Endogenous Monetary Policy with Unobserved Potential Output

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Abstract

This paper characterizes endogenous monetary policy when policymakers are uncertain about the extent to which movements in output and inflation are due to permanent changes in potential output or to temporary, but persistent, demand and cost shocks. We refer to this informational limitation as the "permanent - temporary confusion" (PTC). Two main results of the paper are: 1. Under reasonable conditions policy is likely to be excessively loose (restrictive) for some time when there is a large decrease (increase) in potential output in comparison to a no PTC benchmark. The framework thus makes a step towards providing a unified explanation for the inflation of the seventies and the price stability of the nineties. 2. Using potential output forecasts in policy formulation leads to an ex-ante welfare superior monetary policy as long as the noise in the potential output indicator is finite.

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1. Introduction

A stabilizing role for monetary policy crucially hinges on some notion of 'potential output', a non-observable economic variable representing the desirable (or target) level at which actual output should be. The conduct of monetary policy requires, therefore, that the central bank forecasts, and continually updates, its forecast of potential output. Orphanides (2000, 2001) argues that the real-time information problem inherent to (a trend notion of) potential output makes it undesirable to have a policy rule that is based on such a measure, or the related output gap measure. In particular he provides persuasive support for the view that a significant overestimation of potential output during the oil shocks of the seventies aggravated inflation at that time by leading to a monetary policy stance which turned out to be, with the benefit of hindsight, excessively loose ex-post. Somewhat symmetrically, the strong productivity gains recorded in the United States during the second half of the 1990s raised the possibility, again with the benefit of hindsight, that the subsequently greater-than-expected increases in potential output could have allowed for a less restrictive monetary policy stance than the stance initially suggested by real time estimates of inflation and the output gap.

The work of Orphanides sheds interesting new light on monetary policy during the seventies and raises an important question about the extent to which such retrospective policy mistakes can be avoided in the future. If they were due to poor but correctable forecasting procedures or to an inefficient specification of the "policy rule", a likely answer to this question is yes.

Assessing the extent to which such mistakes were due to "bad policies" rather than to "bad luck" requires a model which identifies optimal monetary policy under imperfect information. Once this benchmark is defined, and its properties are established, one can proceed to evaluate the extent to which (retrospective) policy errors were avoidable. This paper contributes to the debate on the effects of imperfect information by proposing such a benchmark model and analyzing its properties.

We show that, given the structure of information, some policy decisions, which are judged ex-post to be mistakes, may be unavoidable even if the central bank utilizes the most efficient forecasting procedures available to it at the time. Moreover, such retrospective mistakes are small during periods in which changes in potential output are small, and large during periods characterized by substantial changes in the long run trend of output. During the latter episodes policy mistakes in a given direction are likely to persist for some time.

Those claims are established in an environment where potential output is a random walk and actual output and inflation are affected by stationary and persistent demand and cost shocks. We assume that the central bank cannot perfectly disentangle (not even ex-post) the changes in inflation and output that are due to changes in potential output from the movements that are due to higher frequency changes in demand and costs. We refer to this inevitable confusion between temporary demand and cost shocks on one hand and permanent (unit root type) shocks to potential output on the other as the "permanent - temporary confusion" or PTC in brief.[1]

The evidence in Orphanides (2001) supports the view that monetary policy during the seventies was excessively loose since a permanent reduction in potential output was interpreted for some time as a negative output gap. The analytical framework of this paper provides an "optimizing" analytical foundation for this mechanism and identifies the conditions under which it operates.[2] Interestingly, a large permanent decrease in potential output does not lead to an excessively loose policy stance under all circumstances. Whether it does or not depends on the relative persistence of demand and of cost shocks, and on other parameters like the degree of conservativeness of the central bank.

The results above are developed within a garden variety macroeconomic model which underlies the conception of many central banks about the transmission process of monetary policy.[3] The paper identifies conditions under which the presence of the PTC leads monetary policy to be *systematically* tighter than under perfect information in periods of permanent increases in potential output and to be too loose relatively to this benchmark in periods of permanent reductions in poten-

[1] The macroeconomic consequences of this confusion were discussed following the oil shocks of the seventies within frameworks in which monetary policy is exogenous (Brunner and Meltzer (1978), Brunner, Cukierman and Meltzer (1980) and Part II of Cukierman (1984)).

[2] Related work in which potential output is specified as a Hodrick-Prescott filter appears in Lansing (2000). Two differences between our paper and that of Lansing are that in our paper the forecast of potential output is derived from the stochastic structure of the economy, and the policy rule is derived from the loss function of policymakers. By contrast, Lansing postulates both of those concepts exogenously.

[3] A compact formulation of this economic structure appears in Svensson (1997).

tial output. The reason is that, even when they filter available information in an optimal manner, policymakers as well as the public at large detect permanent changes in potential output only *gradually*. When, as was the case in the seventies, there is a permanent decrease in potential output, policymakers interpret part of this reduction as a negative output gap and loosen monetary policy too much in comparison to the no PTC benchmark.[4] Thus, in periods of large permanent decreases in productivity, inflation accelerates because of the relatively expansionary monetary policy stance. Conversely, when – as might have been the case in the US during the nineties – a "new economy" permanently raises the potential level of output, inflation goes down since, as policymakers interpret part of the permanent increase in potential output as a positive output gap, policy is tighter than under perfect information.

In summary the paper provides a unified framework for understanding some of the reasons for the inflation of the seventies, as well as for the remarkable price stability of the nineties. It illustrates how the speed of learning by policymakers and policy deviations from an ideal full-information-benchmark depend on the stochastic structure of the various economic shocks.[5] Identification of such conditions is a necessary first step for empirically testing the hypothesis that imperfect information is quantitatively important for monetary policy and inflation. Moreover, the finding that during periods of large changes in potential output (retrospectively) monetary policy errors are committed does not necessarily imply that an optimally devised forecast of potential output should not be used to formulate policy. The analysis implies that, except for the limiting case in which the variance of demand shocks is infinite, the utilization of such forecasts improves the expected value of policy objectives.

The paper is organized as follows. Section 2 presents a reduced-form model of endogenous monetary policy in the presence of imperfect information about

[4] During periods of large changes in potential output retrospective monetary policy errors will appear to be serially correlated expost in spite of the fact that central bank forecasts are formed rationally. The analytical argument at the base of this claim is analogous to the one developed in the context of tests of market efficiency by Cukierman and Meltzer (1982).

[5] We do not attempt to discriminate between this interpretation of the inflation history and the alternative hypotheses explored by Sargent (1999; (focusing on changing policymakers' beliefs about the Phillips curve tradeoff), Ireland (1999; who relates inflation to the dynamics of structural unemployment) or Albanesi, Chari and Christiano (2000, who show the existence of sunspots equilibria with different steady state inflation rates).

the permanence of shocks to potential output and characterizes optimal monetary policy choices in this environment. The consequences for the behavior of real rates of interest, inflation and the output gap in comparison to their full information counterparts are analyzed in Section 3. Section 4 develops the real time optimal forecasts of potential output and discusses the consequences for the magnitude and persistence of retrospective monetary policy errors during periods of large permanent shocks. This is followed by (preliminary) concluding remarks.

