A Price Setting in the frictionless case

We consider two simple static problems for a monopolist firm. The first has a linear system, the second is a log approximation to a general case.

A.1 Linear Demand

Let the demand for a product be $q(p) = a - b\ p$ where q are quantities and p is the price, a the intercept, b is the slope. Assume that the marginal cost is constant at the value of c. Thus profits are $\Pi(p) = (a - b\ p)(p - c)$, and the static monopoly price is given by

$$p^* = \frac{a + c\ b}{2b} \ ,$$

and the maximized profits can be written as $\Pi^* = b\ (p^*)^2 - c\ a$, or

$$\Pi(p) = \Pi^* + \frac{1}{2}(-2\ b)(p - p^*)^2 = b\ (p^*)^2 - c\ a - b\ (p - p^*)^2 \ .$$

A.2 Log approximation of monopolist profit function

Let $\Pi(P)$ be the profit function of the monopolist as a function of the price P. Let P^* be the optimal price, satisfying $\Pi_P(P^*) = 0$. Consider a second order approximation of the log of Π around $P = P^*$ obtaining:

$$\log \Pi(P) = \log \Pi(P^*) + \frac{1}{2} \frac{\partial^2\ \log \Pi(P)}{\partial (\log P)^2}\bigg|_{P=P^*} (\log P - \log P^*)^2 + o\left((\log P - \log P^*)^2\right) \ .$$

A useful example for this approximation is the case with a constant elasticity of demand equal to $\eta > 1$ where: $q(P) = E\ P^{-\eta}$ where E is a demand shifter and where the monopolist faces a constant marginal cost C, where:

$$P^* = \frac{\eta}{\eta - 1}\ C \quad \text{and} \quad \frac{\partial^2\ \log \Pi(P)}{\partial (\log P)^2}\bigg|_{P=P^*} = -\eta\ (\eta - 1)$$

So, letting $B = -\eta\ (\eta - 1)$, $p = \log P$ and $p^* = \log P^*$ we obtain the problem in the body of the paper.

A.3 Non-linear Price Setting Model

In this section we describe a fully non-linear price setting model. Let the instantaneous profit of a monopolist be as in Appendix A, where the cost C and the relative price is P. We solve this model numerically and compare its predictions with the simple tracking problem of the paper. [TO BE REPORTED IN FUTURE DRAFTS]. This model also helps to interpret ψ and ϕ in the tracking problem as cost in proportion to the period profit.

At the time of the observation we let the general price level be one. This price level increases at the rate π per unit of time. The log of constant margnal cost in real terms evolve as a random walk with innovation variance σ^2 and with drift μ. The demand has constant elasticity η with respect to the price P relative to the general price level. Thus the real demand t periods afger observing a real cost C with a price P is given by $A\left(\hat{P}e^{-\pi t}\right)^{-\eta}$ where A is a constant the determined the level of demand. The nominal mark-up t period after is then $\left(\hat{P} - Ce^{\pi t+\mu t+s(t)\sigma\sqrt{t}}\right)$, where $s(t)$ is a standard normal random variable. Thus the real profitst are given by $A\left(\hat{P}e^{-\pi t}\right)^{-\eta}\left(\hat{P}e^{-\pi t} - Ce^{\mu t+s(t)\sigma\sqrt{t}}\right)$. The profit level, if prices are chosen to maximize the instantaneous profit when real cost are C and the general price level is one are given by $\Pi^*(C) \equiv A\ C^{1-\eta}\left(\eta/(\eta-1)\right)^{-\eta}\left(1/(\eta-1)\right)$. The corresponding Bellman equation is:

$$
\begin{aligned}
v(P,C) &= \max\left\{\hat{v}(C)\ ,\ \bar{v}(P,C)\right\} \\
\hat{v}(C) &= -(\phi+\psi)\Pi^*(C) + \\
&\quad \max_{\hat{\tau},\hat{P}} \int_0^{\hat{\tau}} e^{-\rho t}\int_{-\infty}^{\infty} A\left(\hat{P}e^{-\pi t}\right)^{-\eta}\left(\hat{P}e^{-\pi t} - Ce^{\mu t+s(t)\sigma\sqrt{t}}\right) dN(s(t))\ dt + \\
&\quad + \ e^{-\rho\hat{\tau}}\int_{-\infty}^{\infty} v\left(\hat{P}e^{-\pi\hat{\tau}}, Ce^{\mu\hat{\tau}+s\sigma\sqrt{\hat{\tau}}}\right) dN(s) \\
\bar{v}(P,C) &= -\phi\Pi^*(C) + \max_{\tau}\int_0^{\tau} e^{-\rho t}\int_{-\infty}^{\infty} A\left(Pe^{-\pi t}\right)^{-\eta}\left(Pe^{-\pi t} - Ce^{\mu t+s(t)\sigma\sqrt{t}}\right) dN(s(t))\ dt + \\
&\quad + \ e^{-\rho\tau}\int_{-\infty}^{\infty} v\left(Pe^{-\pi\tau}, Ce^{\mu\tau+s\sigma\sqrt{\tau}}\right) dN(s)
\end{aligned}
$$

This Bellman equation is very similar to the one we solve numerically in Alvarez et al. (2009) for a saving and portfolio problem for households. As in the problem of that paper, it is easy to show that the value function is homogenous of degree $1-\eta$. In this case we can simplfy the problem considering only one state, say P/C. We can develop the expectations and collect terms to obtain:

$$
\begin{aligned}
\hat{v}(C) &= -(\phi+\psi)\Pi^*(C) + \\
&\quad \max_{\hat{\tau},\hat{P}} \int_0^{\hat{\tau}} e^{-\rho t+(1-\eta)(\mu+\sigma^2/2)t} A\ C^{1-\eta}\ \left(\frac{\hat{P}e^{-\pi t}}{Ce^{(\mu+\sigma^2/2)\ t}}\right)^{-\eta}\left(\frac{\hat{P}e^{-\pi t}}{Ce^{(\mu+\sigma^2/2)\ t}} - 1\right) dt \\
&\quad + \ e^{-\rho\hat{\tau}+(1-\eta)(\mu+\sigma^2/2)\hat{\tau}}\int_{-\infty}^{\infty} e^{-(1-\eta)(\mu+\sigma^2/2)\hat{\tau}} v\left(\hat{P}e^{-\pi\hat{\tau}}, Ce^{\mu\hat{\tau}+s\sigma\sqrt{\hat{\tau}}}\right) dN(s)
\end{aligned}
$$

and likewise for $\bar{v}$. Letting a modified discount factor to be $\tilde{\rho} \equiv \rho - (1-\eta)(\mu+\sigma^2/2)$ and using that v is homogeneous of degree $1-\eta$:

$$\begin{aligned}
\frac{\hat{v}(1)}{\Pi^*(1)} &= -(\phi+\psi) + \max_{\hat{\tau},\hat{P}/C} \int_0^{\hat{\tau}} e^{-\tilde{\rho}t} \frac{A}{\Pi^*(1)} \left(\frac{(\hat{P}/C)\ e^{-\pi t}}{e^{(\mu+\sigma^2/2)\ t}} \right)^{-\eta} \left(\frac{(\hat{P}/C) e^{-\pi t}}{e^{(\mu+\sigma^2/2)\ t}} - 1 \right) dt \\
&+ e^{-\tilde{\rho}\hat{\tau}} \int_{-\infty}^{\infty} e^{(1-\eta)(s\sigma\sqrt{\hat{\tau}}-(\sigma^2/2)\hat{\tau})}\ v\left(\frac{(\hat{P}/C)e^{-\pi\hat{\tau}}}{e^{\mu\hat{\tau}+s\sigma\sqrt{\hat{\tau}}}}, 1 \right) \frac{1}{\Pi^*(1)} dN(s) \\
\frac{\bar{v}(P/C,\ 1)}{\Pi^*(1)} &= -(\phi+\psi) + \max_{\tau} \int_0^{\tau} e^{-\tilde{\rho}t} \frac{A}{\Pi^*(1)} \left(\frac{(P/C) e^{-\pi t}}{e^{(\mu+\sigma^2/2)\ t}} \right)^{-\eta} \left(\frac{(P/C) e^{-\pi t}}{e^{(\mu+\sigma^2/2)\ t}} - 1 \right) dt \\
&+ e^{-\tilde{\rho}\hat{\tau}} \int_{-\infty}^{\infty} e^{(1-\eta)(s\sigma\sqrt{\hat{\tau}}-(\sigma^2/2)\hat{\tau})}\ v\left(\frac{(P/C)e^{-\pi\hat{\tau}}}{e^{\mu\hat{\tau}+s\sigma\sqrt{\hat{\tau}}}}, 1 \right) \frac{1}{\Pi^*(1)} dN(s) \\
v(P/C,1) &= \max\left\{ \hat{v}(1)\ , \bar{v}(P/C,1) \right\} .
\end{aligned}$$

B Hazard Rate of Menu Cost Model

In this appendix we described the details for the characterization of the hazard rate of price adjustments of the menu cost model of Section 4.2. Section 2.8.C formula (8.24) of Karatzas and Shreve (1991) displays the density of the distribution for the first time that a brownian motion hit either of two barriers, starting from an arbitrary point inside the barriers. In our case, the initial value is the price gap after adjustment, namely zero, and the barriers are symmetric, given by $-\bar{p}$ and $\bar{p}$. We found more useful for the characterization of the hazard rate to use a transformation of this density, obtained in Kolkiewicz (2002), section 3.3, as the sum of expressions (15) and (16). In our case we set the initial condition $x_0 = 0$ and the barriers $a < x_0 < b$ are thus given by $a = -\bar{p}$ and $b = \bar{p}$, thus obtaining the density $\boldsymbol{f}(t)$:

$$\boldsymbol{f}(t) = \frac{\boldsymbol{\pi}}{2\ (\bar{p}/\sigma)^2} \sum_{j=0}^{\infty} (2j+1)(-1)^j\ \exp\left(-\frac{(2j+1)^2 \boldsymbol{\pi}^2}{8\ (\bar{p}/\sigma)^2}\ t \right) . \tag{A-1}$$

The hazard rate $\boldsymbol{h}(t)$ is then defined as:

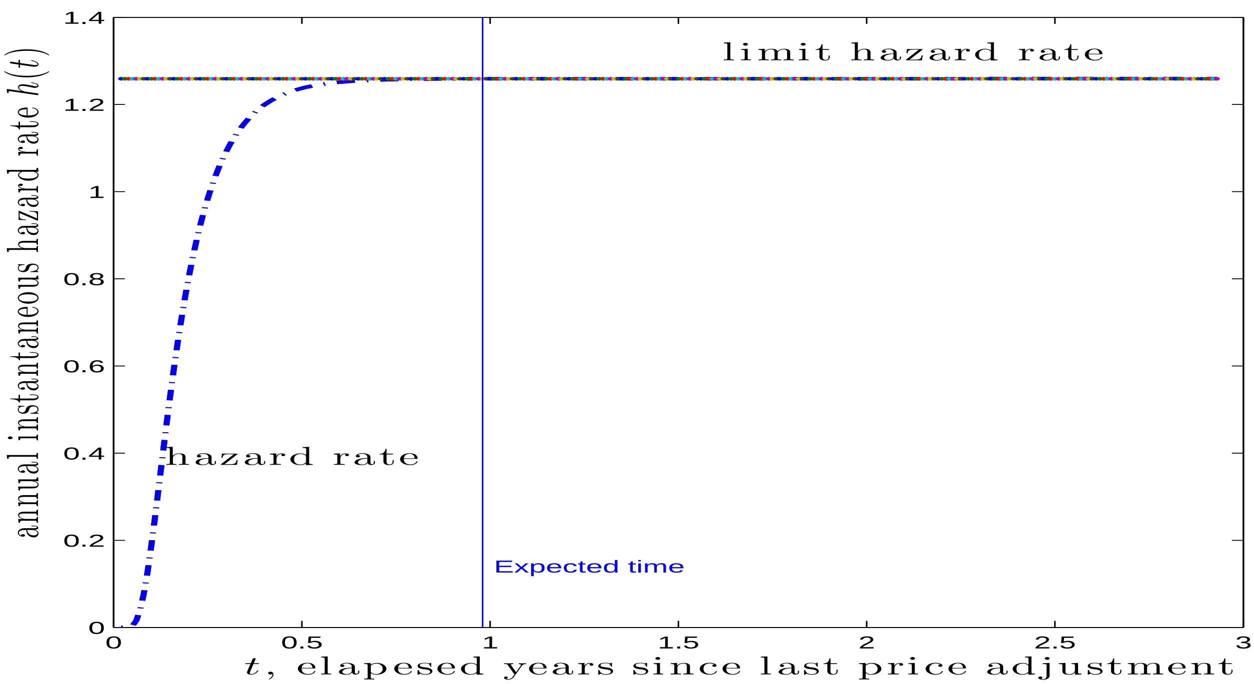
$$\boldsymbol{h}(t) = \frac{\boldsymbol{f}(t)}{\int_t^{\infty} \boldsymbol{f}(s) ds} . \tag{A-2}$$

Notice that since equation (A-1) is a sum of exponentials evaluated at the product of $-t$ times a positive quantity, each of them larger. Thus, for large values of t the first term in the sum dominates, and hence the expression for $\boldsymbol{f}(t)$ becomes

$$\boldsymbol{f}(t) \approx \frac{\boldsymbol{\pi}}{2\ (\bar{p}/\sigma)^2} \exp\left(-\frac{\boldsymbol{\pi}^2}{8\ (\bar{p}/\sigma)^2}\ t \right) \ \text{ for large }\ t\ . \tag{A-3}$$

and hence $\lim_{t\to\infty} \boldsymbol{h}(t) = \frac{\boldsymbol{\pi}^2}{8\ (\bar{p}/\sigma)^2}$. Indeed, the shape of this hazard rate is independent of $\bar{p}/\sigma$, this value only scales it up and down. Moreover, as Figure A-1 shows, the asymptote is approximately attained well before the expected value of the time.

