is $w_1 = \inf(\Omega)$ and the highest wage is $w_M = \sup(\Omega)$. Extend the notion of wages to include the unemployment benefit w_0 $(= b)$ and denote the extended wage support as $\Omega_0 = \Omega \cup \{w_0\}$. The equilibrium distribution of workers Ω_0 is denoted $N(.)$, with a frequency (density) function $n(.)$. Clearly, $n(b) = u$. The distributional density of employed workers over Ω is $n(.)/(1-u)$. Vacancies are distributed over Ω according to the cumulative function $V(.)$, with a density $v(.)$. This is also the density of offer (posted) wages. The total number of vacancies is K, and so the total number of vacancies at w is $v(w)K$. Define $k = K/L$. To unify the notation, let $\lambda(w) = \lambda$ for all $w \neq w_0$ and $\lambda(w_0) = \lambda_0$. Call a w-worker who receives a job application opportunity a w-applicant.

We will analyze firms' and workers' strategies for a large and finite L first and then take the limit $L \to \infty$.[7] When $L \to \infty$, the expected number of workers at w-applicants, $\lambda(w)n(w)L$, is non-stochastic. Assume that this number is an integer, without loss of generality.

Each w-applicant $(w \in \Omega_0)$ maximizes the expected surplus of application, which is the product of the net gain from application and the probability of getting the job. The applicant chooses the subset of wages $T(w) \subseteq \Omega$, to which he applies with positive probability, and the application probabilities $P(w) = \{p(w', w)\}_{w' \in T(w)}$. The worker applies to each job opening of wage w' with probability $p(w', w)$. These probabilities must add up to one, i.e.,

$$\sum_{w' \in T(w)} p(w', w) v(w') K = 1.$$

Each firm maximizes the expected surplus from advertising the job, which is the product of the probability of successfully hiring a worker and the ex post profit the job yields. A firm posts a wage offer w and an ex post selection rule Z. Because workers use mixed strategies to apply and there is no coordination among the applicants, the number of applicants (of each type) that a particular firm receives is a random variable. Let $R(w')$ be the number of w'-applicants that a particular recruiting firm receives and $R = (R(w'))_{w' \in \Omega_0}$ the vector of such numbers. The firm's selection rule describes the probability $Z(w', R)$ with which the selected worker is a w'-worker, conditional on the composition of received applicants R. Given R, each particular w'-worker who applied to the firm is chosen with probability $Z(w', R)/R(w')$. Clearly, the firm cannot select a w'-worker if it has not received any such applicant, and the selection probabilities must add up to one if the firm has received one or more applicant. That is, the following restrictions must hold:

$$Z(w', R) = 0 \text{ if } R(w') = 0, \tag{2.1}$$

$$\sum_{w' \in \Omega_0} Z(w', R) = 1 \text{ if } R \neq 0, \text{ and } 0 \text{ otherwise.} \tag{2.2}$$

[7] We view continuous wage distributions in the market as the limits of discrete distributions. This procedure avoids possible measurability problems associated with analyzing individuals' strategies directly in an environment with a continuous wage distribution.

In addition, we exclude lexicographic selection rules by imposing the following restriction:

$$\begin{aligned}&\text{If } Z(w', R) > 0 \text{ for a particular } R \text{ such that } R(w') > 0,\\&\text{then } Z(w', R) > 0 \text{ for all such } R \text{ that } R(w') > 0.\end{aligned} \tag{2.3}$$

That is, if a firm selects a type of applicants with positive probability in some cases, then it must select such applicants with positive probability in all cases, provided that they show up at the match. The reason for imposing this restriction is that lexicographic selection rules may induce pure strategy equilibria which feature implicit coordination. Note that the firm can rank the applicants probabilistically. In particular, we do *not* restrict the selection probabilities to be equal for all types of received applicants, although this is a result we will establish later.

The ex post employment probabilities are cumbersome, because they involve all possible realizations of the firm's received applicants. Fortunately, we can reformulate the decision problems as if the firm offers ex ante, rather than ex post, employment probabilities. The ex ante probability, denoted $q(w, w')$ for each $w' \in \Omega_0$, specifies the expected or "average" probability with which a w'-worker gets the job when he applies to the firm posting w. It is calculated by aggregating ex post employment probabilities over all possible realizations of R. We refer to q simply as the *employment probability* and denote $Q(w) = \{q(w, w')\}_{w' \in \Omega_0}$. In Appendix A we will show that a firm's decision problem can be reformulated equivalently using Q rather than Z.

We examine only equilibria that are symmetric in the following sense: All workers whose current wages are the same use the same strategies and all firms that post the same wage use the same employment probabilities. Thus, all w'-workers apply to all jobs offering wage w with the same probability $p(w, w')$ and all firms offering w assign the same employment probability $q(w, w')$ to all w'-applicants.

Under symmetry, the probability with which a worker applies to each firm approaches zero when the market becomes large, i.e., $p(w', w) \to 0$ for all w and w'. To facilitate the analysis, we replace $p(w', w)$ in the description of workers' strategies by another object, $a(w', w)$, which stands for the expected number, or the *queue length*, of w-applicants that a firm offering w' receives. Since the number of w-applicants is $\lambda(w)n(w)L$ and each applies to a particular job opening w' with probability $p(w', w)$, the queue length of w-workers to such a firm is

$$a(w', w) = p(w', w)\lambda(w)n(w)L, \quad \forall w \in \Omega_0. \tag{2.4}$$

In the symmetric equilibrium, $p(w', w) = 1/[v(w')K]$ if $w' \in T(w)$ and 0 otherwise. So, $a(w', w) = \lambda(w)n(w)/[v(w')k]$ if $w' \in T(w)$ and 0 otherwise.

Definition 2.1. *An equilibrium in the labor market consists of the distributional characteristics $(\Omega, N(.), V(.), k)$, the recruiting firms' strategies $(w, Q(w))_{w\in\Omega}$, and the workers' strategies $(T(w), A(w))_{w\in\Omega_0}$, where $A(w) = \{a(w', w)\}_{w' \in T(w)}$, such that the following requirements are met: (i) Given the distributional characteristics and other firms' strategies, each firm's decisions are optimal and its posted wage belongs to Ω; (ii) Given the firms' decisions and the distributional characteristics, each worker's application decisions are optimal; (iii) The firms' and workers'* *strategies are symmetric in the sense described earlier; (iv) There is free entry of firms: the expected value of a vacancy is zero for every recruiting firm; (v) The distributions $N(.)$ and $V(.)$ are stationary.*

Because workers can observe firms' offers before application, the decisions (A, T) are functions of (w, Q) and each firm takes such dependence into account when choosing (w, Q).

2.2. Strategies and Payoffs

To characterize a firm's strategy, we calculate a firm's hiring probability, denoted $h(w)$ for a firm recruiting at w. The hiring probability is a function of workers' application strategies. Because a firm fails to recruit only when all potential applicants have applied to other firms, we have

$$h(w) = 1 - \prod_{w' \in \Omega_0} [1 - p(w, w')]^{\lambda(w')n(w')L}.$$

Using the queue length defined in (2.4) and the fact that $(1-p)^{1/p} \to e^{-1}$ when $p \to 0$, we have

$$h(w) = 1 - \exp\left[- \sum_{w' \in \Omega_0} a(w, w')\right]. \tag{2.5}$$

We will formulate a firm's decisions as choices of (w, q, a). For this purpose, it is convenient to rewrite the firm's hiring probability as follows (see Appendix A for a derivation):

$$h(w) = \sum_{w' \in \Omega_0} q(w, w')a(w, w'). \tag{2.6}$$

Loosely speaking, the quantity $q(w, w')a(w, w')$ is the expected number of w'-workers hired by the firm, because the firm attracts an expected number $a(w, w')$ of w'-workers and it selects one of such applicants with probability $q(w, w')$. The sum of such expected hires over the applicants' types is equal to h, because the firm hires exactly one worker if it succeeds in hiring.

Putting (2.5) and (2.6) together, we have

$$1 - \exp\left[- \sum_{w' \in \Omega_0} a(w, w')\right] - \sum_{w' \in \Omega_0} q(w, w')a(w, w') = 0. \tag{2.7}$$

When a and q are treated as separate decision variables of the firm (in addition to w), the above condition serves as a constraint on the decisions.

Now consider a firm's decisions. Let $J_f(w)$ be the value function of the firm that employs a worker at w, and J_v of a vacancy. Let $J_e(w)$ be the value function of a worker employed at wage w, and J_u of an unemployed worker. Sometimes we write $J_u = J_e(w_0)$. These value functions will be calculated later. Taking other firms' decisions as given, an individual firm chooses (w, Q) and the associated A to solve the following problem:

$$(\mathcal{P}) \quad \max \left[1 - \exp\left(- \sum_{w' \in \Omega_0} a(w, w')\right)\right] [J_f(w) - J_v]$$

subject to (2.7) and

$$q(w,w')\,[J_e(w)-J_e(w')] \geq E(w'), \text{ for all } w' \text{ such that } T(w') \ni w. \tag{2.8}$$

The objective function in $(\mathcal{P})$ is the firm's expected surplus, where we have substituted $h(w)$. The constraint (2.8) states that the firm attracts w'-workers only if it provides such a worker with an expected surplus that is greater than or equal to the market surplus. A w'-worker's market surplus, denoted $E(w')$, is taken as given by each individual agent.[8]

An applicant's decision maximizes expected surplus $q(w,w')\,[J_e(w)-J_e(w')]$, given the firms' strategies. For a worker at wage $w' \in \Omega_0$, the set of wages to apply to is $T(w') = \{w \in \Omega : a(w,w') > 0\}$. Given the expected surplus in the market, the decision $a(w,w')$ satisfies:

$$a(w,w') = \begin{cases} =\infty, & \text{if } q(w,w')\,[J_e(w)-J_e(w')] > E(w') \\ \in [0,\infty), & \text{if } q(w,w')\,[J_e(w)-J_e(w')] = E(w') \\ =0, & \text{otherwise.} \end{cases} \tag{2.9}$$

In the first case of (2.9), a job at wage w yields the w'-workers expected surplus higher than the market surplus. Each w'-worker applies to the firm with probability one, which gives the firm an infinite expected number of such applicants when $L \to \infty$. In the opposite case where the job yields expected surplus strictly lower than the market surplus, no w'-worker will apply to the firm. A firm gets a positive and finite expected number of w'-applicants only when the job provides the market surplus to such applicants. This implies that (2.8) must hold with equality in equilibrium for all w' such that $a(w,w') > 0$.[9]

Note that an applicant makes a trade-off between two characteristics of an offer – the employment probability and the worker's ex post surplus. Each level of the expected surplus can be generated by many combinations of the two characteristics, with a low wage being compensated by a high employment probability.

Now we compute the value functions. To do so, we need to calculate the rate at which a w-worker separates endogenously from the job, denoted $\rho(w)$. Endogenous separation occurs when the worker gets an application opportunity and is chosen by the firm he applies to. Thus,

$$\begin{aligned} \rho(w) &= \lambda(w)\textstyle\sum_{w'\in T(w)} q(w',w)p(w',w)v(w')K \\ &= \textstyle\sum_{w'\in T(w)} q(w',w)\frac{a(w',w)}{n(w)}v(w')k, \quad \forall w \in \Omega_0. \end{aligned} \tag{2.10}$$

In equilibrium, the value functions are given as follows:

$$rJ_v = -C + h(w)\,[J_f(w) - J_v]\,, \quad \forall w \in \Omega, \tag{2.11}$$

$$rJ_f(w) = (y-w) - [\sigma + \rho(w)]\,[J_f(w) - J_v]\,, \quad \forall w \in \Omega, \tag{2.12}$$

[8]See Burdett et al. (2001) and Cao and Shi (2000) for analyses of directed search in markets with finite sizes. They show explicitly that the effect of each individual agent's actions on the worker's market surplus is negligible when the size of the market approaches infinity.

[9]To see this, suppose that (2.8) holds with ">" for a particular w'. Then $a(w,w') = \infty$. To satisfy (2.7), it must be the case that $q(w,w') = 0$, which contradicts the supposition that (2.8) holds with ">".

$$
rJ_e(w) = \begin{cases} w - \sigma\left[J_e(w) - J_u\right], & \text{if } w = w_M, \\ \left[\begin{array}{l} w - \sigma\left[J_e(w) - J_u\right] - \lambda S \\ + \sum_{w' \in T(w)} q(w', w)\left[J_e(w') - J_e(w)\right] \frac{a(w',w)}{n(w)} v(w')k \end{array} \right], & \text{if } w < w_M, \end{cases} \tag{2.13}
$$

$$
rJ_u = b - \lambda_0 S + \sum_{w' \in T(b)} q(w', b)\left[J_e(w') - J_u\right] \frac{a(w', b)}{u} v(w')k. \tag{2.14}
$$

We explain (2.13) for example. Eq. (2.13) equates the permanent income of a worker employed at wage w, $rJ_e(w)$, to the expected "cash flow" in such employment. When the worker is currently earning the highest wage, all his future separation is exogenous, in which case the cash flow is equal to the wage w minus the expected loss from exogenous separation, $\sigma[J_e(w) - J_u]$. If the current wage is below the highest, the cash flow also consists of the expected gain from endogenous separation. The sum in (2.13) is the expected gain to the worker from applying to other wages and λS is the expected cost of application.[10]

3. The Configuration of an Equilibrium

In this section, we establish some features of the equilibrium and then make a conjecture about the structure of the equilibrium.

Lemma 3.1. *(**Equal-employment**) All applicants have the same employment probability from a job opening. That is, $q(w, w') = q(w)$ for all such w' that $a(w, w') > 0$ and for all $w \in \Omega$.*

Proof. For all such w' that $a(w, w') > 0$, the choice $a(w, w')$ satisfies the following first-order condition of $(\mathcal{P})$:

$$
\exp\left(- \sum_{w' \in \Omega_0} a(w, w') \right) \left[J_f(w) - J_v + \mu\right] - \mu q(w, w') = 0,
$$

where μ is the Lagrangian multiplier of (2.7). Clearly, the above equation implies that $q(w, w')$ is independent of w'. **QED**

This lemma is stronger than the mere statement that a firm is indifferent between the applicants whom the firm wants to attract, which requires only that $q(w, w') > 0$ for all w' such that $a(w, w') > 0$. It is optimal for the firm to give equal employment probability to all targeted applicants because the firm's hiring probability is concave in the queue lengths of applicants (see (2.5)). By giving equal employment probability to all targeted workers, the firm can maximize the hiring probability.[11]

[10] Here we presume that, for all $w < w_M$ and $w \in \Omega$, a worker employed at w is willing to pay the cost S to apply to higher wages. This will be verified later in equilibrium.

[11] When workers are heterogeneous in productivity, recruiting firms may give different priorities to different workers, see Shi (forthcoming) and Shimer (2001).

