Speculative Growth

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Abstract

We propose a framework for understanding recurrent historical episodes of vigorous economic expansion accompanied by extreme asset valuations, as exhibited by Japan in the 1980s and the U.S. in the 1990s. We interpret this phenomenon as a high-valuation equilibrium with a low effective cost of capital based on optimism about the future availability of funds for investment. The key to the sustainability of such an equilibrium is feedback from increased growth to an increase in the supply of funding. We show that such feedback naturally arises when the expansion is concentrated in a "new economy" sector and when it is supported by sustained fiscal surpluses — which would both constitute an integral part, as cause and consequence, of a "speculative growth" equilibrium. The high-valuation equilibrium we analyze may take the form of a *stock market bubble*. In contrast to classic bubbles on non-productive assets, the bubbles in our model *encourage* real investment, boost long-run savings, and may appear in dynamically *efficient* economies.

1 Introduction

The last two decades have witnessed stark "speculative growth" episodes of extreme stock market valuations accompanied by brisk economic growth. Most notable was the experience of Japan in the 1980s and the United States in the 1990s. Figure 1a illustrates the sharp rise in the Nikkei in the 1980s, followed by the two-year slide of 1990-1992; and the sharp rise in the NASDAQ in the 1990s, followed by the collapse of 2000-2001. The extremes reached by valuations shown in figure 1b, and their collapse in the absence of obvious changes in fundamentals, are suggestive of a stock market bubble. Figures 1c and 1d illustrate the growth and investment boom-and-bust that accompanied the market's gyrations.
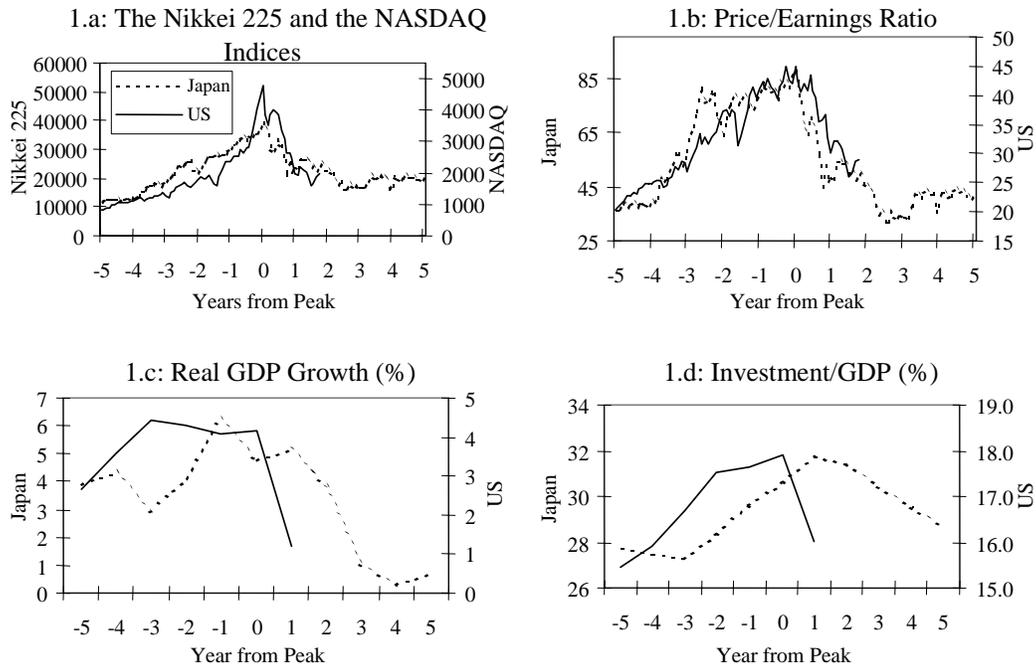
Note: Panel b: numerator is real (inflation-adjusted) Datastream Total Market Index; denominator is moving average over preceding ten years of real earnings corresponding to the index.
Sources: Panel a: Nasdaq Composite Index from The Nasdaq Stock Market, Inc; Nikkei 225 from Datastream. Panel b: Datastream Total Market Index for US and Japan; CPIs from OECD Main Economic Indicators (all items). Panel c: Bureau of Economic Analysis (BEA), NIPA Table 1.2; Economic and Social Research Insitute (ESRI). Panel d: BEA, NIPA Table 1.1; Annual report on national accounts (CD-ROM) 1998

Figure 1: Speculative Growth Episodes

The nature and policy dilemmas of speculative expansions have attracted much attention (e.g., International Monetary Fund (2000), Shiller (2000), Cecchetti et al. (2000)), but our formal understanding of the macroeconomic mechanisms that underlie the relation between stock market speculation and real economic activity remains quite limited. In fact,

the classic literature on bubbles in general equilibrium bears little relation to speculative growth episodes. In this literature, bubbles —understood as asset prices that deviate from fundamental values— can only appear in a "dynamically inefficient" economy, that is one whose structure is conducive to the over-accumulation of capital. Their emergence helps to absorb saving away from investment and to alleviate the over-accumulation problem. Not only have empirical tests of dynamic inefficiency thus far been negative, but also the notion that the emergence of a bubble would crowd out investment, while its bursting would increase investment, is contrary to the patterns depicted in figure 1.[1]

This paper proposes an alternative framework for understanding speculative expansions that is better able to address the facts. Our interpretation of this phenomenon rests on two premises: *(i)* the rise in market valuations in speculative growth episodes results from a low effective long-term cost of capital; and *(ii)* the boom in investment and growth during such episodes is a response to increased asset prices along the standard q-theory line. The central question that arises under this interpretation is: How are expectations of a low long-term cost of capital consistent with a high-investment equilibrium? A key component of our equilibrium account is a *growth-saving feedback* by which the future supply of savings increases as a *result* of the conditions created by a speculative expansion, and ends up lowering the effective cost of capital.

The kind of stock market bubbles that can arise in the presence of growth-saving feedback have very different implications from standard bubbles. First, rather than crowding out capital accumulation, they naturally encourage investment and growth. Conversely, their bursting leads to an investment slump as the economy rids itself of what appears *ex post* as widespread overcapacity. Second, because they induce greater saving, such bubbles are expected to reduce the long-run cost of capital; this, one can show, allows them to arise in a dynamically efficient context. Finally, such bubbles and the resulting speculative expansions are manifestations of a more general class of *high-valuation equilibria* that are the result of the feedback from growth to saving. Such equilibria are qualitatively similar to bubbly equilibria, in that they exhibit high asset valuations and can involve a collapse in asset prices in the absence of any news. However, these equilibria may or may not incorporate a bubble in asset markets, depending on how extreme valuation levels are. In other words, although consistent with a stock market bubble, our account of speculative expansions does not hinge on ascertaining the presence of a bubble.

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[1] Abel et al. (1989) show that dynamic inefficiency essentially implies that the net cash flow to investors from the aggregate market for assets is negative. They find no evidence of such a feature in OECD economies during their period of study.

In the particular case of the U.S, in the 1990s, we argue, at least two factors created the conditions for a speculative growth episode: the emerging information technology sector and conservative fiscal policy. Both factors created favorable conditions for growth-saving feedback and for the possibility of a speculative equilibrium characterized by extreme stock market valuations and a potential crash.

How does the information technology sector generate feedback from growth to saving? This is an emerging sector of the economy, subject to a lesser degree of diminishing returns than more mature sectors. As the economy expands, such a "new economy" sector naturally will expand proportionally more than the rest of the economy and will have a greater ability to generate income for saving. Moreover, in an endogenous growth version of this model, we show that a bubble not only might be necessary for productivity growth to accelerate, but that the resulting expansion in productivity, income, and saving also might be critical to creating the growth-saving feedback necessary to support the bubble. Thus, a fundamental expansion in productivity and stock market speculation are not necessarily mutually competitive explanations. A technological revolution might form an integral part — both as cause and consequence — of a speculative growth equilibrium.

What about fiscal policy? We note that a significant share of increased saving in the recent U.S. speculative expansion is attributable to the public sector. To the extent that pro-cyclical government revenues increase public saving, they reinforce the feedback from growth to saving.[2] In the short run, fiscal surpluses can arise as a consequence of the stock market boom and can facilitate the initial rise in investment. More importantly, fiscal surpluses can play a central role in making the speculative equilibrium feasible by providing the funding necessary to sustain high investment in the long run. This consideration has a particularly stark implication for fiscal policy during a speculative growth episode. The possibility of using the fiscal surpluses that result from such expansions to cut taxes and raise spending might be illusory, as these surpluses might be necessary to sustain the speculative growth equilibrium that generated them.

The speculative growth episodes to which our theory relates are a recurrent phenomenon. Earlier episodes of vigorous economic expansion under speculative asset valuations have

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[2]There are other mechanisms that can generate feedback from growth to saving which might play an important role in supporting a speculative growth equilibrium. For example: *(i)* the expansion might disproportionately benefit agents who have a high marginal propensity to save; *(ii)* when partial self-finance is required by financial constraints, the expansion might incentivize saving by increasing the likelihood of achieving self-finance requirements; *(iii)* when the expansion is associated with major technological developments, it might generate future external spillovers that increase the income of savers.

been documented by economic historians. In the case of the U.S., this phenomenon also can be observed in the expansions of the turn of the 20th century, the 1920s, and the 1960s (see e.g., Shiller 2000). Equally important, our theory can be brought to bear on sustained low-cost-of-capital equilibria, such as the prolonged expansions exhibited in an number of East Asian economies in the post-war period. In fact, the key feedback from growth to saving in these economies has been documented recently. Examining the aggregate relationship between income growth and saving in a cross-country panel of 64 countries over the period 1958-1987, Caroll and Weil (1994) find that growth Granger causes saving with a positive sign, but that saving does not Granger cause growth. The pattern of an acceleration in growth followed by strong increases in saving rates is particularly clear in the high-growth, high-saving East Asian economies of Japan, South Korea, Singapore, and Hong Kong. Gavin et al. (1997) elaborate on this evidence and show that the estimated impact of growth on saving is not only statistically significant but also very large in economic terms.

Relation to the literature

In its economic theme, our paper is part of a long tradition of studies of speculative growth episodes (e.g., Kindleberger 1989). In terms of the recent U.S. experience, this literature is divided between those who see a case of "irrational exuberance" (e.g., Shiller 2000), and those who, based on the significant acceleration in underlying U.S. productivity growth, conclude that the expansion was driven by a technological revolution that affected real fundamentals (e.g., Greenwood and Jovanovic, 1999; Hobijn and Jovanovic, 2000). From this perspective, our contribution is to provide a unified perspective under which these two views need not be mutually exclusive. On the contrary, technological opportunities and stock market speculation may be inseparable complements.

On the methodological side, our paper belongs to the literature on bubbles in general equilibrium. From this perspective, the main distinctive feature of our framework is the centrality of the growth-saving mechanism. The latter overturns standard conclusions by breaking the crowding-out mechanism of equilibrium bubbles.

As we mentioned earlier, the classic theory of bubbles in general equilibrium (see Tirole 1985) is not designed well for addressing speculative growth episodes. It implies that a bubble crowds out investment and only can arise in the dynamically inefficient region of the economy. Neither of these elements is observed during these episodes.

More recently, several papers have demonstrated that in the presence of externalities that create a wedge between private and social returns on investment, bubbles can arise even if the bubbleless economy is dynamically efficient (e.g., Saint-Paul, 1992; Grossman

and Yanagawa, 1993; King and Ferguson, 1993) That is because dynamic efficiency requires that the *social* rate of return exceed the rate of growth of the economy; the possibility of a bubble instead requires that the *private* rate of return be below the rate of growth. These two conditions are simultaneously feasible in the presence of a wedge between private and social returns. The cash-flow test developed by Abel et al. (1989) for dynamic inefficiency applies to private returns, and is inconsistent with standard bubbles even in the presence of externalities. In contrast, our mechanism does not depend on a wedge between private and social returns, and a bubble can arise even if the speculative path starts with a *private* return that exceeds the rate of growth of the economy. In particular, our mechanism passes Abel et al.'s test.

In standard models, bubbles reduce investment and growth by crowding out saving. Olivier (2000) makes an important contribution to this literature by developing a model of bubbles on equity which, like ours, leads to more rather than less investment. However, his model does not escape the crowding-out mechanism and requires large ex-ante dynamic inefficiency for the bubble to be price increasing (see Weil 1990) — i.e., for the negative effect on asset prices caused by the rise in the interest rate to be dominated by the positive effect of the bubble.[3] This issue does not arise in our model because the growth-saving mechanism eventually lowers rather than raises the effective cost of capital.

The rest of this paper is organized as follows. In section 2, we present a prototypical model of speculative growth. Our analysis is based on the Diamond (1965) overlapping-generations model, to which we add adjustment costs and a special assumption about the saving function. Adjustment costs allow us to characterize stock market booms and their key role in facilitating the speculative path from low- to high-investment equilibria. Our special saving function generically captures the growth-saving feedback that is central to the existence of a high-valuation equilibrium. Section 3 presents a specific instance of the generic mechanisms of section 2 in a model of the "new economy." In section 4, we extend our analysis to encompass fiscal surpluses and current account deficits as short- and long-run funding mechanisms. Section 5 concludes and is followed by an appendix containing the proofs of the main propositions.