2. Endogenous monetary policy in the presence of uncertainty about potential output

This section presents a simple model of the macroeconomy that reflects the views of many central banks about the transmission process of monetary policy. This model is a version of the backward looking sticky-price model presented in Svensson (1997). Although the model is not rooted in explicit microfoundations it is widely used, sometimes only implicitly, by decision makers at most central banks to reach decisions about interest rate policy. Since our main objective is to highlight the consequences of imperfect information about potential output in the presence of optimally chosen monetary policy we maintain the assumption that this reduced form model captures the actual behavior of the economy.

2.1. The economy

In this framework (the logarithm of) output (y_t) and inflation (π_t) are determined, respectively, as follows:

$$y_t = z_t - \varphi r_t + g_t \tag{2.1}$$

$$\pi_t = \pi_{t-1} + \lambda(y_t - z_t) + u_t. \tag{2.2}$$

Here z_t denotes (the log of) potential output as of period t, r_t is a *real* short term interest rate, g_t is a demand shock and u_t a cost-push shock.[6] This framework

[6] This simple economic structure allows the basic consequences of imperfect information to be highlighted in a simple manner. A richer economic structure, incorporating transmission lags or forward looking variables, does not eliminate the effects described in the paper but may introduce new ones. Although such models may be preferable for theoretical and an empirical

postulates that potential output z is a fundamental long run determinant of actual output. But, in addition, actual output is also affected by a demand shock and by the real rate of interest, which for given inflationary expectations, is determined in turn by the (nominal) interest rate policy of the central bank.

We assume the economy is subject to two types of temporary but persistent shocks and to a permanent shock to the level of potential output. The temporary shocks are the aggregate demand shock, g_t, and the cost-push shock, u_t. In line with conventional macroeconomic wisdom we postulate that the demand and cost shocks are less persistent than changes in potential output which are affected by long run factors like technology and the accumulation of physical and human capital.[7] The permanence of shocks to potential output is modeled by assuming that z_t is a random walk. More specifically we postulate the following stochastic processes for the shocks:

$$g_t = \mu g_{t-1} + \hat{g}_t \qquad 0 < \mu < 1; \qquad \hat{g}_t \sim N(0, \sigma_g^2) \tag{2.3}$$

$$u_t = \rho u_{t-1} + \hat{u}_t \qquad 0 < \rho < 1; \qquad \hat{u}_t \sim N(0, \sigma_u^2) \tag{2.4}$$

$$z_t = z_{t-1} + \hat{z}_t \qquad \hat{z}_t \sim N(0, \sigma_z^2). \tag{2.5}$$

To reiterate, the main purpose of this simple model is to characterize the macroeconomic consequences of optimally chosen monetary policy (i.e. a sequence for r_t) when policymakers cannot identify with certainty (not even retrospectively) the sources of output changes.

2.2. Monetary Policy

The policy instrument is the nominal interest rate. But since prices are temporarily sticky the policymaker can bring about the real rate he desires by setting the nominal rate. For convenience and without loss of generality we can therefore consider the policymaker as setting the real interest rate r_t. This policy instrument is set at the *beginning* of period t before output, inflation (y_t and π_t) and period t shocks are realized. The policy objective is to minimize the objective

reasons, we prefer to introduce our arguments by means of a parsimonious specification.

[7] The notion that demand shocks are relatively less persistent than shocks to potential output underlies the empirical identification of demand and of supply factors in Blanchard and Quah (1989).

function:

$$L_t \equiv \frac{1}{2} E \left\{ \sum_{j=0}^{\infty} \beta^j \left[\alpha (x_{t+j})^2 + (\pi_{t+j})^2 \right] \mid J_t \right\} \qquad \alpha > 0 \tag{2.6}$$

where $x_t \equiv y_t - z_t$ denotes the output gap (defined as the difference between (the logarithms) of actual and of potential output) and J_t is the information set available at the beginning of period t, when r_t is chosen. The first order condition for the discretionary (time-consistent) monetary policy ($\min_{r_t} L_t$) implies

$$x_{t|t} = -\frac{\lambda}{\alpha} \pi_{t|t}. \tag{2.7}$$

Here $x_{t|t}$ and $\pi_{t|t}$ are the expected values of inflation and of the output gap conditional on the information available at the beginning of period t. At this stage we note that the information set, J_t, available to policymakers at the beginning of period t contains, among other, observations on actual inflation and output up to and including period $t-1$. A fuller specification of J_t appears below. Since period's t values of inflation and of the output gap are not known with certainty at the beginning of period t, those variables (which are indirectly controlled by policy) appear in equation (2.7) in expected terms.

The equilibrium outcomes for the interest rate, output and inflation obey:[8]

$$r_t = \frac{1}{\varphi} \left[g_{t|t} + \frac{\lambda}{\alpha} q (\pi_{t-1} + u_{t|t}) \right] \qquad \text{where } q \equiv \frac{\alpha}{\alpha + \lambda^2} \in (0,1) \tag{2.8}$$

$$y_t = z_t + (g_t - g_{t|t}) - \frac{\lambda}{\alpha} q (\pi_{t-1} + u_{t|t}) \tag{2.9}$$

$$\pi_t = q(\pi_{t-1} + u_t) + \lambda \left(g_t - g_{t|t} \right) + \frac{\lambda^2}{\alpha} q \left(u_t - u_{t|t} \right). \tag{2.10}$$

2.3. The structure of information and optimal policy

The interest rate rule in (2.8) implies that the optimal real interest rate policy for period $t+1$, r_{t+1}, requires the policymaker to form expectations about the values of the demand shock and the cost push shocks, g_{t+1} and u_{t+1}. Although he does not observe those shocks directly, the policymaker possesses information about

[8] These expressions are obtained by expressing expected inflation in terms of the variables $\pi_{t-1}$and $u_{t|t}$ which are known at the beginning of period t. Details appear in Appendix A.

economic variables from which noisy, but optimal, forecasts of the shocks can be derived. In particular we assume that policymakers know the true structure of the economy: $\Omega \equiv \left\{\varphi, \lambda, \rho, \mu, \sigma_u^2, \sigma_g^2, \sigma_z^2\right\}$ but do not know the precise stochastic sources of fluctuations in output and inflation.

