

LIQUIDITY, BUSINESS CYCLES, AND MONETARY POLICY*

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1. Introduction

In this paper and its companion,¹ we provide a simple framework for modelling differences in liquidity across assets. Our purpose is to understand the interactions between asset prices and aggregate activity, and to explain liquidity premia. In so doing, we want to find out what role government policy might have through open market operations that change the mix of assets held by the private sector.

The present paper takes fiat money to be one of the assets under consideration. We investigate under what circumstances money is essential to the smooth allocation of resources. We show that certain apparent anomalies of the real economy are in fact normal features of an economy where money is essential. Among the well-known puzzles we have in mind are: the low risk-free rate puzzle; the excess volatility of asset prices; the anomalous savings behaviour of certain households, and their low participation in asset markets. Before describing our monetary economy, we should start with some remarks about modelling strategy.

In broad terms, there are two ways of getting fiat money into a competitive macroeconomic model. One approach is to endow money with some special function -- for example, cash-in-advance.² The other approach is to starve agents of alternatives to money -- as in an overlapping generations framework where money is the sole means of saving.³ Although the first approach, in particular the cash-in-advance model, has proved important to

¹Kiyotaki-Moore (2001), "Liquidity and Asset Prices".

²See R.W.Clower (1967), "A Reconsideration of the Microfoundations of Monetary Theory", Western Economic Journal, 6, 1-8; and R.E.Lucas and N.L.Stokey (1987), "Money and Interest in a Cash-in-Advance Economy", Econometrica, 55, 491-513.

³P.A.Samuelson (1958), "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money", Journal of Political Economy, 66, 467-482. Bewley also models money in a context where there are no alternative assets with which to save; see T.Bewley (1980), "The Optimum Quantity of Money", in J.Kareken and N.Wallace (eds) Models of Monetary Economics, 169-210.

monetary economics and policy analysis, it is not well-suited to answering larger questions to do with liquidity. By endowing money with a special function, one is imposing rather than explaining the use of money, which precludes the possibility that other assets may substitute for money. And, clearly, the second approach rules out any general discussion of liquidity if there are no alternative assets to money.

There are many noncompetitive models of money, leading with the random matching framework.⁴ In principle, such models are suited to analyzing liquidity. But they are necessarily special, and it is difficult to incorporate them into the rest of macroeconomics. We believe there is a need for a workhorse model of money and liquidity, with competitive markets, which does not stray too far from the other workhorse, the real business cycle model.

In our framework, markets are competitive, money is not endowed with any special function, and there are other assets traded besides money. The basic model presented in Section 2 has two kinds of agent, entrepreneurs and workers, one consumption good, and three assets: fiat money, physical capital, and human capital. The supply of fiat money is fixed. The supply of capital changes through investment and depreciation. A worker's human capital is inalienable, which means that he or she cannot borrow against future labour income: in any period, the only labour market is a spot market for that period's labour services. There is a commonly available technology for combining labour with capital to produce consumption good.

In each period a fraction of the entrepreneurs (but none of the workers) can invest in producing new capital from the consumption good. The arrival of such an investment opportunity is randomly distributed across entrepreneurs through time. This heterogeneity is an important component of the model: because not all entrepreneurs can invest in each period, there is a need to transfer resources from those entrepreneurs who don't have an investment opportunity (that period's savers) to those who do (that period's investors).

⁴For example, N.Kiyotaki and R.Wright (1989), "On Money as a Medium of Exchange", Journal of Political Economy, 97, 927-954.

To acquire consumption good as input for the production of new capital, investing entrepreneurs sell capital -- newly produced capital as well as capital from their existing holdings.

The other crucial feature of the model is that, although capital can change hands, it cannot be traded as quickly as money. Specifically, we make the non-standard assumption that, in any given period, an agent can only sell a fraction θ of his capital holding. This liquidity constraint applies equally to the sale of new capital and existing capital. Of the capital holding the agent does not sell this period, he can sell a further fraction θ in the next period; and so on. Compare it to peeling an onion slowly, layer by layer.⁵

The question is to what extent does this liquidity constraint inhibit the efficient transfer of resources from savers to investors. There may be a role for money to lubricate the transfer of additional resources. Whether or not agents use money is determined endogenously. We show that for high enough values of θ , money is not used and has no value. Indeed, the first-best may be attained, even though $\theta < 1$. But for lower values of θ , money plays an essential role. In the latter case, we call the economy a monetary economy.

We find that a necessary feature of a monetary economy is that the investment of entrepreneurs is limited by liquidity constraints. He cannot raise the entire cost of investment externally, given that the liquidity constraint binds for the sale of new capital. That is, he has to make a downpayment for each unit of investment from his own internal funds. But in trying to raise funds to make this downpayment, he is constrained by how much of his existing capital stock can be sold in time: the liquidity constraint binds here too. In this sense, an investing entrepreneur finds money more valuable than capital, because he can use all of his money to finance new investment whereas he can use only a fraction θ of his capital: money is more liquid than capital.

⁵We can justify this assumption with a moral hazard argument: it takes overnight to deliver capital to a purchaser, and the selling agent can abscond with a fraction $1-\theta$ of his capital prior to dispatch.

Described in this way, our model appears to have only two assets traded: money and capital. There is no paper other than fiat money. This raises the questions: Are we not ruling out all intertemporal contracting? And are we not guilty of starving our agents of other means of saving -- just as in other monetary models like the overlapping generations framework? We do not think so. There is another interpretation of our model which should be borne in mind. Physical capital need not change hands. Suppose that after an entrepreneur undertakes an investment project to produce capital, he is still needed to run the project to produce consumption good throughout its life. Then, rather than sell part of the project's capital, he sells an equity claim against the future return from that capital. For moral hazard reasons, in each period there may be an upper bound, θ^e say, on the fraction of future return that he is able to pledge.⁶ Once issued, these paper claims circulate among savers, but here too there may be a limit on the speed of resale. For example, one saver buying paper from another may need to check the authenticity of the underlying investment project, and verification takes time.⁷ Let us make the crude assumption that an agent can only resell a fraction θ^r , say, of his current paper holding in any period. A priori, there is no reason to think that θ^e should equal θ^r . But in the special case where $\theta^e = \theta^r (= \theta)$, the model is analytically the same as the model in which physical capital rather than paper changes hands.

This alternative interpretation brings out an important idea. Within our single θ -constraint are embedded two related kinds of liquidity constraint. First, a traditional borrowing constraint: the constraint faced by investing entrepreneurs who try to issue new claims against new projects. Second, a saleability constraint: the constraint faced by holders of existing claims who try to resell them to third parties. Both constraints stem from a lack of trust: a limit on people's ability to commit. Of course, if the person who initially bought a claim from an investing entrepreneur were

⁶Let us continue to adopt the onion model, so that he can continually sell equity, even after he has undertaken an investment project.

⁷In practice, there are clearly differences between kinds of paper -- e.g. between the shares of a large publicly-traded company and the stock of a small privately-held business.

content to wait for the flow of dividends -- i.e. if there were a "coincidence of wants in dated goods" between the issuer and the initial buyer -- then saleability wouldn't matter. But in our context, someone who buys claims in one period will want to sell claims in some future period in order to finance his own productive investment, so there is no coincidence of wants in dated goods. To put it grandly, we see the lack of coincidence of wants in dated goods, combined with limited commitment, as the keys to understanding money and liquidity.⁸

Rightly or wrongly, we have chosen to capture both the borrowing and the saleability constraints under the umbrella of our single θ -constraint. The reader may feel that this constraint is too crude: it is tantamount to a peculiar transaction cost per period: zero for the first fraction θ of capital sold, and then infinite. We have sympathy with such a criticism. Our θ -assumption is not grounded in deep theory; it is a pragmatic modelling device. In the final count, it must be judged by what it delivers. So let us turn to our findings.

We find that in a monetary economy, the rate of return on money is very low, less than the return on capital. Nevertheless, a saving entrepreneur chooses to hold some money in his portfolio, because, in the event that he has an opportunity to invest in the future, he will be liquidity constrained, and money is more liquid than capital. The gap between the return on money and the return on capital is a liquidity premium. This may help explain the low risk-free rate puzzle.

We also find that the return on capital is low, less than the rate of time preference. This means that agents such as workers, who don't anticipate having investment opportunities, will choose to hold neither capital nor money. They will simply consume their labour income, period by period. This may help explain why certain households do not save. It is not that they don't have access to asset markets, but rather that the return on assets isn't enough to attract them.

⁸In an earlier paper, Kiyotaki-Moore (2000), "Inside Money and Liquidity", we brought out these ideas more explicitly, in the context of a deterministic model where entrepreneurs take turns to produce.

We show that the liquidity constraints facing investing entrepreneurs cause too few resources to flow from savers. The latter group consumes too much, with the result that aggregate output and investment are too low relative to their first-best levels.

In most real business cycle models, there is no feedback from asset prices to output. That is not true in our monetary economy. A fall, say, in the price of capital and money lowers the net worth of investing entrepreneurs, and so reduces the amount they can afford as downpayment for investment. Moreover, a fall in the price of capital raises the size of the required downpayment per unit of new investment. Altogether, investment suffers from falls in asset prices. This feedback mechanism causes asset prices to be more volatile than they would be in a standard general equilibrium asset pricing model without liquidity constraints.

In Section 3 of the paper we introduce government. We make no attempt at explaining government policy. Our interest is in seeing the effect of policy on the behaviour of the private economy. We assume that the government can own capital, and use it to make a return, in exactly the same way as private entrepreneurs. Unlike the entrepreneurs, the government cannot produce new capital, but it can costlessly change the supply of fiat money. We consider an open market operation where the government buys capital using newly-printed money.

We find that an open market operation like this does affect the economy, but in a way that depends on the future path of policy. Because of the liquidity premium, purchasing more capital in an open market operation increases the subsequent flow of income to the government. How it spends this income will determine the effect of the initial open market operation. One way to spend the additional income stream is to retire money. This leads to a steady rise in the price of money (deflation), and hence to an increase in the return on money. Think of it as paying a "dividend" on money as "equity". In effect, the government is acting as a banker to the entrepreneurs, converting an illiquid income stream (the return on the capital it purchases) into a liquid income stream (the return on money). This provides the entrepreneurial sector with greater lubrication for the transfer of resources from savers to

investors, and hence increases aggregate investment and output.

Another way for the government to spend its additional income stream following the open market operation would be to make transfers, say to the workers. This would be contractionary, because workers cannot borrow against their future income. The government would in effect be converting an illiquid income stream on capital into a completely nontradeable income stream -- thus depriving entrepreneurs of liquidity and causing their investment and output to shrink. Our conclusion, then, is that an open market operation -- buying illiquid capital by issuing fiat money -- in and of itself need not be good for the economy. The benefit only comes from what the government does next with its surplus income.⁹

A word of caution is in order. In practice, an open market operation constitutes a tiny change in the composition of asset holding in the economy, so it is difficult to see why this change should have significant effects. The answer may lie in a more layered model of banking, where the government supplies extremely liquid assets for banks to use, who in turn supply somewhat less liquid assets for use by the rest of the economy. We conjecture that the effects of government policy may be amplified in such a multi-layered model.

In Section 4 of the paper we consider the dynamic response of the economy, both to technological shocks and to monetary policy shocks. To simplify the analysis, we consider a continuous-time approximation of the model, and assume that each of the shocks follows a two-point Markov process. By so doing, we are able to get a qualitative idea of the dynamics of the economy. In due course, our analytical exercise should be complemented by a numerical simulation, calibrated using reasonable parameter values, to get a quantitative understanding. This awaits further research.

⁹This idea can be traced back to L. Metzler (1951), "Wealth, Saving, and the Rate of Interest", Journal of Political Economy, 59, 93-116.

2. The basic model without government

Consider an infinite-horizon, discrete-time economy with four objects traded: a nondurable consumption good, capital, labour and fiat money. Fiat money is intrinsically useless, and is in fixed supply M .

There are two populations of agents, entrepreneurs and workers, each with unit measure. Let us start with the entrepreneurs, who are the central actors in the drama. At date t , a typical entrepreneur has expected discounted utility

$$(1) \quad E_t \sum_{s=0}^{\infty} \beta^s \log c_{t+s}$$

of consumption path $\{c_t, c_{t+1}, \dots\}$, where $0 < \beta < 1$. He has no labour endowment. All entrepreneurs have access to a constant-returns-to-scale technology for producing the consumption good from capital and labour. An entrepreneur holding k_t capital at the start of period t can employ l_t labour to produce

$$(2) \quad A_t (k_t)^\phi (l_t)^{1-\phi}$$

consumption goods, where $0 < \phi < 1$. Production is completed within the period t , during which time capital depreciates to λk_t , $0 < \lambda < 1$. There is no overnight depreciation. We assume that the productivity parameter, $A_t > 0$, which is common to all entrepreneurs, follows a stationary stochastic process that lies near to some constant level, A -- so that we can speak of the economy being in the neighbourhood of the steady state.

The entrepreneur may also have an opportunity to produce new capital stock. Specifically, at each date t , with probability π he has access to a constant-returns technology that produces one unit of capital from one unit of consumption good. The arrival of such an investment opportunity is independently distributed across entrepreneurs and through time, and is

independent of aggregate shocks. Again, production is completed within the period t -- although newly-produced capital does not become available as an input to the production of consumption goods until the following period $t+1$. Let i_t denote the entrepreneur's production of new capital during period t (i_t is zero if he does not have an investment opportunity), which augments his depreciated capital holding of λk_t .

We make the reasonable assumption that the arrival rate of investment opportunities exceeds the depreciation rate, which in turn exceeds the subjective discount rate:

$$\text{(Assumption 1)} \quad \pi > 1-\lambda > 1-\beta.$$

This restriction is not at all essential, but will help to limit the number of cases we need to analyze later on.