Figure A-1: Hazard Rate of Menu Cost Model



Note: $B = 25, \sigma = 0.1$ and $\psi = 0.04$.

C Proofs

C.1 Proof of Proposition 1.

Proof. We let $x^* = \hat{p}/\pi$. Note that as $\rho \to 0$ we have:

$$\frac{\rho e^{-\rho t}}{1-e^{-\rho\tau}} \quad \to \quad \frac{1}{\tau} \ \text{ so } \ x^* \to \frac{1}{\tau}\int_0^\tau t\,dt = \frac{\tau}{2} \quad , \text{ and } \quad \int_0^\tau e^{-\rho t}\,t\,dt \to \tau \int_0^\tau t\,\frac{1}{\tau}dt = \frac{\tau^2}{2} \ .$$

$$v(\tau) \quad \to \quad \int_0^\tau \left(\frac{\tau}{2}-t\right)^2 dt = \tau \int_0^\tau \left(\frac{\tau}{2}-t\right)^2 \frac{1}{\tau}dt = \tau\frac{\tau^2}{12} = \frac{\tau^3}{12} \ .$$

Thus

$$\lim_{\rho\to 0} \rho V(\tau) \quad = \quad \lim_{\rho \to 0}\left(\frac{\pi^2}{\sigma^2}\frac{\rho\tau^3}{12\ (1-e^{-\rho\tau})} + \frac{\rho\ \tau^2}{2\ (1-e^{-\rho\tau})} + \frac{\tilde{\phi}\rho}{(1-e^{-\rho\tau})}\right) = \frac{\pi^2}{\sigma^2}\frac{\tau^2}{12} + \frac{\tau}{2} + \frac{\tilde{\phi}}{\tau}$$

The f.o.c. for τ is:

$$\frac{\tilde{\phi}}{(\tau^*)^2} = \frac{\pi^2}{\sigma^2}\frac{\tau^*}{6} + \frac{1}{2} \quad \text{or} \quad \tilde{\phi} = \frac{\pi^2}{\sigma^2}\frac{(\tau^*)^3}{6} + \frac{(\tau^*)^2}{2}$$

From here we see that the optimal inaction interval τ^* is a function of 2 arguments, it is increasing in the normalized cost $\tilde{\phi}$, and decreasing in the normalized drift $(\pi/\sigma)^2$. Keeping the parameters B, ϕ and π constant we can write:

$$\frac{\phi}{B} = \pi^2\ \frac{(\tau^*)^3}{6} + \sigma^2\frac{(\tau^*)^2}{2}$$

which implies that τ is decreasing in σ.

Note that for $\pi = 0$ we obtain a square root formula on the cost ϕ and with elasticity minus one on σ:

$$\tau^* = \sqrt{2\ \tilde{\phi}} = \sqrt{2\ \frac{\phi}{B\ \sigma^2}} = \sqrt{\frac{2}{B}}\ \phi^{1/2}\ \frac{1}{\sigma} \ .$$

The total differential of the foc for τ gives:

$$\tau^*(\tilde{\phi})\left(\frac{\pi^2}{\sigma^2}\frac{\tau^*(\tilde{\phi})}{2} + 1\right)\left(\frac{\partial \tau^*(\tilde{\phi})}{\partial\tilde{\phi}}\right) \quad = \quad 1$$

since $\lim_{\tilde{\phi}\to 0}\tau^*(\tilde{\phi}) = 0$, then one obtains the same expression than in the case of $\pi = 0$, and thus the elasticity is 1/2, or: $\lim_{\tilde{\phi}\to 0}\frac{\tilde{\phi}}{\tau^*(\tilde{\phi})}\frac{\partial\tau^*(\tilde{\phi})}{\partial\tilde{\phi}} = \frac{1}{2}$. To see the result for $\sigma = 0$, let us write:

$$\tilde{\phi}\ \sigma^2 \equiv \frac{\phi}{B} = \pi^2\ \frac{(\tau^*)^3}{6} + \sigma\frac{(\tau^*)^2}{2} \ ,$$

then we let $\sigma^2 \to 0$ to get $\frac{\phi}{B} = \pi^2\ \frac{\tau^3}{6}$ which implies a cubic root formula on the cost ϕ and with elasticity $-2/3$ on π:

$$\tau^* = \left(\frac{6\ \phi}{B\ \pi^2}\right)^{1/3} = \left(\frac{6}{B}\right)^{1/3}\ \phi^{1/3}\ \pi^{-2/3}.$$

That $\partial\tau^*/\partial\pi = 0$ evaluated at $\pi = 0$ follows from totally differentiating $\frac{\phi}{B} = \pi^2\ \frac{(\tau^*)^3}{6} + \sigma^2 \frac{(\tau^*)^2}{2}$.

Now consider the case when $\rho > 0$ and $\pi = 0$ then $\rho V(\tau)$ equals

$$\rho V(\tau) = \frac{\rho\tilde{\phi}}{1 - e^{-\rho\tau}} - \frac{\tau e^{-\rho\tau}}{1 - e^{-\rho\tau}} + \frac{1}{\rho}\ .$$

The first order condition with respect to τ implies that the optimal choice satisfies:

$$\tilde{\phi} = \frac{\rho\tau^* - 1 + e^{-\rho\tau^*}}{\rho^2}. \tag{A-4}$$

A third order expansion of the right hand side of equation (A-4), gives:

$$\tilde{\phi} = \frac{1}{2}\tau^2 - \frac{1}{6}\rho\tau^3 + o(\rho^2\tau^3).$$

The expression shows that if $\rho = 0$ we obtain a square root formula: $\tau^* = \sqrt{2\ \tilde{\phi}}$, and that the optimal τ^* is increasing in ρ provided ρ or $\tilde{\phi}$ are small enough. ■

C.2 Proof of Proposition 2.

Proof. Differentiating the Bellman equation and evaluating it at zero we obtain:

$$\rho V''(0) = 2B + \sigma^2/2V''''(0) \tag{A-5}$$

and evaluating this expression for $\rho = 0$ we have

$$V''''(0) = -\frac{2B}{\sigma^2/2}\ . \tag{A-6}$$

Differentiating the quartic approximation equation (11), evaluating at $\bar{p}$ and imposing the smooth pasting equation (9) we obtain:

$$0 = V''(0)\bar{p} + \frac{1}{3\ 2}V''''(0)\ \bar{p}^3\ . \tag{A-7}$$

Replacing into this equation the expression for $V''''(0)$ in equation (A-6) and solving for $V''(0)$ we obtain

$$V''(0) = -\frac{1}{3\ 2}V''''(0)\bar{p}^2 = \frac{1}{3\ 2}\frac{2B}{\sigma^2/2}\bar{p}^2\ . \tag{A-8}$$

Using the quartic approximation into the (levels) of equation (9) we obtain:

$$\psi = \frac{1}{2} V''(0)\bar{p}^2 + \frac{1}{4!} V''''(0)\bar{p}^4 \ , \tag{A-9}$$

replacing into this equation $V''(0)$ from equation (A-8) and $V''''(0)$ from equation (A-6) we obtain

$$\psi = \left(\frac{1}{2\ 3\ 2} - \frac{1}{4\ 3\ 2} \right) \frac{2B}{\sigma^2/2}\ \bar{p}^4 = \frac{B}{6\ \sigma^2}\ \bar{p}^4 \ , \tag{A-10}$$

thus solving for $\bar{p}$ we obtain the desired expression. ■

C.3 Proof of Proposition 3.

Proof. We start with a simple preliminary result to set up the analysis showing that the fixed point of V coincides with the solution of the sequence problem. Denote by $V_{\text{info},\phi+\psi}$ the solution of the problem with observation cost only of Section 4.1, but where the observation cost has been set to $\phi + \psi$, and denote the optimal value of the time between observations and price changes as τ_i. We interpret this as the value of following the feasible policy of observing and adjusting ($\chi_{T_i} = 1$ all T_i) every τ periods, and hence $V(\tilde{p}) \leq V_{\text{info},\phi+\psi}$. Also denote by $V_{\text{menu},\psi}(\cdot)$ the solution of the problem with menu cost ψ and no observation cost analyzed in Section 4.2. Since the observation cost is set to zero, this provides a lower bound for the value function: $V(\tilde{p}) \geq V_{\text{menu},\psi}(\tilde{p})$. Letting $\mathbf{T}$ the operator defined by the right side of equation (16), equation (17) and equation (18), and by $\mathbf{T}^n V_0$ the outcome of n successive applications of $\mathbf{T}$ to an initial function V_0, we have that for each $\tilde{p}$:

$$V_{\text{menu},\psi}(\tilde{p}) \leq \mathbf{T}^n V_{\text{menu},\psi}(\tilde{p}) \leq \mathbf{T}^n V_{\text{info},\phi+\psi}(\tilde{p}) \leq V_{\text{info},\phi+\psi}(\tilde{p}) \ ,$$

and the two sequence of functions converge pointwise. Since they converge to a finite value, their limit must be the same, by an adaptation of Theorem 4.14 in Stokey and Lucas (1989). Thus:

$$V(\tilde{p}) = \lim_{n\to\infty} \mathbf{T}^n V_{\text{info},\phi+\psi}(\tilde{p}) = \lim_{n\to\infty} \mathbf{T}^n V_{\text{menu},\psi}(\tilde{p}) \ .$$

pointwise. Furthermore, since $V_{\text{info},\phi+\psi}(\tilde{p})$ is a constant function, i.e. independent of $\tilde{p}$, and since $V_{\text{menu},\psi}(\tilde{p}) > 0$ for all $\tilde{p}$, then we have that the value function $V(\tilde{p})$ is uniformly bounded.

We sketch the argument to show that the value function V is continuous on $\tilde{p}$. Suppose not, that there is jump down at $\bar{p}$ so that $V(\bar{p}) > \lim_{p\downarrow\bar{p}} V(\bar{p})$. Then, fixed the policies that correspond to a value of $p > \bar{p}$ in terms of stopping times and prices $\hat{p}(\cdot)$ as defined in the sequence formulation of equation (14). Thus, the agent will observe, starting with $\bar{p}$, after the same cumulated value of the innovations on p^*, and it will adjust to the value Let $\epsilon = p - \bar{p}$ the difference between these two prices at time zero, and denote by $\{p(t)\}, \{\bar{p}(t)\}$ the stochastic process for the price that follow from the two initial prices. Notice that $\bar{p}(t) - p(t) = \epsilon$ for all $t > 0$. By following that policy when the initial price is $\bar{p}$, the expected discounted value of fixed cost paid are exactly the same for the two initial prices. Thus, the difference of the value function at $p(0)$ and at $\bar{p}(0)$ is given by the

$$\frac{\epsilon^2}{\rho} + \epsilon\, \mathbb{E}_0\Big[\int_0^\infty e^{-\rho t} \left(p(T_i) - p^*(t)\right) dt \Big] .$$

Since this difference is clearly continuous on ϵ and goes to zero as $\epsilon \downarrow 0$, then the value function is continuous.

Now we use the fact that V is bounded above uniformly and continuous to show that for $\phi > 0$, then the optimal policy for $\tau(\tilde{p})$, is uniformly bounded away from zero. Here is a sketch of the argument. Suppose, as a way of contradiction, that for any $\epsilon > 0$, we can find a $\tilde{p}$ for which $\tau(\tilde{p}_\epsilon) < \epsilon$. We will argue that for small enough ϵ it is cheaper to double $\tau(\tilde{p}_\epsilon)$. The main idea is that this decreases the fixed by $e^{-\rho\epsilon}\phi$, and increases the cost due to different information gathering in a quantity that is a continuous function of ϵ. The reason why the second part is continuous as function of ϵ is that the distribution of value of $p^*(t+\epsilon)$ and $p^*(t+2\epsilon)$ have most of the mass concentrated in a neighborhood of $p^*(t)$. The effect due to the increase in cost due to evaluation of the period return objective function is small, since it is the expected value of the integral of a bounded function between 0 and ϵ or between 0 and 2ϵ. Thus, for small ϵ this difference is small. The effect on the value function of the mass $p^*(t)$ is small, because the value function is uniformly bounded and this probability is small, i.e. goest to zero as $\epsilon \downarrow 0$. For the mass that is in the neighborhood of $p^*(t)$, the effect in the value function is small because the value function is continuous.

Using that $\inf \tau(\tilde{p}) \equiv \underline{\tau} > 0$, by Blackwell's sufficient conditions, we obtain that $\mathbf{T}$ is a contraction of modulus $\exp\left(-\underline{\tau}\rho\right)$ in the space of continuous and bounded functions. ■

C.4 Proof of Proposition 4.