Thus, when the interest rate r_{t+1} is chosen, at the beginning of period $t+1$, the policymaker forms expectations about g_{t+1} and u_{t+1} using historical data. The latter consists of observations on output and inflation up to and including period t. The information available at the beginning of period $t+1$ is summarized by the information set

$$J_{t+1} = \{\Omega, y_{t-i}, \pi_{t-i,} \mid i = 0, 1, 2, ...\} \tag{2.11}$$

which is used to form the conditional expectations: $g_{t+1|t+1}$ and $u_{t+1|t+1}$. Past observations on output and inflation are equivalent to past observations on the two signals, $s_{1,t}$ and $s_{2,t}$ (obtained by rearranging (2.9) and (2.10)):

$$s_{1,t} \equiv y_t + g_{t|t} + \frac{\lambda}{\alpha} q(\pi_{t-1} + u_{t|t}) = z_t + g_t \tag{2.12}$$

$$s_{2,t} \equiv \pi_t - q\pi_{t-1} + \lambda g_{t|t} + \frac{\lambda^2}{\alpha} q u_{t|t} = \lambda g_t + u_t \tag{2.13}$$

where variables to the left of the equality sign are observed separately while those to the right are not.[9] Clearly, $s_{1,t}$ and $s_{2,t}$ contain (noisy) information on g_t and u_t which can be used to make inference on g_{t+1} and u_{t+1}, using the fact that $g_{t+1|t+1} = \mu g_{t|t+1}$ and $u_{t+1|t+1} = \rho u_{t|t+1}$.

Notice how the optimal estimates of g_t and u_t conditional on J_{t+1}, $g_{t|t+1}$ and $u_{t|t+1}$ respectively, follow immediately from the two signals (2.12) and (2.13), once the optimal filter of potential output, $z_{t|t+1}$, is known.[10] Therefore, the signal extraction (or filtering) problem solved by the policymaker reduces to an inference problem concerning the level of potential output.

[9] In particular, the construction of the signals, s_{1t} and s_{2t} needed for the formation of the filtered values $u_{t+1|t+1}$, $g_{t+1|t+1}$ and $z_{t+1|t+1}$ utilizes the previous period's filters $u_{t|t}$ and $g_{t|t}$ (based on J_t), which are known at the beginning of period $t+1$.

[10] This follows from the fact that: $g_{t|t+1} = s_{1,t} - z_{t|t+1}$ and $u_{t|t+1} = s_{2,t} - \lambda(s_{1,t} - z_{t|t+1})$.

2.4. Mismeasurement of potential output and policymakers' views about the state of the economy

Let policy makers' forecast errors concerning the variables $z_t, g_t,\ u_t$ given the information set J_{t+1} be:

$$\tilde{u}_{t|t+1} \equiv u_t - u_{t|t+1} \tag{2.14}$$

$$\tilde{g}_{t|t+1} \equiv g_t - g_{t|t+1} \tag{2.15}$$

$$\tilde{z}_{t|t+1} \equiv z_t - z_{t|t+1} \tag{2.16}$$

Using (2.12) and (2.13) the following useful relationship between these errors can be derived :

$$\lambda\tilde{z}_{t|t+1} = -\lambda\tilde{g}_{t|t+1} = \tilde{u}_{t|t+1}. \tag{2.17}$$

The last equation shows that overestimation of potential output ($\tilde{z}_{t|t+1} < 0$) simultaneously *implies* an overestimation of the cost-push shock and an underestimation of the demand shock.[11] This is summarized in the following remark.

Remark 1. *Potential output overestimation ($\tilde{z}_{t|t+1} \equiv z_t - z_{t|t+1} < 0$) implies:*
(i) demand shock underestimation ($\tilde{g}_{t|t+1} \equiv g_t - g_{t|t+1} > 0$)
(ii) cost-push shock overestimation ($\tilde{u}_{t|t+1} \equiv u_t - u_{t|t+1} < 0$)
Inequalities with opposite signs hold when $\tilde{z}_{t|t+1} > 0$.

The intuition underlying this result can be understood by reference to equations (2.12) and (2.13). The first equation implies that an increase in $s_{1,t}$ is partially **and optimally** interpreted as being due to an increase in z_t and the remainder is interpreted as being due to an increase in g_t.[12] Similarly, an increase in $s_{2,t}$ is partially interpreted, for the same reason, as being due to an increase in g_t and the remainder is interpreted as being due to an increase in u_t. Thus, when only z_t increases, part of this increase is interpreted as an increase

[11] This can be seen immediately by rewriting the expressions for the estimates of g and u as

$$g_{t|t+1} = g_t - \tilde{z}_{t|t+1} \tag{2.18}$$

$$u_{t|t+1} = u_t + \lambda\tilde{z}_{t|t+1}. \tag{2.19}$$

[12] Such an interpretation of the increase in z_t is optimal by the minimum mean square error criterion. This is demonstrated rigorously in section 4 below.

in potential output, but the remainder is interpreted as an increase in g_t. As a consequence the error in forecasting z_t is positive and the error in forecasting g_t is negative, producing a **negative** correlation between the forecast errors in those two variables. Since $s_{2,t}$ does not change the (erroneously) perceived increase in g_t is interpreted as a decrease in u_t, producing a positive forecast error for this variable, and therefore, a **positive** correlation between the forecast errors in u_t and in z_t.

3. Consequences of forecast errors for monetary policy, inflation and the output gap

Remark 1 shows how mismeasurement of potential output distorts policymakers' perceptions about cyclical conditions (cost-push and demand shocks). The purpose of this subsection is to answer the following question: How do such noisy perceptions about the phase of the cycle affect monetary policy, inflation and the output gap? We do this by comparing the values of those variables in the presence of the permanent transitory confusion (PTC) with their values in the benchmark case in which there is no such confusion. In the benchmark case policymakers possess in each period *direct information* about the realizations of the shocks up to and including the previous period. Formally, in the absence of the PTC policymakers possess, at the beginning of period $t+1$, the information set J^*_{t+1} that is defined by

$$J^*_{t+1} = \{J_{t+1}, g_{t-i}, u_{t-i,} \mid i = 0, 1, 2, ...\} . \tag{3.1}$$

3.1. Consequences for monetary policy

We begin by studying the determinants of the difference between the settings of monetary policy in the presence and in the absence of the PTC. Using equations (2.8), (2.18), (2.19) and (2.17) the *deviation* of the optimal interest rate in the presence of the PTC from its optimal value in the absence of this confusion (i.e.

