A Ramsey Theory of Financial Distortions

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Abstract

The interest rate on government debt is significantly lower than the rates of return on other assets. From the perspective of standard models of optimal taxation, this empirical fact is puzzling: typically, the government should finance expenditures either through contingent taxes, or by previously-issued state-contingent debt, or by labor taxes, with only minor effects arising from intertemporal distortions on interest rates. We study how this answer changes in the presence of financial frictions such that the entrepreneurs’ net worth has a direct effect on their ability to invest. In a deterministic environment, a stark result emerges. Provided this is feasible, optimal policy calls for the government to increase its debt, up to the point at which it provides sufficient liquidity to avoid financial constraints. In this case, capital-income taxes are zero in the long run, and the returns on government debt and capital are equalized. However, if the fiscal space is insufficient, a wedge opens between the rate of return on government debt and capital. In this case, optimal long-run tax policy is driven by a trade-off between the desire to mitigate financial frictions by subsidizing capital and the incentive to exploit the quasi-rents accruing to producers of capital by taxing capital instead. This latter incentives magnifies the wedge between rates of return on publicly and privately-issued assets.

JEL classification: E22, E44, E62;

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1 Introduction

How should governments finance their expenditures in the least costly way when capital is present? This question has attracted much interest in the past. Judd (1985), Chamley (1986), Chari, Christiano, and Kehoe (1994), and Siu (2004), and a large literature that follows, argues that taxing capital in the long run is a bad idea and that the interest rate on government debt, which is a perfect substitute for capital, should not be distorted as well. But Berndt, Lustig, and Yeltekin (2012) and Reinhart and Sbrancia (2015) find that, in the aftermath of bad fiscal shocks, such as World War II, when government finances are tight, interest rates on government debt remain persistently low and help in reducing the required future primary surpluses needed to stabilize government finances.

Previous research does not focus on the difference between government bonds and capital in terms of liquidity. We view that the imperfect substitutability among assets could be important for government financing, which can potentially explain many prominent features in reality and shed light on optimal policy: is the capital income tax, observed in practice, optimal? Why is the interest rate on government debt lower than the rate of return on other low-risk privately-issued assets? What is the implication for government financing when the interest rate is lower than the growth rate of the economy, as often observed in many developed economies in the past three decades?

In this paper, we revisit the original question in a deterministic model in which liquidity frictions feature prominently. We adopt an otherwise standard neoclassical growth model, but now investment is undertaken by entrepreneurs whose net worth affects their ability to access external sources of finance. This new feature provides two twists to the traditional answers of the Ramsey optimal taxation literature. First, government debt is now a source of liquidity in the economy. Second, the classic no-capital taxation result is no longer true when capital prices are endogenous because they enter directly into the liquidity constraints of entrepreneurs.

In the model, private agents face idiosyncratic investment opportunities, as in Kiyotaki and Moore (2012). Some of them have investment projects, while others do not. When private agents have investment projects, they seek outside financing. But, because of asset liquidity frictions, these agents are constrained in issuing new claims and/or reselling old claims to capital. Government bonds are fully liquid instead, so that potential entrepreneurs have a precautionary motive to buy them. The government aims at an exogenous stream of expenditures; it can tax income from labor and capital and issues debt.

We first illustrate the optimal policy in a simple 2-period deterministic model, with exogenous asset liquidity frictions and then with endogenous asset liquidity frictions through search and match-
ing. Both suggests that when entrepreneurs hold scarce liquidity, an optimal policy calls for distorting financial markets. Financial constraints drive a wedge between the rate of return accruing to buyers of capital, and that perceived by the constrained entrepreneurs; this reduces the elasticity of the supply of capital to its after-tax rate of return. When the government starts from a very strong balance sheet, it finds it optimal to undo the financial distortions by subsidizing capital. Conversely, when the government is desperate for funds, its labor-income tax policy may depress the labor supply so much that investment drops to the point where financial constraints cease to bind, in which case the Chamley-Judd result reemerges. In an intermediate range, the government finds it optimal to exploit the low elasticity of the supply of capital and raise revenues through positive capital taxation.

We later extend the analysis into an infinite-horizon economy, and we study the long-run properties of an optimal allocation. A stark result emerges. If the government is able to issue enough debt to completely eliminate financial frictions, it will choose to do so, and capital-income taxes will be zero in the limit. However, if this level of debt cannot be sustained by raising enough labor-income tax revenues, then the economy converges to a steady state with binding financing constraints, a positive capital tax, and a lower interest rate than the rate of time preference. In this case, government debt commands a liquidity premium. This premium can be so large that the interest rate on government debt becomes negative (i.e., lower than the growth rate).

Our paper builds on a large literature that introduced financial frictions in the form of imperfect asset liquidity. In addition to Kiyotaki and Moore (2012), similar economic environments appear in Shi (2015), Nezafat and Slavik (2010), Del Negro, Eggertsson, Ferrero, and Kiyotaki (2017), Ajello (2016), and Bigio (2012), among many others. We further explore the role of endogenizing these frictions through the search framework of Cui and Radde (2016a, b) and Cui (2016), so that the supply of government debt can affect the participation in asset markets. Search frictions exist in many markets, such as those for corporate bonds, IPO, and acquisitions. They can also capture many aspects of frictional financial markets with endogenous market participation (see e.g., Vayanos and Wang, 2013; Rocheteau and Weill, 2011), while still keeping the simple structure of neoclassical macro framework. This tractability is crucial since one can use all the insight from a standard Ramsey plan. In particular, we use the “primal approach” (see e.g., Lucas and Stokey, 1983; Chari and Kehoe, 1999) to show the allocations chosen by a Ramsey planner.

The presence of liquidity constraints opens the possibility for government bonds or even fiat money to circulate, as in Holmström and Tirole (1998). If private liquidity is not enough, public liquidity can improve efficiency. In this paper, government debt provides liquidity services and has

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3Recent work by Lagos and Rocheteau (2008), Rocheteau (2011), and Cao and Shi (2014) also use search models to endogenize liquidity and asset prices, but they do not study the individual trade-off that agents face between asset liquidity and prices. This channel gives rise to different degrees of liquidity constraints and risk sharing.

4There are thus both fully liquid government issued assets and partially liquid private claims. Changing the portfolio
a “crowding-in” effect, similar to Woodford (1990). At the same time, the need to raise distortionary taxes limits the government’s ability to flood the market with liquidity so that an optimal supply of public liquidity emerges.

Our work is complementary to Angeletos, Collard, Dallas, and Diba (2013), who study a model where non-state-contingent government bonds also may crowd in private investment. In their context, capital-income taxes are ruled out, and a secondary-market price for capital is not present. In that case, the main link between interest rates and government debt goes from the former to the latter: shocks that decrease returns call for additional issuance. We also study how increased financial frictions affect optimal policy, but we mostly analyze the opposite force: the fiscal needs of the government may lead the optimal policy to interfere with intertemporal prices.

Azzimonti and Yared (2017, 2019) consider optimal supply of public liquidity with lump-sum taxes when agents differ in their income. The issuance of public debt is at the stage when agents know their types. Their framework also generates an incentive for the government to manipulate debt prices, keeping interest rate low and some agents liquidity constrained. Compared to their work, we consider the role of idiosyncratic risk and precautionary savings in determining investment. This opens up the discussion of the interaction between low interest rates, capital-income taxes, and capital accumulation.

Finally, Farhi (2010) adds capital to Aiyagari, Marcet, Sargent, and Seppälä (2002) and shows that the results from Aiyagari, Marcet, Sargent, and Seppälä (2002) extend to this case, so that no further major role emerges for distorting intertemporal prices. In the context of a deterministic model, our analysis highlights important differences that emerge when uninsurable idiosyncratic risks and private liquidity frictions are present. In this case, a force emerges that calls for greater issuance of debt. Incorporating aggregate uncertainty and risk-free debt as in Aiyagari, Marcet, Sargent, and Seppälä (2002) would generate a countervailing reason for the government to accumulate assets for self-insurance, generating potentially interesting interactions in government debt management that we plan to explore in future work.

2 A Simple Two-period Framework

We start our analysis with a two-period model. Both the provision of public liquidity and the degree of illiquidity of private assets are exogenous in this section. We analyze how liquidity frictions affect compositions of the two assets can potentially affect the real economy. More recent papers enriched the basic structure by explicitly introducing financial intermediaries that are subject to independent frictions. See, for example, Gertler and Karadi (2011) and Gertler and Kiyotaki (2010).

5 This aspect is in contrast with Aiyagari and McGrattan (1998), where government debt is a perfect substitute to private assets (or capital stock). Therefore, government debt relaxes agents’ borrowing constraints but also crowds out capital accumulation.
the choice of distorting return on capital and interest rates, and how this choice in turn depends on the fiscal constraints faced by the government. Throughout the paper, we use lowercase variables for individual choices, and uppercase for aggregate allocations, except for prices and taxes.

2.1 The Environment

In period $t = 1$, a continuum of firms can produce output by using labor using a constant returns technology, with one unit of labor normalized to produce 1 unit of output. In period 2, the firms have a technology $F(K_1, L_2)$, where $K_1$ and $L_2$ are capital and labor utilized in period 2. We assume that $F$ satisfies Inada conditions, so we can neglect corner solutions. Firms hire labor and rent capital in competitive markets at the wage rates $w_1$ and $w_2$, and the rental rate $\bar{r}_2$.

The economy is populated by a continuum of families, each of which is composed by a continuum of agents. In period 1, they start with some (exogenous) government debt $B_0$. A fraction $\chi$ of agents from each household are revealed to be entrepreneurs and the remainder $1 - \chi$ are workers. Entrepreneurs and workers are separated at the beginning of the period. The entrepreneurs have in total $B_0^e$ units of government bonds, whereas workers have $B_0^w$ units,\(^6\) and we define total per-capita bonds to be\(^7\)

$$B_0 \equiv B_0^e + B_0^w.$$ 

Period 1

Workers supply labor to the firms. Entrepreneurs do not supply labor. Rather, in period 1, they can turn one unit of the firms’ output into one unit of new capital to be used in the subsequent period. This ability will only be used in the first period, since the economy ends after period 2. The amount that each entrepreneur invests is $k_1^e$, the amount of capital available at the beginning of period 2.

Entrepreneurs cannot sell the capital directly, but they can sell claims to the capital $k_1^e$ in a frictional competitive market, in the amount $s_1^e$:

$$s_1^e \leq \phi_1 k_1^e,$$

where $\phi_1$ is asset liquidity. An entrepreneur has internal funds arising from her holdings of government debt, equal to $R_1 B_0^e / \chi$, where $R_1$ is an exogenous initial return on government debt which we only include for symmetry of notation with the second period. The entrepreneur’s budget constraint

\(^{6}\)The per-entrepreneur level of initial bonds that entrepreneurs have is therefore $B_0^e / \chi$, and the per-worker amount owned by workers is $B_0^w / (1 - \chi)$.

\(^{7}\)In multi-period versions, the identity of entrepreneurs will not be known ex ante and $B_0^e / \chi = B_0^w / (1 - \chi)$. We keep the two initial conditions separate because this allows us to study how the problem changes as a function of the entrepreneurs’ initial net worth.
entrepreneurs can only “borrow” by selling claims to capital at the price \( q_1 \), and any left-over funds after investment has taken place are brought back to the family at the end of the period. We will typically be interested in equilibria where constraint (2) is binding and entrepreneurs use all of their available funds to undertake new investment.

Workers use some of their income to purchase new claims to capital from entrepreneurs and new government debt \( b_1^{w} \), and return the remaining funds to the family. Their period-1 budget constraint is

\[
q_1^w s_1^w + b_1^{w} \leq R_1 B_0^w/(1 - \chi) + w_1 \ell_1,
\]

where \( s_1^w \geq 0 \) is the end-of-period private claims on capital that they purchase, \( \ell_t \) is their labor supply, \( q_1^w \) is the price at which claims to capital can be bought.

At the end of the first period, entrepreneurs and workers rejoin their family, pool their capital and their left-over funds, pay taxes, and consume. Their constraint is

\[
c_1 = (1 - \tau_{1}^f)w_1 \ell_1(1 - \chi) + R_1 B_0 - (1 - \chi)b_1^{w} - (1 - \chi)q_1^w s_1^w - \chi(k_e^1 - q_1^e s_1^e),
\]

where \( c_t \) is the family’s consumption in period \( t \), and \( \tau_{1}^f \) is the tax rate on labor income.