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[3]See Cozzi (1998) for a related model applied to education decisions (culture as a bubble).

2 Speculative Growth: The Mechanics

In this section we present a prototypical model of speculative growth. Our analysis is based on the Diamond (1965) overlapping-generations model, to which we add adjustment costs to capital and a special assumption about the saving function. Adjustment costs are important in any model in which changes in stock market valuations are central; otherwise the price of capital would be constant. The special assumption about the saving function is intended to capture the growth-saving feedback that is central to our interpretation of speculative expansions. Although this section introduces the feedback generically through the functional form of the saving function, the next two sections pay closer attention to the specific macroeconomic mechanisms that may be at work, and their relationship to observed conditions during recent speculative growth episodes.

2.1 Model Setup

Consider a standard Diamond (1965) overlapping-generations structure with no population growth and a unit mass of young and old agents who coexist at any date t. Each generation is born with a unit of labor, $L_t = \overline{L} \equiv 1$, to be used when young, for which it receives a total wage W_t determined in a competitive, full-employment labor market. The economy's single consumption good is used as a numéraire.

Technology. Consumption goods are produced with capital, K_t, and labor, L. The production function at any time t is determined by the level of technology, A_t, which grows at an exogenous rate γ:

$$A_{t+1} = (1 + \gamma)A_t.$$

Production is given by a constant returns technology:

$$Y_t = F(K_t, A_t L) = A_t L f(k_t),$$

where $k_t \equiv K_t/A_t L$, $f' > 0$, $f'' < 0$, and $\lim_{k \downarrow 0} f'(k) = +\infty$. To simplify our discussion, we introduce further (conventional) assumptions on production through the wage function. Purely technical assumptions about this and other functions are stated in the appendix. The labor market is competitive, and the wage $w_t \equiv W_t/A_t L_t$ per unit of effective labor is

$$w_t \equiv w(k_t) = f(k_t) - k_t f'(k_t), \qquad w' \geq 0, w'' \leq 0. \tag{1}$$

Capital goods are accumulated with a constant returns technology:

$$K_{t+1} - K_t = G(I_t, K_t) = g(x_t)K_t, \tag{2}$$

where I_t is the investment of consumption goods in capital goods production, $x_t \equiv I_t/K_t$. The adjustment technology $g(x)$ has $g' > 0$, $g'' < 0$, $g(0) = 0$, $\lim_{x \uparrow \infty} g(x) = +\infty$, and $\lim_{x \uparrow \infty} g'(x) = 0$. Normalizing both sides of the accumulation equation by effective labor, $A_t L$, yields

$$(1 + \gamma)k_{t+1} = (1 + g(x_t))\, k_t. \tag{3}$$

Preferences and the saving function. Lifetime utility is a concave function of consumption during youth and old age. Each member of generation t chooses the level of saving when young, S_t, that maximizes lifetime utility. The relevant features of preferences are summarized in the level of saving, $s_t \equiv S_t/A_t L$, which we assume is given by

$$s_t = s(w_t, r_t), \tag{4}$$

where r_t denotes the interest rate between periods t and $t+1$, $s_w > 0$, and $0 < s_r < \infty$.

Capital market equilibrium. Following production in period t, the old sell their capital to the young at a price q_t. The young allocate their savings to the purchase of the existing stock of capital and to investment in new capital. Thus the capital market equilibrium condition is

$$s_t = (q_t + x_t)\, k_t. \tag{5}$$

For any level of saving, the young take the price q_t as given and decide how much of their savings will be allocated to purchase the existing stock of capital, k_t, and how much to invest in new capital. They do so to maximize their stock of capital when old:

$$\max_{\{x_t, k_t\}} k_{t+1} \quad \text{s.t.} \quad \text{(3) and (5),}$$

which yields the standard q-theory condition for investment:

$$q_t = \frac{1 + g(x_t)}{g'(x_t)} - x_t.$$

This equation implicitly defines the investment function

$$x_t = x(q_t), \quad x' > 0. \tag{6}$$

The equilibrium interest rate can be expressed as a function of the stock of capital and its price. Replacing the saving and investment functions, (4) and (6), into the capital market equilibrium condition, (5), and substituting the wage from (1), yields:

$$r_t = r(k_t, q_t). \tag{7}$$

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It is straightforward to verify that $r_q > 0$. We postpone a discussion of the sign of r_k until it is needed.

Equilibrium dynamics. Equilibrium dynamics are given by the capital accumulation equation, (3), and the arbitrage condition for holding a unit of capital from t to $t+1$:

$$(1 + r_t) \, q_t = (1 + G_K (I_t, K_t)) \left(q_{t+1} + F_K (K_{t+1}, A_{t+1} \overline{L}) \right) . \tag{8}$$

The right hand side of this expression is the gross return from holding the unit of capital. The only somewhat unusual term in this return derives from the adjustment cost function, as one unit of capital at time t yields $1 + G_K$ units of capital at $t+1$. Since $G_K = g(x) - g'(x)x$, we will write with some abuse of notation $G_K = G_K \langle x \rangle$ as a function of x. Replacing the interest rate function, (7), into the arbitrage condition, (8), and noting that $F_K = f'$, the system governing equilibrium dynamics can be written as a two-dimensional system in (k_t, q_t)-space:

$$\frac{k_{t+1}}{k_t} = \frac{1 + g(x(q_t))}{1 + \gamma}; \tag{9}$$

$$\frac{q_{t+1}}{q_t} = \frac{1 + r(k_t, q_t)}{1 + G_K \langle x(q_t) \rangle} - \frac{1}{q_t} f' \left(\frac{1 + g(x(q_t))}{1 + \gamma} k_t \right) . \tag{10}$$

2.2   The Growth-Saving Feedback and Speculative Growth

A generic way to introduce feedback from expansion to saving is through the functional form of the saving function, $s(w, r)$. We assume that if the economy expands past a level k^o of the capital stock, then the saving of the young increases sharply. We will analyze specific instances of economic phenomena that can give rise to such feedback in subsequent sections. Here we write the saving function as

$$s(w, r) = \begin{cases} \widetilde{s}(w, r), & w < w(k^o); \\ \widetilde{s}(w, r + \Delta^r), & w \geq w(k^o), \end{cases} \tag{11}$$

where $\Delta^r > 0$ and $\widetilde{s}(w, r)$ is a continuous function in both arguments. Savings jump by $s_r \Delta^r$ when the wage reaches $w(k^o)$. From equilibrium in the capital market we see that the jump in the saving function translates into a dip in the interest rate function, $r(k, q)$, at k^o. Under weak conditions, $r_k > 0$ in all other regions where such sign matters (see the appendix). If the function $\widetilde{r}(q, k)$ corresponds to the equilibrium interest rate, given q and k, were the saving function $\widetilde{s}$ rather than s, we have:

$$r(k, q) = \begin{cases} \widetilde{r}(k, q), & k < k^o; \\ \widetilde{r}(k, q) - \Delta^r, & k \geq k^o, \end{cases} \tag{12}$$

In this context,

Proposition 1 *(Equilibria)*

There exist $\Delta^r > 0$ and $k^o > 0$ such that dynamic system (9)-(10) has precisely two steady states, (k_L^*, q^*) and (k_H^*, q^*), where $k_H^* > k_L^*$.
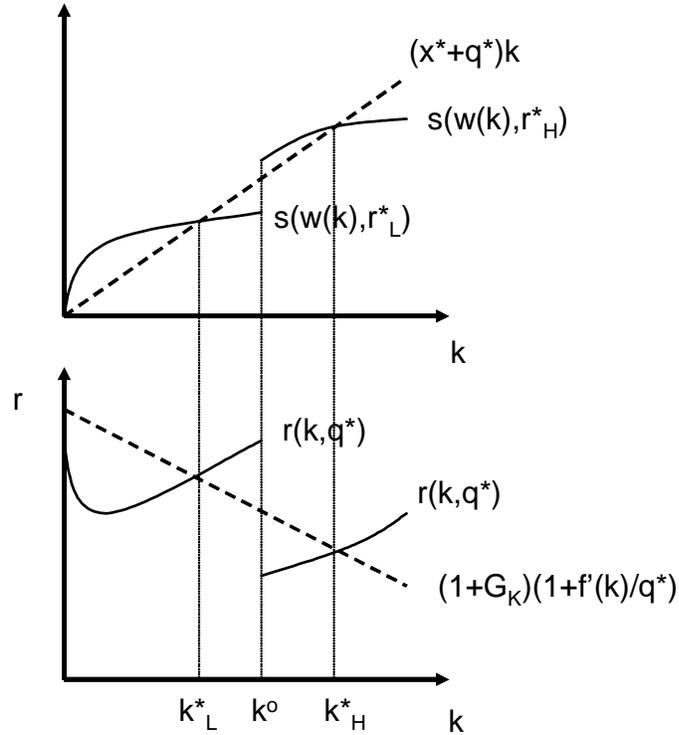
Proof: See the appendix.

Figure 2: Equilibria

Figure 2 illustrates these two equilibria. The solid line in the top panel shows saving (for simplicity, plotted for a given interest rate r_L^* in the first segment, and r_H^* in the second one) while the dashed line represents investment demand for given q^*; both as a function of k. For low levels of k, saving rise more rapidly than investment requirements as k expands. More importantly, because of the concavity of wages, this inequality turns around and eventually yields the low valuation equilibrium. The capital market equilibrium interest rate function portrayed with a solid line in the bottom panel (to the left of k^o) captures the dual of this

pattern. The dashed downward sloping line represents the return obtained from investing in a unit of k, when q remains fixed at q^*. In equilibrium, this return is equal to the interest rate that equilibrates the capital market. Since $s_w w' < (x^* + q^*)$ at k_L^*, this equilibrium is locally stable. However a significant increase in k eventually triggers the jump in the saving function. This is what makes the high-valuation equilibrium possible.

The heuristic phase diagram in figure 3 provides a more complete illustration of equilibria and the corresponding dynamics.

We start by identifying the system's steady states, which correspond to the intersection of the two stable arms, $\{(k_t, q_t) \text{ s.t. } k_{t+1} = k_t\}$ and $\{(k_t, q_t) \text{ s.t. } q_{t+1} = q_t\}$, given respectively by

$$g(x(q_t)) = \gamma; \tag{13}$$

$$1 + r(k_t, q_t) = (1 + G_K \langle x(q_t) \rangle) \left[1 + \frac{1}{q_t} f' \left(\frac{1 + g(x(q_t))}{1 + \gamma} k_t \right) \right]. \tag{14}$$

From equation (13), the k-constant arm is a horizontal line that corresponds to the single level of q_t that balances the capital accumulation rate with the economy's growth rate. The slope of the q-constant arm reflects equilibrium in the capital market: starting at low levels of k, a higher stock of capital will reduce the price q at which young savers are willing to purchase capital since the marginal product of capital declines sharply. At higher levels of k, the dominant reason for the decline in q is that the young's ability to pay for the extra capital does not rise proportionally with the additional investment requirement. Just the opposite happens at $k_t = k^o$, when the price of capital jumps because saving jumps.

To gain further intuition on the behavior of q along its stable arm, let us re-write (14) as an asset pricing equation:

$$q_t = \frac{1}{r - G_K} \left[(1 + G_K) f' \left(\frac{1 + g}{1 + \gamma} k_t \right) \right] \equiv Q(k_t, q_t). \tag{15}$$

The price/earnings ratio, $q / (1 + G_K) f'$, is a measure of market valuation and corresponds to the first term in (15) — it is equal to the inverse of the effective cost of capital. The second term is the marginal product from a unit of capital installed at date t. Both are adjusted to account for the marginal improvement, G_K, in capital accumulation capacity. At low levels of k, the second term drives the implications for q_t of an increase in k_t. The valuation term drives them at high levels of k_t, and $r_k > 0$ in this region.