$r_t^* = \frac{1}{\varphi}\left[\mu g_{t-1} + \frac{\lambda}{\alpha} q(\pi_{t-1} + \rho u_{t-1})\right])$[13] can be written as

$$Dr_{t+1} \equiv r_{t+1} - r_{t+1}^* = -\frac{1}{\varphi}\left[\tilde{g}_{t+1|t+1} + \frac{\lambda}{\alpha} q \tilde{u}_{t+1|t+1}\right] \tag{3.2}$$

$$= \frac{\left(\mu - \rho\frac{\lambda^2}{\alpha+\lambda^2}\right)}{\varphi} \tilde{z}_{t|t+1}. \tag{3.3}$$

It follows immediately from (3.2) that if demand shocks are sufficiently persistent in comparison to cost shocks (i.e. $\mu > \frac{\rho\lambda^2}{\alpha+\lambda^2}$) the deviation of the real interest rate from its full information counterpart moves in the same direction as the forecast error in potential output ($\tilde{z}_{t|t+1}$).[14] Although one cannot rule out the possibility that, when the persistence in cost shocks is sufficiently larger than that of demand shocks, the opposite occurs, it appears that the first case seems more likely a-priori. The reason is that the persistence parameter of the cost shocks is multiplied by a fraction implying that Dr_{t+1} and $\tilde{z}_{t|t+1}$ are positively related even if ρ is larger than μ, but not by too much. Note that the smaller the (Rogoff (1985) type) conservativeness of the central bank (the higher α), the more likely it is that Dr_{t+1} and $\tilde{z}_{t|t+1}$ are positively related even when ρ is larger than μ. Hence, for central banks which are (using Svensson's (1997) terminology) relatively flexible inflation targeters the case in which Dr_{t+1} and $\tilde{z}_{t|t+1}$ are positively related is definitely the more likely one for most or all values of ρ and μ in the range between zero and one. The various possible effects of imperfect information are summarized in the following proposition:

Proposition 1. *(i) When the persistence of demand shocks is sufficiently high ($\mu > \frac{\rho\lambda^2}{\alpha+\lambda^2}$) monetary policy is driven mainly by "demand shocks" considerations. This implies that potential output over/under-estimation (causing the demand shock to be under/over-estimated) leads to real rates which are lower/higher than the rate which is optimal in the absence of the PTC.*

(ii) When the persistence of demand shocks is sufficiently low ($\mu < \frac{\rho\lambda^2}{\alpha+\lambda^2}$) monetary policy is driven mainly by "cost-push shocks" considerations. This implies that potential output over/under-estimation (causing the cost-push shock

[13] Here use has been made of the fact that the information set J_t^* available to policymakers in the absence of the PTC implies $g_{t|t} = \mu g_{t-1}$ and $u_{t|t} = \rho u_{t-1}$.

[14] Note that $\frac{\partial Dr_{t+1}}{\partial \tilde{z}_{t|t+1}} > 0$.

to be over/under-estimated) leads to a real rate which is higher/lower than the rate that is optimal in the absence of the PTC.

To understand the intuition underlying the proposition it is useful to consider the case in which there is, in period t, a negative shock to potential output and no changes in the cyclical shocks, g and u. This leads, as of the beginning of period $t+1$, to overestimation of potential output in period t ($\tilde{z}_{t|t+1} < 0$). Remark 1 implies that this overestimation is associated with an overestimation of the cost shock and an underestimation of the demand shock of period t.

The policy chosen at the beginning of period $t+1$ aims to offset the (presumed) deflationary impact of the demand shock on the output gap and the (presumed) inflationary impact of the cost shock on inflation. In comparison to the no PTC benchmark, the first objectives pushes policy towards expansionism while the second pushes it towards restrictiveness. If demand shocks are relatively persistent the first effect dominates since policymakers believe that most of what they perceive to be a negative demand shock in period t is going to persist into period $t+1$ while what they perceive to be a positive cost shock in period t is not going to persist into period $t+1$.[15] Hence, in this case monetary policy is more expansionary than in the no PTC benchmark and Dr_{t+1} and $\tilde{z}_{t|t+1}$ are positively related (case (i) in the proposition). But if the reverse is true (cost shocks are relatively more persistent) beliefs about the cost shock in period $t+1$ dominate policy pushing it towards tightening. As a consequence monetary policy is more restrictive than in the no PTC benchmark and Dr_{t+1} and $\tilde{z}_{t|t+1}$ are negatively related and case (ii) of the proposition obtains.

3.2. Consequences for the output-gap and inflation

We turn next to the consequences of mismeasurement of potential output for the output-gap and inflation. The objective is, as in the previous subsection, to analyze the deviations of the outcomes obtained in the presence of the PTC from those that arise in the absence of this confusion. Using (2.9) and (2.10) it is immediate to relate these deviations to the interest rate deviations studied above.

[15] This remark follows directly from the fact that $g_{t+1|t+1} = \mu g_{t|t+1}$ and $u_{t+1} = \rho u_{t|t+1}$.

This yields:

$$Dx_{t+1} \equiv x_{t+1} - x^*_{t+1} = -\varphi Dr_{t+1} \tag{3.4}$$

$$D\pi_{t+1} \equiv \pi_{t+1} - \pi^*_{t+1} = -\varphi\lambda Dr_{t+1} \tag{3.5}$$

where x^*_{t+1} and π^*_{t+1} are the values of the output gap and inflation under optimal monetary policy in the absence of the PTC. These equations show that when the interest rate is below (above) its value in the absence of the PTC both inflation and the output gap are above (below) their no PTC values.

The case of over-expansionary monetary policy (case (i) of proposition 1) is consistent with Orphanides (2000, 2001) empirical results according to which, during the seventies US monetary policy was overly expansionary due to an over-estimation of potential output and an associated underestimation of the output gap. Obviously, this underestimation could have been due to inefficient forecasting procedures on the part of the Fed. A main message of this paper is that this effect is present even if monetary policy is ex-ante optimal and forecasting procedures are as efficient as possible. In normal times during which the change in potential output is not too far from its mean this effect is likely to be small and short lived. But when large permanent shocks to potential output occur this effect is likely to be large and more persistent. This point is demonstrated and amplified in the next section.

4. Optimal forecasts of potential output and the effect of large changes in potential output on policy

This section describes the solution to the signal extraction, or filtering, problem faced by policymakers. To convey the intuition of the basic mechanisms at work we focus in the text on the particular (but simpler) case in which demand and cost push shocks are equally persistent ($\mu = \rho$), which yields a tractable closed form solution. The solution for the case in which the degrees of persistence differ ($\rho \neq \mu$) appears in Appendix B.2.

4.1. Filtering under equally persistent demand and cost-push shocks

This section describes the signal extraction problem faced by policymakers. To convey the intuition of the basic mechanisms at work we focus in the following on the simpler case in which demand and cost push shocks are equally persistent ($\mu = \rho$), which yields tractable closed form solutions (the case in which $\rho \neq \mu$ is treated in Appendix B.2). The conditional expectation of z_t based on J_{t+1}, $z_{t|t+1}$, is given by (see Appendix B.1 for the derivation):[16]

$$
\begin{aligned}
z_{t|t+1} &= aS_t + (1-a)(1-\kappa)\sum_{i=0}^{\infty}\kappa^i S_{t-1-i} && (4.1)\\
\text{where} &: \\
\kappa &\equiv \frac{2}{\phi+\sqrt{\phi^2-4}} \in (0,1) \qquad \phi \equiv \frac{2+T(1+\mu^2)}{1+\mu T} \geq 2;\ T \equiv \left(\frac{\sigma_z^2}{\sigma_g^2} + \frac{\lambda^2\sigma_z^2}{\sigma_u^2}\right) \\
a &\equiv \frac{[(1-\mu)+(1-\kappa)+T(1-\mu\kappa)]T}{[T(1-\mu-\mu\kappa)+(1-\mu-\kappa)](1+T)+(T+\mu)(1+\mu T)} \in (0,1) \\
S_{t-i} &\equiv s_{1,t-i} - \frac{\lambda\sigma_g^2}{\sigma_u^2+\lambda^2\sigma_g^2} s_{2,t-i} = z_{t-i} + \frac{\sigma_u^2 \cdot g_{t-i} - \lambda\sigma_g^2 \cdot u_{t-i}}{\sigma_u^2+\lambda^2\sigma_g^2} && (4.2)
\end{aligned}
$$