Claims to capital are subject to an intermediation cost. Intermediaries are competitive and their cost is \( \eta \) per unit of capital intermediated; therefore we have

\[
q_1^w = \eta + q_1.
\]

In period 1, the government budget constraint ensures that its revenues from labor-income taxation and new borrowing cover debt repayments that become due as well as any government spending \( G_1 \).

\[
G_1 + R_1 B_0 = B_1 + \tau_{1}^f w_1 L_1.
\]

**Period 2**

The second and final period is similar to the first, except that no new investment takes place, so that entrepreneurs no longer have any role. We can then collapse the two subperiods, and simply write

\[
k_e^1 \leq R_1 B_0^e/\chi + q_1 s_e^e:
\]

\[\text{Note that the individual labor supply is normalized in per-worker terms, while the aggregate labor supply is in per-capita terms. So, an aggregate labor supply } L_1 \text{ corresponds to } L_1/(1 - \chi) \text{ for each worker. Similar normalizations occur for aggregate capital } K_1, \text{ bonds } B_1, \text{ and intermediated capital } S_1.\]
the joint family budget constraint as
\[
c_2 = (1 - \tau_2^\ell)w_2(1 - \chi)\ell_2 + \left[(1 - \tau_2^k)\tilde{r}_2\right] \left[\chi(k_1^e - s_1^e) + (1 - \chi)s_1^w\right] + R_2(1 - \chi)b_1^w, \tag{7}
\]
where $\tau_2^\ell$ is the labor income tax in period 2, and $\tau_2^k$ is the capital income tax in period 2. $R_2$ is the return of government bonds between period 1 and period 2.

The government budget constraint is:
\[
G_2 + R_2B_1 = \tau_2^k\tilde{r}_2K_1 + \tau_2^\ell w_2L_2, \tag{8}
\]
with $G_2$ being government spending in the second period.

Contrary to period 1, the government is allowed to tax (or subsidize) capital in the second period at a rate $\tau_2^k$, and our goal is to study how this power is used in the presence of financial frictions, together with interest rate $R_2$.

The household preferences are represented by:
\[
\sum_{t=1}^2 \beta^{t-1} \left[u(c_t) - v((1 - \chi)\ell_t)\right], \tag{9}
\]
where $u$ and $v$ are strictly increasing and continuously differentiable functions, $u$ is weakly concave, and $v$ is strictly convex.\(^9\)

### 2.2 Competitive Equilibrium

We next characterize a competitive equilibrium.

The household maximizes (9), subject to (1), (2), (4), and (7), taking prices and taxes as given. We note that, in any equilibrium in which $q_1 < 1$, there would be no sales of capital.\(^{10}\) With this observation, we can limit our analysis to $q_1 \geq 1$ without loss of generality.\(^{11}\)

From the intermediaries’ and firms’ optimality conditions, we obtain (5),
\[
w_1 = 1, \quad w_2 = F_L(K_1, L_2), \tag{10}
\]
and $\tilde{r}_2 = F_K(K_1, L_2)$.\(^{11}\)

\(^9\)The particular choice of scale for the function $v$ is a pure normalization that is convenient to obtain simpler expressions when studying the aggregate allocation.

\(^{10}\)To see this, consider a family whose entrepreneurs are selling capital. By reducing investment and capital sales one for one, the family can simultaneously relax the constraints (1), (2), and (4). The last budget constraint is necessarily binding, since families would otherwise increase their consumption, hence the original plan cannot be optimal.

\(^{11}\)For any competitive equilibrium in which $q_1 < 1$, there exists a competitive equilibrium with the same allocation and the same prices, except for $q_1 = 1$ and $q_1^w = 1 + \eta$. 

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From the families’ necessary and sufficient first-order conditions we obtain:

- Labor supply in period \( t = 1, 2 \):
  \[
  (1 - \tau_t^\ell)w_t u'(C_t) = v'(L_t); \quad (12)
  \]

- Demand for government bonds:
  \[
  1 = \frac{\beta u'(C_2)}{u'(C_1)} R_2; \quad (13)
  \]

- Demand for claims on capital:
  \[
  q_1^w \geq \frac{\beta u'(C_2)}{u'(C_1)} (1 - \tau_2^k) \tilde{r}_2, \quad (14)
  \]
  with equality if \( S_1 > 0 \);

- Investment and supply of claims:
  \[
  u'(C_1) \leq \frac{\beta u'(C_2)}{u'(C_1)} (1 - \tau_2^k) \tilde{r}_2, \quad (15)
  \]
  with equality if \( S_1 = 0 \), and the financing constraint of an entrepreneur implies
  \[
  q_1 = \max \left\{ 1, \frac{K_1 - R_1 B_0^e}{\phi_1 K_1} \right\}. \quad (16)
  \]

In addition, a competitive-equilibrium allocation must satisfy the government budget constraints (6) and (8) and the resource constraints:

\[
L_1 = C_1 + K_1 + (q_1^w - q_1) S_1 + G_1, \quad (17)
\]

and
\[
F(K_1, L_2) = C_2 + G_2. \quad (18)
\]

By Walras’ law, the household budget constraints (4) and (7) must be satisfied as an equality by the aggregate allocation (chosen by the representative family). Furthermore, consumption is always strictly positive and it is always weakly preferable for a household to first use the workers’ resources to fund consumption and only after these are exhausted to potentially cut back on the entrepreneurs’ investment. Because of this, equation (3) does not bind in equilibrium: this equation constrains workers not to invest more than all of their earnings and assets and is slack if something is left for consumption.

**Definition.** A competitive equilibrium is an allocation \( \{C_t\}_{t=1}^2 \), \( \{L_t\}_{t=1}^2 \), and \( \phi_1 \), asset prices \( q_1^w \) and \( q_1 \), wage rate \( \{w_t\}_{t=1}^2 \), capital income rate \( \tilde{r}_2 \), and an interest rate \( \tilde{R}_2 \), such that (5), (6), (8), and
(10)-(18) are satisfied, given exogenous labor income tax rate \(\{\tau^\ell_t\}_{t=1}^2\), capital tax rate \(\tau^k_2\), and bond supply \(B_1\).

In any competitive equilibrium, market clearing implies that \(S_1 \equiv S^w_1 = S^e_1\), where \(S_1\) is the per-capita level of intermediated capital. If \(q_1 > 1\), then both the financing constraint and the entrepreneurs’ budget constraint (1) and (2) bind; if \(q_1 = 1\), (1) is certainly slack and (2) may or may not bind.

### 2.3 Optimal Policy with Zero Intermediation Costs

We start our analysis from the case in which intermediation costs are absent \((\eta = 0)\), so that the financing constraint is the only departure from a standard neoclassical growth model. We also set \(R_1 = 1\). We return to the role of these two parameters in the next section, where we endogenize intermediation costs and allow the government to determine the supply of public liquidity by setting \(R_1\).

**Forming the Policy Problem**  
We study the Ramsey outcome, that is, the best competitive equilibrium that maximizes (9). To do so, we follow the primal approach, deriving a set of necessary and sufficient conditions for an allocation to be part of a competitive equilibrium, without reference to prices and tax rates. These conditions include a restriction that allows us to derive intermediated capital \(S_1\) given the other variables (equation (19) below), and it is thus convenient to also substitute out this variable from the policy problem.

Given any allocation, we can ensure that (10) and (11) hold by setting factor prices \(w_t\) and \(r_t\) to the appropriate marginal product. Similarly, we can ensure that (12) hold with a suitable choice of \(\tau^\ell_t\), for \(t = 1, 2\); (13) holds for the appropriate choice of \(R_2\).

Next, in order for (1) and (2) to hold and for \(S_1\) to be optimally chosen, we must have

\[
S_1 = \begin{cases} 
0 & \text{if } K_1 \leq (1 - \phi_1)K^*, \\
K_1 - (1 - \phi_1)K^* & \text{if } K_1 \in ((1 - \phi_1)K^*, K^*], \\
\phi_1 K_1 & \text{if } K_1 > K^*, 
\end{cases}
\]  

(19)

where

\[
K^* := \frac{B^e_0}{1 - \phi_1}.
\]

\(K^*\) is the maximum level of investment that entrepreneurs can finance when \(q_1 = 1\), and \((1 - \phi_1)K^*\) is the maximum that they can finance using internal funds only.

\(\tau^k_2\) can then be chosen so that either (14) or (15) hold as an equality, depending on whether \(S_1\) is 0 or positive, with the remaining of the two equations holding as the appropriate inequality. Finally,
equation (16) can be used to determine \( q_1 \) and (5) to determine \( q_1^w \).

The remaining conditions that characterize a competitive equilibrium are the following:

- The resource constraints (17) and (18); and

- The household budget constraints evaluated at the aggregate allocation, (2), (4), and (7).

Substituting prices and tax rates from the first-order conditions, we can aggregate the household budget constraints into the following implementability constraint for period 1 and period 2:

\[
\sum_{t=1}^{2} \beta^{t-1} [u'(C_t)C_t - v'(L_t)L_t] = u'(C_1)B_0 + \begin{cases} 
0 & \text{if } K_1 \leq K^* \\
\left(\frac{1}{\phi_1} - 1\right)u'(C_1)(K_1 - K^*) & \text{if } K_1 > K^* 
\end{cases}
\]  

(20)

As usual in Ramsey problems, the implementability constraint represents the cost for the government not to have access to lump-sum taxation.\(^\text{13}\)

The implementability constraint has two branches, corresponding to the two possible types of equilibria in our economy. In the first case, the financing constraint is slack and the price of capital \( q_1 \) is 1. This happens when either the entrepreneurs are sufficiently wealthy to finance all of the investment internally, or when they issue claim to capital that fall short of the constraint (1). In this case, our economy behaves as a standard neoclassical growth model. In the second case, when the financing constraint is binding, a new term appears in Equation (20); this term captures the fact that, when financing constraints are binding, entrepreneurs face a different intertemporal trade-off than workers. When the present-value budget constraint is evaluated at the trade-off faced by workers, who are the unconstrained agents in the family, capital appears as an extra source of revenues. This can be understood in two equivalent ways:

- The entrepreneurs require only one unit of period-1 good to produce 1 unit of capital, but the price of capital is \( q_1 > 1 \), and the last term in (20) captures the family’s profits from investment.

- The physical rate of return on capital is greater than the family’s marginal rate of substitution between the two periods.

**Remark:** For our results, the fact that the price of capital appears in the financing constraint plays an important role. We therefore contrast our results with those that would apply if the entrepreneurs’

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\(^{12}\)The financing constraint (1) holds by construction when (19) holds.

\(^{13}\)We assume here that any lump-sum transfers would be paid to the households after investment has taken place, so that they do not relax the financing constraint (2). In this case, the binding side of the constraint is that the left-hand side must be no smaller than the right-hand side, as is the case in standard Ramsey problems.
constraints (1) and (2) were replaced by a single constraint that only involves the entrepreneurs’ initial net worth and does not include the price of capital:

\[ \theta k_1^e \leq R_1 B_0^e / \chi. \]  

(21)

In this case, the implementability constraint becomes

\[ \sum_{t=1}^{2} \beta^{t-1} [u'(C_t)C_t - v'(L_t)L_t] = u'(C_1)B_0 + \begin{cases} 0 & \text{if } \theta K_1 < R_1 B_0^e \\ [\beta u'(C_2) \tilde{\tau}_2 (1 - \tau_2^k) - u'(C_1)] K_1 & \text{if } \theta K_1 < R_1 B_0^e. \end{cases} \]

(22)

When the implementability constraint is given by (22), whenever the financing constraint is binding, taxing capital is necessarily optimal: as long as \( \beta u'(C_2) \tilde{\tau}_2 - u'(C_1) > 0 \), an increase in the capital-income tax raises revenues for the government without affecting the allocation, since investment remains fixed at the value implied by the constraint (21): locally, capital is a factor in fixed supply. When instead the financing constraint depends on the price of claims to capital, the price changes driven by changes in the tax rate spill over to the allocation.

**Characterization**  We now derive the first-order conditions that must hold if the Ramsey plan is interior. However, it is possible that the plan will be at the kink, which needs to be checked separately. We are particularly interested in studying comparative statics when the financing constraint is binding and Tobin’s q responds to investment, which will be the case when the entrepreneurs’ wealth is sufficiently low relative to the return on capital and the government’s resources are sufficiently scarce relative to its spending.