We make two critical assumptions. First, we assume that capital, unlike money or labour, takes overnight to deliver; and that the entrepreneur cannot credibly make a promise in period t to deliver overnight any more than a fraction θ of his end-of-period capital holding, $i_t + \lambda k_t$. In effect, there is a lower bound on k_{t+1} , the stock of capital he holds at the start of date $t+1$:

$$(3) \quad k_{t+1} \geq (1-\theta)(i_t + \lambda k_t).$$

One justification for this is that the entrepreneur could choose to abscond overnight with a fraction $1-\theta$ of his capital and start a new life at date $t+1$ with a clear record.¹⁰ In this context, inequality (3) is an incentive constraint: instead of absconding, the entrepreneur prefers to honour the

¹⁰ Anonymity would rule out the possibility that social sanctions -- such as a bar on future access to markets -- could be imposed on him for renegeing on a promise to deliver.

promise he makes in period t to deliver $i_t + \lambda k_t - k_{t+1}$ units of capital overnight. It is tantamount to a liquidity constraint. θ is a measure of the liquidity of capital. Note that we are not assuming that the retained stock, $(1-\theta)(i_t + \lambda k_t)$, can never be sold. Rather, we are assuming that there is a limit on the speed of sale. If he chose to, the entrepreneur could eventually sell off all his capital -- but only at a rate θ of his current holding at the end of any single period.

Second, we assume that the entrepreneur's money holdings at the start of date $t+1$, m_{t+1} , can never be negative:

$$(4) \quad m_{t+1} \geq 0.$$

This second liquidity constraint can be explained on the grounds that, without collateral, there is no scope for getting the entrepreneur to repay debt -- he could always abscond.¹¹

Everyone has access to perfectly competitive markets for the consumption good, capital, money and labour. However, there is no insurance market: we assume that an entrepreneur cannot insure against whether or not he has an investment opportunity. This assumption can be justified in a variety of ways. For example, it may not be possible to verify that someone has an investment opportunity; or verification may take so long that the opportunity has gone by the time the claim is paid out. A long-term insurance contract based on self-reporting does not work here because the people are able to trade assets covertly.¹² Each of these justifications warrants formal modelling. But we are reasonably confident that even if

¹¹If the liquidity constraint (3) were slack, then the entrepreneur might be able to borrow against it. In our economy, however, it turns out that liquidity constraints (3) and (4) are simultaneously binding for an entrepreneur at the time of investment. Thus we can ignore such borrowing.

¹²Moreover, in a mutual insurance scheme among entrepreneurs who don't trust each other, there may be a shortage of collateral that can be used to secure payouts. Without collateral, is the mutual insurance company to be trusted any more than the individual entrepreneurs are trusted?

partial insurance were possible our broad conclusions would still hold. So rather than clutter up the model, we simply assume that no insurance scheme is feasible.

At each date t , let q_t be the price, in terms of the date t consumption good, of a unit of capital promised for delivery by the start of date $t+1$. And let p_t and w_t be the prices of fiat money and labour, again both in terms of the date t consumption good. (Warning! p_t is customarily defined as the inverse: the price of consumption in terms of money. But, a priori, money may not have value, so we prefer not to make it the numeraire.) The entrepreneur's flow of funds constraint at date t is then given by

$$(5) \quad c_t + i_t + q_t(k_{t+1} - i_t - \lambda k_t) + p_t(m_{t+1} - m_t) \\ = A_t(k_t)^\phi (\ell_t)^{1-\phi} - w_t \ell_t.$$

The LHS is his expenditure on consumption, investment, and net purchases of capital and money. The RHS is his "gross profit", i.e. output net of labour costs.

Turn now to the workers. At date t , a typical worker has expected discounted utility

$$(6) \quad E_t \sum_{s=0}^{\infty} \beta^s u \left(c'_{t+s} - \frac{\omega}{1+\gamma} (\ell'_t)^{1+\gamma} \right),$$

of consumption path $\{c'_t, c'_{t+1}, \dots\}$ given his labour supply path $\{\ell'_t, \ell'_{t+1}, \dots\}$, where $\omega > 0$, $\gamma > 0$ and $u(\cdot)$ is increasing and strictly concave.

Workers do not have investment opportunities, and cannot borrow against their future labour income. Later we will show that in the neighbourhood of the steady state, a worker will choose to hold neither capital nor money. That is, the worker simply consumes his labour income at each date:

$$(7) \quad c'_t = w_t l'_t.$$

An equilibrium sequence of prices $\{p_t, q_t, w_t\}$ is such that at each date t : entrepreneurs choose consumption c_t , investment i_t , labour demand l_t , and start-of-next-period capital and money holdings, k_{t+1} and m_{t+1} , to maximize (1) subject to (3), (4) and (5); workers choose consumption c'_t and labour supply l'_t to maximize (6) subject to (7); and the markets for the consumption good, capital, money and labour all clear.

Before we characterize equilibrium, it helps to clear the decks a little by suppressing reference to the workers. Given that their population has unit measure, it follows from (6) and (7) that their aggregate labour supply equals $(w_t/\omega)^{1/\gamma}$. To find aggregate labour demand, first maximize the gross profit of a typical entrepreneur holding capital k_t -- the RHS of (5) -- with respect to l_t . This yields his labour demand, $k_t \left[(1-\phi)A_t/w_t \right]^{1/\phi}$, which is proportional to k_t . So if the aggregate stock of capital held by entrepreneurs at the start of date t is K_t , labour-market clearing requires that

$$(w_t/\omega)^{1/\gamma} = K_t \left[(1-\phi)A_t/w_t \right]^{1/\phi}.$$

Substituting back the equilibrium wage w_t into the RHS of (5), we find that the individual entrepreneur's maximized gross profit equals $r_t k_t$ -- also proportional to his capital holding k_t -- where

$$(8) \quad r_t = a_t (K_t)^{\alpha-1},$$

and the parameters a_t , α are derived from A_t , ϕ , ω and γ :

$$(8a) \quad a_t = \phi \left(\frac{1-\phi}{\omega} \right)^{\frac{1-\phi}{\phi+\gamma}} \left(A_t \right)^{\frac{1+\gamma}{\phi+\gamma}}$$

$$(8b) \quad \alpha = \frac{\phi(1+\gamma)}{\phi+\gamma}.$$

The upshot of this side-analysis of the labour market is that we can work with a reduced form of (5), where the RHS is simply $r_t k_t$. Note from (8b) that α lies between 0 and 1, so that r_t -- which is parametric for the individual entrepreneur -- declines with the aggregate stock of capital K_t , because the wage increases with K_t . But for the entrepreneurial sector as a whole, gross profit $r_t K_t$ increases with K_t . Also note from (8a) that r_t is increasing in the productivity parameter A_t , through a_t .

We are now in a position to characterize the equilibrium behaviour of the entrepreneurs. Consider an entrepreneur holding capital k_t and money m_t at the start of date t .

First, suppose he has an investment opportunity: let this be denoted by a superscript i on his choice of consumption, c_t^i , and start-of-next-period capital and money holdings, k_{t+1}^i and m_{t+1}^i . He has two ways of acquiring capital k_{t+1}^i : either produce it at unit cost 1, or buy it in the market at price q_t . (See the LHS of the flow-of-funds constraint (5), where, recall, investment i_t corresponds to production.) If q_t is less than 1, the agent will not produce. If q_t equals 1, he will be indifferent. If q_t is greater than 1, he will produce by selling as much capital as he can subject to the constraint (3). The entrepreneur's production choice is similar to Tobin's q theory of investment.

In the following we want to restrict attention to an equilibrium in which q_t is greater than 1. We also want money to have value in equilibrium. Let us assume

$$(Assumption 2) \quad \theta < \hat{\theta},$$

where $\hat{\theta}$ is the positive root of the quadratic equation

$$0 = \lambda\beta^2\pi(1-\pi)\left((1-\lambda)(1-\pi) - \hat{\theta}(1-\lambda+\lambda\pi)\right) \\ + (1-\lambda+\lambda\pi)\left((\beta-\lambda)(1-\pi) - \hat{\theta}(1-\lambda+\lambda\pi)\right)\left(\lambda(1-\beta)(1-\pi) + \hat{\theta}(1-\lambda+\lambda\pi+\lambda\beta-\lambda\beta\pi)\right)$$

The upper bound on θ in Assumption 2 is tight enough to ensure that the following claim holds.

Claim 1 Under Assumption 2, in the neighbourhood of the steady state:

- (i) the price of money, p_t , is strictly positive;
- (ii) the price of capital, q_t , is strictly greater than 1;
- (iii) an entrepreneur with an investment opportunity will not choose to hold money: $m_{t+1}^i = 0$.

We will be in a position to prove the claim once we have laid out the equilibrium conditions.

There is a caveat to Claim 1(i). Fiat money can only be valuable to someone if other people find it valuable, hence there is always a non-monetary equilibrium in which the price of fiat money is zero. Thus when there is a monetary equilibrium in addition to the non-monetary equilibrium, we restrict attention to the monetary equilibrium: $p_t > 0$.

Apropos Claim 1(ii), the price of capital, q_t , cannot be too high above the cost of production, 1, otherwise an investing entrepreneur would be able to expand without limit. Of each new unit of capital produced at date t , a fraction θ can be sold for θq_t . Hence, to keep i_t finite, it must be the case that $\theta q_t < 1$. The entrepreneur has to draw on his own funds to finance the difference. One can interpret $1 - \theta q_t$ as the required downpayment per unit of investment, maximally levered in the sense that the most funds are raised against an investment project i_t by mortgaging (selling off) the greatest possible amount, θi_t .

Claim 1(iii) says that the entrepreneur prefers investment with the maximum leverage to holding money, even though the return is in the form of capital stock which at date $t+1$ is less liquid than money. (Incidentally, even though the investing entrepreneurs don't want to hold money for liquidity purposes, the non-investing entrepreneurs do -- see below. This is why Claim 1(i) holds.)

Thus, for an investing entrepreneur, the liquidity constraints (3) and (4) are both binding. His flow of funds constraint (5) can be rewritten

$$(9) \quad c_t^i + \left(\frac{1-\theta q_t}{1-\theta} \right) k_{t+1}^i = r_t k_t + \lambda k_t + p_t m_t.$$

The RHS of (9) corresponds to his net worth: gross profit generated from k_t during period t , plus the value of his depreciated capital λk_t , plus the value of his money m_t . Notice he values the depreciated capital at its unit cost of production, 1, not at the market price, q_t , because his sales constraint (3) is binding. Also notice that on the LHS of (9), his end-of-period capital holding k_{t+1}^i is valued at $(1-\theta q_t)/(1-\theta)$ per unit: at the margin, a unit of consumption good can be used as a downpayment to produce $1/(1-\theta q_t)$ units of capital, of which he retains a fraction $1-\theta$. This capital holding constitutes all the saving that he takes into period $t+1$.

Given the discounted logarithmic preferences (1), the entrepreneur saves a fraction β of his net worth, and consumes a fraction $1-\beta$:

$$(10) \quad c_t^i = (1-\beta)(r_t k_t + \lambda k_t + p_t m_t).^{13}$$

¹³ Compare (1) to a Cobb-Douglas utility function, where the expenditure share of present consumption out of total wealth is constant and equal to $1/(1 + \beta + \beta^2 + \dots) = 1 - \beta$.

And so, from (3) as an equality, (9) and (10), we obtain an expression for his production of new capital in period t :

$$(11) \quad i_t = \frac{r_t k_t + \theta q_t \lambda k_t + p_t m_t - (1-\beta)[r_t k_t + \lambda k_t + p_t m_t]}{1 - \theta q_t}$$

In the numerator are the resources he has at hand: gross profit from his capital holding, plus receipts from selling a fraction θ of his depreciated capital, plus the value of his money holding, minus his consumption. The denominator is the required downpayment per unit of investment.

Next, suppose the entrepreneur does not have an investment opportunity: denote this by a superscript n . The flow-of-funds constraint (5) reduces to

$$(12) \quad c_t^n + q_t k_{t+1}^n + p_t m_{t+1}^n = r_t k_t + q_t \lambda k_t + p_t m_t$$

For the moment, let us assume that constraints (3) and (4) do not bind. Then the RHS of (12) corresponds to the entrepreneur's net worth. It is the same as the RHS of (9), except that now his depreciated capital is valued at the market price, q_t . From this net worth he consumes a fraction $1-\beta$:

$$(13) \quad c_t^n = (1-\beta)(r_t k_t + q_t \lambda k_t + p_t m_t)$$

The remainder is split across a savings portfolio of k_{t+1}^n and m_{t+1}^n .

To determine the optimal portfolio, consider the choice of sacrificing one unit of consumption c_t^n to purchase either $1/q_t$ units of capital or $1/p_t$ units of money, which are then used to augment consumption at date $t+1$. The first-order condition is

$$\begin{aligned}
(14) \quad & \frac{1}{q_t} \beta E_t \left(\frac{\pi(r_{t+1} + \lambda)}{(1-\beta)[r_{t+1}k_{t+1}^n + \lambda k_{t+1}^n + p_{t+1}m_{t+1}^n]} \right. \\
& \quad \left. + \frac{(1-\pi)(r_{t+1} + q_{t+1}\lambda)}{(1-\beta)[r_{t+1}k_{t+1}^n + q_{t+1}\lambda k_{t+1}^n + p_{t+1}m_{t+1}^n]} \right) \\
= & \frac{1}{p_t} \beta E_t \left(\frac{\pi p_{t+1}}{(1-\beta)[r_{t+1}k_{t+1}^n + \lambda k_{t+1}^n + p_{t+1}m_{t+1}^n]} \right. \\
& \quad \left. + \frac{(1-\pi)p_{t+1}}{(1-\beta)[r_{t+1}k_{t+1}^n + q_{t+1}\lambda k_{t+1}^n + p_{t+1}m_{t+1}^n]} \right).
\end{aligned}$$

The LHS of (14) is the gain in expected discounted utility from holding $1/q_t$ additional units of capital at date $t+1$. Per unit, this additional capital yields r_{t+1} gross profit, plus its depreciated value. With probability π the entrepreneur has an investment opportunity at date $t+1$, in which case he values depreciated capital at its replacement cost, 1; but with probability $1-\pi$ the entrepreneur does not have an investment opportunity and the depreciated capital is valued at the market price, q_{t+1} -- see the numerators of the two terms inside the expectations operator. The denominators reflect the respective marginal utilities of consumption. (With logarithmic preferences, marginal utility is $1/c_{t+1}$ -- where c_{t+1} is given by (10) and (13) respectively, each forwarded to date $t+1$.) The RHS of (14) is the gain in expected discounted utility from holding $1/p_t$ additional units of money at date $t+1$. It is similar to the LHS, except for the fact that money yields p_{t+1} at date $t+1$, irrespective of whether or not the entrepreneur has an investment opportunity.¹⁴

¹⁴It is easily confirmed that both sides of (14) equal the foregone utility of consumption at date t , $1/c_t^n$. Let $\xi = q_t k_{t+1}^n / (q_t k_{t+1}^n + p_t m_{t+1}^n) \in (0,1)$. Then a convex combination of ξ times the LHS of (14) plus $1-\xi$ times the RHS equals $\beta / [(1-\beta)(q_t k_{t+1}^n + p_t m_{t+1}^n)]$ -- which, from (12) and (13), equals $1/c_t^n$. This

We are now in a position to consider the aggregate economy. The great merit of the expressions for an investing entrepreneur's consumption and investment choices, c_t^i and i_t , and a non-investing entrepreneurs' consumption and savings choices, c_t^n , k_{t+1}^n and m_{t+1}^n , is that they are all linear in start-of-period capital and money holdings k_t and m_t .¹⁵ Hence aggregation is easy: we do not need to keep track of the distributions. Indeed, since investment opportunities are independently distributed, we can work with the total capital and money holdings in the economy, K_t and M . At the start of date t , a fraction π of K_t and M is held by entrepreneurs who have an investment opportunity. From (11), total investment, I_t , in new capital therefore satisfies

$$(15) \quad (1 - \theta q_t) I_t = \pi \left(\beta r_t K_t + \theta q_t \lambda K_t + \beta p_t M - (1 - \beta) \lambda K_t \right).$$

corroborates our assertion that the entrepreneur chooses to consume a fraction $1 - \beta$ of his net worth.