S_{t-i} is a combined signal that summarizes all the relevant information from period's $t-i$ data. Note that it is positively related to that period's potential output and demand shock, and negatively related to that period's cost shock. As a consequence the optimal predictor generally responds positively to current, as well as to all past, shocks to demand, and potential output, and responds negatively to current, as well as to all past cost shocks.

The conditional expectation (4.1) possesses several key properties. First, since a and κ are both bounded between zero and one, the current optimal predictor is positively related to the current, as well as to all past signals. Second, the weight given to a past signal is smaller the further in the past is that signal. Third, since $a < 1$, when a positive (negative) innovation to current potential output (z_t) occurs the potential output *estimate* increases (decreases) *by less* than actual potential output. Fourth, the sum of the coefficients in the optimal predictor in (4.1) is equal to one.

[16] This corresponds to the predictor of (the unit root) potential output, z_t, that minimizes the mean square forecast error.

4.2. Gradual learning and optimal monetary policy in the aftermath of large changes in potential output

The form of the optimal predictor in (4.1), in conjunction with the fact that all coefficients are positive and sum up to one implies that when a single shock to potential output occurs (say) in period t and persists forever without any further shocks to potential output, policymakers do not recognize its full impact immediately. Although their forecasting is optimal policymakers learn about the shock gradually. Initially (in period $t+1$) they adjust their perception of potential output by the fraction a. In period $t+2$ they internalize the larger fraction $a+(1-a)(1-\kappa)$, in period $t+3$ they internalize the, even larger, fraction $a+(1-a)(1-\kappa)+(1-a)(1-\kappa)\kappa$, and so on. After a large number of periods this fraction tends to 1, implying that after a sufficiently large number of periods the full size of the shock is ultimately learned. Thus, equation (4.1) implies that there is gradual learning about potential output and that forecast errors are on the same side of zero during this process.

Consider now the case in which a relatively large permanent change to potential output occurs in period t but other smaller and more frequent changes do occur subsequently. Since it is relatively large and since learning is gradual, period's t change dominates the learning process for some time. Hence the qualitative picture following that period is similar to the one above in which all subsequent shocks to potential output had been artificially shut off. In particular during the time period in the aftermath of period t, forecast errors are on the same side of zero as in the example above. As a consequence when looking backward forecast errors in potential output and the resulting monetary policy "errors" will appear to be serially correlated for some time.[17] The main lessons from these remarks are summarized in the following proposition.

Proposition 2. *Following a period characterized by the realization of a large*

[17] This **does not** occur in periods that do not follow the recent realization of a large change in potential output. The reason is that in the absence of a recent large permanent change the subsequent realizations of additional shocks to potential output assure that forecast errors are usually small and not systematically positioned on one side of zero. By contrast a large, even if locally isolated, change in potential output leads to large, and persistent for some time, forecast and policy errors. A formal proof of this statement in the context of tests of market efficiency in the presence of a permanent - transitory confusion appears in Cukierman and Meltzer (1982).

permanent change to potential output optimal monetary policy appears ex-post as being systematically biased in a certain direction for some time.

(i) When the shock is negative policy is too loose in comparison to the no PTC benchmark in which there is no uncertainty about the sources of change in output. Although optimal at the time, this policy stance is retrospectively judged as being too loose.

(ii) When the shock is positive policy is too restrictive in comparison to the no PTC benchmark. In particular, a large increase in potential output induces policymakers to behave in a way that overemphasizes the concern for price stability. Although optimal at the time, this policy stance is retrospectively judged as being too restrictive.

The first part of the proposition corresponds to the retrospectively loose monetary policy of the seventies identified by Orphanides (2000, 2001). This retrospective policy error was triggered by overestimation of potential output and underestimation of the output gap. The second part of the proposition appears to fit the "new economy" of the nineties. The large positive technological shock to potential output during the nineties was initially partly interpreted as a positive output gap and triggered a policy response that was judged retrospectively to be overly restrictive.

4.3. Should noisy indicators of potential output be used to guide monetary policy?

Orphanides appears to conclude from his empirical results concerning the seventies that, since they were found to be subject to large errors ex-post, real time estimates of potential output should not be used to guide monetary policy. The framework of this paper can be used to shed some light on this question. Obviously, if forecasting procedures are inconsistent with the stochastic structure of the economy such a conclusion is certainly warranted. The problem, however, is that errors in measuring potential output arise both in the case in which forecasts are optimal, as well as in the case in which they are not. Furthermore, the discussion in this section suggests that following large changes in potential output "policy errors" will be large and sustained even if forecasts are optimal. It is therefore impossible to conclude from the fact that forecasts of potential output are noisy, and at times large and systematically biased in one direction, that they should not be used as indicator variables for monetary policy.

The analytical exercise in the paper suggests (since the policy characterized here is optimal) that it is almost always better to utilize information about actual output and inflation to form forecasts of potential output and to use them to guide monetary policy, than to ignore those forecasts. The only exception is when the variance, σ_g^2, of demand shocks is infinite. In this case all the coefficients in the optimal predictor in equation (4.1) are zero and the predictor is a constant that is independent of macroeconomic developments concerning output and inflation. The reason, of course, is that in this case there is no information about potential output in either output or inflation. But, except for this extreme case, utilization of optimal forecasts of potential output improves the performance of monetary policy.[18] This is summarized in the following proposition.

Proposition 3. *Except for the extreme case in which σ_g^2 tends to infinity the outcomes of monetary policy are better when the policy rule utilizes noisy, but optimal, forecasts of potential output than when it does not.*

5. Concluding remarks

This paper provides a unified explanation for part of the inflation of the seventies, as well as for part of the remarkable price stability of the nineties. This is done by showing that, even if monetary policy is optimal and forecasts of potential output efficient, large permanent changes in potential output trigger excessively loose monetary policy when those changes are positive, and excessively tight policy when the changes are negative.