Let \( \beta^{t-1} \lambda_t \) be the Lagrange multiplier on the resource constraint, for \( t = 1 \) and \( t = 2 \), and let \( \Psi_1 \) be the Lagrange multiplier on the implementability constraint. The planner’s first-order conditions for consumption \( C_1 \) and \( C_2 \) are

\[ u'(C_1)(1 + \Psi_1) + \Psi_1 u''(C_1)C_1 - \lambda_1 - \Psi_1 u''(C_1)B_0 = \begin{cases} 0 & \text{if } K_1 < K^* \\ \Psi_1 (1/\phi_1 - 1) u''(C_1)(K_1 - K^*) & \text{if } K_1 > K^* \end{cases} \]

(23)

\[ u'(C_2)(1 + \Psi_1) + \Psi_1 u''(C_2)C_2 - \lambda_2 = 0 \]

(24)

The planner’s first-order conditions for labor supply \( L_1 \) and \( L_2 \) are

\[ v'(L_1)(1 + \Psi_1) + \Psi_1 v''(L_1)L_1 = \lambda_1 A \]

(25)
\[ v'(L_2)(1 + \Psi_1) + \Psi_1 v''(L_2) L_2 = \lambda_2 F_L(K_1, L_2) \] (26)

The first-order conditions for capital \( K_1 \) is

\[-\lambda_1 + \beta \lambda_2 AF_K(K_1, L_2) = \begin{cases} 0 & \text{if } K_1 \leq K^* \\ \Psi u'(C_1)(1/\phi_1 - 1) & \text{if } K_1 > K^* \end{cases} \] (27)

### 2.4 A Special Case

To clarify the role of different distortions, we consider the special case of a Cobb-Douglas production function, \( F(K_{t-1}, L_t) = AK_{t-1}^{\alpha}L_t^{1-\alpha} + (1 - \delta)K_{t-1} \), where \( \alpha \in (0, 1) \), and preferences given by

\[ u(c_t) - v(\ell_t) = c_t - \frac{\mu \ell_t^{1+\nu}}{1 + \nu}, \]

where \( \mu > 0 \) and \( \nu > 0 \). These preferences are convenient because they abstract from the usual incentive to distort intertemporal prices and devalue the families’ initial claims, as emphasized by Armenter (2008). We can then instead focus on the new channels of intertemporal distortions that arise from financial frictions.

Notice that now the marginal utility of consumption is one. From the planner’s choice of consumption, we have \( \lambda_1 = \lambda_2 = 1 + \Psi_1 \). The labor supply in period 1 is simply a function of \( \Psi_1 \), and we can express the other two first-order conditions for labor supply and capital used in period 2:

\[ \mu L_2^{\frac{1 + \Psi_1(1 + \nu)}{1 + \Psi_1}} = A(1 - \alpha) \left( \frac{K_1}{L_2} \right)^{\alpha} \Rightarrow L_2 = \left[ \frac{A(1 - \alpha)}{\mu} \frac{1 + \Psi_1}{1 + (1 + \nu)\Psi_1} \right]^{\frac{1}{\alpha + \nu}} K_1^{\frac{\alpha}{\alpha + \nu}} \] (28)

and

\[ \beta A\alpha(K_1/L_2)^{\alpha - 1} = 1 - \beta(1 - \delta) + \begin{cases} 0 & \text{if } K_1 \leq K^* \\ \frac{\Psi_1 \phi_1^{-1 - 1}}{1 + \Psi_1} & \text{if } K_1 > K^* \end{cases} \] (29)

Consumption \( C_1 \) and \( C_2 \) can be derived from the resource constraints.

Comparing the planner’s optimality condition for capital (29) with the household optimality conditions (14) and (15) (also using the fact that \( \tilde{r}_2 = A\alpha(K_1/L_2)^{\alpha - 1} + 1 - \delta \)), we can establish the following:

- If the allocation is such that the financing constraint of the entrepreneurs is not binding, then capital-income taxes are optimally set to zero, independently of the tightness of the government budget constraint (captured by the multiplier \( \Psi_1 \)). In this case, we recover the standard result that it is not optimal to tax capital, which is an intermediate input. This case can arise either when entrepreneurs have enough wealth to finance investment internally, in which case...
the private cost of investment is 1 and the social cost is $1 + \psi_1$, or when they need to sell part of their capital, but not to the point at which $q$ need to exceed 1. In both cases, the private reward in the second period is $\beta \tilde{r}_2$ and the social reward is $\beta \tilde{r}_2 (1 + \psi_1)$. Thus, private and social costs are proportional to each other and capital-income taxes are zero; moreover, in both cases the trade-off coincides with the marginal rate of transformation coming from technology alone, taking into account the costs of intermediation.\textsuperscript{14}

- When entrepreneurs are sufficiently poor that the financing constraint binds, we obtain a very different result. In this case, in the absence of capital taxes or subsidies, the private rate of return does not coincide with the marginal rate of transformation. Furthermore, changes in the level of investment have an effect on the price of capital, and a higher price of capital tightens in turn the implementability constraint, forcing the government to raise more funds through distortionary taxes.\textsuperscript{15} If the government has abundant resources and $\psi_1 \approx 0$, comparing (29) and (14) (taking into account $K_1 > K^*$) we can see that the optimal policy calls for a capital subsidy, which encourages workers to buy more capital. This is an important difference from the case in which the financing constraint is independent of price. In that case, local changes in capital-income taxes have no effect on investment. Here, by subsidizing capital income in the second period, the government can raise the price of period-1 claims to capital, which in turn relaxes the entrepreneurs’ constraints and allows the economy to attain a higher (and more efficient) level of investment. However, as the cost of public funds $\psi_1$ increases, the factors at play in the exogenous financing case becomes more important: financing constraints may weaken the link between investment and future capital income, so that capital-income taxes may be less distortionary than they would be in a world of perfect capital markets. For this reason, it eventually becomes ambiguous whether a government strapped for cash would subsidize or tax capital.

By assuming linear preferences, we automatically imposed from equation (13) that $R_2 = 1/\beta$, that is, the government choice of taxes or subsidies has no effect on the rate of return on government debt. A further channel at work when preferences are not linear is that a capital-income tax reduces the after-tax return on capital, and hence further favors government debt, which is a further beneficial force in the case of a constrained government. This effect appears on the right-hand side of equation (23) and we will analyze it later in the infinite-horizon economy.

The above simplification implies that we can express all equilibrium outcomes in closed form. There are three cases, with the last case having $K_1$ at the kink $K^*$. Rather than varying $B_0$ and

\textsuperscript{14}Positive intermediation costs $\eta > 0$ would not change this result, because both private/social rewards and private/social costs would be multiplied by the same factor $1 + \eta$.

\textsuperscript{15}The Lagrange multiplier $\psi$ can be viewed as the cost to the planner of starting with an extra unit of government debt in period 0.
finding the implied value of $\Psi_1$, we find it more intuitive to graph directly the optimal solution treating $\Psi_1$ as a parameter, and then backing out the corresponding $B_0$ from the resulting allocation and prices and the government budget constraint.16

Case 1: when $K_1 \leq K^*$. This case occurs when the financing constraint is slack. The planner’s first-order condition for capital becomes $\beta \tilde{r}_2 = 1$ and the price $q_i^w = q_1 = 1$. This condition and the household’s first-order condition for capital (Equation (14)) imply no taxes on capital: $\tau_k^w = 0$. From the first-order conditions, we know that $K_1 = K^u_1(\Psi_1)$, where

$$K^u_1(\Psi_1) := \left[\frac{1 + \Psi_1}{1 + (1 + \nu)\Psi_1} \frac{A(1 - \alpha)}{\mu}\right]^{\frac{1}{2}} \left[\frac{1}{A\alpha} \left[\frac{1}{\beta} - (1 - \delta)\right]\right]^{\frac{\alpha + 1}{(\alpha - 1)^\nu}}, (30)$$

which is a decreasing function of $\Psi_1$. A higher $\Psi_1$ implies higher labor-income taxes, which (given our preferences) reduce the labor supply and discourage investment.

Case 2: when $K_1 > K^*$. We can express labor supply $L_2$ and capital stock $K_1$ as functions of $\Psi_1$, we know that $K_1 = K^c_1(\Psi_1)$ where

$$K^c_1(\Psi_1) := \left[\frac{1 + \Psi_1}{1 + (1 + \nu)\Psi_1} \frac{A(1 - \alpha)}{\mu}\right]^{\frac{1}{2}} \left[\frac{1}{A\alpha} \left[\frac{\phi_1 + \Psi_1}{\phi_1(1 + \Psi_1)} - (1 - \delta)\right]\right]^{\frac{\alpha + 1}{(\alpha - 1)^\nu}}, (31)$$

which is also a decreasing functions of $\Psi_1$ after noticing that $\phi_1 \in (0, 1)$ and $\alpha \in (0, 1)$.

Case 3: $K_1 = K^*$, in which case $\beta A\alpha(K_1/L_2)^{\alpha - 1} \in ((\phi_1 + \Psi_1)/\phi(1 + \Psi_1), 1]$, according to (29). At this kink, labor in period 2 can be still expressed as in (28) by setting $K_1 = K^*$. At this point, the incentive for the government to tax or subsidize capital undergoes a jump represented by the two branches of Equation (29). As $\Psi_1$ increases and the government budget constraint becomes tighter, the tax on labor must increase, discouraging labor supply in the second period. However, because of the kink, the optimal level of capital stays constant for a range of values of $\Psi_1$, with capital taxation adjusting to ensure that this is the case.

From (30) and (31), we can compute the level of capital that satisfy the first-order conditions that apply in the unconstrained and constrained regions respectively, which are two functions of $\Psi_1$: $K^u_1(\Psi_1)$ and $K^c_1(\Psi_1)$ fall when $\Psi_1$ increases. We also know that

$$K^u_1(0) = \left[\frac{A(1 - \alpha)}{\mu}\right]^{\frac{1}{2}} \left[\frac{1}{A\alpha} \left[\frac{1}{\beta} - (1 - \delta)\right]\right]^{\frac{\alpha + 1}{(\alpha - 1)^\nu}} = K^u_1(0);$$

16In this experiment, $B_0^e$ is kept fixed, so that $B_0$ affects the shadow cost of public funds, but not the entrepreneurs’ financing constraints; all the residual bonds are allocated to the workers.
\[ K_c^1(\infty) = \left[ \frac{A(1-\alpha)}{(1+\nu\mu)} \right]^\frac{1}{\nu} \left[ A\alpha \left[ \frac{1}{\phi_1\beta} - (1-\delta) \right] \right]^{-\frac{\alpha+1}{(\alpha-1)\nu}} \]

\[ < \left[ \frac{A(1-\alpha)}{(1+\nu\mu)} \right]^\frac{1}{\nu} \left[ \frac{1}{\beta} - (1-\delta) \right]^{-\frac{\alpha+1}{(\alpha-1)\nu}} = K_1^u(\infty). \]

It follows that \( K_c^1(0) = K_u^1(0) > K_u^1(\infty) > K_c^1(\infty) \). When \( \Psi_1 = 0 \), the government has sufficient wealth at the beginning that the shadow cost of resources in the government budget constraint is zero. In this case, the government can undo the effect of financial constraints by subsidizing the return on capital in the second period, thereby raising the price of capital \( q_1 \) to a level which replicates the efficient level of investment in the absence of constraints.

Since \( \phi_1(\Psi_1)/\phi_1(1+\Psi_1) > 1 \), \( K_u^1(\Psi_1) > K_c^1(\Psi_1) \) for any \( \Psi_1 > 0 \): when the government is forced to raise revenues through distortionary means, it will never choose to fully undo the effect of financing constraints. For any combination of parameters, we can thus find the relevant case for the Ramsey outcome by comparing \( K_u^1(\Psi_1) \), \( K_c^1(\Psi_1) \), and \( K^* \). If \( K^* > K_u^1(\Psi_1) > K_c^1(\Psi_1) \), the solution is necessarily given by \( K_u^1(\Psi_1) \) and the financing constraint is not binding: by contradiction, if it were binding, the planner would choose an even lower level of capital, which would then make the financing constraint even more slack. If \( K_u^1(\Psi_1) > K^* > K_c^1(\Psi_1) \), then the Ramsey outcome is at the kink: to the left of \( K^* \), the government faces the first-order conditions of the unconstrained economy, which would require pushing capital to a higher level, but to the right the government faces the first-order conditions of the constrained economy, which would call for a smaller level of capital. Finally, when \( K_u^1(\Psi_1) > K_c^1(\Psi_1) > K^* \), the Ramsey outcome occurs in the region in which financial constraints are binding.