It also ought to be confirmed that the solution $(c_t^n, k_{t+1}^n, m_{t+1}^n)$ to (12), (13) and (14) satisfies constraints (3) and (4). We will show later, when proving Claim 1(i), that $m_{t+1}^n > 0$ in the neighbourhood of the steady state; i.e. that (4) holds as a strict inequality. However, this presupposes that (3) holds, which it may not. The critical instance is of an entrepreneur who does not have an investment opportunity at date t , but who invested at date $t-1$, and so, by Claim 1(iii), does not hold money. It may be the case that to achieve a balanced savings portfolio -- i.e. one that satisfies (14) -- he has to sell more capital than (3) allows. We could make additional assumptions on the parameters to ensure that this is not the case. Alternatively, we could separate the non-investing entrepreneurs into two groups: those who haven't invested recently, for whom (3) is slack, and those who have invested recently, who have not yet rebalanced their portfolios. The latter group would become negligible if we made the period length sufficiently short; see Section 3 below. With this in mind, we cut a corner and simply ignore the question of whether (3) binds for certain non-investing entrepreneurs.

¹⁵From (12) and (13), the value of savings, $q_t k_{t+1}^n + p_t m_{t+1}^n$, is linear in k_t and m_t , and (the reciprocal of) the portfolio equation (14) is homogeneous in (k_{t+1}^n, m_{t+1}^n) .

Goods-market clearing requires that total output (net of labour costs, which equals the consumption of workers), $r_t K_t$, equals investment plus consumption of entrepreneurs-- which, using (10) and (13), yields

$$(16) \quad r_t K_t = I_t + (1-\beta) \left(r_t K_t + [\pi + (1-\pi)q_t] \lambda K_t + p_t M \right).$$

It remains to find the aggregate counterpart to the portfolio equation (14). During period t , the investing entrepreneurs sell a fraction θ of their investment I_t , together with a fraction θ of their depreciated capital holdings $\pi \lambda K_t$, to the non-investing entrepreneurs. So the stock of capital, K_{t+1}^n , held by the group of non-investing entrepreneurs at the end of the period is given by $(1-\pi)\lambda K_t + \theta I_t + \theta \pi \lambda K_t$. And, by claim 1(iii), we know that this group also hold all the money stock, M . The group's savings portfolio (K_{t+1}^n, M) satisfies (14), which can be simplified to:

$$(17) \quad (1-\pi) E_t \left(\frac{(r_{t+1} + \lambda q_{t+1})/q_t - p_{t+1}/p_t}{r_{t+1} K_{t+1}^n + q_{t+1} \lambda K_{t+1}^n + p_{t+1} M} \right) \\ = \pi E_t \left(\frac{p_{t+1}/p_t - (r_{t+1} + \lambda)/q_t}{r_{t+1} K_{t+1}^n + \lambda K_{t+1}^n + p_{t+1} M} \right),$$

$$\text{where } K_{t+1}^n = \theta I_t + (1 - \pi + \pi \theta) \lambda K_t.$$

Equation (17) lies at the heart of the model. When there is no investment opportunity at date $t+1$, so that the partial liquidity of capital doesn't matter, the return on capital, $(r_{t+1} + q_{t+1} \lambda)/q_t$, exceeds the return on money, p_{t+1}/p_t : the LHS of (17) is positive. However, when there is an investment opportunity, the effective return on capital, $(r_{t+1} + \lambda)/q_t$, is less than the return on money: the RHS of (17) is positive. These return differentials have to be weighted by the respective probabilities and marginal utilities.

Aside from the technology parameter A_t and expectations over future parameters $\{A_{t+s} | s \geq 1\}$ -- embedded in $r_{t+s} = a_{t+s} (K_{t+s})^{\alpha-1}$ through a_{t+s} -- the only state variable in this system is K_t , which evolves according to

$$(18) \quad K_{t+1} = \lambda K_t + I_t.$$

If A_t is a stationary Markov process, then, restricting attention to stable price paths, the competitive equilibrium can be defined recursively as functions (I_t, p_t, q_t, K_{t+1}) of the aggregate state $(K_t; A_t)$ that satisfy (8), (15), (16), (17) and (18), together with the law of motion of A_t .¹⁶

In steady state, when $a_{t+s} \equiv a$ (the RHS of (8a) with $A_t = A$), capital stock K , investment I , and prices p and q , satisfy $I = (1-\lambda)K$ and

$$(19) \quad \pi \beta a K^{\alpha-1} = 1 - \lambda + \pi \lambda (1-\beta) - (1-\lambda+\pi \lambda) \theta q - \pi \beta (pM/K)$$

$$(20) \quad \beta a K^{\alpha-1} = 1 - \lambda + \pi \lambda (1-\beta) + (1-\pi) \lambda (1-\beta) q + (1-\beta) (pM/K)$$

$$(21) \quad \frac{a K^{\alpha-1} + \lambda q}{q} - 1 = \pi \lambda \frac{q-1}{q} \frac{q + (pM/\nu K)}{a K^{\alpha-1} + \lambda + (pM/\nu K)}$$

where $\nu = (1-\pi)\lambda + (1-\lambda+\pi\lambda)\theta$, the steady-state fraction of capital stock held by non-investing entrepreneurs at the end of a period.

Equations (19), (20) and (21) can be viewed as a simultaneous system in three unknowns: the price of capital, q ; the gross profit rate on capital,

¹⁶The state space can easily be enlarged to encompass more general stochastic processes. For example, if A_t were second-order auto-regressive, the state space would need to include A_{t-1} .

$r = aK^{\alpha-1}$; and the value of the money stock as a fraction of total capital, pM/K . (19) and (20) can be solved for $aK^{\alpha-1}$ and pM/K , each as affine functions of q , which when substituted into (21) yield a quadratic equation in q with a unique positive solution. Assumption 2 is sufficient to ensure that this solution lies strictly above 1 (but below $1/\theta$). We can also show that Assumption 2 is the necessary and sufficient condition for money to have value: $p > 0$.

As a prelude to the dynamic analysis that we undertake later on, notice that the technology parameter A only affects the steady-state system through the gross profit term $aK^{\alpha-1}$. That is, a rise in the steady state value of A increases the capital stock, K , but does not affect q , the price of capital. The price of money, p , increases to leave pM/K unchanged.

It is interesting to compare our economy, in which the liquidity constraints (3) and (4) bind for investing entrepreneurs, to a "first-best" economy without such constraints. Consider steady states. In the first-best economy, the price of capital would equal its cost, 1; and the capital stock, K^* say, would equate the return on capital, $a(K^*)^{\alpha-1} + \lambda$, to the agents' common subjective return, $1/\beta$. We can show that, in our constrained economy, the level of activity -- measured by the capital stock K -- is strictly below K^* . Hence, by continuity, the same is true in the neighbourhood of the steady state:

Claim 2 In the neighbourhood of the steady state, the stock of capital, K_t , is less than in the first-best (unconstrained) economy.

Because of the partial liquidity of capital, the economy fails to transfer enough resources to the investing entrepreneurs to achieve the first-best level of investment.

At first sight, the effect of these distortions on the expected return on capital looks ambiguous. Relative to first-best, the price of capital is too high, which decreases the return; whereas the stock of capital is too low, so the marginal product too high, which increases the return. In fact,

the former effect dominates, and we have the following ranking:

Claim 3 In the neighbourhood of the steady state,

$$E_t \frac{p_{t+1}}{p_t} < E_t \frac{a_{t+1} K_{t+1}^{\alpha-1} + \lambda q_{t+1}}{q_t} < \frac{1}{\beta}$$

expected
expected
subjective
return
return
return
on money
on capital

The right-hand inequality says that the expected rate of return on capital is strictly less than the time preference rate.

The left-hand inequality follows directly from (21), given that in steady state $q > 1$. This difference between the expected return on capital and money reflects a liquidity premium. It equals the nominal interest rate on capital.¹⁷ Because entrepreneurs are constrained when they have an investment opportunity, they have to be compensated for holding less liquid capital stock in their savings portfolios. If there were no binding liquidity constraints, money would have no value.¹⁸

The fact that the expected rates of return on capital and money are both lower than the time preference rate justifies our earlier assertion that

¹⁷By the Fisher equation, the nominal interest rate on capital equals the net real return on capital plus the inflation rate. But minus the inflation rate equals the net real return on money. Hence the nominal interest rate on capital equals the real return on capital minus the return on money, i.e. the liquidity premium.

¹⁸In fact, for values of θ slightly higher than $\hat{\theta}$ in Assumption 2, money has no value even though the liquidity constraint (3) still binds: the premium on capital is great enough to deter anyone from wanting to hold money. Eventually, for θ above some higher critical value, θ^* , the economy attains first-best and the liquidity constraints don't bind (even though θ^* is strictly less than 1). To streamline the paper, we have chosen not to give an exhaustive account of the equilibria throughout the parameter space.

workers will not choose to save by holding capital or money. (Of course, if workers could borrow against their future labour income they would do so. But we have ruled this out.) In steady state, workers enjoy a constant consumption equal to their wages.

The reason why an entrepreneur saves, and workers do not, is because the entrepreneur is preparing for his next investment opportunity. And the entrepreneur saves using money as well as capital, despite money's particularly low return, because he anticipates that he will be liquidity constrained at the time of investment. Along a typical time path, he experiences episodes without investment, during which he consumes part of his saving. As the return on saving -- on both capital and money -- is less than his subjective return, the value of his net worth gradually shrinks, as does his consumption. He only expands again at the time of investment. In the aggregate picture, we do not see all this fine grain. But it is important to realize that, even in steady state, the economy is made up of a myriad of such individual histories.

3. Introducing government

We make no attempt to explain government behaviour. Our goal here is simply to explore the effects on equilibrium of an exogenous government policy.

At the start of date t , suppose the government holds K_t^g capital stock. Like all the entrepreneurs, the government has access to the technology (2): K_t^g earns a gross profit of r_t per unit and depreciates by a factor λ . Unlike entrepreneurs, the government cannot produce new capital. However, it can engage in open market operations, to buy (sell) capital by issuing (taking in) money -- it has sole access to a costless money-printing technology. Any sales of capital are subject to the same constraint as (3):

$$K_{t+1}^g \geq (1-\theta)\lambda K_t^g.$$

Finally, the government can make lump sum transfers of consumption goods, which may be negative. Let G_t denote the total net transfer of consumption goods from the government at date t . We assume that G_t is only given to the workers; if negative, G_t corresponds to a lump sum tax on workers. This leaves intact our analysis of entrepreneurs' behaviour. We assume that K_t^g and G_t are not so large that the private economy switches regimes. That is, we are still in an equilibrium in which the liquidity constraints bind for investing entrepreneurs, and money is valuable.

If M_t is the stock of money privately held by entrepreneurs at the start of date t , then the government's flow-of-funds constraint is given by

$$(22) \quad G_t + q_t(K_{t+1}^g - \lambda K_t^g) = r_t K_t^g + p_t(M_{t+1} - M_t).$$

That is, government transfers plus the cost of capital purchases (including the replacement of depreciated capital) must equal gross profit on capital plus seignorage revenues. Since the government is a large agent, at least

relative to each of the atomless private citizens, open market operations will affect the prices p_t and q_t .

All of our earlier analysis goes through, but with obvious adjustments. The transfer G_t changes workers' consumption, but, given the form of their preferences in (6), does not affect their labour supply. So if we let K_t^P denote the capital stock privately held (by entrepreneurs) at the start of date t , equation (8) is modified to

$$(23) \quad r_t = a_t \left(K_t^P + K_t^G \right)^{\alpha-1},$$

where a_t and α are given by (8a) and (8b); and equations (15)-(18) are modified to:

$$(24) \quad (1 - \theta q_t) I_t = \pi \left(\beta r_t K_t^P + \theta q_t \lambda K_t^P + \beta p_t M_t - (1-\beta) \lambda K_t^P \right)$$

$$(25) \quad r_t (K_t^P + K_t^G) = I_t + G_t + (1-\beta) \left(r_t K_t^P + [\pi + (1-\pi)q_t] \lambda K_t^P + p_t M_t \right)$$

$$(26) \quad (1-\pi) E_t \left(\frac{(r_{t+1} + \lambda q_{t+1})/q_t - p_{t+1}/p_t}{r_{t+1} K_{t+1}^n + q_{t+1} \lambda K_{t+1}^n + p_{t+1} M_{t+1}} \right) = \pi E_t \left(\frac{p_{t+1}/p_t - (r_{t+1} + \lambda)/q_t}{r_{t+1} K_{t+1}^n + \lambda K_{t+1}^n + p_{t+1} M_{t+1}} \right)$$

$$\text{where } K_{t+1}^n = \theta I_t + (1-\pi+\pi\theta) \lambda K_t^P - K_{t+1}^G + \lambda K_t^G$$

$$(27) \quad K_{t+1}^P + K_{t+1}^G = \lambda (K_t^P + K_t^G) + I_t.$$

Suppose we take both the policy variables (K_t^g, G_t) and the technology parameter A_t to be exogenous stationary Markov processes. Then, restricting attention to stable price paths, the competitive equilibrium can be defined recursively as functions $(I_t, p_t, q_t, K_{t+1}^p, M_{t+1})$ of the aggregate state $(K_t^p, M_t; K_t^g, G_t, A_t)$ that satisfy (22)-(27), together with the laws of motion of (K_t^g, G_t, A_t) .