The framework in the paper also leads to two wider conclusions that are likely to transcend the particular model used to illustrate them. The first is that even if monetary policy is chosen optimally and even if, given the stochastic structure of shocks, available information is used as efficiently as possible, retrospective policy errors are unavoidable. During periods in which changes in potential output are moderate those errors are not too important, nor are they persistent. As

[18] As shown by Svensson and Woodford (2000) this conclusion is substantially more general than the settings of the particular model used here.

a consequence they do not draw much attention ex-post. But during periods following large sustained changes in potential output retrospective errors appear, with the benefit of hindsight, to be substantial and to be serially correlated. This makes them noticeable and draws public attention. Thus, even central banks that forecast and behave optimally will sometimes be judged retrospectively as having committed serious policy errors. But, since they had behaved efficiently at the time, it does not follow from this statement that (given the information structure) such errors can be avoided in the future.

Obviously, it does not necessarily follow from the above conclusion that policy and forecasting procedures during the seventies were as efficient as possible at the time. The point, however, is that it is not possible to conclude from the ex-post identification of policy errors that such errors were avoidable in real times. The real challenge facing policymakers and economists is to distinguish between avoidable (in real time) and unavoidable policy errors. We believe that a model like the one proposed here, where policy is consistent with the economic structure and information is efficiently processed can be helpful in facing such a challenge.

The second conclusion is that, with the exception of extreme cases, the fact that, during periods following large and sustained changes in potential output policymakers commit serious errors in forecasting potential output, does not imply that noisy, but optimally devised, forecasts of potential output should not be used as indicator variables for monetary policy.

A. Appendix: Model Solution

Condition (2.7) implies the interest rate rule:

$$r_t = \frac{1}{\varphi}\left[g_{t|t} + \frac{\lambda}{\alpha}\pi_{t|t}\right] \tag{A.1}$$

which yields the following output and inflation outcomes:

$$y_t = z_t + (g_t - g_{t|t}) - \frac{\lambda}{\alpha}\pi_{t|t} \tag{A.2}$$

$$\pi_t = \pi_{t-1} + \lambda\left[\left(g_t - g_{t|t}\right) - \frac{\lambda}{\alpha}\pi_{t|t}\right] + u_t \tag{A.3}$$

Note that the inflation equation contains an expected inflation term which, by the rational expectations hypothesis, is:

$$\pi_{t|t} = q(\pi_{t-1} + u_{t|t}) \qquad \text{where } q \equiv \frac{\alpha}{\alpha + \lambda^2} \in (0,1) \tag{A.4}$$

B. Appendix: The Filtering Problem

At time $t+1$ the policy maker's problem is to estimate z_t based on $J_{t+1,}$ i.e. using all the information contained in the observed sequence of signals $s_{1,t-i}$ and $s_{2,t-i}$ $(i = 0, 1, 2, ...)$. To this end, it is convenient to define the new signal $s_{3,t-i} \equiv s_{1,t-i} - s_{2,t-i}$. Let us write the linear predictor for z_t conditional on J_{t+1}as:

$$z_{t|t+1} \equiv \sum_{i=0}^{\infty} a_i \cdot s_{1,t-i} + \sum_{i=0}^{\infty} b_i \cdot s_{3,t-i} \tag{B.1}$$

$$\text{where } s_1 = z_t + g_t \text{ and } s_{3,t} = z_t - (1/\lambda)u_t$$

and the last line follows immediately from (2.12) and (2.13). We seek to determine optimal weights a_i and b_i that minimize the mean square forecast error of the z_t predictor (it follows from this property that the predictor z_t^* equals the expectation of z_t conditional on J_{t+1} i.e. $z_{t|t+1}$). This amounts to solving $\min_{a_i, b_i} Q$,where:

$$\begin{aligned} Q &\equiv E\left\{\left[z_t - z_{t|t+1}\right]^2 \mid J_{t+1}\right\} = \qquad \text{(B.2)}\\ &= \sigma_z^2 \left\{ \begin{array}{c} [1-(a_0+b_0)]^2 + [1-(a_0+b_0)-(a_1+b_1)]^2 + .. \\ .... + [1-(a_0+b_0)-(a_1+b_1) - ... - (a_i+b_i)]^2 + ... \end{array} \right\} + \\ &\quad + \sigma_g^2[(a_0^2 + (\mu a_0 + a_1)^2 + (\mu^2 a_0 + \mu a_1 + a_2)^2 + ... + (\mu^i a_0 + ... + a_i)^2 + ...] + \end{aligned}$$

$$+\frac{\sigma_u^2}{\lambda^2}[(b_0^2+(\rho b_0+b_1)^2+(\rho^2 b_0+\rho b_1+b_2)^2+...+(\rho^i b_0+...+b_i)^2+.....]$$

The first order conditions with respect to the generic a_i and b_i, for $i = 1, 2, ..$yield respectively:

$$\begin{aligned} o &= -\sigma_z^2 \left\{ \begin{array}{c} [1-(a_0+b_0)-..-(a_i+b_i)]+ \\ +[1-(a_0+b_0)-...-(a_i+b_i)-(a_{i+1}+b_{i+1})]+.... \end{array} \right\} + \\ &\quad +\sigma_g^2 \left[(\mu^i a_0+...+a_i)+\mu(\mu^{i+1}a_0+...+a_{i+1})+\mu^2(\mu^{i+2}a_0+...+a_{i+2})+.... \right] \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} o &= -\sigma_z^2 \left\{ \begin{array}{c} [1-(a_0+b_0)-..-(a_i+b_i)]+ \\ +[1-(a_0+b_0)-...-(a_i+b_i)-(a_{i+1}+b_{i+1})]+.... \end{array} \right\} + \\ &\quad +\frac{\sigma_u^2}{\lambda^2} \left[(\rho^i b_0+...+b_i)+\rho(\rho^{i+1}b_0+...+b_{i+1})+\rho^2(\rho^{i+2}b_0+...+b_{i+2})+.... \right]. \end{aligned} \tag{B.4}$$

Note that the two FOC have an identical first term inside the curly bracket and a similar form for the term in the second curly bracket, which only differ in that μ (a_i) is replaced by ρ (b_i). Leading (B.3) by one step, multiplying the resulting expression by μ and subtracting it from (B.3) yields:

$$\begin{aligned} o &= -\sigma_z^2 \left\{ \begin{array}{c} [1-(a_0+b_0)-..-(a_i+b_i)]+ \\ +(1-\mu)\left[(1-(a_0+b_0)-...-(a_i+b_i)-(a_{i+1}+b_{i+1}))+....\right] \end{array} \right\} + \\ &\quad +\sigma_g^2(\mu^i a_0+...+a_i) \end{aligned} \tag{B.5}$$

Leading (B.5) by one step and subtracting the resulting expression from (B.5) yields

$$\begin{aligned} o &= -\sigma_z^2 \left\{ \mu(a_{i+1}+b_{i+1})+(1-\mu)\left[(1-(a_0+b_0)-...-(a_i+b_i))\right] \right\} + \\ &\quad +\sigma_g^2 \left[(1-\mu)(\mu^i a_0+...+a_i)-a_{i+1} \right] \end{aligned} \tag{B.6}$$