The properties of the functions \( K_u^1(\Psi_1) \) and \( K_c^1(\Psi_1) \) that we established above allow us to describe how the Ramsey allocation changes with \( \Psi_1 \), fixing other parameters. This is summarized in the following proposition:

**Proposition 1.** The Ramsey allocation can be characterized as follows:

- The economy under the planner’s allocation is going to be financially unconstrained, regardless of \( \Psi_1 \), if \( K^* \geq K_c^1(0) = K_u^1(0) \).

- If \( K_u^1(0) > K^* > K_c^1(\infty) \), then the economy is financially constrained for small levels of \( \Psi_1 \) and capital is given by \( K^1 = K_c^1(\Psi_1) \); the economy is financially unconstrained for large values of \( \Psi_1 \) and capital is given by \( K^1 = K_u^1(\Psi_1) \), and there is an intermediate range of values of \( \Psi_1 \) for which the Ramsey allocation has capital exactly at the kink.
• If
\[ K_u^1(\infty) \geq K^* > K_c^1(\infty), \]
then the economy is financially constrained for small levels of \( \Psi_1 \) and capital is given by \( K_1 = K_c^1(\Psi_1) \), and it is at the kink for higher values of \( \Psi_1 \).

• Finally, the economy is always financially constrained for any \( \Psi_1 \geq 0 \) and capital is \( K_1 = K_c^1(\Psi_1) \) if
\[ K_c^1(\infty) \geq K^*. \]

A Numerical Example  We use a numerical example to illustrate the previous results. We consider parameter combinations that lead to a binding financing constraint and also others that make it slack. First, consider the following parameters: \( \beta = 0.96 \) (discount factor), \( \alpha = 0.33 \) (capital share), \( \delta = 0.95 \) (depreciation rate), \( \mu = 1 \) (disutility parameter of labor supply), \( \nu = 1 \) (labor supply elasticity), \( A = 1 \) (productivity), and \( \phi = 0.5 \) (asset liquidity). With linear utility in consumption, the Ramsey allocation depends on government spending only through the multiplier \( \Psi_1 \), and we thus do not need to specify it explicitly as explained above.\(^\text{17}\) We plot \( K_1, \tau_2, L_2, \) and \( q_1 \) against \( \Psi_1 \).

Figure 1: The functions \( K_u^1(\Psi_1) \) and \( K_c^1(\Psi_1) \)

We first plot the functions of \( K_u^1(\Psi_1) \) and \( K_c^1(\Psi_1) \). As shown before, both are downward sloping, and \( K_u^1(0) = K_c^1(0) \). However, the two curves converge to different levels with \( K_u^1(\infty) > K_c^1(\infty) \). The critical value \( K^* = B_0^c/(1 - \phi_1) \) is a horizontal line whose position depends on the parameter values. As shown by the proposition, four possibilities emerge. The first case is trivial: when

\(^\text{17}\)The level of initial debt \( B_0 \) that corresponds to a given \( \Psi_1 \) is of course different based on the spending process.
When the liquidity is insufficient, the economy is always constrained. This happens when $K^* \leq K^c_1(\infty)$. Figure 2 illustrates this case, for three possible values of $K^*$, equal to 100%, 90% and 80% of the critical threshold $K^c_1(\infty)$. As $\Psi_1$ goes from zero to infinity, the planner implements initially a capital subsidy (between 80% and 90%), but as the budget becomes tighter this turns into a capital tax. The capital tax converges to a level in the range of 20%-25% as $\Psi_1 \to \infty$. For this special example, the allocation in terms of capital and labor used in period 2 is independent of the initial level of internal funds in the hands of the entrepreneurs, as long as they remain such that $K^* \leq K^c_1(\infty)$. Taxes and asset prices adjust to exactly offset the effect of tighter financing constraints. The less liquidity is given to entrepreneurs, the more capital is subsidized when $\Psi_1$ is small and the less it is taxed for large values of $\Psi_1$.

(2). For higher values of liquidity held by entrepreneurs, we enter the range illustrated in Figure 3). The orange line shows what happens when the Ramsey solution hits the kink and remains there as $\Psi_1 \to \infty$. As in the previous case, it is optimal to implement subsidies for low values of $\Psi_1$ and taxes for higher values. The difference is that the capital income tax becomes decreasing.

Figure 2: Economies with always binding financing constraints.

(1). When the liquidity is insufficient, the economy is always unconstrained. We now show the remaining three cases.

The specific value that we choose to illustrate this case is $K^* = K^c_1(\infty)$.
once the government hits $K^*$. When $\Psi_1$ goes up, the government needs to tax more. As the labor-income tax increases, in order to keep investment at $K^*$, it is necessary to reduce the capital-income taxes.

Figure 3: Economies with possible kinks and slack financing constraints.

(3). For even higher levels of liquidity for the entrepreneurs, (e.g., when $K^* = [K_1^u(0) + K_1^u(\infty)]/2$ represented by the blue line in Figure 3), we get $K_1 = K_1^*$ for intermediate ranges of $\Psi_1$. When public resources are abundant and $\Psi_1$ is low, the planner always finds it better to choose the allocation in the constrained region, which calls for a capital subsidy. As the government budget becomes tighter, the optimal capital level drops, and the subsidy eventually turns into a tax, until the kink $K^*$ is attained. As $\Psi_1$ grows further, the government keeps increasing labor taxes, but it maintains capital at the kink $K^*$ by lowering capital-income taxes. Finally, for even higher values of $\Psi_1$, the labor-income tax is so large that the optimal level of capital is below the one at which financing constraints bind. From here on, there are no further quasi-rents to be extracted, and the optimal capital-income tax is zero. As the economy becomes unconstrained, $q_1$ falls to unity.


3 Endogenous Asset Liquidity

In this section, we allow asset liquidity to be endogenously determined. That is, asset liquidity $\phi$ and the cost of trading $\eta$ will be endogenous. The rest of the model is kept the same. Endogenizing asset liquidity is done through directed search. This allows the government policy to affect the asset liquidity, which could potentially avoids the kink solution as shown before. Additionally, Cui and Radde (2016b) have shown that endogenizing asset liquidity is crucial to generate the positive co-movement of $\phi$ and $q$, which is empirically supported and crucial for amplifying financial shocks.\(^{19}\)

3.1 Competitive Equilibrium with Search-and-Matching

As in the original paper by Moen (1997) with directed search in labor markets, we assume that intermediaries set up markets where trade occurs and compete by offering a given cost of trading (measured by the bid/ask spread) and market tightness. The competition among market makers implies that they will offer contracts that are Pareto optimal for buyers and sellers. Beyond that, the price will reflect market clearing.

Because we have only one type of buyers and one type of sellers, in equilibrium only one financial market opens. In this market, financial intermediaries need to pay a cost $\eta = \eta(\phi_1)$ for intermediating one unit of capital. We assume that $\eta(0) = \eta'(0) = 0$, $\eta(.)$ is convex and twice continuously differentiable. Cui and Radde (2016b) and Cui (2016) provide a microfoundations for these assumptions in a world in which intermediation is subject to search frictions which prevent a full match between buyers and sellers of capital.

Under these assumptions, the competitive financial intermediation sector will set the bid/ask spread according the following revised version of (5):

$$q^w_1 - q_1 = \eta(\phi_1),$$

where the left-hand side is the revenue for intermediating capital and the right-hand side is the cost. The additional feature is that (given $q^w_1$) there is a trade-off between asset liquidity $\phi_1$ and the price $q_1$ at which entrepreneurs can sell their capital. The assumption that $\eta(0) = \eta'(0) = 0$ implies that there is no kink at the point in which entrepreneurs stop selling capital. At this point, there are no intermediation costs and both $q_1$ and $q^w_1$ converge smoothly to 1.

Combining the entrepreneurs’ budget and financing constraints, we obtain that the claims to

\(^{19}\)In a general equilibrium framework like this one, an exogenous negative shock to asset liquidity pushes up asset price $q$ to reflect the scarcity of assets. The key is that entrepreneurs’ financing constraint is also tied to asset liquidity so part of $q$ reflects the tightness of the financing constraint. See Shi (2015) for a critique of models relying on exogenous financial shocks. A negative co-movement can also stabilize financial shocks, because $\phi q$ together matters for investment financing.
capital that an entrepreneur brings back to the household $k_1^e - s_1^e$ satisfy
\[ q_1^r (k_1^e - s_1^e) \leq R_1 B_0^e/\chi, \]
where
\[ q_1^r \equiv \frac{1 - \phi_1 q_1}{1 - \phi_1}. \tag{33} \]

$q_1^r$ can be interpreted as the replacement cost of capital. To bring back a claim to one unit of capital to the household, an entrepreneur produces $1/(1 - \phi_1)$ units; this investment is financed by selling claims to $\phi_1/(1 - \phi_1)$ units at a price $q_1$ and by the initial assets.\(^\text{20}\)

In a directed search environment, an entrepreneur chooses the market which will offer her the lowest value of $q_1^w$, which maximizes the amounts of claims to capital that she can bring to the household at the end of the period. Market makers will only offer contracts on the Pareto frontier. We index the Pareto frontier by the price $q_1^w$ at which workers can buy claims to capital. We can then trace it by solving
\[
\min_{(\phi_1, q_1)} q_1^w = \frac{1 - \phi_1 q_1}{1 - \phi_1}, \quad \text{subj. to (32)}.
\]

Substituting the constraint and taking first-order conditions, we obtain
\[ q_1^w = 1 + \eta(\phi_1) + (1 - \phi_1)\phi_1 \eta'(\phi_1). \tag{34} \]

Equation (34) defines an implicit positive relationship between the price workers are willing to pay for a claim to one unit of capital and the entrepreneurs’ search intensity, which maps into the fraction of capital that they sell. When $q_1^w = 1$, $\phi_1 = 0$ and entrepreneurs retain all of the capital that they produce. As $q_1^w$ increases above 1, $\phi_1 > 0$, so that entrepreneurs sell some of their capital. Under endogenous liquidity constraints, the competitive equilibrium is characterized by conditions (10)-(18) plus (32) and (34). Equations (32) and (34) imply that $q_1 \geq 1$, so that the relevant term in the maximum in equation (16) is always the second one. In a competitive equilibrium, either the financing constraint is slack, in which case $\phi_1 = 0$ and $q_1 = q_1^w = 0$, or $\phi_1 < 1$, as we now show.

If $\eta'(1) > 1$, equations (32), (33), and (34) imply that there exists a unique value $\hat{\phi}$ such that $q_1^w = \hat{\phi}[1 + (1 - \hat{\phi})\hat{\phi} \eta'(\hat{\phi})] = 1$. For $\phi_1 \geq \hat{\phi}$, entrepreneurs would have access to an arbitrage: by producing an extra unit of capital at a unit cost in terms of the period-1 consumption good, they would be able to sell a fraction $\phi_1$ and receive a payment $q_1 \phi_1 \geq 1$, while retaining the extra $1 - \phi_1$ units of capital. In this case, a competitive equilibrium will necessarily have $\phi_1 < \hat{\phi}$.

If $\eta'(1) \leq 1$, the same equations imply $q_1 \phi_1$ remains below 1 even as $\phi_1 \to 1$; by continuity, we can define $\hat{\phi} = 1$, since $\lim_{\phi_1 \to 1} q_1 \phi_1 = 1$. In this case, note that $\phi_1 > 0$ implies that the financial

\(^{20}\)This equation remains valid even when entrepreneurs do not sell any claims to their new investment. In this case, $\phi_1 = 0$ and $q_1 = 1$.\]
constraint is binding, so that \( K_1 = R_1B_0^e/(1 - q_1\phi_1) \). As \( \phi_1 \to 1 \), the amount of capital that entrepreneurs optimally produce diverges to infinity. This would violate the feasibility constraint (and workers would not find it optimal to buy claims to such a large amount of capital), proving that in this case too any competitive equilibrium will feature \( \phi_1 \leq \hat{\phi} < 1 \).

### 3.2 The Ramsey Outcome with Endogenous Financial Constraints

Combining equations (16) and (34) we obtain

\[
(1 - \phi_1) \left[ 1 - \phi_1^2 \eta'(\phi_1) \right] K_1 \equiv x(\phi_1)K_1 \leq R_1B_0^e. \tag{35}
\]

We have \( x(0) = 1, x'(\phi) < 0 \) for \( \phi \in [0, \hat{\phi}] \), and \( x(\hat{\phi}) = 0 \).

This equation links \( K_1 \) and \( \phi_1 \), and replaces equation (19) in the previous section, along with \( S_1 = \phi_1K_1 \). When \( K_1 < R_1B_0^e \), the constraint is slack, entrepreneurs finance investment only through internal funds, and \( \phi_1 = 0 \).