For the moment, let us restrict attention to steady states. (In Section 3 we examine dynamics.) The steady state equilibrium is affected by government policy. There are two degrees of freedom. First, the government can allow the stock of money to grow at a constant rate, μ :

$$\frac{M_{t+1}}{M_t} \equiv \mu \quad \text{for all } t.$$

In steady state, the price of money shrinks at the same rate to leave the value of the money stock, $p_t M_t$, constant -- so $1/\mu$ is the return on money. μ may be greater or less than 1. Second, the government can choose its stock of capital, K^g , and the level of transfers, G -- although these latter two are co-determined with μ through the government's steady-state flow-of-funds constraint:

$$(28) \quad G = [r - (1-\lambda)q]K^g + (\mu - 1)p_t M_t.$$

How does government policy affect private activity in the long run? To provide a benchmark, start with the case where the government is inactive: $K^g = G = 0$, $\mu = 1$, and $M_t \equiv M$. This is the basic economy we looked at in Section 1, where we saw that the return on capital, $(r + \lambda q)/q$, exceeds the return on money, 1. The coefficient of K^g in (28) is positive.

Now suppose instead that the government holds a positive stock of capital, $K^g > 0$ -- small enough that in the new steady state the coefficient

of K^G is still positive. Then the government earns a net surplus on its capital holding, which it can spend either on retiring money ($\mu < 1$), or on transfers to the workers ($G > 0$), or on a mixture of the two. Claims 4(i) and 4(ii) deal with the polar cases:

Claim 4(i) In a steady state with a government that holds capital to finance reductions in the money stock ($K^G > 0$ and $\mu < 1$), and makes no transfers ($G = 0$), relative to an inactive government:

total activity, measured by $K^P + K^G$, is higher,
the price of capital q is lower, and
the value of the money stock $p_t M_t$ is higher.

Claim 4(ii) In a steady state with a government that holds capital to finance transfers ($K^G > 0$ and $G > 0$), and keeps the money stock constant ($M_t \equiv M$), relative to an inactive government:

total activity, measured by $K^P + K^G$, is lower,
the price of capital q is higher, and
the price of money p is higher.

Comparing (i) and (ii), we see that there is a marked difference in the effects of the two policies. If the government uses the income stream from its capital holding to finance reductions in the money stock (case (i)), aggregate activity expands. Why? Think of money as "equity". If the government retires money, so that the price of money rises, the government is in effect paying a "dividend" to the remaining money holders. With the dividend, the value of the money stock is higher. There is more liquidity to lubricate the transfer of resources from savers to investors. And as a result, investment, capital stock and output are all higher. The liquidity premium is lower, as is the price of capital.¹⁹

¹⁹As an alternative to retiring money, the government could pay interest on money. These two policies would be entirely equivalent.

A simple way to understand this is that, by holding capital and retiring money, the government is acting like a banker to the entrepreneurs. The government is converting a partially-liquid stream of income from capital into a fully-liquid stream of income (dividends) on money. Being more liquid, the latter income stream is a more effective instrument for transferring resources among entrepreneurs.

If instead the government uses the income stream from its capital holding to make transfers to the workers (case (ii)), a partially-liquid stream is converted into a nontradeable stream (workers cannot borrow against their future income). The group of entrepreneurs are deprived of an income stream which, although only partially liquid, would otherwise help to lubricate their resource allocation. As a result, the liquidity premium is higher -- as are the prices of capital and money. Investment, capital stock and output are all lower.

For completeness, we should consider the third polar case, where the government doesn't buy any capital, but prints money to finance transfers to workers:

Claim 4(iii) In a steady state with a government that increases the money supply to finance transfers ($\mu > 1$ and $G > 0$), and holds no capital ($K^G = 0$), relative to an inactive government:

total activity, measured by $K^P + K^G$, is lower,
the price of capital q is higher, and
the value of the money stock $p_t M_t$ is lower.

This policy is the worst of both worlds: a combination of the contractionary policy in Claim 4(ii) with the opposite of the expansionary policy in Claim 4(i). The government is creating a nontradeable income stream (transfers given to workers) by depriving entrepreneurs of a liquid income stream (reducing the return on money). Because aggregate output is lower, workers may have lower consumption, despite of a larger share of income.

Finally, before we move on to consider dynamics, it may be worth noting

that if a proportional "helicopter drop" of money were feasible -- anyone holding money at date t is given $x\%$ more -- then it would be neutral. That is, p_t, p_{t+1}, \dots would fall by $x\%$ to keep the value of the money stock the same, but nothing else would change. However, our framework does not allow for helicopter drops. Changes to the money supply occur only through open market operations or as a result of transfers to the workers -- always satisfying the government's flow-of-funds constraint (22).

4. Dynamics

To study the dynamics of the system (22)-(27), we find it easiest to take the continuous-time limit. In so doing, we also obtain a useful expression for the size of the liquidity premium (Proposition 5 below).

Specifically, take the period length to be Δ instead of 1, and consider the limit $\Delta \rightarrow 0$. Let $\delta > 0$ and $\rho > 0$ be the flow rates of depreciation and subjective discounting. And with a slight change in notation, let $\pi > 0$ now denote the flow arrival rate of investment opportunities. (That is, in a discrete time model with period length Δ , our earlier parameters λ , β and π would equal $e^{-\delta\Delta}$, $e^{-\rho\Delta}$ and $1 - e^{-\pi\Delta}$ respectively.)

Equation (23) is unchanged, except that the productivity parameter, a_t , is now measured per unit of time. We assume that a_t is a Markov process with two states, $a(1 + \Delta_a)$ and $a(1 - \Delta_a)$, where the arrival rate of a switch equals $\eta > 0$ and $\Delta_a \geq 0$ is small.

Let us suppose that government's holding of capital stock K_t^g also follows an exogenous Markov process with two states, $K^g(1 + \Delta_{kg})$ and $K^g(1 - \Delta_{kg})$, where the arrival rate of a switch equals $\eta_{kg} > 0$ and $\Delta_{kg} \geq 0$ is small. In continuous time, the government flow-of-funds constraint (22) becomes

$$(29) \quad G_t dt + q_t dK_t^g = (r_t - \delta q_t) K_t^g dt + p_t dM_t,$$

where $dx_t \equiv x_{t+dt} - x_t$: the change of variable x_t in an infinitesimal time interval $[t, t+dt]$. To concentrate on monetary policy, we will hold government transfers G_t constant at G . For simplicity, let us choose G so that in steady state the money supply M_t satisfying (29) is constant.²⁰ When the government

²⁰In steady state, $\Delta_a = \Delta_{kg} = 0$. The particular level of G is $(r - \delta)K$, where r and K are the steady-state values of r_t and K_t .

conducts an open market operation, when K_t^g jumps by dK_t^g , the money supply M_t also jumps, by $dM_t = q_t dK_t^g / p_t$. (q_t and p_t will also jump. The open market operation takes place at the new prices.) Otherwise, M_t moves continuously.

The continuous-time counterparts to (24)-(27) are

$$(30) \quad (1 - \theta q_t) I_t = \pi \left(\theta q_t K_t^P + p_t M_t \right)$$

$$(31) \quad r_t K_t = I_t + G_t + \rho \left(q_t K_t^P + p_t M_t \right)$$

$$(32) \quad \frac{r_t}{q_t} - \delta + \lim_{dt \rightarrow 0} \frac{1}{dt} E_t \left\{ \left(\frac{q_{t+dt}}{q_t} - \frac{p_{t+dt}}{p_t} \right) \frac{q_t K_t^P + p_t M_t}{q_{t+dt} K_t^P + p_{t+dt} M_t} \right\}$$

$$= \pi \left(\frac{q_t - 1}{q_t} \right) \frac{q_t K_t^P + p_t M_t}{K_t^P + p_t M_t},$$

$$(33) \quad \dot{K}_t = I_t - \delta K_t,$$

where

$$(34) \quad K_t = K_t^P + K_t^g.$$

The LHS of (32) is the liquidity premium adjusted for risk aversion. (The expectations operator is with respect to the underlying stochastic processes a_t and K_t^g .) From (32), it therefore follows that, provided q_t is not too far from 1:

Claim 5 In the neighbourhood of steady state,

$$\text{risk-adjusted liquidity premium} \cong \pi(q_t - 1).$$

The size of the liquidity premium can be significant.

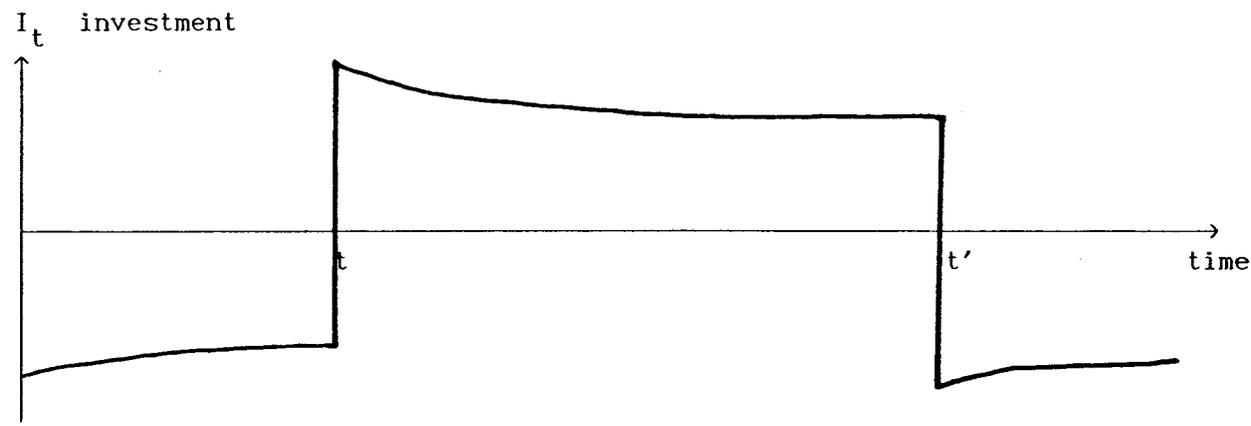
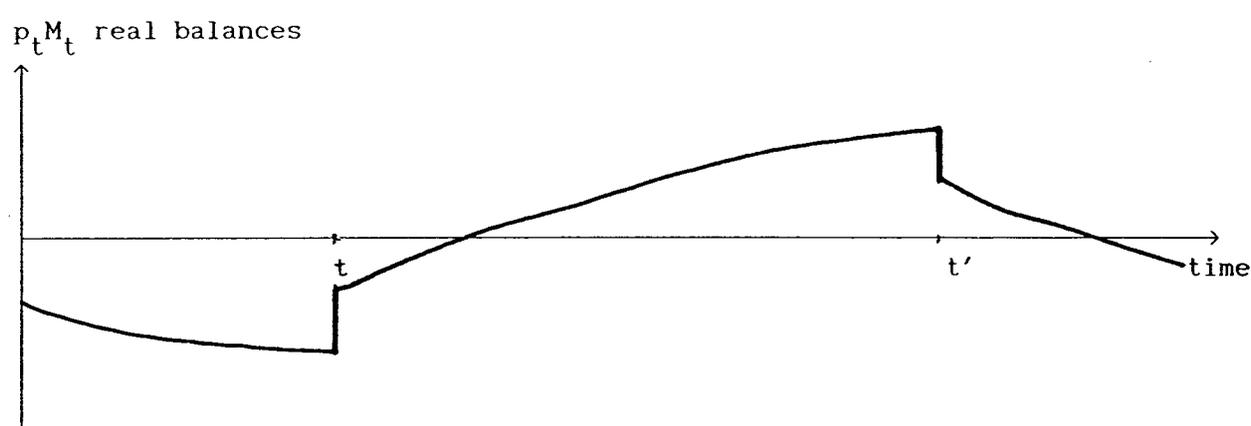
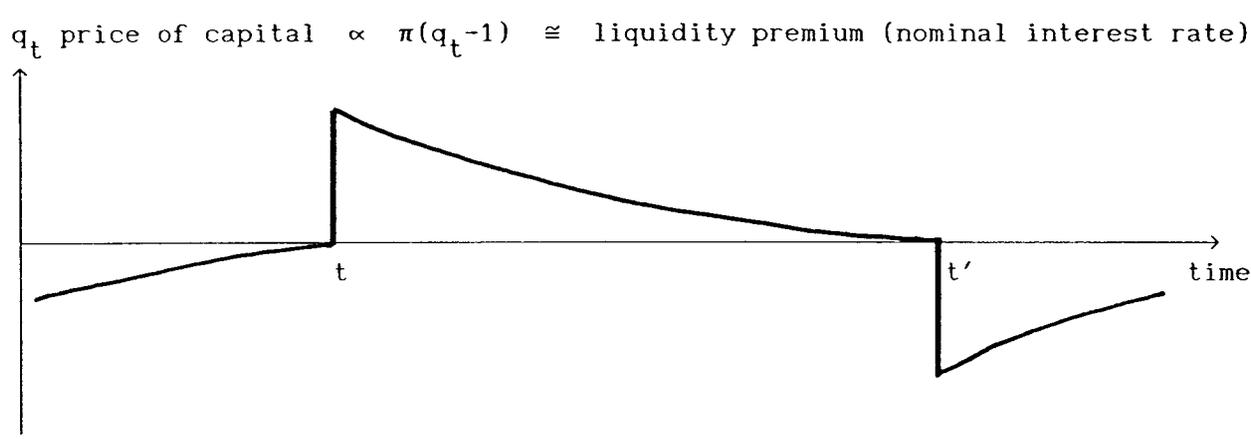
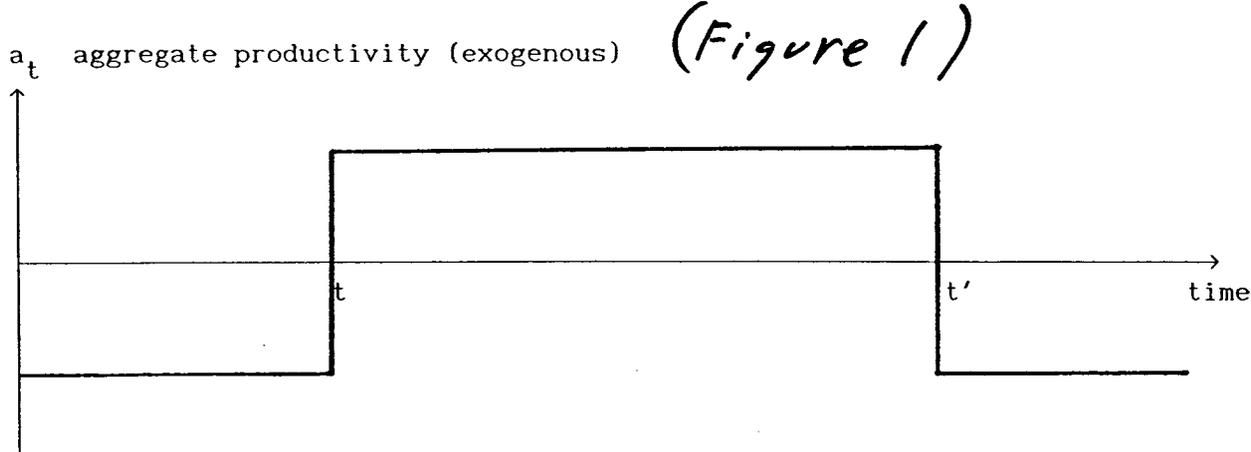
We end by considering two special cases, so as to understand better how the dynamic system behaves: (1) pure productivity shocks; and (2) pure monetary shocks.