Proof. Under the conjecture that $\hat{p} = 0$ and that $V(\cdot)$ is symmetric around zero, and by the symmetry of the normal density, we can rewrite the Bellman equations (17) and (18) using only the positive range for $\tilde{p} \in [0\ ,\ \infty)$ as:[38]

$$\begin{aligned}\bar{V}(\tilde{p}) = \phi + \min_{\tau} B \int_0^\tau e^{-\rho t}\left[\tilde{p}^2 + \sigma^2 t\right] dt \ + \\ e^{-\rho\tau} \int_{-\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V\left(\tilde{p} + s\sigma\sqrt{\tau}\right) dN(s) \quad + e^{-\rho\tau} \int_{\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V\left(-\tilde{p} + s\sigma\sqrt{\tau}\right) dN(s) \quad \text{(A-11)}\end{aligned}$$

$$\hat{V} = \psi + \phi + \min_{\hat{\tau}} B \int_0^{\hat{\tau}} e^{-\rho t}\left[\sigma^2 t\right] dt + e^{-\rho\hat{\tau}}\ 2 \int_0^\infty V\left(s\sigma\sqrt{\hat{\tau}}\right) dN(s) \quad \text{(A-12)}$$

We use the corollary of the contraction mapping theorem. First, notice that if the V in the right side of equation (20) is symmetric around $\tilde{p} = 0$, with a minimum at $\tilde{p} = 0$, then it is optimal to set $\hat{p} = 0$. Second, notice that if the function V in the right side of equation (19) is symmetric with a minimum at $\tilde{p} = 0$, then the value function in the left side of this equation is also symmetric, and hence V in equation (21) is symmetric. Third, using the symmetry, we show that if $V(\tilde{p})$ is weakly increasing, then the right side of equation (21) is weakly

[38]The second line in equation (A-11) uses that $\int_{-\infty}^{\infty} V(p-s)dN(s) = \int_{-p}^{\infty} V(p+s)dN(s) + \int_{p}^{\infty} V(-p+s)dN(s)$.

increasing. It suffices to show that $\bar{V}(\tilde{p})$ given by the right side of (A-11) is increasing in $\tilde{p}$ for a fixed arbitrary value of τ. We do this in two steps. The first step is to notice that the expression containing $\tilde{p}^2$ in (A-11) is obviously increasing in $\tilde{p}$. For the second step, without loss of generality, we assume that V is differentiable almost everywhere and compute the derivative with respect to $\tilde{p}$ of the remaining two terms involving the expectations of $V(\cdot)$ in (A-11). This derivative is:

$$
\begin{aligned}
& \frac{e^{-\rho\tau}}{\sigma\sqrt{\tau}}\left[V(0)\ dN\left(\frac{-\tilde{p}}{\sigma\sqrt{\tau}}\right) - V(0)\ dN\left(\frac{\tilde{p}}{\sigma\sqrt{\tau}}\right)\right] \\
+ \ & e^{-\rho\tau}\left[\int_{-\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V'(\tilde{p}+s(\sigma\sqrt{\tau}))dN(s) - \int_{\tilde{p}/(\sigma\sqrt{\tau})}^{\infty} V'(-\tilde{p}+s(\sigma\sqrt{\tau}))dN(s)\right] \\
= \ & e^{-\rho\tau}\left[\int_0^{\infty} V'(z)\frac{1}{\sigma\sqrt{\tau}}\left(dN\left(\frac{z-\tilde{p}}{\sigma\sqrt{\tau}}\right) - dN\left(\frac{z+\tilde{p}}{\sigma\sqrt{\tau}}\right)\right)\right] \\
= \ & e^{-\rho\tau}\left[\int_0^{\infty} V'(z)\frac{1}{\sigma\sqrt{\tau}\sqrt{2\,\boldsymbol{\pi}}}\left(e^{-\frac{1}{2}\left(\frac{z-\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} - e^{-\frac{1}{2}\left(\frac{z+\tilde{p}}{\sigma\sqrt{\tau}}\right)^2}\right)\ dz\right] \geq 0
\end{aligned}
$$

where the term involving $V(0)$ is zero due symmetry of $dN(s)$, and where the inequality follows since $e^{-\frac{1}{2}\left(\frac{x-\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} - e^{-\frac{1}{2}\left(\frac{x+\tilde{p}}{\sigma\sqrt{\tau}}\right)^2} > 0$ for $x > 0$ and $\tilde{p} > 0$. Notice that the inequality is strict if $\tilde{p} > 0$ and $V'(x) > 0$ in a segment of strictly positive length. If $\tilde{p} = 0$, then the slope is zero.

Finally, differentiating the value function twice, and evaluating at $\tilde{p} = 0$ we get

$$
V''(0) = 2\ B\ \frac{1-e^{-\rho\hat{\tau}}}{\rho} + 2\ \frac{e^{-\rho\hat{\tau}}}{\sigma\sqrt{\hat{\tau}}}\int_0^{\bar{p}} V'(z)\ z\ \frac{e^{-\frac{1}{2}\frac{z^2}{\sigma^2\hat{\tau}}}}{\sigma\sqrt{\hat{\tau}}\ 2\ \boldsymbol{\pi}}\ dz > 0\ .
$$

■

C.5 Proof of Proposition 5.

Proof. First we notice that using the quadratic approximation into the definition of $\bar{p}$ given by $\bar{V}(\bar{p}) = \hat{V}$ implies

$$
\psi = \frac{1}{2}V''(0)\,(\bar{p})^2\ . \tag{A-13}
$$

Second we derive equation (23) as the first order condition for $\hat{\tau}$. To this end, use the Bellman equation (20) for a fixed $\hat{\tau} > 0$ evaluated at the optimal $\hat{p} = 0$, the symmetry of $V(\tilde{p})$, and the approximation

$$
V(\tilde{p}) = \min\{\hat{V}\ ,\ V(0) + \frac{1}{2}V''(0)\ (\tilde{p})^2\}
$$

to write:

$$
\begin{aligned}
V(0) = \hat{V} - \psi &= \phi + B\sigma^2 \int_0^{\hat{\tau}} e^{-\rho t} t \; dt + e^{-\rho \hat{\tau}} \int_{-\infty}^{\infty} V\left(s \sigma \sqrt{\hat{\tau}}\right) dN(s) \\
&= \phi + B\sigma^2 \int_0^{\hat{\tau}} e^{-\rho t} t \; dt + e^{-\rho \hat{\tau}} \; V(0) + \psi \; e^{-\rho \hat{\tau}} \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right)\right] \\
&+ e^{-\rho \hat{\tau}} \; \frac{V''(0)}{2} \int_{-\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}}^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} (\sigma^2 \hat{\tau} s^2) \; dN(s) \\
&= \phi + B\sigma^2 \int_0^{\hat{\tau}} e^{-\rho t} t \; dt + e^{-\rho \hat{\tau}} \; V(0) + \psi \; e^{-\rho \hat{\tau}} \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right)\right] \\
&+ e^{-\rho \hat{\tau}} \; V''(0) \; \sigma^2 \hat{\tau} \int_{0}^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2 \; dN(s)
\end{aligned}
$$

Thus

$$
\rho V(0) = \frac{\phi + B\sigma^2 \int_0^{\hat{\tau}} e^{-\rho t} t \; dt + \psi \; e^{-\rho \hat{\tau}} \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right)\right] + e^{-\rho \hat{\tau}} \; V''(0) \; \sigma^2 \hat{\tau} \; \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2 \; dN(s)}{\left(1 - e^{-\rho \hat{\tau}}\right)/\rho}
$$

letting $\rho \downarrow 0$ gives

$$
\lim_{\rho \downarrow 0} \; \rho V(0) = \frac{\phi + \psi \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right)\right]}{\hat{\tau}} + B\sigma^2 \frac{\hat{\tau}}{2} + V''(0) \; \sigma^2 \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2 \; dN(s)
$$

Maximizing the right side of this expression gives

$$
\begin{aligned}
0 = -\frac{\phi + \psi \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right)\right]}{\hat{\tau}^2} + \frac{B\sigma^2}{2} + \left(\psi \; 2 \; n\left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right) \; \left(\frac{\bar{p}}{\sigma}\right) (\hat{\tau})^{-3/2}\right) \frac{1}{\hat{\tau}} \\
- \; V''(0) \; \sigma^2 \; \left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right)^2 n\left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right) \left(\frac{\bar{p}}{\sigma}\right) (\hat{\tau})^{-3/2}
\end{aligned}
$$

where we use $n(\cdot)$ for the density of the standard normal. Using that $V''(0) = 2 \; \psi \; / \; \bar{p}^2$, this expression simplifies to

$$
0 = -\frac{\phi + \psi \; 2 \left[1 - N\left(\frac{\bar{p}}{\sigma \sqrt{\hat{\tau}}}\right)\right]}{\hat{\tau}^2} + \frac{B\sigma^2}{2}
$$

rearranging and using the definition of $\hat{\varphi}$ gives $\sigma^2 \hat{\tau} = h(\hat{\varphi})$ of equation (23).

Third, we obtain an expression for $V''(0)$. Differentiating the value function twice, and evaluating it at $\tilde{p} = 0$ we get

$$
V''(0) = 2 \; B \; \frac{1 - e^{-\rho \hat{\tau}}}{\rho} + 2 \; \frac{e^{-\rho \hat{\tau}}}{\sigma \sqrt{\hat{\tau}}} \int_0^{\bar{p}} V'(z) \; z \; \frac{e^{-\frac{1}{2} \frac{z^2}{\sigma^2 \hat{\tau}}}}{\sigma \sqrt{\hat{\tau} \; 2 \; \boldsymbol{\pi}}} \; dz
$$

With a change in variable $s = z/(\sigma\sqrt{\hat{\tau}})$ we have:

$$V''(0) = 2\ B\ \frac{1 - e^{-\rho\hat{\tau}}}{\rho} + 2\ e^{-\rho\hat{\tau}} \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} V'\left(\sigma\sqrt{\hat{\tau}}\ s\right)\ s\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \pi}}\ ds\ .$$

Using the third order approximation $V(\tilde{p}) = V(0) + \frac{1}{2}V''(0)\,(\tilde{p})^2$ around $\tilde{p} = 0$ we obtain:

$$V''(0) = 2\ B\ \frac{1 - e^{-\rho\hat{\tau}}}{\rho} + e^{-\rho\hat{\tau}}\ V''(0)\ 2\ \sigma\sqrt{\hat{\tau}}\ \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \pi}}\ ds$$

or collecting terms:

$$V''(0) = \frac{2\ B\ \frac{1-e^{-\rho\hat{\tau}}}{\rho}}{1 - e^{-\rho\hat{\tau}}\ 2\ \sigma\sqrt{\hat{\tau}}\ \int_0^{\frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}} s^2\ \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\ \pi}}\ ds}$$

and letting $\rho \downarrow 0$, using the definition of $\hat{\varphi}$ and N for the CDF of a standard normal:

$$V''(0) = \frac{2\ B\ \hat{\tau}}{1 -\ 2\ \sigma\sqrt{\hat{\tau}}\ \int_0^{\hat{\varphi}} s^2\ dN(s)}\ . \tag{A-14}$$

Using equation (A-13) to replace $V''(0)$ into equation (A-14), using the definition of $\hat{\varphi}$, and using $\sigma^2\hat{\tau} = h(\hat{\varphi})$ to replace $\hat{\tau}$ and $\sqrt{\hat{\tau}}$ we obtain equation (22). ■

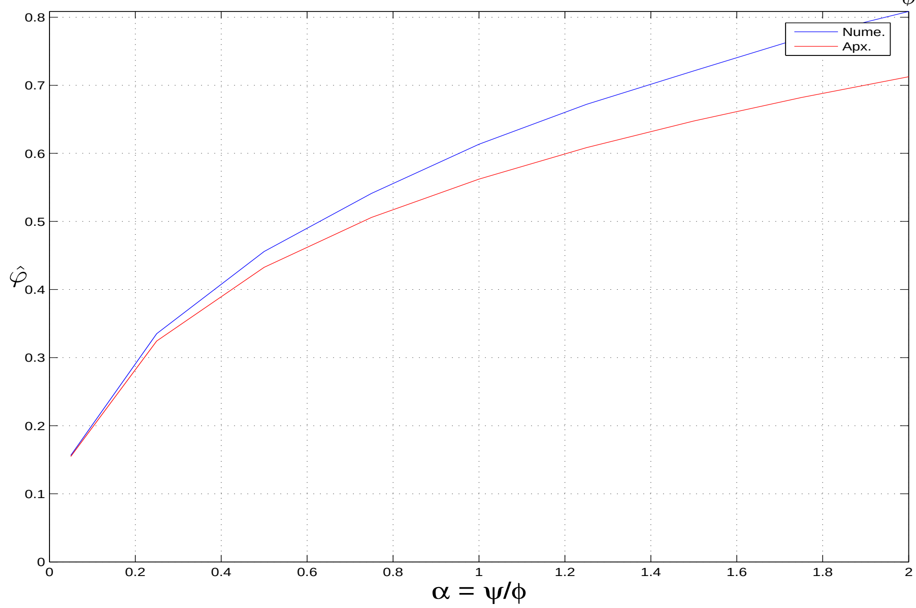
C.6 Numerical evaluation of Proposition 5.

In this section we evaluate the accuracy of the approximated solution in Proposition 5. To do so, we solve the model numerically on a grid for $\tilde{p}$ and obtain the numerical counterparts to the policy rule derived in Proposition 5. In doing so we approximate $\bar{V}(\cdot)$ through either a cubic spline or a sixth order polynomial. Results are invariant to the latter.

As Figure A-2 - Figure A-4 show, the solution for $\bar{p}$ (and as a consequence for $\hat{\varphi}$) in Proposition 5 diverges from its numerical counterpart the more, the larger the ratio $\alpha \equiv \frac{\psi}{\phi}$ is. In particular, the approximated solution tend to understate the value of $\bar{p}$ relatively to the numerical solution.