With the above lemma, (2.7) gives

$$q(w) = \left[1 - \exp\left(-\sum_{w' \in \Omega_0} a(w, w')\right)\right] \Big/ \sum_{w' \in \Omega_0} a(w, w') \,. \tag{3.1}$$

Because the function $(1 - e^{-x})/x$ is decreasing for all $x > 0$, the above formula says that an applicant's chance of getting a job is lower if there are more workers applying to the job. On the other hand, the firm's hiring probability is an increasing function of the total queue length (see (2.5)).

Lemma Equal-employment says nothing about the set of workers that the firm should attract or target. To examine this set, we establish the following lemma.

Lemma 3.2. **(Singleton)** *In equilibrium, a job opening attracts at most one type of applicants. More precisely,* $a(w^*, w_i)a(w^*, w_j) = 0$ *for all* $w_i, w_j, w^* \in \Omega$ *with* $w^* > w_j > w_i$*, provided that* $J_e(.)$ *is an increasing function.*

Proof. The lemma is trivially true for $w^* = w_1$. So, consider a firm posting $w^* > w_1$, whose choice problem is formulated as $(\mathcal{P})$ with w being replaced by w^*, etc.. For any w' such that $a(w^*, w') > 0$, the wage w^* yields the same expected surplus as the market does. So,

$$q(w^*, w') = \frac{E(w')}{J_e(w^*) - J_e(w')}, \text{ whenever } a(w^*, w') > 0. \tag{3.2}$$

Now suppose $a(w^*, w_i) > 0$ and $a(w^*, w_j) > 0$, with $w_j > w_i$. Inverting (3.1), we can write the total queue length of applicants for the firm as $X(q(w^*, w_t))$ for $t = i, j$. Then, the firm's optimal choice of w^* maximizes $\left[1 - e^{-X(q(w^*, w_t))}\right][J_f(w^*) - J_v]$. The first-order condition immediately shows that $\partial q(w^*, w_t)/\partial w^*$ must be the same for $t = i$ and j. However,

$$-\frac{\partial q(w^*, w_t)}{\partial w^*} = \frac{J_e'(w^*)E(w_t)}{[J_e(w^*) - J_e(w_t)]^2} = \frac{q(w^*)J_e'(w^*)}{J_e(w^*) - J_e(w_t)},$$

where we have used the result $q(w^*, w_t) = q(w^*)$ for $t = i, j$. Because $J_e(w_j) > J_e(w_i)$, the absolute value of the above derivative is larger for $t = j$ than for $t = i$. Therefore, it is not an equilibrium feature that $a(w^*, w_i)$ and $a(w^*, w_j)$ are both positive. **QED**

The economics behind this lemma will be repeatedly used for constructing an equilibrium later.[12] To understand the lemma, notice that an applicant's ex post surplus from a new job depends on his current wage. The higher the applicant's current wage, the lower his surplus from any given offer. For an applicant whose current wage is high, a given amount of increase in the wage offer represents a larger *proportional* increase in his expected surplus than for an applicant whose current wage is low. Put differently, a high-wage applicant cares more about the wage

[12]The required feature that $J_e(w)$ increases with w holds in all previous search models and, as verified later, it holds in our model as well.

level and less about its employment probability than a low-wage applicant does. Thus, there is no combination of the employment probability and wage that is most attractive to both high- and low-wage applicants and yet maximizes the firm's own expected surplus from recruiting. The equilibrium entails endogenous separation of workers in the application process by their current wages.

Figure 1 illustrates the above explanation. We draw two indifference curves, one for an applicant at wage $w' = w_i$ and the other for $w' = w_j$, where $w_j > w_i$. The indifference curves, characterized by (3.2), are downward-sloping because a lower wage must be accompanied by a higher employment probability in order to make an applicant indifferent. The two indifference curves have the single-crossing property, with the indifference curve of the high-wage applicant (w_j-worker) crossing that of the low-wage applicant from above. This graphic property reflects the difference between the two workers in the trade-off between the employment probability and the wage level. If the firm increases the wage level from the intersection of the two indifference curves, point A, it can reduce the employment probability significantly to keep the high-wage applicant on his indifference curve, by more than the amount that is needed to keep the low-wage applicant on his indifference curve.

If the firm attracts both types of applicants, the firm's offer must be at the intersection of these two indifference curves, because $q(w, w_i) = q(w, w_j)$ by Lemma Equal-employment. For this offer to be optimal for the firm, the firm's iso-profit curve must be tangent to the applicants' indifference curves. This is not possible. By targeting only one of the two types, instead, the firm's expected surplus increases, as point B in Figure 1 illustrates.

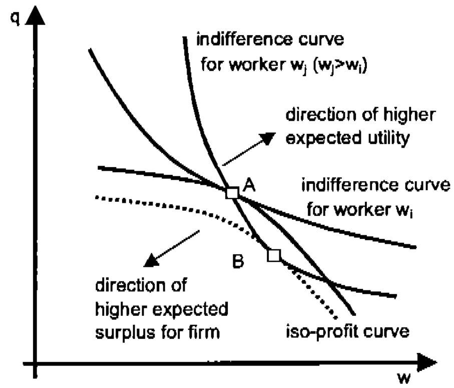

Figure 1

Lemma Singleton provides guidance to constructing the equilibrium. First, the lemma shows that the equilibrium constructed by Burdett and Mortensen (1998) in an undirected search model is not an equilibrium here. In the Burdett-Mortensen model, the assumption of undirected search implies that a firm will receive applicants from all wage levels with positive probability, regardless of the firm's offer. Although an applicant is *not* indifferent between the offers, he cannot choose ex ante which wage to apply to. The equilibrium then features a continuous wage distribution, where the firms that offer very similar wages can end up with applicants whose previous wages are far apart. With directed search, in contrast, workers will choose to apply only to the wages which yield the highest expected surplus to them. Anticipating these choices by the applicants, a firm will not use one wage to attract two different types of workers.

Second, the support of equilibrium wages must contain only a finite number of wage levels, even when $L \to \infty$. To see this, recall that there is a vacancy cost $C > 0$ and, in equilibrium, the expected surplus to a firm must be equal to this vacancy cost for all offer wages. Because each wage attracts only one type of applicants, two wages that are arbitrarily close to each other can both yield expected surpluses to the firm equal to C only if the workers they attract have wages that are arbitrarily close to each other. In turn, these latter workers must have been recruited from two lower wages that are arbitrarily close to each other. Repeating the argument, one finds that there must be two wages that are arbitrarily close to each other and that are both optimal for attracting only workers employed at w_1. This cannot occur, as the optimal pair of (q, w) for each given type of workers is unique (see Figure 1). So, two adjacent wages must be sufficiently apart from each other. Because wages are bounded above by y and below by 0, there must be only a finite number of wages in equilibrium.[13]

Third, a wage ladder is the most likely configuration of the equilibrium. From the explanation for Lemma Singleton, we know that a high-wage applicant is more willing than a low-wage to sacrifice the employment probability for an increase in the wage level. Thus, an applicant whose current wage is low will prefer to apply to the jobs whose wages are just one level above his current wage if these jobs provide a sufficiently high employment probability while other jobs at higher wages provide a low employment probability. This allows the firms to separate the different types of applicants.

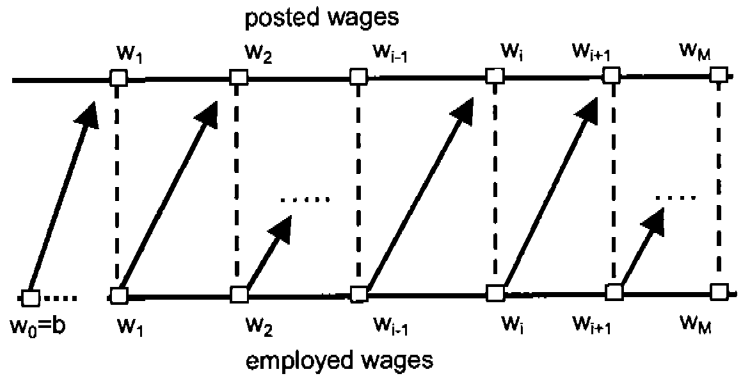

Figure 2

[13]In Burdett and Mortensen (1998), the equilibrium wage distribution cannot contain a mass point. If there were a mass point, a firm could increase the wage offer slightly above the mass point, which increases the number of applicants the firm gets by a discrete amount (as a result of undirected search) and hence increases expected profit. This argument becomes invalid when search is directed, because it is not optimal for the applicants to respond to a marginal increase in a firm's wage offer by a discrete increase in the application probability to the firm – doing so by a large number of applicants would reduce their employment probability by a discrete amount.

We depict the wage ladder in Figure 2. The set of equilibrium wages is $\Omega = (w_1, w_2, ..., w_M)$, where $w_i < w_{i+1}$ for $i = 1, 2, ..., M-1$. Each wage w_i attracts only the w_{i-1}-workers, i.e., $T(w_{i-1}) = \{w_i\}$ for all $i \in \{1, 2, ..., M\}$. The arrows in Figure 2 indicate the directions of job application and hence of endogenous separation. Exogenous separation is not depicted here, which takes the separated workers to unemployment.

With this configuration, we denote $n_i \equiv n(w_i)$ and $v_i \equiv v(w_i)$ for each $w_i \in \Omega$, with $n_0 = u$. For each firm posting w_i, let q_i be the employment probability for each applicant, h_i the firm's hiring probability, and a_i the queue length of applicants the firm attracts. Denoting q and h as functions of a, rather than of w as we have done so far, (2.5) and (3.1) become

$$h_i = h(a_i) \equiv 1 - e^{-a_i}, \quad q_i = q(a_i) \equiv (1 - e^{-a_i})/a_i. \tag{3.3}$$

As stated before, $h'(a) > 0$ and $q'(a) < 0$. Also, let ρ_i be the rate at which a w_i-worker endogenously separates from the job. Because a w_{i-1} worker applies to all vacancies at w_i with the same probability (under symmetry), $p(w_i, w_{i-1}) = 1/(v_i K)$ in equilibrium, and so

$$a_i = \lambda n_{i-1}/(v_i k), \quad \rho_{i-1} = \lambda q_i, \quad \text{for all } i \geq 2. \tag{3.4}$$

For $i = 1$, $a_1 = \lambda_0 u/(v_1 k)$ and $\rho_0 = \lambda_0 q_1$.

The task in the remainder of this paper is to establish the configuration in Figure 2 as an equilibrium. To do so, we need to show that the configuration leaves no incentive for firms and applicants to deviate. Because one firm's deviation to a wage outside Ω sends its potential employee off the equilibrium path for the application for higher wages in the future, we need to specify what firms do to the applicants whose wages happen to lie outside Ω.[14] Lemma Equal-employment indicates that the following restriction is most reasonable:

> **(Off-eqm)** For each i, $q(w_i, w') = q_i \; \forall w'$. That is, for each $w_i \in \Omega$, if a firm recruiting at w_i receives an applicant whose wage is different from w_{i-1}, the firm will select the applicant with the same ex ante probability q_i as it selects a w_{i-1}-worker.

4. Characterizing and Computing the Equilibrium

There are many potential types of deviations from the configuration depicted in Figure 2. In this section, we characterize and compute the equilibrium under the following additional restriction:

> **(One-rung)** For all $i \in \{0, ..., M-1\}$ and all deviation $w^d \in (w_i, w_{i+1})$, either (i) the firm posting w^d attracts only w_{i-1}-applicants and after receiving the job, such a worker's next application is to w_{i+1}, or (ii) the firm posting w^d attracts only w_i workers and after receiving the job, such a worker's next application is to w_{i+2}.

[14] If the belief off the equilibrium path is unrestricted, an arbitrary set of wages may be supported as an equilibrium. For example, consider an arbitrary set of wages Ω and suppose that for each wage w_i in this set, the firms recruiting at w_i give positive employment probability only to w_{i-1}-workers. Then, even a slight deviation from w_{i-1} will reduce the recruit's future employment probability for higher wages to zero. Knowing this, workers may not apply to the deviating firm at all, and this may successfully support Ω as an equilibrium.

This restriction asks that a deviation should disturb the wage ladder by only one rung, in either the applicants it attracts or the direction of its worker's future application. We will eliminate this restriction in section 6 but, for now, we will impose it to simplify the task of characterizing equilibrium wages.

4.1. Wages Lower Than the Highest Level

Examine the wage w_i for $1 \leq i \leq M-1$. Consider a deviation $w^d \in (w_{i-1}, w_{i+1})$. Without loss of generality, suppose that the deviating firm intends to attract only w_{i-1} workers. In this case One-rung requires that a w_{i-1}-worker who gets the job w^d will apply next to w_{i+1} if he receives an application opportunity, and so w^d is a deviation from w_i.[15] Let a^d be the queue length of w_{i-1} workers that w^d attracts. Then, the deviating firm's hiring probability is $h(a^d)$ and each applicant's employment probability is $q(a^d)$, given by (3.3) with a^d replacing a. After an applicant gets the job, the probability of endogenous future separation is λq_{i+1} (note that we invoked Off-eqm here). If the deviating firm successfully hires a worker, the firm's and the employee's value functions can be calculated from (2.12) and (2.13) as follows:

$$J_f(w^d) = \frac{y - w^d + (\sigma + \lambda q_{i+1})\, J_v}{r + \sigma + \lambda q_{i+1}}, \tag{4.1}$$

$$J_e(w^d) = \frac{w^d + \sigma J_u - \lambda S + \lambda q_{i+1} J_e(w_{i+1})}{r + \sigma + \lambda q_{i+1}}. \tag{4.2}$$

Taking $(q_{i+1}, J_e(w_{i+1}), J_v)$ and other firms' strategies as given, the deviator chooses (w^d, a^d) to solve the following problem similar to $(\mathcal{P})$:

$$\begin{array}{ll} (\mathcal{P}d) & \max\ h(a^d)\left[J_f(w^d) - J_v\right] \\ s.t. & q(a^d)\left[J_e(w^d) - J_e(w_{i-1})\right] = E(w_{i-1}). \end{array}$$

For w_i to be an equilibrium wage, the solution to $(\mathcal{P}d)$ must be $(w^d, a^d) = (w_i, a_i)$. The first-order conditions and the constraint of $(\mathcal{P}d)$ yield:

$$\begin{array}{l} J_e(w_i) - J_e(w_{i-1}) = \frac{a_i}{e^{a_i} - 1 - a_i}\left[J_f(w_i) - J_v\right], \\ q_i\left[J_e(w_i) - J_e(w_{i-1})\right] = E(w_{i-1}), \end{array}$$

where $J_f(w_i)$ and $J_e(w_i)$ obey (4.1) and (4.2), respectively, with w_i replacing w^d. The first equation states that the applicant's surplus after getting the job, $J_e(w_i) - J_e(w_{i-1})$, is a share $a_i/(e^{a_i} - 1)$ of the total surplus. This is a decreasing function of a_i and so, if the wage attracts more applicants, the worker gets a smaller share of the surplus.