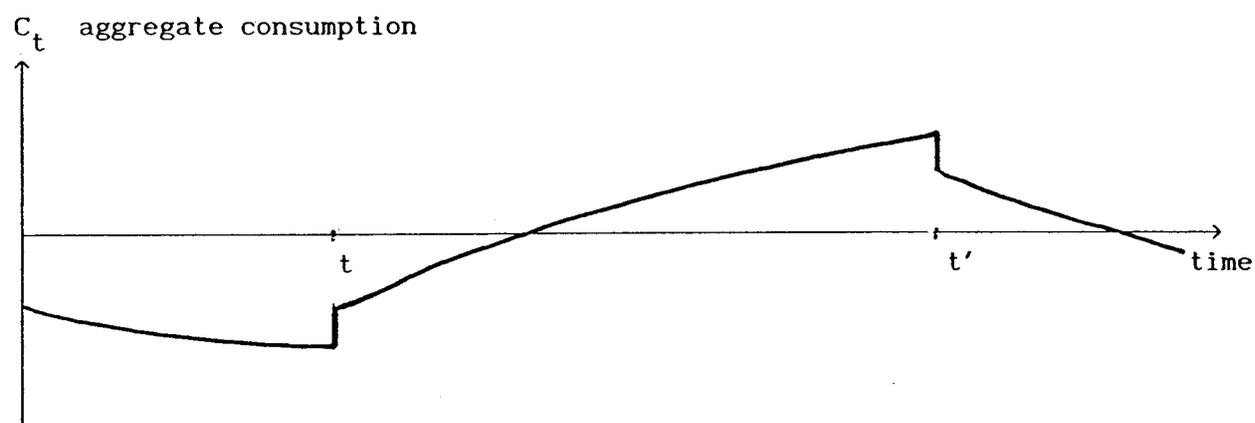
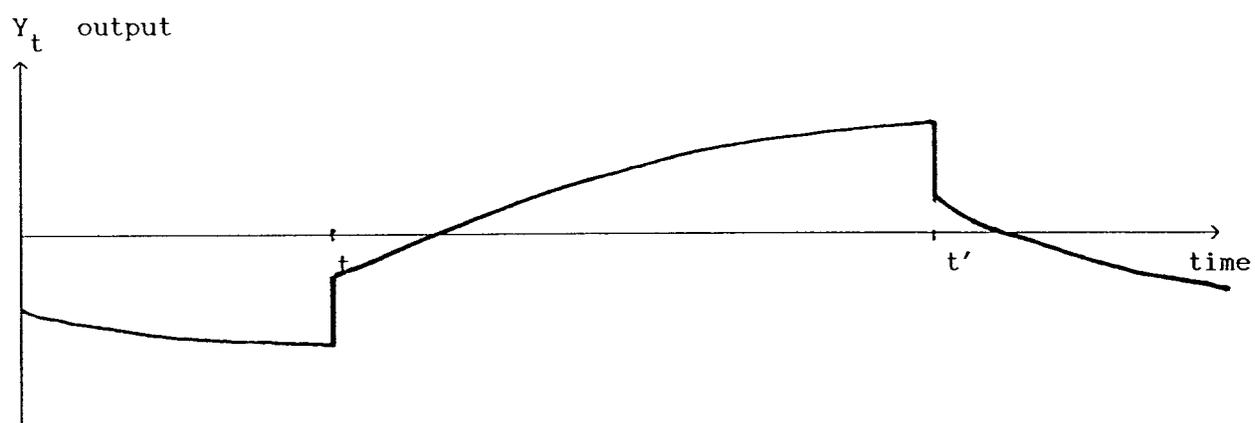
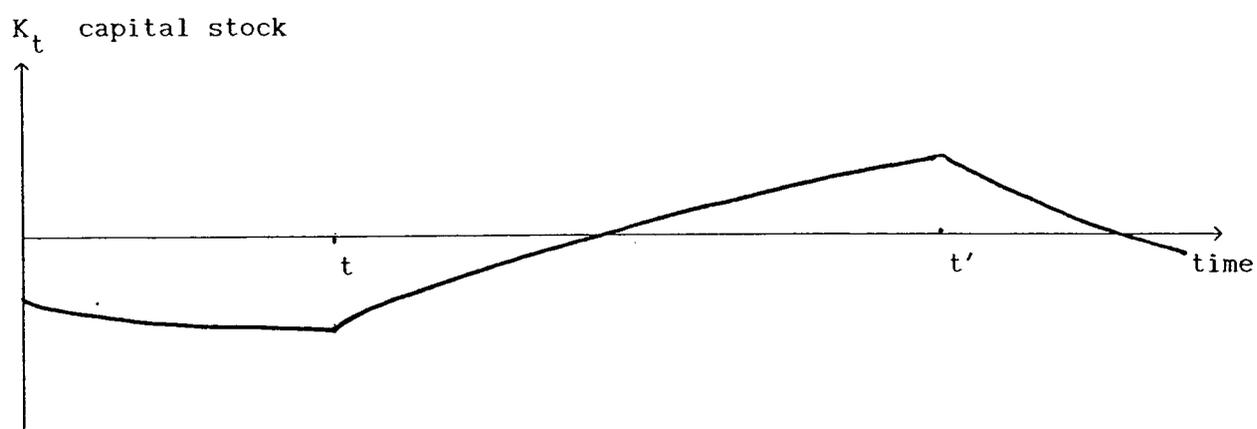
4.1 Productivity shocks

The graphs that appear on the following two pages show how the economy reacts to productivity shocks a_t . Monetary policy is held constant ($\Delta_{kg} = 0$). The arrival of a productivity change is rare and discontinuous, so when productivity jumps up -- suppose it does at time t , say -- asset prices and investment also jump. Because the return on capital increases with productivity, the price of capital jumps up. So too does the liquidity premium (by Claim 5). Anticipating a tight liquidity constraint in the future, entrepreneurs without an investment opportunity want to hold more liquid asset, which leads to a jump up in real balances.

After time t , capital stock starts accumulating with the greater investment. Aggregate output, which rose instantaneously with the jump in productivity at t , continues to rise with capital accumulation. The return on capital falls (with the higher wage), and the price of capital falls back towards normal levels, as does the liquidity premium. The value of money continues to increase as the economy expands. At time t' , say, productivity jumps back down, and these processes reverse: the price of capital jumps down, the liquidity constraint loosens. Investment and real balances jump down too, and the stock of capital starts to fall.

Overall, we conclude that if productivity shocks are driving the fluctuations then the price of capital, the liquidity premium, and real





balances are all procyclical, moving together with output. Investment is procyclical and quite volatile, because it is affected by the net worth of the investing entrepreneurs and the required downpayment, which means that the movements in the prices of capital and money combine to magnify the fluctuation. Consumption is also procyclical, given that the consumption of workers is equal to their wage income and the consumption of the entrepreneurs is proportional to their net worth.

4.2 Monetary policy shocks

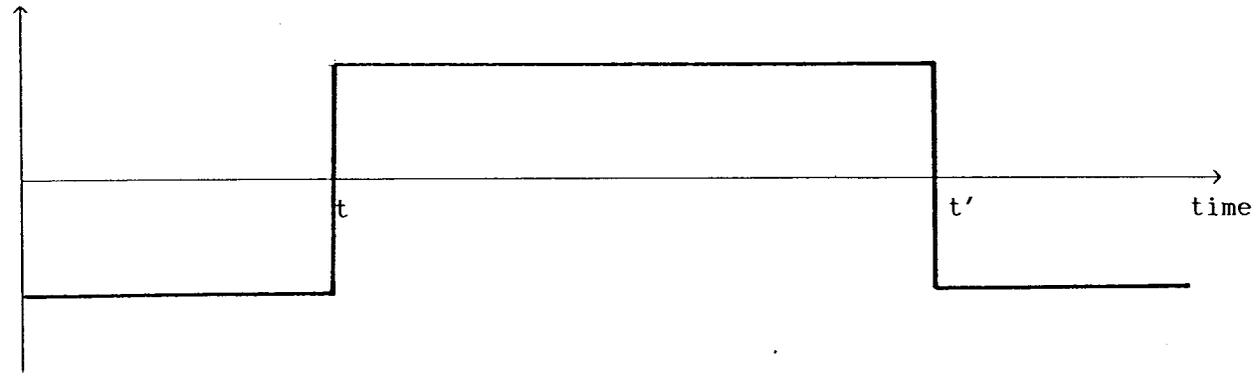
The graphs on the final two pages show how the economy reacts to monetary policy shocks. Productivity is held constant ($\Delta_a = 0$). Here, the government occasionally undertakes an open market operation to change its holding of capital, by changing the supply of money. Between times, the government balances its flow of funds by changing the money supply smoothly (keeping transfers to the workers constant). Suppose the government purchases capital stock at time t by issuing money. Thereafter, it is expected to use the additional income from its increased capital holding to retire money, so the price of money is expected to rise over time. As we argued in Section 2, this is like paying a dividend on money. Anticipating the higher return, the entrepreneurs want to hold more money at time t : real balances jump up. The direction of jump in the price of money is ambiguous, because the demand for real balances may or may not increase as much as the money supply. With larger real balances, the liquidity constraint is looser, the liquidity premium and the price of capital jump down, and investment jumps up.

After time t , capital stock starts accumulating, and output rises. Real balances and the price of money also rise. The price of capital falls, because the return is falling (with the higher wage), and so too does the liquidity premium. The expansion continues until time t' , when the government reverses the open market operation.

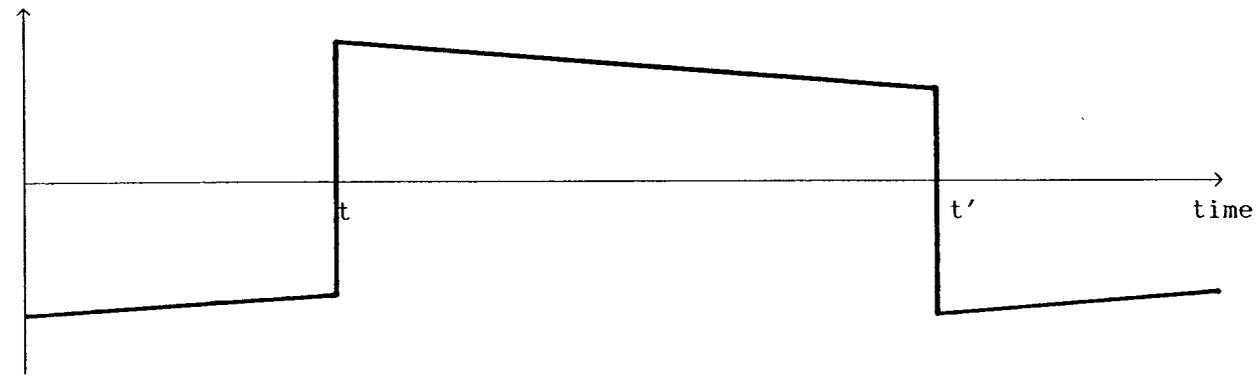
Overall, when the government uses additional capital income to retire money, open market operations lead to persistent expansion in investment and output. The liquidity premium and the price of capital are counter-cyclical, whereas real balances are procyclical.