As stated in Proposition 1, if the growth-saving feedback is strong enough, the two stable arms intersect twice, once to the right and once to the left of k^o. In what follows, we assume this is the case for the economy.
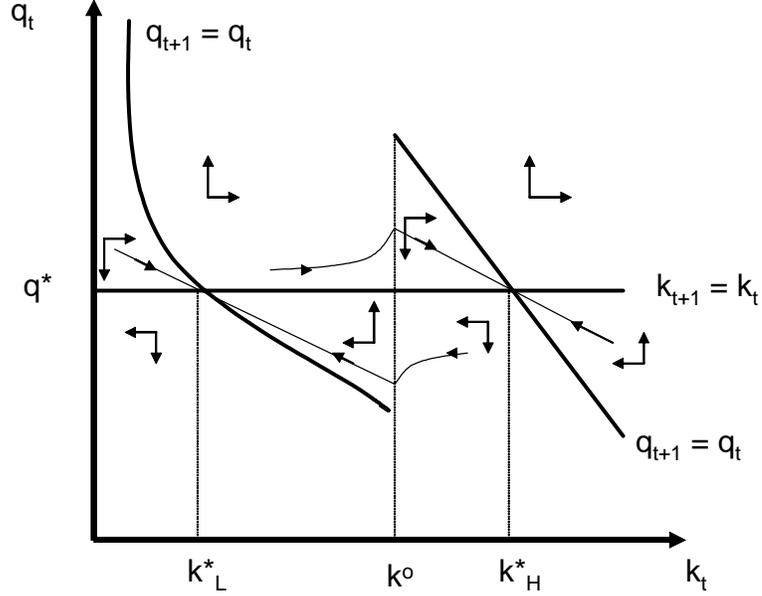
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Figure 3: Dynamics in (k_t, q_t)-space

Turning to out-of-steady-state dynamics, figure 3 illustrates the dynamics generated by the system of difference equations, (9)-(10). There are six distinct regions, characterized by the direction of movement of k_t and q_t over time. Each of the two steady states, k_L^* and k_H^*, is saddle-path stable. Moreover, if Δ^r is large enough, the economy exhibits multiple equilibria. Starting from the same initial value for the state variable, k, it is possible for two equilibrium paths to arise: one that takes the economy to the high-level steady state, (k_H^*, q^*), and one that converges to the low-level steady state, (k_L^*, q^*). We focus on this case for now because it allows us to highlight our insights more clearly.

2.3   Discussion

The nature of equilibrium dynamics shown in figure 3 can be gauged from the simulation illustrated in figure 4. The specific functional forms and parameter values underlying the

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simulation are presented in the appendix. The specific numbers are unimportant. Starting from the initial conditions of steady-state equilibrium (k_L^*, q^*), figure 4 illustrates an unanticipated jump at time $t = 1$ to the equilibrium path that converges to the steady state (k_H^*, q^*). In the same order as the empirical time series presented in figure 1, panels 4a-4d present time series for the stock market index, q_t, the price/earnings ratio $(q_t / (1 + G_K \langle x \rangle) f'(k))$, the growth of output Y_t, and the investment/output ratio (I_t / Y_t).
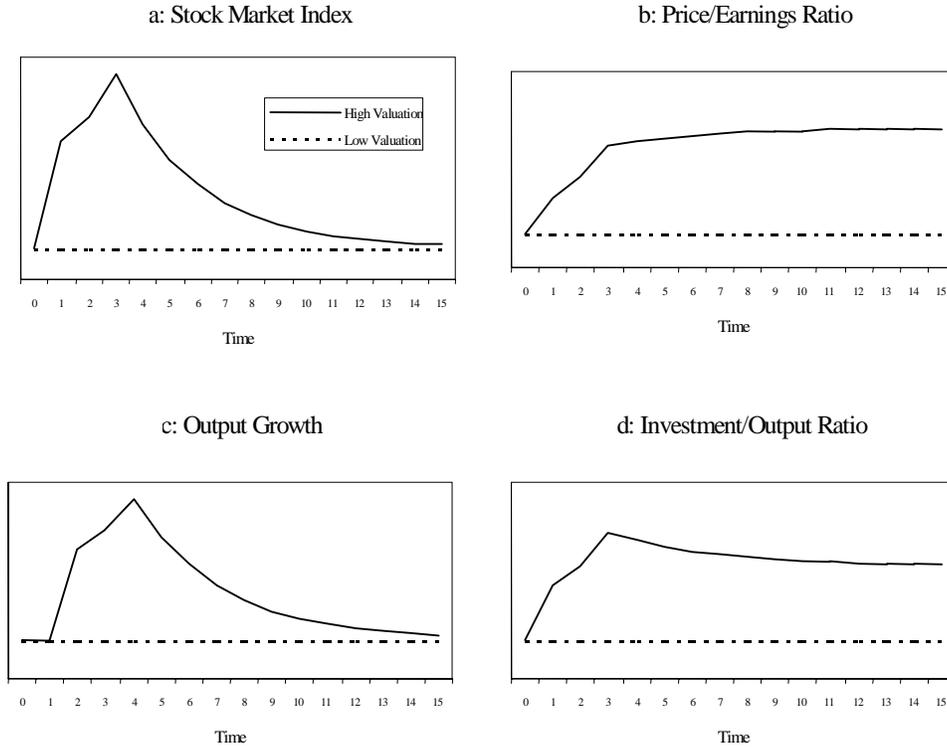

Figure 4: High Valuation Equilibrium Path

Market valuation. Starting from steady state (k_L^*, q^*), the economy's unanticipated jump to the new equilibrium takes the form of an immediate rise in q and stock market gains (panel 4a). The price of capital remains above its original level along the equilibrium path, but it converges back to that level over time. On the other hand, the market's capitalization, qk, converges to a higher level, since $k_H^* > k_L^*$.

When the economy moves to the new equilibrium, the key phenomenon is the rise in market *valuations*. We do not mean by this the absolute price level of capital or market capitalization, but rather their increase *relative* to the earnings generated by capital. As can be seen in panel 4b, the P/E ratio rises when the economy moves to the new equilibrium and continues to rise along the equilibrium path. Compared to (k_L^*, q^*), the new steady state, (k_H^*, q^*), exhibits a higher P/E ratio and, therefore, a lower effective cost of capital.

The factor behind the increase in market valuations is the feedback from economic expansion to saving, incorporated in the jump in the saving function at k^o. This jump creates an abundance in the supply of funds and results in a decline in the effective cost of capital. The stock market boom that takes place when the economy moves to the new equilibrium reflects optimism about the future availability of funds and the cost of capital. With this in mind, we will refer to the equilibrium associated with (k_L^*, q^*) as the *low-valuation* equilibrium, and to the equilibrium associated with (k_H^*, q^*) as the *high-valuation* equilibrium.

Investment and growth. The high-valuation equilibrium exhibits higher investment and growth than the low-valuation steady state. Since q is above its steady-state level along the high-valuation equilibrium path, $x(q)$ is also above its steady state level and the capital stock grows to k_H^*. The growth in the capital stock causes an expansion in income growth during the transition (panel 4c). Over time, the growth rates of capital and income return to normal, but the high-valuation steady state is characterized by permanently higher investment as a share of income (panel 4d).

Of course, from the speculative growth path, the economy could experience an unanticipated dip to a recessionary equilibrium path that takes it back to the low-valuation steady state. Along that path, the stock market crashes and triggers a slump in investment and growth as the economy rids itself of what appears ex post as widespread overcapacity.

2.4 Bubbles as a High-Valuation Equilibrium

There is no reference to bubbles in the discussion above. Such omission was deliberate. It turns out that in our framework an equilibrium with a *stock market bubble* is not qualitatively different from the high-valuation equilibrium we described. This kind of bubble will arise if high investment results in capital accumulation beyond the "satiation" point where the marginal product of capital is zero.

Proposition 2 *(Stock Market Bubbles)*

Suppose the production function has a finite satiation point, $\overline{k} = \min\{k\colon f'(k) = 0\}$. There exist $\Delta^r > 0$ and $k^0 > 0$ such that dynamic system (9)-(10) has precisely two steady states (k_L^*, q^*) and (k_H^*, q^*), where $k_L^* < k^0 < k_H^*$ and $k_H^* \geq \overline{k}$. Along the speculative path, the price of capital has a bubble component, and it becomes a pure bubble for all $k > \overline{k}$.

Proof: See the appendix.

Intuitively, the feedback from expansion to saving can be increased by increasing the size of the jump, Δ^r, in saving function (11). This rise in saving adds to the capital stock, k_H^*, in the high-valuation steady state. If the increase in saving, Δ^r, is large enough, k_H^* crosses the satiation level, $\overline{k}$. At that point, a rational bubble emerges in the price of capital. To see this, recall that the price of capital, q^*, remains the same across all steady states, given by (13). However, the marginal product of capital for any $k_H^* \geq \overline{k}$ is zero. All output in steady state goes to labor, and the aggregate stock market generates no earnings for its owners. Thus, the positive value of q^* corresponds to a rational bubble that is only sustained by the possibility of trading with other agents.[4]

Equilibrium quantities and prices are continuous as we move from the bubbleless steady state to the left of $\overline{k}$ to the bubbly steady state to the right. Both q^* and x^* are the same across all steady states, and k_H^* crosses $\overline{k}$ continuously. Looking at present value expression (15), it is apparent that as earnings in the numerator decline to zero, the effective cost of capital, $r - G_K$, in the denominator also must decline to zero in order to leave their ratio, q^*, unchanged. Moreover, if the high-valuation steady state is bubbly, then the price of capital along the equilibrium path that converges to it also incorporates a bubble. Thus, a bubble emerges in this economy as a limit case of high stock market valuations based on optimism about the future availability of saving and a low effective cost of capital.

The fact that numerator and denominator converge to zero should not lead to the conclusion that the limit price is a fundamentals' price. To see this, we note that (15) can be re-written as the present value of two dividends, $(1 + G_K)f'$ and qG_K, discounted at rate r rather than $r - G_K$. The limit we consider is one in which the first of these present values is zero, and the positive value of capital is sustained entirely by the present value of the speculative dividend.[5]

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[4]See Chirinko and Schaller (2001) for evidence on the response of investment to bubbles in Japan.

[5]Note also that the stock market bubble in our model does not need to grow at the rate of interest. The reason is that the existence of a bubble in q increases the value qG_K of the marginal improvement in capital-accumulation capacity associated with existing units of capital. Essentially, a bubble on capital generates additional bubbles on the capital this asset helps to produce.

The type of bubble that arises in our model is quite different from traditional bubbles on the price of non-productive assets:

- First, while bubbles on non-productive assets absorb saving away from investment, bubbles on capital incentivize capital accumulation.

- Second, a central feature of the classical theory of bubbles on non-productive assets is that they can only arise in economies whose bubbleless equilibrium is dynamically inefficient. Asymptotically, the argument goes, a bubble cannot grow faster than the economy's growth rate, γ. Otherwise, there would come a time when no agent is wealthy enough to hold it. Since a bubble grows at the interest rate, r, its existence is only compatible with $r \leq \gamma$. Moreover, since the introduction of a bubble crowds out investment and, therefore, increases r, we must have $r < \gamma$ in the bubbleless equilibrium. This condition implies dynamic inefficiency: capital must be accumulated at a rate γ that exceeds its return r, and therefore absorbs more resources than it generates.[6] Contrary to this argument, the bubbleless equilibrium in our model need not be dynamically inefficient for a bubble to arise. The main difference comes from the fact that introducing a bubble *increases* investment and, eventually, *reduces r*. Thus, it is possible to start from a dynamically efficient ($r > \gamma$) bubbleless situation, that turns dynamically inefficient ($r < \gamma$) as a bubble eventually lowers r.

In our model, what makes a bubble emerge in the price of capital as opposed to other, non-productive assets? If the bubbleless economy is not dynamically inefficient, the only asset that might support a bubble is capital. However, if a bubble on capital does arise, then it makes room for other bubbles on non-productive assets. The reason is that, in our model, a bubbly equilibrium can be shown to always drive the economy into a dynamically inefficient region.[7]

In sum, in our model stock market bubbles are just a case of high market valuations and can arise in a dynamically efficient context. Qualitatively, a bubbly equilibrium is not

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[6]See the seminal discussion in Tirole (1985).

[7]To see this, recall that the effective cost of capital, $r - G_K$, is zero in a bubbly steady state. Since $G_K = g - g'x < g$ and since, by (13), $g = \gamma$, we must have $r < \gamma$.

Also note that because of the latter, non-productive bubbles will vanish asymptotically if the stock market bubble is to survive. If that was not the case, then $r = \gamma$ and the stock market bubble could not exist. Without the latter bubble it is still possible for the non-productive one to exist, but it must leave enough space for capital to drive the economy into the dynamically inefficient region. If this is not the case, then the steady-state non-productive bubble is not feasible and only an equilibrium with a stock market bubble could survive.

very different from any other fundamentals-based high-valuation equilibrium.