[15]There are two other cases, but the same analysis applies with i being renumbered. The first is that the deviating firm intends to attract w_i workers with $w^d \in (w_i, w_{i+1})$. In this case, the restriction One-rung requires the w_i-worker who gets the job w^d to apply next to w_{i+2}, and so w^d can be treated as a downward deviation from w_{i+1}. The second is that the deviating firm intends to attract w_{i-2} workers with $w^d \in (w_{i-1}, w_i)$. In this case, One-rung requires the w_{i-2}-worker who gets the job w^d to apply next to w_i, and so w^d can be treated as an upward deviation from w_{i-1}.

In addition, the value of every vacancy is zero in equilibrium because of free entry. That is, $J_v = h_i [J_f(w_i) - J_v] - C = 0$. Then,

$$\frac{C}{h_i} = J_f(w_i) - J_v = J_f(w_i) = \frac{y - w_i}{r + \sigma + \lambda q_{i+1}}. \tag{4.3}$$

With this condition, we rewrite the first-order conditions of $(\mathcal{P}d)$ as follows:

$$J_e(w_i) - J_e(w_{i-1}) = C/f_i, \tag{4.4}$$

$$E(w_{i-1}) = q_i [J_e(w_i) - J_e(w_{i-1})] = Cq_i/f_i, \tag{4.5}$$

where

$$f_i = f(a_i) \equiv q(a_i)(e^{a_i} - 1 - a_i). \tag{4.6}$$

Finally, for w_{i-1}-workers to incur the fixed cost of application, the market surplus $E(w_{i-1})$ must be greater than or equal to S. With (4.5), this requirement becomes

$$a_i \le \bar{a}, \quad \text{where } e^{\bar{a}} - 1 - \bar{a} = C/S. \tag{4.7}$$

Under One-rung, the conditions (4.3) – (4.7) ensure that w_i, together with the queue length a_i, is optimal for attracting w_{i-1}-workers among all wages in (w_{i-1}, w_{i+1}), for any $i \in \{1, ..., M-1\}$.

4.2. The Highest Wage in Equilibrium

The highest wage w_M must satisfy (4.3) – (4.7), for $i = M$, in order to ensure that it is the optimal wage for attracting w_{M-1}-workers. In addition, there should not be any incentive for a firm to post a wage higher than w_M. A necessary and sufficient condition for this requirement is that it is not profitable to post any $w^* > w_M$ to attract w_M-workers.[16] To investigate this, consider a single firm's deviation to $w^* > w_M$ with an intention to attract w_M-workers. The deviation will attract w_M-workers until a further increase in the queue length would reduce an applicant's expected surplus below the fixed cost of application. Thus,

$$q(a^*) [J_e(w^*) - J_e(w_M)] = S, \tag{4.8}$$

where a^* is the queue length of applicants for w^*, and $J_e(w) = (w + \sigma J_u)/(r + \sigma)$ for both $w = w^*$ and w_M. The deviator's expected surplus is $h(a^*) [J_f(w^*) - J_v]$, where $J_v = 0$ and $J_f(w^*) = (y - w^*)/(r + \sigma)$. This deviation is not profitable if and only if the firm's maximum expected surplus is less than what is earned by posting equilibrium wages, i.e., less than C. Solving the deviator's maximization problem subject to (4.8), this requirement becomes:

$$w_M > y - S(r + \sigma)e^{\bar{a}}, \tag{4.9}$$

[16]This condition guarantees that it is not profitable to post $w^* > w_M$ to attract w_{M-j}-workers, for $j \ge 1$. The proof consists of two stages. First, by construction, w_M yields the highest expected surplus to w_{M-1}-applicants and hence those applicants prefer to apply to w_M to all other wages $w^* > w_M$. Second, if w_{M-1}-applicants prefer w_M to $w^* > w_M$, so do all other workers at wages w_{M-j} $(j \ge 2)$. The proof for the second stage is similar to that of Lemma 6.4 later and hence omitted.

where $\bar{a}$ is defined in (4.7).

We now explain why a positive application cost S is necessary for supporting an equilibrium. If $S = 0$, then the highest wage must exceed y in order to satisfy (4.9) but in this case, the firm posting w_M makes a negative net profit. To explain why this is so, suppose $S = 0$ and $w_M < y$. A firm that deviates to a slightly higher wage $w_M + \varepsilon$ ($\varepsilon > 0$) can always attract w_M-workers, and so it can succeed in hiring almost surely. Relative to posting w_M, the deviation gives the firm a slightly lower ex post surplus but a discrete increase in the hiring probability. Thus, the deviation is profitable. To prevent such profitable deviations, w_M must be equal to or greater than y, which yields negative expected profit after the vacancy cost is deducted.

For future use, it is useful to express (4.7) and (4.9) for $i = M$ as requirements on the hiring probability h_M, as follows:[17]

$$1 - (1 + \bar{a})e^{-\bar{a}} < h_M \leq 1 - e^{-\bar{a}}. \tag{4.10}$$

4.3. Computing the Equilibrium

The conditions in previous two subsections provide a recursive procedure for computing the equilibrium. Pick up a number h_M that satisfies (4.10). Note that $q_{M+1} = 0$ and $J_v = 0$ in equilibrium. Then,

$$\begin{aligned} &a_M = -\ln(1 - h_M), \quad q_M = h_M / a_M, \\ &w_M = y - (r + \sigma)C/h_M, \quad J_e(w_M) = (w_M + \sigma J_u)/(r + \sigma). \end{aligned}$$

The result for w_M comes from setting $i = M$ in (4.3), and the result for $J_e(w_M)$ from setting $i = M$ and $w^d = w_M$ in (4.2). We have the following proposition.

Proposition 4.1. *(**Recursive**) Given h_M, q_{M-j+1}, w_{M-j} and $J_e(w_{M-j})$, the following conditions hold in equilibrium for $j = 0, 1, 2, ..., M - 2$:*

$$h_{M-j} = \frac{(r + \sigma + \lambda q_{M-j+1})\, C}{y - w_{M-j}}, \tag{4.11}$$

$$a_{M-j} = -\ln(1 - h_{M-j}), \quad q_{M-j} = h_{M-j}/a_{M-j}, \tag{4.12}$$

$$w_{M-j-1} = w_M + \lambda S - \frac{\lambda q_{M-j} C}{f_{M-j}} - (r + \sigma)C \sum_{t=0}^{j} \frac{1}{f_{M-t}}, \tag{4.13}$$

$$J_e(w_{M-j-1}) = \frac{\sigma J_u + w_M}{r + \sigma} - C \sum_{t=0}^{j} \frac{1}{f_{M-t}}. \tag{4.14}$$

In addition, (4.11) and (4.12) hold for $j = M - 1$, and J_u satisfies

$$J_u = \frac{1}{r}\left[b - \lambda_0 S + \lambda_0 C \frac{q_1}{f_1}\right]. \tag{4.15}$$

[17] For $i = M$, (4.7) is equivalent to $h_M \leq h(\bar{a}) = 1 - e^{-\bar{a}}$. To rewrite (4.9), note that $h_M\,[J_f(w_M) - J_v] = C$ in equilibrium and $J_f(w_M) - J_v = (y - w_M)/(r + \sigma)$.

Proof. We prove the proposition by induction. For $j = 0$, we have already verified (4.11) and (4.12). To verify (4.13) and (4.14) for $j = 0$, set $i = M - 1$ (and $w^d = w_{M-1}$) in (4.2) to obtain an equation for $J_e(w_{M-1})$. Using this equation and substituting $J_e(w_M)$, we compute

$$J_e(w_M) - J_e(w_{M-1}) = \frac{w_M - w_{M-1} + \lambda S}{r + \sigma + \lambda q_M}.$$

Combining this equation with (4.4) for $i = M - 1$, one obtains (4.13) and (4.14) for $j = 0$.

Now suppose that (4.11) – (4.14) hold for an arbitrary $j \in \{0, 1, ..., M - 3\}$. We show that they also hold for $j + 1$ and so, by induction, the proposition holds. For $j + 1$, (4.11) comes from setting $i = M - (j + 1)$ in (4.3), and (4.12) from the definitions of $h_{M-(j+1)}$ and $q_{M-(j+1)}$. To verify (4.13) and (4.14) for $j + 1$, set $i = M - j - 2$ (and $w^d = w_{M-j-1}$) in (4.2) to obtain an equation for $J_e(w_{M-j-2})$. Substituting this result, we get:

$$\begin{aligned} & J_e(w_{M-(j+1)}) - J_e(w_{M-(j+2)}) \\ = & \tfrac{1}{r+\sigma+\lambda q_{M-(j+1)}} \left[(r + \sigma) J_e(w_{M-(j+1)}) - w_{M-(j+2)} - \sigma J_u + \lambda S \right] \\ = & \tfrac{1}{r+\sigma+\lambda q_{M-(j+1)}} \left[w_M - w_{M-(j+2)} + \lambda S - (r + \sigma) C \textstyle\sum_{t=0}^{j} \frac{1}{f_{M-t}} \right]. \end{aligned}$$

The second equality comes from substituting (4.14) for j, which holds by supposition. Combining the above result with (4.4) for $i = M - (j + 1)$, we obtain (4.13) and (4.14) for $j + 1$.

Finally, the zero-profit condition (4.3) must hold for a firm posting w_1. By the above derivation, this implies that (4.11) and (4.12) must hold for $j = M - 1$. In contrast, (4.13) and (4.14) need be modified for $j = M - 1$. By definition, $w_0 = b$ and $J_e(w_0) = J_u$. To derive (4.15), use the wage ladder to simplify (2.14) as $rJ_u = b - \lambda_0 S + \lambda_0 q_1 [J_e(w_1) - J_u]$. Substituting $[J_e(w_1) - J_u]$ from (4.4) (with $i = 1$), we obtain (4.15). **QED**

For given h_M, the recursive method generates the sequence $(h_i, a_i, q_i, w_i, J_e(w_i))$. For the computed sequence to be an equilibrium, the value of h_M must be such that the generated value of $J_e(w_1)$ satisfies (4.4) for $i = 1$. We will examine such existence in the next subsection.

Once the equilibrium sequence of (h, a, q) is determined, we can compute the distributions of workers and vacancies. First, because the equilibrium is stationary, the measure of workers who separate from w_i must be equal to the measure of workers newly recruited at wage w_i. That is,

$$\begin{aligned} & (\sigma + \lambda q_{i+1}) n_i = \lambda n_{i-1} q_i, \text{ for all } 2 \leq i \leq M, \\ & (\sigma + \lambda q_2)\, n_1 = \lambda_0 u q_1 = \sigma(1 - u) \text{ and } u = 1 - \textstyle\sum_{i=1}^{M} n_i. \end{aligned} \tag{4.16}$$

These equations solve for u and $(n_i)_{i=1}^{M}$. Second, (3.4) implies

$$v_i = \lambda n_{i-1}/(a_i k) \text{ for } i \geq 2 \text{ and } v_1 = \lambda_0 u/(a_1 k). \tag{4.17}$$

Together with $\sum_{i=1}^{M} v_i = 1$, these equations solve for k and $(v_i)_{i=1}^{M}$. The distributional density of posted wage is $(v_i)_{i=1}^{M}$ and of employed wages $(n_i/(1 - u))_{i=1}^{M}$.

4.4. The Number of Rungs and the Hiring Probability at the Highest Wage

The equilibrium values of M and h_M are such that, starting with h_M, the computed sequence of J_e satisfies $J_e(w_1) - J_u = C/f_1$, as (4.4) requires. For existence of equilibrium (M, h_M), we maintain the following assumption throughout this paper.

Assumption 1. *(**Regularity**) Define $\bar{a}$ by (4.7). Assume that the following conditions hold:*

$$b \leq y + \lambda_0 S - C\left[\frac{(r+\sigma)\,e^a + \lambda_0}{e^a - 1 - a}\right]_{a=\bar{a}-\ln(1+\bar{a})}, \tag{4.18}$$

$$(r+\sigma)/\lambda > f(\bar{a})/\bar{a}. \tag{4.19}$$

The condition (4.18) ensures that there is at least one wage level that yields higher present value to the workers than unemployment (see later discussion), while (4.19) is a technical condition necessary for exploring Proposition Recursive.

Denote $\Delta = J_e(w_1) - J_u - C/f_1$. Setting $j = M-2$ in (4.14) to obtain $J_e(w_1)$ and substituting (4.15) for J_u, we get:

$$\Delta = \frac{w_M - b + \lambda_0 S}{r+\sigma} - \frac{C\lambda_0 q_1}{(r+\sigma)\,f_1} - C\sum_{t=0}^{M-1}\frac{1}{f_{M-t}}. \tag{4.20}$$

By Proposition Recursive, the computed a sequence and w_M depend on the chosen values of (M, h_M), but not on J_u directly. Hence, we write $\Delta = \Delta(M, h_M)$. Equilibrium values of (M, h_M) solve $\Delta(M, h_M) = 0$. The following lemma is proven in the last part of Appendix D.

Lemma 4.2. *Fix $h_M = h^*$, where h^* is any value that satisfies (4.10), and compute the a sequence from Proposition Recursive. Then, there exists an integer $M^* \geq 1$ such that $\Delta(M', h^*) \leq 0$ for all $M' \leq M^*$ and $\Delta(M', h^*) > 0$ for all $M' \geq M^* + 1$.*

Now we can find the lowest equilibrium value of h_M as follows. Choose h^* in the above lemma to be the lower bound of h_M in (4.10) and obtain the corresponding M^*. Then, $\Delta(M^*, h^*) \leq 0$. If $\Delta(M^*, h^*) = 0$, then the chosen value of h^* is the lowest equilibrium value of h_M. Suppose $\Delta(M^*, h^*) < 0$. Fix $M = M^*$ and increase h^*. By Proposition Monotone established later, the a sequence is an increasing function of the chosen value of h_M. So is w_M. Thus, $\Delta(M, h^*)$ is an increasing function of h^* (for given M). For there to be an equilibrium solution for h_M, $\Delta(M, h^*)$ must increase to cross 0 when h^* increases to the upper bound in (4.10). The first crossing gives the lowest equilibrium value of h_M, where $M = M^*$ is the equilibrium value of M.

Similarly, we can find the highest equilibrium value of h_M. To do so, choose the upper bound of h_M, $1 - e^{-\bar{a}}$, to be the starting value of h^* in the above lemma and compute the corresponding M^*. If $\Delta(M^*, h^*) = 0$, then the starting value of h^* is the highest equilibrium value of h_M. If $\Delta(M^*, h^*) < 0$, then fix $M = M^* + 1$ and reduce h^*. For there to be an equilibrium solution for h_M, $\Delta(M, h^*)$ must decrease to cross 0 when h^* decreases to the lower bound of h_M in (4.10).

The first crossing gives the highest equilibrium value of h_M, where $M = M^* + 1$ is the equilibrium value of M. Therefore, the following proposition holds:

Proposition 4.3. (**M-exists**) *Set* $h^* = 1 - (1+\bar{a})\, e^{-\bar{a}}$ *and compute* M^* *as in Lemma 4.2. There exists an equilibrium value of* h_M *if and only if*

$$\Delta(M^*, 1 - e^{-\bar{a}}) \geq 0. \tag{4.21}$$

Under this condition, there exist h_L *and* h_H*, which possibly coincide with each other, such that all equilibrium values of* h_M *lie in* $[h_L, h_H]$*. The equilibrium value of* M *is either* M^* *or* $M^* + 1$.