Substituting prices and taxes from the first-order conditions, we can aggregate the household budget constraints into the following implementability constraint:

\[
\sum_{t=1}^{2} \beta^{t-1} [u'(C_t)C_t - v'(L_t)L_t] = u'(C_1)R_1B_0 + u'(C_1) \left[ (q_1^w - 1)K_1 - (q_1^w - q_1)\phi_1K_1 \right]
= u'(C_1)R_1B_0 + u'(C_1)K_1(1 - \phi_1) \left[ \eta(\phi_1) + \phi_1\eta'(\phi_1) \right]. \tag{36}
\]

Let us define \( z(\phi_1) \equiv (1 - \phi_1) \left[ \eta(\phi_1) + \phi_1\eta'(\phi_1) \right] \). Notice that \( z(0) = z'(0) = 0 \).

Equations (35) and (36) generate two regions in which competitive equilibria can be found, depending on whether \( K_1 \leq R_1B_0^e \) or \( K_1 > R_1B_0^e \). These regions have the same interpretation that applied in the case of exogenous constraints: when investment is small or bond holdings are large, the economy behaves as in the standard neoclassical growth model, whereas an extra term appears when the financing constraint is binding and a wedge appears between the after-tax rate of return on capital and the intertemporal marginal rate of substitution of the households.\(^{21}\) While the implementability constraint (20) features a jump at \( K^* \), in the case of endogenous liquidity constraints equations (35) and (36) imply a smooth transition of \( \phi_1 \) and \( K_1 \) at \( K_1 = R_1B_0^e \); this greatly simplifies the numerical analysis that will follow.

The planner maximizes the household utility (9), subject to the resources constraints (17) and (18) (with \( S_1 = \phi_1K_1 \) as the amount of transaction of claims), the implementability constraint (36), and the equilibrium relationship between \( \phi_1 \) and \( K_1 \), equation (35). Let \( \beta^{t-1}\lambda_t \) be the Lagrange multiplier on the resource constraint for period \( t = 1, 2 \) and \( \Psi_1 \) be the Lagrange multiplier on the imple-

\(^{21}\)Mathematically, note that, when \( K_1 \leq R_1B_0^e, \eta(\phi_1) = \phi_1 = 0 \).
mentability constraint (as before), and let $\gamma_1 u'(C_1)$ be the Lagrange multiplier on equation (35).

Appendix A1 derives the necessary first-order conditions.

3.3 The Special Case Again

To clarify the role of different distortions, we consider again the preferences and technology of Section 2.4. We again express $K_1$ and $L_2$ as functions of $\Psi_1$ as before.

First, if we know $K_1$, we can express labor in period 2 as in (28). Second, from (55) and by using the planner’s FOC for capital (56)

$$
\beta A \alpha \left( \frac{K_1}{L_2} \right)^{\alpha-1} = 1 - \beta (1 - \delta) + \phi_1 \eta(\phi_1) + \frac{\Psi_1}{1 + \Psi_1} h(\phi_1) - g(\phi_1),
$$

where $h(\phi_1) \equiv z(\phi_1) - \frac{x(\phi_1) x'(\phi_1)}{x'(\phi_1)}$ and $g(\phi_1) = \frac{\eta(\phi_1) + \phi_1 \eta'(\phi_1)}{x'(\phi_1)} x(\phi_1)$. Finally, using the relationship between $L_2$ and $K_1$, we have

$$
\beta A \left[ \frac{\mu}{(1 - \alpha) A} \frac{1 + (1 + \nu) \Psi_1}{1 + \Psi_1} K_1^\nu \right]^{\frac{\alpha - 1}{\alpha + \nu}} = 1 - \beta (1 - \delta) + \phi_1 \eta(\phi_1) + \frac{\Psi_1}{1 + \Psi_1} h(\phi_1) - g(\phi_1). \tag{37}
$$

Fixing $\Psi_1$ (which in turn can be mapped into initial government debt $B_0$), the Ramsey outcome can be found by jointly solving (35) and (37) for $K_1$ and $\phi_1$, taking into account that $\phi_1 = 0$ whenever (35) holds as an inequality. Now, $\eta$ is a function of $\phi_1$.

We first establish conditions under which $\phi_1 = 0$ and internal financing is sufficient for entrepreneurs to achieve the Ramsey level of $K_1$. In this case, the capital level is given by $K_u(\Psi_1)$, as defined in equation (30). Note that $K_u$ is strictly decreasing in $\Psi$, for the same reasons previously identified. We thus have three possibilities:

1. If $K_u(0) \leq B_0^c$, the financial constraint does not bind, independently of the state of government finances. In this case, the solution of the standard neoclassical growth model applies.

2. If $K_u(0) > B_0^c \geq K_u(\infty)$, then there exists a threshold $\Psi_1^*$ such that the Ramsey outcome is not affected by financial constraints if $\Psi_1 > \Psi^*$, and is otherwise constrained.

3. Finally, if $K_u(\infty) > B_0^c$, financing constraints are binding, no matter how tight government finances are.

When $K_u(\Psi_1) > R_1 B_0^c$, we can use (35) and substitute $K_1 = B_0^c / x(\phi_1)$ into equation (37), thereby obtaining a single equation to be solved for $\phi_1$. It is straightforward to prove that this equation has at least one solution in $(0, \hat{\phi})$, since the equation holds as different inequalities at the extrema, but in principle it may have multiple solutions, in which case each one should be checked to obtain the global maximum. In our numerical examples, we find a unique solution.
Another Numerical Example  We illustrate the previous discussion with a numerical example. We plot $K_1$, $\phi_1$, $L_2$, and $\tau_2$ as functions of $\Psi_1$. The new addition is the cost function of financial intermediation, set to

$$\eta(\phi) = \eta_0 \phi^{\eta_1}$$

where parameters $\eta_0 = 0.2$ and $\eta_1 = 2$. We have experimented with different sets of values for $\eta_0$ and $\eta_1$ and the qualitative results shown below are robust. An increase of $\eta_0$ leads to higher capital taxes, while an increase of $\eta_1$ does the opposite. Intuitively, an increase of $\eta_1$ makes the intermediation cost more elastic to the quantity to be sold, which calls for smaller taxes and the planner should reduce intervention. $\eta_0$ increases financial frictions and the quasi-rents accruing to entrepreneurs.

Figure 4: Ramsey outcome with Endogenous Asset Liquidity

The solid lines in Figure 4 correspond to the economy when $B_e^0 = 0.5[K^u(0) + K^u(\infty)]$, so that the amount of liquidity held by entrepreneurs satisfies $K^u(\infty) < B_e^0 < K^u(0)$. The general pattern is that, as $\Psi_1$ increases from zero, the planner initially subsidizes capital, then it taxes it in an intermediate range; the capital tax eventually vanishes when $\Psi_1$ becomes large enough. Therefore, we have a hump-shaped pattern, a smoothed outcome of the economy with exogenous asset liquidity.
Before reaching the cut-off level $Ψ^*_{1}$, asset liquidity $φ$ falls as the search activity drops and eventually the economy is going to rely purely on public liquidity to finance capital investment (i.e., when the demand for capital is low). This numerical example illustrates how the key variables react in the constrained and unconstrained regions. We can also confirm these constrained and unconstrained regions from $q_{1}$ and the spread $η(φ_{1})$ schedules.

To further understand the importance of liquidity held by entrepreneurs, when $B_{0}^{e}$ falls about 10% (the red dashed lines), the region of $Ψ_{1}$ indicating financing-constrained economy is widened, as the cut-off $Ψ^*_{1}$ is almost doubled. The capital tax implemented is higher (and of course so is capital subsidy). Because of the further shortage in public liquidity, the private liquidity measured by $φ_{1}$ is higher for any given $Ψ_{1}$. That is, private liquidity works again as an imperfect substitute for public liquidity, although $q_{1}$ and the spread $η = η(φ_{1})$ are higher for any given $Ψ_{1}$.

In summary, when $K^{u}(∞) < B_{0}^{e} < K^{u}(0)$, the planner may implement binding financing constraints when $Ψ_{1}$ is small. As $Ψ_{1}$ becomes larger, the planner does not stick to the financing constrained region, reflecting the interaction between $Ψ_{1}$ and the degree of financial frictions (determined by the initial liquidity $B_{0}^{e}$ held by entrepreneurs). Labor supply in period 1 (not plotted) is unaffected by this consideration though as it does not depend on the financing constraint. Labor supply in period 2 is less when the planner implements a financing constrained allocation, reflecting less capital and a lower wage rate.

## 4 The Infinite Horizon Economy

To discuss the long-run properties of the Ramsey policy, we now extend the model to an infinite horizon. We retain the assumption of directed search, which reduces the number of kinks.

### 4.1 The Setup

We adopt the same notation of the previous sections. The household’s utility in (9) is now:

$$\sum_{t=0}^{∞} \beta^{t} [u(c_{t}) - v((1 - χ)ℓ_{t})]$$  \hspace{1cm} (38)

All households start with some initially given capital $K_{-1}$ and bonds $B_{-1}$. In our two-period economy, we distinguished between the bonds issued by the government and those held by entrepreneurs, so that we could better explain the economic forces at work by discussing independently the consequences of tightening government finances (by increasing $B_{0}$) and loosening financing constraints (by increasing $B_{0}^{e}$). We now assume that each member of a household has an i.i.d. chance $χ$ of being an entrepreneur and a $1 - χ$ chance of being a worker in each period, and this opportunity is realized
after the household has allocated bonds, so that $b^w_t = b^e_t$ at the individual level.\textsuperscript{22} Similarly, each member of a household will start period $t$ with $k_{t-1}$ units of capital. An entrepreneur can finance new investment by selling her government bonds as well as claims to capital; we treat existing and new capital symmetrically, with both subject to intermediation costs. We thus have

$$k^e_t \leq R_t b_{t-1} + q_t s^e_t$$

and

$$s^e_t = \phi_t [k^e_t + (1 - \delta)k_{t-1}] .$$

These two constraints can be combined as

$$(1 - \phi_t q_t)k^e_t \leq R_t b_{t-1} + \phi_t q_t (1 - \delta)k_{t-1} . \tag{39}$$

The household budget constraint is

$$c_t + (1 - \chi) b^w_t + (1 - \chi) q^w_t s^w_t + \chi (k^e_t - q_t s^e_t) = (1 - \tau^k_t) w_t (1 - \chi) \ell_t + R_t b_{t-1} + (1 - \tau^k_t) r_t k_{t-1} .$$

The asset positions evolve according to

$$b_t = (1 - \chi) b^w_t \text{ and } k_t = (1 - \delta) k_{t-1} + (1 - \chi) s^w_t - \chi s^e_t + \chi k^e_t .$$

As before, only workers accumulate government bonds. A household’s claims to capital at the beginning of period $t + 1$ (which are $k_t$) include claims to undepreciated capital from the previous period, which are $(1 - \delta) k_{t-1}$, new purchases from workers $(1 - \chi) s^w_t$, and physical investment by entrepreneurs $\chi k^e_t$ net of claims sold $\chi s^e_t$.\textsuperscript{23}

For convenience, we will work with the following budget constraint, which uses the budget constraint above and the evolution of assets, so that $b_t$ and $k_t$ show up on the left-hand side:

$$c_t + b_t + q^w_t k_t = (1 - \tau^k_t) w_t (1 - \chi) \ell_t + R_t b_{t-1} + (1 - \tau^k_t) r_t k_{t-1}$$

$$+ [q^w_t - \chi \phi_t (q^w_t - q_t)] (1 - \delta) k_{t-1} + [q^w_t - 1 - \phi_t (q^w_t - q_t)] \chi k^e_t . \tag{40}$$

The intermediation of assets follow directed search implemented by financial intermediaries with free entry:

$$q^w_t - q_t = \eta(\phi_t) . \tag{41}$$

\textsuperscript{22}In per-capita aggregate terms, entrepreneurs will thus have $\chi B_t$ units of government debt.

\textsuperscript{23}In a symmetric equilibrium, $(1 - \chi) s^w_t = S^w_t$, $\chi s^e_t = S^e_t$, and market clearing requires $S^w_t = S^e_t$. Capital evolves according to $K_t = (1 - \delta) K_{t-1} + K^e_t$, where $K^e_t$ is the aggregate investment undertaken by entrepreneurs.
The government again has an exogenous stream of spending $G_t$ for any $t \geq 0$. Finally, the production follows a constant-returns-to-scale technology $A_t F(K_{t-1}, L_t)$ with respect to capital and labor for any $t \geq 0$. Following the tradition in an infinite-horizon economy, the production function $F(K, L)$ in this section does not include depreciation and the aggregate productivity is written outside of $F(K, L)$.