Leading (B.6) by one step and subtracting the resulting expression from (B.6) yields

$$\begin{aligned} o &= -\sigma_z^2 \left[(a_{i+1}+b_{i+1})-\mu(a_{i+2}+b_{i+2}) \right] + \\ &\quad +\sigma_g^2 \left[(1-\mu)^2(\mu^i a_0+...+a_i)-(2-\mu)a_{i+1}+a_{i+2} \right] \end{aligned} \tag{B.7}$$

Leading (B.7) by one step, multiplying the resulting expression by $1/\mu$ and subtracting it from (B.7) yields

$$o = \sigma_z^2 \left[(a_i+b_i)\mu-(a_{i+1}+b_{i+1})(1+\mu^2)+(a_{i+2}+b_{i+2})\mu \right] + \sigma_g^2 \left[a_i-2a_{i+1}+a_{i+2} \right] \tag{B.8}$$

Applying to the FOC for b_i (B.4) algebraic transformations of identical nature of those used to establish (B.8) leads to

$$o = \sigma_z^2 \left[(a_i + b_i)\rho - (a_{i+1} + b_{i+1})(1+\rho^2) + (a_{i+2} + b_{i+2})\rho \right] + \frac{\sigma_u^2}{\lambda^2} \left[b_i - 2b_{i+1} + b_{i+2} \right] \tag{B.9}$$

where both (B.8) and (B.9) hold for $i = 1, 2, 3, ...$ These two equations constitute a system of two homogenous linear second order difference equations in the unknowns a_i and b_i. We next solve the simpler case in which $\mu = \rho$ and then present the general solution.

B.1. The case of equally persistent demand and cost-push shocks

When $\mu = \rho$ the difference equations (B.8) and (B.9) can be decoupled. It is immediate to see that in such case the a_i and b_i are related by the linear relationship

$$b_i = a_i \frac{\lambda^2 \sigma_g^2}{\sigma_u^2} \qquad \text{for } i = 0, 1, 2, ... \tag{B.10}$$

where the equality for $i = 0$ is established from the first order conditions for a_0 and b_0 (not reported). Substituting the expression for the generic b_i into (B.8) yields

$$\begin{aligned} o &= a_i - \phi a_{i+1} + a_{i+2} \qquad \text{for } i = 1, 2, ... \\ \text{where } \phi &\equiv \frac{2 + T(1+\mu^2)}{1 + T\mu} \qquad \text{and } T \equiv \left(\frac{\sigma_z^2}{\sigma_g^2} + \frac{\lambda^2 \sigma_z^2}{\sigma_u^2} \right) \end{aligned} \tag{B.11}$$

Equation (B.11) has one non-explosive solution which is given by

$$a_i = a_1 \kappa^{i-1} \qquad \text{for } i = 1, 2, ... \tag{B.12}$$

where a_1 is a constant term to be determined and κ is the "stable" root (i.e. smaller than one) of the second order equation in κ: $\kappa^2 - \phi\kappa + 1$ (from B.11). The values of a_0 and of a_1 remain to be determined. Using the first order conditions for a_0 and a_1 (where the latter is obtained from (B.3) for $i = 1$) the following linear relation is established (after some algebraic transformations of identical nature of those used to establish (B.8)):

$$a_1 \equiv \frac{(1-\mu)(1+T)a_0 - \frac{\sigma_z^2}{\sigma_g^2}(1-\mu)}{(1+\mu T)}. \tag{B.13}$$

A second linear relation between a_0 and a_1 is established after analogous algebraic transformations are applied to equation (B.5) for $i = 1$. This yields

$$a_1 \equiv \frac{(1-\mu)\left[\frac{\sigma_z^2}{\sigma_g^2} - (T+\mu)a_0\right]}{T(1-\mu-\mu\kappa) + (1-\mu-\kappa)}. \tag{B.14}$$

The solutions for a_0 and a_1 are determined by the system: (B.13), (B.14). The value for a_0 is reported in the main text. Using (B.10), (B.12) and the expression for the optimal predictor (B.1) the conditional expectation of z_t can thus be written as

$$z_{t|t+1} = a_0 S_t^{'} + a_1 \sum_{i=0}^{\infty} \kappa^i S_{t-1-i}^{'} \tag{B.15}$$

where :

$$a_0 \equiv \frac{[(1-\mu)+(1-\kappa)+T(1-\mu\kappa)]\frac{\sigma_z^2}{\sigma_g^2}}{[T(1-\mu-\mu\kappa)+(1-\mu-\kappa)](1+T)+(T+\mu)(1+\mu T)} \in (0, 1+\frac{\lambda^2\sigma_g^2}{\sigma_u^2})$$

$$a_1 \equiv \frac{(1-\mu)(1+T)a_0 - \frac{\sigma_z^2}{\sigma_g^2}(1-\mu)}{(1+\mu T)}$$

$$S_{t-i}^{'} \equiv s_{1,t-i} + \frac{\lambda^2\sigma_g^2}{\sigma_u^2}(s_{1,t-i} - \tfrac{1}{\lambda}s_{2,t-i}) = \left(1+\frac{\lambda^2\sigma_g^2}{\sigma_u^2}\right) z_t + g_{t-i} - \frac{\lambda\sigma_g^2}{\sigma_u^2}u_{t-i}$$

Some algebra reveals that $a_0 + \frac{a_1}{1-\kappa} = \left(1+\frac{\lambda^2\sigma_g^2}{\sigma_u^2}\right)$, which suggests the convenient reformulation of the filter used in the main text which is based on the modified signal $S_{t-i} \equiv S_{t-i}^{'}\left(1+\frac{\lambda^2\sigma_g^2}{\sigma_u^2}\right)^{-1}$. Under this formulation, rewrite (B.15) using $a \equiv a_0\left(1+\frac{\lambda^2\sigma_g^2}{\sigma_u^2}\right)^{-1}$ and $a_1^{'} \equiv a_1\left(1+\frac{\lambda^2\sigma_g^2}{\sigma_u^2}\right)^{-1}$. Since $a + a_1^{'}/(1-\kappa) = 1$, this implies $a_1^{'} = (1-\kappa)(1-a)$ used in (4.1) in the main text.