It is analytically difficult to verify (4.21) or to check whether the solution for h_M is unique. In section 5.2 we will provide some numerical examples.

5. Properties of the Equilibrium

In this section, we investigate the properties of the equilibrium, presuming One-rung. In section 6 we will show that, under certain conditions, deviations that violate One-rung are not profitable.

5.1. Analytical Properties

To study the properties of the equilibrium, we examine the properties of the sequence computed with any given value of h_M that satisfies (4.10). Suppress the index $M - j$. Use the subscript $+t$ to stand for $M - j + t$ and $-t$ for $M - j - t$, where $t \geq 1$. We start by deriving some useful equations from Proposition Recursive. Subtracting (4.13) for j and $j+1$, we get:

$$w - w_{-1} = \frac{(r + \sigma + \lambda q)C}{f} - \frac{\lambda C q_{+1}}{f_{+1}}, \tag{5.1}$$

where the suppressed index is $M - j$, and the condition holds for all $j \in \{1, 2, ..., M-2\}$. For $j = 0$, replace the last term by $\lambda C q(\bar{a})/f(\bar{a}) = \lambda S$. Also, for all $j \in \{1, ..., M-2\}$, we have:

$$\begin{aligned} \tfrac{r+\sigma+\lambda q}{h_{-1}} &= \tfrac{1}{C}(y - w_{-1}) = \tfrac{1}{C}(y - w) + \tfrac{1}{C}(w - w_{-1}) \\ &= \tfrac{r+\sigma+\lambda q_{+1}}{h} + \tfrac{1}{C}(w - w_{-1}) = \tfrac{r+\sigma+\lambda q_{+1}}{h} + \tfrac{r+\sigma+\lambda q}{f} - \tfrac{\lambda q_{+1}}{f_{+1}}. \end{aligned}$$

The first equality comes from using (4.11) for $j+1$, the second equality from rewriting, the third equality from using (4.11) for j, and the last equality from substituting (5.1). Thus,

$$h_{-1} = (r + \sigma + \lambda q) \Big/ \left(\frac{r + \sigma + \lambda q_{+1}}{h} + \frac{r + \sigma + \lambda q}{f} - \frac{\lambda q_{+1}}{f_{+1}} \right). \tag{5.2}$$

A similar equation holds for $j = 0$, with the first q_{+1} being replaced by 0 and the term $\lambda q_{+1}/f_{+1}$ by $\lambda q(\bar{a})/f(\bar{a}) = \lambda S/C$.

We establish the following propositions, in Appendices C and D, respectively.

Proposition 5.1. **(Monotone)** *For any given h_M that satisfies (4.10), the sequence constructed in Proposition Recursive has the following properties for all $0 \leq j \leq M-2$ (where the subscript $M-j$ is suppressed):*

$$a_{-1} < a \leq \bar{a}, \; h_{-1} < h, \; q_{-1} > q. \tag{5.3}$$

$$(r+\sigma)/\lambda > f(a_{-1})/a_{-1}. \tag{5.4}$$

$$a_{-1} > a - \ln(1+a), \tag{5.5}$$

$$da/dh_M > 0, \quad dw/dh_M > 0. \tag{5.6}$$

Proposition 5.2. **(W-property)** *For any given h_M that satisfies (4.10), the computed sequence satisfies: (i) $w > w_{-1}$ and (ii) $E(w_{-1}) > E(w) \geq S$. In addition: (iii) $w - w_{-1} > w_{+1} - w$, if the condition in Lemma No-leap holds which is described later as a necessary condition for the equilibrium.*

These propositions reveal some interesting properties of the equilibrium, as listed below:

- A firm is more likely to succeed in hiring at a higher wage than at a lower wage, while an applicant is more successful getting a low-wage job than a high-wage job. This result arises because the endogenous queue length of applicants increases with the wage level.
- The quit rate (endogenous separation) of a worker at a high wage is lower than of a worker at a low wage. The quit rate of a worker at wage w is $\rho(w) = \lambda q(a_{+1})$. Because the employment probability ($q(a_{+1})$) for the next wage decreases as the wage increases, the quit rate falls.
- The ex post value of employment to a worker, $J_e(w)$, increases with the wage level. That is, $J_e(w) > J_e(w_{-1})$, which we used in Lemma Singleton. Also, $E(w_i) \geq S$ for all $1 \leq i \leq M-1$. So, all workers employed below the highest wage are indeed willing to incur the fixed cost S to apply for higher wages.
- The gap between two rungs of the ladder shrinks as a worker climbs up the ladder.
- A worker's expected surplus diminishes as the worker moves up the wage ladder, i.e., $E(w_{-1}) > E(w)$, despite that the ex post value of employment increases. This implies that an applicant's employment probability must decrease more rapidly than the increase in the wage level along the wage ladder.

All these properties are realistic features of the labor market. The last property is a necessary (although not a sufficient) condition for the wage ladder to be an equilibrium in a directed search environment. Since an applicant observes all offers, to induce him to apply only to the next wage level, the expected surplus might be higher from applying for such jobs than for other jobs at higher wages.

The next proposition describes some features of the wage density in the equilibrium (see the middle part of Appendix D for a proof):

Proposition 5.3. (**W-density**) *The density of offer wages decreases with the wage. A sufficient condition for the density of employed wages to be decreasing at the upper end of the wage support is:*

$$\frac{\sigma}{\lambda} > \left[1 - (1 + \bar{a})\, e^{-\bar{a}}\right] / \left[\bar{a} - \ln(1 + \bar{a})\right]. \tag{5.7}$$

When r is sufficiently close to 0, a sufficient condition for the above inequality is $C/S > 2.373$. A sufficient condition for the density of employed wages to be increasing at the upper end of the wage support is $\sigma/\lambda < q(\bar{a})$.

The density of offer wages is a decreasing function. This result is opposite to the increasing density that Burdett and Mortensen (1998) established with undirected search on the job. The decreasing density in our paper can be easily understood by counting the flows of workers in and out of each wage w. The inflow is the number of new hires, hv, i.e., the number of vacancies recruiting at wage w times the probability that each hires successfully. The outflow of workers consists of exogenous separation and endogenous separation. Because endogenous separation from w is the only source of new hires for the next wage w_{+1}, it is equal to $h_{+1}v_{+1}$. In a stationary equilibrium, the flows in and out of w must be equal to each other, which requires $hv > h_{+1}v_{+1}$. Because each job opening at w has a lower hiring probability than an opening at w_{+1}, the requirement can hold only if $v > v_{+1}$, thus the decreasing density of offer wages.

The density of employed wages can also be decreasing, but it is not always so analytically. This is because the density of employed wages depends on both the inflow and the outflow of workers. Although there is a larger flow of workers into a low wage than into a high wage, the quit rate is also higher for a low wage than for a high wage. There are more workers employed at a low wage than at a high wage if and only if the difference between the inflows into the two wages is larger than the difference between the outflows. This is satisfied at the upper end of the wage distribution if the hiring cost is large relative to the application cost. In general, however, the density of employed wages may not be monotonic with respect to wages and if it is monotonic, it may not be increasing.

5.2. A Numerical Example

We provide a numerical example to illustrate the equilibrium. Consider the following values of the parameters: $y = 200$, $b = 40$, $C = 5$, $S = 0.5$, $r = 0.005$, $\lambda = 0.017$, $\lambda_0 = 0.222$, and $\sigma = 0.083$. These parameter values satisfy all requirements for an equilibrium (see Proposition 6.7). Under these parameter values, the number of rungs on the wage ladder is equal to or smaller than 4. An equilibrium with $M = 4$ exists, where the unemployment rate is $u = 28.7\%$ and the overall vacancy-worker ratio is $k = 0.48$.[18] Other characteristics of this equilibrium are summarized in Table 2.

Table 2. Equilibrium in a numerical example

i	w_i	v_i (%)	$\frac{n_i}{1-u}$ (%)	a_i	q_i (%)	h_i (%)
1	196.1	96.2	87.0	0.14	93.4	13.0
2	198.9	3.5	11.8	0.62	74.7	46.0
3	199.4	0.1	1.1	1.52	51.4	78.1
4	199.5	<0.1	<0.1	2.60	35.6	92.6

These numerical results confirm the analytical properties of (w, a, q, h, v) established in Propositions Monotone and W-property. A notable feature is that, although the difference between two adjacent wages is small, it induces large differences between two adjacent rungs in the employment probability, the hiring probability and the density of offer wages. For example, when the wage increases from 196.1 to 198.9, the employment probability falls sharply from 93.4% to 74.7%, the hiring probability increases from 13% to 46% and the density of offer wages falls from 96.2% to 3.5%. A predominant fraction of firms recruit at the lowest wage.

The density of employed wages is also a decreasing function of wages in this equilibrium. A large fraction of workers are employed at the lowest wage, although the distribution is less skewed toward low wages than the offer wage distribution.

6. Optimality of the Wage Ladder

In this section, we eliminate the restriction One-rung imposed in section 4 and show that the wage ladder can be optimal. We divide deviating wages that violate One-rung into two categories, those that belong to the equilibrium wage support and those that lie outside. The two subsections below examine them in turn. The restriction Off-eqm is maintained throughout.

6.1. Workers Apply Only to the Next Wage Level

In this subsection, we confine deviating wages to the set Ω. With this confinement, a wage $w_i \in \Omega$ is a deviation violating One-rung if it attracts workers other than the w_{i-1}-workers. For such deviations to be not profitable, the expected surplus that a w_{i-1}-worker obtains from applying to wage w_i must be larger than or equal to that from applying to any other wage in Ω. That is,

$$q\left[J_e(w) - J_e(w_{-1})\right] \geq q_{+t}\left[J_e(w_{+t}) - J_e(w_{-1})\right]. \tag{6.1}$$

This must hold for all $t \in \{1, 2, ..., j\}$ and all $j \in \{1, 2, ..., M-1\}$, where the suppressed index is $i = M - j$. Notice that we have invoked Off-eqm to compute the payoff of a w_{-1}-worker who applies to wages other than w, because such application is off the equilibrium path.

[18] The only other equilibrium has $M = 3$, which has very similar properties to the equilibrium with $M = 4$.

Lemma 6.1. *For all j and all $t \geq 2$, if the w_{-1}-workers prefer applying to a job at wage w to jobs at w_{+1}, then so do the w_{-t}-workers.*

Proof. Suppose that the w_{-1}-workers prefer w to w_{+1}, i.e.,

$$q\left[J_e(w) - J_e(w_{-1})\right] \geq q_{+1}\left[J_e(w_{+1}) - J_e(w_{-1})\right].$$

For all $t \geq 2$, we have

$$\begin{aligned} & q\left[J_e(w) - J_e(w_{-t})\right] - q_{+1}\left[J_e(w_{+1}) - J_e(w_{-t})\right] \\ = \; & \{q\left[J_e(w) - J_e(w_{-1})\right] - q_{+1}\left[J_e(w_{+1}) - J_e(w_{-1})\right]\} + (q - q_{+1})\left[J_e(w_{-1}) - J_e(w_{-t})\right]. \end{aligned}$$

The difference in $\{.\}$ is non-negative by the supposition. The last term on the right-hand side is also positive, because $q > q_{+1}$ and $J_e(w_{-1}) > J_e(w_{-t})$ for all $t \geq 2$. Thus, the above deference is positive, implying that applying to w yields a higher expected surplus for a w_{-t}-worker than applying to w_{+1}. **QED**

The intuition for the above lemma is the same as that for Lemma Singleton. An applicant at wage w_{-1} is more willing to sacrifice the employment probability for the wage level than any worker at lower wages w_{-t} $(t \geq 2)$. If the high employment probability offered by an opening at w is so attractive that an applicant w_{-1} prefers to apply to it rather than to the higher wage w_{+1}, it must be even more attractive to an applicant at lower wages w_{-t}.

The above lemma significantly reduces the number of inequalities we need to verify for (6.1). For each $j \in \{1, 2, ..., M-1\}$, it is sufficient to verify (6.1) for only $t = 1$. However, there are still $M - 1$ such inequalities. To reduce the number further, we use (4.4) to rewrite (6.1) for $t = 1$ as follows:

$$\frac{q(a)}{q(a_{+1})} - 1 - \frac{f(a)}{f(a_{+1})} \geq 0. \tag{6.2}$$

For given a_{+1}, define $\phi(a_{+1})$ as the solution for a to the equality form of (6.2). Because the left-hand side of (6.2) is a decreasing function of a, the inequality is equivalent to $a \leq \phi(a_{+1})$.

We prove the following lemma in Appendix E:

Lemma 6.2. *The function $\phi(.)$ exists, is unique for given a_{+1}, and has the following properties: (i) $\phi' > 0$; (ii) $a_{+1} > \phi(a_{+1}) > a_{+1} - \ln(1 + a_{+1})$; and (iii) if $a \leq \phi(a_{+1})$, then $a_{-1} < \phi(a)$.*

Property (iii) in Lemma 6.2 says that an applicant at wage w_{M-2} has the strongest incentive among all applicants to leap on the wage ladder. Thus, to verify (6.2) for all $j \in \{1, 2, ..., M-1\}$, it is sufficient to verify it for $j = 1$. For $j = 1$, (6.2) becomes $a_{M-1} \leq \phi(a_M)$, which is equivalent to $h_{M-1} \leq h(\phi(a_M))$. Since (5.2) holds for $j = 0$ after replacing q_{+1} by 0 and $\lambda q_{+1}/f_{+1}$ by $\lambda S/C$, we use it to substitute for h_{M-1} and rewrite the condition $h_{M-1} \leq h(\phi(a_M))$ as follows:

$$0 \leq \frac{r+\sigma}{\lambda}\left(\frac{1}{h_M} + \frac{1}{f_M} - \frac{1}{h(\phi(a_M))}\right) - q_M\left(\frac{1}{h(\phi(a_M))} - \frac{1}{f_M}\right) - \frac{S}{C}.$$

Rewriting this condition further, we have:

Lemma 6.3. (**No-leap**) *Given the wage levels computed*[19] *in Proposition Recursive, it is optimal for workers to apply only to the next wage level, rather than higher ones, if and only if*

$$\frac{r+\sigma}{\lambda} \geq \left(\frac{q_M}{h(\phi(a_M))} - \frac{q_M}{f_M} + \frac{S}{C}\right) \Big/ \left(\frac{1}{h_M} + \frac{1}{f_M} - \frac{1}{h(\phi(a_M))}\right). \tag{6.3}$$

6.2. Deviations That Lie Outside the Equilibrium Wage Support

Now we examine deviations that violate One-rung and lie outside Ω. Consider such a deviation $w^d \in (w_{-1}, w)$. This deviation can violate One-rung either in the type of applicants it attracts, or in the future direction of application by its prospect employee, or in both. The following lemma narrows down the types of such deviations we need to consider.