### 4.2 Competitive Equilibrium

A typical household maximizes (38), subject to the financing constraint (39) and the budget constraint (40). The wage rate and the rental rate of capital are the marginal products of labor and capital

\begin{align}
  w_t &= A_t F_L(K_{t-1}, L_t); \\
r_t &= A_t F_K(K_{t-1}, L_t).
\end{align}

For asset intermediation, we can immediately extend the result from (34) in the two-period model to any arbitrary period $t$:

\begin{equation}
  q_t = 1 + (1 - \phi_t)\phi_t\eta'(\phi_t).
\end{equation}

In equilibrium, the aggregate quantities are the same as individual quantities, because all households are identical: that is, $K_t = k_t$, $B_t = b_t$, $L_t = (1 - \chi)\ell_t$, and $C_t = c_t$. Additionally, the total assets being intermediated are

\begin{equation}
  S_t = (1 - \chi)s_t^w = \phi_t [K_t - (1 - \delta)K_{t-1}] + \phi_t\chi(1 - \delta)K_{t-1}.
\end{equation}

entrepreneurs sell a fraction $\phi_t$ of new investment and of their holdings of previous undepreciated capital. The goods market clearing condition is thus

\begin{equation}
  C_t + G_t + \eta(\phi_t)\phi_t[K_t - (1 - \delta)K_{t-1}] = A_t F(K_{t-1}, L_t) + (1 - \delta)K_{t-1},
\end{equation}

where $G_t$ is the exogenous stream of government expenditures. Substituting (45), this becomes

\begin{equation}
  C_t + G_t + [1 + \phi_t\eta(\phi_t)] K_t = A_t F(K_{t-1}, L_t) + [1 + (1 - \chi)\phi_t\eta(\phi_t)] (1 - \delta)K_{t-1}.
\end{equation}

Given our assumption of a representative household, the aggregate allocation must satisfy the

\footnote{To be more specific, an entrepreneur maximizes the amount of claims to capital brought to the household, which is $(1 - \phi_t) [k_t^e + (1 - \delta)k_{t-1}]$ because a fraction $\phi_t$ of $k_t^e + (1 - \delta)k_{t-1}$ is issued. The financing constraint (53) can be rewritten as

\[
\frac{1 - \phi_t q_t}{1 - \phi_t} (k_t^e + (1 - \delta)k_{t-1}) \leq R_t b_{t-1} + (1 - \delta)k_{t-1}
\]

so that the entrepreneurs will again minimize $q_t^e = (1 - \phi_t q_t)/(1 - \phi_t)$ to achieve his/her goal.}
individual households’ optimality conditions. The first-order condition for labor is

\[(1 - \tau^t_t) w_t u'(C_t) = v'(L_t), \quad (47)\]

for any \( t \geq 0 \). Let \( \beta^t u'(C_t) \chi \rho_t \) be the Lagrange multiplier attached to the financing constraint (39), where the scaling \( u'(c_t) \chi \) simplifies the derivation in the following. \( \rho_t \) is determined from the first-order condition for \( k_t^k \)

\[q^w_t - 1 - \phi_t (q^w_t - q_t) = \rho_t (1 - \phi_t q_t) \rightarrow \rho_t = \frac{q_t - 1 + (1 - \phi_t) \eta(\phi_t)}{1 - \phi_t q_t} = \frac{\phi_t \eta'(\phi_t) + \eta(\phi_t)}{1 - \phi_t^2 \eta'(\phi_t)}, \quad (48)\]

for any \( t \geq 0 \). \( \rho_t \) reflects the liquidity service provided by government debt. It is only positive when entrepreneurs’ financing constraints are binding.

The household first-order condition for government bonds \( b_t \) implies

\[1 = \frac{\beta u'(C_{t+1})}{u'(C_t)} R_{t+1} (1 + \chi \rho_{t+1}) . \quad (49)\]

The term \( \chi \rho_{t+1} \) in equation (49) represents the liquidity services that government bonds offer to the entrepreneurs, arising from the fact that bonds can be liquidated with no intermediation costs by the fraction \( \chi \) of household members that turn out to be entrepreneurs in any given period. This liquidity service pushes down the interest rate \( R_{t+1} \).

The first-order condition for capital \( k_t \) implies

\[q^w_t = \frac{\beta u'(C_{t+1})}{u'(C_t)} \left\{ (1 - \tau^k_t ) r_{t+1} + (1 - \delta) q^w_{t+1} + \chi (1 - \delta) \phi_{t+1} [q_{t+1} (1 + \rho_{t+1}) - q^w_{t+1}] \right\} . \quad (50)\]

\( q^w_t \) represents the cost for a worker to acquire one unit of capital.\(^{25}\) In the next period, the household receives a payoff \((1 - \tau^k_t ) r_{t+1} + (1 - \delta) q^w_{t+1}\) from the investment. In addition, the fraction \( \chi \phi_{t+1} \) of undepreciated capital that entrepreneurs will sell to finance further investment has an extra liquidity value captured by \( q_{t+1} (1 + \rho_{t+1}) - q^w_{t+1} \), the difference between the price at which entrepreneurs sell their capital, adjusted for the shadow value of liquidity, and the price at which workers can buy the capital back.

**Definition.** A competitive equilibrium is an allocation \( \{ C_t, L_t, K_t, K_t^k, \phi_t \}_{t=0}^{\infty} \), a sequence of asset market prices \( \{ q^w_t, q_t, r_t, R_t \}_{t=0}^{\infty} \), wage rates \( \{ w_t \}_{t=0}^{\infty} \), government policies \( \{ G_t, B_t, \tau^k_t, \tau^t_t \}_{t=0}^{\infty} \), shadow values of liquidity \( \{ \rho_t \}_{t=0}^{\infty} \), and an exogenous sequence of productivity \( \{ A_t \}_{t=0}^{\infty} \) such that

\(^{25}\)When the financing constraint is slack, \( q^w_t = 1 \) and an individual household is indifferent whether to purchase an extra unit in the market or to increase its own entrepreneurs’ investment. Hence, \( q^w_t \) remains the correct shadow cost of acquiring an extra unit of capital. This is true even though in the aggregate we must have \( \phi_t = 0 \) and hence no trade in capital claims takes place.
(39) – (44) and (46) – (50) are satisfied, and capital evolves according to $K_t = (1 - \delta)K_{t-1} + K^e_t$.

### 4.3 The Ramsey Outcome

To find the best equilibrium, in a frictionless economy it is possible to write a planner problem that collapses all the constraints into feasibility (equation (46)) and a single present-value implementability condition. The presence of financing constraints implies that we cannot collapse the implementability constraints into a single present-value condition, but rather we have a sequence of them. To simplify notation, from here on we will write $\eta_t, q^w_t, q_t, and \rho_t$ to denote the functions of $\phi_t$ that are defined by $\eta(\phi_t)$, and equations (41), (44), and (48).\footnote{26In computing an optimum, we will take into account that in a competitive equilibrium these variables are functions of $\phi_t$ and of no other variable that enters into the planner’s maximization problem.}

We also define

$$d_t := 1 - q^w_t + \phi_t \eta_t - \chi \rho_t \phi_t q_t,$$

which is also a function of $\phi_t$ alone.

Using the household budget constraint and the first-order conditions, the implementability constraint at $t \geq 1$ can be written as:

$$u'(C_t)C_t - u'(L_t)L_t + u'(C_t)B_t + u'(C_t) (1 + \phi_t \eta_t) K_t = u'(C_{t-1}) \frac{B_{t-1}}{\beta(1 + \chi \rho_t)} + u'(C_{t-1}) \frac{q^w_{t-1} K_{t-1}}{\beta} + u'(C_t) d_t (1 - \delta) K_{t-1}. \quad (51)$$

The implementability constraint at $t = 0$ is

$$u'(C_0)C_0 - u'(L_0)L_0 + u'(C_0)B_0 + u'(C_0) (1 + \phi_0 \eta_0) K_0 = u'(C_0)R_0 B_{-1} + u'(C_0)(1 - \tau_0^*) A_0 F_K(K_{-1}, L_0) + u'(C_0) [1 + (1 - \chi) \phi_0 \eta_0](1 - \delta) K_{-1}, \quad (52)$$

with $B_{-1}$, $K_{-1}$, $R_0$, and $\tau_0^*$ exogenously given. We follow the tradition of exogenously limiting capital-income taxation in period 0, since this would be otherwise a lump-sum tax. Using the individual entrepreneur’s financing constraint (39) and the first-order condition for bonds, we obtain that a competitive equilibrium satisfies the following condition in the aggregate for any period $t > 0$

$$(1 - \phi_t q_t) [K_t - (1 - \delta)K_{t-1}] \leq \chi \left[ \frac{u'(C_{t-1})}{\beta u'(C_t)(1 + \chi \rho_t)} B_{t-1} + \phi_t q_t (1 - \delta) K_{t-1} \right]. \quad (53)$$

In period 0, the financing constraint is

$$(1 - \phi_0 q_0) [K_0 - (1 - \delta)K_{-1}] \leq \chi [R_0 B_{-1} + \phi_0 q_0 (1 - \delta) K_{-1}]. \quad (54)$$
Therefore, the planner maximizes the household utility (38) subject to the sequence of resource constraints represented by (46), the implementability constraints (51) and (52), and the financing constraints (53) and (54). The planner chooses the allocation \( \{C_t, L_t, K_t, B_t, \phi_t\}_{t=0}^{\infty} \), which are consumption, labor hours, capital stock, government bonds, and asset liquidity. We can back out the taxes and prices from the allocation and the other necessary conditions for a competitive equilibrium.

4.4 Long-run Public Liquidity Provision, Capital Tax, and Interest Rates

Let \( \beta^t \Psi_t \) and \( \beta^t \gamma_t u'(C_t) \) be the Lagrange multipliers attached to implementability constraints and the financing constraints. The Appendix contains the derivation of the planner’s first-order conditions. In particular, the planner’s first-order condition for bonds is

\[
\Psi_t - \frac{\Psi_{t+1}}{1 + \chi \rho_{t+1}} + \frac{\chi \gamma_{t+1}}{1 + \chi \rho_{t+1}} = 0 \implies \Psi_{t+1} = (1 + \chi \rho_{t+1}) \Psi_t + \chi \gamma_{t+1}.
\]

An additional unit of debt issuance relaxes the current government budget (or implementability constraint) measured by \( \Psi_t \). Without frictions, this would be exactly offset by a tighter budget constraint in period \( t + 1 \), leading to \( \Psi_t = \Psi_{t+1} \). This is what happens if the financing constraint is slack in period \( t + 1 \). If instead the financing constraint is binding, two forces lead to \( \Psi_t < \Psi_{t+1} \).

First, since bonds can be liquidated without incurring intermediation costs, households are willing to hold them at a lower interest rate, which accounts for the term \( 1 + \chi \rho_{t+1} \) as in equation (49). Second, the additional liquidity provided by the increased supply of bonds directly relaxes the financing constraint of the entrepreneurs in period \( t + 1 \), which justifies the term \( \chi \gamma_{t+1} \).

When the financing constraint is slack, \( \Psi_t = \Psi_{t+1} \) corresponds to the standard tax-smoothing principle. In contrast, with \( \Psi_{t+1} < \Psi_t \), the tightness of government budget is increasing over time. We thus obtain the following result:

**Proposition 2.** Assume that the economy converges to a steady state with finite allocations (finite \( C, K, L, \) and \( B \), given finite \( G \) and \( A \)).

- If the government finds it feasible to flood the economy with public liquidity, it is optimal to do so. More precisely, the government issues enough debt to fully relax the financing constraints in the limit. In this case, \( \Psi_t \) converges to a constant, capital-income taxes are zero in the limit, and the interest rate on government debt is \( 1/\beta \) in the limit.

- If the amount of debt that fully relaxes financing constraints exceeds the fiscal capacity of the government, \( \Psi_t \) grows without bounds and the economy converges to a dynamic equivalent of the top of the Laffer curve. If the utility is quasilinear or if \( \beta \) is sufficiently close to 1, taxes on capital are strictly positive in the limit, that is, \( \lim_{t \to \infty} \tau^k_t > 0 \), and the interest rate on government debt is lower than \( 1/\beta \) in the limit.
For the infinite horizon economy we cannot obtain analytical expressions even with quasi-linear utility. We illustrate our results with numerical exercises.