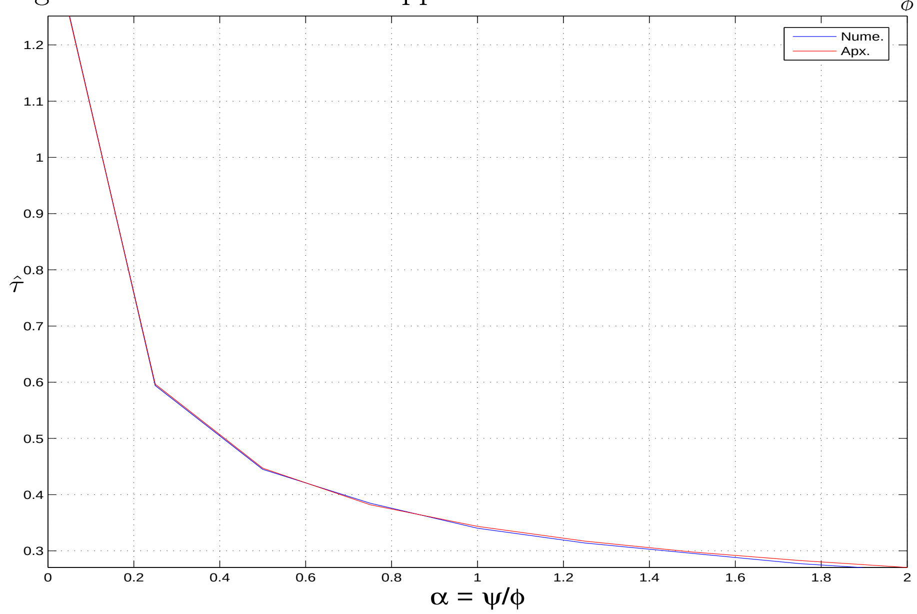
This discrepancy is due to the nature of our approximation which relies on a second order approximation of $\bar{V}(\cdot)$, while higher orders (the fourth one in particular) become more relevant as ψ/ϕ increases, causing the inaction range to widen. To document this effect, we show the following computations. We solved the model numerically, assuming a polynomial of order sixth for $\bar{V}(\cdot)$, on a grid of values for $\tilde{p}$ for values of $\alpha = 0.1$ and $\alpha = 2$. We used the symmetry property of $\bar{V}(\cdot)$ to set the value of all the odd derivatives evaluated at $\tilde{p} = 0$ equal to zero. We then compared the numerical solution for $\bar{V}(\cdot)$ with the approximated one given by Proposition 5, but having an intercept (i.e. $\bar{V}(0)$) equal to the constant term in the sixth order polynomial. As Figure A-5 shows, the quadratic approximation for $\bar{V}(\cdot)$ works better for low values of $\tilde{p}$, and more generally for low values of α. The second order approximation for $\bar{V}(\cdot)$ tends to overstate the value of the function for values away from zero, as it ignores the fourth derivative $V''''(0)$, which is negative. While the difference in the approximation will also affect the value of $\hat{V}$, we find this effect much smaller in our computations. Therefore,

Figure A-2: Numerical and approximated $\hat{\varphi}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$



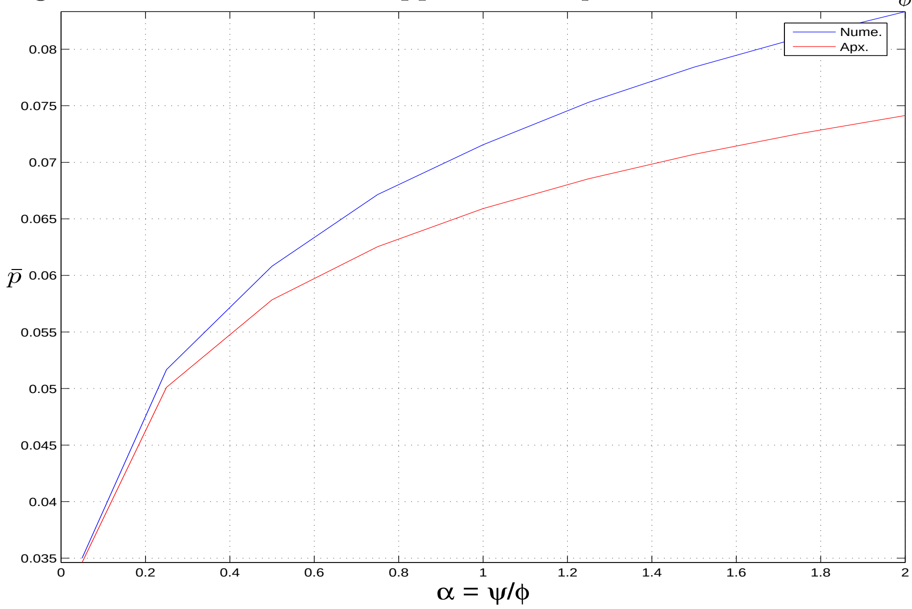
Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \psi = 0.03$. We let ϕ to vary.

Figure A-3: Numerical and approximated $\hat{\tau}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$



Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \psi = 0.03$. We let ϕ to vary.

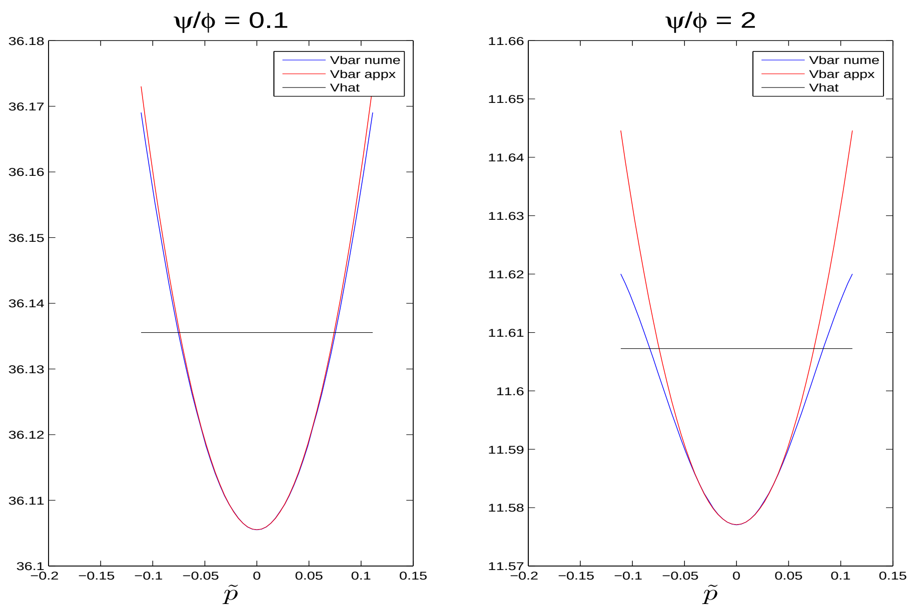
Figure A-4: Numerical and approximated $\bar{p}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$



Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \psi = 0.03$. We let ϕ to vary.

the quadratic approximation tends to understate the inaction range, i.e. to produce values of $\bar{p}$ that are smaller. This is consistent with the values displayed in Figure A-4.

Figure A-5: Numerical and approximated $\bar{V}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$



Note: parameter values are $B = 20,\ \rho = 0.02,\ \sigma = 0.2, \psi = 0.03$. We let ϕ to vary.

C.7 Proof of Proposition 6.

Proof. Begin defining

$$\hat{f}(\hat{\varphi}) = \frac{\hat{h}(\hat{\varphi})\ \hat{\varphi}^2}{1-\ 2\sqrt{h(\hat{\varphi})}\int_0^{\hat{\varphi}} s^2 dN(s)} ,$$
$$\hat{h}(\hat{\varphi}) = 2\ \ (\phi +\ 2\psi\ (1-N(\hat{\varphi})))\ \text{, so that}$$
$$\hat{f}(\hat{\varphi}) = \frac{2\ \hat{\varphi}^2\ \ (\phi +\ 2\psi\ (1-N(\hat{\varphi})))}{1-\ 2\left[2\sigma^2\frac{\phi}{B}+\ 4\sigma^2\frac{\psi}{B}\ (1-N(\hat{\varphi}))\right]^{1/4}\int_0^{\hat{\varphi}} s^2 dN(s)} ,$$

and noting that $\psi = \hat{f}(\varphi)$ is the same as the solution of equation (22) and equation (23).

First we turn to the existence and uniqueness of the solution. We show that it follows from an appliction of the intermediate function theorem, toghether with monotonicity. We show that if $\frac{\phi}{\psi} > 1/2 - 2(1-N(1)) \approx 0.1827$ then: there is a value $0 < \hat{\varphi}' \leq 1$ so that: i) the function $\hat{f}$ is continuous and increasing in $\hat{\varphi} \in [0, \hat{\varphi}')$, ii) $\hat{f}(0) = 0$, iii) $\hat{f}(\hat{\varphi}') > \psi$, iv) $\hat{f}(\hat{\varphi}) < 0$ for $\hat{\varphi} \in (\hat{\varphi}', 1]$.

The value of $\hat{\varphi}'$ is given by the minimum of 1 or the solution to

$$1 =\ 2\left[2\sigma^2\frac{\phi}{B}+\ 4\sigma^2\frac{\psi}{B}\ (1-N(\hat{\varphi}'))\right]^{1/4}\int_0^{\hat{\varphi}'} s^2 dN(s)\ , \tag{A-15}$$

so that if $\hat{\varphi}' < 1$, the function $\hat{f}$ as a discontinuity going from being positive and tending to $+\infty$ to being negative and tending to $-\infty$.

The rest of the proof fills in the details: Step (1): Show that $\hat{h}(\hat{\varphi})^2 \cdot (\hat{\varphi})^2$ is increasing in $\hat{\varphi}$ if $\phi/ > \psi 0.1667$ for $\hat{\varphi} < 1$. Step (2): Show that $\sqrt{h(\hat{\varphi})} \cdot \int_0^{\hat{\varphi}} s^2 dN(s)$ is increasing in $\hat{\varphi}$ if $\hat{\varphi} < 1$. Step (3): Using (1) and (2) the function $\hat{f}$ is increasing in $\hat{\varphi}$ for values of φ that are smaller than 1, provided that its denominator is positive.

Step (1) follows from totally differentiating $h(\hat{\varphi})^2 \cdot (\hat{\varphi})^2$ with respect to $\hat{\varphi}$. Collecting terms we obtain that the derivative is proportional to $\phi + 2\cdot\psi(1-N(\hat{\varphi}) - \hat{\varphi}\cdot N'(\hat{\varphi}))$. Since the function $1-N(\hat{\varphi}) - \hat{\varphi}\cdot N'(\hat{\varphi})$ is positive for small values of $\hat{\varphi}$ and negative for large values, we evaluate it at its upper bound for the relevant region, obtaining: $\phi + 2\psi(1-N(1)-N'(1)) > 0$ or $\phi > \psi 2[N(1)+N'(1)-1] \approx \psi\ 0.1667$. But notice that this condition is implied by our previous restriction $\phi > \psi[1/2 - 2(1-N(1))] \approx \phi\ 0.1827$.

Step (2) follows from totally differentiating $\sqrt{h(\hat{\varphi})} \cdot \int_0^{\hat{\varphi}} s^2 dN(s)$ with respect to $\hat{\varphi}$. Collecting terms we obtain that the derivative is proportional to $\hat{\varphi}^2 - \int_0^{\hat{\varphi}} s^2 dN(s)\ \psi/2(\phi + \psi\ (1 - N(\hat{\varphi})))$. This expression is greater than $\hat{\varphi}^2 - \int_0^{\hat{\varphi}} s^2 dN(s)\ /2((1-N(\hat{\varphi})))$, which is obtained by setting ϕ to zero. This integral is positive for the values of $\hat{\varphi}$ in $(0,1)$.

Now we turn to the comparative statics results. That $\hat{\varphi}^*$ is decreasing in ϕ follows since $\hat{f}$ is increasing in ϕ. That $\sigma^2\hat{\tau}^*$ is increasing it follows from the previous result and inspection of h. That $\hat{\varphi}^*$ is decreasing in σ^2/B follows since $\hat{f}$ is increasing in σ^2/B. That $\sigma^2\hat{\tau}^*$ is increasing it follows from the previous result and inspection of h. That $\partial\hat{\varphi}^*/\partial\frac{\sigma^2}{B} = 0$ at $\sigma^2/B = 0$ follows from differentiating $\hat{f}$ with respect to σ^2/B and verifying that that derivative is zero when evaluated at at $\sigma^2/B = 0$. That $\hat{\varphi}^*$ is strictly increasing in ψ when σ^2/B is small relative to

ϕ it follows from differenting $\hat{f}/\psi$ with respect to ψ. That derivative is strictly negative and continuous on the parameters, when evaluated at $\phi > 0$ and $\sigma^2/B = 0$. ∎

C.8 Proof of Proposition 7.