Lemma 6.4. *The following statements are true regarding any deviation $w^d \in (w_{-1}, w)$. (i) If an applicant gets the w^d-job, then his future application is to either w or w_{+1}. (ii) If w_{-2}-workers do not have incentive to apply to w^d, then neither do w_{-t}-workers, where $t \geq 3$.*

Proof. For (i), we show that a w^d-worker does not have incentive to apply to w_{+t}, for all $t \geq 2$. Suppose that a worker gets the job w^d. Let $J_e^d(w^d)$ be the value function of such a worker employed at w^d. Because this worker is not restricted to applying to w_{+1} next, $J_e^d(w^d)$ may not obey (4.2). However, whatever job opportunities this worker might have in the future, a worker employed at wage w will have the same probability (under Off-eqm). Thus, $J_e^d(w^d) < J_e(w)$. For the worker employed at w^d, applying to w_{+1} next yields a higher expected surplus than to any higher wage w_{+t} $(t \geq 2)$, as shown below:

$$\begin{aligned}
& q_{+t}\left[J_e(w_{+t}) - J_e^d(w^d)\right] - q_{+1}\left[J_e(w_{+1}) - J_e^d(w^d)\right] \\
= \; & q_{+t}\left[J_e(w_{+t}) - J_e(w)\right] + q_{+t}\left[J_e(w) - J_e^d(w^d)\right] - q_{+1}\left[J_e(w_{+1}) - J_e^d(w^d)\right] \\
< \; & q_{+1}\left[J_e(w_{+1}) - J_e(w)\right] + q_{+t}\left[J_e(w) - J_e^d(w^d)\right] - q_{+1}\left[J_e(w_{+1}) - J_e^d(w^d)\right] \\
< \; & q_{+1}\left[J_e(w_{+1}) - J_e(w)\right] + q_{+1}\left[J_e(w) - J_e(w^d)\right] - q_{+1}\left[J_e(w_{+1}) - J_e(w^d)\right] \\
= \; & 0.
\end{aligned}$$

The first inequality follows from our previous result that a w-worker prefers w_{+1} to w_{+t} for all $t \geq 2$. The second inequality comes from the facts that $q_{+t} < q_{+1}$ for all $t \geq 2$ and $J_e(w) > J_e^d(w^d)$.

For (ii), suppose that w_{-2}-workers do not have incentive to apply to w^d. Then,

$$q^d\left[J_e^d(w^d) - J_e(w_{-2})\right] \leq q_{-1}\left[J_e(w_{-1}) - J_e(w_{-2})\right],$$

[19]To rewrite the condition as a lower bound on $(r+\sigma)/\lambda$, note that $\frac{1}{h} + \frac{1}{f} = \left[1 - (1+a)e^{-a}\right]^{-1}$. So, the fact $\phi(a) > a - \ln(1+a)$ implies $\frac{1}{h(a_M)} + \frac{1}{f(a_M)} > \frac{1}{h(\phi(a_M))}$. Moreover, if the right-hand side of the condition is an increasing function of a_M (which seems true from numerical examples), then we can replace (6.3) by the following sufficient condition (using the facts that $a_M \leq \bar{a}$ and $S/C = 1/(e^{\bar{a}} - 1 - \bar{a})$):

$$\frac{r+\sigma}{\lambda} \geq \frac{q(\bar{a})}{h(\phi(\bar{a}))\left[\frac{1}{h(\bar{a})} + \frac{1}{f(\bar{a})}\right] - 1}.$$

where q^d is the probability that an applicant to w^d gets the job. Using an argument similar to that established $J_e(w) > J_e^d(w^d)$, we have $J_e^d(w^d) > J_e(w_{-1})$. Then, the above inequality implies $q^d < q_{-1}$. Now, for all $t \geq 3$, a w_{-t}-worker gets a higher expected surplus from applying to w_{-1} than to w^d, as shown below:

$$\begin{aligned}
& q^d \left[J_e^d(w^d) - J_e(w_{-t})\right] - q_{-1}\left[J_e(w_{-1}) - J_e(w_{-t})\right] \\
= \; & q^d \left[J_e^d(w^d) - J_e(w_{-2})\right] + q^d\left[J_e(w_{-2}) - J_e(w_{-t})\right] - q_{-1}\left[J_e(w_{-1}) - J_e(w_{-t})\right] \\
\leq \; & q_{-1}\left[J_e(w_{-1}) - J_e(w_{-2})\right] + q^d\left[J_e(w_{-2}) - J_e(w_{-t})\right] - q_{-1}\left[J_e(w_{-1}) - J_e(w_{-t})\right] \\
< \; & q_{-1}\left[J_e(w_{-1}) - J_e(w_{-2})\right] + q_{-1}\left[J_e(w_{-2}) - J_e(w_{-t})\right] - q_{-1}\left[J_e(w_{-1}) - J_e(w_{-t})\right] \\
= \; & 0.
\end{aligned}$$

The first inequality follows from the supposition about w_{-2}-workers, and the second inequality from $q^d < q_{-1}$. **QED**

The intuition for the above lemma is again similar to that for Lemma Singleton. This is obviously for part (ii), because part (ii) extends Lemma 6.1 to wages outside the equilibrium support and Lemma 6.1 relies on the intuition for Lemma Singleton. To see the link between part (i) and Lemma Singleton, notice that $w > w^d$. According to the intuition for Lemma Singleton, an applicant at w is more willing to sacrifice the employment probability for the wage level than an applicant at w^d. Because the high employment probability at an opening w_{+1} attracts even a w-applicant to apply to it rather than to higher wages, it must be even more attractive to an applicant at the lower wage w^d.

Under the above lemma, only the following two types of deviations still need be ruled out:

(I) The deviation $w^d \in (w_{-1}, w)$ attracts w_{-1}-workers and, after getting the job with w^d, such a worker will apply to w in the future.

(II) The deviation $w^d \in (w_{-1}, w)$ attracts w_{-2}-workers and, after getting w^d, such a worker will apply to w_{+1}.

Figure 3 depicts these two deviations, where the dashed arrows indicate the deviating firm's source of applicants and its employee's future application directions. A type I deviation is profitable only when the support of the wage distribution is too sparse, while a type II deviation is profitable only when the support is too dense. Note that a type-II deviation is possible only when the number of rungs on the wage ladder is three or more. We will investigate these two deviations in turn.[20]

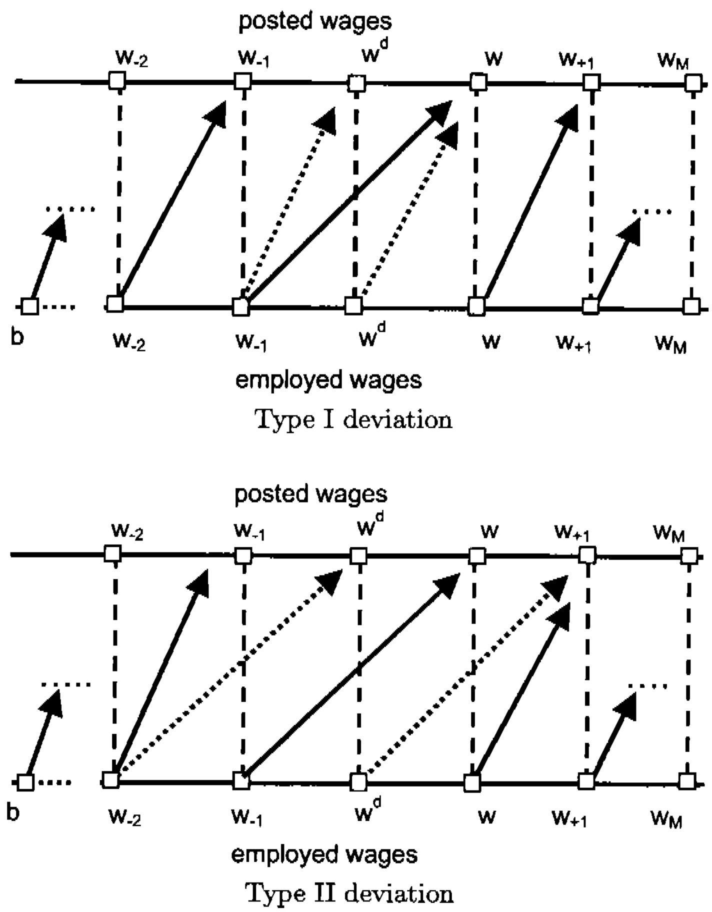

Figure 3

Consider first a type I deviation $w^d \in (w_{-1}, w)$. Let $J_f^d(w^d)$ be the deviating firm's value function after successfully recruiting a worker and $J_e^d(w^d)$ the value function of a worker who gets the job w^d, conditional on that the worker's future application is to w as required in a type I deviation. Then,

$$J_f^d(w^d) = \frac{y - w^d}{r + \sigma + \lambda q},$$

$$J_e^d(w^d) = \frac{w^d + \sigma J_u - \lambda S + \lambda q J_e(w)}{r + \sigma + \lambda q}.$$

[20]Other deviations and the following two types of deviations, in particular, are not profitable. The first is like the one depicted in the upper panel of Figure 3, but with the employee at w^d applying to w_{+1} in the future. This deviation can be viewed as a downward deviation from w that satisfies One-rung and, by construction, the deviation is dominated by w. The second deviation is like the one depicted in the lower panel of Figure 3, but with the employee at w^d applying to w in the future. This deviation can be viewed as an upward deviation from w_{-1} that satisfies One-rung and, by construction, the deviation is dominated by w_{-1}.

These function are different from those in (4.1) and (4.2), because the worker's future application directions are different here from the ones depicted in Figure 2 (again, we invoked Off-eqm).

For the deviation w^d to be profitable, it must satisfy the following conditions:

(Ia) By applying to w^d, a w_{-1}-worker's expected surplus is equal to $E(w_{-1})$;

(Ib) The deviating firm earns an expected surplus greater than C.

These two conditions cannot be both satisfied, as stated in the following lemma:

Lemma 6.5. (Type-I) *A type I deviation is not profitable.*

Proof. Suppose that the deviation satisfies (Ia). Let q^d be the employment probability of a worker applying to w^d. Then, (Ia) implies

$$q^d \left[J_e^d(w^d) - J_e(w_{-1}) \right] = E(w_{-1}) = Cq/f. \tag{6.4}$$

Substituting $J_e^d(w^d)$ and $J_e(w_{-1})$, we solve w^d as follows:

$$w^d = w_{-1} + (r + \sigma + \lambda q)\, Cq/(fq^d). \tag{6.5}$$

The deviating firm's expected surplus is $\pi(a^d) = h^d \left[J_f^d(w^d) - J_v \right]$, where h^d is the firm's hiring probability. Substituting w^d and $J_v = 0$, the deviating firm's expected surplus becomes:

$$\begin{aligned} \pi(a^d) &= h^d \tfrac{y - w^d}{r+\sigma+\lambda q} = h^d \left(\tfrac{y - w_{-1}}{r+\sigma+\lambda q} - \tfrac{w^d - w_{-1}}{r+\sigma+\lambda q} \right) \\ &= h^d \left(\tfrac{C}{h_{-1}} - \tfrac{qC}{fq^d} \right) = C \left(\tfrac{h^d}{h_{-1}} - \tfrac{a^d q}{f} \right). \end{aligned}$$

To obtain the third equality, we substituted $(y - w_{-1})$ using (4.11) and $(w^d - w_{-1})$ using (6.5).

The surplus $\pi(a^d)$ is maximized at $a^d = a^*$, where $a^* = \ln\left(\frac{f}{qh_{-1}}\right)$ solves $\pi'(a^*) = 0$. The maximum is

$$\pi(a^*) = C \left(\frac{h(a^*)}{h_{-1}} - \frac{a^* q}{f} \right) = \frac{Cq}{f} \left(e^{a^*} - 1 - a^* \right) = \frac{e^{a^*} - 1 - a^*}{e^a - 1 - a} C,$$

where the second equality comes from substituting $h_{-1} = e^{-a^*} f/q$, and the third equality from the definition of f. Because $e^a - 1 - a$ is an increasing function, a necessary condition for the deviation to be profitable is $a^* > a$. However, a queue length $a^d > a$ will not attract any w_{-1} applicants. To see this, note that a w-worker is employed at a higher wage than a w^d-worker and has all future job opportunities that a w^d-worker will have. Thus, $J_e(w) > J_e^d(w^d)$. If $a^d > a$, then $q^d < q$, which would lead to the following contradiction to (6.4):

$$E(w_{-1}) = q\left[J_e(w) - J_e(w_{-1})\right] > q\left[J_e^d(w^d) - J_e(w_{-1})\right] > q^d \left[J_e^d(w^d) - J_e(w_{-1})\right].$$

This establishes the lemma. **QED**

The explanation is as follows. A type-I deviation $w^d \in (w_{-1}, w)$ competes against w for the same applicants (i.e., w_{-1}-workers). In comparison with the equilibrium wage w, the deviation w^d offers a w_{-1}-applicant not only a lower wage but also a lower value for future application. For the deviation to attract this applicant, it must provide a significantly higher employment probability than a job opening at w does. This implies that the deviator's hiring probability must be significantly lower than that of a firm recruiting at w. In this case, the deviator's expected surplus from recruiting will not be high enough to cover the vacancy cost.

Now consider a type II deviation $w^d \in (w_{-1}, w)$, depicted in the lower panel in Figure 3. With this deviation, the deviating firm's ex post value $J_f^d(w^d)$ and the employee's value $J_e^d(w^d)$ are:

$$J_f^d(w^d) = \frac{y - w^d}{r + \sigma + \lambda q_{+1}},$$

$$J_e^d(w^d) = \frac{w^d + \sigma J_u - \lambda S + \lambda q_{+1} J_e(w_{+1})}{r + \sigma + \lambda q_{+1}}.$$

In contrast to a type-I deviation, a type-II deviation changes not only the applicant's current payoff but also the future payoff, by changing the potential employee's future application. For this deviation to be not profitable, we need to impose additional conditions.

Suppose that the deviation is profitable. Then it must satisfy the following conditions:

(IIa) By applying to w^d, a w_{-2}-worker's expected surplus is equal to $E(w_{-2})$;

(IIb) The deviating firm earns an expected surplus greater than C;

(IIc) A w^d-worker's future application is indeed to w_{+1} instead of w.