The baseline parameters we choose are $\beta = 0.96$, $\alpha = 1/3$, $\delta = 0.1$, and $\nu = 1/1.5$, which are all standard parameters for a yearly calibration for a macroeconomic model. $\mu$ is set to one as a normalization. We set $\eta_0 = 0.5$ and $\eta_1 = 2$, and then we experiment with different values for $\eta_0$. The $\chi$ parameter is set to 0.2, corresponding to the frequency of firm-investment spikes found in empirical studies (which is also used in many work studying uninsurable idiosyncratic investment risks, e.g., Shi (2015)). The degree of risk-aversion is set to $\sigma = 0.2$, and we then experiment with different values. The steady-state productivity $A$ is normalized to one.

We choose government spending to be the lowest level at which the financing constraint holds as an equality in the limiting steady state. This turns out to be $G = G^* = 0.2147$, and it corresponds to a ratio of government expenditure to output of 29.94%. By construction, we thus have $q = q^w = 1$, $\phi = 0$, and $R = 1/\beta$ in the limit.

Table 1: Steady state of the Ramsey allocation for different government expenditures

<table>
<thead>
<tr>
<th>$G/Y$</th>
<th>29.93%</th>
<th>33.33%</th>
<th>34.44%</th>
<th>34.88%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital: $K$</td>
<td>100%</td>
<td>91.74%</td>
<td>87.00%</td>
<td>84.57%</td>
</tr>
<tr>
<td>Capital tax: $\tau^k$</td>
<td>0%</td>
<td>2.46%</td>
<td>4.04%</td>
<td>4.90%</td>
</tr>
<tr>
<td>Labor tax: $\tau^\ell$</td>
<td>52%</td>
<td>51.23%</td>
<td>50.57%</td>
<td>49.99%</td>
</tr>
<tr>
<td>Interest rate:</td>
<td>4.17%</td>
<td>3.22%</td>
<td>2.37%</td>
<td>1.51%</td>
</tr>
<tr>
<td>Debt-to-output: $B/Y$</td>
<td>117.69%</td>
<td>56.13%</td>
<td>32.80%</td>
<td>14.97%</td>
</tr>
<tr>
<td>Asset Liquidity $\phi$</td>
<td>0</td>
<td>0.1792</td>
<td>0.2465</td>
<td>0.2988</td>
</tr>
</tbody>
</table>

Table 1 displays the results from the baseline calibration, along with higher values of $G$. The corresponding $G$s are chosen such that the growth rate of the Langrange multiplier $\Psi_{t+1}/\Psi_t$ converges to 1 (when $G = G^*$), 1.01, 1.02, and 1.03, respectively. As $G$ increases, the maximum sustainable level of debt decreases. Notice that the government cannot flood the economy with liquidity, while a reduction of debt-to-output increases the liquidity premium and reduces the interest rate on government debt, and is accompanied by higher capital tax burdens. The benefit is such that taxes on labor actually decrease. With smaller amounts of public liquidity, entrepreneurs increasingly rely on financial intermediaries to sell some of their capital and fund their investment: the fraction $\phi$ of capital that is intermediated increases.

27 This is higher than the value in the United States, but our model does not feature transfers. If some of the government expenditure takes the form of transfers, the wealth effect is lessened, which lowers the maximum sustainable level of debt.
Next, we study the role of financial intermediation costs. Specifically, we consider what happens as \( \eta_0 \) increases. To do this, we need intermediation to take place at steady state, and so we set \( G \) to be a level higher than \( G^* \), and this corresponds to the economy when \( G/Y = 1/3 \).\(^{28}\)

Perhaps surprisingly, when intermediation is more costly it is used more in the limit. The reason is that the fiscal capacity of the economy contracts, so the government is less able to issue debt. As a substitute for the inability to relax financing constraints by providing public debt, the government increases the capital-income tax instead.

Table 2: The long-run economies with different financial intermediation

<table>
<thead>
<tr>
<th>( \eta_0 )</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital: ( K )</td>
<td>100%</td>
<td>97.03%</td>
<td>93.36%</td>
<td>87.31%</td>
</tr>
<tr>
<td>Capital tax: ( \tau^k )</td>
<td>2.46%</td>
<td>3.45%</td>
<td>4.78%</td>
<td>7.03%</td>
</tr>
<tr>
<td>Labor tax: ( \tau^\ell )</td>
<td>51.23%</td>
<td>50.94%</td>
<td>50.53%</td>
<td>49.37%</td>
</tr>
<tr>
<td>Interest rate:</td>
<td>3.22%</td>
<td>2.95%</td>
<td>2.60%</td>
<td>1.64%</td>
</tr>
<tr>
<td>Debt-to-output: ( B/Y )</td>
<td>56.13%</td>
<td>53.03%</td>
<td>48.47%</td>
<td>34.52%</td>
</tr>
<tr>
<td>Asset Liquidity ( \phi )</td>
<td>0.1792</td>
<td>0.1851</td>
<td>0.1947</td>
<td>0.2310</td>
</tr>
</tbody>
</table>

As a final illustration in Table 3, we show the consequences of changing \( \sigma \). As we vary \( \sigma \), we also adjust \( G \) to keep it at the lowest level of government expenditures such that the financing constraint remains binding in the limit. The main message of this table is that, as \( \sigma \) increases to a number close to one, the fiscal capacity of the government expands, and the economy remains constrained only at higher and higher levels of spending.

Table 3: The long-run economies with different intertemporal substitution

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital: ( K )</td>
<td>100%</td>
<td>99.31%</td>
<td>99.90%</td>
<td>89.64%</td>
</tr>
<tr>
<td>Capital tax: ( \tau^k )</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>Labor tax: ( \tau^\ell )</td>
<td>40.00%</td>
<td>52%</td>
<td>64%</td>
<td>88%</td>
</tr>
<tr>
<td>Interest rate:</td>
<td>4.17%</td>
<td>4.17%</td>
<td>4.17%</td>
<td>4.17%</td>
</tr>
<tr>
<td>Debt-to-output: ( B/Y )</td>
<td>117.66%</td>
<td>117.69%</td>
<td>117.70%</td>
<td>117.36%</td>
</tr>
<tr>
<td>( G )-to-output: ( G/Y )</td>
<td>21.90%</td>
<td>29.93%</td>
<td>37.98%</td>
<td>54.07%</td>
</tr>
<tr>
<td>Asset Liquidity ( \phi )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The resulting higher limit taxes imply that the economy accumulates a smaller amount of capital, so that entrepreneurs need less public liquidity to finance their investment. However, since we look at the lowest level of government expenditures that the economy is just financing constrained, the

\(^{28}\)We experimented with different values and the results are qualitatively robust.
debt-to-output ratio remains almost unchanged while $G/Y$ ratio increases by a factor of almost 2.5. The rise of government expenditures are funded by significant increases of labor income taxes.

5 Conclusion

Within the context of a Ramsey model of capital taxation, we identified a force that operates as in Sargent and Wallace (1982) and pushes the government to increase its indebtedness to mitigate frictions in private asset markets. We also showed that, when it is impossible to completely undo those frictions in the long run, it is optimal to tax capital even though its provision is already inefficiently low: this happens because the frictions that prevent efficient investment also alter the elasticity of the supply of capital. We considered here an economy with no aggregate risk, where no force counters the upward drift in government debt. In a stochastic economy with non-contingent debt, Aiyagari, Marcet, Sargent, and Seppälä (2002) identify an opposite force, that induces the government to accumulate assets for self insurance. In our next step, we plan to study how capital-income taxes and government debt are optimally chosen when both of these forces are present.

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Appendix

A.1 The Two-period Planner’s Problem (with Endogenous Asset Liquidity)

Here, we write down the planner’s problem in detail and derive the first-order necessary conditions. First, the planner’s objective can be stated as the following Lagrangian:

\[
L = \sum_{t=1}^{2} \beta^{t-1} \left\{ u(C_t) - v(L_t) + \Psi_1 [u'(C_t)C_t - v'(L_t)L_t] \right\} - \Psi_1 u'(C_1)z(\phi_1)K_1 + \gamma_1 u'(C_1) [R_1B_0^t - x(\phi_1)K_1] \\
+ \lambda_1 [L_1 - C_1 - K_1 - \eta(\phi_1)\phi_1K_1 - G_1] + \beta\lambda_2 [F(K_1, L_2) - C_2 - G_2]
\]

where we use the household’s utility, the implementability constraint, the financing constraint, and the resource constraints. Thanks to the smoothness of function \( \eta(\cdot) \), we do not need to impose the constraint \( \phi_1 \geq 0 \). Second, we derive all the planner’s first-order necessary conditions here. The first-order conditions for consumption \( C_1 \) and \( C_2 \) are

\[
u'(C_1)(1 + \Psi_1) + \Psi_1 u''(C_1)C_1 - \lambda_1 = \Psi_1 u''(C_1)K_1z(\phi_1); \]

\[
u'(C_2)(1 + \Psi_1) + \Psi_1 u''(C_2)C_2 - \lambda_2 = 0. \]

The first-order conditions for labor supply \( L_1 \) and \( L_2 \) are

\[
v'(L_1)(1 + \Psi_1) + \Psi_1 v''(L_1)L_1 = \lambda_1; \]

\[
v'(L_2)(1 + \Psi_1) + \Psi_1 v''(L_2)L_2 = \lambda_2 F_L(K_1, L_2). \]

The first-order condition for asset liquidity \( \phi_1 \) is

\[- \lambda_1 [1 + \phi_1\eta(\phi_1)] + \beta\lambda_2 F_K(K_1, L_2) = u'(C_1) [\Psi_1 z(\phi_1) + \gamma_1 x(\phi_1)]. \quad (55)\]

The first-order condition for capital \( K_1 \) is

\[
\lambda_1 [\eta(\phi_1) + \phi_1\eta'(\phi_1)] + u'(C_1) [\Psi_1 z'(\phi_1) + \gamma_1 x'(\phi_1)] = 0. \quad (56)\]

Notice that \( \phi_1 = 0 \) if and only if \( \gamma_1 = 0 \): this happens when the planner optimally chooses an allocation such that the financing constraint is slack.

A.2 The Infinite-horizon Planner’s Problem

Let \( \beta^t \lambda_t \) be the Lagrange multiplier on constraint (46), \( \Psi_0 \) be the Lagrange multiplier on constraint (52), \( \beta^t \Psi_t \) the Lagrange multiplier on (51), \( u'(C_0)\gamma_0 \) on (54), and \( \beta^i u'(C_i)\gamma_i \) on (53). For brevity, we denote by \( \eta'_i, (q^i)^', q'_i, \rho'_i, \) and \( d'_i \) the derivatives of each (previously defined) function with respect to \( \phi_i \).