Proof. We rewrite the solution of $\bar{p}$ and $\hat{\tau}$ using the solution of equation (24) into equation (22) and equation (23). The approximation in equation (24) is based on the zero derivative found in Proposition 6 in item 3. We now characterize the elasticity of $\hat{\varphi}_0$. First we write the equation defining $\hat{\varphi}_0$ as

$$\tilde{\alpha} = 2\tilde{\varphi}(\tilde{\alpha}) + \log\left(2 + e^{\tilde{\alpha}} 4N\left(e^{\tilde{\varphi}(\tilde{\alpha})}\right)\right) \tag{A-16}$$

where we let $\tilde{\alpha} = \log \alpha$ and $\tilde{\varphi} = \log \hat{\varphi}_0$. Differentiating $\tilde{\varphi}$ this expression with respect to $\tilde{\alpha}$ and collecting terms we obtain:

$$\frac{\partial \tilde{\varphi}}{\partial \tilde{\alpha}} \equiv \frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} = \frac{1 - \frac{4\alpha(1-N(\hat{\varphi}_0))}{2+4\alpha(1-N(\hat{\varphi}_0))}}{2 - \frac{4\alpha(1-N(\hat{\varphi}_0))}{2+4\alpha(1-N(\hat{\varphi}_0))} \frac{n(\hat{\varphi}_0)\hat{\varphi}_0}{(1-N(\hat{\varphi}_0))}} \tag{A-17}$$

Since, $\hat{\varphi}_0 \to 0$ as $\alpha \to 0$, then $\frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} \to 0$. For values of $\alpha > 0$, we have that

$$\frac{\partial \log \hat{\varphi}_0}{\partial \log \alpha} < \frac{1}{2} \iff \frac{n(\hat{\varphi}_0)\hat{\varphi}_0}{(1 - N(\hat{\varphi}_0))} < 2, \tag{A-18}$$

which is a property of the normal distribution for values of $\hat{\varphi}_0 < 1$. Finally, the first inequality follows because $2\ \hat{\varphi}_0/\alpha = [1 - \hat{\varphi}_0^2 4(1 - N(\hat{\varphi}_0))] < 1$. The second inequality follows becasue $\hat{\varphi}_0 < 1$. ∎

C.9 Proof of Proposition 8.

Proof. The expression is based on a second order expansion of τ around $\tilde{p} = 0$. The first order condition for τ can be written as:

$$F(\tau; \tilde{p}) \equiv e^{-\rho\tau}\left(B\left(\tilde{p}^2 + \sigma^2\tau\right) - \rho \int_{-\infty}^{\infty} V\left(\tilde{p} - s\sigma\sqrt{\tau}\right) dN(s) + \int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right) \frac{-s\ \sigma}{2\ \sqrt{\tau}}\, dN(s)\right) \ .$$

At a minimum $F(\tau(\tilde{p})\ ;\ \tilde{p}) = 0$ and $F_\tau(\tau(\tilde{p})\ ;\ \tilde{p}) \geq 0$. We have $\left.\frac{\partial \tau(\tilde{p})}{\partial \tilde{p}}\right|_{\tilde{p}=0} = -\frac{F_{\tilde{p}}}{F_\tau} = 0$. That $\partial\tau/\partial\tilde{p} = 0$ follows from the symmetry of $\tau(\cdot)$ around $\tilde{p}$, which is verified directly by checking that $F_{\tilde{p}} = 0$ (see below). Totally differentiating $F_\tau \tau' + F_{\tilde{p}}$ we obtain:

$$0 = F_{\tau\tau}(\tau')^2 + F_{\tau\tilde{p}}\tau' + F_\tau \tau'' + F_{\tilde{p}\tau}\tau' + F_{\tilde{p}\tilde{p}}\ ,$$

using that $\tau' = 0$ we get the second derivative:

$$\left.\frac{\partial^2 \tau(\tilde{p})}{(\partial \tilde{p})^2}\right|_{\tilde{p}=0} = -\frac{F_{\tilde{p}\tilde{p}}}{F_\tau} = 0$$

To compute this second derivative we first compute:

$$F_\tau(\tau;\tilde{p}) = -\rho F(\tau;\tilde{p}) + e^{-\rho\tau}\left(B\sigma^2 - \rho\int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\frac{-s\sigma}{2\sqrt{\tau}}dN(s)\right.$$
$$\left. - \int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma\tau^{-3/2}}{4}\ dN(s) + \int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s)\right)$$

Taking $\rho \downarrow 0$, using that at the optimum $F = 0$, that in the approximation $\bar{V}'(\tilde{p}) = V''(0)\ \tilde{p}$ and that $\bar{V}''(\tilde{p}) = V''(0)$ we obtain:

$$\begin{aligned} F_\tau(\tau;0) &= B\ \sigma^2 - \int_{-\infty}^{\infty} V'\left(-s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma\ \tau^{-3/2}}{4}\ dN(s) + \int_{-\infty}^{\infty} V''\left(-s\sigma\sqrt{\tau}\right)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s) \\ &= B\ \sigma^2 - \int_{-\infty}^{\infty} V''(0)\ \frac{s^2\sigma^2}{4\ \tau}\ dN(s) + \int_{-\infty}^{\infty} V''(0)\frac{s^2\sigma^2}{4\ \tau}\ dN(s) \\ &= B\ \sigma^2\ . \end{aligned} \tag{A-19}$$

We also have:

$$F_{\tilde{p}}(\tau;\tilde{p}) = e^{-\rho\tau}\left(2B\tilde{p} - \rho\int_{-\infty}^{\infty} V'\left(\tilde{p} - s\sigma\sqrt{\tau}\right)dN(s) + \int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma}{2\ \sqrt{\tau}}\ dN(s)\right)$$
$$F_{\tilde{p}\tilde{p}}(\tau;\tilde{p}) = e^{-\rho\tau}\left(2B - \rho\int_{-\infty}^{\infty} V''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)dN(s) + \int_{-\infty}^{\infty} V'''\left(\tilde{p} - s\sigma\sqrt{\tau}\right)\ \frac{-s\ \sigma}{2\ \sqrt{\tau}}\ dN(s)\right)$$

Evaluating $F_{\tilde{p}\tilde{p}}$ at $\tilde{p} = 0$ for $\rho \downarrow 0$ and the approximation with $V'''(0) = 0$ gives:

$$F_{\tilde{p}\tilde{p}}(\tau;0) = 2B\ . \tag{A-20}$$

Expanding $\tau(\cdot)$ around $\tilde{p} = 0$, using that its first derivative is zero, and that the second derivative is the negative of the ratio of the expressions in equation (A-19) and equation (A-20) we obtain:

$$\tau\left(\tilde{p}\right) = \tau\left(0\right) + \tau'\left(0\right)\left(\tilde{p}\right) + \frac{1}{2}\tau''\left(0\right)\left(\tilde{p}\right)^2 = \hat{\tau} - \frac{1}{2}\frac{F_{\tilde{p}\tilde{p}}}{F_\tau}\ \left(\tilde{p}\right)^2 = \hat{\tau} - \left(\frac{\tilde{p}}{\sigma}\right)^2\ .$$

which appears in the proposition. ■

C.10 Proof of Proposition 9.

Proof. We start by computing the second derivative of $\mathcal{T}_a(p)$ at $p = 0$ (for notation simplicity we use p in the place of $\tilde{p}$). To further simplify notation rename the extreme of integration as:

$$s_1 \equiv \frac{p - \bar{p}}{\sigma\sqrt{\tau(p)}} \quad , \quad s_2 \equiv \frac{p + \bar{p}}{\sigma\sqrt{\tau(p)}}$$

which depend on p with derivatives

$$\frac{\partial s_1}{\partial p} = \frac{\sqrt{\tau(p)} - (p-\bar{p})\frac{\tau'(p)}{2\sqrt{\tau(p)}}}{\sigma\tau(p)} \quad , \quad \frac{\partial s_2}{\partial p} = \frac{\sqrt{\tau(p)} - (p+\bar{p})\frac{\tau'(p)}{2\sqrt{\tau(p)}}}{\sigma\tau(p)} \; .$$

The first order derivative is:

$$\begin{aligned} \mathcal{T}_a'(p) = \tau'(p) + \int_{s_1}^{s_2} \mathcal{T}_a'\left(p - s\sigma\sqrt{\tau(p)}\,\right)\left(1 - \frac{\sigma\tau'(p)}{2\sqrt{\tau(p)}}\; s\right) dN(s) \\ - \mathcal{T}_a(\bar{p})\; n\left(s_1\right) \frac{\partial s_1}{\partial p} + \mathcal{T}_a(-\bar{p})\; n\left(s_2\right) \frac{\partial s_2}{\partial p} \end{aligned}$$

where $n(s)$ denotes the density of the standard normal. The second order derivative is:

$$\begin{aligned} \mathcal{T}_a''(p) = \tau''(p) &+ \int_{s_1}^{s_2} \mathcal{T}_a''\left(p - s\sigma\sqrt{\tau(p)}\right)\left(1 - \frac{\sigma\tau'(p)}{2\;\sqrt{\tau(p)}}\; s\right)^2 dN(s) \\ &+ \int_{s_1}^{s_2} \mathcal{T}_a'\left(p - s\sigma\sqrt{\tau(p)}\right) \frac{s\sigma}{2\tau(p)} \left(-\tau''(p)\sqrt{\tau(p)} + \frac{\left(\tau'(p)\right)^2}{2\sqrt{\tau(p)}}\right) dN(s) \\ &- \mathcal{T}_a'(\bar{p})\left(1 - \frac{\sigma\tau'(p)}{2\sqrt{\tau(p)}}\; s_1\right)\; n(s_1)\; \frac{\partial s_1}{\partial p} + \mathcal{T}_a'(-\bar{p})\left(1 - \frac{\sigma\tau'(p)}{2\sqrt{\tau(p)}}\; s_2\right)\; n(s_2)\; \frac{\partial s_2}{\partial p} \\ &- \mathcal{T}_a(\bar{p})\left(n'(s_1)\;\left(\frac{\partial s_1}{\partial p}\right)^2 + n(s_1)\; \frac{\partial^2 s_1}{(\partial p)^2}\right) + \mathcal{T}_a(-\bar{p})\left(n'(s_2)\;\left(\frac{\partial s_2}{\partial p}\right)^2 + n(s_2)\; \frac{\partial^2 s_2}{(\partial p)^2}\right) \; . \end{aligned}$$

To evaluate this expression note that at $p = 0$ we have $\tau'(0) = 0$, $\tau(0) = \hat{\tau}$, $-s_1 = s_2 = \hat{\varphi}$, $\frac{\partial s_1}{\partial p} = \frac{\partial s_2}{\partial p} = \frac{1}{\sigma\sqrt{\hat{\tau}}}$ (recall the notation already used above $\hat{\varphi} \equiv \frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}$). Hence the second to last line in the previous formula is $-2\mathcal{T}_a'(\bar{p})\; n(s_1)\frac{\partial s_1}{\partial p}$ by the symmetry of $\mathcal{T}_a(p)$. Note moreover that at $p = 0$

$$\frac{\partial^2 s_1}{(\partial p)^2} = -\frac{\partial^2 s_2}{(\partial p)^2} = \frac{\bar{p}\;\tau''(0)\;\sqrt{\hat{\tau}}}{2\sigma\;\hat{\tau}^2}$$

Thus we get:

$$\begin{aligned} \mathcal{T}_a''(0) = \tau''(0) + \int_{-\hat{\varphi}}^{\hat{\varphi}} \mathcal{T}_a''\left(-s\sigma\sqrt{\hat{\tau}}\right) dN(s) - \frac{\sigma\tau''(0)}{2\sqrt{\hat{\tau}}} \int_{-\hat{\varphi}}^{\hat{\varphi}} \mathcal{T}_a'\left(-s\sigma\sqrt{\hat{\tau}}\right)\; s\; dN(s) \\ - 2\mathcal{T}_a'(\bar{p})\; \frac{n\left(\hat{\varphi}\right)}{\sigma\sqrt{\hat{\tau}}} - 2\mathcal{T}_a(\bar{p})\left(-\frac{n'(\hat{\varphi})}{\sigma^2\;\hat{\tau}} + n(\hat{\varphi})\frac{\bar{p}\;\tau''(0)\;\sqrt{\hat{\tau}}}{2\sigma\;\hat{\tau}^2}\right) \end{aligned} \tag{A-21}$$

Using that $\tau''(0) = -2/\sigma^2$ (from Proposition 8), the last term in the previous equation can be rewritten as $-2\frac{\mathcal{T}_a(\bar{p})}{\sigma^2\hat{\tau}}\left(-n'(\hat{\varphi}) - \hat{\varphi}\; n(\hat{\varphi})\right)$, which is zero since $n'(x) + x\; n(x) = 0$ for a standard normal density.