B.2. Solution for the general case

When $\mu \neq \rho$ the second-order difference equations system given by (B.8) and (B.9) can not be decoupled. Defining the new variables $c_i = a_{i+1}$ and $d_i = b_{i+1}$, the 2-equation second-order system can be rewritten as the following 4-equation first-order system:

$$\mathbf{0} = \mathbf{A}\mathbf{s}_{i+1} + \mathbf{B}\mathbf{s}_i \tag{B.16}$$

where

$$\mathbf{A} \equiv \begin{bmatrix} -(2+M) & -M & 1+N & N \\ -W & -(2+W) & U & 1+U \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \mathbf{B} \equiv \begin{bmatrix} 1+N & N & 0 & 0 \\ U & 1+U & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{s}_i \equiv \begin{bmatrix} a_i \\ b_i \\ c_i \\ d_i \end{bmatrix} \text{ and } M \equiv (1+\mu^2)\frac{\sigma_z^2}{\sigma_g^2};\; N \equiv \mu\frac{\sigma_z^2}{\sigma_g^2};\; W \equiv (1+\rho^2)\frac{\sigma_z^2}{\sigma_u^2};\; V \equiv \rho\frac{\sigma_z^2}{\sigma_u^2}.$$

To solve (B.16) we begin by rewriting the system as:

$$\mathbf{s}_{i+1} = \mathbf{C}\mathbf{s}_i \tag{B.17}$$

$$\text{where } \mathbf{C} \equiv \mathbf{A}^{-1}(-\mathbf{B}) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -\frac{-2-2U-M-MU+WN}{1+U+N} & -\frac{-M-MU+2N+WN}{1+U+N} \\ 0 & -1 & \frac{-2U-MU+W+WN}{1+U+N} & \frac{-MU+2+W+2N+WN}{1+U+N} \end{bmatrix}$$

Let us denote the eigenvalues and eigenvectors of C respectively as λ_i and X^i ($i = 1,2,3,4$), and define the two following (4x4) matrices: $\Lambda \equiv \sum_{j=1}^{4}(\lambda_j I)$ (i.e. a diagonal matrix with the eigenvalues on the main diagonal) and $X \equiv [\mathbf{X}^1 \mid \mathbf{X}^2 \mid \mathbf{X}^3 \mid \mathbf{X}^4]$ (i.e. a matrix the columns of which are given by the eigenvectors of C). Let us define the orthogonal variables $\tilde{s}_i \equiv X^{-1}s_i$. Using that $XCX^{-1} = \Lambda$ into (B.17) it follows immediately that these variables solve $\tilde{s}_{i+1} = \Lambda\tilde{s}_i$. The solution of the latter system is therefore given by $\tilde{s}_{j,i} = \tilde{s}_j\lambda_j^{i-1}$, for $j = 1,2,3,4$ (which indexes the 4 arguments of the orthogonalized vector $\tilde{s}_i$) and $i = 1,2,3,...$(i.e. 'period' indices from 1 onwards). The undetermined parameter $\tilde{s}_j$ (a real number) represent variable $\tilde{s}_{j,i}$ initial condition (i.e. its value for $i = 1$). The solution of the original system is obtained using $s_i \equiv X\tilde{s}_i$. The solution for the first element of the s_i vector is

$$\begin{aligned} a_i &= \tilde{s}_1\lambda_1^{i-1}\cdot X_1^1 + \tilde{s}_2\lambda_2^{i-1}\cdot X_1^2 + \tilde{s}_3\lambda_3^{i-1}\cdot X_1^3 + \tilde{s}_4\lambda_4^{i-1}\cdot X_1^4 \\ \text{for } i &= 1,2,3,... \end{aligned} \tag{B.18}$$

where X_1^j is the first element of the j-th eigenvector. The eigenvalues of C allow to rewrite (B.18) (and the analogous solutions for (b_i, c_i, d_i)) as[19]

$$\begin{aligned} a_i &= \tilde{s}_1X_1^1 + \tilde{s}_2X_1^2 + \tilde{s}_4\lambda_4^{i-1}\cdot X_1^4 \\ b_i &= \tilde{s}_1X_2^1 + \tilde{s}_2X_2^2 + \tilde{s}_4\lambda_4^{i-1}\cdot X_2^4 \end{aligned} \tag{B.19}$$

[19] The eigenvalues of $\mathbf{C}$ are equal to: $\lambda_1 = \lambda_2 = 1$,
$\lambda_3 = \frac{1}{2(1+U+N)}\left(W+M+2+\sqrt{(W^2+2WM+4W+M^2+4M-8U-8N-4U^2-8UN-4N^2)}\right)$,
$\lambda_4 = \frac{1}{2(1+U+N)}\left(W+M+2-\sqrt{(W^2+2WM+4W+M^2+4M-8U-8N-4U^2-8UN-4N^2)}\right)$.
Tedious algebra reveals that $\lambda_3 > 1$ and that $\lambda_4 \in (0,1)$ (the analytical expressions for the eigenvectors $\mathbf{X}^j$are not reported for reasons of space).

$$
\begin{aligned}
c_i &= \tilde{s}_1 X_3^1 + \tilde{s}_2 X_3^2 + \tilde{s}_4 \lambda_4^{i-1} \cdot X_3^4 \\
d_i &= \tilde{s}_1 X_4^1 + \tilde{s}_2 X_4^2 + \tilde{s}_4 \lambda_4^{i-1} \cdot X_4^4 \\
\text{for } i &= 1, 2, 3, ...
\end{aligned}
$$

where we imposed that the $\tilde{s}_3$ coefficient is nil in order to rule out an explosive solution for a_i (since the associated eigenvalue is larger than unity).

To complete the solution for the optimal a_i and b_i weights ($i = 0, 1, 2, 3, ...$) in the optimal predictor (B.1) we still need to determine $\tilde{s}_1$, $\tilde{s}_2$, $\tilde{s}_4$ and the initial values for a_0 and b_0.[20] We therefore need 5 linearly independent relations to determine these unknowns. Two relations are immediately obtained from the definitions $c_i = a_{i+1}$ and $d_i = b_{i+1}$. Another two are obtained from (B.7) and from the corresponding equation obtained from the FOC for b_i for $i = 1$.[21] The first order conditions for a_0 yields a fifth linear relationship.[22] This solves the optimal predictor's problem.

---

[20] Once a_1 is determined all other coefficients $a_{2,}a_{3,}..$ are known. To see this note that (B.19) implies: $a_{i+1} = (\tilde{s}_1 X_1^1 + \tilde{s}_2 X_1^2)(1 - \lambda_4) + \lambda_4 a_i$ (for $i = 1, 2, 3, ..$). The same holds for the b_i's.

[21] The two relations are:

$$
\begin{aligned}
o &= -\sigma_z^2 \left[(a_2 + b_2) - \mu(a_3 + b_3)\right] + \sigma_g^2 \left[(1-\mu)^2(\mu a_0 + a_1) - (2-\mu)a_2 + a_3\right] \\
o &= -\sigma_z^2 \left[(a_2 + b_2) - \rho(a_3 + b_3)\right] + \frac{\sigma_u^2}{\lambda^2} \left[(1-\rho)^2(\rho b_0 + b_1) - (2-\rho)b_2 + b_3\right].
\end{aligned}
$$

[22] The relation (obtained after algebraic transformations similar to those used to establish (B.8) is:

$$
o = -\sigma_z^2 \left[(a_1 + b_1) - \mu(a_2 + b_2)\right] + \sigma_g^2 \left[(1-\mu)^2 a_0 - (2-\mu)a_1 + a_2\right]
$$

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