Under (IIa), $q^d \left[J_e^d(w^d) - J_e(w_{-2})\right] = E(w_{-2}) = Cq_{-1}/f_{-1}$. That is,

$$\begin{aligned}
\tfrac{q_{-1}}{q^d f_{-1}} &= \tfrac{1}{C}\left[J_e^d(w^d) - J_e(w_{-2})\right] \\
&= \tfrac{1}{C}\left[J_e(w) - J_e(w_{-1})\right] + \tfrac{1}{C}\left[J_e(w_{-1}) - J_e(w_{-2})\right] + \tfrac{1}{C}\left[J_e^d(w^d) - J_e(w)\right] \\
&= \tfrac{1}{f} + \tfrac{1}{f_{-1}} - \tfrac{(w-w^d)/C}{r+\sigma+\lambda q_{+1}}.
\end{aligned}$$

The last equality comes from substituting $(J_e^d(w^d), J_e(w))$ and using (4.4). Solve for w^d:

$$w^d = w - C(r + \sigma + \lambda q_{+1})\left(\frac{1}{f} + \frac{1}{f_{-1}} - \frac{q_{-1}}{q^d f_{-1}}\right). \tag{6.6}$$

The deviating firm's expected surplus is $\pi(a^d) = h^d\left[J_f^d(w^d) - J_v\right]$. Substituting $(J_f^d(w^d), w^d)$ and using (4.3) for $(y - w)$, this surplus becomes:

$$\pi(a^d) = \frac{C\left[(y-w) + \left(w - w^d\right)\right]}{r + \sigma + \lambda q_{+1}} = C\left[h^d\left(\frac{1}{h} + \frac{1}{f} + \frac{1}{f_{-1}}\right) - \frac{a^d q_{-1}}{f_{-1}}\right].$$

The expected surplus $\pi(a^d)$ is maximized at $a^d = A$ that solves $\pi'(A) = 0$. So,

$$A = \ln\left(\frac{1}{q_{-1}}\left(1 + f_{-1}\left(\frac{1}{h} + \frac{1}{f}\right)\right)\right). \tag{6.7}$$

Because $a_{-1} > a - \ln(1+a)$ by Proposition Monotone, it can be shown that $A > a_{-1}$. The unconstrained maximum of $\pi(a^d)$ is:

$$\pi(A) = C\left(e^A - 1 - A\right) / \left(e^{a_{-1}} - 1 - a_{-1}\right) > C.$$

Thus, a type II deviation is not profitable if and only if the constraint (IIc) keeps a^d a sufficient distance away from A.

The constraint (IIc) requires $q_{+1}\left[J_e(w_{+1}) - J_e^d(w^d)\right] \geq q\left[J_e(w) - J_e^d(w^d)\right]$. Using (4.4) to substitute $J_e(w_{+1})$, noting that $J_e(w) - J_e^d(w^d) = (w - w^d)/(r + \sigma + \lambda q_{+1})$, and substituting w^d, we can rewrite (IIc) as follows:

$$q^d \leq q_{-1} \Big/ \left[1 + f_{-1}\left(\frac{1}{f} - \frac{q_{+1}}{(q - q_{+1})\, f_{+1}}\right)\right]. \tag{6.8}$$

This constraint requires a^d to be large. Let β be the level of a^d that satisfies (6.8) as equality. Define β^* by:

$$h(\beta^*) = 1 \Big/ \left[\frac{1}{h} + \frac{q_{+1}}{(q - q_{+1})\, f_{+1}}\right]. \tag{6.9}$$

The following lemma describes when (6.8) is sufficient to prevent a type II deviation from being profitable (see Appendix F for a proof).

Lemma 6.6. (**Type-II**) *A type II deviation is not profitable if and only if $\beta^* \geq \beta$, which is equivalent to*

$$\beta^* - h(\beta^*)e^{a_{-1}} + \frac{r + \sigma + \lambda q_{+1}}{r + \sigma + \lambda q}\left(e^{a_{-1}} - 1 - a_{-1}\right) \geq 0. \tag{6.10}$$

The condition (6.10) is difficult to verify analytically, because it involves three variables (a_{-1}, a, a_{+1}). However, it is satisfied in the numerical example in section 5. We conclude this section by summarizing the results on existence in the following proposition.

Proposition 6.7. *Maintain Assumptions Regularity and Off-eqm. An equilibrium with the described wage ladder exists under the conditions in Proposition M-exists, Lemma No-leap and (if $M \geq 3$) Lemma Type-II.*

7. Conclusion

In this paper we have studied the equilibrium in a large labor market where employed workers search on the job and firms direct workers' search intentionally using wage offers and selection rules. All workers observe all posted job openings before the application. There is wage dispersion among workers, despite the fact that all workers (and all jobs) are homogeneous. Moreover, equilibrium wages form a ladder and workers choose to climb up the ladder over time, one rung at a time. This occurs without the familiar elements that prevent workers from jumping over the ladder, such as a gradual increase in productivity, differential information among workers, and intentional discrimination by firms according to workers' current wages. Other properties of the wage ladder are as follows: (i) a low-wage job is easier to be obtained than a high-wage job, and so a low-wage job experiences a higher quit rate than a high-wage job; (ii) The distance between two adjacent rungs on the wage ladder becomes smaller as wage increases; (iii) the density of offer wages is a decreasing function; and (iv) the wage density is decreasing at high-wage levels when the hiring cost is larger relative to the application cost.

Appendix

A. Simplifying a Firm's Strategy

In this appendix we establish the following proposition that simplifies the decision problems:

Proposition A.1. *Suppose that firms post (w, Z), where Z satisfies (2.1) – (2.3). Agents' decision problems can be formulated equivalently using Q rather than Z, where $Q = (q(w, w'))_{w' \in \Omega_0}$. Moreover, (2.6) is satisfied.*

Proof. Suppress the particular firm's wage offer w in various notation. Let $\mathcal{R}$ be the set containing the realizations of R, which is the composition of applicants that the firm receives. Let $\Gamma(.)$ be the distributional function of R, with a density $\gamma(.)$. Because a w'-worker applies to the firm with probability $p(w')$, we have:

$$\gamma(R) = \prod_{w' \in \Omega_0} \left[\left(\begin{array}{c} R(w') \\ \lambda(w')n(w')L \end{array} \right) [p(w')]^{R(w')} \left[1 - p(w')\right]^{\lambda(w')n(w')L - R(w')} \right],$$

where the expression in [.] is the probability with which the firm receives exactly $R(w')$ number of w'-workers.

We reformulate the firm's problem. Under (w, Z), the firm's objective function, i.e., the expected surplus, is as follows:

$$\sum_{w^* \in \Omega_0} \sum_{R \in \mathcal{R}} Z(w^*, R) \left[J_f(w) - J_v\right] \gamma(R) = \left[J_f(w) - J_v\right] \sum_{w^* \in \Omega_0} \sum_{R \in \mathcal{R}} Z(w^*, R) \gamma(R).$$

The equality follows from the fact that the wage to be paid is independent of the type of the applicant that the firm will select ex post and of the realization of R. The double summation in the last expression is the firm's hiring probability, as demonstrated below:

$$\begin{aligned} &\textstyle\sum_{w^* \in \Omega_0} \sum_{R \in \mathcal{R}} Z(w^*, R) \gamma(R) = \sum_{R \in \mathcal{R}} \gamma(R) \left[\sum_{w^* \in \Omega_0} Z(w^*, R)\right] \\ &= \textstyle\sum_{R \in \mathcal{R} \setminus \{0\}} \gamma(R) \left[\sum_{w^* \in \Omega_0} Z(w^*, R)\right] = \sum_{R \in \mathcal{R} \setminus \{0\}} \gamma(R) = 1 - \Gamma(0) = h. \end{aligned} \tag{A.1}$$

The first equality comes from switching the order of the two summations. The second and third equalities come from (2.2), i.e., that the sum of $Z(w^*, R)$ over w^* is 0 when $R = 0$ and 1 when $R \neq 0$. The fourth equality comes from the definition of $\Gamma(0)$ and the last equality from the meaning of the hiring probability.

Therefore, the firm's objective function is $h\left[J_f(w) - J_v\right]$, as we used in subsection 2.2. If we can show that h depends on Q as in (2.6), then the choices of (w, Q) give the same outcome to the firm as the choices of (w, Z) do. We will verify (2.6) later.

Second, we reformulate the applicant's decision problem. Consider a particular w^*-worker who contemplates applying to the firm. Let $q(w^*)$ be the shortened notation for the worker's ex ante employment probability, denoted $q(w, w^*)$ in the text. To calculate q, we need the distribution function of the composition of the firm's received applicants other than the particular w^*-worker in discussion. This is the distribution function of R conditional on that the particular w^*-worker applies to the firm, and hence it is different from the unconditional distribution Γ. Denote the composition of the firm's received applicants other than the particular w^*-worker by $\bar{R}$. Let $\mathcal{R}^*$ be the set of vectors containing the possible values of $\bar{R}$ and $\bar{\gamma}(., w^*)$ the density function of $\bar{R}$. Conditional on that the particular w^*-worker applies to the firm, the composition of applicants that the firm receives is R, where $R(w') = \bar{R}(w')$ if $w' \neq w^*$ and $R(w^*) = \bar{R}(w^*) + 1$ otherwise. With a realization of $\bar{R}$ in addition to the particular w^*-worker, the firm chooses a w^*-worker with probability $Z(w^*, R)$, out of which the particular w^*-applicant is the chosen one with probability $1/R(w^*)$. Therefore,

$$q(w^*) = \sum_{\bar{R}\in\mathcal{R}^*} \frac{Z(w^*, R)}{R(w^*)} \bar{\gamma}(\bar{R}, w^*). \tag{A.2}$$

When applying to the job, a w^*-worker's expected surplus is

$$\begin{aligned} &\textstyle\sum_{\bar{R}\in\mathcal{R}^*} \frac{Z(w^*,R)}{R(w^*)} \left[J_e(w) - J_e(w^*)\right] \bar{\gamma}(\bar{R}, w^*) \\ &= \left[J_e(w) - J_e(w^*)\right] \textstyle\sum_{\bar{R}\in\mathcal{R}^*} \frac{Z(w^*,R)}{R(w^*)} \bar{\gamma}(\bar{R}, w^*) \\ &= q(w^*) \left[J_e(w) - J_e(w^*)\right]. \end{aligned}$$

The first equality follows from the fact that the wage the applicant gets is independent of the realization of $\bar{R}$, and the second equality from the previous formula of q. The above result shows that what matters for a worker's application decision is the wage offer and the ex ante employment probability q, as we used in subsection 2.2.

Finally, we show that h depends on Q as in (2.6). To do so, we find the relationship between the two densities, $\bar{\gamma}$ and γ. Compute

$$\begin{aligned} \bar{\gamma}(\bar{R}, w^*) &= \prod_{w'\neq w^*} \left[\begin{pmatrix} R(w') \\ \lambda(w')n(w')L \end{pmatrix} [p(w')]^{R(w')} [1 - p(w')]^{\lambda(w')n(w')L - R(w')} \right] \\ &\quad \times \begin{pmatrix} \bar{R}(w^*) \\ \lambda(w^*)n(w^*)L - 1 \end{pmatrix} [p(w^*)]^{\bar{R}(w^*)} [1 - p(w^*)]^{\lambda(w^*)n(w^*)L - 1 - \bar{R}(w^*)}. \end{aligned}$$

Here we have isolated w^*-applicants and used the fact that $R(w') = \bar{R}(w')$ for all $w' \neq w^*$. Rewrite:

$$\begin{aligned} &\begin{pmatrix} \bar{R}(w^*) \\ \lambda(w^*)n(w^*)L - 1 \end{pmatrix} [p(w^*)]^{\bar{R}(w^*)} [1 - p(w^*)]^{\lambda(w^*)n(w^*)L - 1 - \bar{R}(w^*)} \\ &= \tfrac{\bar{R}(w^*)+1}{\lambda(w^*)n(w^*)Lp(w^*)} \begin{pmatrix} \bar{R}(w^*) + 1 \\ \lambda(w^*)n(w^*)L \end{pmatrix} [p(w^*)]^{\bar{R}(w^*)+1} [1 - p(w^*)]^{\lambda(w^*)n(w^*)L - (1 + \bar{R}(w^*))} \\ &= \tfrac{R(w^*)}{a(w^*)} \begin{pmatrix} R(w^*) \\ \lambda(w^*)n(w^*)L \end{pmatrix} [p(w^*)]^{R(w^*)} [1 - p(w^*)]^{\lambda(w^*)n(w^*)L - R(w^*)}. \end{aligned}$$

Here we have used the result, $a(w^*) = p(w^*)\lambda(w^*)n(w^*)L$, and the fact $R(w^*) = \bar{R}(w^*) + 1$. Substituting the above result into the formula of $\bar{\gamma}$ and using the formula of γ, we have:

$$\bar{\gamma}(\bar{R}, w^*) = \frac{R(w^*)}{a(w^*)} \gamma(R).$$

Since $a(w^*)$ is independent of the realization of R, (A.2) then implies

$$q(w^*) = \sum_{R \in \mathcal{R}} \left[\frac{Z(w^*, R)}{R(w^*)} \cdot \frac{R(w^*)}{a(w^*)} \gamma(R) \right] = \frac{1}{a(w^*)} \sum_{R \in \mathcal{R}} Z(w^*, R)\gamma(R).$$

Therefore, (2.6) holds, as shown below:

$$\sum_{w^* \in \Omega_0} a(w^*) q(w^*) = \sum_{w^* \in \Omega_0} \sum_{R \in \mathcal{R}} Z(w^*, R)\gamma(R) = h.$$

The second equality uses (A.1). This completes the proof of the proposition. **QED**

B. Properties of $f(a)$ and $g(a)$

In this appendix we establish the following lemma.

Lemma B.1. *Define $f(.)$ as in (4.6) and $g(.)$ as follows:*

$$g(a) \equiv (r + \sigma + \lambda q) \left(\frac{1}{f(a)} - \frac{1}{h(a)} \right). \tag{B.1}$$

For all $a > 0$, $f'(a) > 0$, $\frac{d}{da}\left(\frac{f(a)}{a}\right) > 0$, $\frac{d}{da}\left(\frac{1}{f(a)} - \frac{1}{aq(a)}\right) < 0$, and

$$(e^a - 1)\left[a\left(e^a - 1\right)^2 - \left(e^a - 1 - a\right)^2\right] - \left(e^a - 1 - a\right)^3 > 0. \tag{B.2}$$

Furthermore, if $(r + \sigma)/\lambda > f(a)/a$ then $g'(a) < 0$ for all $a > 0$.

Proof. First, we show that $\frac{d}{da}\left(\frac{f(a)}{a}\right) > 0$ implies $f'(a) > 0$ and $\frac{d}{da}\left(\frac{1}{f(a)} - \frac{1}{aq(a)}\right) < 0$. Since $\frac{d}{da}\left(\frac{f(a)}{a}\right) = \frac{1}{a^2}\left(af'(a) - f\right) > 0$ implies $f'(a) > f(a)/a$, clearly it implies $f'(a) > 0$. Also, it implies that

$$\begin{aligned}
&\tfrac{d}{da}\left(\tfrac{1}{f(a)} - \tfrac{1}{aq(a)}\right) = -\tfrac{f'}{f^2} + \tfrac{q + aq'}{(aq)^2} < -\tfrac{1}{af} + \tfrac{q+aq'}{(aq)^2} \\
&\sim -aq + \tfrac{f}{q}(q + aq') \\
&= -(1 - e^{-a}) + (e^a - 1 - a)\left(\tfrac{1-e^{-a}}{a} - \tfrac{1-(1+a)e^{-a}}{a}\right) \\
&= -(1 - e^{-a}) + (e^a - 1 - a)e^{-a} = -ae^{-a} < 0,
\end{aligned}$$

where the symbol $\sim$ means "having the same sign as".