Given that $\mathcal{T}_a'(0) = \mathcal{T}_a'''(0) = 0$, and $\mathcal{T}_a''(0) < 0$, we approximate $\mathcal{T}_a(p)$ with a quadratic function on the interval $[-\bar{p}\ ,\ \bar{p}]$:

$$\mathcal{T}_a(p) = \mathcal{T}_a(0) + \frac{1}{2}\mathcal{T}_a''(0)\,(p)^2\,.$$

Using the first and the second derivative of this quadratic approximation into the right hand side of equation (A-21), and $\tau''(0) = -2/\sigma^2$, gives:

$$\mathcal{T}_a''(0) = -2/\sigma^2 + \mathcal{T}_a''(0)\,(2\ N(\hat{\varphi}) - 1) - 2\mathcal{T}_a''(0)\int_0^{\hat{\varphi}} s^2 dN(s) - 2\ \mathcal{T}_a''(0)\ \hat{\varphi}\ n(\hat{\varphi})$$

or

$$\mathcal{T}_a''(0) = \frac{-1/\sigma^2}{1 - N(\hat{\varphi}) + \int_0^{\hat{\varphi}} s^2 dN(s) + \hat{\varphi} n(\hat{\varphi})}\quad . \tag{A-22}$$

To solve for $\mathcal{T}_a(0)$ let us evaluate $\mathcal{T}_a(p)$ at $\bar{p}$ obtaining:

$$\mathcal{T}_a(\bar{p}) = \tau(\bar{p}) + \int_0^{2\bar{\varphi}} \mathcal{T}_a\left(\bar{p} - s\sigma\sqrt{\tau(\bar{p})}\right) dN(s) \qquad \text{, where} \quad \bar{\varphi} \equiv \frac{\bar{p}}{\sigma\sqrt{\tau(\bar{p})}} = \frac{\hat{\varphi}}{\sqrt{1-\hat{\varphi}^2}}\ .$$

Using the quadratic approximation for $\mathcal{T}_a$ in the previous equation we get

$$\begin{aligned}
\mathcal{T}_a(\bar{p}) &= \tau(\bar{p}) + \int_0^{2\bar{\varphi}} \left(\mathcal{T}_a(0) + \frac{1}{2}\mathcal{T}_a''(0)\left(\bar{p} - s\sigma\sqrt{\tau(\bar{p})}\right)^2\right) dN(s) \\
&= \tau(\bar{p}) + \left(N\left(2\bar{\varphi}\right) - \frac{1}{2}\right)\mathcal{T}_a(0) + \frac{1}{2}\mathcal{T}_a''(0)\int_0^{2\bar{\varphi}} \left(\bar{p} - s\sigma\sqrt{\tau(\bar{p})}\right)^2 dN(s)
\end{aligned}$$

Replacing $\mathcal{T}_a(\bar{p}) = \mathcal{T}_a(0) + \frac{1}{2}\mathcal{T}_a''(0)\ \bar{p}^2$ on the left hand side, and collecting terms gives

$$\begin{aligned}
\mathcal{T}_a(0) =& \frac{\tau(\bar{p}) + \frac{1}{2}\mathcal{T}_a''(0)\left(\int_0^{2\bar{\varphi}}\left(\bar{p} - s\sigma\sqrt{\tau(\bar{p})}\right)^2 dN(s) - \bar{p}^2\right)}{1.5 - N\left(2\bar{\varphi}\right)} \\
&= \tau(\bar{p})\ \frac{1 + \frac{\sigma^2}{2}\ \bar{\varphi}^2\ \mathcal{T}_a''(0)\left(\int_0^{2\bar{\varphi}}\left(1 - \frac{s}{\bar{\varphi}}\right)^2 dN(s) - 1\right)}{1.5 - N\left(2\bar{\varphi}\right)} \\
&= \hat{\tau}\left(1 - \hat{\varphi}^2\right)\ \frac{1 + \frac{\sigma^2}{2}\ \bar{\varphi}^2\ \mathcal{T}_a''(0)\left(\int_0^{2\bar{\varphi}}\left(1 - \frac{s}{\bar{\varphi}}\right)^2 dN(s) - 1\right)}{1.5 - N\left(2\bar{\varphi}\right)}
\end{aligned} \tag{A-23}$$

where the last line uses the equality $\tau(\bar{p}) = \hat{\tau} - \left(\frac{\bar{p}}{\sigma}\right)^2 = \hat{\tau}\,(1 - \hat{\varphi}^2)$. Substituting equation (A-22) into equation (A-23) gives

$$\mathcal{T}_a(0) = \hat{\tau}\frac{(1-\hat{\varphi}^2)}{1.5 - N\left(2\bar{\varphi}\right)}\ \left(1 - \frac{1/2\ \left(\int_0^{2\bar{\varphi}}\left(\bar{\varphi} - s\right)^2 dN(s) - \bar{\varphi}^2\right)}{1 - N(\hat{\varphi}) + \int_0^{\hat{\varphi}} s^2 dN(s) + \hat{\varphi} n(\hat{\varphi})}\ \right) \tag{A-24}$$

which gives the approximation for the expression $\mathcal{T}_a(0) = \hat{\tau} \cdot \mathcal{A}(\hat{\varphi})$ in the proposition. A numerical study of the function $\mathcal{A}(\hat{\varphi})$ shows that $\mathcal{A}(0) = 1$, and that the function approximation is accurate and increasing for $\hat{\varphi} \in (0, 0.75)$, that $\mathcal{A}(0.75) \cong 1.78$ and decreasing thereafter.

Next we show that, given equation (30), the average frequency of price adjustment can be written as $n_a = 1/\mathcal{T}_a(0)$ where $\mathcal{T}_a(0) = \hat{\tau}\tilde{\mathcal{A}}(\hat{\varphi})$. Rewrite equation (31) in terms of $\varphi(p) = \frac{p}{\sigma\sqrt{\tau(p)}}$,

$$\mathcal{T}(\tilde{p}) = \tilde{\mathcal{T}}(\tilde{\varphi}) = \tau(\tilde{\varphi}) + \int_{-\bar{\varphi}}^{\bar{\varphi}} \tilde{\mathcal{T}}(\varphi) n \left(\varphi \frac{\sqrt{\tau(\varphi)}}{\sqrt{\tau(\tilde{\varphi})}} - \tilde{\varphi} \right) \frac{\sqrt{\tau(\varphi)}}{\sqrt{\tau(\tilde{\varphi})}} d\varphi, \tag{A-25}$$

$$= \frac{\hat{\tau}}{1+\tilde{\varphi}^2} + \int_{-\bar{\varphi}}^{\bar{\varphi}} \tilde{\mathcal{T}}(\varphi) n \left(\varphi \frac{\sqrt{1+\tilde{\varphi}^2}}{\sqrt{1+\varphi^2}} - \tilde{\varphi} \right) \frac{\sqrt{1+\tilde{\varphi}^2}}{\sqrt{1+\varphi^2}} d\varphi, \tag{A-26}$$

where the first equality follows from strict monotonicity of $\varphi(p)$. Then we can write $\mathcal{T}_a(0) = \tilde{\mathcal{T}}(0) = \hat{\tau}\tilde{\mathcal{A}}(0, \hat{\varphi}) \equiv \mathcal{A}(\hat{\varphi})$, where

$$\tilde{\mathcal{A}}(\tilde{\varphi}, \hat{\varphi}) = \frac{1}{1+\tilde{\varphi}^2} + \int_{-\bar{\varphi}}^{\bar{\varphi}} \tilde{\mathcal{A}}(\varphi, \hat{\varphi}) n \left(\varphi \frac{\sqrt{1+\tilde{\varphi}^2}}{\sqrt{1+\varphi^2}} - \tilde{\varphi} \right) \frac{\sqrt{1+\tilde{\varphi}^2}}{\sqrt{1+\varphi^2}} d\varphi. \tag{A-27}$$

We use a grid of values for $\tilde{\varphi}$ to solve recursively for $\tilde{\mathcal{A}}(\tilde{\varphi}, \hat{\varphi})$. ■

C.11 Proof of Proposition 10.

Proof. Let $q(\varphi)$ and $Q(\varphi)$ be respectively the density and CDF of

$$\varphi(p) \equiv \frac{p}{\sigma\sqrt{\tau(p)}} .$$

Notice that $\varphi(p)$ is a monotonic transformation of p, $\frac{d\varphi}{dp} = \frac{1}{\sigma\sqrt{\tau(p)}} - \varphi \frac{\tau'(p)}{2\tau(p)} > 0$. Notice that using $p(\varphi)$ to denote the inverse function, we compute

$$\tau(p) = \hat{\tau} - \left(\frac{p}{\sigma}\right)^2 = \frac{\hat{\tau}}{1 + (\varphi(p))^2}$$

which, abusing notation, defines the new function

$$\tau(\varphi) = \frac{\hat{\tau}}{1+\varphi^2} \tag{A-28}$$

The monotonicity of the transformation also gives that $Q(\varphi(p)) = G(p)$ at all p, implying

$$g(p) \frac{dp}{d\varphi} d\varphi = q(\varphi(p)) d\varphi \tag{A-29}$$

Using equation (A-29) and the change of variables from p to φ in equation (33) we write:

$$g(\tilde{p}) = \int_{-\bar{\varphi}}^{\bar{\varphi}} g(p(\varphi))\ n\left(\frac{\tilde{p} - p(\varphi)}{\sigma\sqrt{\tau(p(\varphi))}}\right) \frac{1}{\sigma\sqrt{\tau(p(\varphi))}} \frac{dp}{d\varphi}\ d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} g(p(\varphi)) \frac{dp}{d\varphi} d\varphi\right]\ n\left(\frac{\tilde{p}}{\sigma\sqrt{\hat{\tau}}}\right) \frac{1}{\sigma\sqrt{\hat{\tau}}}$$
$$= \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi)\ n\left(\frac{\tilde{p}}{\sigma\sqrt{\tau(p(\varphi))}} - \varphi\right) \frac{1}{\sigma\sqrt{\tau(p(\varphi))}} d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n\left(\frac{\tilde{p}}{\sigma\sqrt{\hat{\tau}}}\right) \frac{1}{\sigma\sqrt{\hat{\tau}}}.$$

For clarity, rewrite the previous equation using the density of $\tilde{p}$ conditional on φ:

$$f(\tilde{p}|\varphi) \equiv n\left(\frac{\tilde{p}}{\sigma\sqrt{\tau(p(\varphi))}} - \varphi\right) \frac{1}{\sigma\sqrt{\tau(p(\varphi))}}$$

This gives:

$$g(\tilde{p}) = \int_{-\bar{\varphi}}^{\bar{\varphi}} f(\tilde{p}|\varphi)\ q(\varphi) d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right]\ f(\tilde{p}|0). \tag{A-30}$$

Now consider the following monotone transformation of the random variable $\tilde{\varphi}$: $\Phi(\tilde{\varphi}, \varphi) \equiv \tilde{\varphi}\frac{\sqrt{\tau(\tilde{\varphi})}}{\sqrt{\tau(\varphi)}}$, where the function $\tau(\varphi)$ is given in equation (A-28).[39] Using the definition of φ and the law of motion for $\tilde{p}$ it follows that $\Phi(\tilde{\varphi}, \varphi) - \varphi$ is a random variable with the standard normal distribution: $n\left(\Phi(\tilde{\varphi}, \varphi) - \varphi\right)$.

By doing the change in variables from $\tilde{p}$ to $\tilde{\varphi}$ on the left-hand side of equation (A-30), and from $\tilde{p}$ to $\Phi(\tilde{\varphi}, \varphi)$ on the right-hand side of equation (A-30), we obtain

$$q(\tilde{\varphi}) \frac{d\tilde{\varphi}}{d\tilde{p}} = \int_{-\bar{\varphi}}^{\bar{\varphi}} n\left(\Phi(\tilde{\varphi}, \varphi) - \varphi\right) \frac{d\Phi(\tilde{\varphi}, \varphi)}{d\tilde{\varphi}} \frac{d\tilde{\varphi}}{d\tilde{p}}\ q(\varphi)\ d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n\left(\tilde{\varphi}\right) \frac{d\Phi(\tilde{\varphi}, \tilde{\varphi})}{d\tilde{\varphi}} \frac{d\tilde{\varphi}}{d\tilde{p}},$$
$$q(\tilde{\varphi}) = \int_{-\bar{\varphi}}^{\bar{\varphi}} n\left(\Phi(\tilde{\varphi}, \varphi) - \varphi\right) \frac{d\Phi(\tilde{\varphi}, \varphi)}{d\tilde{\varphi}}\ q(\varphi)\ d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n\left(\tilde{\varphi}\right) \quad .$$

We notice that $q(\cdot)$ attains its maximum at $\tilde{\varphi} = 0$, and that it is symmetric, so that $q'(0) = q'''(0) = 0$ and $q''(0) < 0$. Furthermore, $q(\bar{\varphi}) > 0$. Then, for small $\bar{\varphi} = \frac{\bar{p}}{\sigma\sqrt{\hat{\tau}}}$, this function can be approximated by a quadratic function with

$$q(\varphi) = q(0) + \frac{1}{2} q''(0)\ \ \varphi^2\ .$$

The value of $q(\cdot)$ and its first and second derivatives with respect to $\tilde{\varphi}$, evaluated at $\tilde{\varphi} = 0$,

[39] The monotonicity holds since Φ is increasing in $\tilde{\varphi}$.

are given by

$$
\begin{aligned}
q(0) &= \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n\left(-\varphi\right) \frac{d\Phi(0,\varphi)}{d\tilde{\varphi}} d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n(0), \\
q'(0) &= \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n'\left(-\varphi\right) \left(\frac{d\Phi(0,\varphi)}{d\tilde{\varphi}}\right)^2 d\varphi + \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n\left(-\varphi\right) \frac{d^2\Phi(0,\varphi)}{(d\tilde{\varphi})^2} d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n'(0), \\
q''(0) &= \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n''\left(-\varphi\right) \left(\frac{d\Phi(0,\varphi)}{d\tilde{\varphi}}\right)^3 d\varphi + \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n'\left(-\varphi\right) 3 \frac{d\Phi(0,\varphi)}{d\tilde{\varphi}} \frac{d^2\Phi(0,\varphi)}{(d\tilde{\varphi})^2} d\varphi \\
&\quad + \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) n\left(-\varphi\right) \frac{d^3\Phi(0,\varphi)}{(d\tilde{\varphi})^3} d\varphi + \left[1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi\right] n''(0).
\end{aligned}
$$

Using equation (A-28) gives

$$
\frac{d\Phi(0,\varphi)}{d\tilde{\varphi}} = \sqrt{1+\varphi^2} \quad , \quad \frac{d^2\Phi(0,\varphi)}{(d\tilde{\varphi})^2} = 0 \quad , \quad \frac{d^3\Phi(0,\varphi)}{(d\tilde{\varphi})^3} = -3\sqrt{1+\varphi^2}.
$$