(Figure 2)

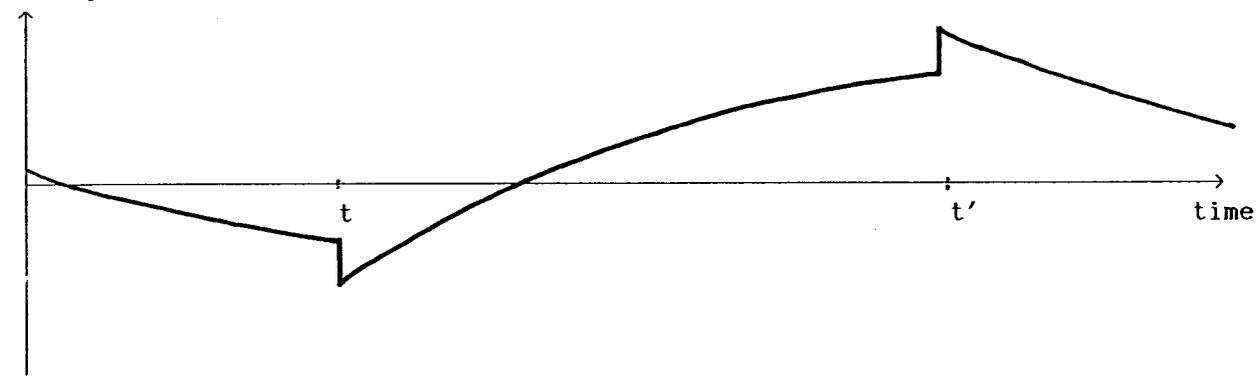
K_t^g government capital holding (exogenous)



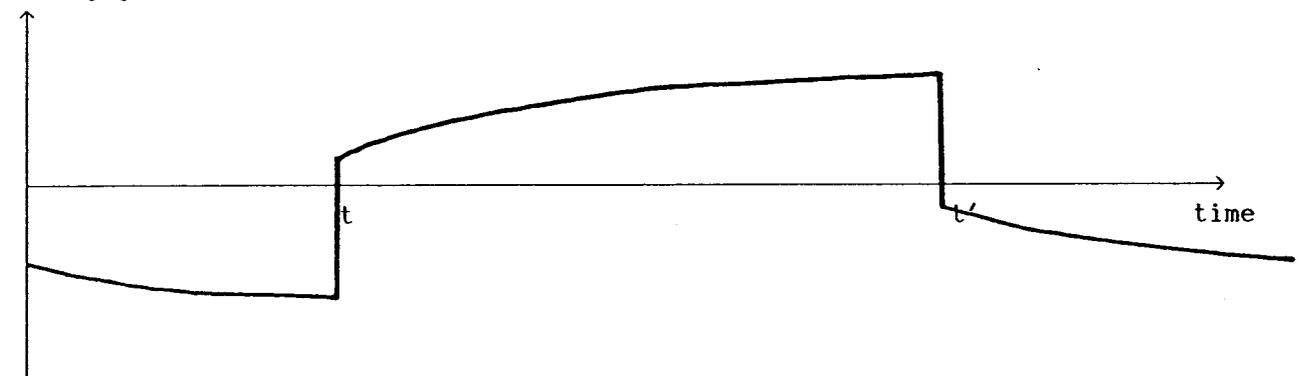
$\ln M_t$ money supply



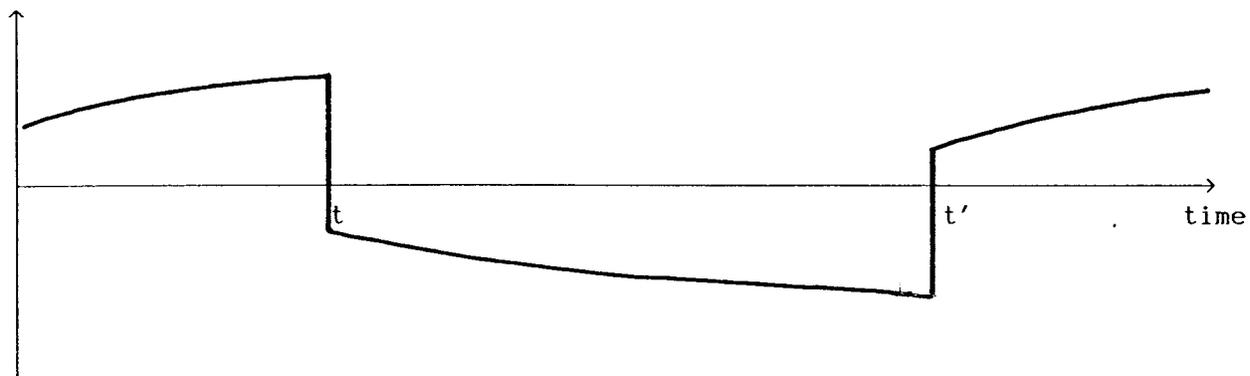
$\ln p_t$ price of money



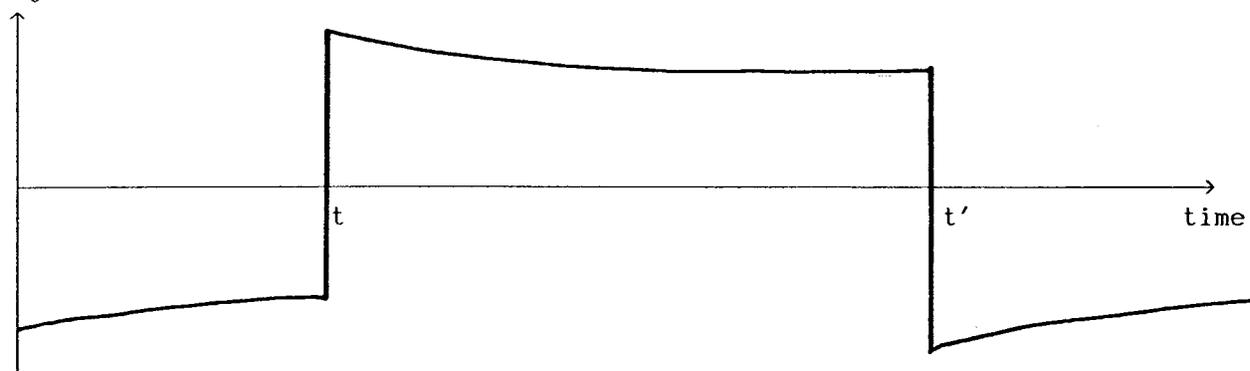
$\ln(p_t M_t)$ real money balances



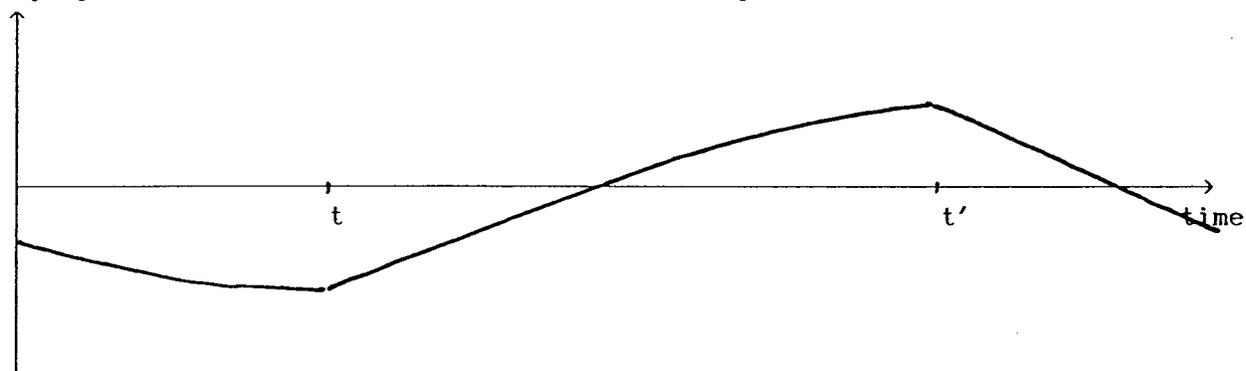
q_t price of capital $\propto \pi(q_t - 1) \cong$ liquidity premium (nominal interest rate)



I_t investemnt



$K_t + K_t^g$ aggregate capital holding \propto output Y_t



Appendix 1: Proof of Claims

Proof of Claim 1: Define $r_t \equiv a_t K_t^{\alpha-1}$: output-capital ratio, and $h_t \equiv p_t M_t / K_t$: the ratio of real balance to capital. From (19,20,21), we have:

$$(A1) \quad \pi\beta r = 1 - \lambda + \lambda\pi(1 - \pi) - \theta'q - \pi\beta h,$$

$$(A2) \quad \beta r = 1 - \lambda + \lambda\pi(1 - \beta) + (1 - \beta)(1 - \pi)\lambda q + (1 - \beta)h,$$

$$(A3) \quad \frac{r}{q} + \lambda - 1 = \pi\lambda \frac{q-1}{q} \frac{q + h/\nu}{r + \lambda + h/\nu},$$

where $\theta' \equiv (1 - \lambda + \lambda\pi)\theta$ and $\nu \equiv K^N / K = \lambda(1 - \pi) + \theta'$. Then we have,

$$(A4) \quad r(q) = \frac{1}{\pi\beta} \left((1 - \beta + \pi\beta)[1 - \lambda + \lambda\pi(1 - \beta)] + (1 - \beta)[\lambda\pi\beta(1 - \pi) - \theta']q \right),$$

$$(A5) \quad h(q) = \frac{1}{\pi} \left((1 - \pi)[1 - \lambda + \lambda\pi(1 - \beta)] - [\lambda\pi(1 - \beta)(1 - \pi) + \theta']q \right),$$

$$(A6) \quad F(q; \theta') \equiv 1 - \lambda - \frac{r(q)}{q} + \pi\lambda \frac{q-1}{q} \frac{q + h(q)/\nu}{r(q) + \lambda + h(q)/\nu} = 0.$$

We know $r'(q) > 0$ if $\lambda\pi\beta(1 - \pi) > \theta'$, and $h'(q) < 0$.

Proof of Claim 1(ii): From (A6), we see F is an increasing function of q , at least for $q > 1$. Since $F(1; \theta') < 0$, we learn there is a unique $q > 1$, which satisfies the equilibrium condition in the steady state, $F(q; \theta') = 0$.
Q.E.D.

Proof of Claim 1(i): From (A5), we know $h = pM/K > 0$, if and only if

$$q < \frac{(1 - \pi)[1 - \lambda + \lambda\pi(1 - \beta)]}{\theta' + \lambda\pi(1 - \pi)(1 - \beta)} \equiv \bar{q}.$$

Since F is an increasing function of q , this is equivalent to:

$$\begin{aligned} 0 < F(\bar{q}; \theta') &= 1 - \lambda - r(\bar{q})/\bar{q} + \pi\lambda \frac{\bar{q} - 1}{r(\bar{q}) + \lambda} \\ &= 1 - \lambda - \frac{\lambda(1 - \pi)(1 - \beta) + \theta'}{\beta(1 - \pi)} + \pi\lambda\beta \frac{(1 - \pi)(1 - \lambda) - \theta'}{[1 - \lambda + \lambda\pi + \lambda\beta(1 - \pi)]\theta' + \lambda(1 - \pi)(1 - \beta)(1 - \lambda + \lambda\pi)}. \end{aligned}$$

This is equivalent to:

$$(A7) \quad 0 < \lambda\beta^2\pi(1-\pi)\left((1-\lambda)(1-\pi) - \theta'\right) \\ + [(\beta-\lambda)(1-\pi)-\theta']\left[\lambda(1-\pi)(1-\beta)(1-\lambda+\lambda\pi) + \theta'[1-\lambda+\lambda\pi+\lambda\beta(1-\pi)]\right] \equiv f(\theta').$$

$f(\theta')$ is a quadratic function of θ' with a negative term of θ'^2 with $f(0) > 0$ from Assumption 1. Therefore, (A7) holds, or $p > 0$, if and only if Assumption 2 is satisfied. *Q.E.D.*

Proof of Proposition 1(iii): We know $m_{t+1}^i = 0$, if and only if the marginal utility of consumption (or marginal cost of saving) exceeds the marginal benefit of saving in for the investing agent at date t :

$$(A8) \quad \frac{1}{c_t^i} > \frac{1}{p_t} \beta E_t \left(\pi \frac{p_{t+1}}{c_{t+1}^{ii}} + (1-\pi) \frac{p_{t+1}}{c_{t+1}^{in}} \right),$$

where c_{t+1}^{ii} is consumption of the agent who invests at both dates t and $t+1$, and c_{t+1}^{in} is consumption of the agent who invests at date t but not at $t+1$.

From (9,10,13), we know that:

$$c_t^i = (1-\beta)b_t^i, \\ c_{t+1}^{ii} = (1-\beta)(r_{t+1}+\lambda)k_{t+1}^i = (1-\beta)(r_{t+1}+\lambda)\frac{1-\theta}{1-\theta q_t} \beta b_t^i, \\ c_{t+1}^{in} = (1-\beta)(r_{t+1}+\lambda q_{t+1})k_{t+1}^i = (1-\beta)(r_{t+1}+\lambda q_{t+1})\frac{1-\theta}{1-\theta q_t} \beta b_t^i.$$

where $b_t^i = r_t k_t + \lambda k_t + p_t m_t$. Thus, the inequality (A8) in the steady state is equivalent to:

$$1 > \frac{1-\theta q}{1-\theta} \left(\frac{\pi}{r+\lambda} + \frac{1-\pi}{r+\lambda q} \right).$$

Or

$$(A9) \quad \frac{1-\theta q}{(1-\theta)q} \left(1 + \pi \lambda \frac{q-1}{r+\lambda} \right) > \frac{r+\lambda q}{q}$$

$$= 1 + \pi\lambda \frac{q-1}{q} \frac{q + h/\nu}{r + \lambda + h/\nu}, \quad (\text{from (A3)}).$$

Multiply both sides by $(1-\theta)q$, (A9) is equivalent to:

$$q-1 > \pi\lambda(q-1) \left(\frac{1-\theta q}{r+\lambda} - (1-\theta) \frac{q + h/\nu}{r + \lambda + h/\nu} \right),$$

or,

$$r+\lambda > \pi\lambda \left(1-\theta q - (1-\theta) \frac{(r+\lambda)(q+h/\nu)}{r + \lambda + h/\nu} \right).$$

But this is always true because $\pi < 1$. *Q.E.D.*

Proof of Claim 2: We have

$$\begin{aligned} \text{LHS of (A3)} &= \left(r + \lambda - \frac{1}{\beta} \right) \left(\frac{1}{q} + \frac{1}{h} \right) - \left(\frac{1}{\beta} - \lambda \right) \left(1 - \frac{1}{q} \right) - \left(r + \lambda - \frac{1}{\beta} \right) \frac{1}{h} + \frac{1-\beta}{\beta} \\ &= \left(r + \lambda - \frac{1}{\beta} \right) \left(\frac{1}{q} + \frac{1}{h} \right) - \left(\frac{1}{\beta} - \lambda \right) \frac{q-1}{q} - \frac{1}{\beta h} \lambda(1-\beta)(1-\pi)(q-1), \quad (\text{from (A2)}). \end{aligned}$$

Since RHS of (A3) > 0 . Together with $q > 1$, we learn $r + \lambda - \frac{1}{\beta} > 0$. *Q.E.D.*

Proof of Claim 3: From $q > 1$, we learn $\frac{r}{q} + \lambda > 1$ from (A3). Thus the expected rate of return on capital dominates money $E_t[(r_{t+1} + \lambda q_{t+1})/q_t] > E_t(p_{t+1}/p_t)$ in the neighborhood of the steady state. Also we have:

$$\begin{aligned} \text{LHS of (A3)} &= \left(\frac{r}{q} + \lambda - \frac{1}{\beta} \right) \left(1 + \frac{q}{h} \right) + \frac{1}{\beta} - 1 - \frac{1}{h} \left(r + q \left(\lambda - \frac{1}{\beta} \right) \right) \\ &= \left(\frac{r}{q} + \lambda - \frac{1}{\beta} \right) \left(1 + \frac{q}{h} \right) + \frac{q-1}{\beta h} [1 - \lambda + \lambda\pi(1-\beta)], \quad (\text{from (A2)}). \end{aligned}$$