Now we show that $\frac{d}{da}\left(\frac{f(a)}{a}\right) > 0$. Substituting $q(a) = \frac{1-e^{-a}}{a} = \frac{e^a - 1}{ae^a}$, we have

$$\frac{f(a)}{a} = \frac{(e^a - 1)(e^a - 1 - a)}{a^2 e^a}.$$

Then, $\frac{d}{da}\left(\frac{f(a)}{a}\right) = \frac{f1(a)}{a^3 e^a}$, where

$$f1(a) = (a - 2)e^{2a} + (a + 4)e^a - (a^2 + 2a + 2)$$

Note that $f1(0) = 0$. Denote the nth order derivative of $f1(a)$ by $f1^{(n)}(a)$. We have:

$$f1^{(1)}(a) = (2a - 3)e^{2a} + (a + 5)e^a - (2a + 2), \text{ with } f1^{(1)}(0) = 0,$$

$$f1^{(2)}(a) = (4a-4)e^{2a} + (a+6)e^a - 2, \text{ with } f1^{(2)}(0) = 0,$$

$$f1^{(3)}(a) = e^a[(8a-4)e^a + a + 7], \text{ with } f1^{(3)}(0) = 3 > 0,$$

$$\frac{d}{da}\left[e^{-a} f1^{(3)}(a)\right] = (8a+4)e^a + 1 > 0 \text{ for all } a \geq 0.$$

The last two results imply that $f1^{(3)}(a) > f1^{(3)}(0) > 0$ for all $a > 0$, which in turn implies that $f1^{(2)}(a) > f1^{(2)}(0) = 0$, $f1^{(1)}(a) > f1^{(1)}(0) = 0$ and $f1(a) > f1(0) = 0$. Therefore, $\frac{d}{da}\left(\frac{f(a)}{a}\right) > 0$ for all $a > 0$.

We can establish (B.2) using the same procedure. Denote the left-hand side of (B.2) temporarily as $LHS(a)$. Then, $LHS^{(i)}(0) = 0$ for $i = 0, 1, 2$, and

$$LHS^{(3)}(a) = 27(a-1)e^{3a} + (16a+72)\,e^{2a} - \left(4a^2 + 31a + 51\right) e^a + 6.$$

Because $(a-1)\,e^a + 1 > 0$ for all $a > 0$, $(a-1)\,e^{3a} > -e^{2a}$. Substituting this result for the first term in $LHS^{(3)}(a)$, we have $LHS^{(3)}(a) > (12a^2 + 30a - 6)\,e^a + 6$. The last expression has a value 0 at $a = 0$ and a positive derivative for all $a > 0$. Thus, $LHS^{(3)}(a) > 0$. Then, for all $a > 0$, we have $LHS^{(2)}(a) > LHS^{(2)}(0) = 0$, $LHS^{(1)}(a) > LHS^{(1)}(0) = 0$ and $LHS(a) > LHS(0) = 0$.

Finally, we show that $g'(a) < 0$ for all $a > 0$ if $(r+\sigma)/\lambda > f(a)/a$. Compute:

$$g'(a) = (r + \sigma + \lambda q)\frac{d}{da}\left(\frac{1}{f(a)} - \frac{1}{aq(a)}\right) + \lambda\left(\frac{1}{f(a)} - \frac{1}{aq(a)}\right) q'.$$

Because $\frac{d}{da}\left(\frac{1}{f(a)} - \frac{1}{aq(a)}\right) < 0$, as shown above, the condition $(r+\sigma)/\lambda > f(a)/a$ implies

$$g'(a) < \left(\frac{\lambda f}{a} + \lambda q\right)\frac{d}{da}\left(\frac{1}{f} - \frac{1}{aq}\right) + \lambda\left(\frac{1}{f} - \frac{1}{aq}\right) q' \sim g1(a),$$

where $g1(a) = (3-a)e^{3a} - (5a+9)e^{2a} + (7a^2 + 9a + 9)e^a - (2a^3 + 7a^2 + 7a + 3)$ (correction to $(3-a)e^{3a} - (5a+9)e^{2a} + (7a^2 + 13a + 9)e^a - (2a^3 + 7a^2 + 7a + 9$; the calculations should also be corrected accordingly.) Verify that $g1(0) = 0$, $g1^{(1)}(0) = -4 < 0$, $g1^{(2)}(0) = -8 < 0$, $g1^{(3)}(0) = -12 < 0$, and $g1^{(4)}(a) = e^a g2(a)$, where

$$g2(a) = (135 - 81a)e^{2a} - (80a + 304)e^a + 7a^2 + 65a + 129.$$

Verify that $g2(0) = -40 < 0$, $g2^{(1)}(0) = -130 < 0$, $g2^{(2)}(0) = -234 < 0$ and $g2^{(3)}(a) = e^a g3(a)$, where

$$g3(a) = (108 - 648a)e^a - (80a + 544).$$

Verify that $g3(0) = -436 < 0$ and $g3'(a) < 0$ for all $a \geq 0$. Tracing all the way back, we have $g1(a) < 0$ for all $a > 0$ and so $g'(a) < 0$ for all $a > 0$. **QED**

C. Proof of Proposition Monotone

First, we verify (5.3) and (5.4) by induction. To begin, we show that they hold for $j = 0$. By (4.10), $a_M \leq \bar{a}$. Since $h(a)$ is an increasing function and $q(a)$ a decreasing function, all three inequalities in (5.3) are equivalent to each other, and so we show $h_{M-1} < h_M$ only. Since (5.2) holds for $j = 0$ after replacing the term q_{+1} by 0 and $\lambda q_{+1}/f_{+1}$ by $\lambda S/C$, $h_{M-1} < h_M$ if and only if

$$0 < \frac{r + \sigma + \lambda q_M}{f_M} - \frac{\lambda S}{C} - \frac{\lambda q_M}{h_M} = \lambda\left(\frac{1}{e^{a_M} - 1 - a_M} - \frac{1}{e^{\bar{a}} - 1 - \bar{a}}\right) + \left(\frac{r+\sigma}{f_M} - \frac{\lambda}{a_M}\right),$$

where we have used the definition of $\bar{a}$ in (4.7) to replace S/C. Because $a_M \leq \bar{a}$ by construction (see (4.10)) and $(e^a - 1 - a)$ is an increasing function, the term in the first (.) is positive. Also, $f(a)/a$ is an increasing function, as shown in Appendix B, and so Assumption Regularity implies $(r+\sigma)/\lambda > f(\bar{a})/\bar{a} \geq f_M/a_M$. That is, the term in the second (.) above is also positive. Thus, $h_{M-1} < h_M$, verifying (5.3) for $j = 0$.

Now that $a_{M-1} < a_M \leq \bar{a}$, and that $f(a)/a$ is an increasing function of a, (4.19) implies $(r+\sigma)/\lambda > f(a_{M-1})/a_{M-1}$. That is, (5.4) holds for $j = 0$.

Suppose that (5.3) and (5.4) hold for some arbitrary $j \in \{0, 2, ..., M-3\}$. We show that they hold for $j + 1$. For (5.3), this amounts to proving $h_{-2} < h_{-1}$. Computing h_{-2} using (5.2), $h_{-2} < h_{-1}$ if and only if

$$0 < \frac{r + \sigma + \lambda q}{h_{-1}} - \frac{\lambda q}{f} + (r + \sigma + \lambda q_{-1})\left(\frac{1}{f_{-1}} - \frac{1}{h_{-1}}\right).$$

Because $h_{-1} < h$ by supposition, a sufficient condition for the above inequality is:

$$0 < \frac{r + \sigma + \lambda q}{h} - \frac{\lambda q}{f} + (r + \sigma + \lambda q_{-1})\left(\frac{1}{f_{-1}} - \frac{1}{h_{-1}}\right).$$

The last term is equal to $g(a_{-1})$. In Appendix B we showed that $g'(a) < 0$ if $(r+\sigma)/\lambda > f(a)/a$. Because $(r+\sigma)/\lambda > f(\bar{a})/\bar{a}$, we have $g'(a) < 0$ for all $a \leq \bar{a}$. Since $a_{-1} < a$ by supposition and $a \leq \bar{a}$, then $g(a_{-1}) > g(a)$. Thus,

$$\begin{aligned} & \tfrac{r+\sigma+\lambda q}{h} - \tfrac{\lambda q}{f} + g(a_{-1}) \\ > \; & \tfrac{r+\sigma+\lambda q}{h} - \tfrac{\lambda q}{f} + (r + \sigma + \lambda q)\left(\tfrac{1}{f} - \tfrac{1}{h}\right) = (r+\sigma)/f(a) > 0. \end{aligned}$$

That is, (5.3) holds for $j + 1$. This in turn implies $a_{-2} < a_{-1}$. Because $f(a)/a$ is an increasing function of a (see Appendix B), the supposition $(r+\sigma)/\lambda > f(a_{-1})/a_{-1}$ implies $(r+\sigma)/\lambda > f(a_{-2})/a_{-2}$. That is, (5.4) also holds for $j + 1$. By induction, (5.3) and (5.4) hold for all $j \in \{0, 1, ..., M-2\}$.

Second, we prove (5.5), which is equivalent to $h_{-1} > h\left(a - \ln(1+a)\right)$. By (5.2), this in turn is equivalent to:

$$\begin{aligned} 0 \; & > \tfrac{r+\sigma+\lambda q_{+1}}{h} - \tfrac{\lambda q_{+1}}{f_{+1}} + (r + \sigma + \lambda q)\left(\tfrac{1}{f} - \tfrac{1}{h(a - \ln(1+a))}\right) \\ & = \tfrac{r+\sigma+\lambda q_{+1}}{h} - \tfrac{\lambda q_{+1}}{f_{+1}} - \tfrac{r+\sigma+\lambda q}{h} = -\lambda\left(\tfrac{q - q_{+1}}{h} + \tfrac{q_{+1}}{f_{+1}}\right). \end{aligned}$$

The equalities follow from calculating f and $h(a - \ln(1+a))$ explicitly. Because $q > q_{+1}$, the above inequality clearly holds, and so does (5.5).

Finally, we show that $da/dh_M > 0$ and $dw/dh_M > 0$ for any h_M that satisfies (4.10). From (4.13) it is easy to see that $da/dh_M > 0$ implies $dw/dh_M > 0$; so, we need to prove only $da/dh_M > 0$. Because $a_M = -\ln(1-h_M)$, it is obvious that $da_M/dh_M > 0$. If $da_{+t}/dh_M \geq 0$ for all $t \geq 1$ implies $da/dh_M > 0$, then $da/dh_M > 0$ by induction. Suppose that $da_{+t}/dh_M \geq 0$ for all $t \geq 1$. By construction, $h = \lambda C \left(\frac{r+\sigma}{\lambda} + q\right) / (y-w)$. Totally differentiating this relationship with respect to h_M (where dw/dh_M can be calculated using (4.13)), we have

$$\begin{aligned} \tfrac{y-w}{\lambda C h} h' \tfrac{da}{dh_M} &= \tfrac{1}{\lambda C}\tfrac{dw_M}{dh_M} + \tfrac{r+\sigma}{\lambda}\sum\nolimits_{t=2}^{j} \tfrac{f'_{+t}}{f^2_{+t}}\left(\tfrac{da_{+t}}{dh_M}\right) \\ &\quad + \left[q'_{+1}\left(\tfrac{1}{h} - \tfrac{1}{f_{+1}}\right) + \left(\tfrac{r+\sigma}{\lambda} + q_{+1}\right)\tfrac{f'_{+1}}{f^2_{+1}}\right]\left(\tfrac{da_{+1}}{dh_M}\right). \end{aligned}$$

Because $w_M = y - (r+\sigma)C/h_M$, $dw_M/dh_M > 0$. Because $da_{+t}/dh_M \geq 0$ for all $t \geq 1$, a sufficient condition for $da/dh_M > 0$ is that the following inequality holds for all j:

$$q'(a)\left(\frac{1}{h_{-1}} - \frac{1}{f}\right) + \left(\frac{r+\sigma}{\lambda} + q\right)\frac{f'(a)}{f^2} > 0.$$

To verify this inequality, temporarily denote the left-hand side of the inequality by LHS. Because $a_{-1} > a - \ln(1+a)$, $q' < 0$, $(r+\sigma)/\lambda > f/a$ and $f' > 0$, we have

$$LHS > q'(a)\left(\frac{1}{h\left(a - \ln(1+a)\right)} - \frac{1}{f}\right) + \left(\frac{f}{a} + q\right)\frac{f'(a)}{f^2}.$$

After substituting (q, f, q', f'), the right-hand side of this inequality has the same sign as the expression, $(e^a - 1)\left[a\left(e^a-1\right)^2 - \left(e^a - 1 - a\right)^2\right] - \left(e^a - 1 - a\right)^3$, which is positive for all $a > 0$ as shown in Lemma B.1 in Appendix B. Thus, the required condition $LHS > 0$ holds. **QED**

D. Proofs of Propositions W-property and W-density, and Lemma 4.2

We prove Proposition W-property first. Using (4.13) and (4.14), it is easy to verify (i) in the position. Property (ii) can be shown by induction, using the facts that $E(w_{-1}) = Cq/f > Cq_{+1}/f_{+1} = E(w)$ and $E(w_{M-1}) \geq S$ (see (4.9) or equivalently under the first part of (4.10)).

To establish (iii), use (5.1) to rewrite it as

$$\frac{R+q}{f} - \frac{R+q_{+1}}{f_{+1}} - \frac{q_{+1}}{f_{+1}} + \frac{q_{+2}}{f_{+2}} > 0,$$

where $R = (r+\sigma)/\lambda$. For the computed sequence to be an equilibrium we need $q > q_{+1}\left(1 + f/f_{+1}\right)$ (see (6.2)), as shown in section 6. Under this condition, the left-hand side of the above inequality is greater than the following expression:

$$(R + q_{+1})\left(\frac{1}{f} - \frac{1}{f_{+1}}\right) + \frac{q_{+2}}{f_{+2}}.$$

This is clearly positive, because $a < a_{+1}$ and $f(.)$ is an increasing function. Thus, Proposition W-property holds.

To prove Proposition W-density, recall that the density of offer wages is (v_i) and of employed wages $(n_i/(1-u))$, where $i = 1, 2, ..., M$. So, the density of offer wages is a decreasing function iff $v_{-1} > v$ and the density of employed wages is decreasing iff $n_{-1} > n$. By (4.16) and (4.17), $n_{-1}/n = \left(\frac{\sigma}{\lambda} + q_{+1}\right)/q$ and $v_{-1}/v = (n_{-2}a)/(n_{-1}a_{-1})$ for all $i \geq 3$. Thus, $v_{-1} > v$ for all $i \geq 3$, as shown below:

$$\frac{v_{-1}}{v} = \left(a\frac{\sigma}{\lambda} + h\right) \Big/ h_{-1} > \frac{h}{h_{-1}} > 1.$$

Similarly, the result holds for $i = 2$; i.e., $v_1/v_2 > h_2/h_1 > 1$.