3   Speculation and "New Economy" Expansions

a: Stock Market Indices

b: US Technology Investment (% of Total)
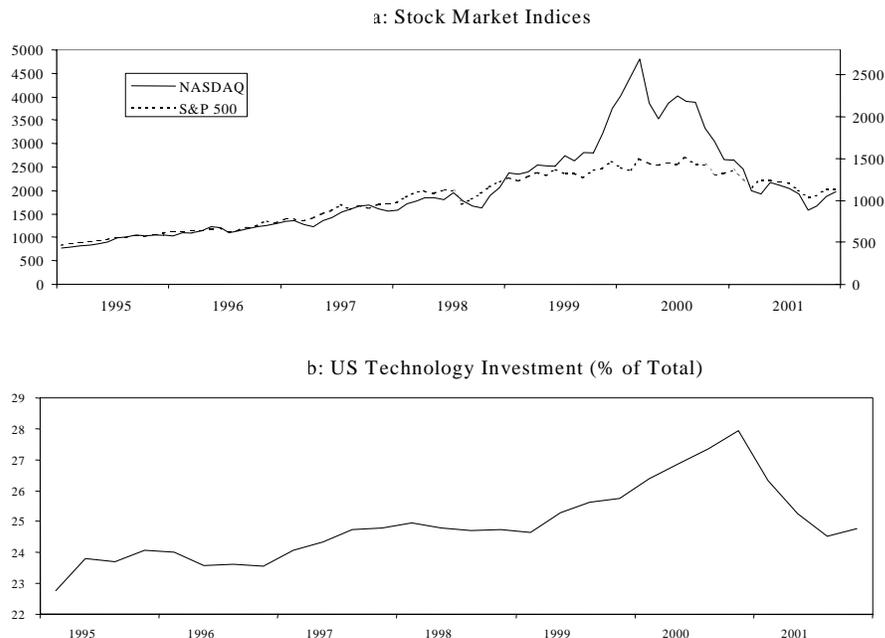
Note: Panel b: technology investment is investment on "Information processing equipment and software",
which consists of computers and peripheral equipment, software, communication equipment, instruments,
photocopy and related equipment and office and accounting equipment.
Sources: Panel a: Nasdaq Composite Index from The Nasdaq Stock Market, Inc; S&P 500 Composite
from Datastream. Panel b: BEA, NIPA Table 5.4.

Figure 5: New Economy Boom

A distinctive characteristic of the U.S. speculative expansion of the 1990s is that it was
concentrated in the new technology sector. As shown in panel 5a, "new economy" stocks of
technology and growth companies boomed, while the price of traditional "old economy" did
not appreciate nearly as much. Panel 5b illustrates that during the same period the share
of aggregate investment that went to technology capital experienced a sharp increase. More
generally, speculative growth episodes typically have been associated with the expansion
of newly emerging sectors of the economy. This was the case during the expansion at the
turn of the twentieth century, which witnessed the growth of such new sectors as railroads,

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steel, and oil, and during the "Roaring Twenties," which witnessed the emergence of mass-production manufacturing, electrification, automotive transportation, etc.[8]

In this section we argue that the emergence of a new production sector is associated with a natural growth-funding feedback that provides a prime instance of the mechanisms that can support a speculative growth equilibrium. We also argue, in an endogenous growth context, that such speculative growth results in increased productivity growth.

3.1 An Emerging-Sector Model

We make two modifications to the model in section 2: we introduce a second production sector; and we drop the ad-hoc assumption that the expansion-saving feedback comes from preferences. The feedback now results from the two-sector structure of the economy.

Recall that production in the model in section 2 uses production function $A_t f(k_t) L_t$, where k_t is subject to adjustment function $g(x_t)$. We now add a second sector that produces the same consumption good using a different type of capital, denoted by Z_t, as the only variable input in the following production function:

$$A_t[h(z_t) + \upsilon L^o], \qquad h' > 0,\ h'' < 0,$$

where $z_t \equiv Z_t/A_t$ and $L^o = \rho L$. The latter is a fixed amount of labor that is combined with a fixed factor which earns positive rents in equilibrium, $\upsilon > w$.

Consumption goods can be transformed one-to-one into Z with no adjustment costs. Thus,

$$h'(z_{t+1}) = r_t. \tag{16}$$

We refer to the k-sector as the "new" sector and to the z-sector as the "old" sector. The main difference between the new sector and the old sector is that the latter has more limited expansion opportunities and reaches maturity earlier than the former. More precisely, expanding capital in the old sector leads to faster diminishing returns on capital and benefits wages less: both are key drivers of our results. We capture the former characteristic by assuming $|h''|$ to be large enough (see below), and the latter by eliminating any interaction between z and labor demand as well as any expansion opportunity in labor.

We can preserve much of the stark features of section 2's phase diagram by making $h(z)$ a step function:

$$h(z) = \begin{cases} r^+ z, & z < z^o; \\ r^+ z^o + (r^+ - \Delta^{ro})(z - z^o), & \text{otherwise}, \end{cases} \qquad 0 < \Delta^{ro} < r^+.$$

---

[8]See, e.g., Rappoport and White (1993).

We now remove the jump in section 2's saving function and assume that agents have undiscounted log utility. We also assume that the fixed factors working with z and L^o do not save. While this assumption is irrelevant for the factor that complements z, it is not so for that which complements L^o. Mechanically, the role of $\rho > 0$ is to leverage the impact of an increase in k on savings, by raising the wages of workers (savers) in both sectors at once without an offseting force from the fixed factors saving behavior. Absent this channel we still obtain multiple steady states but not multiple equilibria. We drop this assumption later on in the endogenous growth model where the externality is sufficient to generate multiple equilibria.

In order to keep the notation as close as possible to that of section 2, we still normalize all quantities by effective labor in the k-sector, AL. Labor employed in this sector now differs from the total labor force which is equal to $(1 + \rho)L$. Thus, after adding savings across workers in the old and new sector, we have:

$$s_t^T \equiv (1 + \rho)s_t = (1 + \rho)\frac{1}{2}w_t, \qquad (17)$$

which in capital market equilibrium must be equal to investment in the two capital stocks:

$$s_t^T = (1 + \gamma)z_{t+1} + (x(q_t) + q_t)k_t. \qquad (18)$$

From (17)-(18), the surplus that can be used to fund the z-sector is

$$z_{t+1} = (1 + \gamma)^{-1} \left[\frac{1}{2}(1 + \rho)w(k_t) - (x(q_t) + q_t)\,k_t \right].$$

Because of the concavity of the wage function, $w(k_t)$, there can be two levels of k, $\widetilde{k^l}(q)$ and $\widetilde{k^u}(q)$, such that this surplus equals z^o. Thus, from (16), the interest rate is given by

$$r(k,q) = \begin{cases} r^+, & k < \widetilde{k^l}(q) \quad \text{and} \quad k > \widetilde{k^u}(q); \\ r^+ - \Delta^{ro}, & \text{otherwise.} \end{cases}$$

From here, the model can be solved exactly as in section 2, with dynamics given by the system, (9)-(10).[9]

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[9]Note that since (16) allows us to solve out for z_{t+1} as a function of the interest rate, we can focus directly on the saving that goes into the new sector, defined as $s_t^{Tk} \equiv s_t^T - (1 + \gamma)z_{t+1}$. By (16)-(18), and for smoother h functions, it is given by $s_t^{Tk} = \frac{1}{2}(1 + \rho)w_t - (1 + \gamma)(h')^{-1}(r_t) \equiv s^{Tk}(w_t, r_t)$, with $s_w^{Tk} > 0$ and $s_r^{Tk} > 0$.

Having solved out for the old-sector capital stock, we obtain a saving function into the new sector that is similar to function (4) in section 2. However, the difference is that the positive interest-elasticity of $s^{Tk}(w_t, r_t)$ does not come from preferences, but from the possibility of reallocating saving from the old to the new sector.

Proposition 3 *(New Economy Equilibria)*

There exist $r^+ > 0$, $\Delta^{ro} > 0$ and $z^o > 0$ such that dynamic system (9)-(10) has precisely two steady states (k_L^*, q^*) and (k_H^*, q^*), where $k_H^* > k_L^*$.

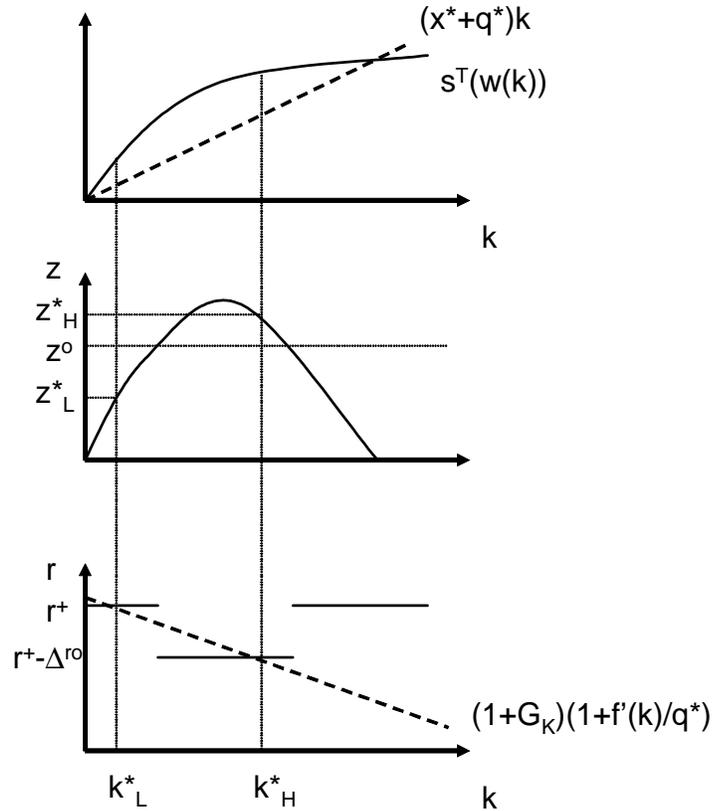**Proof:** See the appendix.

Figure 6: Equilibria

The key mechanism in this two-sector model is the endogenous relation between expansion and saving. The top panel in figure 6 illustrates the feedback on saving, $s_t^T = (1 + \rho)w(k_t)/2$, from expanding k_t. Given q_t (which in the figure we take as being q^*), the effect of this feedback on the capital market and the interest rate is measured by the surplus $(1 + \rho)w(k_t)/2 - (x(q_t) + q_t)\,k_t$ that it generates. This surplus then funds capital in the z-sector (middle panel) and, therefore, determines the interest rate by (16) (step function in the bottom panel). By the concavity of the wage function, the expansion of

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the new sector raises this surplus when k_t is low and reduces it when k_t is high. Thus, the impact on the interest rate of increasing k_t is negative for $k_t < \underline{k}(q_t)$ and positive for $k_t > \underline{k}(q_t)$, where $\underline{k}(q)$ is defined implicitly from $(1+\rho)w'(\underline{k}(q))/2 = x(q) + q$. For low k_t, expanding the new sector generates more saving than it uses and reduces the interest rate; for high k_t, it generates less saving than it uses and increases the interest rate. Note that the effect of k_t on the interest rate is greater, as the degree of diminishing returns increases, $|h''|$, in the old sector, here captured by Δ^{ro}.[10]

The dashed line in the bottom panel of figure 6 represents the return on investing in one unit of k, taking q^* as given. It is downward sloping because of decreasing returns in the new sector. If decreasing returns in the old sector are pronounced relative to those of the new sector —that is, if z^0 is small and Δ^{ro} is large relative to $|f''|$— then a high-valuation equilibrium is possible. This is the case illustrated in the figure and captures the idea that the new sector is relatively immature. If this were not the case, the old sector would compete for saving with the new sector and would cause the interest rate to rise relative to the new sector marginal product. This would prevent a sufficient fall in the effective cost of capital that is behind the high-valuation steady state.

In this case, as illustrated in figure 7, the phase diagram around the high-valuation and low-valuation steady states is similar to that in figure 3 of section 2.[11] However, because the mapping from k to z depends on the value of q, —the higher is the latter, the higher is the value of k corresponding to z^o— there is a region where the q-constant arm does not exist. This is just a technical issue which does not affect our analysis of equilibrium or dynamics.[12] As before, there are two saddle-path stable steady states: (k_L^*, q^*) and (k_H^*, q^*). Both are associated with the same price of capital, q^*. However, (k_H^*, q^*) exhibits a lower marginal product of capital and a lower effective cost of capital than (k_L^*, q^*).

The dominant forces that determine the low-valuation steady state is diminishing returns in the two sectors. Equilibrium is reached when marginal products are equal in both sectors,

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[10] Of course, the "increasing" and "reducing" interest happens only discontinuously in our simple step-function model. More generally,

$$r_k(k_t, q_t) = \frac{h''(z_{t+1})}{1+\gamma} \left[\tfrac{1}{2}(1+\rho)w'(k_t) - (x(q_t) + q_t) \right]\Bigg|_{z_{t+1} \text{ given by } (17)-(18)}. \tag{19}$$

[11] If $k_1^* < \widetilde{k}^l(q^*)$, which we assume, there can be no equilibrium in the region $k > \widetilde{k}^u(q)$. The reason is that in this region the q-stable arm is the continuation of the same arm for $k < \widetilde{k}^u(q)$, which is monotonically decreasing and has already intersected the k-stable arm at k_1^*.