The necessary first-order conditions for a Ramsey outcome are the following:
• Liquidity in period 0:

\[
\frac{\lambda_0A_0F_K(K_{t-1},L_0) + \Psi_0u'(C_0) + \Psi_0u''(C_0) + (1 + \phi_0\eta_0)K_0}{1 + \chi p_t}
\]

• Consumption in period 0:

\[
(1 + \Psi_0)u'(C_0) + \Psi_0u''(C_0)[C_0 + B_0 + (1 + \phi_0\eta_0)K_0] - \Psi_0u''(C_0)[R_0B_{-1} + (1 - \tau_0^k)A_0F_K(K_{-1},L_0) + (1 + (1 - \chi)\phi_0\eta_0)(1 - \delta)K_{-1}]
\]

\[
+ \gamma_0u''(C_0)[(R_0B_{-1} + \phi_0q_0(1 - \delta)K_{-1} - (1 - \phi_0q_0)(K_0 - (1 - \delta)K_{-1})] - \lambda_0
\]

\[
= - \frac{\chi B_0}{\beta(1 + \chi p_t)};
\]

• Consumption in period \(t \geq 1\):

\[
(1 + \Psi_t)u'(C_t) + \Psi_tu''(C_t)C_t + \Psi_tu''(C_t)[B_t + (1 + \phi_t\eta_t)K_t] - \Psi_tu''(C_t)d_t(1 - \delta)K_{t-1}
\]

\[
+ \gamma_tu''(C_t)[[1 - (1 - \chi)\phi_tq_t](1 - \delta)K_{t-1} - (1 - \phi_tq_t)K_t] - \lambda_t
\]

\[
= - \gamma_{t+1}u''(C_t) + \frac{\chi B_t}{1 + \chi p_{t+1}} + \Psi_{t+1}u''(C_t)
\]

• Leisure in period 0:

\[
v'(L_0)(1 + \Psi_0) + \Psi_0u''(L_0)L_0 = \lambda_0A_0F_K(K_{t-1},L_0) + \Psi_0u'(C_0)(1 - \tau_0^k)A_0F_K(K_{t-1},L_0);
\]

• Leisure in period \(t \geq 1\):

\[
v'(L_t)(1 + \Psi_t) + \Psi_tu''(L_t)L_t = \lambda_tA_tF_K(K_{t-1},L_t);
\]

• Liquidity in period 0:

\[
[\Psi_0u'(C_0) - \lambda_0](\eta_0 + \phi_0\eta_0) + \gamma_0(q_0 + \phi_0q_0) = 0;
\]

• Liquidity in period \(t \geq 1\):

\[
\Psi_tu'(C_t)K_t(\eta_t + \phi_t\eta_t) - \gamma_tu'(C_{t-1})\frac{\chi B_{t-1}}{\beta(1 + \chi p_t)^2} + \gamma_tu'(C_t)[K_t - (1 - \chi)(1 - \delta)K_{t-1}] + \phi_tq_t
\]

\[
+ \lambda_t[(1 - (1 - \chi)(1 - \delta)K_{t-1} - K_t)(\eta_t + \phi_t\eta_t) + \Psi_tu'(C_{t-1})\frac{\chi B_{t-1}}{\beta(1 + \chi p_t)^2} - \Psi_tu'(C_t)(1 - \delta)K_{t-1}d_t]
\]

\[
= \Psi_{t+1}u'(C_t)K_t(q_t^w);
\]

• Capital in period \(t \geq 0\):

\[
\lambda_t(1 + \phi_t\eta_t) - \Psi_tu'(C_t)(1 + \phi_t\eta_t) + \gamma_tu'(C_t)(1 - \phi_tq_t)
\]

\[
= \beta\lambda_{t+1}[A_{t+1}F_K(K_{t+1},L_{t+1}) + [1 + (1 - \chi)\phi_{t+1}\eta_{t+1}](1 - \delta)] - \Psi_{t+1}u'(C_t)q_t^u
\]

\[
- \beta\Psi_{t+1}u'(C_{t+1})d_{t+1}(1 - \delta) + \beta\gamma_{t+1}u'(C_{t+1})[1 - (1 - \chi)\phi_{t+1}q_{t+1}](1 - \delta);
\]

• Bond choice for \(t \geq 0\):

\[
\Psi_t = \frac{\Psi_{t+1}}{1 + \chi p_{t+1}} - \frac{\chi\gamma_{t+1}}{1 + \chi p_{t+1}} \rightarrow \Psi_{t+1} = (1 + \chi p_{t+1})\Psi_t + \chi\gamma_{t+1}.
\]
A.3 Proof to Proposition 2

We denote steady-state allocations by a bar over each variable. From the first-order conditions for bonds, equation (61), we know that $\Psi_t$ is weakly increasing. Moreover, it is constant if and only if $\rho_{t+1} = 0$ and $\gamma_{t+1} = 0$, which happens if and only if the financing constraint is slack. If the Ramsey allocation converges to a constant, we then have two possibilities as follows.

Case 1: $\Psi_t$ converges to a finite constant $\bar{\Psi} > 0$. In this case, the Lagrange multiplier of the financing constraint converges to zero in the limit and so does the financial-market trading in (claims to) capital, that is $\phi_t \to 0$. The limiting first-order conditions look like those of a standard neoclassical growth model. In particular, the limit of the planner’s first-order condition with respect to capital becomes

$$\beta [\bar{A} F_K(\bar{K}, \bar{L}) + 1 - \delta] = 1,$$

which coincides with the first-order condition for capital of the households with $\tau_t^k = \bar{\tau}^k = 0$. With $\bar{\rho} = 0$, the households’ first-order condition for bonds evaluated at steady state implies that $\bar{R} = 1/\beta$.

Case 2: $\Psi_t$ diverges to infinity. In this case, we use equations (58) and (61) to substitute for $\lambda_t$ and $\gamma_t$ in equations (57), (59), and (60). If the Ramsey allocation converges to a steady state, these three equations in the limit turn into linear second-order difference equations in $\Psi_t$. These equations are generically distinct. In order for the system to have a solution, it must be that the 5 variables $(\bar{C}, \bar{L}, \bar{K}, \bar{B}, \bar{\phi})$ are such that equations (46), (51), and (53) (the resources, implementability, and financing constraints respectively) are satisfied in the steady state, and such that the three difference equations share at least one root. This gives us 5 (nonlinear) conditions to solve for the 5 variables. In addition, $\Psi_{t+1}/\Psi_t$ must converge to a constant $\zeta$. Expressing the second-order difference equations as two-equation systems of first-order difference equations for the vector $(\Psi_{t+1}, \Psi_t)$, the constant $\zeta$ corresponds to the ratio $\Psi_{t+1}/\Psi_t$ in the eigenvector associated with the common eigenvalue across the three systems. Note that this eigenvalue must be real; if the systems had complex eigenvalues, matching eigenvalues would imply 2 additional constraints, giving us 7 conditions for 5 variables and implying that generically there would be no solution.

\[\text{30}\]

\[\text{31}\]

\[\text{29}\] If $\Psi_t = 0$ at any time $t$, it is straightforward to show that it must be the case that $\Psi_t = 0$ in all periods and that the Ramsey solution attains the first best. In this case, capital is subsidized if the financing constraint is binding, as we discussed in the context of the two-period example.

\[\text{30}\] That $\lambda_t$ converges to a constant follows from the first-order conditions with respect to consumption or labor.

\[\text{31}\] Expressing the second-order difference equations as two-equation systems of first-order difference equations for the vector $(\Psi_{t+1}, \Psi_t)$, the constant $\zeta$ corresponds to the ratio $\Psi_{t+1}/\Psi_t$ in the eigenvector associated with the common eigenvalue across the three systems. Note that this eigenvalue must be real; if the systems had complex eigenvalues, matching eigenvalues would imply 2 additional constraints, giving us 7 conditions for 5 variables and implying that generically there would be no solution.
\[
\frac{u'(C)}{u''(C)} + C + B + (1 + \phi \eta)K + \gamma \frac{[1 - (1 - \chi)\phi q]}{1 - (1 - \phi q)} K
\]
\[
=d(1 - \delta)K + \frac{1}{u''(C)} - \tilde{\gamma} \zeta \frac{B}{1 + \chi \rho} + \zeta \left( q''w K + \frac{B}{1 + \chi \rho} \right)
\]
and after we use the financing constraint
\[
\frac{u'(C)}{u''(C)} + C + B + (1 + \phi \eta)K - \tilde{\gamma} \zeta \frac{1 - \beta \zeta}{1 + \chi \rho} = d(1 - \delta)K + \frac{1}{u''(C)} + \zeta \left( q''w K + \frac{B}{1 + \chi \rho} \right).
\]

The FOC for capital:

\[
\tilde{\lambda} (1 + \phi \eta) - u'(C) (1 + \phi \eta - \zeta q''w) + \tilde{\gamma} u'(C)(1 - \phi q)
\]
\[
= \beta \lambda \zeta [AF_K(K, L) + [1 + (1 - \chi)\phi \eta] (1 - \delta)] - \beta u'(C) d(1 - \delta) \zeta + \beta \tilde{\gamma} \zeta u'(C) [1 - (1 - \chi)\phi q] (1 - \delta).
\]

The FOC for bonds:

\[(1 - \chi \tilde{\gamma}) \zeta = 1 + \chi \rho.
\]

In such a steady state, the entrepreneurs’ financing constraint binds, so that \( \phi > 0 \) and \( \rho > 0 \). This means that the interest rate \( R = \frac{1}{\beta(1 + \chi \rho)} < \frac{1}{2} \). We are also ready to show that capital tax \( \tau^k > 0 \), which can be seen from comparing the planner’s first-order condition for capital in (65) and the household’s first-order condition for capital (50):

\[
\beta \left[ F_K(K, L) + [1 + (1 - \chi)\phi \eta] (1 - \delta) \right] = \frac{u'(C)}{\lambda} q''w + \frac{\tilde{\gamma}}{\lambda} u'(C)(1 - \phi q) \left[ 1 - \frac{u'(C)}{\lambda} \right] \frac{1 + \phi \eta}{\zeta}
\]
\[
+ \frac{\beta u'(C)}{\lambda} d(1 - \delta) - \beta \frac{\tilde{\gamma}}{\lambda} u'(C) [1 - (1 - \chi)\phi q] (1 - \delta);
\]

\[
\beta \left[ (1 - \tau^k) F_K(K, L) + (q''w - \chi \phi \eta + \chi \rho \phi q) (1 - \delta) \right] = q''w.
\]

Taking the difference of the two and using the relationship \( d_i = d(\phi_t) = 1 + \phi_t \eta t - q''w - \chi \rho_t \phi_t q_t \), we obtain

\[
\tau^k \beta F_K(K, L) = \left[ \frac{u'(C)}{\lambda} - 1 \right] \left[ q''w - \frac{1 + \phi \eta}{\zeta} + \beta (1 - \delta) d + \frac{\tilde{\gamma}}{\lambda} u'(C) \left[ \frac{1 - \phi q}{\zeta} - \beta (1 - \delta) [1 - (1 - \chi)\phi q] \right] \right]
\]
\[
> \left[ \frac{u'(C)}{\lambda} - 1 \right] \left[ q''w - \frac{1 + \phi \eta}{\zeta} + \beta (1 - \delta) d + \frac{\tilde{\gamma}}{\lambda} u'(C) \left[ \frac{1 - \phi q}{\zeta} - \beta (1 - \delta) [1 - (1 - \chi)\phi q] \right] \right]
\]
\[
> \left[ \frac{u'(C)}{\lambda} - 1 \right] \rho + \frac{\tilde{\gamma}}{\lambda} u'(C) \beta [(1 - \phi q) - (1 - \delta) [1 - (1 - \chi)\phi q]],
\]

where the first inequality uses the fact that \( \zeta > 1 \), and the second inequality uses \( \zeta < 1/\beta \) and the relationship \( \rho = (q''w - 1 - \phi q) / (1 - \phi q) \). Notice that the second term (after \( \beta \)) in (66) is positive, because the financing constraint (62) implies that

\[ [1 - \phi q - (1 - \delta) [1 - (1 - \chi)\phi q]] = \chi RB/K > 0. \]

We only need to look at the term before \( \beta \) in (66).

Firstly, if the utility function is quasi-linear, \( u'(C) = 1 \), \( u''(C) = 0 \), and \( \tilde{\lambda} \to 1 \) according to (64). Equation (66) then implies that \( \tau^k > 0 \).

Second, if the utility function is concave, then in general \( \tau^k \) is not zero; we further prove that capital tax \( \tau^k > 0 \) as
long as $\beta \to 1$. To see the last claim, we use the planner’s first-order condition for consumption (64) and rearrange:
\[
\frac{u'(C)}{\lambda} - 1 = \frac{u''(C)}{\lambda} \left[ d(1 - \delta)K + \zeta \left( q^w K + \frac{B}{1 + \chi \rho} \right) - C - B - (1 + \phi \eta)K + \frac{\chi B}{1 + \chi \rho} \frac{1 - \beta \zeta}{\beta} \right].
\] (67)

Then, we can show that $\frac{u'(C)}{\lambda} - 1 > 0$ because $u$ is concave so that $u''(C) < 0$ and the expression in the middle bracket on the right-hand side is negative as long as $\beta \to 1$. To do so, we use the implementability condition (63)
\[
C - \frac{u'(L)}{u'(C)} L + B + (1 + \phi \eta) K = \frac{B}{\beta(1 + \chi \rho)} + \frac{q^w}{\beta} K + d(1 - \delta)K.
\]

Therefore, as $1 < \zeta < 1/\beta$, the implementability condition implies that in (67)
\[
d(1 - \delta)K + \zeta \left( q^w K + \frac{B}{1 + \chi \rho} \right) - C - B - (1 + \phi \eta) K < \frac{\chi B}{1 + \chi \rho} \frac{1 - \beta \zeta}{\beta} - \frac{u'(L)}{u'(C)} L < 0,
\]
as $\beta \to 1$. This means that $\frac{u'(C)}{\lambda} - 1 > 0$ as $\beta \to 1$ in (67). Because $\rho > 0$, the whole right-hand side of (66) is positive and we thus prove that $\tau^k > 0$. \qed