Using that $n''(-\varphi) = n(-\varphi)(\varphi^2 - 1)$, we rewrite $q(0)$ and $q''(0)$ as

$$
\begin{aligned}
q(0) &= 2\int_0^{\bar{\varphi}} q(\varphi) n(\varphi) \sqrt{1+\varphi^2} d\varphi + \left(1 - 2\int_0^{\bar{\varphi}} q(\varphi) d\varphi\right) n(0), \\
q''(0) &= 2\int_0^{\bar{\varphi}} q(\varphi) n(\varphi) \sqrt{1+\varphi^2} (\varphi^4 - 4) d\varphi - \left(1 - 2\int_0^{\bar{\varphi}} q(\varphi) d\varphi\right) n(0).
\end{aligned}
$$

These two equations and the quadratic approximation for $q(\cdot)$ give a system of 2 equations in 2 unknowns: $q(0)$ and $q''(0)$:

$$
\begin{aligned}
q(0) + q''(0) &= 2\int_0^{\bar{\varphi}} \left(q(0) + \frac{1}{2} q''(0)\ \varphi^2\right) n(\varphi) \sqrt{1+\varphi^2} (\varphi^4 - 3) d\varphi \\
q(0) &= 2\int_0^{\bar{\varphi}} \left(q(0) + \frac{1}{2} q''(0)\ \varphi^2\right) n(\varphi) \sqrt{1+\varphi^2} d\varphi + \left(1 - 2\int_0^{\bar{\varphi}} \left(q(0) + \frac{1}{2} q''(0)\ \varphi^2\right) d\varphi\right) n(0) \quad .
\end{aligned}
$$

The equations above imply

$$
\begin{aligned}
\frac{q''(0)}{q(0)} &= -\frac{1 - 2\int_0^{\bar{\varphi}}\ n\left(\varphi\right) \sqrt{1+\varphi^2} (\varphi^4 - 3) d\varphi}{1 - \int_0^{\bar{\varphi}}\ n\left(\varphi\right) \varphi^2 \sqrt{1+\varphi^2} (\varphi^4 - 3) d\varphi} \equiv \theta(\bar{\varphi}) \\
q(0) &= \frac{n(0)}{1 + 2\int_0^{\bar{\varphi}} \left(n(0) - n(\varphi)\sqrt{1+\varphi^2}\right) d\varphi + \theta(\bar{\varphi}) \int_0^{\bar{\varphi}} \left(n(0) - n(\varphi)\sqrt{1+\varphi^2}\right) \varphi^2 d\varphi} \quad .
\end{aligned}
$$

Using these results into equation (34) to obtain

$$\mathcal{R}(\hat{\varphi}) = 2\int_0^{\bar{\varphi}} \tau(\varphi)q(\varphi)d\varphi + \left[1 - 2\int_0^{\bar{\varphi}} q(\varphi)d\varphi\right]\hat{\tau} \tag{A-31}$$

$$\approx \hat{\tau} - 2\hat{\tau}\left[-\int_0^{\bar{\varphi}} (1+\varphi^2)^{-1}\left(q(0) + \frac{1}{2}q''(0)\varphi^2\right)d\varphi + \int_0^{\bar{\varphi}}\left(q(0) + \frac{1}{2}q''(0)\varphi^2\right)d\varphi\right] \tag{A-32}$$

$$\approx \hat{\tau} - 2\hat{\tau}\left[\int_0^{\bar{\varphi}}\left(q(0) + \frac{1}{2}q''(0)\varphi^2\right)\frac{\varphi^2}{(1+\varphi^2)}d\varphi\right] \tag{A-33}$$

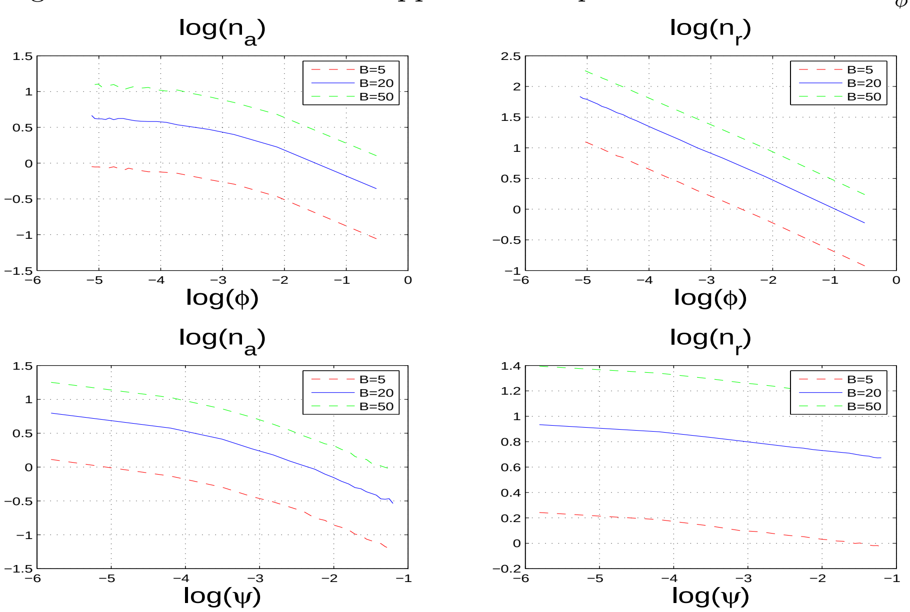
where we use the quadratic approximation of $q(\cdot)$, the definition of $\varphi(p)$, and the quadratic approximation of $\tau(p)$.

Notice that, given equation (30), the average frequency of price review can be always written as $n_r = 1/\mathcal{T}_r(0)$ where $\mathcal{T}_r(0) = \hat{\tau}\mathcal{R}(\hat{\varphi})$. This result follows directly from substituting equation (30) into equation (A-32). ■

C.12 Proof of Proposition 11.

In this section we report the numerical solution of the model to the following experiments: (i) a change in ψ holding ϕ fixed; (ii) a change in ϕ holding ψ fixed. These experiments are meant to capture the elasticities of n_a and n_r with respect to ψ and ϕ.

Figure A-6: Numerical and approximated $\bar{p}$ as a function of $\alpha \equiv \frac{\psi}{\phi}$



Note: fixed parameter values are $\rho = 0.02$, $\sigma = 0.2$; $\psi = 0.03$ when ϕ is allowed to change; $\phi = 0.06$ when ψ is allowed to change.

We show results for different parameterizations of $B = 5, 20, 50$. The first row in Figure A-6 display results for $\log(n_a), \log(n_r)$ to changes in $\log(\phi)$ holding $\psi = 0.03$ as in our benchmark calibration. The larger ϕ, the closer the value of α to zero. As we can see, the elasticity of n_r with respect to ϕ is roughly equal to - 1/2, independently of the level of B and the level of α. Similarly, the elasticity of n_a with respect to ϕ is not changing much to changes in the level of B, however it is sensitive to the level of α, being roughly equal to - 1/2 for large values of ϕ, i.e. for α closer to zero, and smaller at smaller values of ϕ.

The second row in Figure A-6 display results for $\log(n_a), \log(n_r)$ to changes in $\log(\psi)$ holding $\phi = 0.06$ as in our benchmark calibration. The smaller ψ, the closer the value of α to zero. The elasticities of n_a and n_r with respect to ψ are smaller at smaller values of ψ.

C.13 Proof of Proposition 12.

Proof. Rewrite equation (40) as

$$w(\Delta p) = \frac{1}{\sigma\sqrt{\hat{\tau}}} \frac{\int_{-\bar{\varphi}}^{\bar{\varphi}} \sqrt{1+\varphi^2}\; n\left(\frac{\Delta p\; \sqrt{1+\varphi^2}}{\sigma\sqrt{\hat{\tau}}} - \varphi\right) q(\varphi) d\varphi}{1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi} + \frac{1}{\sigma\sqrt{\hat{\tau}}}\; n\left(\frac{\Delta p}{\sigma\sqrt{\hat{\tau}}}\right),$$

usign the change of variable $\varphi = \tilde{p}/(\sigma\sqrt{\tau(\tilde{p})})$, using the approximation for the optimal policty: $\tau(p) = \hat{\tau} - (p/\sigma)^2 = \frac{\hat{\tau}}{1+\varphi^2}$, which implies that $p(\varphi) = \sigma\sqrt{\hat{\tau}}\; \varphi/\sqrt{1+\varphi^2}$, and hence this change of variables gives the density of the normalized prices: $q(\varphi) = g(p(\varphi)) dp(\varphi)/d\varphi$. Thus letting the normalized price changes be:

$$x \equiv \frac{\Delta p}{\sigma\sqrt{\hat{\tau}}} \;, \tag{A-34}$$

we define the density of the normalized price changes x as $v(\cdot)$, satisfying $v(x)/(\sigma\sqrt{\hat{\tau}})$ and then the distribution of normalized price adjustment have density given by the change of variable formula: $v(x) = w\left(\frac{\Delta p}{\sigma\; \sqrt{\hat{\tau}}}\right)\; \sigma\sqrt{\hat{\tau}}$

$$v(x) \;\; = \;\; \frac{\int_{-\bar{\varphi}}^{\bar{\varphi}} \sqrt{1+\varphi^2}\; n\left(x\; \sqrt{1+\varphi^2} - \varphi\right) q(\varphi) d\varphi}{1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} q(\varphi) d\varphi} + \; n\left(x\right) \quad \text{for } |x| > \bar{\varphi} \;. \tag{A-35}$$

Finally we can use the approximation for $q(\varphi) \approx q(0) + \frac{1}{2} q''(0)\varphi^2$ and the formulas for $q(0)$ and $q(0)''$ developed in the proof of Proposition 10. The expressions obtained there for $q(0)$ and $q''(0)$ are a function of $\bar{\varphi}$. Thus we can write:

$$\begin{aligned} v(x; \bar{\varphi}) \;\; &= \;\; \frac{\int_{-\bar{\varphi}}^{\bar{\varphi}} \left[q(0) + \frac{1}{2} q''(0)\varphi^2\right] \sqrt{1+\varphi^2}\; n\left(x\; \sqrt{1+\varphi^2} - \varphi\right) d\varphi}{1 - \int_{-\bar{\varphi}}^{\bar{\varphi}} \left[q(0) + \frac{1}{2} q''(0)\varphi^2\right] d\varphi} \\ &+ \;\; n\left(x\right) \quad \text{for } |x| > \bar{\varphi} \;. \end{aligned} \tag{A-36}$$

where we include $\bar{\varphi}$ as an argument to emphasize that this density does not depend on any other parameter. This is the expression in the proof. Finally, we notice that the mode and minimum are equal to $\bar{\varphi}$. ■

C.14 Proof of Proposition 14.

Proof. Consider

$$G(\hat{p}\ ; \bar{T}, \pi) \equiv \mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\left[(\pi t - \hat{p}) + \sigma W(t)\right]^2 dt\right) \tag{A-37}$$

where $W(t)$ is a standard brownian motion. If $\hat{p}$ and the $\bar{T}$ are optimal, then $G(\cdot; \bar{T}, \pi)$ should be maximized at $\hat{p}$. We will show that, provided that the stopping time is positive and finite,

$$\frac{\partial G(0\ ; \bar{T}, \pi)}{\partial \hat{p}} < 0 \text{ if } \pi > 0\ . \tag{A-38}$$

We write equation (A-37) as

$$G(\hat{p}\ ; \bar{T}, \pi) \equiv \mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\left[(\pi t - \hat{p})^2 + 2\sigma W(t)\,(\pi t - \hat{p}) + \sigma^2 W(t)^2\right] dt\right)$$

and thus

$$\begin{aligned}
\frac{\partial G(\hat{p}\ ; \bar{T}, \pi)}{\partial \hat{p}} &= \mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t}\left[-2\,(\pi t - \hat{p}) - 2\sigma W(t)\right] dt\right) \\
&= -\mathbb{E}\left(\int_0^{\bar{T}} e^{-\rho t} 2\,(\pi t - \hat{p})\, dt\right) - \sigma\ 2\mathbb{E}\left[\int_0^{\bar{T}} e^{-\rho t} W(t) dt\right]
\end{aligned}$$

Equating this first order condition to zero, rearranging, and taking ρ to zero:

$$\hat{p} \;=\; \frac{\pi\ \mathbb{E}\left(\frac{\bar{T}^2}{2}\right) + \sigma\ \mathbb{E}\left[\int_0^{\bar{T}} W(t)dt\right]}{\mathbb{E}\left(\bar{T}\right)}$$

C.15 Proof of Proposition 15.

