Expectations and the law of iterated expectations: an intuitive guide

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Expectations

- Suppose we have a population with a random variable taking on values y_i, for i=1,2,…,N with probability p_i. The population mean is

$$\mu_N = \sum_{i=1}^{N} p_i y_i \equiv E(y_i)$$

Expectations with a continuous random variable

- Suppose we have a population with a random variable taking on continuous values $y_i = g(i)$, for i = [0,1] with probability density dy.
- The probability of any range of values is

$$\Pr(i \le y \le j) = \int_i^j dy$$

- The population mean is

$$\mu_y = \int_i^j g(y)\,dy \equiv E(y_i)$$

Joint expectations

		Price		
		1	2	3
Yield	5	0.16	0.03	0.083
	10	0.14	0.14	0.16
	15	0.03	0.16	0.083

$$E(\mathrm{Re}\,venue) = E(py) = \sum_{y=5,10,15} \sum_{p=1,2,3} p \cdot y \cdot d(py)$$

$$\begin{aligned}
E_p\left(E(y|p)p\right) &= \sum_{p=1,2,3} \sum_{y=5,10,15} p \cdot y \cdot d(py) \\
&= \sum_{p=1,2,3} \left(\sum_{y=5,10,15} y \cdot d(y|p) \right) pd(y) \\
&= (1)(5)(0.16) + (1)(10)(0.14) + (1)(15)(0.03) + \\
&\quad (2)(5)(0.03) + (2)(10)(0.14) + (2)(15)(0.16) + \\
&\quad (3)(5)(0.083) + (3)(10)(0.16) + (3)(15)(0.083) \\
&= \left[(1)(5)(0.5) + (1)(10)(0.4) + (1)(15)(0.1)\right]\left[\tfrac{1}{3}\right] + \\
&\quad \left[(2)(5)(0.1) + (2)(10)(0.4) + (2)(15)(0.5)\right]\left[\tfrac{1}{3}\right] + \\
&\quad \left[(3)(5)(0.25) + (3)(10)(0.5) + (3)(15)(0.25)\right]\left[\tfrac{1}{3}\right] +
\end{aligned}$$

Recall we claimed

$$\begin{aligned}\tau &= E(Y_{1i}) - E(Y_{0i}) \\ &= \Pr(T_i = 1)E(Y_{1i} \mid T_i = 1) + \Pr(T_i = 0)E(Y_{1i} \mid T_i = 0) \\ &\quad - \left[\Pr(T_i = 1)E(Y_{0i} \mid T_i = 1) + \Pr(T_i = 0)E(Y_{0i} \mid T_i = 0)\right]\end{aligned}$$

or

$$E(Y_{1i}) = \Pr(T_i = 1)E(Y_{1i} \mid T_i = 1) + \Pr(T_i = 0)E(Y_{1i} \mid T_i = 0)$$

$$E(Y_{0i}) = \Pr(T_i = 1)E(Y_{0i} \mid T_i = 1) + \Pr(T_i = 0)E(Y_{0i} \mid T_i = 0)$$

Why?

Law of iterated expectations

- $E(z) = E_x(z \mid x)$
- Taking the expectation of a random variable, e.g. *z*, you can condition on some other random variable, *x*, as long as you then take the expectation over *x* as well.
- Easiest to see using summations rather than expectations.

Look at E(Y_{1i})

$$E(Y_{1i}) = \frac{1}{N}\sum_{i=1}^{N} Y_{1i}$$

$$= \frac{1}{N}\left(\sum_{i=1}^{N_1}(Y_{1i}|T_i = 1) + \sum_{i=1}^{N_0}(Y_{1i}|T_i = 0)\right)$$

$$= \frac{N_1}{N}\left(\frac{1}{N_1}\sum_{i=1}^{N_1}(Y_{1i}|T_i = 1)\right) + \frac{N_0}{N}\left(\frac{1}{N_0}\sum_{i=1}^{N_0}(Y_{1i}|T_i = 0)\right), \text{ where } N_1=\#(T_i=1 \text{ obs}), N_0=\#(T_i=0 \text{ obs})$$

$$= \Pr(T_i = 1)E(Y_{1i} \mid T_i = 1) + \Pr(T_i = 0)E(Y_{1i} \mid T_i = 0)$$

$$= E_T(E(Y_{1i} \mid T_i))$$

Why the last step? Recall that if x is a binary variable taking on values x_1 and x_2 then E(x) = Pr(x=x_1)x_1 + Pr(x=x_2)x_2