[12] For the latter, there is still a well defined value of q, for any given k in that region, such that dynamics change from those below an (imaginary) q-constant arm to those above it.

after taking account of adjustment costs. However, the two sectors would respond very differently in an expansion beyond (k_L^*, q^*). It is the relative immaturity of the new sector compared to the old that makes the high-valuation steady state, (k_H^*, q^{**}), possible. It implies that the new sector can generate the saving necessary to fund an expansion. In our specific model, the new sector is able to generate saving by increasing wages. More generally, it can do so through other forms of income, such as profits — or, if it is technologically intensive, through externalities.
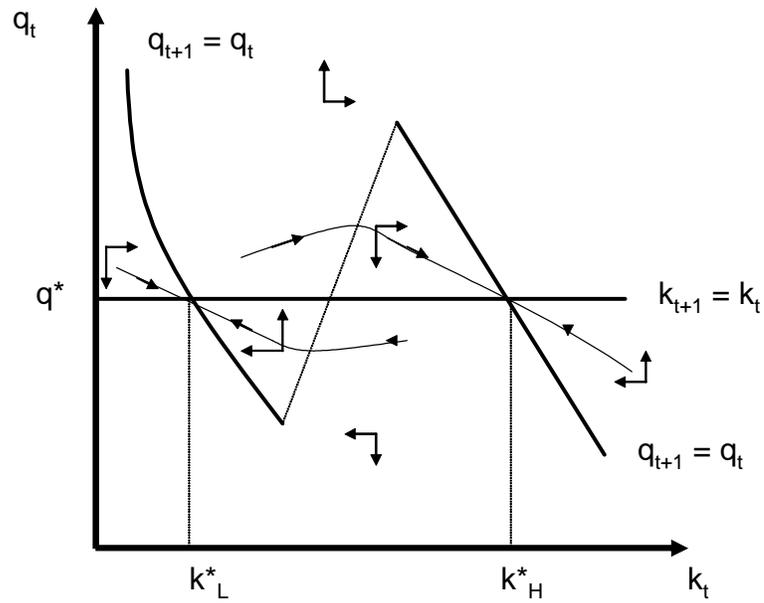

Figure 7: The Two-sector Model

What determines transitional dynamics along a speculative growth expansion? Suppose the economy starts at the low-valuation steady state and experiences an unanticipated jump onto the equilibrium path that converges to the high-valuation steady state. Sector reallocation plays an important role in this transition. The immediate increase in capital

Table 1: US Multifactor Productivity Growth

time period	growth rate (%, annualized)
1981-1985	0.51
1986-1990	0.52
1991-1995	0.58
1996-2000	1.21

Note: Multifactor Productivity is for the Private Nonfarm Business sector.
Source: Bureau of Labor Statistics.

accumulation in the new sector is caused by the reallocation of capital from the old sector to the new, accompanied by a jump in q, combined with an initial increase in the interest rate. As the new sector expands, it starts generating a saving surplus that helps to restore resources to the old sector and reverses the hike in the interest rate (increases z toward z^o, in our piece-wise linear model). Ultimately, as the economy converges to (k_H^*, q^*), the old sector expands beyond its initial capital stock and the cost of capital falls below its initial level.

3.2 Technology Bubbles

The U.S. speculative expansion of the 1990s not only was concentrated in the technology sector, but also was associated with an increase in productivity growth beyond what is attributable to pro-cyclical productivity (e.g., Baily and Lawrence (2001)). Table 1 presents average multifactor productivity growth for the U.S. economy in subsequent five-year intervals. It illustrates the acceleration of productivity growth in the second half of the 1990s. This type of evidence formed the basis for the case that the U.S. expansion was not caused by market speculation, but rather by underlying fundamentals that pointed to a technological revolution (e.g., Greenwood and Jovanovic, 1999; Hobijn and Jovanovic, 2000). To address this issue, we endogenize growth in our two-sector model and show that the speculative growth equilibrium is associated with higher productivity growth. Moreover, we show that increased productivity growth provides increased future income which fuels the key feedback from growth to saving. Thus, increased productivity growth is not only a result of market speculation, but also can play a central role in the mechanisms that make a speculative equilibrium possible. In our model, a technological revolution can be considered an integral part — both as cause and consequence — of a speculative growth equilibrium.

Endogenizing growth. In order to endogenize growth in our two-sector model, we assume

linear production functions in the two sectors, set $\rho = 0$, and write aggregate output as

$$Y_t = \overline{A}K_t^e L + \overline{r}Z_t, \quad \overline{A}, \overline{r} > 0.$$

As before, an expansion in K benefits labor (savers) while an expansion in Z does not. We interpret K_t as "technology" capital, subject to standard external spillovers, and make the extreme assumption that K_t enters as a pure externality in production, which is indicated by the superscript e on K_t^e. Because K_t enters as a pure externality, its private fundamental value is $q^f \equiv 0$ (let investment be irreversible as well). However, its price may be higher than q^f and incorporate a bubble. As before, K_t is accumulated according to (2), subject to adjustment costs; and Z_t is subject to no adjustment costs.

Accumulation of K_t is still determined by the q-theory relationship, $x_t = x(q_t)$. In the absence of a private dividend, arbitrage equation (8) governing q_t becomes

$$(1 + r_t)\, q_t = (1 + G_K \langle x(q_t) \rangle) q_{t+1}. \tag{20}$$

As long as $Z_t > 0$, the interest rate is equal to the marginal product of Z_t:

$$r_t = \overline{r}. \tag{21}$$

Finally, since the wage is $\overline{A}K_t^e$, our assumption of undiscounted log-utility implies that saving is $\frac{1}{2}\overline{A}K_t^e$, and the capital-market equilibrium condition is:

$$\tfrac{1}{2}\overline{A}K_t^e = Z_{t+1} + (x(q_t) + q_t)\,K_t^e. \tag{22}$$

Speculative growth. Starting from an initial stock $K_0^e > 0$, this economy can exhibit two equilibria: a *low-valuation* equilibrium that corresponds to fundamental-value pricing of technology, $q^f = 0$; and a *high-valuation* equilibrium that incorporates a bubble $q_t^* > 0$ in the price of technology. Under the former equilibrium, $x_L^* = x(0) = 0$ and $K_L^{e*} \equiv K_0^e$ is not accumulated. $Z_L^* = \frac{1}{2}\overline{A}K_0^e$ is determined by capital-market equilibrium condition (22) and, by the normalization $L = 1$, aggregate output is

$$Y_L^* = \left(1 + \tfrac{1}{2}\overline{r}\right)\overline{A}K_0^e.$$

This equilibrium exhibits no productivity growth and no capital accumulation over time. It is dynamically efficient since $\overline{r} > \gamma = 0$.

Under the high-valuation equilibrium, as long as Z_{Ht}^* is positive, (20) and (22) imply that $G_K \langle x(q_t^*) \rangle \equiv \overline{r}$. The bubble is therefore equal to $q_t^* \equiv x^{-1}\left(G_K^{-1}\langle\overline{r}\rangle\right)$ and $x_{Ht}^* \equiv G_K^{-1}\langle\overline{r}\rangle$. By (2), K_t^e grows at rate $\gamma_H^* = g(x_H^*)$. Capital-market equilibrium condition (22) implies

that $Z^*_{Ht+1} = \left(\frac{1}{2}\overline{A} - x^*_H - q^*\right) K^e_t$ and grows at the same rate as K^{e*}_{Ht}. Z^*_{Ht} is positive if $\overline{A} > 2\left(x^*_H - q^*\right)$, which we assume. Aggregate output is given by

$$Y^*_{Ht} = \left[\overline{A} + \frac{\overline{r}}{1 + \gamma^*_H}\left(\tfrac{1}{2}\overline{A} - x^*_H - q^*\right)\right] K^{e*}_{Ht}.$$

Thus, in the high-valuation equilibrium, productivity and output grow at endogenous rate γ^*_H.

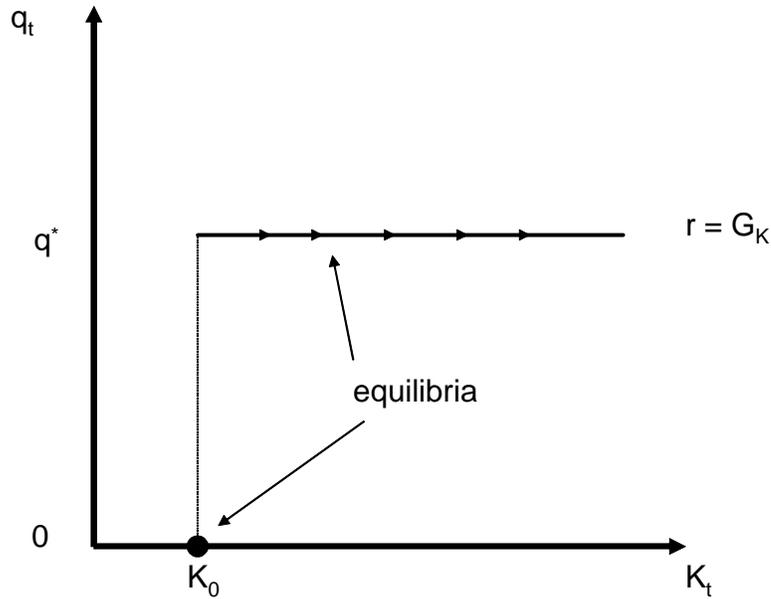Figure 8 illustrates these two equilibria.

Figure 8: Endogenous Growth Model

Discussion. This model provides a stark example of speculative equilibrium that results in sustained endogenous growth. Even though neither the K-sector nor the Z-sector are subject to social diminishing returns, only the former can be the engine of growth. The reason is that it is the only one that can generate the feedback from growth to saving necessary to fund investment along the balanced growth path. In the low-valuation trap, all saving is

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trapped in the Z-sector, which does not generate more saving as it grows. If the economy moves to the high-valuation equilibrium and incentivizes technology accumulation, saving shifts to the technology sector which will create more saving as it grows. This growth-saving feedback emerges through the technological spillover associated with technology investment, as productivity growth increases the income of savers. The feedback reduces the effective cost of capital, $r - G_K$, and gives rise to a high-valuation equilibrium.

Two aspects of the high-valuation equilibrium are worth noting. First, unlike our previous models, here the fall in the effective cost of capital does not occur through a fall in the interest rate, which is equal to a constant, $\bar{r}$. High valuations in the stock market will not be reflected by high valuations in the bond market. The reduction in the effective cost of capital takes the form of an increase in the flow-value, qG_K, offered by capital in opening opportunities for profitable investment. This value increases along with the increase in the investment ratio, x. Put differently, the speculative increase in the valuation of technology is not associated with a fall in the interest rate, but rather with an increase in the value of future investment opportunities opened by technology. This increase in value is intimately associated with the high pace of investment in the high-valuation equilibrium.

Second, the effective cost of capital in the high-valuation equilibrium falls to an extreme level, $\bar{r} - G_K = 0$, consistent with the emergence of a technology bubble. In the presence of externalities, such a bubble — if sustainable — can enhance welfare. Speculation of this type can substitute for well-functioning markets and can encourage investment and growth. Obviously, the risk is that the bubble may crash and trigger a crisis, an aspect we do not model in this paper.

4 Public and Foreign Saving

Starting from the initial conditions of the low-valuation steady state in the models of sections 2 and 3, the high-valuation equilibrium implies an investment boom. For this investment boom to materialize, it is important that it be supported by an increase in saving both in the short term and in the long term.[13]

Both models, whether we consider the short or the long run, rely on an increase in *private domestic* saving. Of course, this is only one of the possible saving channels that can fund investment. Figure 7 depicts the saving channels that funded the investment booms associated with the Nikkei and the Nasdaq bubbles. The investment boom in both

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[13]Strictly, the model in section 3 does not need an increase in aggregate saving at impact since reallocation from old to new sector is possible.

a: United States

b: Japan
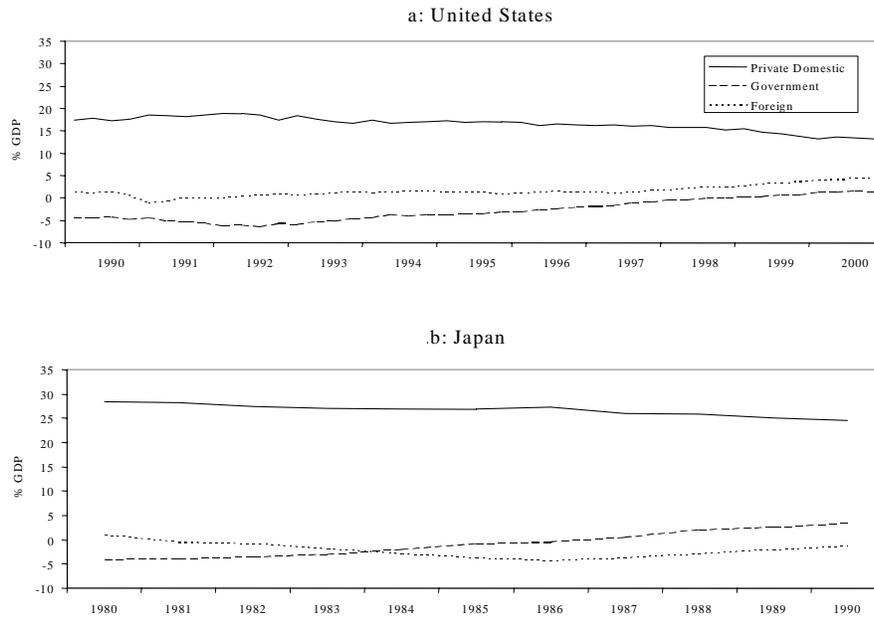
Notes: Panel a: private domestic saving is "Gross Private Saving"; government saving is "Gross Government Saving" minus "Gross Government Investment"; foreign saving is minus "Net Foreign Investment". Panel b: for each domestic sector, gross saving is used which is defined as the sum of "Saving", "Consumption of fixed capital" and "Capital transfer etc. net"; foreign saving is minus "Net lending to the rest of the world".
Sources: Panel a: BEA, NIPA Table 5.1. Panel b: Annual report on national accounts (CD-ROM) 1998.