Since we already proved $(r+\lambda q)/q > 1$, we know $(r+\lambda)/q < 1$ from (17) in the steady state. Also because $K^n < K$, we learn:

$$\text{RHS of (A3)} < \pi\lambda \frac{q-1}{q} \frac{q + h}{r+\lambda+h}.$$

Therefore, the sufficient condition for $(r+\lambda q)/q < 1/\beta$ is:

$$\pi\lambda \frac{q+h}{r+\lambda+h} < q \frac{1-\lambda+\lambda\pi(1-\beta)}{\beta h},$$

Using (A2), this is equivalent to:

$$\begin{aligned} \text{(A10)} \quad \frac{\lambda\beta\pi[1+(h/q)]}{1-\lambda+\lambda\pi(1-\beta)} &< 1 + \frac{\lambda}{h} + \frac{1}{h} \left(\frac{1-\lambda+\lambda\pi(1-\beta)}{\beta} + \frac{(1-\beta)(1-\pi)\lambda q}{\beta} \right) + \frac{1-\beta}{\beta}, \\ &= \frac{1}{\beta} + \frac{1}{\beta h} + (q-1) \frac{(1-\beta)(1-\pi)\lambda}{\beta h}. \end{aligned}$$

From (A5), we know:

$$h(q) < h(1) = \frac{1}{\pi} \left((1-\pi)(1-\lambda) - \theta' \right) < \frac{(1-\pi)(1-\lambda)}{\pi}.$$

Then

$$\text{RHS of (A10)} > \frac{1}{\beta} \left(1 + \frac{\pi}{(1-\pi)(1-\lambda)} \right) = \frac{1-\lambda+\lambda\pi}{\beta(1-\pi)(1-\lambda)}.$$

$$\text{LHS of (A10)} < \frac{\beta\lambda(1-\lambda+\lambda\pi)}{1-\lambda+\lambda\pi(1-\beta)}.$$

But because $\beta^2\lambda(1-\pi)(1-\lambda) < 1-\lambda+\lambda\pi(1-\beta)$, we learn LHS of (A10) < RHS of (A10), which implies $(r+\lambda q)/q < 1/\beta$. *Q.E.D.*

Proof of Claim 4: Here, we prove the claim in the continuous time version of the model. By continuity, the claims hold for the discrete time model, when the length of the period is sufficiently short so that the discrete time model is approximated well by the continuous time model. Define $\kappa_t \equiv K_t^P/K_t$: fraction of capital held by the private agents, $g_t \equiv G_t/K_t$: the ratio of the government transfer to capital, and $\tau_t \equiv \dot{M}_t/M_t$: the growth rate of money supply (or, the rate of inflation tax). Consider the steady state equilibrium of the continuous time model with the government (29-34). In the following, we analyze the economy with small government:

$$\text{(A11)} \quad \kappa \cong 1, \quad g \cong 0, \quad \text{and} \quad \tau \cong 0.$$

Then the steady state equilibrium (29-33) can be written as:

$$(A12) \quad g = (r - \delta q)(1 - \kappa) + \tau h,$$

$$(A13) \quad (1 - \theta q)\delta = \pi(\theta q \kappa + h),$$

$$(A14) \quad r = \delta + g + \rho(q\kappa + h)$$

$$(A15) \quad \frac{r}{q} - \delta + \tau = \pi \frac{q-1}{q} \frac{q\kappa + h}{\kappa + h},$$

and $i = \delta$. (A12) is the budget constraint of the government, (A13) is the flow of funds constraint of the investing agents, (A14) is goods market clearing condition, and (A15) is the asset market clearing condition. From (A13), we have:

$$(A16) \quad \pi h = \delta - \theta(\delta + \pi \kappa)q.$$

Substituting this into (A14) and (A15), we have:

$$(A17) \quad r = \frac{1}{\pi} \left((\pi + \rho)\delta + \pi g + [(1 - \theta)\pi \kappa - \theta \delta] \rho q \right),$$

$$(A18) \quad \frac{r}{q} - \delta + \tau = \pi \frac{q-1}{q} \frac{\delta + [(1 - \theta)\pi \kappa - \theta \delta]q}{(\delta + \pi \kappa)(1 - \theta q)}.$$

Proof of Claim 4(i): Since $g = 0$, we have from (A12) as:

$$(A19) \quad -\tau = (r - \delta q) \frac{1 - \kappa}{h} = (r - \delta q) \frac{\pi(1 - \kappa)}{\delta - \theta(\delta + \pi \kappa)q}, \quad (\text{from (A16)}).$$

Substituting this into (A18), we have:

$$(A20) \quad 0 = \pi \frac{q-1}{q} \frac{\delta + [(1 - \theta)\pi \kappa - \theta \delta]q}{(\delta + \pi \kappa)(1 - \theta q)} - \left(\frac{r}{q} - \delta \right) \left(1 - \frac{\pi(1 - \kappa)q}{\delta - \theta(\delta + \pi \kappa)q} \right) \equiv \psi(q, r, \kappa).$$

From (A20), we learn $\partial \psi / \partial q > 0$, $\partial \psi / \partial r < 0$, and $\partial \psi / \partial \kappa < 0$ with small government. The goods market clearing condition is (A17) with $g = 0$.

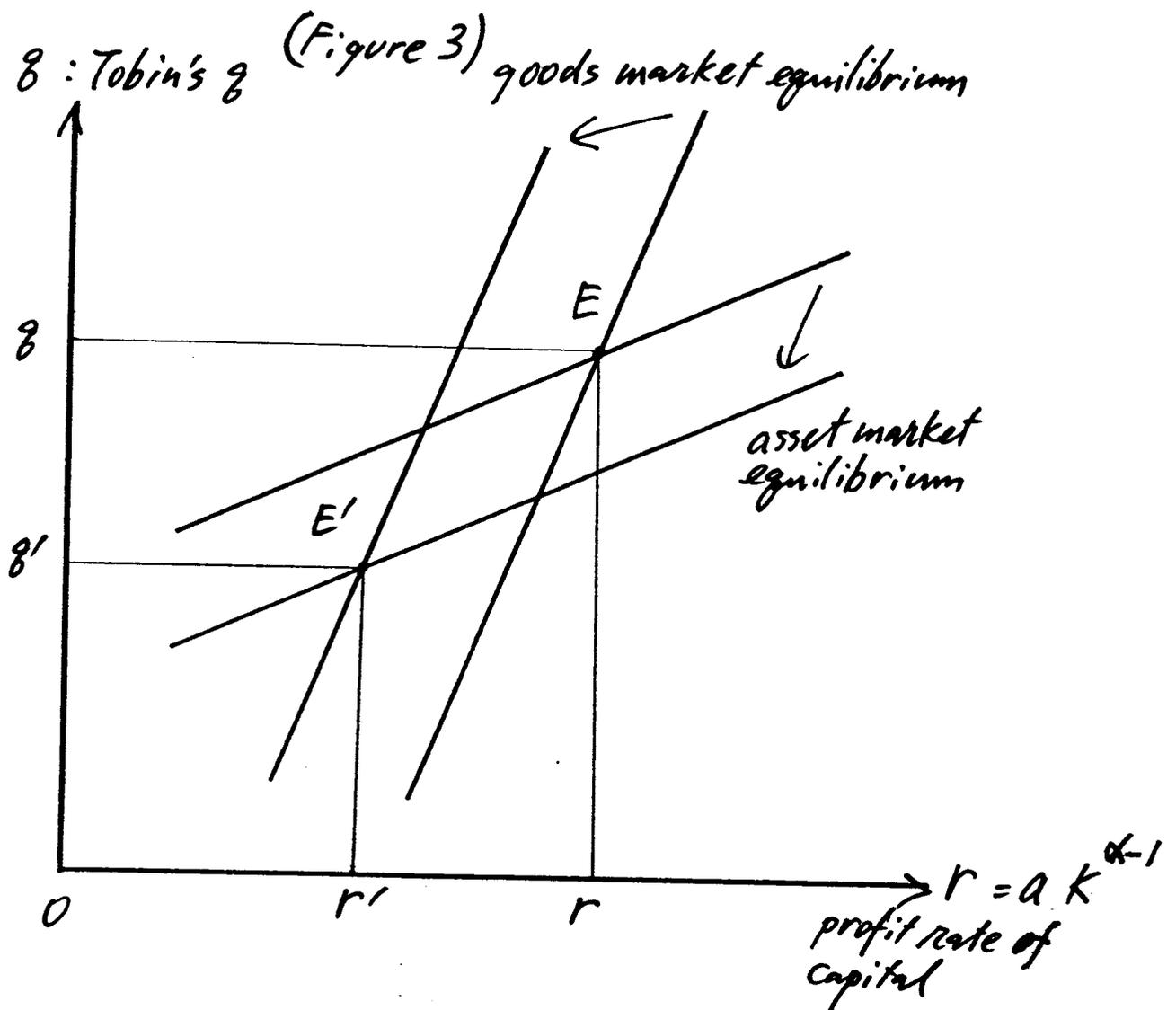
Assumption 1 in the text in continuous time is:

$$(A21) \quad \pi > \delta > \rho.$$

Then in the monetary equilibrium with small government, we have

$$(A22) \quad (1-\theta)\pi\kappa - \theta\delta > \delta - \theta(\pi\kappa + \delta) > 0,$$

from (A16) with $h > 0$ and $q > 1$. Thus in (A17), r is an increasing function of q . Figure 3 describes the equilibrium of the goods market (A17) and the assets market (A20). Both schedules are upward sloping. Also, from (A17) and (A20), we learn that $\psi(q, r, \kappa) > 0$ when the economy moves up along the goods market equilibrium, and thus the goods market clearing schedule is steeper than the asset market clearing schedule.



When the government purchases capital by issuing money in the open market, and will use the revenue from the capital to retire money in future, then the fraction of the private holding of capital κ decreases with a lower inflation tax rate. Then, the goods market clearing condition shifts to the left from (A17), and the asset market clearing condition shifts down from (A20). Thus, at the new equilibrium, Tobin's q is lower (the liquidity constraint is loosen), the profit rate of capital is lower, and capital stock is larger. Also, from (A16), we learn the real balance is larger because the ratio of real balance to capital stock is higher as well as capital stock is larger. *Q.E.D.*

Proof of Claim 4(ii): When there is no change in money supply, $\tau = 0$,

$$(A23) \quad g = (r - \delta q)(1 - \kappa),$$

from the budget constraint of the government (A12). Then from (A17) and (A18), the market clearing conditions are:

$$(A24) \quad r = \frac{1}{\pi\kappa} \left\{ (\pi + \rho)\delta + \left[\rho[(1 - \theta)\pi\kappa - \theta\delta] - \delta(1 - \kappa) \right] q \right\} \equiv \tilde{r}(q, \kappa),$$

$$(A25) \quad 0 = \pi \frac{q-1}{q} \frac{\delta + [(1 - \theta)\pi\kappa - \theta\delta]q}{(\delta + \pi\kappa)(1 - \theta q)} - \left(\frac{r}{q} - \delta \right) \equiv \tilde{\psi}(q, r, \kappa).$$

From (A17), we know $\partial\tilde{r}/\partial q > 0$ and $\partial\tilde{r}/\partial\kappa > 0$. Also we learn from (A25), we learn that $\partial\tilde{\psi}/\partial q > 0$, $\partial\tilde{\psi}/\partial r < 0$, and $\partial\tilde{\psi}/\partial\kappa > 0$. Thus when the government purchase capital and will use the revenue to transfer to the workers in lump sum, then the schedules of market clearing shift as in Figure 4. The equilibrium moves from E to \tilde{E} . Tobin's q is higher (the liquidity constraint tightens), the profit rate of capital is higher, and the capital stock is smaller in the new steady state. Also, from (A16), we know the real balance is smaller. *Q.E.D.*

Proof of Proposition 4(iii): When the government finance lump-sum transfer to the workers by inflation tax, we have $\kappa = 1$ and:

$$(A26) \quad \tau = g/h,$$

$$(A27) \quad h = \frac{1}{\pi} \left[\delta - \theta(\delta + \pi)q \right],$$

from (A12) and (A13). Substituting these into (A17) and (A18), the equilibrium conditions are:

$$(A28) \quad r = \frac{1}{\pi} \left\{ (\pi + \rho)\delta + \pi g + [(1 - \theta)\pi - \theta\delta]\rho q \right\} \equiv \bar{r}(q, g),$$

$$(A29) \quad 0 = \pi \frac{q-1}{q} \frac{\delta + [(1 - \theta)\pi - \theta\delta]q}{(\delta + \pi)(1 - \theta q)} - \left(\frac{r}{q} - \delta \right) - \frac{\pi g}{\delta - \theta(\pi + \delta)q} \equiv \bar{\psi}(q, r, g).$$

From these, we learn $\partial \bar{r} / \partial q > 0$, $\partial \bar{r} / \partial g > 0$, $\partial \bar{\psi} / \partial q > 0$, $\partial \bar{\psi} / \partial r < 0$, and $\partial \bar{\psi} / \partial g < 0$. Thus after this open market operation, the equilibrium shifts from E to \bar{E} in Figure 5. In the new equilibrium, Tobin's q is higher, the profit rate of capital is higher, capital stock is smaller, and the real balance is smaller. Q.E.D.