The density of employed wages is a decreasing function iff $\sigma/\lambda > q - q_{+1}$. Because $q_{M+1} = 0$, the density of employed wages is decreasing at the upper end of the wage support (i.e. $n_{M-1} > n_M$) iff $\sigma/\lambda > q_M$. Because $q(.)$ is a decreasing function and $a_M \geq \bar{a} - \ln(1+\bar{a})$ by (4.10), a sufficient condition for $n_{M-1} > n_M$ is $\sigma/\lambda > q(\bar{a} - \ln(1+\bar{a}))$, which can be rewritten as (5.7). When r is sufficiently close to 0, this condition is satisfied iff $(r+\sigma)/\lambda > q(\bar{a} - \ln(1+\bar{a}))$. Because $(r+\sigma)/\lambda \geq f(\bar{a})/\bar{a}$ by Assumption Regularity, (5.7) is satisfied if $f(\bar{a})/\bar{a} > q(\bar{a} - \ln(1+\bar{a}))$, which is equivalent to $\bar{a} > 1.605$ and hence to $C/S > 2.373$. Similarly, because $a_M \leq \bar{a}$ by (4.10), a sufficient condition for $n_{M-1} < n_M$ is $\sigma/\lambda < q(\bar{a})$. This completes the proof of Proposition W-density.

To prove Lemma 4.2, pick up an arbitrary but sufficiently large integer m. Starting with $h_m = h^*$, where h^* is any value that satisfies (4.10), and compute the sequence $(a_{m-t})_{t \geq 0}$ according to Proposition Recursive. Define

$$\delta_i(m) = \frac{w_m - b + \lambda_0 S}{r + \sigma} - \frac{C\lambda_0 q_i}{(r+\sigma) f_i} - C \sum_{t=0}^{m-i} \frac{1}{f_{m-t}}.$$

Note that $\delta_1(M) = \Delta(M, h^*)$, where Δ is defined by (4.20). By Proposition Monotone (described later), $a_{m-t-1} < a_{m-t} \leq \bar{a} < \infty$ for all $t \geq 0$. Since $1/f_{m-t}$ and q_{m-t}/f_{m-t} are both decreasing functions of a_{m-t}, and $a_i < a_{i+1}$, we get:

$$\delta_{i+1}(m) - \delta_i(m) = \frac{C\lambda_0}{r+\sigma}\left(\frac{q_i}{f_i} - \frac{q_{i+1}}{f_{i+1}}\right) + \frac{C}{f_i} > \frac{C}{f_i} \geq \frac{C}{f(\bar{a})}.$$

Because $C/f(\bar{a})$ is bounded away from 0, the sequence δ_i decreases by a strictly positive amount each time when i decreases. If $\delta_m \geq 0$, then the sequence δ_i will decrease to cross the level 0 when i decreases. That is, there exists i^* such that $\delta_i \leq 0$ for all $i \leq i^*$ and $\delta_i > 0$ for all $i \geq i^* + 1$. Let $M^* = m - i^* + 1$. Lemma 4.2 holds, where $M^* \geq 1$ is ensured by $\delta_m(m) \geq 0$.

We rewrite the requirement $\delta_m \geq 0$. After substituting $w_m = y - C(r+\sigma)/h_m$, this condition becomes

$$b \leq y + \lambda_0 S - \frac{C\left[(r+\sigma)e^{a_m} + \lambda_0\right]}{e^{a_m} - 1 - a_m}.$$

The right-hand of this inequality is an increasing function of a_m. Because a_m is bounded from below by $\bar{a} - \ln(1+\bar{a})$ according to (4.10), a sufficient condition for the above inequality is that it holds for this lower bound of a_m, which is imposed as (4.18). **QED**

E. Proof of Lemma 6.2

To show that $\phi(a_{+1})$ is well-defined for given a_{+1} by the equality form of (6.2), we use the definition of f to rewrite the equality as

$$\frac{q(a)}{q(a_{+1})}\left[1-\frac{e^{a}-1-a}{e^{a_{+1}}-1-a_{+1}}\right]=1. \tag{E.1}$$

The left-hand side of (E.1) is a decreasing function of a and an increasing function of a_{+1} (note that $a_{+1} > a$). If $\phi(a_{+1})$ is a solution for a, then the solution is unique and satisfies $\phi' > 0$, verifying part (i) of the lemma. When $a = a_{+1}$, the left-hand side of (E.1) is 0, which is less than the right-hand side. When $a \to 0$, the left-hand approaches $1/q(a_{+1}) > 1$. Thus, the solution for a, $\phi(a_{+1})$, indeed exists and is unique. This argument also establishes the inequality $\phi(a_{+1}) < a_{+1}$ in part (ii) of the lemma.

For the inequality $\phi(a_{+1}) > a_{+1} - \ln(1 + a_{+1})$ in part (ii), we show that the left-hand side of (E.1) is greater than 1 (the right-hand side) when $a = a_{+1} - \ln(1 + a_{+1})$. Substituting this particular value of a and re-arranging terms, the condition to be established becomes $\ln(1+a_{+1}) - \frac{a_{+1}}{1+a_{+1}} > 0$. The left-hand side of this inequality is equal to 0 when $a_{+1} = 0$, and its derivative with respect to a_{+1} is $a_{+1}/(1+a_{+1})^2 > 0$. Thus, the desired inequality holds for all $a_{+1} > 0$.

Before establishing part (iii), we conjecture the following inequalities:

$$\frac{d}{da}\left[\frac{1}{f(a)}-\frac{1}{h(\phi(a))}\right]\leq 0, \tag{E.2}$$

$$\left[\frac{f(a)}{a}+q(a)\right]\frac{d}{da}\left[\frac{1}{f(a)}-\frac{1}{h(\phi(a))}\right]+q'(a)\left[\frac{1}{f(a)}-\frac{1}{h(\phi(a))}\right]\leq 0. \tag{E.3}$$

The expressions in these conditions are single-variable functions which do not have any parameter. So, it is easy to verify these conditions using computer. (However, it is difficult to prove them using pen and paper.)

Now, suppose $a \leq \phi(a_{+1})$. We show $a_{-1} < \phi(a)$ or, equivalently, $h_{-1} < h(\phi(a))$. Under (5.2), this desired condition is equivalent to:

$$\frac{r+\sigma+\lambda q_{+1}}{h}-\frac{\lambda q_{+1}}{f_{+1}}+(r+\sigma+\lambda q)\left(\frac{1}{f}-\frac{1}{h(\phi(a))}\right)>0.$$

Because $a \leq \phi(a_{+1})$ by supposition, $h \leq h(\phi(a_{+1}))$, and so a sufficient condition for the above inequality is

$$\frac{r+\sigma+\lambda q_{+1}}{h(\phi(a_{+1}))}-\frac{\lambda q_{+1}}{f_{+1}}+(r+\sigma+\lambda q)\left(\frac{1}{f}-\frac{1}{h(\phi(a))}\right)>0.$$

Under (E.2) and (E.3), we have

$$\begin{aligned}
& \frac{d}{da}\left\{(r+\sigma+\lambda q)\left[\frac{1}{f}-\frac{1}{h(\phi(a))}\right]\right\} \\
= \; & (r+\sigma+\lambda q)\frac{d}{da}\left[\frac{1}{f}-\frac{1}{h(\phi(a))}\right]+\lambda q'(a)\left[\frac{1}{f}-\frac{1}{h(\phi(a))}\right] \\
< \; & \lambda\left(\frac{f}{a}+q\right)\frac{d}{da}\left[\frac{1}{f}-\frac{1}{h(\phi(a))}\right]+\lambda q'(a)\left[\frac{1}{f}-\frac{1}{h(\phi(a))}\right] \\
\le \; & 0.
\end{aligned}$$

The first inequality comes from (E.2) and the result $(r+\sigma)/\lambda > f(a)/a$ in Proposition Monotone, and the second inequality from (E.3). Because $a < a_{+1}$, the above result implies

$$\begin{aligned}
& \left[\frac{r+\sigma+\lambda q_{+1}}{h(\phi(a_{+1}))}-\frac{\lambda q_{+1}}{f_{+1}}\right]+(r+\sigma+\lambda q)\left[\frac{1}{f}-\frac{1}{h(\phi(a))}\right] \\
\ge \; & \left[\frac{r+\sigma+\lambda q_{+1}}{h(\phi(a_{+1}))}-\frac{\lambda q_{+1}}{f_{+1}}\right]+(r+\sigma+\lambda q_{+1})\left[\frac{1}{f_{+1}}-\frac{1}{h(\phi(a_{+1}))}\right] \\
= \; & \frac{r+\sigma}{f_{+1}} > 0.
\end{aligned}$$

This is the desired result. **QED**

F. Proof of Lemma Type-II

In the text we have shown that $\max \pi(a^d) = \pi(A) > C$ and $\pi'(A) = 0$, where A is defined in (6.7). Because $\pi'(a^d) > 0$ iff $a^d < A$, there exist A_1 and A_2, with $A_1 < A < A_2$, such that $\pi(A_i) = C$ for $i = 1, 2$ and $\pi(a^d) > C$ iff $a^d \in (A_1, A_2)$. Clearly, $\pi'(A_1) > 0 > \pi'(A_2)$. Because a type II deviation must satisfy $a^d \ge \beta$ (i.e., the constraint (IIc)), the deviation is not profitable if and only if either $\beta \ge A_2$ or $\beta \le a^d \le A_1$. We rewrite these conditions to obtain the condition $\beta \le \beta^*$ or (6.10) in the lemma. Let us denote $Y = q_{+1}/\left[(q - q_{+1})\, f_{+1}\right]$ in this appendix.

First, we show that $\beta > A_1$, and so the case $\beta \le a^d \le A_1$ never occurs. The inequality $\beta > A_1$ holds iff $q(\beta) < q(A_1)$ and hence iff

$$q(A_1) > \frac{q_{-1}}{1+f_{-1}\left(\frac{1}{f}-Y\right)} = \frac{q(A_1)\left[1+f_{-1}\left(\frac{1}{h}+\frac{1}{f}\right)\right]-f_{-1}/A_1}{1+f_{-1}\left(\frac{1}{f}-Y\right)}.$$

Here we have used the definition of $q(\beta)$ first and then the definition of A_1 to substitute for q_{-1}. Re-arranging terms and using the definition of β^*, the above inequality is equivalent to $h(A_1) < h(\beta^*)$. So, $\beta > A_1$ is equivalent to $\beta^* > A_1$. Because $a < \phi(a_{+1})$, $Y < 1/f$ and so

$$h(\beta^*) > \left(\frac{1}{h}+\frac{1}{f}\right)^{-1} = 1-(1+a)\,e^{-a}.$$

A sufficient condition for $\beta^* > A_1$ is then $A_1 < a - \ln(1+a)$. Because $a - \ln(1+a) < a_{-1}$ by Proposition Monotone and $a_{-1} < A$ as shown in the text, $a - \ln(1+a) < A$. Because $\pi'(a^d) > 0$ for all $a^d < A$ and $\pi(A_1) = C$, then $A_1 < a - \ln(1+a)$ iff $\pi(a - \ln(1+a)) > C$. Calculating $\pi(a - \ln(1+a))$ and re-arranging terms, the latter condition becomes $q(a - \ln(1+a)) > q_{-1}$, which is satisfied because $q(.)$ is a decreasing function and $a - \ln(1+a) < a_{-1}$. Now that $\beta \le A_1$, a type II deviation is not profitable iff $\beta \ge A_2$.

Second, we show that $\beta \geq A_2$ iff $\beta^* \geq \beta$. Similar to the above procedure that showed $\beta > A_1$ iff $A_1 < \beta^*$, we can show that $\beta \geq A_2$ iff $\beta^* \geq A_2$. Because $\beta^* > A_1$, as shown above, and $\pi(A_2) = C$, the inequality $\beta^* \geq A_2$ holds iff $\pi(\beta^*) \leq C$. Substituting $\pi(\beta^*)$, we rewrite the latter condition as

$$\begin{aligned} 0 &\leq \tfrac{1}{h(\beta^*)} - \left(\tfrac{1}{h} + \tfrac{1}{f} + \tfrac{1}{f_{-1}}\right) + \tfrac{\beta^*}{h(\beta^*)} \tfrac{q_{-1}}{f_{-1}} \\ &= \left(\tfrac{1}{h} + Y\right) - \left(\tfrac{1}{h} + \tfrac{1}{f} + \tfrac{1}{f_{-1}}\right) + \tfrac{q_{-1}}{q(\beta^*) f_{-1}} \\ &= \tfrac{q_{-1}}{q(\beta^*) f_{-1}} - \left(\tfrac{1}{f_{-1}} + \tfrac{1}{f} - Y\right). \end{aligned}$$

Using the equation that defines β to substitute for q_{-1}, we can rewrite the above inequality further as $q(\beta^*) \leq q(\beta)$. Thus, $\beta \geq A_2$ holds iff $\beta^* \geq \beta$.

Finally, we show that $\beta^* \geq \beta$ is equivalent to (6.10). To do so, rewrite (5.2) as

$$\frac{1}{f} - Y = \frac{1}{h_{-1}} - \frac{r + \sigma + \lambda q_{+1}}{r + \sigma + \lambda q} \left(\frac{1}{h} + Y\right).$$

Then, $\beta^* \geq \beta$ iff $1/q(\beta^*) \geq 1/q(\beta)$ and hence iff

$$\begin{aligned} 0 &\leq \tfrac{1}{q(\beta^*)} - \tfrac{1}{q_{-1}} \left\{1 + f_{-1}\left[\tfrac{1}{h_{-1}} - \tfrac{r+\sigma+\lambda q_{+1}}{r+\sigma+\lambda q}\left(\tfrac{1}{h} + Y\right)\right]\right\} \\ &= \tfrac{1}{q(\beta^*)} - \tfrac{1}{q_{-1}}\left(1 + \tfrac{f_{-1}}{h_{-1}}\right) + \left(\tfrac{r+\sigma+\lambda q_{+1}}{r+\sigma+\lambda q}\right) \tfrac{f_{-1}}{q_{-1} h(\beta^*)} \\ &= \tfrac{1}{q(\beta^*)} - e^{a_{-1}} + \left(\tfrac{r+\sigma+\lambda q_{+1}}{r+\sigma+\lambda q}\right)\left(e^{a_{-1}} - 1 - a_{-1}\right)/h(\beta^*). \end{aligned}$$

The inequality comes from substituting the definition of β and the term $\left(\frac{1}{f} - Y\right)$; the two equalities come from substituting the definitions of $h(\beta^*)$ and f. Multiplying the last expression by $h(\beta^*)$ yields (6.10). **QED**

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