Figure 9: Saving Channels

episodes actually was accompanied by a *decline* in private domestic saving. The main funding sources were government *budget surpluses* and the *current account*. This evidence relates to the funding of investment at the inception and "early" stages of the speculative expansion. Since the bubbles burst, we have no direct evidence on the long-term funding issues.

Guided by the short-run evidence and its relevance for the policy debate, this section focuses on two alternatives to domestic private saving in supporting speculative growth: government saving and the current account.

4.1   Fiscal Policy and Surplus Illusion

The fiscal surpluses generated during the Japanese and U.S. speculative growth experiences were the combined result of fiscal consolidation measures and the automatic effect of procyclical tax revenues in a booming environment. The reduction in public debt had a moderating effect on interest rates in the short run and substituted for increased private saving to fund the investment boom. It is in this sense that a tight policy of paying down the national debt supported of the boom in the short term. Turning to the more novel issue of longer-term sustainability, and hence to a necessary condition for a rational speculative growth episode, we argue that a continuing policy rule of generating fiscal surpluses provides critical support to the high-valuation equilibrium.

To develop our argument, we add a government to the first model of section 3 (with $\rho = 0$). We assume the government taxes wage income at a rate τ_t and spends $g_t A_t L$ on goods that do not enter agents' utility function. Thus the government's budget constraint is

$$(1+\gamma)d_{t+1} = (1+r_t)d_t - (\tau_t w_t - g_t), \tag{23}$$

where $d_t \equiv D_t/A_t L$ denotes public debt per unit of effective labor, and we have set $\rho = 0$.

Adding the public sector alters two equations in the model. The saving of the young net of investment in z, s_t^k, is now a function of after-tax wages:

$$s_t^k = s^k \left((1 - \tau_t)w(k_t), r_t\right), \tag{24}$$

while the capital-market equilibrium condition is now given by

$$(\tau_t w_t - g_t - r_t d_t) + s_t^k = d_t + (q_t + x(q_t)) k_t. \tag{25}$$

The saving of the government and of the young must fund the purchase of the existing public debt and capital stock from the old as well as funding new investment. Combined, equations (24)-(25) implicitly define a new interest rate function $r_t = r(k_t, q_t, d_t, \tau_t, g_t)$. The economy's dynamics are described by system (9)-(10) with the new interest rate function together with the government budget constraint, (23), and a fiscal policy rule.

Let us consider a benchmark *fixed-parameters* policy rule under which detrended government spending and the tax rate are fixed at $\overline{g} > 0$ and $\overline{\tau} > 0$. To facilitate comparisons, we assume that this and other policy rules result in a balanced budget in the low-valuation steady state — which requires $\overline{g} = \overline{\tau} w_L^*$ — and, therefore, result in the same low valuation steady state. The important point to notice is that this fixed-parameters policy creates primary surpluses during expansions beyond k_L^* and primary deficits during contractions.

To see this, note that the fiscal rule implies that the primary government surplus, $(\overline{\tau}w_t - \overline{g})$, increases with w_t in an expansion. Thus the response of the combined "gross" saving of government and the young to an increase in wages is given by $\left(\overline{\tau} + s_w^k(1 - \overline{\tau})\right)dw_t$. As long as the young's marginal propensity to save is less than one $(s_w^k < 1)$, which we assumed, the fixed-parameter rule generates surpluses as the wage rises.

The increase in saving generated by the fixed-parameters rule not only facilitates the funding of investment in the short run but, more importantly, plays a central role in facilitating the emergence of a speculative growth scenario. This feasibility point is made most clearly by focusing on the high-valuation steady state rather than on the entire path. One can show easily that if, instead of being fixed, government spending is raised with the endogenous increase in the wage, then aggregate saving falls and so does the high-valuation steady-state capital stock, k_H^*. In fact this fiscal expansion experiment is similar to reducing Δ^s in section 2. For a high enough indexation of fiscal spending to wages, the high investment equilibrium is no longer feasible. In other words, fiscal surpluses are not only a symptom of the speculative growth episode, but also can be a central element of the factors that support it.

The notion that the fiscal surpluses generated by the high-investment equilibrium can be partly spent or rebated to the taxpayer may be an illusion. The surpluses could be a pillar of the high-investment equilibrium, and might swiftly disappear if this equilibrium unravels — giving rise to what one might term as a *surplus illusion*.

Proposition 4 *(Surplus Illusion)*
Let $g_t = \overline{g} + \alpha\overline{\tau}(w_t - w_L^*)$. If the economy has two saddle path steady state equilibria for $\alpha = 0$, there exists an $\alpha^m > 0$ such that for any $\alpha \geq \alpha^m$, the high valuation equilibrium disappears.

Proof: See the appendix.

4.2 The Current Account

The second major source of funding for the investment boom in Japan and the U.S. was the current account. As a short-term funding mechanism, international capital flows can moderate the rise in interest rates needed to fund the investment boom.[14] Over time, the

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[14] Ventura (2001) describes an interesting alternative portfolio channel connecting the current account and a domestic bubble. In his model, the main effect of the bubble is to raise domestic wealth. As domestic agents attempt to rebalance their portfolios by investing in non-bubbly domestic equity, external borrowing

whole world may be dragged into a speculative path, with the home economy becoming the chronic recipient of external resources.

A simple reinterpretation of the model in section 2 (or 3) helps isolate these short and long run aspects of capital inflows. For this purpose, let domestic residents now correspond to the old at time 0, and assume they owe foreigners $b_0 A_0 L = q^* k_L^* A_0 L$. All future (active) generations are foreigners that live abroad and export to the domestic economy intermediate inputs with 100 percent labor content, measured by L. Only the domestic economy accumulates capital, K_t. Except for the labeling of the intergenerational flows, this model can be analyzed in exactly the same way as the model in section 2. Net exports correspond to $A_0 L f(k_0) - I_0 - (q_0 K_0 - b_0 A_0 L)$ at date 0 and $A_t L f(k_t) - I_t$ thereafter.

Let us use as a benchmark the low-valuation steady state, where the economy experiences de-trended trade surpluses $f(k_L^*) - x^* k_L^*$ for the indefinite future. Now suppose that, starting at the same level of capital, k_L^*, the price of capital jumps to a speculative growth path. Domestic residents sell the spread, $(q_0 - q^*) k_L^* A_0 L_0$, to foreigners and consume the proceeds. Thus the stock market rise fuels a domestic consumption boom financed by foreign capital inflows. The new high-investment equilibrium is then financed fully by foreigners. If the high-valuation equilibrium is dynamically inefficient, then net exports are asymptotically negative, and the domestic economy becomes a resource sink for foreigners. The current account remains negative throughout and, again, becomes an integral part of the speculative growth episode.

5   Conclusion

This paper attempts to build a theoretical framework for thinking about episodes of speculative growth. We characterize this phenomenon as a high-valuation equilibrium based on optimism about the future availability of funds for investment and, therefore, about a low effective cost of capital. Our framework highlights the key short-term and long-term funding mechanisms necessary to sustain a speculative growth equilibrium. The emergence of a new sector, fiscal surpluses, or current account deficits not only are possible consequences of a high-investment equilibrium, but also are central to generating the saving necessary to support it.

The principles we uncovered are more general than the models we discussed. To keep matters focused, we highlighted purely neoclassical factors behind the supply and demand for capital. It is not difficult to enrich the model and to add financial frictions, which

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rises to finance the investment that is required to build domestic equity.

can substitute for some of our mechanisms. For example, in the model in section 3 we could eliminate the decreasing returns in the old sector and instead split the young into entrepreneurs and savers. The latter can invest in old-capital and lend to entrepreneurs. The multiplicity can be generated by the different values of entrepreneurs' internal funds and collateral along the different paths. A speculative path in this context is driven by the endogenous relaxation of financial constraints.

Similarly, the role of fiscal rules in developing the conditions for a speculative growth episode also can be achieved by private sector practices. For example, stock options reallocate income flows toward managers and some key workers along a speculative growth path. If these are natural entrepreneurs or savers, then stock options will create a fertile ground for speculative growth episodes.

Finally, ours is not a framework of "irrational exuberance," although it offers a natural interpretation of such episodes. These episodes occur when the long-run funding mechanism is not strong enough to eventually validate the explosive capital gains dynamics that are needed in the short run to reallocate resources toward the sectors that drive the boom. A prototypical example of such a scenario is when a bubble attaches to the "wrong" sector. Importantly, whether such a situation is present or not bears little relation to the price-earnings ratio and related statistics often discussed by analysts — these may be useful in assessing relative valuations but miss the point when we assess the feasibility of an aggregate stock market boom. Of course, it is hard to verify in real time whether future conditions will support a rational speculative growth episode or not, but one could in principle check whether expectations of such a mechanism are consistent with the path. To this effect, we point out that the U.S. episode of the 1990s ocurred together with a significant decline in the slope of the yield curve, much of which was due to the fall in long rates. Exploring this lead seems worthwhile.

A Parametric Example underlying Figure 4

This appendix provides the functional forms and parameter values underlying the simulations presented in figure 4. The following functional forms were used:

$$
\begin{aligned}
f(k) &= k^{\xi}, \\
g(x) &= x - \phi x^2, \\
\widetilde{s}(w, r) &= \sigma(r)w,
\end{aligned}
$$

where

$$
\sigma(r) = \frac{\beta^{\theta}(1+r)^{\theta-1}}{1 + \beta^{\theta}(1+r)^{\theta-1}}.
$$

With $\xi \in (0,1)$ the production function satisfies all the assumptions stated in the text. The functional form for g yields the investment function

$$
x(q) = -q + \sqrt{q^2 + \frac{q-1}{\phi}}.
$$

The domain of x is $[\underline{q}, \infty)$ and its range is given by $\left[-\underline{q}, \frac{1}{2\phi}\right)$ where

$$
\underline{q} = -\frac{1}{2\phi} + \sqrt{\frac{1}{4\phi^2} + \frac{1}{\phi}}.
$$

Note that the function g satisfies $g'(x) > 0$ only for $x < \frac{1}{2\phi}$, but as $x(q) < \frac{1}{2\phi}$ we have $g'(x(q)) > 0$ as well as $g''(x(q)) < 0$ for all $q \in [\underline{q}, \infty)$.

The functional form for g also yields

$$
G_K \langle x \rangle = \phi x^2.
$$

The saving function $\widetilde{s}$ is derived from a CES utility function with elasticity of substitution θ and discount factor β. It satisfies $\widetilde{s}_w > 0$ and setting $\theta > 1$ insures $\widetilde{s}_r > 0$.

Using equation (12) we obtain the interest rate function

$$
r(k,q) = \begin{cases} \sigma^{-1}\left(\frac{(x(q)+q)k^{1-\xi}}{1-\xi}\right), & k < k^o; \\ \sigma^{-1}\left(\frac{(x(q)+q)k^{1-\xi}}{1-\xi}\right) - \Delta^r, & k \geq k^o. \end{cases}
$$

Clearly $r_k > 0$ for $k < k^o$ as well as for $k > k^o$.

The parameter values used are given in table A.1.

Table A.1: parameters

parameter	value
ξ	0.05
ϕ	0.4
β	0.76
θ	20
k^o	0.157
Δ^r	0.054
γ	0.25

B Proof of Propositions in Section 2

First we give a summary of the assumption made in the text, supplementing them by some technical conditions.

Assumption 1 *The production function $f : \mathbf{R}_+ \to \mathbf{R}_+$ is three times differentiable and satisfies*

1. *$\lim_{k\downarrow 0} f'(k) = +\infty$, $\lim_{k\uparrow\infty} f'(k) = 0$, $\lim_{k\downarrow 0} w'(k) = +\infty$, $\lim_{k\uparrow\infty} w'(k) = 0$, and*

2. *$f'(k) \geq 0$, $f''(k) \leq 0$, $w'(k) \geq 0$, $w''(k) \leq 0$ for all $\mathbf{R}_+$,*

3. *for $k \in \mathbf{R}_+$, $f'(k) > 0$ implies $f''(k) < 0$ and $w'(k) > 0$ implies $w''(k) < 0$,*

   where $w(k) = f(k) - f'(k)k$.

Assumption 2 *The function $\widetilde{s} : \mathbf{R}_+ \times [-1, +\infty) \to \mathbf{R}_+$ is continuously differentiable and satisfies*

1. *$0 < \widetilde{s}_w(w, r) \leq 1$, $0 < \widetilde{s}_r(w, r) < \infty$,*

2. *$\lim_{r\to\infty} \widetilde{s}(w, r) = w$ for all $w \in \mathbf{R}_+$,*

3. *$\widetilde{s}(w, -1) = 0$ for all $w \in \mathbf{R}_+$.*

In the main text it was pointed out that under weak conditions $r_k(k, q) > 0$ in all relevant regions except at points with $k = k^o$. We will simplify the analysis further here with a condition that ensures that this holds everywhere but at points with $k = k^o$.