Figure 4

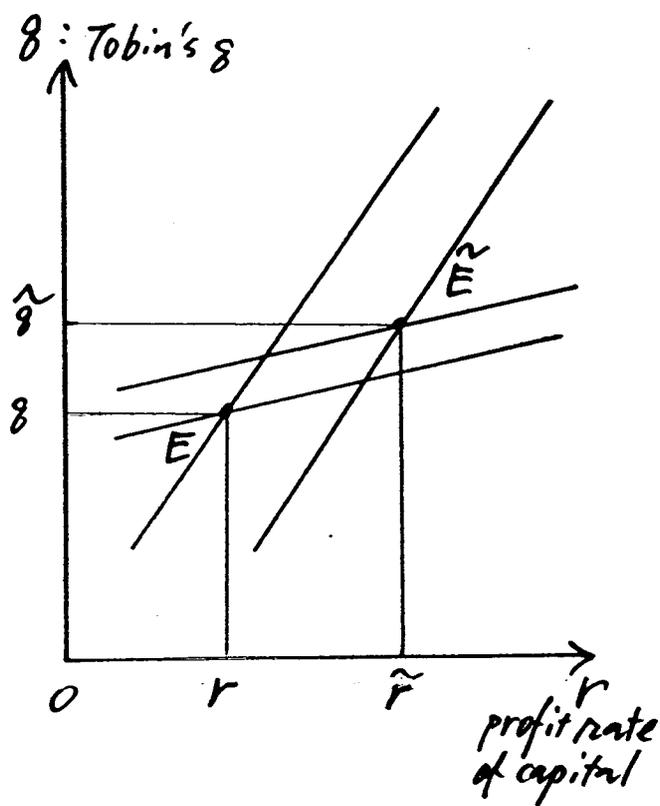
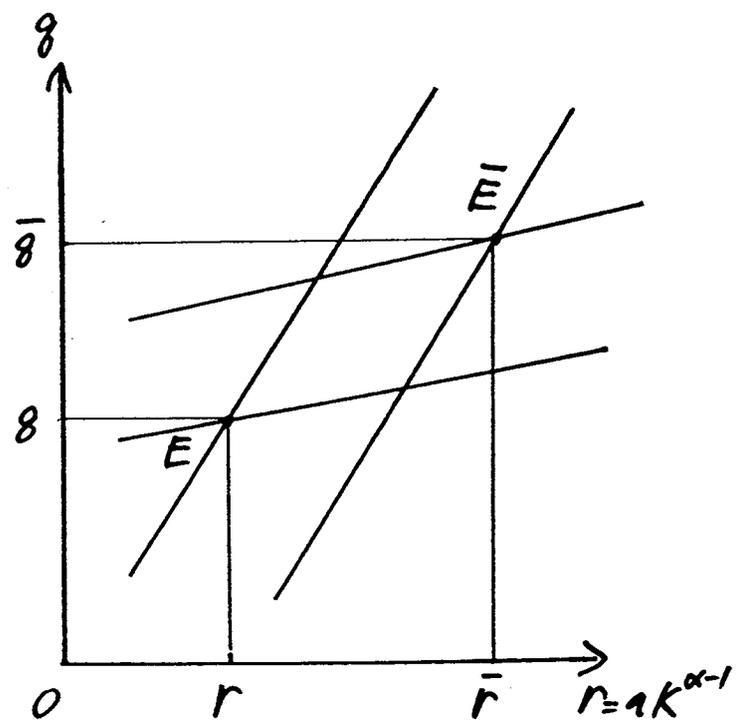


Figure 5



Appendix 2: Derivation of Local Dynamics:

Here, we only consider the open market operation in which the government adjusts the inflation tax with the change in capital income without any change in transfer ($G_t=0$) as in Claim 4(i). We also consider in the neighborhood of the small government, (A11). Define $i_t \equiv I_t/K_t$: investment rate, and $n_t \equiv M_t/K_t$: ratio of nominal money supply to capital. Then, we can express the equilibrium conditions (30-33) by scaling by capital stock as:

$$(A30) \quad (1-\theta q_t)i_t = \pi(\theta q_t \kappa_t + p_t n_t),$$

$$(A31) \quad r_t = i_t + \rho(q_t \kappa_t + p_t n_t)$$

$$(A32) \quad \frac{r_t}{q_t} - \delta + \lim_{t \rightarrow 0} E_t \left(\frac{1}{dt} \left(\frac{q_{t+dt}}{q_t} - \frac{p_{t+dt}}{p_t} \right) \frac{q_t \kappa_t + p_t n_t}{q_{t+dt} \kappa_t + p_{t+dt} n_t} \right) = \pi \frac{q_t^{-1} q_t \kappa_t + p_t n_t}{\kappa_t + p_t n_t}.$$

$$(A33) \quad \frac{dr_t}{r_t} = \frac{da_t}{a_t} - (1-\alpha)(i_t - \delta)dt,$$

where $dx_t \equiv x_{t+dt} - x_t$. In deriving (A33), we use $r_t = a_t K_t^{\alpha-1}$ and $\dot{K}_t/K_t = i_t - \delta$. The budget constraint of the government (29) is:

$$(A34) \quad 0 = q_t d\kappa_t + p_t dn_t + \left((r_t - q_t i_t)(1-\kappa_t) + p_t n_t (i_t - \delta) \right) dt.$$

The aggregate state of the economy is summarized by (K_t, a_t, κ_t) , or equivalently by (r_t, a_t, κ_t) . Denotes $\hat{x}_t \equiv (x_t - \bar{x})/\bar{x}$: proportional deviation of variable x_t from its steady state value. Linearly approximating the equilibrium condition around the steady state, we want to derive the endogenous variables as the function of the state variables as:

$$(A35) \quad \hat{i}_t = \chi \hat{r}_t + \xi \hat{a}_t + \omega \hat{\kappa}_t,$$

$$(A36) \quad \hat{q}_t = \chi_q \hat{r}_t + \xi_q \hat{a}_t + \omega_q \hat{\kappa}_t,$$

$$(A37) \quad \hat{h}_t = \chi_h \hat{r}_t + \xi_h \hat{a}_t + \omega_h \hat{\kappa}_t,$$

where $h_t \equiv p_t M_t / K_t = p_t n_t$: the ratio of real balance to capital. From (A34), when there is no open market operation during the interval $[t, t+dt]$, we have:

$$(A38) \quad \dot{\hat{n}}_t \equiv \frac{d\hat{n}_t}{dt} = \frac{r-\delta q}{h} \hat{\kappa}_t - \delta \hat{i}_t,$$

in the neighborhood of the steady state, because $\dot{K}_t / K_t = i_t - \delta = \delta \hat{i}_t$. When the government does the open market operation in $[t, t+dt]$, we have:

$$(A39) \quad \dot{\hat{n}}_t = -\frac{q}{h} d\hat{\kappa}_t,$$

around the steady state with a small government (A11). Then from (A33), we have:

$$(A40) \quad \lim_{t \rightarrow 0} E_t \left(\frac{1}{dt} \left(\frac{q_{t+dt}}{q_t} - \frac{p_{t+dt}}{p_t} \right) \frac{q_t + p_t n_t}{q_{t+dt} + p_{t+dt} n_t} \right)$$

$$= \left(\frac{\dot{q}_t}{q_t} - \frac{\dot{p}_t}{p_t} \right)_{da_t = d\kappa_t = 0} + \left(\frac{q_{t+dt}}{q_t} - \frac{p_{t+dt}}{p_t} \right)_{da_t \neq 0} + \left(\frac{q_{t+dt}}{q_t} - \frac{p_{t+dt}}{p_t} \right)_{d\kappa_t \neq 0}$$

$$= -(\chi_q - \chi_h)(1-\alpha)\delta \hat{i}_t + \left(\frac{r-\delta q}{h} \hat{\kappa}_t - \delta \hat{i}_t \right) + (\chi_q - \chi_h + \xi_q - \xi_h) \eta_a (-2\hat{a}_t) + (\omega_q - \omega_h - \frac{q}{h}) \eta_{kg} (-2\hat{\kappa}_t).$$

The first term of RHS of (A40) is the difference in the rate of capital gains between capital and money due to capital accumulation. The second term is the effect of continual change in money-capital ratio. The third term is the difference in the expected rates of capital gains associated with the productivity change, noting that the arrival rate of the change is η_a and the proportional size of the change ($d\hat{a}_t$) is equal to $-2\hat{a}_t$ by the assumption of two-point Markov process. The last term is the difference in the expected rates of capital gains associated with the open market operation. Thus linearizing (A30), (A31) and (A32) around the steady state with no government, we have:

$$(A41) \quad (1-\theta q)\delta \hat{i}_t = \pi h \hat{h}_t + \theta(\pi+\delta)q \hat{q}_t + \pi \theta q \hat{\kappa}_t,$$

$$(A42) \quad r \hat{r}_t = \delta \hat{i}_t + \rho h \hat{h}_t + \rho q \hat{q}_t + \rho q \hat{\kappa}_t,$$

$$(A43) \quad \frac{r}{q}(\hat{r}_t - \hat{q}_t) - [1 + (\chi_q - \chi_h)(1 - \alpha)]\delta\hat{i}_t + \frac{r - \delta q}{h}\hat{\kappa}_t - 2(\chi_q - \chi_h + \xi_q - \xi_h)\eta_a\hat{a}_t \\ - 2(\omega_q - \omega_h \frac{q}{h})\eta_{kg}\hat{\kappa}_t = \pi \left[\frac{1}{q} \frac{h+q}{h+1} \hat{q}_t^2 - \frac{h}{q} \left(\frac{q-1}{h+1} \right)^2 \hat{h}_t^2 + \frac{h}{q} \left(\frac{q-1}{h+1} \right)^2 \hat{\kappa}_t^2 \right]$$

Substituting (A35, A36, A37) into (A41, A42, A43), these equations must hold for any \hat{r}_t , \hat{a}_t , and $\hat{\kappa}_t$. Thus we have 9 unknown variables ($\chi, \chi_p, \chi_q, \xi, \xi_p, \xi_q, \omega, \omega_p, \omega_q$) which satisfy 9 equations, three terms of \hat{r}_t , \hat{a}_t and $\hat{\kappa}_t$ in each of (A41), (A42) and (A43).

Solving (A41, A42) for \hat{h}_t and \hat{q}_t with respect to \hat{i}_t , \hat{r}_t and $\hat{\kappa}_t$, we have:

$$(A44) \quad \begin{pmatrix} \hat{h}_t \\ \hat{q}_t \end{pmatrix} = \frac{1}{\det} \begin{pmatrix} [\rho(1-\theta q) + \theta(\pi + \delta)]\delta q & -\theta(\pi + \delta)qr & \delta\theta\rho q^2 \\ -[\rho(1-\theta q) + \pi]\delta h & \pi hr & -(1-\theta)\pi\rho hq \end{pmatrix} \begin{pmatrix} \hat{i}_t \\ \hat{r}_t \\ \hat{\kappa}_t \end{pmatrix} \\ = \begin{pmatrix} D_{hi} & D_{hr} & D_{h\kappa} \\ D_{qi} & D_{qr} & D_{q\kappa} \end{pmatrix} \begin{pmatrix} \hat{i}_t \\ \hat{r}_t \\ \hat{\kappa}_t \end{pmatrix},$$

where $\det \equiv qh\rho[\pi - \theta(\pi + \delta)] > 0$ under the continuous time limit of Assumption 1, with $h > 0$. From (A35, A36, A37, A44), we learn:

$$(A45) \quad \begin{pmatrix} \chi_h(\chi) & \xi_h(\xi) & \omega_h(\omega) \\ \chi_q(\chi) & \xi_q(\xi) & \omega_q(\omega) \end{pmatrix} = \begin{pmatrix} D_{hi} & D_{hr} & D_{h\kappa} \\ D_{qi} & D_{qr} & D_{q\kappa} \end{pmatrix} \begin{pmatrix} \chi & \xi & \omega \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Substituting (A44, A45) into (A43), we have three equations for the terms of \hat{r}_t , \hat{a}_t and $\hat{\kappa}_t$ as:

$$(A46) \quad 0 = \left(1 + [\chi_q(\chi) - \chi_h(\chi)](1 - \alpha) \right) \delta\chi - \frac{r}{q} \\ + \pi \frac{1}{q} \frac{h+q}{h+1} \chi_q(\chi) - \pi \frac{h}{q} \left(\frac{q-1}{h+1} \right)^2 \chi_h(\chi) \equiv \Phi(\chi) \equiv \phi_2\chi^2 + \phi_1\chi + \phi_0,$$

$$(A47) \quad 0 = \left(1 + [\chi_q(\lambda) - \chi_h(\lambda)](1-\alpha)\right) \delta \xi + 2\eta_a [\chi_q(\lambda) - \chi_h(\lambda) + (D_{qi} - D_{hi})\xi] \\ + \pi \frac{1}{q} \frac{h+q}{h+1} D_{qi} \xi^2 - \pi \frac{h}{q} \left(\frac{q-1}{h+1}\right)^2 D_{hi} \xi \equiv \Psi(\xi, \lambda) \equiv \psi_1(\lambda)\xi + \psi_0(\lambda),$$

$$(A48) \quad 0 = \left(1 + [\chi_q(\lambda) - \chi_h(\lambda)](1-\alpha)\right) \delta \omega + 2\eta_{kg} [\omega_q(\omega) + \omega_h(\omega) - \frac{q}{h}] \\ + \pi \frac{1}{q} \frac{h+q}{h+1} \omega_q(\omega)^2 - \pi \frac{h}{q} \left(\frac{q-1}{h+1}\right)^2 \omega_h(\omega) + \pi \frac{h}{q} \left(\frac{q-1}{h+1}\right)^2 - \frac{r-\delta q}{h} \equiv J(\xi, \lambda) \equiv j_1(\lambda)\xi + j_0(\lambda).$$

Under Assumption 1 and 2, we learn $\Phi(\lambda) = 0$ is quadratic equation of λ , with $\phi_2 < 0$ and $\phi_0 > 0$. Therefore, there is a unique λ which satisfies the the saddle-point stability condition $\lambda > 0$, (stability requires \hat{i}_t is a decreasing function of \hat{K}_t , or an increasing function of $r_t = a_t K_t^{1-\alpha}$). From (A44, A45, A46), we can derive the properties:

$$(A49) \quad 0 < \lambda, \quad 0 < \chi_h, \quad \chi_q < \chi_h.$$

Also from (A47), we see that $\Psi(\xi, \lambda) = 0$ is a linear equation of ξ . We also learn ξ is negative and proportional to η_a . Again from (A44, A45, A46, A47), we can learn the properties:

$$(A50) \quad \xi < 0, \quad \xi_h < 0, \quad \xi_q > 0,$$

$$(A51) \quad 0 < \lambda + \xi, \quad 0 < \chi_h + \xi_h, \quad 0 < \chi_q + \xi_q,$$

and all ξ , ξ_h , and ξ_q are proportional to η_a .

For the effects of the open market operation, from (A45, A48), we learn:

$$(A52) \quad \omega < 0, \quad \omega_h < 0, \quad \text{and } \omega_q > 0.$$