Proof. To show that $\frac{\partial \hat{p}}{\partial \pi}|_{\pi=0} > 0$ we note that $\hat{p} = 0$ for zero inflation, and -by equation (47) in Proposition 14- strictly positive for positive inflation and strictly negative for negative inflation. To show that $\frac{\partial \hat{\tau}}{\partial \pi}|_{\pi=0} = 0$ we let $F(\hat{\tau}, \hat{p}, \pi) = 0$ denote the first order condition of the problem equation (17) with respect to $\hat{\tau}$:

$$\begin{aligned}
F(\hat{\tau}, \hat{p}, \pi) = B\,(\hat{p} - \pi\hat{\tau})^2 - \rho \int_{-\infty}^{\infty} V\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right) dN(s) \\
+ \int_{-\infty}^{\infty} V'\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right)\left(-\pi - \frac{s\sigma}{2\sqrt{\hat{\tau}}}\right) dN(s)
\end{aligned}$$

Totally differentiating $F(\hat{\tau}(\pi), \hat{p}(\pi), \pi)$ with respect to π, we can solve for $\partial \hat{\tau}/\partial \pi$ as function of $\partial \hat{p}/\partial \pi$ and derivatives of the function F as:

$$0 = \frac{\partial F}{\partial \hat{\tau}} \frac{\partial \hat{\tau}}{\partial \pi} + \frac{\partial F}{\partial \hat{p}} \frac{\partial \hat{p}}{\partial \pi} + \frac{\partial F}{\partial \pi} \; .$$

We now show that $\partial F(\hat{\tau}, 0, 0)/\partial \hat{p} = \partial F(\hat{\tau}, 0, 0)/\partial \pi = 0$ and $\partial F(\hat{\tau}, 0, 0)/\partial \hat{\tau} > 0$, hence $\partial \hat{\tau}/\partial \pi = 0$. Direct computations give:

$$\begin{aligned}\frac{\partial F(\hat{\tau}, \hat{p}, \pi)}{\partial \pi} &= -2B\left(\hat{p} - \pi\hat{\tau}\right)\hat{\tau} + \rho\hat{\tau}\int_{-\infty}^{\infty} V'\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right) dN(s) \\ &- \hat{\tau}\int_{-\infty}^{\infty} V''\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right)\left(-\pi - \frac{s\sigma}{2\sqrt{\hat{\tau}}}\right) dN(s) - \int_{-\infty}^{\infty} V'\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right) dN(s)\end{aligned}$$

and evaluating this expression at $\pi = \hat{p} = 0$, using the symmetry of $V(\tilde{p})$ and of the normal distribution $N(s)$ around zero, we obtain that this derivative is zero. Likewise we obtain that:

$$\begin{aligned}\frac{\partial F(\hat{\tau}, \hat{p}, \pi)}{\partial \hat{p}} &= 2B\left(\hat{p} - \pi\hat{\tau}\right) - \rho\hat{\tau}\int_{-\infty}^{\infty} V'\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right) dN(s) \\ &+ \int_{-\infty}^{\infty} V''\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right)\left(-\pi - \frac{s\sigma}{2\sqrt{\hat{\tau}}}\right) dN(s)\end{aligned}$$

which evaluated at $\pi = \hat{p} = 0$ also vanishes. Finally

$$\begin{aligned}\frac{\partial F(\hat{\tau}, \hat{p}, \pi)}{\partial \hat{\tau}} &= 2B\left(\hat{p} - \pi\hat{\tau}\right)\pi - \rho\int_{-\infty}^{\infty} V'\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right)\left(-\pi - \frac{s\sigma}{2\sqrt{\hat{\tau}}}\right) dN(s) \\ &+ \int_{-\infty}^{\infty} V''\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right)\left(-\pi - \frac{s\sigma}{2\sqrt{\hat{\tau}}}\right)^2 dN(s) \\ &+ \int_{-\infty}^{\infty} V'\left(\hat{p} - \pi\hat{\tau} - s\sigma\sqrt{\hat{\tau}}\right)\left(\frac{s\sigma}{4\ \hat{\tau}^{3/2}}\right) dN(s)\end{aligned}$$

which is strictly positive evaluated at $\pi = \hat{p} = 0$ since $V''(0) > 0$ and since $V(\tilde{p})$ and $dN(s)$ is symmetric around zero.

C.16 Proof of Proposition 16.

Proof. It is mathematically simpler to solve the menu cost model as concentrating in the steady state case, i.e. when $\rho = 0$. This problem corresponds to the the limit as $\rho \downarrow 0$. Thus when $\pi > 0$ and $\sigma = 0$ we can write the objective function as:

$$
\begin{aligned}
& \min_{\hat{p},\tau} \frac{1}{\tau}\left[\int_0^\tau B\left(\pi t-\hat{p}\right)^2\ dt+(\psi+\phi)\right] = \min_{\hat{p},\tau} \frac{1}{\tau}\left[B\int_0^\tau \left(\pi^2 t^2 - 2\hat{p}\pi t+\hat{p}^2\right)\ dt+(\psi+\phi)\right] \\
= & \min_{\hat{p},\tau} \frac{1}{\tau}\left[B\left(\frac{\pi^2\tau^3}{3}-\frac{2\hat{p}\pi\tau^2}{2}+\hat{p}^2\tau\right)+(\psi+\phi)\right] = \min_{\tau} \frac{1}{\tau}\left[B\left(\frac{\pi^2\tau^3}{3}-\frac{\pi^2\tau^3}{2}+\frac{\pi^2\tau^3}{4}\right)+(\psi+\phi)\right] \\
= & \min_{\tau} \frac{1}{\tau}\left[B\frac{\pi^2\tau^3}{12}+(\psi+\phi)\right] = \min_{\tau}\left[B\frac{\pi^2\tau^2}{12}\right]+\frac{(\psi+\phi)}{\tau}
\end{aligned}
$$

where we use that $\frac{\tau\pi}{2} = \arg\min_{\hat{p}} \frac{2\hat{p}\tau^2}{2}+\hat{p}^2\tau$. The first order condition for τ gives

$$\frac{2\ B\ \pi^2\ \tau}{12} = \frac{(\psi+\phi)}{\tau^2}\ .$$

This implies the following optimal rules:

$$\tau^* = \left(\frac{6\ (\psi+\phi)}{B\ \pi^2}\right)^{1/3},\ \hat{p} = \left(\frac{3\ \pi\ (\psi+\phi)}{4\ B}\right)^{1/3}, \text{and } \hat{p}-\underline{p} = \pi\tau = \left(\frac{6\ \pi\ (\psi+\phi)}{B}\right)^{1/3} \quad \text{(A-39)}$$

The value of n_a is obtained as $n_a = 1/\tau^*$. The values for $\tau(\tilde{p})$ are obtained by requiring that the review happens exactly at the time of an adjustment: $\tau(\tilde{p})\pi = \tilde{p}-\underline{p}$ in the range of inaction. The optimal policy has this form because, due to the deterministic evolution of $\tilde{p}(t)$, if $\phi > 0$ it is optimal to review only at the time of an adjustment.

Now consider the menu cost version of this model for $\rho > 0$, $\sigma = 0$ and $\pi > 0$. We use this model to show that for all $\rho > 0$, the optimal return point and boundaries satisfy $\bar{p}-\hat{p} < \hat{p}-\underline{p}$. In the range of inaction $\tilde{p} \in (\underline{p}, \bar{p})$ the value function satisfy the ODE:

$$\rho V(\tilde{p}) = B\tilde{p}^2 - \pi V'(\tilde{p})$$

with value matching, optimality of $\hat{p}$ and smooth pasting conditions:

$$V(\underline{p}) = V(\hat{p})+\psi+\phi,\ V'(\hat{p}) = 0 \text{ and } V'(\underline{p}) = 0\ .$$

The solution of the ODE in the range of inaction is:

$$V(\tilde{p}) = \frac{B}{\rho}\tilde{p}^2 - \frac{\pi}{\rho}\frac{2\ B}{\rho}\tilde{p}+\left(\frac{\pi}{\rho}\right)^2\frac{2\ B}{\rho}+A\ e^{-\rho/\pi\ \tilde{p}}$$

for some constant A to be determined. Let $a_0\tilde{p}+a_1\tilde{p}+a_2\tilde{p}^2$ be the particular solution of the ODE. Its coefficients must solve:

$$\rho\left(a_0+a_1\tilde{p}+a_2\tilde{p}^2\right) = B\tilde{p}^2-\pi\left(a_1+2a_2\tilde{p}\right)$$

and thus

$$a_2 = \frac{B}{\rho}\ ,\ a_1 = -\frac{\pi}{\rho}\frac{2\ B}{\rho}\ ,\ a_0 = \left(\frac{\pi}{\rho}\right)^2\frac{2\ B}{\rho}$$

Notice that the quadratic function showing the particular solution of the ODE is the value of a policy where the fixed cost is never paid. Hence in the range of inaction, where the solution to this ODE must hold, the value function has to be smaller. This implies that the constant A has to be negative. We now use that $A < 0$ implies that $\bar{p} - \hat{p} < \hat{p} - \underline{p}$. To see this, denote the quadratic particular solution of the ODE as $Q(\tilde{p})$, so we have $V'(\tilde{p}) = Q'(\tilde{p}) - A(\rho/\pi)^{-\rho/\pi\tilde{p}}$. This derivative is zero at $\hat{p}$ where the function attains its minimum, and it is positive for higher values and negative for lower ones. From $V'(\hat{p}) = 0$ we obtain:

$$2\ \frac{B}{\rho}\ \hat{p} - \frac{\pi}{\rho}\frac{B}{\rho}\ 2 = \frac{\rho}{\pi}\ A\ e^{-\frac{\rho}{\pi}\ \hat{p}}$$

so that

$$A = \left(\hat{p} - \frac{\pi}{\rho}\right)\ 2\ \frac{B}{\rho}\ \frac{\pi}{\rho}\ e^{\frac{\rho}{\pi}\ \hat{p}}$$

From $V(\bar{p}) - V(\hat{p}) = \phi + \psi$ we obtain:

$$\frac{B}{\rho}(\bar{p}^2 - \hat{p}^2) - \frac{\pi}{\rho}\frac{B}{\rho}2\ (\bar{p} - \hat{p}) + A\ \left(e^{-\frac{\rho}{\pi}\bar{p}} - e^{-\frac{\rho}{\pi}\ \hat{p}}\right) = \phi + \psi$$

and substituting A into this expression we get:

$$\frac{B}{\rho}(\bar{p}^2 - \hat{p}^2) - \frac{\pi}{\rho}\frac{B}{\rho}2\ (\bar{p} - \hat{p}) + \left(\hat{p} - \frac{\pi}{\rho}\right)2\ \frac{B}{\rho}\frac{\pi}{\rho}\ \left(e^{\frac{\rho}{\pi}(\hat{p}-\bar{p})} - 1\right) = \phi + \psi$$

We now obtain an expression for $\bar{p}$ as a function of $\hat{p}$ and the parameters $\pi(\phi + \psi)/B$ by letting ρ to go to zero. To do so, we use that $e^x - 1 = x + (1/2)x^2 + o(x^2)$ and apply it to $e^{\frac{\rho}{\pi}(\hat{p}-\bar{p})} - 1$ obtaining:

$$\bar{p}^2 - \hat{p}^2 - \frac{\pi}{\rho}2\ (\bar{p} - \hat{p}) + \left(\hat{p} - \frac{\pi}{\rho}\right)2\frac{\pi}{\rho}\ \left(\frac{\rho}{\pi}(\hat{p} - \bar{p}) + \frac{1}{2}(\frac{\rho}{\pi})^2(\hat{p} - \bar{p})^2 + o(\rho^2)\right) = \frac{\phi + \psi}{B}\rho$$

In this expression we can simplify the terms that are multiplied by $1/\rho$. To see this we can developed the third term into:

$$\begin{aligned}\frac{\phi + \psi}{B}\rho &= \bar{p}^2 - \hat{p}^2 - \frac{\pi}{\rho}2\ (\bar{p} - \hat{p}) + \left(\hat{p} - \frac{\pi}{\rho}\right)2\ \ (\hat{p} - \bar{p}) \\ &+ \left(\hat{p} - \frac{\pi}{\rho}\right)2\frac{\pi}{\rho}\left(\frac{1}{2}(\frac{\rho}{\pi})^2(\hat{p} - \bar{p})^2 + o(\rho^2)\right)\end{aligned}$$

or canceling the terms multiplied by $1/\rho$

$$\frac{\phi + \psi}{B}\rho = \bar{p}^2 - \hat{p}^2 + \hat{p}\ 2\ \ (\hat{p} - \bar{p}) + \left(\hat{p} - \frac{\pi}{\rho}\right)\left(\frac{\rho}{\pi}(\hat{p} - \bar{p})^2 + o(\rho)\right)$$

developing the parenthesis

$$\frac{\phi + \psi}{B}\rho = \bar{p}^2 - \hat{p}^2 + \hat{p}\ 2\ \ (\hat{p} - \bar{p}) + \hat{p}\frac{\rho}{\pi}(\hat{p} - \bar{p})^2 - (\hat{p} - \bar{p})^2 + o(\rho)$$

developing the square

$$\frac{\phi+\psi}{B}\rho = \bar{p}^2 - \hat{p}^2 + 2\hat{p}^2 - 2\hat{p}\bar{p} + \hat{p}\frac{\rho}{\pi}(\hat{p}-\bar{p})^2 - \hat{p}^2 - \bar{p}^2 + 2\bar{p}\hat{p} + o(\rho)$$

canceling the common terms

$$\frac{\phi+\psi}{B}\rho = \hat{p}\frac{\rho}{\pi}(\hat{p}-\bar{p})^2 + o(\rho)$$

multiplying π/ρ:

$$\frac{\phi+\psi}{B}\pi = \hat{p}\ (\hat{p}-\bar{p})^2 + \frac{o(\rho)}{\rho}$$

and taking ρ to zero:

$$\frac{\phi+\psi}{B}\pi = \hat{p}\ (\hat{p}-\bar{p})^2$$

We can write $\bar{p} = a\ \hat{p}$ for a constant $a > 1$ to be determined:

$$\frac{\phi+\psi}{B}\pi = \hat{p}^3\ (1-a)^2$$

and replacing the expression for $\hat{p}^3$:

$$\frac{\phi+\psi}{B}\pi = \left(\frac{3}{4}\frac{\phi+\psi}{B}\pi\right)(1-a)^2$$

or solving for the positive value of a:

$$\sqrt{\frac{4}{3}} + 1 = a$$

or

$$\bar{p} = \left(\sqrt{\frac{4}{3}} + 1\right)\hat{p}$$

Now we compare $\bar{p} - \hat{p}$ with $\hat{p} - \underline{p}$:

$$\frac{\bar{p}-\hat{p}}{\hat{p}-\underline{p}} = \frac{\sqrt{\frac{4}{3}}}{8^{\frac{1}{3}}} = \frac{1}{\sqrt{3}}$$