Assumption 3 *The production function f and the function $\tilde{s}$ satisfy*

$$\left(\frac{\tilde{s}_w(w(k),r)w(k)}{\tilde{s}(w(k),r)}\right) \cdot \left(\frac{w'(k)k}{w(k)}\right) < 1$$

for all $k \in \mathbf{R}_+$ and $r \in (-1,+\infty)$.

Assumption 4 *The adjustment cost function $g : \mathbf{R} \to \mathbf{R}$ is twice continuously differentiable and satisfies*

1. *$g(0) = 0$, $\lim_{x \uparrow \infty} g(x) = \infty$,*

2. *$g'(x) > 0$ for all $x \in \mathbf{R}$ and $g'(0) = 1$,*

3. *$\lim_{x \uparrow \infty} g'(x) = 0$,*

4. *$g''(x) < 0$ for all $x \in \mathbf{R}$.*

Assumption 5 *The rate of technological progress is nonnegative, that is $\gamma \geq 0$.*

Proposition 1 *Under assumptions 1–5, there exists a $\Delta^r > 0$ and a $k^o > 0$ such that the dynamic system (9)–(10) has precisely two steady states (k_L^*, q^*) and (k_H^*, q^*), where $k_H^* > k_L^*$.*

Lemma 5 *Under assumption 4 the investment choice problem of the young agents has a unique solution $x(q)$ for all $q > \underline{q}$ and no solution for $q \leq \underline{q}$, where $\underline{q} \in (0,1)$ is the unique solution of the equation $g(-q) = -1$. The function $x : (\underline{q},\infty) \to \mathbf{R}$ is continuously differentiable, strictly increasing and satisifies $\lim_{x \to \infty} x(q) = \infty$. If assumption 5 holds as well, then there exists a unique q^* such that $g(x(q^*)) = \gamma$ and we have $q^* \geq 1$.*

Proof: As $g(0) = 0$, $g'(0) = 1$ and $g''(x) < 0$, we have $g(x) \leq x$ with strict inequality for $x \neq 0$. In particular we have $\lim_{x \to -\infty} g(x) = -\infty$. Then the equation $g(-q) = -1$ has precisely one solution $\underline{q}$, which must lie the interval $(0,1)$.

Now consider the problem of a young agent with savings $s > 0$ of choosing his investment optimally, facing a price of capital of $q > 0$. His problem is

$$\max_{(x,k) \in \mathbf{R}^2} (1 + g(x))k \qquad \text{s.t.} \quad (x+q)k \leq s \ \wedge \ k \geq 0.$$

Clearly this problem has no solution if $g(-q) > -1$, that is if $q < \underline{q}$. In this case the agent could set $x = -q$. Then the first constraint is satisfied for all $k \geq 0$ and the objective $(1 + g(-q))k$ can be made arbitrarily large.

So consider the case $g(-q) \leq -1$, that is $q \geq \underline{q}$. In this case the agent can achieve a positive value of the objective by setting $x = 0$ and $k = \frac{s}{q}$. As choosing $x \leq -q$ makes the objective nonpositive and setting $k = 0$ makes the objective zero, we only need to consider $x > -\underline{q} \geq -q$ and $k > 0$. Among these choices, it is clear that $(x + q)k < s$ cannot be optimal. So if

$$\frac{1 + g(x)}{x + q} \tag{26}$$

has a unique maximizer $x(q)$ in $(-q, +\infty)$, then $\left(x(q), \frac{s}{x(q)+q}\right)$ is the unique solution to the maximization problem of the agent. The derivative of this function with respect to x is

$$\frac{g'(x)(x + q) - (1 + g(x))}{(x + q)^2}.$$

Consider the numerator

$$D_q(x) = g'(x)(x + q) - (1 + g(x)).$$

We have

$$D_q(-\underline{q}) = g'(-\underline{q})(q - \underline{q}).$$

and

$$D_q'(x) = g''(x)(x + q) < 0$$

for all $x \in (-q, +\infty)$. If $q = \underline{q}$, then $D_q(-\underline{q}) = 0$. In this case the agent would like to set x arbitrarily close to $-\underline{q}$, so the problem does not have a solution. However, if $q > \underline{q}$, we have $D_q(-\underline{q}) > 0$.

Moreover, as g is concave,

$$D_q(x) \leq g'(x)q - 1,$$

so

$$\lim_{x \to \infty} D_q(x) \leq -1.$$

Thus the equation $D_q(x) = 0$ has a unique solution $x(q)$ which is the unique maximizer on $(-q, +\infty)$ of the expression in equation (26).

So now we have constructed the function $x : (\underline{q}, +\infty) \to \mathbf{R}$, and it satisfies

$$g'(x(q))(q + x(q)) = (1 + g(x(q))) \tag{27}$$

for all $q \in (\underline{q}, +\infty)$. By the implicit function theorem, the function $x(q)$ is continuously differentiable and we have

$$x'(q) = \frac{-g'(x(q))}{g''(x(q))(x(q) + q)} > 0,$$

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so $x(q)$ is strictly increasing. Suppose $\lim_{q \to \infty} x(q) = \bar{x} < \infty$. As $g'(\bar{x}) > 0$, taking limits of (27) as $q \to \infty$ yields infinity for the left hand side but $1 + g(\bar{x}) < \infty$ for the right hand side. Thus it must be the case that $\lim_{q \to \infty} x(q) = \infty$.

By substituting $q = 1$ into equation (27) we see that $x(1) = 0$.

Finally consider the equation

$$g(x(q)) = \gamma. \tag{28}$$

We have

$$g(x(1)) = 0,$$

and since $\lim_{q \to \infty} x(q) = \infty$ and $\lim_{x \to \infty} g(x) = \infty$ we have $\lim_{q \to \infty} g(x(q)) = \infty$. Thus if $\gamma \geq 0$ there is a unique q^* solving equation (28) and clearly $q^* \geq 1$. $\qquad\square$

Lemma 6 *Let assumptions 1–5 be satisfied. Then for all $q > \underline{q}$, the equation*

$$\tilde{s}(w(k), r) = (x(q) + q)k \tag{29}$$

has a unique solution $\tilde{r}(k, q)$ for all $k \in (0, \bar{k}(q))$ where $\bar{k}(q) > 0$ is the unique solution to the equation

$$w(k) = (x(q) + q)k. \tag{30}$$

The function $\tilde{r}(k, q)$ is continuously differentiable and satisifes $\lim_{k \uparrow \bar{k}(q)} \tilde{r}(k, q) = \infty$.

Proof: Let $q > \underline{q}$ and $k > 0$. We begin by analyzing equation (30). Consider the function $E_q : \mathbf{R}_+ \to \mathbf{R}$ where

$$E_q(k) = w(k) - (x(q) + q)k.$$

Then $\lim_{k \downarrow 0} E_q(0) = 0$ and

$$E_q'(k) = w'(k) - (x(q) + q).$$

The assumption that $\lim_{k \downarrow 0} w'(k) = +\infty$ ensures that $E_q(k)$ is strictly positive for sufficiently small k. As long as $w'(k)$ is strictly positive, it is strictly decreasing. Taking into account the limits of $w'(k)$, this means that the equation $E_q'(k) = 0$ has a unique solution $k_{\max}$ which is the global maximizer of E_q. Then we have $E_q(k_{\max}) > 0$ and $E_q'(k) < 0$ for $k > k_{\max}$. Moreover, $\lim_{k \uparrow \infty} w'(k) = 0$ implies that $E_q(k)$ eventually turns negative. Hence there is a unique $\bar{k}(q) \in (0, +\infty)$ such that $E_q(\bar{k}(q)) = 0$. The function $\bar{k} : (\underline{q}, \infty) \to \mathbf{R}_+$ is continuously differentiable by the implicit function theorem.

Now consider equation (29). The left hand side is increasing in r and satisfies $\tilde{s}(w(k), -1) = 0$ and $\lim_{r \to \infty} \tilde{s}(w(k), r) = w(k)$. So the equation has at most one solution $\tilde{r}(k, q)$, and this

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solution exists if and only if $k < \bar{k}(q)$. The function $\tilde{r}(k, q)$ is continuously differentiable by the implicit function theorem and

$$\tilde{r}_k(k, q) = \frac{1 - \left(\frac{\tilde{s}_w(w(k), \tilde{r}(k,q)) w(k)}{\tilde{s}(w, \tilde{r}(k,q))} \right) \cdot \left(\frac{w'(k)k}{w(k)} \right)}{\frac{\tilde{s}_r(w(k), \tilde{r}(k,q))k}{\tilde{s}(w(k), \tilde{r}(k,q))}} > 0.$$

It is also easy to see that $\lim_{k \uparrow \bar{k}(q)} \tilde{r}(k, q) = +\infty$. Suppose $\lim_{k \uparrow \bar{k}(q)} \tilde{r}(k, q) = \bar{r} < \infty$, then taking limits of

$$\tilde{s}(w(k), \tilde{r}(k, q)) = (x(q) + q)k$$

as $k \uparrow \bar{k}(q)$ yields

$$\tilde{s}(w(\bar{k}(q)), \bar{r}) = (x(q) + q)\bar{k}(q),$$

which contradicts the fact that for finite $\bar{r}$ we have

$$\tilde{s}(w(\bar{k}(q)), \bar{r}) < w(\bar{k}(q)) = (x(q) + q)\bar{k}(q).$$

$\square$

Lemma 7 *Let assumptions 1– 5 be satisfied. Then for all $q > \underline{q}$, the equation*

$$s(w(k), r) = (x(q) + q)k \tag{31}$$

has a unique solution $r(k, q)$ for all $k \in (0, \bar{k}(q))$ and

$$r(k, q) = \tilde{r}(k, q) - I(k \geq k_0)\Delta^r$$

where the function $\bar{k}$ and $\tilde{r}$ are as in lemma 6. The function $r(k, q)$ is continuously differentiable except at points (q, k) with $k = k^o$, and $\lim_{k \uparrow \bar{k}(q)} r(k, q) = \infty$.

Proof: Suppose $q > \underline{q}$ and $k \in (0, \bar{k}(q))$. Then the equation

$$\tilde{s}(w(k), r) = (x(q) + q)k$$

has the unique solution $\tilde{r}(k, q)$. Suppose $k < k^o$, then $s(w(k), r) = \tilde{s}(w(k), r)$ for all $r \in [-1, \infty)$ and consequently $r(k, q) = \tilde{r}(k, q)$ is the unique solution of equation (31). If $k \geq k^o$ equation (31) can be written as

$$\tilde{s}(w(k), r + \Delta^r) = (x(q) + q)k,$$

which has the unique solution $r(k, q) = \tilde{r}(k, q) - \Delta^r$. The properties of the function $r(k, q)$ then follow from lemma 6. $\square$

Proof of proposition 1: Given the results of the preceding lemmas, (k^*, q^*) is a steady state if and only if k^* satisfies the equation $m(k) = 0$ where

$$m(k) = \frac{\tilde{r}(k, q^*) - I(k \geq k_0) \cdot \Delta^r - G_K \langle x(q^*) \rangle}{1 + G_K \langle x(q^*) \rangle} - f'(k). \tag{32}$$

The function $m(k)$ is continuous and strictly increasing except for a downward jump at k^o. Hence the equation $m(k) = 0$ has at most two solutions. We will choose k^o and Δ^r so that there are precisely two solutions. Consider the function

$$\tilde{m}(k) = \frac{\tilde{r}(k, q^*) - G_K \langle x(q^*) \rangle}{1 + G_K \langle x(q^*) \rangle} - f'(k). \tag{33}$$

It is continuous on $(0, +\infty)$ and has $\tilde{m}'(k) > 0$ for all $k \in (0, +\infty)$, $\lim_{k \downarrow 0} m(k) = -\infty$ and $\lim_{k \uparrow \bar{k}(q^*)} m(k) = +\infty$. Thus there exists a unique $k_L^* \in (0, \bar{k}(q^*))$ such that $\tilde{m}(k_L^*) = 0$. Choosing $k^o \in (k_L^*, \bar{k}(q^*))$, we also have $m(k_L^*) = 0$.
Next consider the function

$$\tilde{m}_{\Delta^r}(k) = \frac{\tilde{r}(k, q^*) - \Delta^r - G_K \langle x(q^*) \rangle}{1 + G_K \langle x(q^*) \rangle} - f'(k). \tag{34}$$

For each $\Delta^r \in$ there is a unique $k_H(\Delta^r)$ such that $\tilde{m}_{\Delta^r}(k_H(\Delta^r)) = 0$. The function $k_H(\Delta^r)$ is strictly increasing and $\lim_{\Delta^r \to \infty} k_H(\Delta^r) = \bar{k}(q^*)$. Choosing Δ^r such that $k_H^* = k_H(\Delta^r) > k^o$, we have $m(k_H^*) = 0$. $\qquad \square$

Assumption 6 *There exists $\bar{k} \in (0, +\infty)$ such that $f'(k) > 0$ for all $k < \bar{k}$ and $f'(k) = 0$ for all $k \geq \bar{k}$. Moreover $w(\bar{k}) < (x(q^*) + q^*)\bar{k}$ where q^* is as in lemma 5.*

Definition 1   *1. A speculative path $\{k_t, q_t\}_{t=0}^{\infty}$ is an equilibrium path with origin $k_0 < k^o$ that converges to the high valuation steady state (k_H^*, q^*).*

   2. The price of capital at time t along a speculative path has a bubble component if there exists N such that $n \geq N$ implies $f'(k_{t+n}) = 0$.

   3. The price of capital at time t along a speculative path is a pure bubble if $f'(k_{t+n}) = 0$ for all $n \geq 1$.

Proposition 2 *Under assumptions 1–6, there exists a $\Delta^r > 0$ and a $k^o > 0$ such that the dynamic system (9)–(10) has precisely two steady states (k_L^*, q^*) and (k_H^*, q^*), where $k_L^* < k^o < k_H^*$ and $k_H^* > \bar{k}$. Along a speculative path, the price of capital has a bubble component, and it becomes a pure bubble for all $k \geq \bar{k}$.*

Proof: The second part of assumption 6 implies $\bar{k} < \bar{k}(q^*)$ where $\bar{k}(q^*)$ is defined as in lemma 5.

From proposition 1 we can choose k^o and Δ^r such that there are precisely two steady states (k_L^*, q^*) and (k_H^*, q^*) with $k_L^* < k^o < k_H^*$.

Recall that in the proof of that proposition we constructed k_H^* by choosing Δ^r such that $k_H(\Delta^r) > k^o$. But as $\lim_{\Delta^r \to \infty} k_H(\Delta^r) = \bar{k}(q^*)$ and both $k^o < \bar{k}(q^*)$ and $\bar{k} < \bar{k}(q^*)$, we can also choose Δ^r such that $k_H(\Delta^r) > \max(k^o, \bar{k})$.

As a speculative path $\{k_t, q_t\}_{t=0}^{\infty}$ converges to (k_H^*, q^*) by definition, there exists N such that $t \geq N$ implies that $k_t \geq \bar{k}$ and thus $f'(k_t) = 0$. Hence q_t has a bubble component for all $t \geq 0$. As $k_{t+1} \geq k_t$ for all $t \geq 0$ along a speculative path, $k_t \geq \bar{k}$ implies that q_t is a pure bubble. $\square$

C Proof of Proposition 3

Assumption 7 *The production function f satisfies assumption 1 and in addition $f''(k) < 0$ for all $k \in \mathbf{R}_+$.*

Proposition 3 *Under assumptions 3–5 and 7 there exist $r^+ > 0$, $\Delta^{ro} > 0$ and $z^o > 0$ such that the dynamic system (9)–(10) has precisely two steady states (k_L^*, q^*) and (k_H^*, q^*), where $k_H^* > k_L^*$.*

Proof: We begin by choosing r^+, Δ^{ro} and $z^o > 0$ such that there are precisely two steady states with positive levels of z. Then we show that for these values of r^+, Δ^{ro} and $z^o > 0$ there is no steady state with $z = 0$. We set $\rho = 0$ as this plays no qualitative role in the proposition.

From lemma 5 the function $x(q)$ is well defined and there is a unique q^* such that $g(x(q^*)) = \gamma$. Consider the function $E_{q^*} : \mathbf{R}_+ \to \mathbf{R}$ with

$$E_{q^*}(k) = \frac{1}{2} w(k) - (x(q^*) + q^*)k.$$

As discussed in the proof of lemma 6 this function has a unique global maximizer $k_{\max}$ such that $E_{q^*}(k_{\max}) > 0$, $E'_{q^*}(k) > 0$ for $k < k_{\max}$ and $E'_{q^*}(k) < 0$ for $k > k_{\max}$. So we can choose z^o such that

$$0 < (1 + \gamma)z^o < E_{q^*}(k_{\max}),$$

and as $\lim_{k \to 0} E_{q^*}(k) = 0$ and $\lim_{k \to \infty} E_{q^*}(k) = -\infty$ the equation $E_{q^*}(k) = (1 + \gamma)z^o$ has precisely two solutions $0 < \tilde{k}^l(q^*) < \tilde{k}^u(q^*)$. Also note that there is a unique $\bar{k}(q^*)$ such

that $E_{q^*}(\bar{k}(q^*)) = 0$.

Now if $z > 0$ then

$$r(k, q^*) = \begin{cases} r^+, & 0 < k < \widetilde{k}^l(q^*) \quad \text{or} \quad \widetilde{k}^u(q^*) < k < \bar{k}(q^*), \\ r^+ - \Delta^{ro}, & \widetilde{k}^l(q^*) \leq k \leq \widetilde{k}^u(q^*), \end{cases}$$

where Δ^{ro} is still to be determined. The capital stock in a steady state with $z > 0$ must satisfy the equation $m(k) = 0$ where

$$m(k) = \frac{r(k, q^*) - G_K \langle x(q^*) \rangle}{1 + G_K \langle x(q^*) \rangle} - f'(k). \tag{35}$$

The function $m(k)$ is strictly increasing and continuous except for a downward jump at $\widetilde{k}^l(q^*)$ and an upward jump at $\widetilde{k}^u(q^*)$. Hence the equation $m(k) = 0$ has at most two solutions. We will now choose r^+ and Δ^{ro} such that there will be precisely two solutions. Pick $k_L^* \in (0, \widetilde{k}^l(q^*))$ and set

$$r^+ = (1 + G_K \langle x(q^*) \rangle) \left(1 + \frac{f'(k_L^*)}{q^*} \right) - 1 > 0.$$

Similarly pick $k_H^* \in (\widetilde{k}^l(q^*), \widetilde{k}^u(q^*))$ and set

$$\Delta^{ro} = 1 + r^+ - (1 + G_K \langle x(q^*) \rangle) \left(1 + \frac{f'(k_H^*)}{q^*} \right).$$

Then $k_H^* > k_L^*$ and both satisfy the equation $m(k) = 0$. Choosing k_H^* sufficiently close to $\widetilde{k}^l(q^*)$ insures that $\Delta^{ro} < r^+$.

Now suppose there is a steady state with $z = 0$. Then the capital stock in this steady state must be equal to $\bar{k}(q^*) > k_L^*$, which implies a steady state interest rate

$$r = (1 + G_K \langle x(q^*) \rangle) \left(1 + \frac{f'(\bar{k}(q^*))}{q^*} \right) - 1 < r^+,$$

but at this interest rate agents would want to invest in z, so $z = 0$ is inconsistent with equilibrium. $\qquad \square$

D Proof of Proposition 4

Assumption 8 *The interest rate $r^+ - \Delta^{ro}$ is strictly less than the rate of technological progress γ.*

Proposition 4 *Let $g_t = \bar{g} + \alpha \bar{\tau}(w_t - w^*)$. Suppose that the economy satisfies assumption 8. Also suppose that for $\alpha = 0$ it has two steady states (k_L^*, q^*, d_L^*) and (k_H^*, q^*, d_H^*) with $k_H^* > k_L^*$. Then there exists $\alpha^m > 0$ such that for any $\alpha \geq \alpha^m$ the high-valuation equilibrium disappears.*

Proof: Let $\alpha \geq 0$ be given and suppose the high-valuation equilibrium exists for this α. The two steady state capital stocks are given by

$$k_L^* = (f')^{-1}\left(\frac{r^+ - G_K \langle x(q^*)\rangle}{1 + G_K \langle x(q^*)\rangle}\right),$$
$$k_H^* = (f')^{-1}\left(\frac{r^+ - \Delta^{ro} - G_K \langle x(q^*)\rangle}{1 + G_K \langle x(q^*)\rangle}\right).$$

Note that k_H^* does not depend on α as there are only two levels for the interest rate in this economy.

Now the interest rate in the high-valuation steady state must be $r^+ - \Delta^{ro}$. This requires that the implied steady state value of z exceeds z^o. This yields the condition

$$\frac{1}{2}(1-\bar{\tau})w(k_H^*) + (\bar{\tau}w(k_H^*) - g_H^*) - (1 + r_H^*)d_H^* - (x(q^*) + q^*)k_H^* > (1+\gamma)z^o \qquad (36)$$

where $g_H^* = \bar{g} + \alpha\bar{\tau}(w(k_H^*) - w(k_L^*))$ is government spending and $r_H^* = r^+ - \Delta^{ro}$ is the interest rate in the high-valuation steady state. Using equation (23), we obtain

$$d_H^* = -\frac{\bar{\tau}w(k_H^*) - g_H^*}{\gamma - r_H^*}.$$

Using the fact that $\bar{g} = \bar{\tau}w(k_L^*)$, we have

$$\bar{\tau}w(k_H^*) - g_H^* = (1-\alpha)\bar{\tau}(w(k_H^*) - w(k_L^*)).$$

Then we can rewrite condition (36) as

$$\frac{1}{2}(1-\bar{\tau})w(k_H^*) + \frac{1+\gamma}{\gamma - r_H^*}(1-\alpha)\bar{\tau}(w(k_H^*) - w(k_L^*)) - (x(q^*) + q^*)k_H^* > (1+\gamma)z^o.$$

By assumption this condition is satisfied for $\alpha = 0$. But as the left hand side is linear in α with a negative coefficient, there exists $\alpha^m > 0$ such for all $\alpha \geq \alpha^m$ the condition is violated and the high-valuation steady state does not exist. $\qquad\square$

References

[1] Abel, A.B., N.G. Mankiw, L. H. Summers and R. J. Zeckhauser (1989): "Assessing Dynamic Efficiency: Theory and Evidence," *Review of Economic Studies*, 56, 1-20.

[2] Baily, M.N., and R.Z., Lawrence (2001): "Do We Have a New E-Economy?" NBER WP# 8243, April.

[3] Blanchard, Oliver, and Stanley Fisher (1989): *Lectures on Macroeconomics*, Cambridge: The MIT Press.

[4] Carroll, Christopher D., and David N. Weil (1994): "Saving and Growth: A Reinterpretation," *Carnegie-Rochester Conference Series on Public policy*, 40, 133-192.

[5] Cecchetti, Stephen G., Hans Genberg , John Lipsky and Sushil B. Wadhwani (2000): *Asset Prices and Central Bank Policy*, Geneva Reports on the World Economy 2, London: Center for Economic Policy Research.

[6] Chirinko, R.S., and H. Schaller (2001), "Business Fixed Investment and "Bubbles": The Japanese Case," *American Economic Review*, June, 663-680.

[7] Cozzi, G. (1998), "Culture as a Bubble," *Journal of Political Economy*, vol. 106(2), 376–394.

[8] Diamond, Peter (1965): "National Debt in a Neoclassical Growth Model," *American Economic Review*, 55, 1126–1150.

[9] Gavin, M., R. Hausmann, E. Talvi (1997), "Saving Behavior in Latin America: Overview and Policy Issues," IADB Wp #346, May.

[10] Greenwood, Jeremy, and Boyan Jovanovic (1999): "The Information-Technology Revolution and the Stock Market," *American Economic Review* (Papers and Proceedings), 89(2), 116-122.

[11] Grossman, G., and N. Yanagawa (1993): "Asset Bubbles and Endogenous Growth," *Journal of Monetary Economics*, 31, 3-19.

[12] Hobijn, Bart, and Boyan Jovanovic (2001): "The Information-Technology Revolution and the Stock Market: Evidence," *American Economic Review*, 91(5), 1203-1220.

[13] International Monetary Fund (2000): *World Economic Outlook - Asset Prices and the Business Cycle*, Washington: International Monetary Fund.

[14] Kindleberger, Charles P., (1989): *Manias, Panics, and Crashes*, (revised edition) Basic Books Inc., Harper Collins, USA.

[15] King, I., and D. Ferguson (1993): "Dynamic Inefficiency, Endogenous Growth and Ponzi Games," *Journal of Monetary Economics,* 79-104.

[16] Olivier, Jacques (2000): "Growth-Enhancing Bubbles," *International Economic Review,* 41(1), 133-151.

[17] Rappoport, P. and E.N. White (1993): "Was there a Bubble in the 1929 Stock Market?" *The Journal of Economic History 53-3*, September, 549-574.

[18] Saint-Paul, Gilles (1992): "Fiscal Policy in an Endogenous Growth Model," *European Economic Review,* 36, 763-781.

[19] Shiller, Robert J. (2000): *Irrational Exuberance,* Princeton: Princeton University Press.

[20] Tirole, Jean (1985): "Asset Bubbles and Overlapping Generations," *Econometrica,* 53(6), 1499-1528.

[21] Ventura, Jaume (2001): "A Portfolio View of the U.S. Current Account Deficit," *Brookings Papers on Economic Activity,*1, 241-253.

[22] ____, (2002): "..", MIT mimeo, May 2002.

[23] Woodford, Michael (1991): "Public Debt as Private Liquidity," *American Economic Review,* 80(2), 382-388.